k-contraction analysis for discrete-time systems

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Abstract:
The definition of k-contraction promises a useful generalization of the classical notion of contraction for dynamical systems. However, most of the k-contraction literature focuses on continuous-time systems. In this work, we derive conditions for k-contractivity of discrete-time dynamics. Our first result follows traditional lines for k-contraction analysis, and provides Lyapunov-like sufficient conditions based on matrix compounds and state-dependent metrics. However, our subsequent results avoid the complexities related to matrix compounds. Inspired by recent findings in the context of k-contraction for continuous-time systems, we provide conditions on the system's dynamics that rely on generalized Lyapunov inequalities and quadratic cone fields. The proposed conditions are also shown to be necessary for linear time-invariant systems.

Keywords: Contraction, Nonlinear systems, Discrete-time systems, Compound matrices.

1. INTRODUCTION

Contraction is a stability property that has been exploited to solve numerous control problems, such as observer design [Sanfelice and Praly, 2012], multi-agent synchronization [Giaccagli et al., 2024], and controller design [Manchester and Slotine, 2017]. In few words, contractivity of system's dynamics implies that the distances between any two trajectories decrease exponentially in time. However, complex asymptotic behavior (e.g., multi-stability and oscillations) cannot be studied via classical contraction theory. This motivated the study of suitable generalizations of such a notion, such as k-contraction [Wu et al., 2022, Cecilia et al., 2023]. k-contraction generalizes the requirement of decreasing distances to the one of decreasing volumes of k-dimensional objects. Hence, k-contraction boils down to classical contraction when $k = 1$. However, it provides interesting additional asymptotic properties for $k > 1$. In 2-contractive systems, every bounded solution converges to an equilibrium point (not necessarily unique) [Li and Muldowney, 1995]. Under some assumptions, bounded trajectories of 3-contractive systems converge to a limit cycle [Cecilia et al., 2023, Lemma 6].

Nonetheless, most of the literature in k-contraction focuses on continuous-time systems. To fill this gap, we aim to define k-contraction in the discrete-time domain, and provide sufficient conditions for its verification. Following the lines of [Wu et al., 2022, Ofir et al., 2022], we first provide a sufficient condition for k-contraction involving matrix compounds of the Jacobian of the dynamics' vector field. However, compound-based conditions are known to rapidly grow in dimension. Moreover, they hinder the derivation of feedback design methodologies [Dalin et al., 2023, Cecilia et al., 2023]. For this reason, we present alternative conditions that do not rely on matrix compounds. Our findings build upon generalized Lyapunov matrix inequalities studied in [Berger and Jungers, 2020, 2021] and are inspired by [Cecilia et al., 2023, Zoboli et al., 2023]. We present necessary and sufficient results for linear dynamics, and sufficient conditions in the nonlinear framework.

Notation: $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. $|\cdot|$ denotes the standard Euclidean norm. $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ depicts the binomial coefficient, with $n!$ the factorial of $n \in \mathbb{N}$. Let $\circ$ be the composition operator. For all $\mathfrak{t} \in \mathbb{N}$, we define the discrete-time flow operator $f^{\mathfrak{t}} := f \circ f \circ \cdots \circ f$, where f is applied $\mathfrak{t}$ times. Then, $\psi^0(x) = x$. The inertia of a matrix with respect to the imaginary axis is denoted by the triplet of integers $\mathrm{In_c}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$, where $\pi_-(P)$, $\pi_+(P)$ and $\pi_0(P)$ denote the numbers of eigenvalues of P with negative, positive and zero real part, resp., counting multiplicities. The inertia of a matrix with respect to the unitary disk is denoted by the triplet of integers $\mathrm{In_d}(P) := (\pi_{<1}(P), \pi_1(P), \pi_{>1}(P))$, where $\pi_{<1}(P)$, $\pi_{>1}(P)$ and $\pi_1(P)$ denote the numbers of eigenvalues of P inside, outside and on the unit circle, resp., counting multiplicities. The cardinality of a set is denoted as $\mathrm{card}(\cdot)$.

2. ON DISCRETE-TIME k-CONTRACTION

2.1 Definition of k-contraction

We consider discrete-time nonlinear systems of the form

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$$x^+ = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

with f being sufficiently smooth vector field and x being the state. In the following, we adapt the definition of k-contraction introduced in [Zoboli et al., 2023, Cecilia et al., 2023] for continuous-time systems to the discrete-time case. In particular, consider a set of sufficiently smooth functions $\mathcal{I}_k$ defined on $[0,1]^k$, namely

$$\mathcal{I}_k := \big\{\Phi : [0,1]^k \to \mathbb{R}^n \mid \ \Phi \text{ is a smooth immersion}\big\}. \tag{2}$$

Let $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a matrix-valued function

$$\bar{p}I \succ P(x) \succ \underline{p}I, \qquad \forall x \in \mathbb{R}^n,$$

satisfying for some positive constants $\bar{p}, \underline{p} > 0$. For each Φ in $\mathcal{I}_k$, we define the k-order volume of Φ as

$$V^k(\Phi) := \int_{[0,1]^k} \sqrt{\det\left\{\frac{\partial \Phi}{\partial r}(r)^\top P(\Phi(r)) \frac{\partial \Phi}{\partial r}(r)\right\}}\, \mathrm{d}r\,. \tag{3}$$

We now define the k-contraction property for systems of the form (1), which will be considered in this article. Throughout the paper, we will assume $k \in [1, \dots, n]$.

