The Cauchy Problem For Quasi-Linear Parabolic Systems Revisited
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To cite this version:
Isabelle Gallagher, Ayman Moussa. The Cauchy Problem For Quasi-Linear Parabolic Systems Revisited. 2024. hal-04642697

HAL Id: hal-04642697
https://hal.science/hal-04642697
Preprint submitted on 10 Jul 2024

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THE CAUCHY PROBLEM FOR QUASI-LINEAR PARABOLIC SYSTEMS REVISITED

ISABELLE GALLAGHER AND AYMAN MOUSSA

Abstract. We study a class of parabolic quasilinear systems, in which the diffusion matrix is not uniformly elliptic, but satisfies a Petrovskii condition of positivity of the real part of the eigenvalues. Local wellposedness is known since the work of Amann in the 90s, by a semi-group method. We revisit these results in the context of Sobolev spaces modelled on $L^2$ and exemplify our method with the SKT system, showing the existence of local, non-negative, strong solutions.

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1. Introduction

1.1. Main results. This article deals with local well-posedness for the following quasilinear parabolic system, set on the $d$-dimensional torus $\mathbb{T}^d$:}

$$
\begin{aligned}
\partial_t U - \sum_{k=1}^{d} \partial_k \left[ A(U) \partial_k U \right] &= F, \\
U|_{t=0} &= U^0.
\end{aligned}
$$

(1.1)

In this system $U^0 : \mathbb{T}^d \to \mathbb{R}^N$ and $F : \mathbb{R}_{\geq 0} \times \mathbb{T}^d \to \mathbb{R}^N$ are given, $A : \mathbb{R}^N \to M_N(\mathbb{R})$ is a smooth matrix field and $U : \mathbb{R}_{\geq 0} \times \mathbb{T}^d \to \mathbb{R}^N$ is the unknown. Our analysis will rely on a detailed study of the linear case in which a matrix field $M$ is given and one searches for $V$ solving

$$
\begin{aligned}
\partial_t V - \sum_{k=1}^{d} \partial_k \left[ M \partial_k V \right] &= F, \\
V|_{t=0} &= V^0.
\end{aligned}
$$

(1.2)

As we shall see later on, this system already hides several difficulties in order to build a well-posedness theory with propagation of Sobolev norms. As a matter of fact, the following spectral condition will be of utmost importance in our construction (we refer to Section C for more on that condition).

Definition 1.1 (Petrovskii condition). A matrix $B \in M_N(\mathbb{R})$ satisfies the Petrovskii condition if it belongs to $\mathcal{P}$, where

$$
\mathcal{P} := \bigcup_{\delta > 0} \mathcal{P}_\delta, \quad \mathcal{P}_\delta := \left\{ B \in M_N(\mathbb{R}) : z \in \text{Sp}(B) \Rightarrow \text{Re}(z) \geq \delta \right\}.
$$

(1.3)

With this Petrovskii condition at hand, we are in position to state our two main results. We anticipate a notation that will be introduced in Paragraph 1.4: for $s \in \mathbb{R}$ and $T > 0$ we note $E^T_s$ for the $\mathcal{H}^s(\mathbb{T}^d)$-energy space $\mathcal{C}_0^0([0, T]; \mathcal{H}^s(\mathbb{T}^d)) \cap L^2(0, T; \mathcal{H}^{s+1}(\mathbb{T}^d))$ that we equip
allows for a direct use of Picard’s fixed-point theorem to establish the following

\[ \|U\|_{E_T^s} := \left( \|U\|_{Z^0([0,T];H^s(T^d))}^2 + \int_0^T \|\nabla U(t)\|_{H^s(T^d)}^2 \, dt \right)^{1/2}. \]

We also define \( Y_T^s := L^2(0,T;H^s(T^d)) \), and the set \( Q_T := [0,T] \times T^d \).

The initial data \( U^0 \) will be chosen in a Sobolev space \( H^s(T^d) \), while the force \( F \) will correspondingly lie in \( Y_T^{s-1} \) for some \( T > 0 \). We denote the functional framework for the data by \( D_T^s := H^s(T^d) \times Y_T^{s-1} \) and the size of the data is measured by

\[ \|(U^0,F)\|_{D_T^s} := \|U^0\|_{H^s(T^d)} + \|F\|_{Y_T^{s-1}} + \int_0^T |\langle F(t) \rangle| \, dt. \]

We have denoted by \( \langle f \rangle \) the average of any function \( f \) over \( T^d \). Our main result is the following.

**Theorem 1** (Local well-posedness). Consider a smooth \( A : \mathbb{R}^N \to \mathcal{P} \), and \( s > d/2 \). For any \((U^0,F)\) belonging to \( D_\infty^s \), there exists \( T > 0 \) and a unique element \( U \) of \( E_T^s \) which solves the parabolic Cauchy problem (1.1) on \( Q_T \). Moreover, if \((U^0_1,F_1),(U^0_2,F_2)\) lie in \( D_\infty^s \), with \( U_1 \) and \( U_2 \) the respective solutions both belonging to \( E_T^s \) for some \( T > 0 \), then there is a constant \( C \) depending on the norms of \((U^0_1,F_1),(U^0_2,F_2)\) such that

\[ \|U_1 - U_2\|_{E_T^s} \leq C\|(U^0_1 - U^0_2,F_1 - F_2)\|_{D_T^s}. \]

With this (local in time) well-posedness setting we can define the lifetime of the solution associated with \((U^0,F)\) in \( D_\infty^s \) for some \( s > d/2 \) by

\[ T^*_s(U^0,F) := \sup \{ T > 0 : \exists U \in E_T^s \text{ solving (1.1) on } Q_T \}. \]

The construction leading to Theorem 1 provides a lifetime \( T^*_s \) which depends on the data \((U^0,F)\) not only through its size but also (in some sense) through its form. Actually it is possible, thanks to a propagation of regularity result, to prove that the lifetime actually only depends on the size of the data. Also, if the data is small enough the lifetime is infinite. We have more precisely the following result.

**Theorem 2** (Lifetime and blow-up). Consider the assumptions of Theorem 1.

(i) There exists \( \varepsilon > 0 \) depending only on \( A \) and \( s \) such that

\[ \|(U^0,F)\|_{D_\infty^s} < \varepsilon \implies T^*_s(U^0,F) = \infty. \]

(ii) There exists a decreasing function \( \varphi \) such that \( T^*_s(U^0,F) \geq \varphi(\|(U^0,F)\|_{D_\infty^s}) \).

(iii) If \( T^*_s(U^0,F) < +\infty \), then \( \lim_{t \to T^*_s(U^0,F)} \|U(t)\|_{H^s(T^d)} = \infty \).

Of course, in the previous results, lower order nonlinearities may be added without changing the conclusion of these statements except point (i) of the second one (unless structural assumptions on the lower order term are added). For instance, estimate (1.4) of Theorem 1 allows for a direct use of Picard’s fixed-point theorem to establish the following corollary, useful for cross-diffusion systems.
Corollary 1.2. Consider a smooth function $R : \mathbb{R}^N \to \mathbb{R}^N$ and the assumptions of Theorem 1. All conclusions of Theorem 1 and Theorem 2 except point (i) of the latter hold for the following system

\[
\begin{aligned}
\partial_t U - \sum_{k=1}^d \partial_k [A(U)\partial_k U] &= F + R(U), \\
U|_{t=0} &= U^0.
\end{aligned}
\] (1.5)

1.2. State of the art. Parabolic systems have been studied for a long time. The pioneer contribution of Petrovskii \cite{19} seems to be the starting point of the story. Back then existence and uniqueness of solution for parabolic linear systems was yet to be explored. The major step of Petrovskii in this context is the discovery of a condition on those linear systems ensuring the existence and uniqueness of a global solution. For a parabolic system in divergence form (non-constant coefficients $a_{ij}^{k\ell}$, unknown $U := (u_i)_{1 \leq i \leq M}$ considered on $\mathbb{R}_{\geq 0} \times \mathbb{T}^d$)

\[
\begin{aligned}
\partial_t u_i - \sum_{k,\ell=1}^d \sum_{j=1}^M \partial_k \left[ a_{ij}^{k\ell} \partial_\ell u_j \right] &= 0,
\end{aligned}
\] (1.6)

Petrovskii’s condition requires that for all vectors $\xi \in \mathbb{R}^d$ of euclidean norm 1, the matrix field $A_\xi := (\sum_{k,\ell} a_{ij}^{k\ell} \xi_k \xi_\ell)_{ij}$ has a spectrum lying in the set \{z $\in \mathbb{C}$ : Re(z) > 0\}. In the literature many references can be found in which this previous condition is replaced by $\langle a_\xi X, X \rangle \geq 0$ which amounts to asking that the symmetric part of the matrix field $A_\xi$ satisfies Petrovskii’s condition. This latter assumption is extremely restrictive and flushes out all the subtlety of the problem, because under this condition the system (1.6) has an obvious energy estimate. We give below (in the quasilinear setting) a natural example of a system which satisfies Petrovskii’s condition but not this restrictive one. We focus therefore here on the references in the literature which treat systems following Petrovskii’s condition and not the previous stronger one. In \cite{19}, the tensor field $A = (a_{ij}^{k\ell})_{i,j,k,\ell}$ depends only on the time variable and the setting is rather regular both for the data and the solution; existence of a solution is obtained by means of a fundamental solution. Let us precise however that Petrovskii’s condition and construction also hold for higher order parabolic systems. In the two decades following \cite{19}, several contributions extended this study of the Cauchy problem to more general systems and in less regular settings, see \cite{2, 14, 8} to cite a few. We also refer to the bibliographical remarks section of the book \cite{9} of Friedman and to the last chapter of the monograph \cite{13} for other references. All those cited works rely on the condition exhibited by Petrovskii, with a construction of the fundamental solution (exception made of \cite{17} which relies on semigroup theory). Further generalizations of this condition encompassing even more general systems and sets of functions have been explored, see for instance the review \cite{18} on the Gel’fand-Shilov theory for parabolic systems.

Leaving the realm of linear systems, the literature is by far less generous. For quasilinear parabolic systems, Amann’s work \cite{1} seems to be the only reference covering a variety of cases comparable to the Petrovskii theory for linear systems. Without surprise, the work
of Amann relies crucially on Petrovskii’s condition (Amann speaks of the normal ellipticity condition). Instead of the linear system \((1.6)\), Amann tackles the following quasilinear one

\[
\partial_t u_i - \sum_{k,\ell=1}^d \sum_{j=1}^M \partial_k \left[ a_{ij}^{k\ell}(U) \partial_\ell u_j \right] = 0,
\]

where the tensor \( A := (a_{ij}^{k\ell})_{i,j,k,\ell} \) now depends on as many variables as the system and takes its values in the set of tensors satisfying Petrovskii’s condition \(^1\). In this setting, Amann’s theory establishes a local well-posedness result for Sobolev-valued solutions. More precisely, given \( p > d \) and any initial data in \( W^{1,p}(\mathbb{T}^d) \) there exists a unique \( W^{1,p}(\mathbb{T}^d) \)-valued solution \( U \) to \((1.7)\), in a vicinity of the origin in \( \mathbb{R}^d \); if the maximal lifetime of this solution is finite, then blow-up occurs in the \( W^{1,p}(\mathbb{T}^d) \) norm. As a matter of fact, Amann’s theory allows for even more complicated systems: it encompasses the boundary-value problem on a domain of \( \mathbb{R}^d \), with extra dependence on \( t \) and \( x \) for the tensor \( A \) and more (lower order) terms in the system. Amann’s theory is a highly complex machinery relying on several non-trivial ingredients: general interpolation, maximal regularity and analytic semigroup theory. We also mention that 10 years ago, Pierre-Louis Lions gave a series of lectures on parabolic systems \([16]\) in which part of the bibliographical material that we have cited here is presented together with a possible strategy to build local solutions.

Systems like \((1.7)\) (quite often with a non-vanishing source term) arise naturally in several contexts as a model of diffusion in a multiphase setting. The choice of a diagonal diffusion tensor \( A \) (that is \( A_{ij}^{k\ell} = 0 \) for \( k \neq \ell \)) corresponds to standard or isotropic diffusion while non-diagonal diffusion tensor corresponds to anisotropic diffusion. The latter case can be preferred when the quantities at stake evolve in a highly heterogeneous environment in which the brownian motion from which \((1.7)\) originates is not completely symmetric in all directions. These types of models do exist (see for instance \([5]\) or \([22]\)) but their use is rather limited in comparison with the isotropic case. For this reason and because this work originates from questions arising in population dynamics (see the SKT model below) we have chosen to focus here only on systems of the form \((1.1)\), that is exactly the case of isotropic diffusion. This class of systems already contains a large number of models, the mathematical analysis of which is highly non-trivial. This includes renowned cases of models describing chemical concentrations, cell density, gas mixtures or population densities. For a nice description of those varieties of models, we refer to \([12, \text{Chapter 4}]\). All those systems, originally introduced in a modelling purpose, offered to the mathematical community genuine and challenging questions about their behavior, be it existence and uniqueness of solution, blow-up or long-time behavior. A common feature shared by those models is their cross-diffusion aspect: even though the diffusion operator used on each component is isotropic, several components of the system undergo the influence of other components on the intensity of its diffusion. It is a remarkable fact that even if the environment in which the components evolve is completely isotropic, the sole mutual influence on the intensity of their diffusion can lead to asymmetric patterns. A spectacular instance of

\(^1\) Just as above we omit here any reference asking for the non-negativeness of the symmetric part of the tensor, because as before this assumption spoils the problem’s interest, in some sense.
this phenomenon is observable in the SKT (for Shigesada, Kawasaki and Teramoto) model introduced in [21]. In this cross-diffusion system (which falls into the scope of (1.1)), even though the diffusion is isotropic, stable segregation steady states are possible corresponding to cases in which each of the species shares out the available space, in some sense. The SKT model and its generalizations are iconic examples of the possible use of Amann’s theory. If global weak solutions are known to exist thanks to the (rather lately discovered) entropy structure for those systems (see [6, 7] and the references therein), as far as our knowledge goes the only way to build (local) strong solutions is to rely on Amann’s theory (as noticed by Amann himself in [1]). From this step, a considerable amount of attempts to prove the existence of \textit{global} strong solutions to the SKT system (or its variants) emerged (see [11, 10] and the references therein for the most recent improvements). In all those works, Amann’s theory is used as a black box and the quest is reduced to the denial of the blow-up criterion which holds in case of finite lifetime, as established by Amann.

This work aims at proposing an alternative approach to the construction of local strong solutions for quasilinear parabolic system like (1.1) (satisfying Petrovskii’s condition), using relatively few elaborate tools, in comparison with Amann’s construction. From this point of view, our approach differs from [1] by the fact that we do not use any abstract result on parabolic equations (no semi-group theory nor maximal regularity) but we rely instead on Fourier analysis and the paraproduct of Bony [4] to treat the most severe non-linearities of the system. In this way, we manage to build solutions in a finer scale of spaces, but yet comparable: our solutions live in $H^s(\mathbb{T}^d)$ for $s > d/2$ whereas Amann’s in $W^{1,p}(\mathbb{T}^d)$ for $p > d$. In the specific example of the SKT model, we hope that this new path will shed some light on the question of the possible blow-up of these solutions, at least in the periodic setting that we consider.

1.3. \textbf{Sign-preserving property and application to the SKT model.} As repeatedly noticed by Amann [1] and contrary to the scalar case, parabolic systems satisfying Petrovskii’s condition do not offer any maximum principle. When dealing with diffusive models aiming at describing the evolution of \textit{densities}, the non-negativity of the solution is a crucial property of the model that one would like to propagate from the initial data. This is not a harmless detail from the point of view of mathematical analysis either as it may happen (see below for some examples) that Petrovskii’s condition is only satisfied on the cone of non-negative vectors. This transference of non-negativeness (component-wise) from the initial data to the solution on its whole lifetime is tightly linked to the structure of the system. We give below a sufficient condition on the matrix field to ensure this propagation. This condition was originally suggested at the formal level in [16]. In order to motivate the following definition, notice that in the case of systems, the non-negativity of the diagonal part of the operator alone does not ensure the preservation of the sign of the solution, due to the presence of lower-order terms. These terms will not affect the preservation of sign only if they are themselves in some sense diagonal, as presented in the coming Definition 1.3. Proposition 1.4 and Theorem 3 below enlight the relevance of that definition in our setting of solutions.
For $V \in \mathbb{R}^N$ the notation $\text{diag}(V)$ refers to the diagonal square matrix of size $N$ with entries given by the components of $V$. The partial order $\geq$ on $\mathbb{R}^N$ or $M_N(\mathbb{R})$ has to be understood component-wise.

**Definition 1.3.** A matrix field $A : \mathbb{R}^N \to M_N(\mathbb{R})$ is said to be **sign-preserving** if there exist smooth maps $D : \mathbb{R}^N \to \text{diag}(\mathbb{R}^N)$ and $B : \mathbb{R}^N \to M_N(\mathbb{R})$ such that

- $A(U) = D(U) + \text{diag}(U)B(U)$;
- for some nonnegative real number $\alpha$ and any $U \geq 0$ one has $D(U) \geq \alpha I_N$.

