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# RETRACTION TO A PARABOLIC SUBGROUP AND APPLICATIONS

FRANÇOIS DIGNE, EDDY GODELLE AND JEAN MICHEL

ABSTRACT. We continue the study of the retraction from an Artin group to a standard parabolic subgroup introduced by Blufstein, Charney, Paris and the second author. Using right and left retractions we obtain new results on minimal parabolic subgroups, intersection of parabolic subgroups, double cosets with respect to parabolic subgroups and conjugacy classes in Artin groups.

## 0. INTRODUCTION

Given a non-empty set  $S$ , a Coxeter matrix is a symmetric matrix  $(m_{s,t})_{s,t \in S}$  such that  $m_{s,s} = 1$  and  $m_{s,t} \in \mathbb{N}_{\geq 2} \cup \{\infty\}$  for  $s \neq t$ . To such a Coxeter matrix one can associate a group, namely its associated Coxeter group, defined by the following presentation

$$(*) \quad W = \left\langle S \mid \begin{array}{ll} s^2 = 1 & \text{for } s \in S \\ \underbrace{sts \cdots}_{m_{s,t} \text{ terms}} = \underbrace{tst \cdots}_{m_{s,t} \text{ terms}} & \text{for } s \neq t \text{ and } m_{s,t} \neq \infty \end{array} \right\rangle$$

Given a copy  $\mathbf{S}$  of  $S$  one can also define the associated Artin group

$$(**) \quad B = \left\langle \mathbf{S} \mid \underbrace{\mathbf{s}t\mathbf{s} \cdots}_{m_{s,t} \text{ terms}} = \underbrace{\mathbf{t}st \cdots}_{m_{s,t} \text{ terms}} \text{ for } s \neq t \text{ and } m_{s,t} \neq \infty \right\rangle$$

We denote by  $B^+$  the submonoid of  $B$  generated by  $\mathbf{S}$ ; its elements are called positive braids. We denote the length on  $W$  with respect to  $S$  by  $\ell_S : W \rightarrow \mathbb{N}$  and we denote by  $\ell_{\mathbb{Z}} : B \rightarrow \mathbb{Z}$  the group morphism which sends  $\mathbf{S}$  to 1. It immediately follows from the presentations that the map  $\mathbf{s} \in \mathbf{S} \mapsto s \in S$  induces a morphism  $\text{pr} : B \rightarrow W$ . Its kernel is called the pure Artin group associated to  $B$  (or the coloured Artin group) and will be denoted by  $P$ . Moreover, the inverse bijection  $s \mapsto \mathbf{s}$  extends by lifting reduced expressions to a canonical section (of sets) from  $W$  to  $B$ . In the sequel  $\mathbf{W}$  will denote the image of this section. This is a subset of  $B^+$ . The elements of  $\mathbf{W}$  are called simple braids. We will use the following notations: for  $\mathbf{w} \in \mathbf{W}$  the notation  $w$  means  $\text{pr}(\mathbf{w})$  and for  $w \in W$  the notation  $\mathbf{w}$  means the lift of  $w$  to  $\mathbf{W}$ . In other words  $\mathbf{w}$  is the unique element in  $B^+$  such that  $\text{pr}(\mathbf{w}) = w$  and  $\ell_{\mathbb{Z}}(\mathbf{w}) = \ell_S(w)$ .

Many papers focus on Artin groups, but whereas Coxeter groups are almost well-understood, this is not the case for Artin groups. Even the word problem remains open in general. The strategy used to prove most of the results obtained is to use a particular family of subgroups, namely the family of standard parabolic subgroups.

This is why in [5] the authors claim that “any result on these subgroups is likely to be useful”. For  $I \subseteq S$ , the standard parabolic subgroup  $B_I$  (*resp.*  $W_I$ ) of  $B$  (*resp.* of  $W$ ) is the subgroup generated by  $\mathbf{I} \subseteq \mathbf{S}$  lifting  $I$  (*resp.* generated by  $I$ ). Such a subgroup is an Artin group (*resp.* a Coxeter group) for the submatrix  $(m_{s,t})_{s,t \in I}$  (*cf.* [19], see also Proposition 2.5). A parabolic subgroup is any subgroup that is conjugate to a standard parabolic subgroup.

The present article studies the properties of a retraction on standard parabolic subgroups which has been defined in [2]. This is a continuation of a series of articles [15, 5, 2, 14] on this topic. A complex  $\text{Sal}(B)$ , called the Salvetti complex, can be associated with each Artin group. Furthermore, for every standard parabolic subgroup  $B_I$  there exists a canonical embedding  $\text{Sal}(B_I) \rightarrow \text{Sal}(B)$ . In [15], the authors prove that this embedding admits a retraction. Then in [5, Theorem 1.2], the authors use the above retraction to prove that standard parabolic subgroups of Artin groups are convex (see also Proposition 4.7). Along the way, they use the above retraction to construct a retraction  $\pi_I$  from the Artin group  $B$  to its standard parabolic subgroup  $B_I$ . The construction was implicit in the proof of the main result of [5] and is made explicit in [2]. In [14] an alternative construction of this retraction is obtained, which is based only on algebraic and elementary arguments; several algebraic properties of the retraction are also obtained.

The present article will take this approach further. One of the aims is to relate the definition of the retraction to root systems and make it sufficiently explicit that it is easy to calculate. The retraction is not a morphism, and a goal is to better understand the retraction of a product. In particular, we extend results of [2, 14]. We recall that  $B^+$  is a locally Garside monoid; each element  $\mathbf{b} \in B^+$  has a so-called *Garside normal form* which is the unique sequence  $(\mathbf{b}_1, \dots, \mathbf{b}_n)$  such that  $\mathbf{b} = \mathbf{b}_1 \dots \mathbf{b}_n$  where  $\mathbf{b}_i$  is the greatest left-divisor of  $\mathbf{b}_i \dots \mathbf{b}_n$  in  $B^+$  which is in  $\mathbf{W}$  and  $n \geq 0$  is minimal. The following proposition summarises some of our results on the retraction of a product (we postpone the definition of reduced elements and of the morphism  $\varphi_{\text{pr}(\mathbf{b})}$  to the next sections).

**Proposition 0.1.** *Let  $B$  be an Artin group and  $B_I$  be a standard parabolic subgroup for some  $I \subseteq S$ . Let  $\mathbf{b}$  be in  $B$ .*

- (1) (*Proposition 2.6(7)*) *If  $\text{pr}(\mathbf{b}) \in W_I$  then for  $\mathbf{b}' \in B$  we have  $\pi_I(\mathbf{b}\mathbf{b}') = \pi_I(\mathbf{b})\pi_I(\mathbf{b}')$ .*
- (2) (*Proposition 2.6(9)*) *Assume for  $J \subseteq S$  that  $\text{pr}(\mathbf{b})$  is  $I$ -reduced- $J$ . Then for  $\mathbf{j} \in B_J$  we have  $\pi_I(\mathbf{b}\mathbf{j}) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_{J_1}(\mathbf{j}))$  where  $J_1 = I^{\text{pr}(\mathbf{b})} \cap J$ .*
- (3) (*Proposition 2.11*) *For  $\mathbf{b} \in B^+$ , let  $(\mathbf{b}_1, \dots, \mathbf{b}_n)$  and  $(\mathbf{i}_1, \dots, \mathbf{i}_m)$  be the Garside normal forms of  $\mathbf{b}$  and  $\pi_I(\mathbf{b})$  respectively. Then,  $m \leq n$  and for  $1 \leq i \leq m$ , the element  $\pi_I(\mathbf{b}_1 \dots \mathbf{b}_i)$  left divides  $\mathbf{i}_1 \dots \mathbf{i}_i$ .*

In the next proposition we consider the case of Artin groups of spherical type, that is when  $W$  is finite. In this case  $W$  has a unique longest element relative to  $\ell_S$ . Its lift in  $\mathbf{W}$ , denoted by  $\Delta$ , is a Garside element of  $B^+$ . For  $I \subseteq S$ , the corresponding element of  $B_I^+$  lifting the longest element of  $W_I$  is denoted by  $\Delta_I$ .

**Proposition 0.2.** (*Proposition 2.12*) *Assume  $W$  is finite. Then for any  $\mathbf{b} \in B$ ,  $I \subseteq S$  and  $i \in \mathbb{Z}$  we have  $\pi_I(\mathbf{b}\Delta^i) = \pi_I(\mathbf{b})\pi_I(\Delta^i) = \pi_I(\mathbf{b})\Delta_I^i$ .*

For  $\mathbf{b} \in B$ , define  $\mathbf{t}_I(\mathbf{b})$  by  $\mathbf{b} = \pi_I(\mathbf{b})\mathbf{t}_I(\mathbf{b})$ . Then, the map  $\mathbf{b} \mapsto \mathbf{t}_I(\mathbf{b})$  gives a transversal of the right coset  $B_I\mathbf{b}$  in  $B$  (see Proposition 2.9). Exchanging left and right, one can define a right retraction  $\pi_I^r(\mathbf{b})$  such that  $\mathbf{b} = t_I^r(\mathbf{b})\pi_I^r(\mathbf{b})$  and

$\mathbf{b} \mapsto t_I^r(\mathbf{b})$  gives a transversal of the left coset  $\mathbf{b}B_I$  in  $B$ . These two retractions are closely related (see Proposition 3.1). We use them to address the question of how to solve the double coset problem in Artin groups. A solution to the double coset problem for Coxeter groups is known since [3]: every double coset has a unique element of minimal length which is easy to compute starting from any (word representing an) element of the double coset. Very few results have been obtained for Artin groups. The only result on the subject known to the authors is in the unpublished article [16]. It gives a partial answer in the case of braid groups, that is when  $W$  is of type  $A$ . Using left and right retractions, we obtain a partial result with the same flavour as in the case of Coxeter groups:

**Proposition 0.3.** *(Corollary 3.9 and Remark 3.10) Let  $I, J \subseteq S$  and  $\mathbf{b} \in B$  be such that  ${}^{\text{pr}(\mathbf{b})}J \cap I = \emptyset$ . Then the element  $\mathbf{b}_0 = \mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b})) = \mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b}))$  is the unique element of  $B_I \mathbf{b} B_J$  such that  $\pi_I(\mathbf{b}_0) = \pi_J^r(\mathbf{b}_0) = 1$ .*

We obtain actually a more general result (see Corollary 3.9).

It is a conjecture that the intersection of any two parabolic subgroups is a parabolic subgroup. The validity of the conjecture is proved for several families of Artin groups [1, 6, 7, 8, 17] but remains open in general. Using the retraction we address the conjecture and we show that starting from any element  $\mathbf{b}$  in  $B$ , the study of  $B_I \cap {}^{\mathbf{b}}B_J$  can be reduced to the case where  $\mathbf{b} = \mathbf{p}$  is in  $P$  and  $J = I$  (see Proposition 3.12). In this case we prove

**Proposition 0.4.** *Let  $I$  be a subset of  $S$  and  $\mathbf{p}$  be in  $P$  such that  $\pi_I(\mathbf{p}) = 1$ ; then*

$$B_I \cap {}^{\mathbf{p}}B_I = C_{B_I}(\mathbf{p})$$

where  $C_{B_I}(\mathbf{p}) = \{\mathbf{b} \in B_I \mid \mathbf{p}\mathbf{b} = \mathbf{b}\mathbf{p}\}$ , the centraliser of  $\mathbf{p}$  in  $B_I$ .