Definition 1 (k-contr.). *System* (1) *is k-contractive on a forward invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist real numbers $\rho \in (0,1)$ and $\lambda > 0$ such that, for every $\Phi \in \mathcal{I}_k$ satisfying* $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$*, the following holds*

$$V^k(f^{\mathfrak{t}} \circ \Phi) \leq \lambda \rho^{\mathfrak{t}} V^k(\Phi), \qquad \forall \mathfrak{t} \in \mathbb{N}. \tag{4}$$

In other words, a system is k-contractive if, for any parametrized k-dimensional submanifold of initial conditions, the volume contracts along the system trajectories. This definition is the discrete-time analog of the definition provided in [Zoboli et al., 2023, Cecilia et al., 2023]. Nonetheless, it presents two major differences. Firstly, the volume $V^k(\cdot)$ can be weighted by a state-dependent metric $P(x)$, while the condition in [Zoboli et al., 2023, Cecilia et al., 2023] only considered the constant metric case. Secondly, the metric $P(x)$ does not need to be differentiable.

2.2 Sufficient conditions based on matrix compounds

For continuous-time systems, sufficient conditions for k-contraction were originally presented in the seminal work [Muldowney, 1990], and they have been recently revisited in [Wu et al., 2022, Cecilia et al., 2023, Angeli et al., 2022]. These conditions strongly depended on the use of a mathematical tool known as matrix compound, see [Bar-Shalom et al., 2023] for a comprehensive review on the topic. In discrete-time k-contraction, multiplicative compounds [Bar-Shalom et al., 2023] turn out to be predominant.

Definition 2 (Mult. Compound). *Consider a matrix $Q \in \mathbb{R}^{n\times m}$ and select an integer $k \in [1, \min\{n,m\}]$. Moreover, define a minor of order k of the matrix Q as the determinant of some $k \times k$ submatrix of Q. The k-th multiplicative compound of Q, denoted as $Q^{(k)}$, is the $\binom{n}{k} \times \binom{m}{k}$ matrix including all the minors of order k of Q in a lexicographic order.*

Bearing this definition in mind, we provide a sufficient condition for k-contraction in the following theorem.

Theorem 1. *System* (1) *is k-contractive on a forward invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist a symmetric matrix-valued function $Q : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$ and two constants $\bar{q}, \underline{q} > 0$ satisfying*

$$\bar{q}I \succ Q(x) \succ \underline{q}I, \qquad \forall x \in \mathbb{R}^n,$$

and a real number $\eta \in (0,1)$ such that for all $x \in \mathcal{S}$

$$\left(\frac{\partial f}{\partial x}(x)^{(k)}\right)^\top Q(f(x)) \left(\frac{\partial f}{\partial x}(x)^{(k)}\right) \preceq \eta^2 Q(x). \tag{5}$$

Proof: Consider any $\Phi \in \mathcal{I}_k$, where $\mathcal{I}_k$ is defined in (2), satisfying $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$. To simplify notation, let us define for all $(r, \mathfrak{t})$ in $[0,1]^k \times \mathbb{N}$

$$\Gamma(r,\mathfrak{t}) := f^{\mathfrak{t}} \circ \Phi(r)\ ,\ \Gamma_r(r,\mathfrak{t}) := \frac{\partial \Gamma}{\partial r}(r,\mathfrak{t}).$$

By definition, we have

$$\Gamma(r, \mathfrak{t}+1) = f(\Gamma(r,\mathfrak{t})), \quad \forall (r,\mathfrak{t}) \in [0,1]^k \times \mathbb{N}. \tag{6}$$

Moreover, since $\mathcal{S}$ is forward invariant and $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$, we have $\Gamma(r,\mathfrak{t}) \in \mathcal{S}$ for all $(r,\mathfrak{t})$ in $[0,1]^k \times \mathbb{N}$. Additionally, by the chain rule, $\Gamma_r(r,\mathfrak{t})$ evolves according to

$$\begin{aligned}\Gamma_r(r,\mathfrak{t}+1) &= \frac{\partial \Gamma}{\partial r}(r,\mathfrak{t}+1) = \frac{\partial f}{\partial r}(\Gamma(r,\mathfrak{t})) \\ &= \frac{\partial f}{\partial x}(\Gamma(r,\mathfrak{t})) \frac{\partial \Gamma(r,\mathfrak{t})}{\partial r} = \frac{\partial f}{\partial x}(\Gamma(r,\mathfrak{t}))\Gamma_r(r,\mathfrak{t})\end{aligned}$$

for all $(r,\mathfrak{t})$ in $[0,1]^k \times \mathbb{N}$. Finally, by the Cauchy-Binet formula [Fallat and Johnson, 2022, Chapter 1], we obtain

$$\Gamma_r(r,\mathfrak{t}+1)^{(k)} = \frac{\partial f}{\partial x}(\Gamma(r,\mathfrak{t}))^{(k)} \Gamma_r(r,\mathfrak{t})^{(k)}. \tag{7}$$

Define $Q(x) := P(x)^{(k)}$ for all $x \in \mathcal{S}$, with $P(x)$ as in (3). Since $\Gamma_r(r,t) \in \mathbb{R}^{n\times k}$, from the Cauchy-Binet formula the following equality holds

$$\begin{aligned}&\det\Big(\Gamma_r(r,\mathfrak{t})^\top P(\Gamma(r,\mathfrak{t}))\,\Gamma_r(r,\mathfrak{t})\Big) \\ &\quad = \Big(\Gamma_r(r,\mathfrak{t})^{(k)}\Big)^\top Q(\Gamma(r,\mathfrak{t}))\,\Gamma_r(r,\mathfrak{t})^{(k)} := v(r,\mathfrak{t}).\end{aligned} \tag{8}$$

Then, the volume of $f^{\mathfrak{t}} \circ \Phi$ computed according to (3) is

$$V^k(f^{\mathfrak{t}} \circ \Phi) = \int_{[0,1]^k} \sqrt{v(r,\mathfrak{t})}\mathrm{d}r\,.$$