The relevance of that definition stems from the following proposition, proved in Section 6.

**Proposition 1.4.** Fix $s > d/2 + 2$. Let $\rho : \mathbb{R}^N \to \mathbb{R}^N$ be a given smooth function and define $R(U) := \text{diag}(U)\rho(U)$. Consider a smooth sign-preserving matrix field $A$, and a solution $U$ to the Cauchy problem (1.5) in $E_T^s$ associated with non-negative $(U^0, F) \in \mathcal{D}_\infty^s$. Then $U$ is non-negative on $[0, T]$.

Finally let us state the following theorem, which is a consequence of our main result and Proposition 1.4, and will be applied to the SKT system below. Its proof can also be found in Section 6.

**Theorem 3.** Fix $s > d/2$. Let $\rho : \mathbb{R}^N \to \mathbb{R}^N$ be a given smooth function and define $R(U) := \text{diag}(U)\rho(U)$. Consider a smooth sign-preserving matrix field $A$ satisfying $A(\mathbb{R}^N_{\geq 0}) \subset \mathcal{P}$, and non-negative $(U^0, F) \in \mathcal{D}_\infty^s$. There exists $T > 0$ and a unique element $U$ of $E_T^s$ which solves the Cauchy problem (1.5). Moreover, we have the stability estimate (1.4) and points (ii) and (iii) of Theorem 2; point (i) holds if $\rho$ vanishes identically. Finally, $U$ is non-negative on its whole lifetime.

**Remark 1.5.** As compared to Theorem 1 and Theorem 2, Petrovskii’s condition is here only required to be satisfied on the cone $\mathbb{R}^N_{\geq 0}$.

Our interest in this question originates from the study of the SKT model [21]. We end this paragraph by an example of use of Theorem 3 on this specific system. In its original form, the SKT model writes

\begin{equation}
\begin{aligned}
\partial_t u_1 - \Delta[(d_1 + a_{11}u_1 + a_{12}u_2)u_1] &= u_1(r_1 - s_{11}u_1 - s_{12}u_2), \\
\partial_t u_2 - \Delta[(d_2 + a_{21}u_1 + a_{22}u_2)u_2] &= u_2(r_2 - s_{21}u_1 - s_{22}u_2),
\end{aligned}
\end{equation}

where the unknowns $u_1, u_2 : \mathbb{R}_{\geq 0} \times \mathbb{T}^d \to \mathbb{R}_{\geq 0}$ are density population and all the coefficients $a_{ij}, r_i, s_{ij}$ are nonnegative while the $d_i$’s are positive. Such a system can be written in the form (1.5) for $U := \mathcal{T}(u_1, u_2)$,

\[
R(U) = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \begin{pmatrix} r_1 - s_{11}u_1 - s_{12}u_2 \\ r_2 - s_{21}u_1 - s_{22}u_2 \end{pmatrix}
\]

and the matrix field

\begin{equation}
A_{\text{SKT}} : \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} a_{11}u_1 & a_{12}u_2 \\ a_{21}u_2 & a_{22}u_2 \end{pmatrix}.
\end{equation}
Writing
\[ A_{SKT}(U) = \begin{pmatrix} d_1 + a_{12}u_2 & 0 \\ 0 & d_2 + a_{21}u_1 \end{pmatrix} + \begin{pmatrix} 2a_{11}u_1 & a_{12}u_1 \\ a_{21}u_2 & 2a_{22}u_2 \end{pmatrix}, \]
we see that this matrix field is indeed sign-preserving in the sense of Definition 1.3. Lastly, we have that \( A_{SKT}(\mathbb{R}_0 \times \mathbb{R}_0) \subseteq \mathcal{P} \): it can be readily checked that \( \det A_{SKT}(u_1, u_2) \) and \( \text{Tr} A_{SKT}(u_1, u_2) \) are both positive for \( u_1, u_2 \geq 0 \) (because the \( d_i \)'s are positive), so either the eigenvalues are not real and share a positive real part, or they are both real and have the same (positive) sign. Theorem 3 therefore applies to produce local strong and non-negative solutions to the SKT system. Let us however note that
\[
\det(A_{SKT} + \frac{1}{2} A_{SKT})(u_1, u_2) = (d_1 + 2a_{11}u_1 + a_{12}u_2)(d_2 + a_{21}u_1 + 2a_{22}u_2) - (a_{12}u_1 + a_{21}u_2)^2,
\]
may become negative on \( \mathbb{R}_0 \times \mathbb{R}_0 \), this expression becomes negative on the two fundamental axes, far from the origin (and therefore also near those axes). This simple example explains why Petrovskii’s condition is indeed crucial for the study of parabolic systems.

1.4. Notations. In the following we denote \( \mathbb{P}_0 := \text{Id} - (\cdot) \) the orthogonal projection from \( L^2(\mathbb{T}^d) \) onto mean free functions. For \( T > 0 \), we note \( Q_T \) the (periodic) closed cylinder \( Q_T := [0, T] \times \mathbb{T}^d \). For \( 1 \leq p \leq \infty \) the \( L^p(\mathbb{T}^d) \) and \( L^p(Q_T) \) norms will be noted \( \| \cdot \|_p \) (if there is no ambiguity), while we will generally use \( \| \cdot \|_X \) for the norm of some functional space \( X \).

For any real number \( s \) we denote \( X^s_T \) the space \( \mathcal{C}^0([0, T]; \mathcal{H}^s(\mathbb{T}^d)) \) and we recall that \( Y^s_T \) is the space \( L^2(0, T; \mathcal{H}^s(\mathbb{T}^d)) \); we then define the energy space \( E^s_T := X^s_T \cap Y^{s+1}_T \) that we equip with the norm \( V \mapsto (\|V\|^2_{X^s_T} + \|\nabla V\|^2_{Y^{s+1}_T})^{1/2} \).

For \( T > 0, \alpha \in [0, 1] \) and \( k \in \mathbb{N} \) we denote by \( \mathcal{C}^k,\alpha(Q_T) \) the space of \( k \) times continuously differentiable functions, whose partial derivatives of order \( k \) are \( \alpha \)-Hölder continuous and we denote by \( \| \cdot \|_{\mathcal{C}^k,\alpha(Q_T)} \) the corresponding norm. We simply note \( \mathcal{C}^k(Q_T) \) when \( \alpha = 0 \) and sometimes precise the set of values \( X \) writing \( \mathcal{C}^k,\alpha(Q_T; X) \).

We fix a norm \( \| \cdot \| \) on \( \mathbb{C}^N \) and the subordinate norm \( \| \cdot \|_\infty \) on \( M_N(\mathbb{C}) \). For a continuous matrix field \( M \in \mathcal{C}^0(Q_T; \mathcal{P}) \), \( \|M\|_\infty \) will refer to the uniform norm of \( M \) (with \( \| \cdot \| \) at arrival). For such a matrix field \( M \), there exists \( \eta(M) > 0 \) such that \( M \in \mathcal{C}^0(Q_T; \mathcal{P}_{\eta(M)}) \). We refer to Appendix Section C for the definition and properties of this function \( \eta \). For \( \alpha \in [0, 1] \) and a matrix field \( M \in \mathcal{C}^0,\alpha(Q_T; \mathcal{P}) \), we will use repeatedly the following notation
\[
[M]_\alpha := \|M\|_{\mathcal{C}^0,\alpha(Q_T)} + \eta(M)^{-1}.
\]
Finally if \( C_1, \ldots, C_n \) is a collection of positive numbers, we write
\[
A \lesssim_{C_1,\ldots,C_n} B
\]
if there is an increasing function \( g \) such that
\[
A \leq g(C_1 + \cdots + C_n)B.
\]
Such a function does not depend on any other relevant variable and it is liable to change from line to line. We will in general not track it.

1.5. **Main results in the linear setting.** Theorem 1 will be obtained thanks to a detailed study of the linear setting, that is of system (1.2) where \( M \) is a given matrix field. We collect in this paragraph some results which will be useful in the proof of Theorem 1 and Theorem 2 and, even though focusing on the linear setting, are interesting for their own sake. We will often use the notation \( L_M \) for the linear differential operator applied to \( V \) in the left-hand side of (1.2):

\[
L_M V := \partial_t V - \sum_{k=1}^d \partial_k [M \partial_k V].
\]

Well-posedness for (1.2) will be established under adequate assumptions on \( M \) and thanks to the following a priori estimates. The first result focuses on the \( L^2(\mathbb{T}^d) \) setting while the second (which requires more regularity on \( M \)) concerns the \( H^s(\mathbb{T}^d) \) setting for \( s > d/2 \). As will be shown later, one can assume without loss of generality that the functions under study are mean free.

**Theorem 4.** Let \( T > 0 \), \( \alpha \in (0,1] \) and consider a matrix field \( M \) which belongs to the space \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \). For any \( V \in E_T^0 \) such that \( L_M V \in Y_T^{-1} \) and satisfying \( \langle V(t) \rangle = 0 \) for all \( t \in [0,T] \), there holds

\[
\|V\|_{E_T^0} \lesssim_{T,|M|,\alpha} \|V^0\|_2 + \|L_M V\|_{Y_T^{-1}}.
\]

**Remark 1.6.** It will be clear from the proof of Theorem 4 that a similar one can be recovered under mere continuity for the matrix field \( M \), but where the symbol \( \lesssim_{T,|M|,\alpha} \) is replaced by one involving the modulus of continuity of \( M \) on \([0,T]\).

**Theorem 5.** Let \( T > 0 \), \( \alpha \in (0,1] \), \( s > d/2 \) and consider a matrix field \( M \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) which belongs furthermore to \( Y_T^{s+1} \). For any \( V \in E_T^0 \) such that \( \langle V(0) \rangle \in H^s(\mathbb{T}^d) \), \( L_M V \in Y_T^{s-1} \) and \( \langle V(t) \rangle = 0 \) for all \( t \in [0,T] \), one has actually that \( V \) belongs to \( E_T^0 \cap \mathcal{C}^{0,\alpha_s}(Q_T) \) for some \( \alpha_s \in (0,1) \) and

\[
\|V\|_{E_T^0} + \|V\|_{\mathcal{C}^{0,\alpha_s}(Q_T)} \lesssim_{T,|M|,\alpha_s Y_T^{s+1}(M,\mathbb{R})} \|V^0\|_{H^s(\mathbb{T}^d)} + \|L_M V\|_{Y_T^{s-1}}.
\]

**Remark 1.7.** The number \( d_{Y_T^{s+1}}(M,\mathbb{R}) \) is the distance (in the Hilbert space \( Y_T^{s+1} \)) between \( M \) and the closed subspace of constant matrix fields. In practice, in this estimate this expression can therefore be replaced by \( \|M - B\|_{Y_T^{s+1}} \) for any fixed matrix \( B \in M_N(\mathbb{R}) \).

**Remark 1.8.** The proofs of Theorem 4 and Theorem 5 may easily be adapted to the case when \( T = \infty \), provided the matrix field \( M \) converges as time goes to infinity towards a stationary matrix field \( \overline{M} \in \mathcal{C}^0(\mathbb{T}^d; \mathcal{P}) \).

Finally let us state a wellposedness result for the Cauchy problem (1.2).
Theorem 6. Fix $T > 0$, $\alpha \in (0,1]$. For $M \in C^{0,\alpha}(Q_T; \mathcal{P})$ and $(V^0, F) \in \mathcal{D}_T^0$, the Cauchy problem (1.2) is well posed in $E_T^0$. For $s \geq d/2$, if furthermore $(V^0, F)$ belongs to $\mathcal{D}_T^s$ and $M$ to $Y_T^{s+1}$, then $V$ actually belongs to $E_T^s \cap C^{0,\alpha_s}(Q_T)$, for some $\alpha_s \in (0,1)$.

1.6. Plan of the paper. In the coming Section 2 we prove Theorems 4 and Theorem 5, which concern a priori estimates. This will lead, in Section 3, to the proof of the linear wellposedness Theorem 6. The proof of the nonlinear wellposedness Theorem 1 is provided in Section 4 while Theorem 2 is proved in Section 5. Finally Theorem 3 is proved in Section 6. Four appendixes are devoted to some classical results on Sobolev spaces, to basics of Littlewood-Paley theory, to important facts related to the Petrovskii condition, and to a technical but useful retraction result, of $\mathbb{R}^N$ on $\mathbb{R}^N_{\geq 0}$, respectively.

2. Estimates in the linear case

We start by studying the case of a matrix field independent of the space variable (see Paragraph 2.2), first in the constant coefficient case (Proposition 2.1), and then in the time-dependent case (Corollary 2.2). We explain then in Paragraph 2.3 how the proof of Theorem 4 can be reduced to a simpler result (Lemma 2.3) and then prove this lemma in Subsection 2.4. We prove then Theorem 5 in Paragraph 2.5, using Theorem 4 on each block of the Littlewood-Paley decomposition of the equation, so as to propagate the $H^s(\mathbb{T}^d)$ norms by a paralinearization argument.

2.1. The case of a constant matrix field. In this paragraph we treat the simplest case in which the matrix field is constant. Well-posedness is obtained in $E_T^s$ for $s \geq 0$ together with an estimate.

Proposition 2.1. Fix $\delta > 0$ and $B \in \mathcal{P}_\delta$, as well as $s \in \mathbb{R}$ and $T > 0$. For $(V^0, F) \in \mathcal{D}_T^s$, having both vanishing spatial mean, the Cauchy problem

$$
\begin{align*}
\partial_t V - B\Delta V &= F, \\
V|_{t=0} &= V^0,
\end{align*}
$$

is well posed in the energy space $E_T^s$ with the following energy estimate

$$
\|V\|_{E_T^s}^2 + \delta \|\nabla V\|_{Y_T^{s-1}}^2 \leq C_{B,\delta} \left( \|V^0\|_{H^s(\mathbb{T}^d)}^2 + \frac{1}{\delta} \|F\|_{Y_T^{s-1}}^2 \right),
$$

with

$$
C_{B,\delta} := a_N \left( 1 + \frac{\|B\|}{\delta} \right)^N,
$$

and $a_N$ a constant depending only on the dimension $N$ of the system.

Proof. We compute the evolution in time of the (spatial) Fourier modes of $V$, denoted for $k \in \mathbb{Z}^d$ by $c_k(V) : \mathbb{R} \to \mathbb{C}^N$. The equation being linear with constant coefficients, $V$ is explicitly given by the behavior of those coefficients, so existence and uniqueness will follow from the a priori estimate (2.2) because, by a standard argument (see Lemma A.1 in
Appendix A), if V belongs $L^\infty(0, T; H^s(\mathbb{T}^d)) \cap Y^{s+1}_T$ with a time derivative in $Y^{s-1}_T$, then V belongs to $X^s_T$ and thus to $E^s_T$. The coefficients take the explicit form

$$c_k(V)(t) = e^{-B|k|^2 t} c_k(V_0) + \int_0^t e^{-B|k|^2 (t-t')} c_k(F)(t') \, dt'.$$

Estimate (C.1) of Lemma C.1 gives the existence of a constant $C_{B, \delta}$ as in (2.3) such that (for $k \neq 0$)

$$|c_k(V)(t)| \leq C_{B, \delta} \left( |c_k(V_0)| e^{-\delta|k|^2 t/2} + \int_0^t e^{-\delta|k|^2 (t-t')/2} |c_k(F)(t')| \, dt' \right).$$

From now on, we allow the constant $C_{B, \delta}$ to change from line to line, only through the irrelevant dimension-dependent constant $a_N$. Multiplying the previous inequality by $|k|^s$, and applying the Cauchy-Schwarz inequality to deal with the time integral, we infer

$$|k|^s |c_k(V)(t)| \leq C_{B, \delta} \left( |k|^s |c_k(V_0)| e^{-\delta|k|^2 t/2} + \frac{|k|^s}{|k|\sqrt{\delta}} \left( \int_0^t |c_k(F)(t')|^2 \, dt' \right)^{1/2} \right).$$

Taking the squares and summing over $k \neq 0$ and then taking the sup norm in $t \in [0, T]$ on both sides we get

$$\|V(t)\|_{X^s_T}^2 \leq C_{B, \delta} \left( \|V_0\|_{H^s(\mathbb{T}^d)}^2 + \frac{1}{\delta} \int_0^T \|F(t')\|_{H^{s-1}(\mathbb{T}^d)}^2 \, dt' \right),$$

On the other hand, multiplying (2.4) by $|k|^{s+1}$ we have pointwise for $t \geq 0$

$$|k|^{s+1} |c_k(V)(t)| \leq C_{B, \delta} \left( |k|^{s+1} |c_k(V_0)| y_k(t) + |k|^{s+1} y_k \ast c_k(F)(t) \right),$$

with $y_k : t \mapsto e^{-\delta|k|^2 t/2} 1_{t>0}$ (where the convolution is on the time variable, extending $c_k(F)$ to 0 on $\mathbb{R}_{\leq 0}$). Since $(|k|^2 y_k)_k$ is bounded uniformly in $L^1(\mathbb{R}_{\geq 0})$ by $2/\delta$ and $(|k| y_k)_k$ is bounded uniformly in $L^2(\mathbb{R}_{\geq 0})$ by $1/\sqrt{\delta}$, Young’s inequality gives

$$\int_0^t |k|^{2(s+1)} |c_k(V)(t')|^2 \, dt' \leq C_{B, \delta} \left( \frac{|k|^{2s}}{\delta} |c_k(V_0)|^2 + \frac{1}{\delta^2} \int_0^t |c_k(F)(t')|^2 |k|^{2(s-1)} \, dt' \right),$$

which implies after summation over $k \neq 0$

$$\delta \int_0^t \|\nabla V(t')\|_{H^{s}(\mathbb{T}^d)}^2 \, dt' \leq C_{B, \delta} \left( \|V_0\|_{H^s(\mathbb{T}^d)}^2 + \frac{1}{\delta} \int_0^T \|F(t')\|_{H^{s-1}(\mathbb{T}^d)}^2 \, dt' \right),$$

and estimate (2.2) is proved.