One of the main long-standing open problems for braid groups is the existence of a polynomial-time algorithm for the conjugacy problem. The partial results obtained on the subject are related to the Nielsen-Thurston classification (reducible, periodic or pseudo-Anosov) of braids. The conjugacy problem, and even the word problem remain open in the general case. In the last part of the article, we focus on the study of the conjugacy class of an element. In the spirit of the Nielsen-Thurston classification, we focus on what might be called “reducible” elements.

Denote by  $\ell_{\mathbf{S}^{\pm 1}} : B \rightarrow \mathbb{N}$  the length function on  $B$  with respect to the generating set  $\mathbf{S}^{\pm 1}$ . For  $\mathbf{b} \in B$ , it is equal to  $\ell(b)$ , the length of any minimal word  $b \in (\mathbf{S}^{\pm 1})^*$  representing  $\mathbf{b}$ . By convexity any such word  $b$  is written on the same set  $\mathbf{I} \subseteq \mathbf{S}$ , the image  $I \subseteq S$  of which we call the support of  $\mathbf{b}$ , denoted by  $\text{supp}(\mathbf{b})$ . The following result is a part of Propositions 4.8, 4.12 and Corollary 4.11.

**Proposition 0.5.** *Let  $\mathbf{b} \in B$  and let  $\text{Conj}_B(\mathbf{b})$  denote the conjugacy class of  $\mathbf{b}$ .*

- (1) *For  $I \subseteq S$ , either  $\text{Conj}_B(\mathbf{b}) \cap B_I$  is empty or it contains an element  $\mathbf{h}$  such that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{h})$  is minimal in  $\text{Conj}_B(\mathbf{b})$ .*
- (2) *If  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}(\mathbf{b})$  and if  $\mathbf{b}' \in \text{Conj}(\mathbf{b})$  satisfies  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{b}') = \ell_{\mathbf{S}^{\pm 1}}(\mathbf{b})$ , then  $\text{supp}(\mathbf{b})$  and  $\text{supp}(\mathbf{b}')$  are conjugate in  $W$ .*
- (3) *If  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}(\mathbf{b})$ , then  $B_{\text{supp}(\mathbf{b})}$  is a minimal parabolic subgroup containing  $\mathbf{b}$  (among all parabolic subgroups, not only the standard ones).*

Note that point (3) above does not state the uniqueness of a minimal parabolic subgroup since as mentioned above, the question remains open whether the intersection of two parabolic subgroups is parabolic.

In Section 4, we will prove that the existence of a unique minimal parabolic subgroup containing a given element is equivalent to the following conjecture:

**Conjecture 0.6.** *Let  $\mathbf{b} \in B$  such that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}(\mathbf{b})$  and let  $I = \text{supp}(\mathbf{b})$ , then any  $\mathbf{p} \in P$  which centralises  $\mathbf{b}$  and is such that  $\pi_I(\mathbf{p}) = 1$  centralises  $B_I$ .*

We also obtain a partial result on the conjugacy problem: one can say that we provide a solution to the conjugacy problem for “reducible” elements in Artin groups with  $S$  finite.

**Proposition 0.7.** *(Propositions 4.17 and 4.18) Assume  $S$  is finite and let  $\Lambda$  be a non-empty set of subsets of  $S$  such that for any  $I$  in  $\Lambda$ , the word problem and the conjugacy problem are solvable in  $B_I$ . Then the two following problems are solvable.*

- (1) *Given  $I$  in  $\Lambda$  and a word  $b$  on  $\mathbf{I}^{\pm 1} = \mathbf{I} \cup \mathbf{I}^{-1}$ , decide whether the word  $b$  represents the unity in  $B$ .*
- (2) *Given two finite subsets  $I, J$  in  $\Lambda$  and words  $i, j$  on  $\mathbf{I}^{\pm 1}$  and  $\mathbf{J}^{\pm 1}$ , respectively, decide whether or not  $i$  and  $j$  represent conjugate elements in  $B$ .*

The paper is structured as follows.

In Section 1 we focus on words. If  $X$  is a set, we denote by  $X^*$  the set of words on  $X$ . Using root systems, we give an alternate definition of the retraction  $\hat{\pi}_I$  from  $(\mathbf{S}^{\pm 1})^*$ , the words on  $\mathbf{S} \cup \mathbf{S}^{-1}$ , to  $(\mathbf{S}_I^{\pm 1})^*$  and extend some results of [14]. We obtain new results for words in  $(\mathbf{S}^{\pm 1})^*$  whose image in  $W$  is  $I$ -reduced- $J$ .

In Section 2, we introduce the retraction  $\pi_I$  on the Artin group and use the results of the previous section to prove Propositions 0.1 and 0.2. Using the notion of biclosed subset and results of Dyer we study the compatibility of the retraction with lcm’s and gcd’s.

In Section 3 we turn to the projection  $\pi_I^r$  and prove Propositions 0.3 and 0.4.

In Section 4 we consider the conjugacy classes, prove Propositions 0.5, 0.7 and state Conjecture 0.6 .

Finally, we provide in Section 5 (assuming  $S$  finite) a new topological version of  $\pi_I$  restricted to  $\text{pr}^{-1}(W_I)$ , using the Tits cone and results of Van der Lek [19].

## 1. RETRACTION ON WORDS

For  $I \subseteq S$ , we recall that an element  $w \in W$  is said to be  $I$ -reduced (*resp.* reduced- $I$ ) if  $\ell_S(iw) = \ell_S(i) + \ell_S(w)$  (*resp.*  $\ell_S(wi) = \ell_S(i) + \ell_S(w)$ ) for any  $i \in W_I$ , or equivalently for any  $i \in I$ . We say that an element is  $I$ -reduced- $J$  if it is both  $I$ -reduced and reduced- $J$ . For  $w \in W$  the unique element of minimum length in the coset  $W_I w$  is  $I$ -reduced; we denote it by  $t_I(w)$ .

**Lemma 1.1.** *If  $w$  is  $I$ -reduced and  $s \in S$ , then the property that  ${}^w s \in W_I$  is equivalent to  $ws$  not being  $I$ -reduced. Furthermore, in this case  ${}^w s \in I$ .*

*Proof.* Assume first that  $\ell_S(ws) > \ell_S(w)$ . If  $ws$  is not  $I$ -reduced, then there exists  $s' \in I$  such that  $\ell_S(s'ws) < \ell_S(ws)$ , which by the exchange lemma implies that  $s'ws$  is either equal to  $w$  or to  $\hat{w}s$  where  $\ell_S(\hat{w}) < \ell_S(w)$ . But  $s'ws = \hat{w}s$  contradicts the fact that  $w$  is  $I$ -reduced, so that  $s'ws = w$ , that is  ${}^w s = s' \in I$ . Conversely if  ${}^w s \in W_I$  then  $ws \in W_I w$  so is not  $I$ -reduced.

Now if  $\ell_S(ws) < \ell_S(w)$ , write  $w = w's$ , then  $ws = w'$  is  $I$ -reduced and by the first case  $w's = w's$  is not in  $W_I$ .  $\square$

**Definition 1.2** (Retraction). Write  $b \in (\mathbf{S}^{\pm 1})^*$  as  $b = \mathbf{s}_1^{\varepsilon_1} | \dots | \mathbf{s}_k^{\varepsilon_k}$  where  $\varepsilon_i = \pm 1$  and  $\mathbf{s}_i \in \mathbf{S}$ . For  $1 \leq j \leq k$  let  $w_j = s_1 \dots s_j$  be the image in  $W$  of the  $j$  first letters of  $b$ . We apply Lemma 1.1 to  $t_I(w_{j-1})$  (with  $w_0 = 1$ ) to deduce that if  $x = {}^{t_I(w_{j-1})}s_j \in W_I$  then  $x \in I$ ; we denote then  $x$  by  $s_{f(j)}$  and say that  $j$  is good; otherwise we say that  $j$  is bad and set  $s_{f(j)} = 1$ . The retraction  $\hat{\pi}_I(b)$  is the word in  $(\mathbf{I}^{\pm 1})^*$  whose letters are the  $\mathbf{s}_{f(j)}^{\varepsilon_j}$  for  $j$  good taken in increasing order.

This definition is equivalent to that in [2, Page 1522] though the notation there makes it a little bit difficult to see.

**Remark 1.3.** The following points are clear from the definition.

- (1)  $\hat{\pi}_I(b) = b$  if and only if  $b \in (\mathbf{I}^{\pm 1})^*$ .
- (2) If the word  $b$  is a prefix of the word  $b'$  then  $\hat{\pi}_I(b)$  is a prefix of  $\hat{\pi}_I(b')$ .
- (3)  $\hat{\pi}_I$  sends  $\mathbf{S}^*$  (positive words) to  $\mathbf{I}^*$ .
- (4)  $\hat{\pi}_I$  commutes with the map  $(\mathbf{S}^{\pm 1})^* \rightarrow (\mathbf{S}^{\pm 1})^*$  induced by  $\mathbf{s} \mapsto \mathbf{s}^{-1}$  for  $\mathbf{s} \in \mathbf{S}^{\pm 1}$ .
- (5) We denote by  $\ell(b)$  the length of  $b \in (\mathbf{S}^{\pm 1})^*$ . For any word  $b$  we have  $\ell(\hat{\pi}_I(b)) \leq \ell(b)$ , with equality if and only if  $b \in (\mathbf{I}^{\pm 1})^*$ .

An efficient way to compute  $t_I(w_j)$  is given by the following lemma.

**Lemma 1.4.** In the setting of Definition 1.2, for  $1 \leq j \leq k$  we have

- (1)  $t_I(w_j)$  is equal to the product of the  $s_l$  such that  $l$  is bad and  $1 \leq l \leq j$ ,
- (2)  $t_I(w_{j-1})s_j = s_{f(j)}t_I(w_j)$ .
- (3) For all  $j$ , we have  $w_j = s_{f(1)} \cdots s_{f(j)}t_I(w_j)$ .

*Proof.* We prove items (1) and (2) at the same time by induction on  $j$ . Assume that  $t_I(w_{j-1})$  is the product of the  $s_l$  for  $l$  bad and  $l \leq j-1$ . We write  $w_{j-1} = vt_I(w_{j-1})$  with  $v \in W_I$ . If  $j$  is good then  $w_j = {}^{w_{j-1}}s_j w_{j-1} = {}^{w_{j-1}}s_j vt_I(w_{j-1})$  with  ${}^{w_{j-1}}s_j v \in W_I$  so that  $t_I(w_j) = t_I(w_{j-1})$ , hence  $t_I(w_{j-1})s_j = s_{f(j)}t_I(w_{j-1}) = s_{f(j)}t_I(w_j)$ . If  $j$  is not good then we write  $w_j = vt_I(w_{j-1})s_j$ . We have to prove that  $t_I(w_{j-1})s_j$  is  $I$ -reduced, so is equal to  $t_I(w_j)$ , which is equal to  $s_{f(j)}t_I(w_j)$  since  $s_{f(j)} = 1$  in this case. If  $t_I(w_{j-1})s_j$  was not  $I$ -reduced, by Lemma 1.1  ${}^{t_I(w_{j-1})}s_j$  would be in  $W_I$  and then also  ${}^{w_{j-1}}s_j \in W_I$ , contradicting  $j$  bad.

Now, (3) results immediately by induction from (2).  $\square$

**Lemma 1.5.** If  $b = \mathbf{s}_1^{\varepsilon_1} | \dots | \mathbf{s}_k^{\varepsilon_k} \in (\mathbf{S}^{\pm 1})^*$ , where  $\varepsilon_i = \pm 1$ , is such that  $s_1 | \dots | s_k$ , is a reduced decomposition of an  $I$ -reduced element of  $W$ , then  $\hat{\pi}_I(b)$  is the empty word.