Moreover, it evolves according to

$$\begin{aligned}V^k(\psi^{\mathfrak{t}+1} \circ \Phi) &= \int_{[0,1]^k} \sqrt{v(r,\mathfrak{t}+1)}\,\mathrm{d}r \\ &= \int_{[0,1]^k} \sqrt{\big(\Gamma_r(r,\mathfrak{t}+1)^{(k)}\big)^\top Q(\Gamma(r,\mathfrak{t}+1))\Gamma_r(r,\mathfrak{t}+1)^{(k)}}\mathrm{d}r.\end{aligned}$$

Hence, adapting to discrete-time the arguments of [Zoboli et al., 2023, Theorem 3] and by means of (6) and (7), for all $(r,\mathfrak{t})$ in $[0,1]^k \times \mathbb{N}$, we obtain

$$\begin{aligned}V^k(\psi^{\mathfrak{t}+1} \circ \Phi) = \int_{[0,1]^k} &\sqrt{\left(\frac{\partial f}{\partial x}(\Gamma(r,\mathfrak{t}))^{(k)}\Gamma_r(r,\mathfrak{t})^{(k)}\right)^\top} \\ &\times \sqrt{Q(f(\Gamma(r,\mathfrak{t})))\frac{\partial f}{\partial x}(\Gamma(r,\mathfrak{t}))^{(k)}\Gamma_r(r,\mathfrak{t})^{(k)}}\,\mathrm{d}r.\end{aligned} \tag{9}$$

Then, invoking inequality (5) and recalling that $\Gamma(r,\mathfrak{t}) \in \mathcal{S}$ for all $(r,\mathfrak{t})$ in $[0,1]^k \times \mathbb{N}$, the previous relation implies

$$V^k(\psi^{\mathfrak{t}+1} \circ \Phi) \leq \eta \int_{[0,1]^k} \sqrt{v(r,\mathfrak{t})}\,\mathrm{d}r \leq \eta V^k(f^{\mathfrak{t}} \circ \Phi)\,.$$

Then, by recursively applying the above relation we obtain

$$\begin{aligned}V^k(\psi^{\mathfrak{t}+i} \circ \Phi) &\leq \eta V^k(\psi^{\mathfrak{t}+i-1} \circ \Phi) \leq \eta^2 V^k(\psi^{\mathfrak{t}+i-2} \circ \Phi) \\ &\leq \cdots \leq \eta^i V^k(\psi^{\mathfrak{t}} \circ \Phi) \quad \forall i \in \mathbb{N}\,,\end{aligned}$$

thus concluding the proof. □

Notably, Theorem 3 presents sufficient conditions for k-contraction based on state-dependent metrics $Q(x)$, while the (continuous-time) conditions in [Cecilia et al., 2023, Wu et al., 2022] only consider constant metrics. This generalization applies to a wider class of system, as hinted by recent results on contraction analysis based on Riemannian metrics, e.g. [Andrieu et al., 2020].

However, we highlight that the presence of multiplicative compounds in (5) reduces the applicability of Theorem 1. Firstly, matrix compounds rapidly explode in dimension for low value of k and systems of large system dimension. This fact is of particular relevance since, in practice, conditions of the form (5) relying on matrix-valued functions are often solved with numerical methods, which may suffer from the curse of dimensionality. Secondly, the use of matrix compounds hinders the derivation of tractable matrix inequality problems for feedback design. More details on these points can be found in [Dalin et al., 2023] and [Cecilia et al., 2023, Section II-B]. Therefore, we devote the next sections to the derivation of sufficient conditions for k-contraction avoiding matrix compounds. To illustrate the rationale behind the proposed conditions, we first consider the linear framework.

3. k-CONTRACTION FOR LINEAR SYSTEMS

We consider discrete-time linear systems of the form

$$x^+ = Ax, \qquad x \in \mathbb{R}^n. \tag{10}$$

Without loss of generality, we will consider a constant metric P in (3). Moreover, we assume the eigenvalues of the matrix A in (10) to be ordered such that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|. \tag{11}$$

We now provide a set of sufficient and necessary conditions guaranteeing that (10) is k-contractive according to Definition 1. This result is based on the following two facts:

1) System (10) is k-contractive if and only if the product of the norms of any combination of k-eigenvalues of A is strictly smaller than 1;
2) Inequalities of the form $A^\top P A \prec \mu^2 P$ admit a symmetric solution P of inertia $\mathrm{In_c}(P) = (p, 0, n-p)$ if and only if A has p eigenvalues with norm larger than μ and $n-p$ with norm smaller than μ.

We start by presenting the relation between the eigenvalues of A and k-contraction.

Lemma 1. *Consider system* (10)*. The following statements are equivalent:*

1) System (10) *is k-contractive.*
2) $A^{(k)}$ is Schur.
3) The eigenvalues of A, ordered according to (11)*, satisfy*

$$\prod_{i=1}^{k} |\lambda_i| < 1 \ . \tag{12}$$

Proof: We start by proving 1) $\iff$ 2), in particular the implication 1) $\implies$ 2). By following the same steps as in the proof of Theorem 1 up to equality (9) and recalling Definition 1, if system (10) is k-contractive there exist $\rho \in (0,1)$ and $\lambda > 0$ such that

$$\begin{aligned} V^k(\psi^{\mathfrak{t}+1} \circ \Phi) &= \int_{[0,1]^k} \sqrt{\left(\Gamma_r(r,0)^{(k)}\right)^\top H(\mathfrak{t}) \Gamma_r(r,0)^{(k)}}\, \mathrm{d}r \\ &\leq \lambda \rho^{\mathfrak{t}+1}\, V^k(\Phi), \end{aligned} \tag{13}$$