2.2. The case of a homogeneous in space matrix field. In this paragraph we focus on the case when M does not depend on the space variable but may depend on time: using Proposition 2.1, we can actually indeed recover a similar result (here only stated and proved in the case $s = 0$) for a class of non autonomous systems.

Corollary 2.2. Let $\alpha > 0$, $M \in C^0,\alpha([0, T]; \mathcal{P})$. For any $V \in E^0_T$ having vanishing spatial mean at all times and such that $L_M V \in Y^{s-1}_T$ there holds

$$\|V\|_{E^0_T} \lesssim_{T, |M|_\alpha} \|V(0)\|_2 + \|L_M V\|_{Y^{s-1}_T}.$$
Proof. We consider a subdivision $t_0 < t_1 < \cdots < t_\kappa = T$ of $[0, T]$, such that each subinterval has size smaller than $T/\kappa$ with $\kappa$ to be determined. Using the notation introduced in Corollary C.3, we see that each matrix $M(t_i)$ belongs to $\mathcal{P}_{\eta(M)}$. So writing

$$\partial_t V - M(t_i) \Delta V = L_M V + (M - M(t_i)) \Delta V,$$

we get from Proposition 2.1 with $s = 0$ and $t \in [t_i, t_{i+1}]$, shifting the initial time to $t_i$

$$\|V\|_{L^\infty([t_i, t]; L^2(\mathbb{T}^d))} + \eta(M) \int_{t_i}^{t} \|\nabla V(t')\|_2^2 \, dt' \leq C_{M(t_i), \eta(M)} \|V(t_i)\|_2^2 + \frac{C_{M(t_i), \eta(M)}}{\eta(M)} \int_{t_i}^{t} \|L_M V(t')\|_{H^{-1}(\mathbb{T}^d)}^2 \, dt' + \frac{C_{M(t_i), \eta(M)}}{\eta(M)} \int_{t_i}^{t} \|M(t') - M(t_i)\|_2^2 \|\nabla V(t')\|_2^2 \, dt'.$$

Now, returning to the definition (2.3) of $C_{B, \delta}$ and recalling notation (1.10), we can rewrite the previous inequality as

$$\|V\|_{L^\infty([t_i, t]; L^2(\mathbb{T}^d))} + \int_{t_i}^{t} \|\nabla V(t')\|_2^2 \, dt' \lesssim_{[M]_\alpha} \|V(t_i)\|_2^2 + \int_{t_i}^{t} \|L_M V(t')\|_{H^{-1}(\mathbb{T}^d)}^2 \, dt' + \int_{t_i}^{t} \|M(t') - M(t_i)\|_2^2 \|\nabla V(t')\|_2^2 \, dt'.$$

We recall that $\lesssim_{[M]_\alpha}$ stands for multiplication by $g([M]_\alpha)$ with $g$ increasing. If we choose $\kappa$ large enough so that

$$(2.7) \quad \left(\frac{T}{\kappa}\right)^{2\alpha} \|M\|_{H^{0, \alpha}([0, T])}^2 \leq \frac{1}{2g([M]_\alpha)},$$

then recalling that $|t_i - t_{i+1}| \leq T/\kappa$ for all $i \in [0, \kappa - 1]$, we have in particular

$$\forall i \in [0, \kappa - 1], \quad \sup_{t \in [t_i, t_{i+1}]} \|M(t) - M(t_i)\|_2^2 \leq \frac{1}{2g([M]_\alpha)}.$$

The inequality on $[t_i, t_{i+1}]$ becomes

$$\|V\|_{L^\infty([t_i, t]; L^2(\mathbb{T}^d))} + \frac{1}{2} \int_{t_i}^{t} \|\nabla V(t')\|_2^2 \, dt' \lesssim_{[M]_\alpha} \|V(t_i)\|_2^2 + \int_{t_i}^{t} \|L_M V(t')\|_{H^{-1}(\mathbb{T}^d)}^2 \, dt'.$$

Summing this estimate with $i = 0$ and $t = t_1$ with the one for $i = 1$ and $t \in [t_1, t_2]$ we get in particular

$$\|V(t_1)\|_2^2 + \|V\|_{L^\infty([t_1, t]; L^2(\mathbb{T}^d))} + \frac{1}{2} \int_{t_0}^{t} \|\nabla V(t')\|_2^2 \, dt' \lesssim_{[M]_\alpha} \|V(0)\|_2^2 + \|V(t_1)\|_2^2 + \int_{0}^{t} \|L_M V(t')\|_{H^{-1}(\mathbb{T}^d)}^2 \, dt'.$$
Recalling that \( \lesssim_{[M]_\alpha} \) stands for multiplication by \( K_M := g([M]_\alpha) \), we have therefore

\[
\|V\|_{L^\infty([t_1,t];L^2(T^d))}^2 + \frac{1}{2} \int_0^t \|\nabla V(t')\|_2^2 \, dt' \\
\leq K_M \|V(0)\|_2^2 + (K_M - 1)\|V(t_1)\|_2^2 + K_M \int_0^t \|L_M V(t')\|_{H^{-1}(T^d)}^2 \, dt',
\]

which implies eventually on \([0,t_2]\)

\[
\|V\|_{L^\infty([0,t];L^2(T^d))}^2 + \frac{1}{2} \int_0^t \|\nabla V(t')\|_2^2 \, dt' \leq K_M^2 \|V(0)\|_2^2 + K_M \int_0^t \|L_M V(t')\|_{H^{-1}(T^d)}^2 \, dt'.
\]

Iterating, we recover on \([0,t_\kappa] = [0,T]\)

\[
\|V\|_{L^\infty([0,T];L^2(T^d))}^2 + \frac{1}{2} \int_0^T \|\nabla V(t')\|_2^2 \, dt' \leq K_M^2 \|V(0)\|_2^2 + K_M \int_0^T \|L_M V(t')\|_{H^{-1}(T^d)}^2 \, dt'.
\]

The proof is over once noticed that the condition (2.7) required on \( \kappa \) can indeed be satisfied choosing \( \kappa = \tilde{g}(T + [M]_\alpha) \) with \( \tilde{g} \) some increasing function.

\[\square\]

2.3. Reduction of Theorem 4 to a single lemma. In this subsection we explain how the estimate of Theorem 4 can be recovered by the following (seemingly) weaker result.

**Lemma 2.3.** Fix \( \alpha \in (0,1) \). For any lipschitz map \( M \in \mathcal{C}^{0,1}(Q_T; \mathcal{P}) \) and any \( V \in E^0_T \) such that \( L_M V \in Y^{-1}_T \) and \( \langle V(t) \rangle = 0 \) for all \( t \in [0,T] \), one has

\[
\|V\|_{E^0_T}^2 \lesssim_{T,[M]_\alpha} \|V(0)\|_2^2 + \|L_M V\|_{Y^{-1}_T}^2 + \left(1 + \|M\|_{\mathcal{C}^{0,1}(Q_T)}^2\right) \int_0^T \|V(t)\|_2^2 \, dt.
\]

Admitting for the moment the previous lemma, Theorem 4 can be proved thanks to an approximation argument. If \( M \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \), usual convolution properties lead to the existence of lipschitz matrix-valued functions \( (M_\varepsilon)_\varepsilon \) for which

\[
\|M_\varepsilon\|_{\mathcal{C}^{0,\alpha}(Q_T)} \leq \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)} ,
\]

\[
\|M - M_\varepsilon\|_\infty \leq \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)}\varepsilon^\alpha ,
\]

\[
\|M_\varepsilon\|_{\mathcal{C}^{0,1}(Q_T)} \leq (1 + \varepsilon^{-1})\|M\|_\infty .
\]

Because of (2.9) and the continuity of \( \eta \) (see Paragraph 1.4 for the definition of \( \eta \), and Corollary C.3 for its properties), for \( \varepsilon \) small enough we have \( \eta(M_\varepsilon) \geq \eta(M)/2 \) so that \( [M_\varepsilon]_\alpha \lesssim [M]_\alpha \) and we thus can infer from Lemma 2.3 that for any \( V \in E^0_T \) such that \( L_{M_\varepsilon} V \in Y^{-1}_T \) and \( \langle V(t) \rangle = 0 \) for all \( t \in [0,T] \),

\[
\|V\|_{E^0_T}^2 \lesssim_{T,[M]_\alpha} \|V(0)\|_2^2 + \|L_{M_\varepsilon} V\|_{Y^{-1}_T}^2 + \left(1 + \|M_\varepsilon\|_{\mathcal{C}^{0,1}(Q_T)}^2\right) \int_0^T \|V(t)\|_2^2 \, dt.
\]

Now since

\[
L_{M_\varepsilon} V - L_M V = \sum_{k=1}^d \partial_k \left[(M_\varepsilon - M) \partial_k V\right],
\]
we have thanks to (2.9)
\[ \| (L_{M_\varepsilon} - L_M)V \|_{Y^{-1}_T} \leq \| M_\varepsilon - M \|_\infty \| \nabla V \|_{Y^2_T} \leq d \varepsilon^\alpha \| M \|_{\mathcal{C}^{0,\alpha}(Q_T)} \| \nabla V \|_{Y^2_T}, \]
so the estimate above becomes
\[
\| V \|_{X^0_T}^2 + \| \nabla V \|_{Y^2_T}^2 \lesssim_{T, [M]_\alpha} \| V(0) \|_2^2 + \| L_M V \|_{Y^{-1}_T}^2 + \left( 1 + \| M_\varepsilon \|_{\mathcal{C}^{0,1}(Q_T)}^2 \right) \int_0^T \| V(t) \|_2^2 \, dt + \varepsilon^{2\alpha} \| \nabla V \|_{Y^2_T}^2.
\]
Now we recall that the multiplicative constant behind \( \lesssim_{T, [M]_\alpha} \) is an increasing function \( g(T + [M]_\alpha) \), so if we take \( \varepsilon \) small enough so as
\[
g(T + [M]_\alpha) \varepsilon^{2\alpha} < \frac{1}{2},
\]
the previous estimate implies
\[
(2.12) \quad \| V \|_{X^0_T}^2 + \frac{1}{2} \| \nabla V \|_{Y^2_T}^2 \lesssim_{T, [M]_\alpha} \| V(0) \|_2^2 + \| L_M V \|_{Y^{-1}_T}^2 + \left( 1 + \| M_\varepsilon \|_{\mathcal{C}^{0,1}(Q_T)}^2 \right) \int_0^T \| V(t) \|_2^2 \, dt.
\]
Using (2.10), we infer
\[
\| V \|_{X^0_T}^2 + \| \nabla V \|_{Y^2_T}^2 \lesssim_{T, [M]_\alpha} \left( \| V(0) \|_2^2 + \| L_M V \|_{Y^{-1}_T}^2 + (1 + \varepsilon^{-2}) \int_0^T \| V(t) \|_2^2 \, dt \right),
\]
so recalling the definition of \( X^0_T \), we are just off one Grönwall lemma of ending the proof of Theorem 4, provided \( \varepsilon \) can be replaced by some decreasing function of \( T + [M]_\alpha \): let us track the precise dependence of \( \varepsilon \) with respect to \( M \). The only two conditions on \( \varepsilon \) are sufficient smallness for (2.11) to hold, and for \( M_\varepsilon \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) to be satisfied. For the first condition, it is clearly satisfied if \( \varepsilon \) is replaced by some decreasing function of \( T + [M]_\alpha \). The second condition is trickier. We rely on (2.8) and Corollary C.4 to infer the existence of an non-increasing function \( f \) such that
\[
\| M - M_\varepsilon \|_\infty \leq f([M]_\alpha) \Rightarrow M_\varepsilon(Q_T) \subset \mathcal{P}_{\eta(M)/2} \subset \mathcal{P}.
\]
Using (2.9) this means that we can replace \( \varepsilon \) by some decreasing function of \([M]_\alpha \) to ensure \( M_\varepsilon \in \mathcal{C}^{0}(Q_T; \mathcal{P}) \). Theorem 4 is proved.

2.4. **Proof of Lemma 2.3.** In this subsection we prove Lemma 2.3 which, due to the argument of Subsection 2.3 implies Theorem 4. The idea is to reduce to the case of a constant in space matrix field considered in Paragraph 2.2, namely Corollary 2.2, by a partition of unity of \( \mathbb{T}^d \).

We start with a localization lemma.
Lemma 2.4. Fix $M \in C^{0,1}(Q_T; \mathcal{P})$ and $\alpha \in (0,1)$. For $\varepsilon$ small enough, and depending decreasingly on $T + [M]_\alpha$, the following holds. For any $V \in E^2_T$ such that $L_M V$ belongs to $Y_T^{-1}$ and any smooth bump function $\theta$ supported in a ball of $T^d$ of radius $\varepsilon$, there holds

$$\|\mathbb{P}_0(\theta V)\|_{E^2_T}^2 \lesssim_{T,[M]_\alpha} \|\mathbb{P}_0(\theta V)(0)\|_{L^2}^2 + \|\theta\|_{C^{0,1}(T^d)}^2 \|L_M V\|_{Y_T^{-1}}^2$$

$$+ \|\theta\|_{C^{1,1}(T^d)}^2 \|M\|_{C^{0,1}(Q_T)}^2 \int_0^T \|V(t)\|_{L^2}^2 dt.$$  

Proof. For the moment, let us start the computation with an arbitrary $\varepsilon > 0$ (yet to be fixed) with $\theta$ supported in the ball of center $x^* \in T^d$ and radius $\varepsilon > 0$. For $M^* : t \mapsto M(t,x^*)$ we have for all $t \in [0,T]$ (2.13)

$$\forall x \in \text{supp} \theta, \quad \|M(t,x) - M^*(t)\| \leq \varepsilon^\alpha \|M\|_{C^{0,\alpha}(Q_T)}.$$  

Next we compute (2.14)

$$L_{M^*}(\theta V) := \partial_t(\theta V) - M^* \Delta(\theta V) = \theta L_M V + \sum_{k=1}^d \left( \partial_k R_k^* + S_k^* \right),$$

where $R_k^*$ is the error due to freezing in space the coefficients of $M$

$$R_k^* := (M - M^*) \partial_k(\theta V),$$

and $S_k^*$ takes into account the commutators between space derivatives and the truncation $\theta$:

$$S_k^* := 2M(\partial_k V)(\partial_k \theta) + MV \partial_k^2 \theta + (\partial_k M)V(\partial_k \theta).$$

Noticing $L_M \mathbb{P}_0(\theta V) = \mathbb{P}_0 L_{M^*}(\theta V)$ and using Corollary 2.2 we get

$$\|\mathbb{P}_0(\theta V)\|_{X^2_T}^2 + \|\nabla(\theta V)\|_{Y^2_T}^2 \lesssim_{T,[M]_\alpha} \left( \|\mathbb{P}_0(\theta V)(0)\|_{L^2}^2 + \|\mathbb{P}_0 L_{M^*}(\theta V)\|_{Y_T^{-1}}^2 \right).$$

Using (2.14) we have thus (2.16)

$$\|\mathbb{P}_0(\theta V)(t)\|_{X^2_T}^2 + \|\nabla(\theta V)\|_{Y^2_T}^2 \lesssim_{T,[M]_\alpha} \|\mathbb{P}_0(\theta V)(0)\|_{L^2}^2$$

$$+ \|\mathbb{P}_0(\theta L_M V)\|_{Y_T^{-1}}^2 + \sum_{k=1}^d \left( \|\partial_k R_k^*\|_{Y_T^{-1}}^2 + \|\mathbb{P}_0 S_k^*\|_{Y_T^{-1}}^2 \right).$$

But we have, using (2.13) and (2.15),

$$\|\partial_k R_k^*\|_{H^{-1}} \leq \|R_k^*\|_2 \leq \varepsilon^\alpha \|M\|_{C^{0,\alpha}(Q_T)} \|\nabla(\theta V)\|_2.$$  