*Proof.* This follows from the definition of  $\hat{\pi}_I$  by using that any prefix of an  $I$ -reduced element is  $I$ -reduced and so, using Lemma 1.1, all indices are bad.  $\square$

We now give another definition of  $\hat{\pi}_I$ .

We call reflections of  $W$  the  $W$ -conjugates of the elements of  $S$  and denote their set by  $T$  (and similarly denote by  $T_I$  the set of reflections of  $W_I$ ).

As in [3, Chapter V, §4] we consider a faithful representation of  $W$  on a vector space with a basis  $\Pi = \{\alpha_s\}_{s \in S}$  whose elements are called simple roots. We call root system of  $W$  the set  $\Phi$  of images of  $\Pi$  under  $W$ . We call positive roots, the

elements of  $\Phi$  which are linear combinations of  $\Pi$  with positive coefficients, and we denote their set by  $\Phi^+$ ; similarly, we denote by  $\Phi_I, \Phi_I^+, \Pi_I$  the subsets associated with  $W_I$ . For any  $\alpha \in \Phi$  we denote the associated reflection by  $s_\alpha$ , and for  $t \in T$  we denote the corresponding positive root by  $\alpha_t$ .

Let  $\Phi^*$  denote the set of finite sequences of elements of  $\Phi$ , and let  $(\pm\Pi)^*$  be the subset of  $\Phi^*$  whose terms are in  $\Pi$  or  $-\Pi$ . For  $\underline{\alpha} = \alpha_1 | \dots | \alpha_n \in \Phi^*$  we denote by  $\text{prod}(\underline{\alpha})$  the product  $s_{\alpha_1} \cdots s_{\alpha_n} \in W$  and for  $w \in W$  we write  $w_{\underline{\alpha}}$  for the sequence  ${}^w\alpha_1 | \dots | {}^w\alpha_n$ . For  $\underline{\alpha} \in \Phi^*$  we denote by  $\underline{\alpha} \cap \Phi_I$  the sequence obtained by keeping only the terms lying in  $\Phi_I$ . We define  $\vec{N} : \Phi^* \rightarrow \Phi^*$  by  $\vec{N}(\alpha_1 | \dots | \alpha_n) = \alpha_1 | {}^{s_{\alpha_1}}\alpha_2 | \dots | {}^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-1}}}\alpha_n = \alpha_1 | {}^{\text{prod}(\alpha_1)}\alpha_2 | \dots | {}^{\text{prod}(\alpha_1 | \dots | \alpha_{n-1})}\alpha_n$ .

**Lemma 1.6.**  *$\vec{N}$  has the following properties for any sequence  $\underline{\alpha}$*

- (1)  $\vec{N}(\underline{\alpha} | \underline{\alpha}') = \vec{N}(\underline{\alpha}) | {}^{\text{prod}(\underline{\alpha})}\vec{N}(\underline{\alpha}')$  for any sequence  $\underline{\alpha}'$ .
- (2)  $\text{prod}(\vec{N}(\underline{\alpha})) = \text{prod}(\underline{\alpha})^{-1}$ .
- (3) For any  $w \in W$  we have  ${}^w\vec{N}(\underline{\alpha}) = \vec{N}({}^w\underline{\alpha})$ .
- (4)  $\vec{N}(\vec{N}(\underline{\alpha})) = \underline{\alpha}$ .

*Proof.* Properties (1) and (3) are clear. It results from the equality  $s_{w\alpha} = {}^w s_\alpha$  that  $\text{prod}(\vec{N}(\underline{\alpha})) = s_{\alpha_1} \cdot {}^{s_{\alpha_1}}s_{\alpha_2} \cdot \dots \cdot {}^{s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{n-1}}}s_{\alpha_n} = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1} = \text{prod}(\underline{\alpha})^{-1}$ , whence (2). Now the  $i$ -th term of  $\vec{N}(\vec{N}(\underline{\alpha}))$  is the image of  ${}^{s_{\alpha_1} \cdots s_{\alpha_{i-1}}}\alpha_i$  by the product  $\prod_{j=1}^{i-1} {}^{s_{\alpha_1} \cdots s_{\alpha_{j-1}}}s_{\alpha_j} = s_{\alpha_{i-1}} s_{\alpha_{i-2}} \cdots s_{\alpha_1}$  hence is equal to  $\alpha_i$ , whence (4).  $\square$

We define  $p : \mathbf{S}^{\pm 1} \rightarrow \pm\Pi$  by  $\mathbf{s} \mapsto \alpha_s$  and  $\mathbf{s}^{-1} \mapsto -\alpha_s$  for  $\mathbf{s} \in \mathbf{S}$  and we extend  $p$  to a map  $p^*$  from  $(\mathbf{S}^{\pm 1})^*$  to  $(\pm\Pi)^*$  in the obvious way.

**Lemma 1.7.** *A positive word  $b \in \mathbf{S}^*$  has a simple image in  $B$  (that is an image in  $W$ ) if and only if all terms in  $\vec{N}(p^*(b))$  are positive.*

*Proof.* Let  $b = \mathbf{s}_1 | \dots | \mathbf{s}_k$  with  $\mathbf{s}_i \in \mathbf{S}$ . As is well known  $l(s_1 \dots s_k) = l(s_1 \dots s_{k-1}) + 1$  if and only if  ${}^{s_1 \cdots s_{k-1}}\alpha_k \in \Phi^+$ . This gives the result by induction on  $k$ .  $\square$

The following proposition is a reformulation of the definition of retraction in [14, Definition 2.1] and of Definition 1.2.

**Proposition 1.8.** *For  $b \in (\mathbf{S}^{\pm 1})^*$ , we can define  $\hat{\pi}_I(b)$  as  $p^{*-1}(\vec{N}(\vec{N}(p^*(b)) \cap \Phi_I))$ ; this is well-defined since  $\vec{N}(\vec{N}(p^*(b)) \cap \Phi_I) \subseteq (\pm\Pi_I)^*$ .*

*Proof.* We retain the notation of Definition 1.2. For  $i = 1, \dots, k$ , let  $\alpha_i = {}^{w_i}p(\mathbf{s}_i^{\varepsilon_i})$ , that is  $\alpha_1 | \dots | \alpha_k = \vec{N}(p^*(b))$ . For any good index  $i$ , let  $A_i = \text{prod}((\alpha_1 | \dots | \alpha_{i-1}) \cap \Phi_I)$ , so that  ${}^{A_i}\alpha_i$  (for  $i$  good) are the terms of the sequence  $\vec{N}(\vec{N}(p^*(b)) \cap \Phi_I)$ . We prove by induction on good indices  $i$  that for  $i$  good we have  $A_i w_i = t_I(w_i) s_i$ . Assume the property for  $i$  and let  $j$  be the next good index. We have  $A_j = A_i t_i$ , hence  $A_j w_j = A_i t_i w_i s_{i+1} \cdots s_j = A_i w_{i-1} s_{i+1} \cdots s_j = t_I(w_i) s_{i+1} \cdots s_j$ , the second equality since  $t_i w_i = {}^{w_i} s_i w_i = w_i s_i = w_{i-1}$ , and the last equality by induction. We have  $A_j w_j = t_I(w_i) s_{i+1} \cdots s_j = t_I(w_j) s_j$ , the second equality by Lemma 1.4, whence the assertion.

We deduce that for  $i$  good we have  ${}^{A_i}\alpha_i = {}^{A_i w_i} p(\mathbf{s}_i^{\varepsilon_i}) = {}^{t_I(w_i)} p(\mathbf{s}_i^{\varepsilon_i})$ , which is in  $\pm\Pi_I$  by Lemma 1.1, since  $t_I(w_i)$  being  $I$ -reduced  ${}^{t_I(w_i)} s_i \in I$  is equivalent to  ${}^{t_I(w_i)} \alpha_{s_i} \in \Pi_I$ . We get the proposition since by definition the retraction  $\hat{\pi}_I(b)$  is  $p^{*-1}$  of the sequence of the  ${}^{t_I(w_i)} p(\mathbf{s}_i^{\varepsilon_i})$  for  $i$  good.  $\square$

**Corollary 1.9.** *For  $I, J$  subsets of  $S$  we have  $\hat{\pi}_I \circ \hat{\pi}_J = \hat{\pi}_{I \cap J}$ . In particular if  $J \subseteq I$  we have  $\hat{\pi}_J \circ \hat{\pi}_I = \hat{\pi}_J$ .*

*Proof.* By Proposition 1.8 we have

$$\begin{aligned} \hat{\pi}_I \circ \hat{\pi}_J(b) &= p^{*-1}(\vec{N}(\vec{N}(p^*(p^{*-1}(\vec{N}(\vec{N}(p^*(b)) \cap \Phi_J))) \cap \Phi_I)) \\ &= p^{*-1}(\vec{N}(\vec{N}(\vec{N}(\vec{N}(p^*(b)) \cap \Phi_J) \cap \Phi_I))) \\ &= p^{*-1}(\vec{N}(\vec{N}(p^*(b)) \cap \Phi_J \cap \Phi_I)) \\ &= \hat{\pi}_{I \cap J}(b) \end{aligned}$$

where the equality before last is from Lemma 1.6(4).  $\square$

We define a map  $\hat{\text{pr}}$  from  $(\mathbf{S}^{\pm 1})^*$  to  $W$  by  $\text{prod} \circ p^*$  (this factors obviously through the map  $\text{pr} : B \rightarrow W$  of the introduction).

**Proposition 1.10.** *For  $b \in (\mathbf{S}^{\pm 1})^*$  we have  $\hat{\text{pr}}(b) = \hat{\text{pr}}(\hat{\pi}_I(b))t_I(\hat{\text{pr}}(b))$ .*

*Proof.* With the notation of Lemma 1.4 we prove by induction on  $j$  that  $\hat{\text{pr}}(b_j) = \hat{\text{pr}}(\hat{\pi}_I(b_j))t_I(\hat{\text{pr}}(b_j))$  where  $b_j = \mathbf{s}_1^{\varepsilon_1} | \cdots | \mathbf{s}_j^{\varepsilon_j}$ , so that  $w_j = \hat{\text{pr}}(b_j)$ . We have  $\hat{\text{pr}}(b_j) = \hat{\text{pr}}(b_{j-1})s_j = \hat{\text{pr}}(\hat{\pi}_I(b_{j-1}))t_I(\hat{\text{pr}}(b_{j-1}))s_j = \hat{\text{pr}}(\hat{\pi}_I(b_{j-1}))s_{f(j)}t_I(\hat{\text{pr}}(b_j))$ , the second equality by induction and the third by Lemma 1.4(2). We get the result since  $\hat{\text{pr}}(\hat{\pi}_I(b_j)) = \hat{\text{pr}}(\hat{\pi}_I(b_{j-1}))s_{f(j)}$  by definition of  $\hat{\pi}_I$ .  $\square$

**Proposition 1.11.** *We have  $\hat{\pi}_I(b|b') = \hat{\pi}_I(b)|p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\hat{\text{pr}}(b))}p^*(b')) \cap \Phi_I))$ . In particular if  $\hat{\text{pr}}(b) \in W_I$  then  $\hat{\pi}_I(b|b') = \hat{\pi}_I(b)|\hat{\pi}_I(b')$ .*