where we defined $H(\mathfrak{t}) := ((A^{(k)})^{\mathfrak{t}+1})^\top P^{(k)} (A^{(k)})^{\mathfrak{t}+1}$. Since (13) holds for any Φ defined in (2), it must hold for the particular case of $\mathrm{Im}(\Phi)$ being a parallelotope with vertices on any set of $k+1$ initial conditions of (10). That is, pick $k+1$ initial conditions $x_0^1, \ldots, x_0^{k+1} \in \mathbb{R}^n$ and define

$$\Phi_p(r) = \sum_{i=1}^{k} r_i x_0^i + \left(1 - \sum_{i=1}^{k} r_i\right) x_0^{k+1}, \tag{14}$$

with $r_i \in [0,1]$ for $i \in \{1, \ldots, k\}$ being the i-th component of r. Under this specific selection, $\Gamma_r(r,0)$ is a term that does not depend on r. Namely,

$$\Gamma_r(r,0) = \frac{\partial \Phi_p}{\partial r}(r) = \begin{bmatrix} x_0^1 - x_0^{k+1} & x_0^2 - x_0^{k+1} & \ldots & x_0^k - x_0^{k+1} \end{bmatrix}.$$

Consequently, since the terms inside the square roots in (13) are positive and independent of r, (13) with the specific immersion (14) implies

$$(\Gamma_r(r,0)^{(k)})^\top (H(\mathfrak{t}) - \lambda^2 \rho^{2(\mathfrak{t}+1)} P^{(k)}) \Gamma_r(r,0)^{(k)} \leq 0, \ \forall \mathfrak{t} \in \mathbb{N}.$$

Since $\lambda, \rho > 0$ and $\mathfrak{t} \in \mathbb{N}$, the above inequality implies

$$(A^{(k)})^\top (P^{(k)})^{\frac{1}{\mathfrak{t}+1}} A^{(k)} \preceq \lambda^{\frac{2}{\mathfrak{t}+1}} \rho^2 (P^{(k)})^{\frac{1}{\mathfrak{t}+1}}.$$

Since $P \succ 0$ and $\mathfrak{t} \in \mathbb{N}$, we have that $(P^{(k)})^{\frac{1}{\mathfrak{t}+1}} \succ 0$, thus showing that $A^{(k)}$ is Schur. To show 2) $\implies$ 1), notice that if $A^{(k)}$ is Schur stable, there exists $\eta \in (0,1)$ and a matrix $Q \succ 0$ such that (5) holds. Then, system (10) is k-contractive by Theorem 1.

We now show 2) $\iff$ 3). A spectral property of the multiplicative compound matrix is that the eigenvalues of the matrix $A^{(k)}$ are all the possible products of the form $\lambda_{i_1} \lambda_{i_2} \cdot \cdots \cdot \lambda_{i_k}$, with $1 \leq i_1 < \cdots < i_k \leq n$ [Wu et al., 2022, Section 2.3]. Hence, $A^{(k)}$ is Schur if and only if the product of any combination of k eigenvalues of A is inside the unit disc. In particular, due to the ordering (11), this holds if and only if the first k eigenvalues satisfy condition (12), thus concluding the proof. □

The following lemma presents inertia properties of Lyapunov inequalities of the form (16a).

Lemma 2. *Given a matrix $A \in \mathbb{R}^{n \times n}$, a positive constant μ, and an integer $p \in \{0, \ldots, n\}$, the following statements are equivalent:*

1) A has p eigenvalues with norm larger than μ and $n-p$ eigenvalues with norm smaller than μ,
2) the matrix $\hat{A} := \mu^{-1} A$ satisfies $\mathrm{In_d}(\hat{A}) = (n-p, 0, p)$,
3) there exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ with inertia $\mathrm{In_c}(P) = (p, 0, n-p)$ satisfying

$$A^\top P A \prec \mu^2 P, \tag{15}$$

4) for any $Q \succ 0$, there exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ with inertia $\mathrm{In_c}(P) = (p, 0, n-p)$ satisfying $\hat{A}^\top P \hat{A} = P - Q$, with $\hat{A} := \mu^{-1} A$.

Proof: The eigenvalues of $\mu^{-1} A$ are the scaled eigenvalues $\mu^{-1}\lambda_1, \ldots, \mu^{-1}\lambda_n$, where eigenvalues are ordered according to (11). If $\mu^{-1} A$ has inertia $(n-p, 0, p)$, it implies that $\mu^{-1}|\lambda_{p+1}| < 1$ and $\mu^{-1}|\lambda_p| > 1$. Due to the ordering (11), this shows that 1) $\Leftrightarrow$ 2). The implication 2) $\Leftrightarrow$ 3) follows from [Lancaster and Tismenetsky, 1985, Section

13.2, Theorem 2] and exploiting the strict positivity of the right-hand side matrix in [Lancaster and Tismenetsky, 1985, Section 13.2, eq. (2)]. Similarly, the implication 2) $\Leftrightarrow$ 4) can be found, e.g., in [Lancaster and Tismenetsky, 1985, Section 13.2, Theorem 2] by considering the $\hat{A}$. $\square$

Combining the properties presented in Lemma 1 and Lemma 2, we derive the following main result.