In particular, denoting by $g_1$ the multiplicative constant behind $\lesssim_{T,[M]_\alpha}$ in (2.16) and defining

$$g_2(z) := \left\{ 2d z^2 g_1(z) \right\}^{\frac{1}{2\alpha}},$$

then if $\varepsilon < g_2(T + [M]_\alpha)^{-1}$ we get

$$\varepsilon^{2\alpha} \|M\|_{C^{0,\alpha}(Q_T)}^2 g_1(T + [M]_\alpha) < \frac{1}{2d},$$
and estimate (2.16) becomes
\[ \|P_0(\theta V)(t)\|_{X_2^p}^2 + \frac{1}{2}\|\nabla(\theta V)\|_{Y_2^p}^2 \lesssim_{T,[M]\alpha} \|P_0(\theta V)(0)\|_2^2 \]
(2.17)
\[ + \|P_0(\theta LMV)\|_{Y_{T-1}}^2 + \sum_{k=1}^d \|P_0 S_k^*\|_{Y_{T-1}}^2. \]

Now let us estimate the other terms on the right-hand side of (2.17). First, we notice that for \( t \in [0, T] \)
\[ \|P_0(\theta LMV)(t)\|_{H^{-1}(\mathbb{T}^d)} \lesssim (\|\theta\|_\infty + \|\nabla\theta\|_\infty)\|LMV(t)\|_{H^{-1}(\mathbb{T}^d)}. \]

For the term involving \( S_k^* \) we write
\[ \|P_0(M(\partial_k V(t))(\partial_k \theta))\|_{H^{-1}(\mathbb{T}^d)} \]
\[ \leq \|V(t)\|_2 \left( \|M\|_\infty \|\nabla \theta\|_\infty + \|\nabla M\|_\infty \|\nabla \theta\|_\infty + \|M\|_\infty \|\Delta \theta\|_\infty \right) \]
\[ \text{and the other terms defining } S_k^* \text{ are estimated similarly. Finally, going back to (2.17) we infer} \]
\[ \|P_0(\theta V)\|_{E_{(2.17)}^2} \lesssim_{T,[M]\alpha} \|P_0(\theta V)(0)\|_2^2 + \|\theta\|_{L^0(\mathbb{T}^d)}^2 \|LMV\|_{Y_{T-1}}^2 \]
\[ + \|\theta\|_{L^{1,1}(\mathbb{T}^d)}^2 \|M\|_{L^{0,1}(Q_T)} \int_0^T \|V(t)\|_2^2 dt, \]
\[ \text{and the proof of Lemma 2.4 is over.} \]

We now proceed to the proof of Lemma 2.3. We fix \( \varepsilon > 0 \) as in Lemma 2.4 and decompose \( \mathbb{T}^d \) into a finite union of essentially disjoint hypercubes denoted \((K_j^\varepsilon)_{1 \leq j \leq J_M}\) centered at points \( x_j \in K_j^\varepsilon \), with sidelengths \( \varepsilon \). This implies that \( J_M \) is of the order of \( 1/\varepsilon^d \). We then consider a partition of unity \((\theta_j^\varepsilon)_{1 \leq j \leq J_M}\) where each \( \theta_j^\varepsilon \) is compactly supported in a ball \( B_j^\varepsilon \) of \( \mathbb{T}^d \) of radius \( \varepsilon \) containing strictly \( K_j^\varepsilon \), and takes it values in \([0, 1]\). We assume in particular that for any multi-index \( \alpha \in \mathbb{N}^d \), there is a constant \( C_\alpha \) such that for any \( 1 \leq j \leq J_M \), any \( \varepsilon > 0 \) and any \( t \geq 0 \)
\[ \|D_\alpha^T \theta_j^\varepsilon\|_\infty \leq C_\alpha \varepsilon^{-|\alpha|}. \]
Finally we note the existence of a constant \( C > 0 \) such that
\[ c \leq \sum_j (\theta_j^\varepsilon)^2 \leq \left( \sum_j \theta_j^\varepsilon \right)^2 = 1. \]

Now let us apply the result of Lemma 2.4 with \( \theta = \theta_j^\varepsilon \), for \( 1 \leq j \leq J_M \). We have thus
\[ \|P_0(\theta_j^\varepsilon V)\|_{E_2^0} \lesssim_{T,[M]\alpha} \|P_0(\theta_j^\varepsilon V)(0)\|_2^2 + \|\theta_j^\varepsilon\|_{L^0(\mathbb{T}^d)}^2 \|LMV\|_{Y_{T-1}}^2 \]
\[ + \|\theta_j^\varepsilon\|_{L^{1,1}(\mathbb{T}^d)}^2 \|M\|_{L^{0,1}(Q_T)} \int_0^T \|V(t)\|_2^2 dt. \]
Note that the constant behind $\lesssim_{T, [M]_\alpha}$ is increasing with $T + [M]_\alpha$ and does not depend on $j$. Using (2.18) and that $\varepsilon$ is a decreasing function of $T + [M]_\alpha$ we find
\[
\|\mathbb{P}_0 (\theta_\varepsilon^j V)\|_{X_T^0}^2 + \|\nabla (\theta_\varepsilon^j V)\|_{Y_T^1}^2 \lesssim_{T, [M]_\alpha} \mathbb{P}_0 (\theta_\varepsilon^j V)(0) \frac{2}{2}
+ \|L_M V\|_{Y_T^{-1}}^2 + \|M\|_{C^{0,1}(Q_T)}^2 \int_0^T \|V(t)\|_{2}^2 dt.
\]
Using again (2.18) we have
\[
\|\theta_\varepsilon^j \nabla V(t)\|_2 \lesssim \frac{1}{\varepsilon} \|V(t)\|_2 + \|\nabla (\theta_\varepsilon^j V(t))\|_2,
\]
so using also that $\mathbb{P}_0$ is a projection and again that $\varepsilon$ is a decreasing function of $T + [M]_\alpha$, we infer
\[
\|\mathbb{P}_0 (\theta_\varepsilon^j V)\|_{X_T^0}^2 + \int_0^T \|\theta_\varepsilon^j \nabla V(t)\|_{Y_T^1}^2 dt \lesssim_{T, [M]_\alpha} \|\theta_\varepsilon^j V(0)\|_2^2 + \|L_M V\|_{Y_T^{-1}}^2
+ \left(1 + \|M\|_{C^{0,1}(Q_T)}^2\right) \int_0^T \|V(t)\|_{2}^2 dt.
\]
We now sum over $1 \leq j \leq J_M$, using (2.19) to deduce
\[
\sum_{j=1}^{J_M} \|\mathbb{P}_0 (\theta_\varepsilon^j V)\|_{X_T^0}^2 + c \int_0^T \|\nabla V(t)\|_{2}^2 dt \lesssim_{T, [M]_\alpha} \|V(0)\|_2^2 + J_M \|L_M V\|_{Y_T^{-1}}^2
+ J_M \left(1 + \|M\|_{C^{0,1}(T)}^2\right) \int_0^T \|V(t)\|_{2}^2 dt.
\]
On the other hand a Cauchy-Schwarz inequality implies for $t \in [0, T]
\[
\left\| \sum_{j=1}^{J_M} \mathbb{P}_0 (\theta_\varepsilon^j V)(t) \right\|_2 \leq J_M \sum_{j=1}^{J_M} \|\mathbb{P}_0 (\theta_\varepsilon^j V(t))\|_2,
\]
from which, using that $(\theta_\varepsilon^j)_j$ is a partition of unity and $\mathbb{P}_0$ linear with $\mathbb{P}_0 V(t) = V(t)$, we deduce
\[
\|V(t)\|_{X_T^0}^2 \leq J_M \sum_{j=1}^{J_M} \|\mathbb{P}_0 (\theta_\varepsilon^j V)\|_{X_T^0}^2.
\]
Putting these inequalities together, we thus
\[
\|V\|_{X_T^0}^2 + \int_0^T \|\nabla V(t)\|_{2}^2 dt \lesssim_{T, [M]_\alpha} J_M \|V(0)\|_2^2 + J_M^2 \|L_M V\|_{Y_T^{-1}}^2
+ J_M^2 \left(1 + \|M\|_{C^{0,1}(Q_T)}^2\right) \int_0^T \|V(t)\|_{2}^2 dt.
\]
Now, recalling that $J_M \sim \varepsilon^{-d}$ and that $\varepsilon$ has been chosen decreasingly depending on $T + [M]_\alpha$, we can absorb $J_M$ and $J_M^2$ in $\lesssim_{T, [M]_\alpha}$ and thus end the proof of Lemma 2.3.
2.5. **Proof of Theorem 5.** Instead of localizing in the physical variable as we did in Theorem 4 our strategy here is to localize in frequency (the Fourier variable) and apply the estimate of Theorem 4 on each local part. Letting $F := L_M V$, we use the Littlewood-Paley decomposition recalled in Appendix B to write for all $j \geq -1$,

\begin{equation}
\partial_t \Delta_j V - \Delta_j \sum_{k=1}^d \partial_k [M \partial_k V] = \Delta_j F.
\end{equation}

Since Theorem 4 focuses on the operator $L_M$, we write instead

\begin{equation}
L_M \Delta_j V := \partial_t \Delta_j V - \sum_{k=1}^d \partial_k [M \partial_k \Delta_j V] = \Delta_j F + [L_M, \Delta_j] V,
\end{equation}

where, since $\Delta_j$ is linear and commutes with partial derivatives, the commutator $[L_M, \Delta_j]$ is actually given by

\[ [L_M, \Delta_j] = \sum_{k=1}^d \partial_k (\Delta_j (M \partial_k V) - M \Delta_j \partial_k V) = \sum_{k=1}^d \partial_k ([\Delta_j, M] \partial_k V), \]

where in the last term we identified $M$ and the corresponding linear multiplication operator. Understanding the commutator $[\Delta_j, M]$ can be done through the paralinearization procedure, which is based on the following identity

\[ [\Delta_j, M] = \Delta_j \sum_{j' \geq -1} (\Delta_{j'} M) S_{j'+2} - \sum_{j' \geq -1} [S_{j'-1} M, \Delta_j] \Delta_{j'} + \sum_{j' \geq -1} (S_{j'-1} M - M) \Delta_j \Delta_{j'}. \]

For a definition of the operators $S_j$ and a proof of this identity, we refer to Appendix B and more precisely to (B.3). Since $\Delta_j$ obviously commutes with constant functions, we have $[\Delta_j, M] = [\Delta_j, M + B]$, for any element $B$ of $M_N(\mathbb{R})$. We fix from now on such a matrix $\hat{B}$ and note $M_B := M + \hat{B}$. It follows that (2.21) writes under the form

\[ L_M \Delta_j V = \Delta_j F + R_j, \]

where

\begin{equation}
R_j := \sum_{k=1}^d \partial_k ([\Delta_j, M_B] \partial_k V),
\end{equation}

can be decomposed into $R_j = \sum_{i=1}^3 R_j^i$ with

\begin{align*}
R_j^1 &:= \Delta_j \sum_{j' \geq j} \partial_k \left( (\Delta_{j'} M_B) \partial_k S_{j'+2} V \right), \\
R_j^2 &:= - \sum_{|j-j'| \leq 1} \partial_k \left( [S_{j'-1} M_B, \Delta_j] \partial_k \Delta_{j'} V \right), \\
R_j^3 &:= \sum_{|j-j'| \leq 1} \partial_k \left( (S_{j'-1} M_B - M_B) \partial_k \Delta_{j'} \Delta_j V \right).
\end{align*}
Notice that \( \Delta_j \Delta_j \) vanishes unless \( |j - j'| \leq 1 \), which explains the restriction on the summation in \( R^1_j \). There is a similar restriction on \( R^2_j \), due to the fact that one can write

\[
R^2_j = -\partial_k \sum_{j' \geq -1} (S_{j'-1} M_B) \partial_k \Delta_j \Delta_j V - \partial_k \Delta_j \sum_{j' \geq -1} (S_{j'-1} M_B) \partial_k \Delta_j V.
\]

The first sum indeed involves only indices for which \( |j - j'| \leq 1 \) because of the product \( \Delta_j \Delta_j \), and so does the second one since it is a paraproduct term: \( (S_{j'-1} M_B) \partial_k \Delta_j V \) is supported in Fourier space in a ring of size \( \sim 2^j \). Finally \( (\Delta_j M_B) \partial_k S_{j'+2} V \) is supported in Fourier space in a ball of size \( \sim 2^{j'} \), whence the restriction \( j' \geq j \) in the definition of \( R^1_j \).

Let us now apply Theorem 4. Since \( V \) belongs to \( E^0 \) and has a vanishing mean for all times \( t \), that is also the case of \( \Delta_j V \). We get therefore for all times \( t \geq 0 \), from (1.11),

\[
\| \Delta_j V(t) \|^2 + \int_0^t \| \nabla \Delta_j V(t') \|^2 \, dt' \lesssim_{T, \|M\|_\alpha} \| \Delta_j V(0) \|^2 + \int_0^t \| R_j(t') \|_{H^{-1}(\mathbb{T}^d)}^2 \, dt' + \int_0^t \| \Delta_j F(t') \|_{H^{-1}(\mathbb{T}^d)}^2 \, dt'.
\]

Since \( \Delta_j \) localizes at \( 2^j \) the frequency variable (see Appendix B), we infer

\[
\| \Delta_j V(t) \|^2 + 2^j \int_0^t \| \Delta_j V(t') \|^2 \, dt' \lesssim_{T, \|M\|_\alpha} \| \Delta_j V(0) \|^2 + \int_0^t 2^{-2j} \| R_j(t') \|^2 \, dt' + \int_0^t 2^{-2j} \| \Delta_j F(t') \|^2 \, dt'.
\]

From now on and until the end of the proof, we will denote \( * \) the discrete convolution on \( \ell^2(\mathbb{Z}) \), a the sequence \( (2^{-j})_j \) and for any positive integer \( J, A_J := (a_j \mathbf{1}_{j \geq J \geq 0})_j \). We understand algebraic operations (power and multiplication) on sequences term-wise. The following (time dependent) elements of \( \ell^2(\mathbb{Z}) \) (recall the convention \( \Delta_j = 0 \) for \( j < -1 \)) will play a central role in the coming lines:

\[
m(t) := (2^{j(s+1)} \| \Delta_j M_B(t) \|_2)_j \quad \text{and} \quad v(t) := (2^{jx} \| \Delta_j V(t) \|_2)_j,
\]

which satisfy

\[
\| v(t) \|_{\ell^2(\mathbb{Z})} \sim \| V(t) \|_{H^s(\mathbb{T}^d)},
\]

\[
\| m(t) \|_{\ell^2(\mathbb{Z})} \sim \| M_B(t) \|_{H^{s+1}(\mathbb{T}^d)}.
\]

We will also use the notation \( r^i(t) := (2^{j(s-1)} \| R^i_j(t) \|_2)_j \) and our goal is to prove that

\[
\forall i \in \{1, 2, 3\}, \quad \| r^i(t) \|_{\ell^2(\mathbb{Z})} \lesssim \| m(t) \|_{\ell^2(\mathbb{Z})} \| v(t) \|_{\ell^2(\mathbb{Z})}.
\]