*Proof.* By Lemma 1.6(1) we get  $\vec{N}(p^*(b|b') \cap \Phi_I) = \vec{N}(p^*(b) \cap \Phi_I)|^{\hat{\text{pr}}(b)}\vec{N}(p^*(b') \cap \Phi_I)$  where we have used that  $\text{prod}(p^*(b)) = \hat{\text{pr}}(b)$ . Using again Lemma 1.6(1) we get  $\vec{N}(\vec{N}(p^*(b|b') \cap \Phi_I)) = \vec{N}(\vec{N}(p^*(b) \cap \Phi_I))|^{\text{prod}(\vec{N}(p^*(b) \cap \Phi_I))}\vec{N}(\hat{\text{pr}}(b)\vec{N}(p^*(b') \cap \Phi_I))$ . Applying  $p^{*-1}$  to this equality and using Proposition 1.8, we get

$$\hat{\pi}_I(b|b') = \hat{\pi}_I(b)|p^{*-1}(\text{prod}(\vec{N}(p^*(b) \cap \Phi_I))\vec{N}(\hat{\text{pr}}(b)\vec{N}(p^*(b') \cap \Phi_I))).$$

The right-hand side becomes  $\hat{\pi}_I(b)|p^{*-1}(\hat{\text{pr}}(\hat{\pi}_I(b))^{-1}\vec{N}(\hat{\text{pr}}(b)\vec{N}(p^*(b') \cap \Phi_I))$  using Lemma 1.6(2). Using Lemma 1.6(3) and that  $\hat{\text{pr}}(\hat{\pi}_I(b))$  normalises  $\Phi_I$  this formula becomes  $\hat{\pi}_I(b)|p^{*-1}(\vec{N}(\hat{\text{pr}}(\hat{\pi}_I(b))^{-1}\hat{\text{pr}}(b)\vec{N}(p^*(b') \cap \Phi_I))$ . We get the proposition using once more Lemma 1.6(3) and then Proposition 1.10.  $\square$

**Corollary 1.12.** *For  $b_1, \dots, b_n \in (\mathbf{S}^{\pm 1})^*$  we have*

$$\hat{\pi}_I(b_1 | \cdots | b_n) = \prod_{i=1}^{i=n} p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\hat{\text{pr}}(b_1 \cdots b_{i-1}))}p^*(b_i)) \cap \Phi_I)).$$

*If all  $b_i$  are positive words that have simple images in  $B$  the same is true for each term of the product on the right-hand side.*

Note that if all  $b_i$  are in  $\mathbf{S}^{\pm 1}$  one recovers Definition 1.2.

*Proof.* The formula is obtained by recursively applying Proposition 1.11. We prove the second assertion. By Lemma 1.7 a positive word  $b$  has a simple image in  $B$  if and only if all terms of  $\vec{N}(p^*(b))$  are positive. Since  $\vec{N}$  is an involution by Lemma 1.6(4),  $\vec{N}({}^{t_I(\hat{\text{pr}}(b_1 \cdots b_{i-1}))}p^*(b_i)) \cap \Phi_I$  is equal to  $\vec{N}(p^*(b'_i))$ , where  $b'_i$  is the  $i$ -th term of the product on the right-hand side of the statement; and all terms



of  $\vec{N}({}^{t_I(\widehat{\text{pr}}(b_1 \cdots b_{i-1}))} p^*(b_i)) \cap \Phi_I = {}^{t_I(\widehat{\text{pr}}(b_1 \cdots b_{i-1}))} \vec{N}(p^*(b_i)) \cap \Phi_I$  are positive by the simplicity assumption on the image of  $b_i$ , using the fact that  $t_I(\widehat{\text{pr}}(b_1 \cdots b_{i-1}))$  is  $I$ -reduced.  $\square$

**Lemma 1.13.** *Let  $I, J \subseteq S$  and let  $w \in W$  be  $I$ -reduced- $J$ ; then  $I_1 = I \cap {}^w J$  is the unique maximal subset of  $I$  such that  $I_1^w \subseteq J$ . Equivalently  $J_1 = I^w \cap J$  is the unique maximal subset of  $J$  such that  ${}^w J_1 \subseteq I$ .*

*Proof.* This is clear.  $\square$

**Definition 1.14.** *Let  $I \subseteq S$ ; an element  $w \in W$  which is  $I$ -reduced and such that  $I^w \subseteq S$  is called an  $I$ -ribbon. If  $J = I^w$  we also call it an  $I$ -ribbon- $J$  or a ribbon- $J$ , in order to specify  $J$ .*

*In this situation  $w$  is also reduced- $J$  and  $w^{-1}$  induces a bijection  $\varphi_w : J \xrightarrow{\sim} I$ , which lifts naturally to a bijection from  $(\mathbf{J}^{\pm 1})^*$  to  $(\mathbf{I}^{\pm 1})^*$  that we still denote by  $\varphi_w$ .*

**Remark 1.15.** *Under the assumptions of Lemma 1.13, the element  $w$  is a  $I_1$ -ribbon- $J_1$ ; indeed we have  $I_1^w = J_1$  and  $w$  is  $I_1$ -reduced- $J_1$ . Moreover  $I_1$  and  $J_1$  are the maximal subsets of  $I$  and  $J$  respectively with that property.*

**Proposition 1.16.** *For  $I, J \subseteq S$ , let  $b, b' \in (\mathbf{S}^{\pm 1})^*$  be such that  $\widehat{\text{pr}}(b)$  is an  $I$ -ribbon- $J$ . Then  $\hat{\pi}_I(b|b') = \hat{\pi}_I(b)|\varphi_{\widehat{\text{pr}}(b)}(\hat{\pi}_J(b'))$ .*

*Proof.* Proposition 1.11 gives  $\hat{\pi}_I(b|b') = \hat{\pi}_I(b)|p^{*-1}(\vec{N}(\vec{N}(\widehat{\text{pr}}(b))p^*(b')) \cap \Phi_I)$ . Using that  $\Phi_I^{\widehat{\text{pr}}(b)} = \Phi_J$  the second term of the right-hand side is equal to

$$p^{*-1}(\widehat{\text{pr}}(b)\vec{N}(\vec{N}(p^*(b')) \cap \Phi_J)) = \varphi_{\widehat{\text{pr}}(b)}(p^{*-1}(\vec{N}(\vec{N}(p^*(b')) \cap \Phi_J))) = \varphi_{\widehat{\text{pr}}(b)}(\hat{\pi}_J(b')).$$

$\square$

If we lift an expression  $w = s_1 \cdots s_k$  to a word  $\hat{w} \in \mathbf{S}^*$ , the image  $N(w)$  in  $\mathbb{Z}/2\mathbb{Z}[T]$  of  $\vec{N}(p^*(\hat{w}))$ , obtained by associating to each root the corresponding reflection, depends only on  $w$ . Indeed  $N(w) = \sum_{i=1, \dots, k} s_1 \cdots s_{i-1} s_i \in \mathbb{Z}/2\mathbb{Z}[T]$  can also be considered as the subset of  $T$  consisting of elements which appear an odd number of times in the sequence  $\{s_1 \cdots s_{i-1} s_i\}_{1 \leq i \leq k}$ ; it is known that  $N$  depends only on  $w$  (see [3, Chapter IV, §1.4, Lemme 2]). It is also shown in loc. cit. that  $N$  is injective and that  $|N(w)| = \ell_S(w)$ . For  $v, v' \in W$  one has  $N(vv') = N(v) + {}^v N(v')$ .

The next proposition is [18, Lemma 2]. The proof we give, in the setting of the elementary Coxeter group theory, is shorter than the original proof.

**Proposition 1.17.** *Let  $I, J$  be subsets of  $S$  and  $w \in W$  be  $I$ -reduced- $J$ . Then  $W_J \cap W_I^w = W_{J_1}$  where  $J_1 = I^w \cap J$ .*

*Proof.* If  $y \in W_J \cap W_I^w$  then  $wy \in W_I w$  so we can write  $wy = xw$  with  $x \in W_I$ . We get thus  $N(w) \coprod {}^w N(y) = N(x) \coprod {}^x N(w)$  where the sums are disjoint because  $w$  is reduced on both sides. But since  $w$  is  $I$ -reduced  $N(w)$  does not meet  $W_I$ , thus  $N(w) \subseteq {}^x N(w)$  and we must have equality since these sets have same cardinality. Thus in turn we must have  ${}^w N(y) = N(x)$ . We claim that by induction on  $\ell_S(y)$  the assertion  ${}^w N(y) \subseteq W_I$  implies  $y \in W_{J_1}$ . Let  $y = sy'$  where  $s \in J$  and  $\ell_S(y') = \ell_S(y) - 1$ . By Lemma 1.1 the fact that  ${}^w s \in W_I$  implies  ${}^w s \in I$  thus  $s \in J_1$ , and  $ws = tw$  for some  $t \in I$ . Using  $N(sy') = s \coprod {}^s N(y')$  we get  ${}^{ws} N(y') \subseteq W_I$ , and since  $ws = tw$  this implies  ${}^w N(y') \subseteq W_I$  and we conclude by induction.  $\square$

**Corollary 1.18.** *Let  $I, J$  be subsets of  $S$  and  $w \in W$  be  $I$ -reduced- $J$ . Then  $\Phi_J \cap \Phi_I^w = \Phi_{J_1}$  where  $J_1 = I^w \cap J$ .*

*Proof.* We have  $\alpha \in \Phi_J \cap \Phi_I^\psi$  if and only if  $s_\alpha \in W_J \cap W_I^\psi = W_{J_1}$  by Proposition 1.17, which in turn is equivalent to  $\alpha \in \Phi_{J_1}$ .  $\square$

**Proposition 1.19.** *Let  $I, J \subseteq S$  and  $b \in (\mathbf{S}^{\pm 1})^*$  be such that  $\widehat{\text{pr}}(b)$  is  $I$ -reduced- $J$ . Let  $J_1 = I^{\widehat{\text{pr}}(b)} \cap J$ ; Then for any word  $b' \in (\mathbf{J}^{\pm 1})^*$ , we have  $\widehat{\pi}_I(b|b') = \widehat{\pi}_I(b)|\varphi_{\widehat{\text{pr}}(b)}(\widehat{\pi}_{J_1}(b'))$ .*

*Proof.* Using that  $t_i(\widehat{\text{pr}}(b)) = \widehat{\text{pr}}(b)$  since  $\widehat{\text{pr}}(b)$  is  $I$ -reduced, and using Lemma 1.6(3) twice, Proposition 1.11 becomes in our case  $\widehat{\pi}_I(b|b') = \widehat{\pi}_I(b)|p^{*-1}(\widehat{\text{pr}}(b)\vec{N}(\vec{N}(p^*(b')) \cap \Phi_I^{\widehat{\text{pr}}(b)}))$ . Now by assumption  $\vec{N}(p^*(b')) \subseteq \Phi_J$ , and by Corollary 1.18 we have  $\Phi_J \cap \Phi_I^{\widehat{\text{pr}}(b)} = \Phi_{J_1}$ . The second term is thus  $p^{*-1}(\widehat{\text{pr}}(b)\vec{N}(\vec{N}(p^*(b')) \cap \Phi_{J_1}))$ , and we conclude by Proposition 1.11 since  $p^{*-1}(\widehat{\text{pr}}(b)\vec{N}(\vec{N}(p^*(b')) \cap \Phi_{J_1}))$  is equal to  $\varphi_{\widehat{\text{pr}}(b)}(p^{*-1}(\vec{N}(\vec{N}(p^*(b')) \cap \Phi_{J_1})))$ , that is to  $\varphi_{\widehat{\text{pr}}(b)}(\widehat{\pi}_{J_1}(b'))$ .  $\square$

## 2. RETRACTION ON BRAIDS

We say that a subgroup of  $W$  is parabolic if it is conjugate to a standard parabolic subgroup of  $W$ .