Theorem 2. *System* (10) *is k-contractive in* $\mathbb{R}^n$ *if and only if there exist:*

- *a positive integer* $\ell \in \mathbb{N}$ *satisfying* $1 \leq \ell \leq k$,
- ℓ *strictly positive real numbers* $\mu_i \in \mathbb{R}_{>0}$, *with* $i \in \{0, \dots, \ell-1\}$,
- ℓ *positive integers* $d_i \in \mathbb{N}$, *with* $i \in \{0, \dots, \ell-1\}$, *satisfying*
$$0 = d_0 < d_1 < \cdots < d_{\ell-1} \leq k-1,$$
- *and* ℓ *symmetric matrices* $P_i \in \mathbb{R}^{n\times n}$ *of respective inertia* $\mathrm{In_c}(P_i) = (d_i, 0, n-d_i)$, *with* $i \in \{0, \dots, \ell-1\}$,

such that

$$A^\top P_i A \prec \mu_i^2 P_i, \qquad \forall i \in \{0, \dots, \ell-1\}, \tag{16a}$$

$$\prod_{i=0}^{\ell-1} \mu_i^{h_i} \leq 1\,, \tag{16b}$$

where $h_0 \geq 1$ *and* $h_i = d_{i+1} - d_i$, *for all* $i = \{0, \dots, \ell-1\}$ *with* $d_\ell \in \mathbb{N}$ *satisfying* $d_{\ell-1} + 1 \leq d_\ell \leq k$.

Proof: Adapting to discrete-time the proof of [Cecilia et al., 2023, Theorem 2], we introduce the following notation to represent the eigenvalues of A and their associated multiplicities relative to the norm. Consider the matrix A in (10) and let $\Pi : \mathbb{C} \to \mathbb{R}$ denote the 2-norm operator, namely $\Pi(\lambda) = |\lambda|$. Let $\sigma(A)$ be the spectrum of A and suppose $\Pi(\sigma(A)) = \{\alpha_1, \alpha_2, \dots, \alpha_q\}$ $(q \leq n)$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_q$. Set $\bar{h}_i = \mathrm{card}\left(\Pi^{-1}(\alpha_{i+1}) \bigcap \sigma(A)\right)$, where eigenvalues have been counted with their multiplicities (so that $\bar{h}_0 + \bar{h}_1 + \cdots + \bar{h}_{q-1} = n$). Finally, let $\bar{d}_0 = 0$ and $\bar{d}_i = \sum_{j=0}^{i-1} \bar{h}_j$, for each $i \in \{1, \dots, q-1\}$.

Sufficiency. In order to prove the sufficiency, we will show that the set of inequalities (16) implies the condition (12). To this end, notice that a solution of (16a) for some $\mu_i \in \mathbb{R}$ and P_i with inertia $\mathrm{In_c}(P_i) = (d_i, 0, n-d_i)$ implies that $\mathrm{In_d}(\mu^{-1}A) = (n-d_i, 0, d_i)$ by Lemma 2. In other words, A has only d_i eigenvalues with norm strictly larger than μ_i or, equivalently,

$$|\lambda_{d_{i+1}}| \leq |\lambda_{d_i+1}| < \mu_i, \quad \forall i \in \{0, \dots, \ell-1\}. \tag{17}$$

Invoking again Lemma 2, we have that $\mu_{i+1} < \mu_i$ for all $i \in \{0, \dots, \ell-2\}$. Therefore, bound (16b) implies $\mu_{\ell-1} \leq 1$ and, in turn, $|\lambda_{d_\ell}| < 1$ by inequality (17). Moreover, due to the eigenvalue ordering (11), the following bound trivially holds for all $i \in \{0, \dots, \ell-1\}$

$$\prod_{j=d_i+1}^{d_{i+1}} |\lambda_j| \leq |\lambda_{d_i+1}|^{d_{i+1}-d_i} = |\lambda_{d_i+1}|^{h_i},$$

where we used the definition $h_i := d_{i+1} - d_i$. As a consequence, since $|\lambda_{d_\ell}| < 1$ and $d_\ell \leq k$, we obtain the following inequalities

$$\prod_{i=1}^{k} |\lambda_i| \leq \prod_{i=1}^{d_\ell} |\lambda_i| = \prod_{i=0}^{\ell-1} \prod_{j=d_i+1}^{d_{i+1}} |\lambda_j| \leq \prod_{i=0}^{\ell-1} |\lambda_{d_i+1}|^{h_i}. \tag{18}$$

Then, combining (17), (18) and (16b) we have

$$\prod_{i=1}^{k} |\lambda_i| < \prod_{i=0}^{\ell-1} \mu_i^{h_i} \leq 1\,, \tag{19}$$

showing k-contractiveness of system (10) by Lemma 1.

Necessity. By recalling that $\bar{d}_{q-1} > \cdots > \bar{d}_0 = 0$, define

$$\begin{aligned} p_k &:= \max\left(\bar{d}_{q-1}, k-1\right), \\ c_k &:= \mathrm{card}\left(\{\bar{d}_0, \bar{d}_1, \dots, \bar{d}_{q-1}\} \bigcap [0, k-1]\right). \end{aligned} \tag{20}$$

Then, the following equality holds

$$\alpha_{c_k}^{k-p_k} \prod_{i=0}^{c_k-2} \alpha_{i+1}^{h_i} = \prod_{i=1}^{k} |\lambda_i|.$$

Hence, by Lemma 1, if the system is k-contractive, the following bound must hold

$$\alpha_{c_k}^{k-p_k} \prod_{i=0}^{c_k-2} \alpha_{i+1}^{h_i} < 1. \tag{21}$$

Then, by continuity, there exists a scalar $\varepsilon > 1$, such that

$$(\varepsilon\alpha_{c_k})^{k-p_k} \prod_{i=0}^{c_k-2} (\varepsilon\,\alpha_{i+1})^{h_i} \leq 1.$$