Let us analyze the right-hand side of (2.23), starting with \( R^1_j \) defined in (2.22). We notice that

\[
\| R^1_j(t) \|_2 \lesssim 2^j \sum_{j' \geq j-1} \left( \| \Delta_{j'} M_B(t) \|_2 \sum_{j'' \leq j'} 2^{j''} \| \Delta_{j''} V(t) \|_\infty \right),
\]
so using the Bernstein inequality (B.2) to write \(|\Delta_j^\nu V(t)|_\infty \lesssim 2^{j''\frac{d}{2}}|\Delta_j^\nu V(t)|_2\) we find
\[
r_j^1(t) := 2^{j(s-1)}\|R_j^1(t)\|_2 \lesssim 2^{js} \sum_{j'' \geq j-1, j'' \leq j} \|\Delta_j^\nu M_B(t)\|_2 2^{j'' 2^{j''(\frac{d}{2} - s)}} m_{j''}(t) v_{j''}(t)
= \sum_{j'' \geq j-1, j'' \leq j} 2^{(j-j'')s} 2^{j''-j'} 2^{j''(\frac{d}{2} - s)} m_{j'}(t) v_{j'}(t)
= A^*_j \left[ A_j \ast (a^{s-d/2} v(t)) \right] m(t),
\]
where the positive integer \(J\) is related to the symbol \(j'' \lesssim j'\). Now using \(s > d/2\) we have \(a^{s-d/2} \leq 1\) component-wise. Using that \(a_j^*\) and \(a_J\) are respectively in \(\ell^1(\mathbb{Z})\) and \(\ell^2(\mathbb{Z})\) we get, thanks to Young’s inequalities in the cases \(\ell^1(\mathbb{Z}) \ast \ell^2(\mathbb{Z})\) and \(\ell^2(\mathbb{Z}) \ast \ell^2(\mathbb{Z})\)
\[
\left\| A^*_j \left[ A_j \ast (a^{s-d/2} v(t)) \right] m(t) \right\|_{\ell^2(\mathbb{Z})} \leq \|v(t)\|_{\ell^2(\mathbb{Z})} \|m(t)\|_{\ell^2(\mathbb{Z})},
\]
and we have therefore proved
\[
\|r_j^1(t)\|_{\ell^2(\mathbb{Z})} \lesssim \|v(t)\|_{\ell^2(\mathbb{Z})} \|m(t)\|_{\ell^2(\mathbb{Z})}.
\]
For \(R_j^2\) we write
\[
\|R_j^2(t)\|_2 \lesssim 2^j \sum_{|j'\rangle \leq 1} \|\Delta_j^\nu [S_{j'} M_B(t), \Delta_j^\nu V(t)]\|_2
\]
and we use the fact (see for instance [3, Lemma 2.97]) that for any two functions \(f\) and \(g\)
\[
\|\Delta_j^\nu [S_{j'} f, \Delta_j^\nu g]\|_{L^2} \lesssim 2^{-j} \|\nabla S_{j'} f\|_\infty \|\Delta_j^\nu g\|_2.
\]
This implies, using again Bernstein’s inequality (B.2) in the third line, that
\[
\|r_j^2(t) := 2^{j(s-1)}\|R_j^2(t)\|_2 \lesssim 2^{js} \sum_{|j'\rangle \leq 1} \|\Delta_j^\nu [S_{j'} M_B(t), \Delta_j^\nu V(t)]\|_2
\lesssim 2^{js} \sum_{|j'\rangle \leq 1} \sum_{j'' \leq j} 2^{-j''} \|\nabla \Delta_j^\nu M_B(t)\|_\infty \|\Delta_j^\nu V(t)\|_2
\lesssim 2^{j(s-1)} \sum_{|j'\rangle \leq 1} \sum_{j'' \leq j} 2^{j''} 2^{j''(\frac{d}{2} - 1)} \|\Delta_j^\nu M_B(t)\|_2 2^{j'} \|\Delta_j^\nu V(t)\|_2
= 2^{j(s-1)} \sum_{|j'\rangle \leq 1} \sum_{j'' \leq j} 2^{j''} 2^{j''(\frac{d}{2} - 1)} 2^{j''(s-1)} 2^{j'} 2^{-j''} m_{j''}(t) v_{j''}(t).
\]
Since \(|j'\rangle \leq 1|\) in the previous sums, we infer by the Cauchy-Schwarz inequality (using the fact that \(s > d/2\))
\[
r_j^2(t) \lesssim \sum_{|j'\rangle \leq 1} v_{j'}(t) \sum_{j''} 2^{j''(\frac{d}{2} - 1)} m_{j''}(t) \lesssim \|m(t)\|_2 \sum_{|j'\rangle \leq 1} v_{j'}(t),
\]
so we recover
\[
\|r_j^2(t)\|_{\ell^2(\mathbb{Z})} \leq \|m(t)\|_{\ell^2(\mathbb{Z})} \|v(t)\|_{\ell^2(\mathbb{Z})}.
\]
Finally
\[ r_j^3(t) := 2^{j(s-1)} \sum_{|j-j'| \leq 1} \| \partial_k ((S_{j-1} - M_B(t)) \partial_k \Delta_j V) \|_2 \]
\[ \lesssim 2^{j(s-1)} \sum_{|j''-j| \leq 1} 2^{j''} \| \Delta_j M_B(t) \|_\infty 2^{j'} \| \Delta_j V(t) \|_2 + \| \Delta_j M_B(t) \|_\infty 2^{2j'} \| \Delta_j V(t) \|_2 \]
\[ \lesssim 2^{j(s-1)} \sum_{|j''-j| \leq 1} 2^{j''} \| \Delta_j M_B(t) \|_2 2^{j''} 2^{j'} \| \Delta_j V(t) \|_2 \]
by Bernstein’s inequality (B.2). It follows that
\[ r_j^3(t) \lesssim 2^{j(s-1)} \sum_{|j''-j| \leq 1} 2^{j''} 2^{j''} 2^{j''} 2^{j''} - j''(s+1) \sum_{|j''-j| \leq 1} 2^{j''} v_j(t) m_{j''}(t) \]
for which we can proceed as for \( r^2(t) \) to recover
\[ \| r_3^3(t) \|_{L^2(\mathbb{Z})} \leq \| m(t) \|_{L^2(\mathbb{Z})} \| v(t) \|_{L^2(\mathbb{Z})} \]
Returning to (2.23) we get for all \( j \geq -1 \)
\[ 2^{2js} \| \Delta_j V(t) \|^2_2 + 2^{2j(s+1)} \int_0^t \| \Delta_j V(t') \|^2_2 \ dt' \lesssim T[M]_\alpha 2^{2js} \| \Delta_j V(0) \|^2_2 \]
\[ + 2^{2j(s-1)} \int_0^t (\| R_j(t') \|^2_2 + \| \Delta_j F(t') \|^2_2) \ dt' . \]
Using (2.24) – (2.25), the previous bounds on the sequences \( r^i(t) \) and the triangular inequality to write \( 2^{j(s-1)} \| R_j(t) \|_2 \leq r_j^1(t) + r_j^2(t) + r_j^3(t) \), we infer, after summing from \( j = -1 \) to some arbitrary integer \( J \in \mathbb{N} \)
\[ \| S_j V(t) \|^2_{H^s(\mathbb{T}^d)} + \int_0^t \| \nabla S_j V(t') \|^2_{H^s(\mathbb{T}^d)} \ dt' \lesssim T[M]_\alpha \| V(0) \|^2_{H^s(\mathbb{T}^d)} \]
\[ + \int_0^t \| S_j V(t') \|^2_{H^s(\mathbb{T}^d)} \| M_B(t') \|^2_{H^{s+1}(\mathbb{T}^d)} \ dt' + \int_0^t \| F(t') \|^2_{H^{s-1}(\mathbb{T}^d)} \ dt' . \]
Using Grönwall’s lemma we obtain first
\[ \sup_{t \in [0,T]} \| S_j V(t) \|^2_{H^s(\mathbb{T}^d)} + \int_0^T \| \nabla S_j V(t') \|^2_{H^s(\mathbb{T}^d)} \ dt' \]
\[ \lesssim T[M]_\alpha \| M_B \|_{H^{s+1}(\mathbb{T}^d)} \| V(0) \|^2_{H^s(\mathbb{T}^d)} + \| L_M V \|^2_{Y^{s-1}_T(\mathbb{T}^d)} . \]
From this follows that \( V \) actually belongs to \( L^\infty([0,T];H^s(\mathbb{T}^d)) \cap Y^{s+1}_T \) (because \( S_j V \), \( j \) is bounded in that space), and therefore that \( V \) belongs to \( E^s_T \) because from the equation we
have $\partial_t V \in Y^s_T$ (see Lemma A.1 for this standard result), and we have the estimate

$$\|V\|_{E^s_T} \lesssim_T |M|_\alpha \|M_B\|_{Y^s_{T+1}} \|V(0)\|_{H^s(\mathbb{T}^d)} + \|LMV\|_{Y^{s-1}_{T+1}(\mathbb{T}^d)}.$$ 

Now, we infer from Lemma A.2 of Appendix A the existence of a real number $\alpha_s \in (0,1)$ such that $\|V\|_{C^{0,\alpha_s}(\mathcal{Q}_T)} \lesssim \|V\|_{X^s_T} + \|LMV\|_{Y^s_{T+1}}$ and we recover therefore estimate (1.12) recalling that here $M_B := M + B$ with $B$ arbitrary. Theorem 5 is proved.

2.6. A useful corollary. We end this series of a priori estimates with a corollary of Theorem 5 which will be useful when $M$ is of the form $A(U)$ with $A : \mathbb{R}^N \to \mathcal{P}$ a smooth matrix field, as this is the case in Theorem 1.

**Corollary 2.5.** Fix $s > d/2$ and $T > 0$. Consider $U : Q_T \to \mathbb{R}^N$ belonging to $Y^{s+1}_T$ and to $C^{0,\alpha}(Q_T)$ for some $\alpha \in (0,1]$, and $A : \mathbb{R}^N \to \mathcal{P}$ a smooth matrix field. For any $V \in E^s_T$ such that $V(0) \in H^s(\mathbb{T}^d)$, $L_A(U)V \in Y_{T+1}^s$ and $(V(t)) = 0$ for all $t \in [0,T]$, one has actually that $V$ belongs to $E^s_T \cap C^{0,\alpha_s}(Q_T)$ for some $\alpha_s \in (0,1)$ and

$$\|V\|_{E^s_T} + \|V\|_{C^{0,\alpha_s}(Q_T)} \lesssim_T \|U\|_{C^{0,\alpha}(Q_T)} \|U\|_{Y^s_T} \|V(0)\|_{H^s(\mathbb{T}^d)} + \|L_A(U)V\|_{Y^s_{T+1}}. \tag{2.26}$$

**Proof.** Of course the proof reduces to justifying the use of Theorem 5 for $M = A(U)$ and to replace the intricate dependence $T + [A(U)]_\alpha + d_{Y^{s+1}_T}(A(U), \mathbb{R})$ by the above simpler one for the symbol $\lesssim T \|U\|_{C^{0,\alpha}(Q_T)} \|U\|_{Y^s_T}$ appearing in the estimate.

We first note that $A$ being smooth, it stabilizes Sobolev spaces and induces locally a lipschitz map. More precisely, since $U \in E^s_T \to L^\infty(Q_T)$, we use Lemma A.3 with $\Phi = A$, $f = U$, $g = 0$ and $\sigma = s + 1$ to recover $A(U) \in Y^{s+1}_T$ with a bound

$$\|A(U) - A(0)\|_{Y^{s+1}_T} \lesssim \|U\|_{C^{0,\alpha}(Q_T)} \|U\|_{Y^s_T}. \tag{2.27}$$

Also, since $A$ is locally lipschitz, the $\alpha$-Hölder regularity of $U$ is inherited by $A(U)$ with an estimate of the form $\|A(U)\|_{C^{0,\alpha}(Q_T)} \leq g(\|U\|_{C^{0,\alpha}(Q_T)})$, with $g$ an increasing function. We are now in position to invoke Theorem 5 which states exactly

$$\|V\|_{E^s_T} + \|V\|_{C^{0,\alpha}(Q_T)} \lesssim_T \|A(U)\|_{C^{0,\alpha}} d_{Y^{s+1}_T}(A(U), \mathbb{R}) \|V(0)\|_{H^s(\mathbb{T}^d)} + \|L_A(U)V\|_{Y^s_{T+1}}.$$

We use Remark 1.7 with $B = A(0)$ to replace $d_{Y^{s+1}_T}(A(U), \mathbb{R})$ by $\|A(U) - A(0)\|_{Y^{s+1}_T}$ and this term is handled thanks to (2.27). Recalling the definition of $[A(U)]_\alpha$ in Paragraph 1.4, we only need to handle $\eta(A(U))^{-1}$. For this, we use Corollary C.3 to see that $A$ maps the ball of radius $R$ of $\mathbb{R}^N$ into some $\mathcal{P}_{\delta_R}$ with $\delta_R$ decreasing in $R$, so that $\eta(A(U))^{-1}$ is indeed bounded by some increasing function of $\|U\|_\infty \leq \|U\|_{C^{0,\alpha}(Q_T)}$. Corollary 2.5 is proved.

3. Existence theory and parabolic regularization in the linear case

In this short section we first prove Theorem 6 thanks to the a priori estimates of Theorem 4 and Theorem 5 established in Section 2, and then state and prove as a corollary of these results a propagation of regularity result.
3.1. **Proof of Theorem 6.** Uniqueness is a straightforward consequence of estimates (1.11) and (1.12), so we focus only on the existence part starting with the case \( s = 0 \), that is the \( E_T^0 \) setting. Also, without loss of generality we only need to establish this existence result replacing \( F \) by \( P_0F \) and \( V^0 \) by \( P_0V^0 \): if a solution is built in this vanishing mean setting, adding to it

\[
\langle V^0 \rangle + \int_0^t \langle F(t') \rangle \, dt',
\]

we will recover a solution in the general case.

The set \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) is path-connected for the \( \mathcal{C}^{0,\alpha}(Q_T; M_N(\mathbb{R})) \) topology. That is indeed the case for \( \mathcal{P} \) within \( M_N(\mathbb{R}) \) since it is starshaped with respect to the identity matrix: the segment \( \sigma \mapsto \sigma M + (1 - \sigma)I_N \) links continuously any element \( B \in \mathcal{P} \) to \( I_N \) without exiting \( \mathcal{P} \). It remains to map any \( M \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) continuously to a constant matrix, for instance using the path \( \sigma \mapsto \{(t, x) \mapsto M(\sigma t, \sigma x)\} \).

Define \( \mathcal{I} \) as the subset of \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) constituted of all \( \alpha \)-Hölder Petrovskii-valued matrix fields for which the problem (1.2) (for arbitrary data \((V^0, F)\) with vanishing mean) has a solution in \( E_T^0 \). The set \( \mathcal{I} \) is closed in \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \). Indeed, should \((M_k)\), \( k \in \mathbb{N} \), only converge uniformly to \( M \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \), this is already sufficient to ensure (see Corollary C.3) that \( \eta(M_k) \) converges to \( \eta(M) > 0 \), so that the whole sequence satisfies \( \eta(M_k) \geq \delta \) for some \( \delta > 0 \). Now, as \((M_k)\) is bounded in \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) and \( \lim_k \eta(M_k) > 0 \), we infer that \( \lim_k[M]_k < +\infty \). We thus infer uniformity in \( k \) for the \textit{a priori} estimate (1.11) of Theorem 4 satisfied by the solutions \( V_k \) associated with \( M_k \) (such solutions \( V_k \) exist precisely because the sequence \((M_k)\) lies in \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \)). The equation being linear, we recover in this way by a weak\((-\star)\) compactness argument first a solution in \( L^\infty(0, T; L^2(\mathbb{T}^d)) \cap Y^1_T \) which in fact belongs to \( E_T^0 \), using the equation to control the time derivative and the standard Lemma A.1. The subset \( \mathcal{I} \) is also open in \( \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) : for \( M \in \mathcal{I} \) and \( \varepsilon > 0 \) to be defined later, if \( \tilde{M} \in \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \) satisfies (the uniform norm is enough) \( \|M - \tilde{M}\|_\infty < \varepsilon \), one can define the map from \( E_T^0 \) to itself which sends \( V \) to the solution \( \tilde{V} \) of (taking \( V^0 \) as initial data)

\[
\partial_t \tilde{V} - \sum_{k=1}^d \partial_k [M \partial_k \tilde{V}] = F + \sum_{k=1}^d \partial_k [(M - \tilde{M})\partial_k V].
\]

The existence of \( \tilde{V} \) is due to the fact that \( M \) has been chosen in \( \mathcal{I} \). Linearity and the \textit{a priori} estimate (1.11) of Theorem 4 provide

\[
\|\tilde{V}_1 - \tilde{V}_2\|^2_{E_T^0} \lesssim_{T,J_{\alpha}} \|M - \tilde{M}\|_{\infty} \int_0^T \|\nabla(V_1 - V_2)(t')\|^2_2 \, dt' \leq \varepsilon \|V_1 - V_2\|^2_{E_T^0}.
\]

Choosing \( \varepsilon \) small enough implies that the map \( V \mapsto \tilde{V} \) is a contraction, thus \( \mathcal{I} \) contains \( \tilde{M} \). Finally, we proved that \( \mathcal{I} \) is open and closed in \( \mathcal{C}^{0}(Q_T; \mathcal{P}) \) which is connected; the set \( \mathcal{I} \) is non empty (constant matrices belong to \( \mathcal{I} \) thanks to Proposition 2.1) so \( \mathcal{I} = \mathcal{C}^{0,\alpha}(Q_T; \mathcal{P}) \).

In the case \( s > d/2 \) in which \((V^0, F) \in D_T^r \), Theorem 5 is enough to obtain the existence of \( \alpha_s \in (0, 1) \) such that \( V \) actually belongs to \( E_T^s \cap \mathcal{C}^{0,\alpha_s}(Q_T) \).
3.2. Propagation of regularity. Let us prove the following result.