As a direct application of Proposition 1.17, we get

**Lemma 2.1.** *the intersection of any two parabolic subgroups  $D$  and  $Q$  of  $W$  is a parabolic subgroup of each of them.*

*Proof.* Up to conjugacy, we may assume that  $Q$  is standard, say  $Q = W_J$ . Assume  $D = W_I^v$ . Write  $w = iwj$  with  $i \in W_I$ ,  $j \in W_J$  and  $w$   $I$ -reduced- $J$ . Then  $D \cap Q = (W_J \cap W_I^w)^v$ , and we conclude by Proposition 1.17.  $\square$

**Lemma 2.2.** *For any parabolic subgroups  $Q, D$  of  $W$  where  $D$  is dihedral the set  $T \cap Q \cap D$  is either empty, or is reduced to one reflection or is the set  $T \cap D$  of all reflections of  $D$ .*

*Proof.* By Lemma 2.1 the intersection of the two parabolic subgroups  $D$  and  $Q$  of  $W$  is a parabolic subgroup of each of them. The possible parabolic subgroups of  $D$  are  $\{1\}$ , a subgroup generated by one reflection, or  $D$  itself. This gives the three cases of the statement.  $\square$

**Lemma 2.3.** *Let  $b, b' \in (\mathbf{S}^{\pm 1})^*$  and assume one of the following for two words  $r$  and  $r'$  in  $(\mathbf{S}^{\pm 1})^*$ :*

- (1)  $r, r'$  are the two members of a braid relation,
- (2)  $r = \mathbf{s}|\mathbf{s}^{-1}$  or  $\mathbf{s}^{-1}|\mathbf{s}$  and  $r'$  is the empty word,

*then the two words  $\widehat{\pi}_I(b|r|b')$  and  $\widehat{\pi}_I(b|r'|b')$  are either equal or differ by a relation in  $(\mathbf{I}^{\pm 1})^*$  of the same type as the relation  $r \equiv r'$ .*

*Proof.* Applying Corollary 1.12 to  $b|r|b'$  and  $b|r'|b'$ , since  $\widehat{\text{pr}}(r) = \widehat{\text{pr}}(r')$ , it is sufficient to compare  $p^{*-1}(\vec{N}(\vec{N}(t_I(\widehat{\text{pr}}(b)))p^*(r)) \cap \Phi_I)$  and  $p^{*-1}(\vec{N}(\vec{N}(t_I(\widehat{\text{pr}}(b)))p^*(r')) \cap \Phi_I)$ . Using Lemma 1.6(3) it is equivalent to compare for any  $I$ -reduced element  $w \in W$  the words  $p^{*-1}(\vec{N}(w\vec{N}(p^*(r)) \cap \Phi_I))$  and  $p^{*-1}(\vec{N}(w\vec{N}(p^*(r')) \cap \Phi_I))$ .

In case (1) the sequences in  $\vec{N}(p^*(r))$  and  $\vec{N}(p^*(r'))$  are reversed from each other and consist of all the positive roots of a standard dihedral parabolic. Hence the roots in  $\vec{N}(p^*(r)) \cap w^{-1}\Phi_I$  and  $\vec{N}(p^*(r')) \cap w^{-1}\Phi_I$  are all the positive roots in the intersection of that dihedral subgroup with  $w^{-1}W_I$ . By Lemma 2.2 there are three

cases for this set of roots: either it is empty or it is reduced to one root or it is the set of all the positive roots of a dihedral parabolic subgroup of  $w^{-1}W_I$ .

In the first two cases we get clearly the equality of the sequences  $\vec{N}(w\vec{N}(p^*(r)) \cap \Phi_I)$  and  $\vec{N}(w\vec{N}(p^*(r')) \cap \Phi_I)$ . In the third case we have  $\vec{N}(w\vec{N}(p^*(r)) \cap \Phi_I) = \vec{N}(w\vec{N}(p^*(r))) = w p^*(r)$  and  $\vec{N}(w\vec{N}(p^*(r')) \cap \Phi_I) = \vec{N}(w\vec{N}(p^*(r'))) = w p^*(r')$  and these two sequences are reversed from each other and are sequences in  $\Pi_J^*$ . Thus applying  $p^{*-1}$  gives the two members of a braid relation in  $B_I$ , whence the result.

In case (2) we have  $w\vec{N}(p^*(r)) = \alpha | - \alpha$  for some root  $\alpha$ . If  $\alpha \notin \Phi_I$ , then  $w\vec{N}(p^*(r)) \cap \Phi_I$  is the empty sequence and  $\hat{\pi}_I(b|r|b')$  and  $\hat{\pi}_I(b|r'|b')$  are equal. If  $\alpha \in \Phi_I$  then the two words  $p^{*-1}(\vec{N}(w\vec{N}(p^*(r)) \cap \Phi_I))$  and  $p^{*-1}(\vec{N}(w\vec{N}(p^*(r')) \cap \Phi_I))$  differ by a relation of type  $\mathbf{s}|\mathbf{s}^{-1} \equiv \emptyset$  or  $\mathbf{s}^{-1}|\mathbf{s} \equiv \emptyset$  with  $\mathbf{s} \in \mathbf{I}$ .  $\square$

**Proposition 2.4.** *For  $b \in (\mathbf{S}^{\pm 1})^*$  of image  $\mathbf{b}$  in  $B$ , the image in  $B$  of  $\hat{\pi}_I(b)$  depends only on  $\mathbf{b}$ . We call it  $\pi_I(\mathbf{b})$ .*

*Proof.* This is an immediate consequence of Lemma 2.3.  $\square$

The following proposition was first proved by Van der Lek. In [19],  $S$  is assumed to be finite but the result for infinite  $S$  is an almost immediate consequence.

**Proposition 2.5.** *For  $I \subseteq S$ , the natural morphism from the braid group of  $W_I$  to  $B_I$  is an isomorphism.*

*Proof.* The morphism is clearly surjective. We show the injectivity. Let  $b$  be a word in  $(\mathbf{I}^{\pm 1})^*$ . We have to show that if  $b \equiv \emptyset$  in  $B$ , then the same equivalence occurs using only relations in  $(\mathbf{I}^{\pm 1})^*$ . Let  $b = b_1 \equiv b_2 \equiv \dots \equiv \emptyset$  be a chain of equivalences in  $B$  by relations of the type considered in Lemma 2.3. Apply  $\hat{\pi}_I$  to this chain. The first and the last term are unchanged, and by Lemma 2.3 each other equivalence is mapped to an equivalence in  $(\mathbf{I}^{\pm 1})^*$ . This proves the result.  $\square$

Statements of Section 1 translate to the following (sometimes weaker) properties of  $\pi_I$ :

**Proposition 2.6.** *For  $\mathbf{b} \in B$  we have the following:*

- (1)  $\pi_I(\mathbf{b}) = \mathbf{b}$  if and only if  $\mathbf{b} \in B_I$ .
- (2)  $\pi_I$  sends  $B^+$  to  $B_I^+$ .
- (3) If  $\mathbf{b} \in B^+$  left-divides  $\mathbf{b}' \in B^+$  then  $\pi_I(\mathbf{b})$  left-divides  $\pi_I(\mathbf{b}')$ .
- (4)  $\pi_I$  commutes with the automorphism  $\text{inv} : B \rightarrow B$  induced by  $\mathbf{s} \mapsto \mathbf{s}^{-1}$  for  $\mathbf{s} \in \mathbf{S}$ .
- (5) If  $\mathbf{b} \in \mathbf{W}$  and  $\text{pr}(\mathbf{b})$  is  $I$ -reduced, then  $\pi_I(\mathbf{b}) = 1$ .
- (6)  $\text{pr}(\mathbf{b}) = \text{pr}(\pi_I(\mathbf{b}))t_I(\text{pr}(\mathbf{b}))$ .
- (7) Assume that  $\text{pr}(\mathbf{b}) \in W_I$ ; then for  $\mathbf{b}'$  in  $B$  we have  $\pi_I(\mathbf{b}\mathbf{b}') = \pi_I(\mathbf{b})\pi_I(\mathbf{b}')$ . In particular the restriction of  $\pi_I$  to  $\text{pr}^{-1}(W_I)$  is a group homomorphism.
- (8) Let  $\mathbf{b}' \in B$  and let  $I, J \subseteq S$  be such that  $\text{pr}(\mathbf{b})$  is an  $I$ -ribbon- $J$ . Then  $\pi_I(\mathbf{b}\mathbf{b}') = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_J(\mathbf{b}'))$ .
- (9) For  $I, J \subseteq S$  such that  $\text{pr}(\mathbf{b})$  is  $I$ -reduced- $J$  and any  $\mathbf{j} \in B_J$ , we have  $\pi_I(\mathbf{b}\mathbf{j}) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_{J_1}(\mathbf{j}))$  where  $J_1 = I^{\text{pr}(\mathbf{b})} \cap J$ .
- (10) If  $\mathbf{b}$  is a product of  $n$  elements of  $\mathbf{W}$ , then  $\pi_I(\mathbf{b})$  is a product of at most  $n$  elements of  $\mathbf{W}_I$ .

*Proof.* (1), and (4) are (1) and (4) of Remark 1.3. (2), (3) follow respectively from (3) and (2) of Remark 1.3. (5) is Lemma 1.5, (6) is Proposition 1.10, (7) is

Proposition 1.11, (8) is Proposition 1.16, and (9) is Proposition 1.19. Finally (10) follows from Corollary 1.12.  $\square$

**Lemma 2.7.** *Let  $\mathbf{b} \in B$  be such that  $\text{pr}(\mathbf{b}) \in N_W(W_I)$ ; then there exists a unique  $w \in W$  such that  $w$  is an  $I$ -ribbon- $I$  and such that  $\text{pr}(\mathbf{b}) \in W_I w$ . For any  $\mathbf{b}' \in B$  we have  $\pi_I(\mathbf{b}\mathbf{b}') = \pi_I(\mathbf{b})\varphi_w(\pi_I(\mathbf{b}'))$ .*

*Proof.* The unicity of  $w$  comes from the fact that  $w$  is the  $I$ -reduced element in  $W_I \text{pr}(\mathbf{b})$ . By [11, Lemme 6.1.7] (the result is implicit in [4]) the  $I$ -reduced element  $w$  in  $W_I \text{pr}(\mathbf{b})$  is an  $I$ -ribbon- $I$ . Let  $\mathbf{w} \in \mathbf{W}$  be the lift of  $w$  and  $\mathbf{b}_1 = \mathbf{b}\mathbf{w}^{-1}$ ; then  $\text{pr}(\mathbf{b}_1) \in W_I$  and, by items (7) and (8) of Proposition 2.6, we have  $\pi_I(\mathbf{b}\mathbf{b}') = \pi_I(\mathbf{b}_1) \pi_I(\mathbf{w}\mathbf{b}') = \pi_I(\mathbf{b}_1)\varphi_w(\pi_I(\mathbf{b}'))$ . The equality of the lemma follows since  $\pi_I(\mathbf{b}) = \pi_I(\mathbf{b}_1)$  by the last equality applied with  $\mathbf{b}' = 1$ .  $\square$

**Definition 2.8.** *Given  $I \subseteq S$ ,*

- *For  $\mathbf{b}$  in  $B$ , we define  $\mathbf{t}_I(\mathbf{b}) = \pi_I(\mathbf{b})^{-1}\mathbf{b}$ .*
- *For  $w \in W$ , we define  $\pi_I(w) = wt_I(w)^{-1}$ .*

The following lemma shows that the definitions  $t_I, \mathbf{t}_i$  and  $\pi_I, \pi_I$  are compatible.