By selecting $\ell = c_k$, $d_\ell = k - p_k + d_{\ell-1}$, $d_i = \bar{d}_i$ for all $i \in \{0, \dots, \ell-1\}$ and defining $\mu_{i-1} := \varepsilon\alpha_i$, for all $i \in \{1, \dots, \ell\}$, we obtain

$$\prod_{i=0}^{\ell-1} \mu_i^{h_i} = (\varepsilon\alpha_{c_k})^{k-p_k} \prod_{i=0}^{c_k-2} (\varepsilon\,\alpha_{i+1})^{h_i} \leq 1,$$

thus showing (16b). Now, define matrices $\hat{A}_i := \mu_i^{-1} A$ with $i \in \{0, \dots, \ell-1\}$. It is clear that, since $\mu_{i-1} > \alpha_i$ by definition, each matrix $\hat{A}_i$ has $\bar{d}_i$ eigenvalues outside the unit disc and $n - \bar{d}_i$ eigenvalues inside the unit disc. That is $\mathrm{In_d}(\hat{A}_i) = (n-\bar{d}_i, 0, \bar{d}_i)$. Then, by Lemma 2, there exist P_i with $\mathrm{In_c}(P_i) = (\bar{d}_i, 0, n-\bar{d}_i)$ such that $A^\top P_i A \prec \mu_i^2 P_i$ for all $i = 0, \dots, \ell-1$, thus concluding the proof. $\square$

4. k-CONTRACTION FOR NONLINEAR SYSTEMS

In this section, we generalize the result of Theorem 2 to nonlinear systems of the form (1). Naturally, the eigenvalue interpretation of Theorem 2 cannot be used for the nonlinear case. Nonetheless, it can be reformulated as an argument on contracting and expanding subspaces of the tangent space along the trajectories of the nonlinear dynamics. This intuition is the foundation of the following theorem, which provides a sufficient condition for k-contractivity of nonlinear systems of the form (1). Our result is restricted to systems evolving in compact sets and having a diffemorphic vector field f. We also highlight that our conditions are presented with state-dependant matrices P_i.

Theorem 3. *Let* $\mathcal{S} \subsetneq \mathbb{R}^n$ *be a compact forward invariant set and assume* f *in* (1) *is a diffeomorphism. Moreover, suppose there exist two symmetric matrix-valued functions* $P_0, P_{k-1} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *of respective inertia* $\mathrm{In_c}(P_0(x)) = (0, 0, n)$ *and* $\mathrm{In_c}(P_{k-1}(x)) = (k-1, 0, n-k+1)$ *for all* $x \in \mathcal{S}$, *which are bounded according to*

$$|x^\top P_i(x) x| \leq p_i |x|^2, \quad i \in \{0, k-1\}, \quad \forall x \in \mathcal{S} \tag{22}$$

for some positive real constants $p_0, p_{k-1} > 0$. Finally, assume there exist strictly positive constants $\mu_0, \mu_{k-1} \in \mathbb{R}_{>0}$ such that

$$\frac{\partial f}{\partial x}(x)^\top P_0(f(x))\frac{\partial f}{\partial x}(x) \prec \mu_0^2 P_0(x), \tag{23a}$$

$$\frac{\partial f}{\partial x}(x)^\top P_{k-1}(f(x))\frac{\partial f}{\partial x}(x) \prec \mu_{k-1}^2 P_{k-1}(x) \tag{23b}$$

$$\mu_{k-1}\mu_0^{k-1} < 1, \tag{23c}$$

for all $x \in \mathcal{S}$. Then, system (1) is k-contractive on $\mathcal{S}$.

The proof of Theorem 3 is derived in the following subsections. We begin by providing preliminary results required to understand and prove Theorem 3. In Section 4.2, we provide the theorem's proof.

4.1 Preliminary technical results

We provide in this section some preliminary results that will be used for the proof of Theorem 3. For these proofs, we will study the linearization of (1) along a trajectory $f^{\mathtt{t}}(x_0)$, that is

$$v^+ = \frac{\partial f}{\partial x}(f^{\mathtt{t}}(x_0))\, v, \quad v \in T_{f^{\mathtt{t}}(x_0)}\mathbb{R}^n = \mathbb{R}^n. \tag{24}$$

We will denote $\frac{\partial f}{\partial x}^{\mathtt{t}}(x_0)v_0$ as a trajectory of (24) at time $\mathtt{t}$ initialized at v_0 at $\mathtt{t} = 0$. We now recall (with a mild reformulation) a result on the quadratic cone field criterion [Berger and Jungers, 2020, Theorem 1]. [1]

Theorem 4. *Consider system (1) and let $\mathcal{S} \subsetneq \mathbb{R}^n$ be a compact forward invariant set. Assume f is a diffeomorphism and suppose there exist a bounded symmetric matrix-valued function $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ of inertia $\mathrm{In_c}(P(x)) = (k-1, 0, n-k+1)$ for all $x \in \mathcal{S}$ and satisfying (22). Moreover, assume there exists a strictly positive real constant $\mu \in \mathbb{R}_{>0}$ such that*

$$\frac{\partial f}{\partial x}(x)^\top P(f(x))\frac{\partial f}{\partial x}(x) \prec \mu^2 P(x), \quad \forall x \in \mathcal{S}. \tag{25}$$

Then, for all $x \in \mathcal{S}$, there exists an invariant splitting $T_x\mathbb{R}^n = \mathcal{V}_x \oplus \mathcal{H}_x$, i.e. there exists a continuous mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ invertible for any $x \in \mathcal{S}$ and satisfying

$$\mathbf{T}(x) := [\mathbf{T_h}(x)\ \mathbf{T_v}(x)], \tag{26}$$

where $\mathbf{T_h} : \mathbb{R}^n \to \mathbb{R}^{n\times n-k+1}$ and $\mathbf{T_v} : \mathbb{R}^n \to \mathbb{R}^{n\times k-1}$ satisfy

$$\mathrm{Im}\ \mathbf{T_h}(x) = \mathcal{H}_x, \quad \mathrm{Im}\ \mathbf{T_v}(x) = \mathcal{V}_x. \tag{27}$$

Moreover, there exist a scalar $c_h > 0$ such that

$$\left|\frac{\partial f}{\partial x}^{\mathtt{t}}(x)\,[\mathbf{T_h}(x)\ 0]\, v\right| \le c_h\mu^{\mathtt{t}}\,|[\mathbf{T_h}(x)\ 0]\, v|, \tag{28}$$

holds for all $(\mathtt{t}, x, v) \in \mathbb{N} \times \mathcal{S} \times T_x\mathbb{R}^n$.