**Corollary 3.1.** Let $s > d/2$ and $(V^0, F) \in \mathcal{D}^s_T$. Consider $V \in E^s_T$ the associate solution to (1.2) as given by Theorem 6. Let $s' \in [s, s+1]$ be given and assume furthermore that $(V^0, F)$ belongs to $\mathcal{D}^{s'}_T$. Then $V$ actually belongs to $E^{s'}_T$ and satisfies

$$\|V\|_{E^{s'}_T} \lesssim_{T,[M], \alpha} \|V^0, F\|_{\mathcal{D}^{s'}_T}. \quad (3.1)$$

**Proof.** Thanks to the well-posedness setting of Theorem 6 we only need to prove the estimate with smooth $V$ and $F$. We have thanks to estimate (1.12)

$$\|V\|_{E^{s'}_T} \lesssim_{T,[M], \alpha} \|V^0, F\|_{\mathcal{D}^{s'}_T}. \quad (1.12)$$

Using the interpolation $H^s(\mathbb{T}^d) = [H^s(\mathbb{T}^d), H^{s+1}(\mathbb{T}^d)]_\theta$ for $s < s' < s+1$, it is enough to prove (3.1) for $s' = s+1$. Now, for any spatial derivative $\partial_\ell$, we note that $Z_\ell := \partial_\ell V$ solves

$$\partial_\ell Z_\ell - \sum_{k=1}^d \partial_\ell [M \partial_k Z_\ell] = \partial_\ell F + \sum_{k=1}^d \partial_\ell [\partial_\ell M \partial_k V].$$

Using the assumption on $M$ and that $\nabla V \in L^2(0,T; H^s(\mathbb{T}^d))$, the previous equality already implies that $\partial_\ell Z_\ell \in L^1(0,T; H^s(\mathbb{T}^d))$ which implies $Z_\ell \in C^0([0,T]; H^s(\mathbb{T}^d))$; this establishes that $V \in C^0([0,T]; H^{s+1}(\mathbb{T}^d))$. Then, we use once more estimate (1.12) for each $\ell$ to infer after summation using the algebra structure of $H^s(\mathbb{T}^d)$, for $t \leq T$

$$\|\nabla V(t)\|_{L^2_T}^2 \lesssim_{T,[M], \alpha} \|\nabla V^0, \nabla F\|_{\mathcal{D}^{s'}_T} + \int_0^T \|\nabla M(t)\|_{H^s(\mathbb{T}^d)} \|V(t)\|_{H^{s+1}(\mathbb{T}^d)}^2 \, dt$$

and the conclusion follows by Gronwall’s inequality. \qed

4. Proof of Theorem 1

Now that we have a clear setting of well-posedness for the linear problem (1.2), in order to prove Theorem 1 we aim to solve (1.1) for a given $(U^0, F) \in \mathcal{D}^s_{\infty}$ on a small interval $[0,T_0]$, by means of a Picard scheme. For $s > d/2$ and $\alpha_s \in (0,1)$ given by Lemma A.2, we will use in this paragraph the notation $G^s_T := E^s_T \cap C^{0,\alpha_s}(Q_T)$ and by a small abuse of notation we will write $G^s_T(U^0)$ for the (closed) affine subspace of $G^s_T$ constituted of those vector fields $U$ satisfying $U(0) = U^0$. Note that $G^s_T(U^0)$ is a complete metric space.

4.1. Existence and uniqueness in a small ball of $G^s_T(U^0)$. Given $(U^0, F) \in \mathcal{D}^s_{\infty}$, we consider the following map

$$\Theta : G^s_T(U^0) \to G^s_T(U^0)$$

$$U \mapsto U^*,$$

where $U^*$ is the only element (existence and uniqueness stem from Theorem 6) of $G^s_T(U^0)$ solving $L_{A(U)} U^* = F$. The use of Theorem 6 is justified because $U \in G^s_T \Rightarrow A(U) \in G^s_T$, using Lemma A.3 with $\Phi = A$, $f = U$ and $g = 0$. Just as we did in the proof of Corollary 2.5.
in Paragraph 2.6 we recover in this way that \( A(U) \in X^{s+1}_T \cap \mathcal{C}^{0,\alpha}(Q_T) \). To see that \( A(U) \) belongs to \( X^s_T \) we rely on Lemma A.3 (with \( f = U \) and \( g = 0 \)) using \( X^s_T \hookrightarrow L^\infty(Q_T) \).

Now that \( \Theta \) is well-defined for all \( T > 0 \), we hope to find a time small enough so as \( \Theta \) becomes a contraction. Since

\[
L_{A(U_1)}(U_1^* - U_2^*) = \sum_{k=1}^d \partial_k \left[ (A(U_1) - A(U_2)) \partial_k U_2^* \right],
\]

we infer from Corollary 2.5

\[
\|U_1^* - U_2^*\|_{G^s_T} \lesssim_{T,\|U_1\|_{G^s_T}} \|A(U_1) - A(U_2)\|_{X^s_T} \|\nabla U_2^*\|_{Y^s_T}.
\]

Thanks to Lemma A.3 we have

\[
\forall U_1, U_2 \in X^s_T,
\|A(U_1) - A(U_2)\|_{X^s_T} \leq \varphi(\|U_1\|_{X^s_T} + \|U_2\|_{X^s_T}) \|U_1 - U_2\|_{X^s_T},
\]

where \( \varphi \) is some increasing function related to \( A \), so

\[
\|U_1^* - U_2^*\|_{G^s_T} \lesssim_{T,\|U_1\|_{G^s_T},\|U_2\|_{G^s_T}} \|U_1 - U_2\|_{X^s_T} \|\nabla U_2^*\|_{Y^s_T}.
\]

It seems clear, due to the presence of the multiplicative constant, that no global contraction rate can be achieved for \( \Theta \) and we need to localize this map on some ball of \( G^s_T \) to hope for a contraction. On the other hand \( \|\nabla U_2^*\|_{Y^s_T}^2 \) will indeed tend to be small as \( T \to 0 \), but with a decay which will depend on \( U_2 \) and not only on the data of the problem. The strategy is thus to choose as fixed profile \( U_F \in G^s_T(U^0) \) around which the fixed-point will be searched. More precisely, we have the following lemma, recalling here the notation

\[
\|(U^0, F)\|_{\mathcal{D}^s_T} := \|U^0\|_{H^s(\mathbb{T}^d)} + \|F\|_{Y^{s-1}_T} + \int_0^T \|\hat{F}(t)\| \, dt,
\]

to keep track of the data’s size.

**Lemma 4.1.** Fix \( s > d/2 \) and data \((U^0, F) \in \mathcal{D}^s_T\). For any \( T > 0 \) there exists a unique \( U_F \in G^s_T(U^0) \) such that \( L_{A(0)} U_F = F \) and it satisfies the following estimate

\[
\|U_F\|_{G^s_T} \lesssim \|(U^0, F)\|_{\mathcal{D}^s_T},
\]

where \( \lesssim \) depends only on the matrix \( A(0) \). Furthermore, there exists an increasing function \( g \) such that, for any \( r \in (0,1] \) and \( T > 0 \), the closed ball \( B_{G^s_T}(U_F, r) \) is stabilized by \( \Theta \) as soon as \( T \) and \( r \) satisfy

\[
g(T + 1 + \|(U^0, F)\|_{\mathcal{D}^s_T}) \|\nabla U_F\|_{Y^s_T} \leq r.
\]

Under this condition \( \Theta \) is lipschitz on \( B_{G^s_T}(U_F, r) \) with a lipschitz constant bounded by

\[
g(T + 1 + \|(U^0, F)\|_{\mathcal{D}^s_T})(r + \|\nabla U_F\|_{Y^s_T}).
\]

**Proof.** We note that \( U_F \) is in fact nothing more than \( \Theta(0) \), so its existence and uniqueness are not new since we already proved that \( \Theta \) is well-defined. However, the symbol \( \lesssim \) in estimate (4.3) is independent of the time variable, and this is important. To obtain this, we rely on the setting for constant matrix fields given in Proposition 2.1, using Lemma A.2 to
add the Hölder norm in the estimate and adding the time evolution of the spatial average as we did in the beginning of Section 3. This proves (4.3).

For any $r \in (0, 1)$, if $U_1$ belongs to $B_{G_T^0}(U_F, r)$, we infer from (4.2) applied with $U_2 = 0$ and $U_2^* = \Theta(0) = U_F$ that

$$
\|U_1^* - U_F\|_{G_T^0} \leq \int_{U_1} \|U_1\|_{X_T^1} \|\nabla U_F\|_{Y_T^1}.
$$

(4.5)

Using $\|U_1 - U_F\|_{G_T^0} \leq r \leq 1$ together with (4.3), we get, for some increasing function $g_1$,

$$
\|U_1^* - U_F\|_{G_T^0} \leq g_1(T + 1 + \|(U^0, F)|_{D_T^0})\|\nabla U_F\|_{Y_T^1}.
$$

Now if indeed $T$ is small enough so as

$$
g_1(T + 1 + \|(U^0, F)|_{D_T^0})\|\nabla U_F\|_{Y_T^1} \leq r,
$$

we have that $\Theta(U_1) = U_1^*$ lies in $B_{G_T^0}(U_F, r)$ so that this closed ball is indeed preserved by $\Theta$. Finally to evaluate the lipschitz constant of $\Theta$ on this ball, we use once more (4.2) with $U_1, U_2 \in B_{G_T^0}(U_F, r)$ and the triangular inequality to infer, for some increasing function $g_2$

$$
\|U_1^* - U_2^*\|_{G_T^0} \leq g_2(T + 2\|U_F\|_{G_T^0} + 2r)\|\nabla U_2\|_{Y_T^1}\|U_1 - U_2\|_{G_T^0}.
$$

$$
\leq g_3(T + 1 + \|(U^0, F)|_{D_T^0})(r + \|\nabla U_F\|_{Y_T^1})\|U_1 - U_2\|_{G_T^0},
$$

where we used (4.3), $r \leq 1$ and $g_3(z) := g_2(2z)$. The proof follows for $g := \max(g_1, g_3)$. □

The proof of Theorem 1 will now follow from Lemma 4.1 and Picard’s fixed-point theorem. For any $T > 0$, if we choose

$$
r = r_T := g(T + 1 + \|(U^0, F)|_{D_T^0})\|\nabla U_F\|_{Y_T^1},
$$

where $g$ is the increasing function of Lemma 4.1, estimate (4.5) is automatically satisfied and $B_{G_T^0}(U_F, r_T)$ is thus preserved by $\Theta$. For this choice $r = r_T$, the bound of the lipschitz constant given in Lemma 4.1 becomes strictly less than 1 as soon as

$$
r_T + \|\nabla U_F\|_{Y_T^1} = [1 + g(T + 1 + \|(U^0, F)|_{D_T^0})]\|\nabla U_F\|_{Y_T^1} < \frac{1}{g(T + 1 + \|(U^0, F)|_{D_T^0})},
$$

which ultimately takes the form

$$
\|\nabla U_F\|_{Y_T^1} < \frac{1}{h(T + 1 + \|(U^0, F)|_{D_T^0})},
$$

(4.6)

for yet another increasing function $h$. This ends the proof of local existence for Theorem 1 because

$$
\|\nabla U_F\|_{Y_T^1}^2 := \int_0^T \|\nabla U_F(t)\|_{H_{\mu}(\mathbb{T}^d)}^2 dt
$$

tends to 0 as $T \to 0$, so we recover indeed for $T$ small enough that $\Theta$ induces a contraction map on $B_{G_T^0}(U_F, r_T)$ and have thus a fixed-point.
4.2. Global uniqueness and stability. In the previous paragraph, we have shown the existence of a solution on some small time interval. We have also, by construction, proved its uniqueness but only in an appropriate neighborhood of $U_F$. In this short paragraph, we establish global uniqueness (as stated in Theorem 1) of this solution in $E_T^s$ by means of the stability estimate (1.4) (which obviously implies uniqueness). To prove this estimate, we rely once more on Theorem 5.

We consider therefore $U_1$ and $U_2$ two solutions, associated with data $(U_1^0, F_1)$ and $(U_2^0, F_2)$ respectively, and let $T > 0$ be a common time of existence; both solutions are in $E_T^s$ by assumption and since their time derivatives belong to $Y_T^{s-1}$, we have of course that both of them are in $G_T^s$ (see Lemma A.2), just as the solutions we built above.

We set $V := U_1 - U_2$ and notice that

$$L_{A(U_1)}V = \sum_{k=1}^d \partial_k ([A(U_1) - A(U_2)]\partial_k U_2) + F_1 - F_2.$$ 

Since $L_{A(U_1)}V = \mathbb{P}_0 L_{A(U_1)}V = L_{A(U_1)}\mathbb{P}_0 V$, we infer from Corollary 2.5, using that $H^s(\mathbb{T}^d)$ is an algebra,

$$\|\mathbb{P}_0 V\|^2_{G_T^s} \lesssim_{T, \|U_1\|_{G_T^s}, \|U_2\|_{G_T^s}} \|\mathbb{P}_0 V(0)\|^2_{H^s(\mathbb{T}^d)} + \int_0^T \|A(U_1)(t) - A(U_2)(t)\|^2_{H^s(\mathbb{T}^d)} \|\nabla U_2(t)\|^2_{H^s(\mathbb{T}^d)} \, dt + \int_0^T \|\mathbb{P}_0(F_1 - F_2)(t)\|^2_{H^{s-1}(\mathbb{T}^d)} \, dt.$$ 

Using once more the lipschitz estimate of Lemma A.3 with $\Phi = A$, $f = U_1$ and $g = U_2$, we infer as $G_T^s \hookrightarrow X_T^s \hookrightarrow L^\infty(Q_T)$, after adding $(V(t))^2 = \left(\langle V(0) \rangle + \int_0^t \langle F_1 - F_2 \rangle(t) \, dt'\right)^2$ on both sides and replacing $T$ by an arbitrary $t \in [0,T]$

$$\|V(t)\|^2_{H^s(\mathbb{T}^d)} + \int_0^t \|\nabla V(t')\|^2_{H^s(\mathbb{T}^d)} \, dt' \lesssim_{T, \|U_1\|_{G_T^s}, \|U_2\|_{G_T^s}} \|V(0)\|^2_{H^s(\mathbb{T}^d)} + \int_0^t \|V(t')\|^2_{H^s(\mathbb{T}^d)} \|\nabla U_2(t')\|^2_{H^s(\mathbb{T}^d)} \, dt + \int_0^t \|\mathbb{P}_0(F_1 - F_2)(t')\|^2_{H^{s-1}(\mathbb{T}^d)} \, dt' + \left(\int_0^t \langle F_1 - F_2 \rangle(t) \, dt'\right)^2.$$ 

Grönwall’s lemma allows to conclude and establish (1.4). Theorem 1 is proved.

5. Proof of Theorem 2

In this last section, we prove Theorem 2. The three coming paragraphs respectively focus on points (i), (ii) and (iii) in the statement of the theorem.
5.1. Global solutions for small data. We rely, just as we did in Subsection 4.1, on a Picard scheme. We use the same map $\Theta : U \mapsto U^*$ introduced at the beginning of Subsection 4.1 and defined on $G^s_T(U^0)$. Instead of $\Theta(0) = U_F$, we shall choose 0 as center of the ball. We first note for any $U \in G^s_T(U^0)$ that

$$L_{A(0)}U^* = \sum_{k=1}^d \partial_k \left( [A(U) - A(0)] \partial_k U^* \right) + F.$$ \hfill (5.1)

Now recall the existence of an increasing function $\varphi$ depending only on $A$ and satisfying (4.1). Together with the algebra structure of $H^s(\mathbb{T}^d)$, we then write

$$\left\| [A(U) - A(0)] \partial_k U^* \right\|_{Y^s_T} \leq \|A(U) - A(0)\|_{X^s_T} \|\nabla U^*\|_{Y^s_T} \leq \varphi(\|U\|_{X^s_T}) \|U\|_{X^s_T} \|U^*\|_{Y^s_T+1} \leq \varphi(\|U\|_{G^s_T}) \|U\|_{G^s_T} \|U^*\|_{G^s_T}.$$ \hfill (5.2)

Then, returning to (5.1), the point is, instead of using Corollary 2.5, to rely on Proposition 2.1, for which the estimate is independent of the time variable. More precisely, just as we did for $U_F$ in the proof of Lemma 4.1, using Lemma A.2 to add the Hölder norm in the estimate and adding the time evolution of the spatial average, we infer

$$\left\| U^* \right\|_{G^s_T} \leq C_{A(0)} \left( \left\| (U^0, F) \right\|_{D^s_T} + \varphi(\|U\|_{G^s_T}) \|U\|_{G^s_T} \|U^*\|_{G^s_T} \right),$$

where $C_{A(0)}$ depends only on the matrix $A(0)$ and $\| (U^0, F) \|_{D^s_T}$ is finite by assumption. Now, fix $r \in (0, 1]$ such that $r C_{A(0)} \varphi(1) < 1$. For any $U \in \mathcal{B}_{G^s_T}(0, r)$ estimate (5.2) implies

$$\left\| U^* \right\|_{G^s_T} \leq \frac{C_{A(0)}}{1 - r \varphi(1) C_{A(0)}} \| (U^0, F) \|_{D^s_T}.$$ \hfill (5.3)

In particular, for any $r$ as above, if

$$\| (U^0, F) \|_{D^s_T} \leq r \left( \frac{1}{C_{A(0)}} - r \varphi(1) \right),$$

we have just proved that for all times $T$ the closed ball $\mathcal{B}_{G^s_T}(0, r)$ is preserved by $\Theta$. What about the lipschitz constant of $\Theta$ in that ball? Just as in (5.1) we rely on the flow of the constant matrix field $A(0)$ writing for $U_1, U_2$ in $\mathcal{B}_{G^s_T}(0, C_{A(0)}/2)$

$$L_{A(0)}(U_1^* - U_2^*) = \sum_{k=1}^d \partial_k \left( [A(U_1) - A(0)] \partial_k [U_1^* - U_2^*] \right) + \sum_{k=1}^d \partial_k \left( [A(U_1) - A(U_2)] \partial_k U_2^* \right).$$

We use as above Proposition 2.1 together with Lemma A.2 to estimate the full $G^s_T$ norm, and (4.1) just as above with the pair $(U_1, 0)$ and $(U_1, U_2)$: since $U_1^* - U_2^*$ vanishes at the initial time, we obtain that for the same increasing function $\varphi$ as before

$$\left\| U_1^* - U_2^* \right\|_{G^s_T} \leq C_{A(0)} \left( \varphi(\|U_1\|_{X^s_T}) \|U_1\|_{X^s_T} \|U_1^* - U_2^*\|_{Y^s_T+1} + \varphi(\|U_1\|_{X^s_T} + \|U_2\|_{X^s_T}) \|U_1 - U_2\|_{X^s_T} \|U_2^*\|_{Y^s_T+1} \right).$$
Since $U_1$, $U_2$ and $U^*_2$ both belong to $\mathcal{B}_{G_2^s} (0, r)$ with $G_2^s \hookrightarrow X_T^{s+1}$ (with operator norm less than 1), and $r \in (0, 1]$ is such that $rC_{A(0)}\varsigma (1) < 1$, we get
\[
\|U^*_1 - U^*_2\|_{G_2^s} \leq r \frac{C_{A(0)}\varsigma (2)}{1 - rC_{A(0)}\varsigma (1)} \|U^*_1 - U^*_2\|_{G_2^s}.
\]
So we first choose $r \in (0, 1 \wedge (C_{A(0)}\varsigma (1))^{-1})$ small enough so as
\[
r \frac{C_{A(0)}\varsigma (2)}{1 - rC_{A(0)}\varsigma (1)} < 1,
\]
and this defines the threshold (5.3) for $\|(U^0, F)\|_{\mathcal{D}^s_T}$ below which we have a solution for all times, thanks to Picard’s fixed-point theorem. Point (i) of Theorem 5.3 is proved.