**Lemma 2.9.** *Let  $I \subseteq S$  and  $\mathbf{b}$  be in  $B$ .*

- (1)  $\pi_I(\mathbf{t}_I(\mathbf{b})) = 1$ .
- (2)  $\text{pr}(\mathbf{t}_I(\mathbf{b})) = t_I(\text{pr}(\mathbf{b}))$ . *In particular,  $\text{pr}(\mathbf{t}_I(\mathbf{b}))$  is  $I$ -reduced.*
- (3)  $\text{pr}(\pi_I(\mathbf{b})) = \pi_I(\text{pr}(\mathbf{b}))$ .
- (4) *If  $\mathbf{b}'$  is in  $B$ , then  $B_I \mathbf{b} = B_I \mathbf{b}' \iff \mathbf{t}_I(\mathbf{b}) = \mathbf{t}_I(\mathbf{b}')$ .*

Note that (3) can be seen as a part of the commutative diagram in Proposition 2.22.

*Proof.* We have  $\pi_I(\mathbf{b}) = \pi_I(\pi_I(\mathbf{b})\mathbf{t}_I(\mathbf{b})) = \pi_I(\pi_I(\mathbf{b})) \pi_I(\mathbf{t}_I(\mathbf{b})) = \pi_I(\mathbf{b}) \pi_I(\mathbf{t}_I(\mathbf{b}))$ , the second equality by Proposition 2.6(7). So  $\pi_I(\mathbf{t}_I(\mathbf{b})) = 1$ . The second and third items come from  $\text{pr}(\pi_I(\mathbf{b})) \text{pr}(\mathbf{t}_I(\mathbf{b})) = \text{pr}(\mathbf{b}) = \text{pr}(\pi_I(\mathbf{b}))t_I(\text{pr}(\mathbf{b}))$  where the first equality is by definition and the second by Proposition 2.6(6). For (4), clearly, if  $\mathbf{t}_I(\mathbf{b}) = \mathbf{t}_I(\mathbf{b}')$  then  $B_I \mathbf{b} = B_I \mathbf{b}'$ . Conversely, assume  $\mathbf{i}\mathbf{b} = \mathbf{i}'\mathbf{b}'$  with  $\mathbf{i}, \mathbf{i}'$  in  $B_I$ . Then  $\mathbf{i}\pi_I(\mathbf{b})\mathbf{t}_I(\mathbf{b}) = \mathbf{i}'\pi_I(\mathbf{b}')\mathbf{t}_I(\mathbf{b}')$ , thus to prove that  $\mathbf{t}_I(\mathbf{b}) = \mathbf{t}_I(\mathbf{b}')$  it suffices to prove that  $\mathbf{i}\pi_I(\mathbf{b}) = \mathbf{i}'\pi_I(\mathbf{b}')$ . But by Proposition 2.6(7) we have  $\mathbf{i}\pi_I(\mathbf{b}) = \pi_I(\mathbf{i}\mathbf{b})$  and  $\mathbf{i}'\pi_I(\mathbf{b}') = \pi_I(\mathbf{i}'\mathbf{b}')$  so they are equal by assumption.  $\square$

Note that  $\mathbf{t}_I(B^+) \not\subseteq B^+$ . For instance, in type  $A_2$  with  $\mathbf{S} = \{\mathbf{s}, \mathbf{t}\}$  and  $I = \{s\}$ , we have  $\pi_I(\mathbf{tts}) = \mathbf{s}$ , thus  $\mathbf{t}_I(\mathbf{tts}) = \mathbf{s}^{-1}\mathbf{tts}$ .

**Proposition 2.10.** *For  $w \in W$ ,  $\pi_I(w)$  is the unique element  $v \in W_I$  such that  $N(w) \cap T_I = N(v)$ .*

*Proof.* Since  $N$  is injective it is sufficient to prove that  $v$  defined by  $w = vt_I(w)$  satisfies  $N(v) = N(w) \cap T_I$ . We have  $N(w) = N(v) + {}^vN(t_I(w))$ . We want to show that  ${}^vN(t_I(w))$  does not meet  $W_I$  or equivalently that  $N(t_I(w))$  does not meet  $W_I$ . But if  $t_I(w) = s_1 \dots s_k$  is a reduced expression this is equivalent to  $s_1 \dots s_j s_{j-1} \dots s_1 \notin W_I$  for all  $j$  or equivalently  $s_1 \dots s_j \notin W_I s_1 \dots s_{j-1}$ , which is a consequence of  $s_1 \dots s_j$  being  $I$ -reduced.  $\square$

Since  $B^+$  is a locally Garside monoid, elements have a greatest common left-divisor (left-gcd) and if they have a common right-multiple they have a right-lcm. Each element has a Garside normal form, which for  $\mathbf{b} \in B^+$  is a sequence

$(\mathbf{b}_1, \dots, \mathbf{b}_n)$  with  $\mathbf{b}_i \in \mathbf{W}$  such that  $\mathbf{b} = \mathbf{b}_1 \cdots \mathbf{b}_n$ , uniquely defined by the property that  $\mathbf{b}_i$  is the greatest left-divisor in  $\mathbf{W}$  of  $\mathbf{b}_i \mathbf{b}_{i+1}$  and by the property that no  $\mathbf{b}_i$  is 1.

**Proposition 2.11.** *For  $\mathbf{b} \in B^+$ , let  $(\mathbf{b}_1, \dots, \mathbf{b}_n)$  and  $(\mathbf{i}_1, \dots, \mathbf{i}_m)$  be the Garside normal forms of  $\mathbf{b}$  and  $\pi_I(\mathbf{b})$  respectively. Then,  $m \leq n$  and for  $1 \leq i \leq m$ , the element  $\pi_I(\mathbf{b}_1 \cdots \mathbf{b}_i)$  left divides  $\mathbf{i}_1 \cdots \mathbf{i}_i$ .*

*Proof.* As a particular case of Proposition 2.6(10) the normal form of  $\pi_I(\mathbf{b}_1 \cdots \mathbf{b}_n)$  has  $n$  terms or less, since the product of  $n$  simple braids has not more than  $n$  terms in its normal form. Now by Proposition 2.6(3) we have that  $\pi_I(\mathbf{b}_1 \cdots \mathbf{b}_i)$  left-divides  $\mathbf{i}_1 \cdots \mathbf{i}_m$  and since it has  $i$  simple terms by Proposition 2.6(10), by [9, III, 1.14] it divides  $\mathbf{i}_1 \cdots \mathbf{i}_i$ .  $\square$

The next two propositions assume  $W$  finite. In this case  $B^+$  is a Garside monoid with Garside element the lift  $\Delta \in \mathbf{W}$  of the longest element of  $W$ . The elements of  $\mathbf{W}$  are both the left-divisors and the right-divisors of  $\Delta$ . Similarly  $B_I^+$  is a Garside monoid with Garside element  $\Delta_I$ , the lift of the longest element of  $W_I$ .

**Proposition 2.12.** *Assume that  $W$  is finite. Then for any  $\mathbf{b} \in B$ ,  $I \subseteq S$  and  $i \in \mathbb{Z}$  we have  $\pi_I(\mathbf{b}\Delta^i) = \pi_I(\mathbf{b})\Delta_I^i$ .*

*Proof.* One deduces the proposition for  $i < 0$  from the result for  $-i$  by writing  $\pi_I(\mathbf{b}) = \pi_I(\mathbf{b}\Delta^i\Delta^{-i}) = \pi_I(\mathbf{b}\Delta^i)\Delta_I^{-i}$ . The proposition for  $i \in \mathbb{N}$  is obtained by iterating the case where  $i = 1$ , so we assume now  $i = 1$ .

We denote by  $\varphi$  the automorphism of  $B$  induced by conjugating by  $\Delta$ . Define  $\mathbf{w}$  by  $\Delta = \Delta_I \mathbf{w}$ ; then  $w = \text{pr}(\mathbf{w})$  is a  $I$ -ribbon- $\varphi(I)$ . We get  $\pi_I(\mathbf{b}\Delta) = \pi_I(\Delta\varphi^{-1}(\mathbf{b})) = \pi_I(\Delta_I \mathbf{w}\varphi^{-1}(\mathbf{b})) = \Delta_I \pi_I(\mathbf{w}\varphi^{-1}(\mathbf{b})) = \Delta_I \varphi_w(\pi_{\varphi(I)}(\varphi^{-1}(\mathbf{b})))$  where the third equality is by Proposition 2.6(7) and the last by Proposition 2.6 items (5) and (8). Now the automorphism  $\varphi$  clearly commutes with  $\pi_I$ , that is for any  $\mathbf{b} \in B$  we have  $\varphi_w \pi_{\varphi(I)}(\varphi^{-1}(\mathbf{b})) = \varphi_I(\pi_I(\mathbf{b}))$  where  $\varphi_I$  is automorphism of  $B_I$  induced by  $\Delta_I$ . The proposition follows since  $\Delta_I \varphi_I(\pi_I(\mathbf{b})) = \pi_I(\mathbf{b})\Delta_I$ .  $\square$

**Proposition 2.13.** *Assume that  $W$  is finite. We recall that for  $\mathbf{b} \in B$  we define  $\text{sup}(\mathbf{b}) = \min\{j \mid \mathbf{b}^{-1}\Delta^j \in B^+\}$  and  $\text{inf}(\mathbf{b}) = \max\{j \mid \Delta^{-j}\mathbf{b} \in B^+\}$ . We have  $\text{sup}(\pi_I(\mathbf{b})) \leq \text{sup}(\mathbf{b})$  and  $\text{inf}(\pi_I(\mathbf{b})) \geq \text{inf}(\mathbf{b})$ .*

*Proof.* The reversed of the word  $\mathbf{b}^{-1}$  is  $\text{inv}(\mathbf{b})$ , where  $\text{inv}$  is as in Proposition 2.6(4), and  $\Delta$  is equal to its reversed, thus the condition  $\mathbf{b}^{-1}\Delta^j \in B^+$  is equivalent to  $\Delta^j \text{inv}(\mathbf{b}) \in B^+$ , which is itself equivalent to  $\text{inv}(\mathbf{b})\Delta^j \in B^+$ . We have  $\pi_I(\text{inv}(\mathbf{b})\Delta^j) = \pi_I(\text{inv}(\mathbf{b}))\Delta_I^j = \text{inv}(\pi_I(\mathbf{b}))\Delta_I^j \in B_I^+$ , the first equality by Proposition 2.6(7) and the second one by Remark 2.6(4); arguing as at the beginning but in  $B_I$  the last is equivalent to  $\pi_I(\mathbf{b})^{-1}\Delta_I^j \in B_I^+$ , which concludes.  $\square$

The proof for  $\text{inf}$  is similar but simpler: we don't have to use reversing.  $\square$

**Lemma 2.14.** *Any  $\mathbf{b} \in B$  can be written  $\mathbf{p}\mathbf{i}\mathbf{w}$  where  $\mathbf{p}$  is pure,  $\mathbf{i} \in \mathbf{W}_I$  (the lift of  $W_I$ ) and  $\text{pr}(\mathbf{w})$  is  $I$ -reduced; in such a decomposition we have  $\pi_I(\mathbf{b}) = \pi_I(\mathbf{p})\mathbf{i}$ .*

*Proof.* It is clear that any element  $\mathbf{b} \in B$  can be written  $\mathbf{p}\mathbf{i}\mathbf{w}$  as above. By Proposition 2.6(7) we have  $\pi_I(\mathbf{b}) = \pi_I(\mathbf{p})\pi_I(\mathbf{i})\pi_I(\mathbf{w}) = \pi_I(\mathbf{p})\mathbf{i}\pi_I(\mathbf{w})$  and by Lemma 2.6(5)  $\pi_I(\mathbf{w}) = 1$ .  $\square$

The following lemma is [2, Proposition 2.3(3)].