Notice that inequality (23b) with μ_{k-1} strictly smaller than 1, implies contraction in a subspace of dimension $n-k+1$ of the tangent space for all $x \in \mathcal{S}$ by Theorem 4. Nonetheless, this is not a sufficient condition for k-contraction, since the unstable subspace could be expanding at faster rate. This motivates (23a). We clarify the effects of (23a) via the following lemma.

[1] In [Berger and Jungers, 2020], Theorem 4 is proposed as a non-strict inequality of the form $\frac{\partial f}{\partial x}(x)^\top P(f(x))\frac{\partial f}{\partial x}(x) \preceq -\mu^2 P(x) - \varepsilon I$ for all $x \in \mathcal{S}$ and some positive constant ε. Nonetheless, since we are considering compact sets and $P_i(x)$ for $i \in \{1, k-1\}$ are bounded, the strict formulation (25) allows for the same conclusions.

Lemma 3. *Consider system (1), suppose f is a diffeomorphism and assume there exist a forward invariant compact set $\mathcal{S} \subsetneq \mathbb{R}^n$, a bounded positive definite matrix-valued function $P_0 : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfying (22), and a scalar μ_0 satisfying (23a) for all $x \in \mathcal{S}$. Then there exists a constant $c_v > 0$ such that*

$$\left|\frac{\partial f}{\partial x}^{\mathtt{t}}(x)v\right| \le c_v\mu_0^{\mathtt{t}}\,|v|, \quad \forall(\mathtt{t}, x, v) \in \mathbb{N} \times \mathcal{S} \times T_x\mathbb{R}^n. \tag{29}$$

Proof: The proof directly follows from Theorem 4 by noting that, since $P_0(x)$ is strictly positive and bounded for all $x \in \mathcal{S}$, we have that $\mathcal{H}_x = T_x\mathbb{R}^n$. □

Condition (23a) can be seen as imposing a bound on the maximum expansion rate of the subspace $\mathcal{V}_x$ in (27), which is related to the contraction rate of the subspace $\mathcal{H}_x$ by (23c). We now link this property to k-contraction. As a first step, we present a technical lemma related to matrix compounds.

Lemma 4. *Consider a time-varying matrix $M(\mathtt{t}) \in \mathbb{R}^{n\times n}$*

$$M(\mathtt{t}) = [H(\mathtt{t})\ V(\mathtt{t})],$$

with $H(\mathtt{t}) \in \mathbb{R}^{n\times n-p}$, $V(\mathtt{t}) \in \mathbb{R}^{n\times p}$ and $p \in [0, n)$. Assume there exist real numbers $c_h, c_v > 0$, a real constant $\alpha \in (0, 1)$ and a real constant $\beta \ge 1$ such that

$$|H(\mathtt{t})| \le c_h\alpha^{\mathtt{t}}, \quad |V(\mathtt{t})| \le c_v\beta^{\mathtt{t}}, \qquad \forall\mathtt{t} \in \mathbb{N}. \tag{30}$$

If $\alpha\beta^{k-1} < 1$ for some integer $k \in [p+1, n]$, there exist some real numbers $c > 0$ and $\varepsilon \in (0, 1)$ such that

$$|M(\mathtt{t})^{(k)}| \le c\varepsilon^{\mathtt{t}}, \qquad \forall\mathtt{t} \in \mathbb{N}. \tag{31}$$

Proof: Each element of $M(\mathtt{t})^{(k)}$ is a k^{th}-order minor of the original matrix $M(\mathtt{t})$, i.e., it is the determinant of a $k \times k$ submatrix of $M(\mathtt{t})$, see Definition 2. Since $k \ge p+1$, each $k \times k$ submatrix contains at least one column composed of elements of $H(\mathtt{t})$. That is, in the minimum case

$$M_k(\mathtt{t}) = [h(\mathtt{t})\ v_1(\mathtt{t})\ \dots\ v_{k-1}(\mathtt{t})], \tag{32}$$

where $M_k(\mathtt{t}) \in \mathbb{R}^{k\times k}$ is a submatrix of $M(\mathtt{t})$, $h(\mathtt{t}) \in \mathbb{R}^k$ is a vector with components of $H(\mathtt{t})$ and $v_i(\mathtt{t}) \in \mathbb{R}^k$ for $i = 1, \dots, k-1$ is a vector with components of $V(\mathtt{t})$. By following [Cecilia et al., 2023, Proof of Lemma 13] up to equation (65), it holds

$$\det(M_k(\mathtt{t})) = \sum_{i_1=1}^{k} \cdots \sum_{i_k=1}^{k} h^{i_1}(\mathtt{t})v_2^{i_2}(\mathtt{t}) \dots v_{k-1}^{i_k}(\mathtt{t})E_k, \tag{33}$$

where $E_k := (e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k})$, e^i are the elements of the canonical vector in $\mathbb{R}^n$ and v_j^i, h^i are the i_{th} element of the vectors v_j, h, respectively. By (30), we have

$$|h^i(\mathtt{t})| \le c_h\alpha^{\mathtt{t}}, \quad |v^i(\mathtt{t})| \le c_v\beta^{\mathtt{t}}.$$