5.2. Finer description of the lifetime. Let us prove point (ii) of Theorem 2. We consider $(U^0, F)$ in $\mathcal{D}^s_T$. We recall the sufficient condition (4.6) for a solution to exist in $E_T^s$ (where $h$ is some increasing function). For any $T > 0$ satisfying this condition, we have $T^*_s \geq T$. Now consider $\sigma$ any real number in $(d/2, s)$ such that $s \leq \sigma + 1$. Using the interpolation $H^s (\mathbb{T}^d) = [H^{s-1} (\mathbb{T}^d), H^s (\mathbb{T}^d)]_{\theta}$ we can write for some $\theta \in (0, 1)$
\[
\|\nabla U_F(t)\|_{H^s (\mathbb{T}^d)} \leq \|\nabla U_F(t)\|_{H^{s-1} (\mathbb{T}^d)} \|\nabla U_F(t)\|_{H^s (\mathbb{T}^d)}^{1 - \theta} \|\nabla U_F(t)\|_{H^s (\mathbb{T}^d)}^{\theta}.
\]
We have thus
\[
\|\nabla U_F\|_{Y_T^s} \leq \|U_F\|_{X_T^s} \|\nabla U_F\|_{Y_T^s}^{1 - \theta} T^{\theta/2},
\]
where we used $\sigma \leq s$ and (4.3). Now let us explore the sufficient condition (4.6) in the $H^s (\mathbb{T}^d)$ setting; we see that it is satisfied as soon as
\[
T^{\theta/2} \leq \frac{1}{\|(U^0, F)\|_{\mathcal{D}^s_T}} \frac{1}{h(T + 1 + \|(U^0, F)\|_{\mathcal{D}^s_T})}.
\]
Using again $\|(U^0, F)\|_{\mathcal{D}^s_T} \leq \|(U^0, F)\|_{\mathcal{D}^s_T}$, the previous inequality is satisfied as soon as
\[
T \leq \Phi(T + 1 + \|(U^0, F)\|_{\mathcal{D}^s_T}),
\]
where $\Phi (z) = z^{-2/\theta} h(z)^{-2/\theta}$. Since $\Phi$ is decreasing, it is also the case of the function $\Phi^{-1}$ and $\Psi : z \mapsto \Phi^{-1}(z) - z - 1$, and we find that
\[
T^*_s (U^0, F) \geq \varphi (\|(U^0, F)\|_{\mathcal{D}^s_T})
\]
where $\varphi := \Psi^{-1}$ is indeed decreasing. It now suffices to prove that $T^*_s (U^0, F) \geq T^*_s (U^0, F)$ (note that the reverse inequality is obvious). But this is actually an immediate consequence of the propagation of regularity result stated in Corollary 3.1, so point (ii) of Theorem 2 is proved.

5.3. Blow-up for finite lifetime. This follows directly from (ii).
6. Sign preservation

In this section we prove Proposition 1.4 and Theorem 3, which in particular leads to a well-posedness result of the SKT system as explained in the introduction of this paper.

6.1. Proof of Proposition 1.4. Let us consider $A$ a smooth sign-preserving matrix field in the sense of Definition 1.3, and $U$ a smooth solution to (1.5) (namely in $E^d_T$ for $s > d/2 + 2$). We assume that the data $U^0 \in H^s(\mathbb{T}^d)$ and $F$ in $Y^{s-1}_\infty$ are non-negative.

Now, we use that $A(U) = D(U) + \text{diag}(U)B(U)$ and $R(U) = \text{diag}(U)\rho(U)$ to write for all $i \in [1, N]$, with obvious notations

$$\partial_t u_i - \sum_{j=1}^d \partial_k [d_i(U)\partial_k u_j] - \sum_{k=1}^d \sum_{j=1}^N \partial_k [u_i b_{ij}(U)\partial_k u_j] = f_i + u_i \rho(U).$$

For a real function $f$, we note $f^- = -f 1_{f<0}$ its negative part and recall the formula $\nabla f^- = -1_{f<0}\nabla f$, which holds for Sobolev regularity. Multiplying the previous equation by $-u_i^-$ we infer after integration on $[0, t] \times \mathbb{T}^d$ for $t \in [0, T]$

$$\frac{1}{2} ||u_i^-(t)||_2^2 + \int_0^t \int_{\mathbb{T}^d} d_i(U)|\nabla u_i^-|^2 \, dx \, ds = \frac{1}{2} ||u_i^- (0)||_2^2 - \int_0^t \int_{\mathbb{T}^d} u_i^- f_i \, dx \, ds$$

$$+ \int_0^t \int_{\mathbb{T}^d} (u_i^-)^2 \rho(U) \, dx \, ds - \sum_{k=1}^d \sum_{j=1}^N \int_0^t \int_{\mathbb{T}^d} u_i^- \partial_k u_i^- b_{ij}(U) \partial_k u_j \, dx \, ds.$$

Using $U^0 = U(0) \geq 0$, $F \geq 0$, $\rho(U) \in L^\infty(Q_T)$, $d_i(U) \geq 0$ and another integration by parts to handle the last term we infer

$$\frac{1}{2} ||u_i^-(t)||_2^2 \leq ||\rho(U)||_{L^\infty(Q_T)} \int_0^t ||u_i^-(s)||_2^2 \, ds + \frac{1}{2} \sum_{k=1}^d \sum_{j=1}^N \int_0^t \int_{\mathbb{T}^d} (u_i^-)^2 \partial_k [b_{ij}(U)\partial_k u_j] \, dx \, ds.$$

The fact that $s > 2 + d/2$ is enough to justify all the previous computations and claim furthermore that $\partial_k [b_{ij}(U)\partial_k u_j]$ belongs to $L^\infty(Q_T)$. This leads eventually to an estimate of the form

$$||u_i^-(t)||_2^2 \lesssim ||u_i^-(s)||_2^2 \, ds,$$

and Grönwall’s inequality leads to the fact that $u_i^- = 0$ on $[0, T]$. Proposition 1.4 is proved.

6.2. Proof of Theorem 3. The proof consists in transforming system (1.5) into one for which one can apply Proposition 1.4.

First let us check that one can assume without loss of generality that $s > d/2 + 2$. A consequence of Corollary 1.2 is indeed the following. Pick $U^0 \in H^s(\mathbb{T}^d)$ and $F$ in $Y^{s-1}_\infty$ with $s > d/2$, and consider smooth approximations $U^0_\varepsilon$ and $F_\varepsilon$ of $U^0$ and $F$ respectively in $H^s(\mathbb{T}^d)$ and $Y^{s-1}_\infty$. Fix any time $T < T^*_\varepsilon(U^0, F)$. Combining estimate (1.4) of Theorem 1 and point (iii) of Theorem 2, we have for $\varepsilon$ small enough $T^*_\varepsilon(U^0_\varepsilon, F_\varepsilon) \geq T$ and $(U_\varepsilon)_{\varepsilon} \to U$.
in $E^s_T$ as $\varepsilon$ goes to zero. As the previous convergence preserves non-negativeness, we can therefore assume without loss of generality that $s$ is as large as needed.

Now, Theorem 3 follows from Proposition D.1 of Appendix Section D. Indeed, since $\mathcal{P}$ is open (see Lemma C.2) that's also the case of $\Omega := A^{-1}(\mathcal{P})$ and by assumption $\Omega$ contains $\mathbb{R}^N_{\geq 0}$. Thanks to Proposition D.1 we have therefore a smooth function $h$ sending $\mathbb{R}^N$ on $\Omega$ and leaving all points of $\mathbb{R}^N_{\geq 0}$ unchanged. Corollary 1.2 applies to find a solution $U$ to the system (1.5) where $A$ is replaced by $A \circ h$. Moreover since $A \circ h$ is sign-preserving in the sense of Definition 1.3, Proposition 1.4 shows that $U \geq 0$ on its lifetime so that $A \circ h(U) = A(U)$ and we have built a non-negative solution to the original problem (1.5). Thanks to the uniqueness offered by our setting, this construction (and the corresponding maximal lifetime) is independent of the map $h$ that we choose to define the solution. □

APPENDIX A. SOBOLEV ESTIMATES

Let us start by stating this very classical lemma, the proof of which is recalled for the convenience of the reader.

Lemma A.1. Fix $s \in \mathbb{R}$. If $f \in L^\infty(0,T; H^s(\mathbb{T}^d))$ with $\partial_t f \in Y^s_T$, then $f \in X^s_T$.

Proof. Recalling the definition of $X^s_T$, we only need to prove continuity in time with values in $H^s(\mathbb{T}^d)$. If $(f_n)_n$ is a sequence of smooth functions approaching $f$ in $Y^s_T$ such that $(\partial_t f_n)_n$ converges to $\partial_t f$ in $Y^s$ and $(f_n)_n$ is uniformly bounded in $L^\infty(0,T; H^s(\mathbb{T}^d))$, a direct computations gives for $n, k \in \mathbb{N}$ and $t, r \in [0,T]$

$$\|f_n(t) - f_k(t)\|^2_{H^s(\mathbb{T}^d)} = \|f_n(r) - f_k(r)\|^2_{H^s(\mathbb{T}^d)} + \int_r^t \langle (f_n - f_k)(t'), (\partial_t f_n - \partial_t f_k)(t') \rangle_{H^s(\mathbb{T}^d)} dt',$$

where $\langle \cdot \rangle_{H^s(\mathbb{T}^d)}$ denotes the scalar product in $H^s(\mathbb{T}^d)$. From this we infer by the Cauchy-Schwarz inequality

$$\|f_n(t) - f_k(t)\|^2_{H^s(\mathbb{T}^d)} \leq \|f_n(r) - f_k(r)\|^2_{H^s(\mathbb{T}^d)} + \|f_n - f_k\|_{Y^s_T} \|\partial_t f_n - \partial_t f_k\|_{Y^{s-1}_T}.$$

Integrating in $r \in [0,T]$ we get

$$T\|f_n(t) - f_k(t)\|^2_{H^s(\mathbb{T}^d)} \leq \|f_n - f_k\|^2_{Y^s_T} + T\|f_n - f_k\|_{Y^{s+1}_T} \|\partial_t f_n - \partial_t f_k\|_{Y^{s-1}_T},$$

from which we infer that $(f_n)_n$ is a Cauchy sequence in $C^0([0,T]; H^s(\mathbb{T}^d))$, which entails this regularity for $f$. □

Lemma A.2. Fix $s > d/2$. There exists $\alpha_s \in (0, 1)$ such that any $f \in X^s_T$ satisfying that $\partial_t f \in Y^s_T$ actually belongs to $C^{0, \alpha_s}(T)$ with an estimate

$$\|f\|_{C^{0, \alpha_s}(T)} \lesssim_s \|f\|_{X^s_T} + \|\partial_t f\|_{Y^{s-1}_T}.$$
Proof. We choose $\sigma \in (d/2, s)$ such that $\sigma > s - 1$ so that by interpolation $H^\sigma(\mathbb{T}^d) = [H^{s-1}(\mathbb{T}^d), H^s(\mathbb{T}^d)]_\theta$. The assumption on the time derivative imply that $f$ belongs to the space $\mathcal{C}^{0,1/2}([0,T]; H^{s-1}(\mathbb{T}^d))$ and thus for $t_1 \neq t_2 \in [0,T]$

$$
\|f(t_1) - f(t_2)\|_{H^\sigma(\mathbb{T}^d)} \leq \|f(t_1) - f(t_2)\|_{H^{s-1}(\mathbb{T}^d)} \|f(t_1) - f(t_2)\|_{H^{1}(\mathbb{T}^d)}^{1-\theta} \\
\leq |t_1 - t_2|^{\theta/2} \|\partial_t f\|_{H^{s-1}(\mathbb{T}^d)}^{\theta} 2^{1-\theta}\|f\|_{1-\theta}^{1-\theta},
$$

so that

$$
\frac{\|f(t_1) - f(t_2)\|_{H^\sigma(\mathbb{T}^d)}}{|t_1 - t_2|^{\theta/2}} \lesssim \|\partial_t f\|_{H^{s-1}(\mathbb{T}^d)} + \|f\|_{1-\theta},
$$

and the conclusion follows using the Sobolev embedding $H^\sigma(\mathbb{T}^d) \hookrightarrow \mathcal{C}^{0,\beta}(\mathbb{T}^d)$ which holds for some $\beta \in (0, 1)$.

□

Lemma A.3. For $\sigma > d/2$ and $\Phi$ a smooth function, there exists an increasing function $\varphi$ for which, for any elements $f, g \in H^\sigma(\mathbb{T}^d)$

$$
\|\Phi(f) - \Phi(g)\|_{H^\sigma(\mathbb{T}^d)} \leq \varphi(\|f\|_{\infty} + \|g\|_{\infty}) \\
\times \left[\|f - g\|_{H^\sigma(\mathbb{T}^d)} + \|f\|_{H^\sigma(\mathbb{T}^d)} + \|g\|_{H^\sigma(\mathbb{T}^d)}\|f - g\|_{\infty}\right].
$$

Proof. See for instance [3, Corollary 2.91]. □

Appendix B. Littlewood-Paley theory

In this section we present the elements of Littlewood-Paley theory that are used in this study. We recall (see for instance [3]) that the basic idea is to consider a dyadic partition of unity in $\mathbb{R}^d$

$$
1 = \hat{\chi} + \sum_{j \geq 0} \hat{\varphi}(2^{-j} \cdot)
$$

where $\hat{\chi}$ and $\hat{\varphi}$ (the Fourier transforms of two smooth functions $\chi$ and $\varphi$) are smooth, radial functions, taking values in $[0, 1]$ and supported respectively in the ball $B(0,4/3)$ and the ring $[3/4,8/3]$. We set for any integer $j \geq 0$ and any function $f$ defined on $\mathbb{T}^d$

$$
\Delta_j f := f \ast 2^{-jd} \varphi(2^{-j} \cdot)
$$

and

$$
\Delta_{-1} f := f \ast \chi.
$$

Note that in particular

$$
\sum_j \Delta_j = \text{Id}.
$$

Finally for $j < -1$ we set $\Delta_j = 0$ and

$$
\forall j \geq 0, \quad S_j := \sum_{j'=-1}^{j-1} \Delta_{j'}.
$$
Writing these formulas in Fourier space we see that the support of the Fourier transform of \( \Delta_j f \) lies in a ring of size \( 2^j \) if \( j \geq 0 \) and in the unit ball if \( j = -1 \) (this corresponds therefore to the average of \( f \)). Moreover the functions \( \hat{\chi} \) and \( \hat{\varphi} \) are designed so as to have

\[
|j - j'| \geq 2 \implies \text{Supp} \; \hat{\varphi}(2^{-j'}) \cap \text{Supp} \; \hat{\varphi}(2^{-j'}) = \emptyset
\]

and

\[
j \geq 1 \implies \text{Supp} \; \hat{\chi} \cap \text{Supp} \; \hat{\varphi}(2^{-j'}) = \emptyset.
\]

The following Bernstein inequality is used many times in this paper (B.2)

\[
\forall \alpha \in \mathbb{N}^d, \quad \forall 1 \leq p \leq q \leq \infty, \quad \|\partial^\alpha \Delta_j f\|_q \lesssim 2^j (|\alpha| + d(\frac{1}{p} - \frac{1}{q})) \|\Delta_j f\|_p.
\]

For the convenience of the reader let us recall how to prove this inequality: we consider a smooth, compactly supported function \( \hat{\varphi} \) such that \( \hat{\varphi} \hat{\varphi} \equiv 1 \) and we note that

\[
\Delta_j f = \Delta_j f \ast 2^{id} \hat{\varphi}(2^j \cdot).
\]

Then we write

\[
\partial^\alpha \Delta_j f = \Delta_j f \ast 2^{id+j|\alpha|} (\partial^\alpha \hat{\varphi})(2^j \cdot)
\]

and we conclude by Young's inequality

\[
\|\partial^\alpha \Delta_j f\|_q \leq 2^{id+j|\alpha|} \|\partial^\alpha \Delta_j f\|_p \|\partial^\alpha \hat{\varphi}(2^j \cdot)\|_r, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}
\]

and the result (B.2) follows.