**Lemma 2.15.** *If  $\mathbf{p}$  is a pure braid then  $\pi_I(\mathbf{p})$  is pure.*

*Proof.* We have  $\mathbf{p} = \pi_I(\mathbf{p})\mathbf{t}_I(\mathbf{p})$  so it is sufficient to show that  $\mathbf{t}_I(\mathbf{p})$  is pure. By Lemma 2.9(2) we have  $\text{pr}(\mathbf{t}_I(\mathbf{p})) = t_I(\text{pr}(\mathbf{p})) = 1$ , whence the result.  $\square$

The following lemma is [14, Lemma 3.2]

**Lemma 2.16.** *Let  $I, J$  be subsets of  $S$  and let  $\mathbf{b} \in B$  be such that  $\text{pr}(\mathbf{b})$  is  $I$ -reduced- $J$ ; then  $\pi_I(\mathbf{b}B_J) = \pi_I(\mathbf{b})B_{I_1}$  where  $I_1 = {}^{\text{pr}(\mathbf{b})}J \cap I$ .*

*Proof.* Applying Proposition 2.6(9) we get  $\pi_I(\mathbf{b}B_J) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_{J_1}(B_J)) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(B_{J_1}) = \pi_I(\mathbf{b})B_{I_1}$  where  $J_1 = I_1^{\text{pr}(\mathbf{b})}$ .  $\square$

The following is [2, Theorem 1.1].

**Proposition 2.17.** *Let  $I, J$  be subsets of  $S$  and let  $\mathbf{b} \in B$  be such that  ${}^{\mathbf{b}}B_J \subseteq B_I$ . Then there exists  $\mathbf{b}' \in B_I$  and  $K \subseteq I$  such that  ${}^{\mathbf{b}}B_J = {}^{\mathbf{b}'}B_K$ . In other terms a parabolic subgroup of  $B$  lying inside  $B_I$  is a parabolic subgroup of  $B_I$ .*

*Proof.* Let  $w = \text{pr}(\mathbf{b})$ . Applying  $\text{pr}$  we get  ${}^wW_J \subseteq W_I$ . Write  $w = iw'j$  where  $i \in W_I, j \in W_J$  and  $w'$  is  $I$ -reduced- $J$ . We still have  ${}^{w'}W_J \subseteq W_I$  and by Lemma 1.1 we have  ${}^{w'}J = K$  for some  $K \subseteq I$ . Lifting  $i, j, w'$  to  $\mathbf{i}, \mathbf{j}, \mathbf{w}' \in \mathbf{W}$  we can write  $\mathbf{b} = \mathbf{i}\mathbf{p}\mathbf{w}'\mathbf{j}$  where  $\mathbf{p}$  is pure. We still have  ${}^{\mathbf{p}\mathbf{w}'}B_J \subseteq B_I$ , and since  ${}^{w'}J = K$  we have  ${}^{\mathbf{w}'}B_J = B_K$ . We thus get  ${}^{\mathbf{p}}B_K \subseteq B_I$ . Thus  ${}^{\mathbf{p}}B_K = \pi_I({}^{\mathbf{p}}B_K) = \pi_I(\mathbf{p})B_K\pi_I(\mathbf{p})^{-1}$ , the last equality from Lemma 3 since  $\mathbf{p}$  is pure and  $\text{pr}(B_K) \subseteq W_I$ . This proves the statement with  $\mathbf{b}' = \mathbf{i}\pi_I(\mathbf{p})$ .  $\square$

**The map  $\mathbf{N}$ .** The following proposition is [10, Proposition 2.1]. We recall it here as it is unpublished.

**Proposition 2.18.**

- (1) *There exists a well defined map  $\mathbf{N} : B \rightarrow \mathbb{Z}[T]$  such that*

$$\mathbf{N}(\mathbf{s}_1^{\varepsilon_1} \cdots \mathbf{s}_k^{\varepsilon_k}) = \sum_{i=1}^{i=k} \varepsilon_i s_1^{\cdots s_{i-1}} s_i$$

*for  $\mathbf{s}_i \in \mathbf{S}$  and  $\varepsilon_i = \pm 1$ .*

- (2) *The map  $\mathbf{N} \rtimes \text{pr}$  is group homomorphism from  $B$  to  $\mathbb{Z}[T] \rtimes W$ , where the action of  $W$  on  $\mathbb{Z}[T]$  extends linearly the conjugation action of  $W$  on  $T$ .*

*Proof.* We prove that the map  $\mathbf{s} \mapsto (s, s)$  for  $\mathbf{s} \in \mathbf{S}$  extends to a group morphism  $f : B \rightarrow \mathbb{Z}[T] \rtimes W$ . Composing this morphism with the first projection will give a map  $\mathbf{N}$  which satisfies (1). It is sufficient to show that the braid relations (\*\*) are satisfied, that is for any  $\mathbf{s}, \mathbf{t} \in \mathbf{S}$ , we have the equality  $f(\mathbf{s})f(\mathbf{t}) \cdots = f(\mathbf{t})f(\mathbf{s}) \cdots$ , where there are  $m_{s,t}$  terms on both sides. In  $\mathbb{Z}[T] \rtimes W$  the first component of  $(s, s)(t, t) \cdots$  (with  $m_{s,t}$  factors) is the sum of all reflections of the dihedral group generated by  $s$  and  $t$  and the second component is the longest element of this dihedral group. So we get the same result by swapping  $s$  and  $t$ , whence the proposition.  $\square$

The map  $\mathbf{N}$  lifts  $N$  to the braid group: for  $\mathbf{b} \in B$  the set  $N(\text{pr}(\mathbf{b}))$  is the reduction modulo 2 of  $\mathbf{N}(\mathbf{b})$ . For  $\mathbf{w} \in \mathbf{W}$  we can identify  $\mathbf{N}(\mathbf{w})$  to  $N(w)$ : one gets  $\mathbf{N}(\mathbf{w})$  from  $N(w)$  by lifting  $\mathbb{Z}/2\mathbb{Z}$  to  $\{0, 1\} \subseteq \mathbb{Z}$ . If we interpret  $\mathbf{N}$  as a map  $B \rightarrow \mathbb{Z}[\Phi^+]$ , then  $\mathbf{N}(\mathbf{b})$  is the sum of the sequence  $\vec{N}(p^*(b))$  for any word  $b$  representing  $\mathbf{b}$ .

**Lemma 2.19.** *We say that  $\mathbf{w} \in \mathbf{W}$  is  $I$ -reduced if  $\text{pr}(\mathbf{w})$  is  $I$ -reduced.*

- *The pure braid group  $P$  is generated by  $P_I = P \cap B_I$  and the elements  $\mathbf{iws}^2\mathbf{w}^{-1}\mathbf{i}^{-1}$  for  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{i} \in \mathbf{W}_I$  and  $\mathbf{w} \in \mathbf{W}$  such that  $\mathbf{ws}$  is an  $I$ -reduced element.*
- *The group  $\text{pr}^{-1}(W_I)$  is generated by  $B_I$  and the elements  $\mathbf{ws}^2\mathbf{w}^{-1}$  with  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{ws} \in \mathbf{W}$  an  $I$ -reduced element.*
- *If  $\mathbf{s} \in \mathbf{S}$  and if  $\mathbf{ws} \in \mathbf{W}$  is an  $I$ -reduced element then  $\pi_I(\mathbf{ws}^2\mathbf{w}^{-1}) = 1$ .*

*Proof.* We first prove that  $P$  is generated by the  $\mathbf{wsw}^{-1}$  with  $\mathbf{w}, \mathbf{s}, \mathbf{ws} \in \mathbf{W}$ : if in the Reidemeister-Shreier method we take  $\mathbf{W}$  as representatives of the  $P$ -cosets in  $B$ , we get that  $P$  is generated by the  $\mathbf{wsv}^{-1}$  where  $\mathbf{w} \in \mathbf{W}$  and  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{v} \in \mathbf{W}$  is the representative of  $\mathbf{ws}$ ; if  $\mathbf{ws}$  is in  $\mathbf{W}$  we have  $\mathbf{v} = \mathbf{w}$ , otherwise  $\mathbf{w} = \mathbf{us}$  for some  $\mathbf{u} \in \mathbf{W}$  and  $\mathbf{v} = \mathbf{u}$  so that  $\mathbf{wsv}^{-1} = \mathbf{usu}^{-1}$ . We get then the first item of the lemma by writing  $\mathbf{w} = \mathbf{iv}$  where  $\mathbf{i} \in \mathbf{W}_I$  and  $\mathbf{v}$  is  $I$ -reduced.

We get the second item from the first using that an element of  $\text{pr}^{-1}(W_I)$  is the product of an element of  $B_I$  by a pure braid.

For the third item, by Proposition 2.6(7) we have  $\pi_I(\mathbf{ws}^2\mathbf{w}^{-1})\pi_I(\mathbf{ws}^{-1}) = \pi_I(\mathbf{ws})$ , and by Proposition 2.6(5) we obtain  $\pi_I(\mathbf{ws}^{-1}) = \pi_I(\mathbf{ws}) = 1$ . Thus,  $\pi_I(\mathbf{ws}^2\mathbf{w}^{-1}) = 1$ .  $\square$

**Lemma 2.20.** *For  $\mathbf{b} \in B$  and  $\mathbf{s} \in \mathbf{S}$ , we have  $\mathbf{N}(\mathbf{bs}^2\mathbf{b}^{-1}) = 2t$  where  $t = \text{pr}(\mathbf{bsb}^{-1})$ .*

*Proof.* Using that  $\mathbf{N} \times \text{pr}$  is a morphism, that is

$$(\mathbf{N} \times \text{pr})(\mathbf{bb}') = (\mathbf{N}(\mathbf{b}) + {}^{\text{pr}(\mathbf{b})}\mathbf{N}(\mathbf{b}'), \text{pr}(\mathbf{bb}')),$$

and setting  $b = \text{pr}(\mathbf{b})$ , we get

$$(2.21) \quad (\mathbf{N}(\mathbf{bs}^2\mathbf{b}^{-1}), 1) = (\mathbf{N}(\mathbf{b}), b)(\mathbf{N}(\mathbf{s}^2), 1)(\mathbf{N}(\mathbf{b}^{-1}), b^{-1})$$

From  $(0, 1) = (\mathbf{N}(\mathbf{b}), b)(\mathbf{N}(\mathbf{b}^{-1}), b^{-1})$  we have  ${}^b\mathbf{N}(\mathbf{b}^{-1}) = -\mathbf{N}(\mathbf{b})$ . Using this and  $\mathbf{N}(\mathbf{s}^2) = 2s$  in equation (2.21) we get

$$(\mathbf{N}(\mathbf{bs}^2\mathbf{b}^{-1}), 1) = (\mathbf{N}(\mathbf{b}) + 2 \cdot {}^b s + {}^b\mathbf{N}(\mathbf{b}^{-1}), 1) = (2 \cdot {}^b s, 1) = (2t, 1).$$

$\square$

**Proposition 2.22.** *The following diagram is commutative:*

$$\begin{array}{ccc} B & \xrightarrow{\mathbf{N} \times \text{pr}} & \mathbb{Z}[T] \rtimes W \\ \downarrow \pi_I & & \downarrow \text{proj}_{T_I} \times \pi_I \\ B_I & \xrightarrow{\mathbf{N} \times \text{pr}} & \mathbb{Z}[T_I] \rtimes W_I \end{array}$$

*Proof.* Writing an element of  $\mathbf{b} \in B$  as  $\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3$  where  $\mathbf{b}_1$  is pure,  $\mathbf{b}_2 \in B_I$  and  $\mathbf{b}_3 \in \mathbf{W}$  is  $I$ -reduced, and applying Proposition 2.6(7), we can reduce to the case of an element of one of the above 3 forms.