Moreover, the factor E_k will be either zero or an element of the canonical basis in $\mathbb{R}^n$ multiplied by plus or minus one. Thus, using the triangle inequality, one obtains

$$|\det(M_k(\mathtt{t}))| \le \kappa c_h c_v^{k-1}(\alpha\beta^{(k-1)})^{\mathtt{t}}$$

where $\kappa > 0$ is a positive constant related to the number of non-zero instances of E_k. Now, since $\alpha\beta^{(k-1)} < 1$ by assumption, by continuity there always exists $\varepsilon \in (0, 1)$ such that $\varepsilon^{-1}\alpha\beta^{(k-1)} < 1$. Then,

$$|M(\mathtt{t})^{(k)}| = |\varepsilon^{\mathtt{t}}\varepsilon^{-\mathtt{t}}M(\mathtt{t})^{(k)}| \le \varepsilon^{\mathtt{t}}|\varepsilon^{-\mathtt{t}}M(\mathtt{t})^{(k)}|.$$

By considering the worst-case (32), we have

$$\varepsilon^{\mathtt{t}}|\det(M_k(t))| \le \bar{c}(\varepsilon^{-1}\alpha\beta^{(k-1)})^{\mathtt{t}},$$

for some $\bar{c} > 0$. Hence, since $\varepsilon^{-1}\alpha\beta^{(k-1)} < 1$, each element of $\varepsilon^{\mathfrak{t}} M(\mathfrak{t})^{(k)}$ is exponentially decreasing and the norm $|\varepsilon^{\mathfrak{t}} M(\mathfrak{t})^{(k)}|$ is uniformly bounded for all $\mathfrak{t} \in \mathbb{N}$. □

Leveraging on the previous lemmas, we provide a bound on the compound of the variational system (24).

Lemma 5. *Consider system* (1) *and assume there exist a forward invariant compact set* $\mathcal{S} \subsetneq \mathbb{R}^n$*, two symmetric matrix-valued functions* $P_0, P_{k-1} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *of respective inertia* $\mathrm{In_c}(P_0) = (0, 0, n)$ *and* $\mathrm{In_c}(P_{k-1}) = (k-1, 0, n-k+1)$ *for all* $x \in \mathcal{S}$ *which are bounded, that is, they satisfy* (22)*, and strictly positive constants* $\mu_0, \mu_{k-1} \in \mathbb{R}_{>0}$ *such that* (23) *is satisfied. Then, there exist a constant* $c > 0$ *and* $\varepsilon \in (0, 1)$ *such that*

$$\left| \frac{\partial f^{\,\mathfrak{t}}}{\partial x}(x)^{(k)} \right| \leq c\varepsilon^{\mathfrak{t}}, \quad \forall (\mathfrak{t}, x) \in \mathbb{N} \times \mathcal{S}. \tag{34}$$

Proof: The proof follows the same steps as [Cecilia et al., 2023, Proof of Lemma 14], by adapting the exponential bounds of the form $ce^{\varepsilon t}$ with the proper $\mathfrak{t}$-powers of scalars of the form $c\varepsilon^{\mathfrak{t}}$, similar to the proof of Lemma 4. □

With these technical results in mind, we can now present the proof of Theorem 3.

4.2 Proof of Theorem 3

Consider $\Phi \in \mathcal{I}_k$ such that $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$, with $\mathcal{I}_k$ being the set of submanifolds defined in (2). By following the first steps of the proof of Theorem 1, dynamics (7) imply

$$\begin{aligned}\Gamma_r(r, \mathfrak{t})^{(k)} &= \prod_{i=0}^{\mathfrak{t}} \frac{\partial f}{\partial x}(\Gamma(r, i))^{(k)} \Gamma_r(r, 0)^{(k)} \\ &= \left(\frac{\partial f^{\,\mathfrak{t}}}{\partial x}(\Gamma(r, 0)) \right)^{(k)} \Gamma_r(r, 0)^{(k)}.\end{aligned}$$

Then, by means of Lemma 5 and (34) we deduce the following bound

$$\begin{aligned}\left|\Gamma_r(r, \mathfrak{t})^{(k)}\right| &\leq \left| \left(\frac{\partial f^{\,\mathfrak{t}}}{\partial x}(\Gamma(r, 0)) \right)^{(k)} \right| \left|\Gamma_r(r, 0)^{(k)}\right| \\ &\leq c\varepsilon^{\mathfrak{t}} \left|\Gamma_r(r, 0)^{(k)}\right|.\end{aligned}$$

Then, by selecting $P(x)$ in (3) as the identity matrix, by (8) we obtain

$$\begin{aligned}V^k(\psi_{\mathfrak{t}} \circ \Phi) &= \int_{[0,1]^k} \left|\Gamma_r(r, \mathfrak{t})^{(k)}\right| \, dr \leq \int_{[0,1]^k} c\varepsilon^{\mathfrak{t}} \left|\Gamma_r(r, 0)^{(k)}\right| \, dr \\ &\leq c\varepsilon^{\mathfrak{t}} \int_{[0,1]^k} \left|\Gamma_r(r, 0)^{(k)}\right| \, dr \leq c\varepsilon^{\mathfrak{t}} V^k(\Phi),\end{aligned}$$

thus concluding the proof.

5. CONCLUSIONS

In this work, we derive sufficient conditions for k-contraction in discrete-time systems, by adapting and expanding some results in [Cecilia et al., 2023]. First, we present a condition based on multiplicative matrix compounds and state-dependent metrics. Second, we provide sufficient conditions based on cone fields. These conditions are proven to be necessary in the linear framework. Future works will focus on exploiting the compound-free conditions to derive k-contraction feedback designs.

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