With this construction, Sobolev spaces can be defined by the equivalent norm

\[
\|f\|_{H^s} \sim \left\| 2^j s \|\Delta_j f\|_{L^2(T^d)} \right\|_{L^2(\mathbb{Z})}.
\]

One major interest of this theory is the paraproduct algorithm due to Bony [4]: decomposing formally any two tempered distributions \( f \) and \( g \) as

\[
f = \sum_j \Delta_j f \quad \text{and} \quad g = \sum_j \Delta_j g
\]

then the product \( fg \) can formally be decomposed into three parts

\[
fg = T_fg + T_gf + R(f, g), \quad T_j f := \sum_j S_{j-1} f \Delta_j g, \quad R(f, g) := \sum_{|j-j'| \leq 1} \Delta_j f \Delta_j g.
\]

On the Fourier side, thanks to the support properties of \( \varphi \) and \( \chi \), each term \( S_{j-1} f \Delta_j g \) of the paraproduct \( T_j f \) is supported in a ring of size \( 2^j \) (hence the sum is well defined under mild assumptions on \( f \) and \( g \): for instance \( f \) bounded and \( g \) in any Sobolev or Hölder space). The remainder term \( R(f, g) \) however is not always well defined. On the Fourier side, each term \( \Delta_j f \Delta_j g \) is supported in a ball of size \( 2^j \) (since \( j \sim j' \)) and the sum only makes sense if the regularities of \( f \) and \( g \) sum up to a positive number. We refer to [3] for instance for more on this.
In this paper we use a less sharp decomposition, writing formally
\[ fg = \sum_j S_{j-1} f \Delta_j g + \sum_j \Delta_j f S_j g. \]

Identifying a function \( a \) with the corresponding linear multiplication operator, the following decomposition of the commutator of \( \Delta_j \) and a function \( a \) is very useful:
\[ [\Delta_j, a] = \Delta_j \sum_{j'} (\Delta_j a) S^j_{j'+2} - \sum_{j'} [S^j_{j'+1} a, \Delta_j] \Delta_j + \sum_{j'} (S^j_{j'-1} a - a) \Delta_j \Delta_j'. \]

Indeed, since the \( \Delta_j \)'s sum to identity, the previous equality is equivalent to
\[ \Delta_j a = \Delta_j \sum_{j'} (\Delta_j a) S^j_{j'+2} - \sum_{j'} [S^j_{j'+1} a, \Delta_j] \Delta_j + \sum_{j'} (S^j_{j'-1} a - a) \Delta_j \Delta_j'. \]

which is even true without the \( \Delta_j \) operators, using
\[ a = \sum_{\ell, j'} (\Delta_j a) \Delta_\ell, \] and distinguishing in this double sum the cases \( \ell \leq j' + 2 \).

**Appendix C. Petrovskii Condition, Hurwitz Matrices and Spectral Radius**

For \( B \) in \( M_N(\mathbb{C}) \) we denote by \( \text{Sp}(B) \) the set of all its eigenvalues; the spectral radius \( \rho(B) \) of \( B \) is then defined by \( \rho(B) = \max_{\lambda \in \text{Sp}(B)} |\lambda| \). For any \( \delta \in \mathbb{R} \) we denote by \( \mathcal{P}_\delta \) the set of matrices \( B \) for which \( \text{Sp}(B) \subset \{ z \in \mathbb{C} : \text{Re}(z) \geq \delta \} \).

The matrix \( B \) is said to satisfy the Petrovskii condition if \( \text{Sp}(B) \subset \mathbb{R}^+ + i\mathbb{R} \), that is if \( B \) belongs to \( \mathcal{P} := \cup_{\delta > 0} \mathcal{P}_\delta \). Note that in control theory and dynamical systems, the denomination Hurwitz matrix also exists, but refers instead to a matrix whose spectrum lies in \( \mathbb{R}^+ + i\mathbb{R} \); we will not use this terminology here.

The results below, even though elementary, are of crucial importance in our analysis.

**Lemma C.1.** Fix a norm \( \| \cdot \| \) on \( M_N(\mathbb{C}) \). For a symbol \( \lesssim \) depending only on \( \| \cdot \| \) and the dimension \( N \) there holds for all \( \delta > 0 \)
\[ \forall B \in \mathcal{P}_\delta, \quad \forall t \geq 0, \quad \| e^{-tB} \| \lesssim (1 + \| B/\delta \|_N^N) e^{-\delta t/2}. \]

**Proof.** We will actually prove the following estimate for \( B \in \mathcal{P}_\delta \) and \( t \geq 0 \)
\[ \| e^{-tB} \| \lesssim (1 + t^N \| B/\delta \|_N^N) e^{-\delta t}, \]
which indeed implies (C.1) because
\[ \sup_{t \geq 0} (1 + t^N \| B/\delta \|_N^N) e^{-\delta t/2} = \sup_{\sigma \geq 0} (1 + \sigma^N \| B/\delta \|_N^N) e^{-\sigma/2} \lesssim (1 + \| B/\delta \|_N^N) \sup_{\sigma \geq 0} (1 + \sigma^N) e^{-\sigma/2}. \]
In fact, replacing \( B \) by \( \tilde{B} := B - \delta I_N \in \mathcal{P}_0 \) for which \( e^{-tB} = e^{-t\tilde{B}}e^{-t\delta} \), we only need to prove (C.2) for \( \delta = 0 \). Finally without loss of generality, due to the equivalence of the norms, we can assume that \( \| \cdot \| \) is the subordinate matrix norm to the \( \ell^\infty \)-norm on \( \mathbb{C}^N \), that we denote \( | \cdot | \) in the sequel.

Assume now that \( B \in \mathcal{P}_0 \). The Schur decomposition ensures that there is a unitary matrix \( U \in U_N(\mathbb{C}) \) such that \( T := UBU^* \) is upper triangular. In particular, since \( U_N(\mathbb{C}) \) is a bounded set, there holds
\[
\| T \| \leq \| B \| \leq \| T \|,
\]
and similarly \( \| e^{-tB} \| \leq \| e^{-tT} \| \). We are therefore reduced to the case of an upper triangular matrix \( T \in \mathcal{P}_0 \). Fix \( z^0 \) a unit vector, and consider the curve \( z : t \mapsto e^{-tT}z^0 \) which solves \( z' = -TZz \). Since \( T = D + R \) with \( D \in \mathcal{P}_0 \) diagonal \( R \) strictly upper diagonal, the Duhamel formula gives
\[
z(t) = e^{-Dt}z^0 + \int_0^t e^{-(t-t')D}R(t')z(t')\,dt',
\]
which gives a control on all the \((z_k(t))_{1 \leq k \leq N}\)
\[
|z_k(t)| \leq |z_k^0| + \max_{ij} |R_{ij}| \int_0^t \sum_{j=k+1}^N |z_j(t')|\,dt'.
\]
Note that \( \max_{ij} |R_{ij}| \leq \max_{ij} |T_{ij}| \leq \| T \| \) by equivalence of norms. Starting with \( |z_N(t)| \leq |z_N^0| \) we obtain therefore after iteration
\[
|z_k(t)| \leq (1 + t^{N-k}\| T \|^{|N-k|})|z_k^0|,\]
so that we in the end we have established
\[
|e^{-tT}z^0|\|_\infty = |z(t)|\|_\infty \lesssim (1 + t^N\| T \|^N)|z^0|\|_\infty.
\]
Finally (again by equivalence of norms) we have \( \| e^{-tT} \| \leq 1 + t^N\| T \|^N \), which is exactly (C.2) with \( \delta = 0 \) for \( B = T \). \( \square \)

**Lemma C.2.** The map \( \gamma : B \mapsto -\ln \rho(e^{-B}) \) is continuous from \( M_N(\mathbb{C}) \) to \( \mathbb{R} \). Furthermore we have \( \mathcal{P} = \gamma^{-1}(\mathbb{R}_{>0}) \) and \( B \in \mathcal{P}_{\gamma(B)} \), for any matrix \( B \).

**Proof.** The map \( B \mapsto \text{Sp}(B) \) is continuous, for the (modified) Hausdorff distance on finite sets \( d_H \) at arrival (see [20, Theorem 5.2]). In particular, since \( \rho(B) = d_H(\text{Sp}(B), \{0\}) \), the spectral radius map \( B \mapsto \rho(B) \) is continuous and therefore so is \( \gamma \) because \( \rho \) is positive on \( \text{GL}_N(\mathbb{R}) \). For the remaining part of the statement, the proof is ended once noticed that for any matrix \( B \) one has \( \rho(e^{-B}) = \max_{\lambda \in \text{Sp}(B)} e^{-\text{Re}(\lambda)} \), so that \( \gamma(B) \) is actually the lowest real part among all eigenvalues of \( B \). \( \square \)

**Corollary C.3.** Consider \( K \) a metric compact set and \( \mathcal{C}^0(K;M_N(\mathbb{R})) \) equipped with the uniform topology. The map
\[
\eta : \mathcal{C}^0(K;M_N(\mathbb{R})) \longrightarrow \mathbb{R}
\]
\[
M \mapsto \min_K \gamma \circ M
\]
is well-defined and continuous. In particular, $\mathcal{C}^0(K;\mathcal{P})$ is open and lies in $\bigcup_{k>0} \mathcal{C}^0(K;\mathcal{P}_k)$, where each $\mathcal{C}^0(K;\mathcal{P}_k)$ is closed.

Proof. The map is well-defined because $\gamma \circ M$ is continuous and reaches therefore its minimum on $K$. If $(M_k)_k$ converges uniformly to $M$, then so does $(\gamma \circ M_k)_k$, to $\gamma \circ M$ because $\gamma$ is continuous and $M_N(\mathbb{R})$ locally compact; this classically implies the convergence of $\eta(M_k)$ towards $\eta(M)$. By continuity if $M \in \mathcal{C}^0(K;\mathcal{P})$, then $\eta(M) > 0$ and $M(K) \subset \mathcal{P}_{\eta(M)}$. In particular $\mathcal{C}^0(K;\mathcal{P}) = \eta^{-1}(\mathbb{R}_{>0})$ is indeed open and each $\mathcal{C}^0(K;\mathcal{P}_k) = \eta^{-1}(\mathbb{R}_{\geq k})$ is closed. □

Corollary C.4. Fix $\alpha \in (0,1]$. There exists a non-increasing function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for any $M,H \in \mathcal{C}^{0,\alpha}(Q_T;M_N(\mathbb{R}))$ such that $\|H\|_{\mathcal{C}^{0,\alpha}(Q_T)} \leq \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)}$ the following implication holds

$$\|M - H\|_{\infty} < f([M]_\alpha) \implies H(Q_T) \subset \mathcal{P}_{\eta(M)/2},$$

where $[M]_\alpha := \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)} + \eta(M)^{-1}$.

Proof. Thanks to Ascoli’s theorem, the closed ball $B_{R_M}$ of the set $\mathcal{C}^{0,\alpha}(Q_T;M_N(\mathbb{R}))$ of radius $R_M := \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)}$ is compact in $\mathcal{C}^0(Q_T;M_N(\mathbb{R}))$ and the continuous function $\eta$ given by Corollary C.3 is thus uniformly continuous on $B_{R_M}$. There exists therefore a in $(0,2R_M]$ for which the following implication holds for $H_1,H_2 \in B_{R_M}$

$$\|H_1 - H_2\|_{\infty} \leq a \implies |\eta(H_1) - \eta(H_2)| \leq \eta(M)/2.$$  

The supremum $a_M$ of those $a$ is well-defined and non-increasing in $R_M = \|M\|_{\mathcal{C}^{0,\alpha}(Q_T)}$ while non-decreasing in $\eta(M)$ and thus non-increasing in $[M]_\alpha$. In particular, for any $H \in B_{R_M}$, if $\|H - M\|_{\infty} \leq a_M$, one has $\eta(H) \geq \eta(M)/2$. □

**Appendix D. Smooth (almost) retraction of $\mathbb{R}^N$ on $\mathbb{R}^N_{\geq 0}$**

We prove in this paragraph the following proposition.

**Proposition D.1.** For any open neighbourhood $\Omega$ of $\mathbb{R}^N_{\geq 0}$, there exists a smooth function $h : \mathbb{R}^N \to \Omega$ such that its restriction to $\mathbb{R}^N_{\geq 0}$ is the identity map.

Let us first recall the smooth Urysohn lemma.

**Lemma D.2** (Smooth Urysohn). For two disjoints and closed sets $F_0$ and $F_1$ of $\mathbb{R}^N$ there exists a smooth function $\varphi : \mathbb{R}^N \to [0,1]$ such that $\varphi^{-1}({1}) = F_1$ and $\varphi^{-1}({0}) = F_0$.

Proof. First use [15, Theorem 2.29] to find for $k \in \{0,1\}$ smooth functions $\psi_k : \mathbb{R}^N \to \mathbb{R}$ (easily chosen non-negative) such that $\psi_k^{-1}({0}) = F_k$ and then letting $\varphi(x) := \psi_1/(\psi_0 + \psi_1)$ does the trick. □

We will deduce Proposition D.1 from the following lemma.

**Lemma D.3.** For any open neighbourhood $\Omega$ of $\mathbb{R}^N_{\geq 0}$ there is an open set $\tilde{\Omega}$ such that $\mathbb{R}^N_{\geq 0} \subset \tilde{\Omega} \subset \Omega$, and which is furthermore infinitely diffeomorphic to $\mathbb{R}^N$.  

Proof. Consider \( \varphi \) the function given by Lemma D.2, associated with the (disjoint) closed sets \( F_0 := \mathbb{R}^N \setminus \Omega \) and \( F_1 := \mathbb{R}^N_{\geq 0} \). Let \( 1 \) be the vector of \( \mathbb{R}^N \) with entries all equal to 1. For any initial data at time \( t = 0 \), the differential equation
\[
\dot{V} = -\varphi(V)1
\]
has a unique maximal solution, which is global since \( \varphi \) takes it values in \([0, 1]\). One can therefore define for all \( v \in \mathbb{R}^N \) a smooth curve \( V_v : \mathbb{R}^N \rightarrow \mathbb{R}^N \) equal to \( v \) at time \( t = 0 \) and solving that equation. Since \( \varphi \) vanishes outside of \( \Omega \), for \( v \in \Omega \) there holds \( V_v(\mathbb{R}^N) \subset \Omega \) and flow lines passing through a point of \( \Omega \) do not exit \( \Omega \). For all \( t > 0 \), the flow \( \Phi_t : \mathbb{R}^N \rightarrow \mathbb{R}^N \) which maps \( v \) to \( V_v(t) \) is a \( C^\infty \)-diffeomorphism. It is easy to see that \( \dot{F}_1 \) is infinitely diffeomorphic to \( \mathbb{R}^N \), and that is therefore also the case for \( t > 0 \) of the open set \( \tilde{\Omega} := \Phi_t(F_1) \).
The set \( \tilde{\Omega} \) is contained in \( \Omega \) (since curves stemming from \( \dot{F}_1 \) do not exit \( \Omega \)) and finally \( \tilde{\Omega} \) contains \( F_1 \) since \( \varphi \) is equal to 1 on \( F_1 \). Lemma D.3 is proved. \( \square \)

**Acknowledgement**

The authors would like to thank Pierre-Louis Lions for his inspiring online lectures [16], and also for several fruitful discussions and his awareness concerning a previous (false) proof of Lemma C.1.

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