For an element of  $B_I$  the proposition is trivial since  $\pi_I$  is the identity on  $B_I$ .

For  $\mathbf{w} \in \mathbf{W}$  an  $I$ -reduced element, we know the commutativity with  $\text{pr}$  by Lemma 2.9(3), and we have  $\pi_I(\mathbf{w}) = 1$ . All that remains is to prove that  $\mathbf{N}(\mathbf{w})$  has trivial coefficients on  $T_I$ . Since  $\mathbf{w} \in \mathbf{W}$ , the lift of  $N(w)$  is  $\mathbf{N}(\mathbf{w})$  and we conclude since  $N(w) \cap W_I = \emptyset$  by Proposition 2.10.

For a pure braid, we can, using Proposition 2.6(7) and Lemma 2.19, reduce to an element of the form  $\mathbf{ws}^2\mathbf{w}^{-1}$  where  $\mathbf{ws}$  is  $I$ -reduced. By Lemma 2.20 we have

$\mathbf{N}(\mathbf{ws}^2\mathbf{w}^{-1}) = 2t$  where  $t = \text{pr}(\mathbf{wsw}^{-1})$ . By Lemma 2.19  $\pi_I(\mathbf{ws}^2\mathbf{w}^{-1}) = 1$  and by Lemma 1.1 we have  $t \notin W_I$ . Thus the commutation holds.  $\square$

**Closed subsets.** For  $w \in W$  the set  $N(w)$  is in bijection with the set  $\Phi_w = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \notin \Phi^+\}$  since  $N(w) = \{s_\alpha \mid \alpha \in \Phi_w\}$ . We now recall results of [13]. We say that a set  $\Gamma \subseteq \Phi^+$  is closed if for any  $\alpha, \beta \in \Gamma$  and  $a, b \in \mathbb{R}_{>0}$ , if  $a\alpha + b\beta \in \Phi$  then  $a\alpha + b\beta \in \Gamma$ . We call closure of  $\Gamma$ , denoted by  $\overline{\Gamma}$ , the smallest closed set containing  $\Gamma$ . We say that  $\Gamma$  is biclosed if it is closed and its complement  $\mathcal{C}\Gamma$  in  $\Phi^+$  is closed. Thanks to the bijection between reflections and positive roots we can use the same words for subsets  $A \subseteq T$  (closed, notation  $\overline{A}$  for the closure, biclosed and notation  $\mathcal{C}A$  for the complement in  $T$ ). The notion of closed for a set  $A \subseteq T$  can be given in purely group-theoretic terms: a set  $A$  is closed if and only if given any two reflections  $s, s' \in W$ , the intersection of  $A$  with the dihedral group  $\langle s, s' \rangle$  is closed. In a dihedral group  $W = \langle s, s' \rangle$ , the group theoretic definition of closed is as follows: we give a total order on  $T$  by  $s < ss's < ss'ss's < \dots < s'ss' < s'$  (exchanging  $s$  and  $s'$  gives the opposite order); then a set  $A$  is closed if and only if whenever it contains  $t, t' \in T$  with  $t < t'$  it contains any  $t''$  such that  $t < t'' < t'$  (see [12, (2.2)] for the equivalence of the group-theoretic condition with closed assuming the existence of a reflection order and see [12, Proposition 2.3] for the existence of a reflection order).

**Lemma 2.23.** *For  $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$  it is equivalent that  $\mathbf{w}$  left-divides  $\mathbf{w}'$  in the monoid  $B^+$  or that  $N(w) \subseteq N(w')$ .*

*Proof.* The assumption that  $\mathbf{w}$  left-divides  $\mathbf{w}'$  is equivalent in  $W$  to the existence of  $v \in W$  such that  $w' = wv$  and  $\ell_S(w) + \ell_S(v) = \ell_S(w')$ . By the formula  $N(wv) = N(w) \cup {}^wN(v)$  it is clear that left-divisibility implies  $N(w) \subseteq N(w')$ . We show the converse by induction on  $\ell_S(w')$ . Let  $\mathbf{s} \in \mathbf{S}$  be a left-divisor of  $\mathbf{w}$ , equivalently  $\ell_S(sw) = \ell_S(w) - 1$  or equivalently  $s \in N(w)$  (see [11, Lemma 2.1.6(ii)]). It follows that  $\mathbf{s}$  is also a left-divisor of  $\mathbf{w}'$ . Let  $w = sw_1$  and  $w' = sw'_1$ . From the formulae  $N(w) = \{s\} \cup {}^sN(w_1)$  and  $N(w') = \{s\} \cup {}^sN(w'_1)$  it follows that  $N(w_1) \subseteq N(w'_1)$  and we conclude by induction.  $\square$

More generally, using that divisibility in  $\mathbf{W}$  projects to the weak order in  $W$ , we have

**Proposition 2.24** (Dyer).

- A set  $A \subseteq T$  is biclosed finite if and only if it is an  $N(w)$  for some  $w \in W$ .
- Let  $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$  be two simple braids which have a common right-multiple in the monoid  $B^+$ . Then their right-lcm is an element of  $\mathbf{W}$  such that

$$N(\text{pr}(\text{right-lcm}(\mathbf{w}, \mathbf{w}'))) = \overline{N(w) \cup N(w')}.$$

- Let  $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$  be two simple braids. Then

$$\mathcal{C}N(\text{pr}(\text{left-gcd}(\mathbf{w}, \mathbf{w}'))) = \overline{\mathcal{C}(N(w) \cap N(w'))}.$$

*Proof.* See [13, 4.1, 1.5].  $\square$

**Proposition 2.25.** *Let  $\mathbf{w}, \mathbf{w}' \in \mathbf{W}$  be two simple braids. Then for  $I \subseteq S$  we have  $\pi_I(\text{right-lcm}(\mathbf{w}, \mathbf{w}')) = \text{right-lcm}(\pi_I(\mathbf{w}), \pi_I(\mathbf{w}'))$  and  $\pi_I(\text{left-gcd}(\mathbf{w}, \mathbf{w}')) = \text{left-gcd}(\pi_I(\mathbf{w}), \pi_I(\mathbf{w}'))$ .*



*Proof.* By Proposition 2.10,  $\pi_I(\mathbf{w})$  is characterised by  $N(\text{pr}(\pi_I(\mathbf{w}))) = N(w) \cap T_I$ , that is  $(\Phi_I)_{\pi_I(w)} = \Phi_w \cap \Phi_I$ .

Thus, for the right-lcm, from Proposition 2.24 we have to prove that taking the closure of a set of positive roots commutes with intersecting with  $\Phi_I^+$ , which is clear since if for  $\alpha, \beta \in \Phi^+$  and  $a, b \in \mathbb{R}_{>0}$ , the root  $a\alpha + b\beta$  is in  $\mathbb{R}_{\geq 0}\Pi_I$  then  $\alpha$  and  $\beta$  also. This can be written as a formula  $\overline{A \cap T_I} = \overline{A} \cap T_I$ .

For the left-gcd, if we set  $A = N(w) \cap N(w')$ , we have  $\overline{\mathbb{C}A} \cap T_I = \overline{\mathbb{C}A \cap T_I} = \overline{\mathbb{C}_I(A \cap T_I)}$  where the first equality is the previous observation and in the second equality  $\mathbb{C}_I$  denotes the complement in  $T_I$ ; this proves the proposition.  $\square$

By Proposition 2.6(3), for any positive braids  $\mathbf{b}$  and  $\mathbf{b}'$ ,  $\text{right-lcm}(\pi_I(\mathbf{b}), \pi_I(\mathbf{b}'))$  is a left-divisor of  $\pi_I(\text{right-lcm}(\mathbf{b}, \mathbf{b}'))$  and  $\pi_I(\text{left-gcd}(\mathbf{b}, \mathbf{b}'))$  is a left-divisor of  $\text{left-gcd}(\pi_I(\mathbf{b}), \pi_I(\mathbf{b}'))$ ; but equality does not generally hold for non-simple braids.

For example, in the braid group of  $\mathfrak{S}_3$  with  $\mathbf{S} = \{\mathbf{s}, \mathbf{t}\}$ , taking  $I = \{s\}$ ,  $\mathbf{b} = \mathbf{stt}$  and  $\mathbf{b}' = \mathbf{t}$  we have  $\pi_I(\mathbf{b}) = \mathbf{s}$ ,  $\pi_I(\mathbf{b}') = 1$ , hence  $\text{right-lcm}(\pi_I(\mathbf{b}), \pi_I(\mathbf{b}')) = \mathbf{s}$  but  $\pi_I(\text{right-lcm}(\mathbf{b}, \mathbf{b}')) = \pi_I(\mathbf{ststs}) = \mathbf{ss}$ .

A counterexample for the gcd is obtained with  $\mathbf{b} = \mathbf{tts}$ ,  $\mathbf{b}' = \mathbf{s}$  and  $I = \{s\}$ . We have  $\pi_I(\mathbf{b}) = \pi_I(\mathbf{b}') = \mathbf{s}$  hence  $\text{left-gcd}(\pi_I(\mathbf{b}), \pi_I(\mathbf{b}')) = \mathbf{s}$  but  $\text{left-gcd}(\mathbf{b}, \mathbf{b}') = 1$  whose retraction is 1.

### 3. RETRACTION ON THE OTHER SIDE

Definition 1.2 of  $\pi_I$  is based on right-cosets  $W_I w \in W_I \backslash W$  and the “left  $I$ -tail”  $t_I(w)$ . It is well-behaved regarding the left-divisibility of words: if  $b \in (\mathbf{S}^{\pm 1})^*$  is a prefix of  $b'$  then  $\hat{\pi}_I(b)$  is a prefix of  $\hat{\pi}_I(b')$ . Using left-cosets  $wW_I \in W/W_I$  and the counterpart  $t_I^r(w)$  of  $t_I(w)$ , we obtain another retraction that we denote by  $\pi_I^r$ . A natural question is the connection between the two retractions. The next lemma gives an answer to this question.

**Lemma 3.1.** *We have  $\pi_I^r(\mathbf{b}) = \text{rev}(\pi_I(\text{rev}(\mathbf{b}))) = (\pi_I(\mathbf{b}^{-1}))^{-1}$ , where  $\text{rev}$  is the unique antiautomorphism of  $B$  that fixes  $\mathbf{S}$ .*

*Proof.* The first equality is clear and the second equality follows from Proposition 2.6(4) and from the equality  $\mathbf{b}^{-1} = \text{inv}(\text{rev}(\mathbf{b}))$ .  $\square$

For  $\mathbf{b} \in B$  we define  $\mathbf{t}_I^r(\mathbf{b}) = \mathbf{b} \pi_I^r(\mathbf{b})^{-1}$ ; it satisfies the counterpart result of Lemma 2.9.

The right-retraction counterpart of Proposition 2.6(9) is

**Proposition 3.2.** *Let  $I, J \subseteq S$  and  $\mathbf{b} \in B$  be such that  $\text{pr}(\mathbf{b})$  is  $I$ -reduced- $J$ ; then for  $\mathbf{i} \in B_I$  and for  $I_1 = I \cap \text{pr}(\mathbf{b})J$ , we have  $\pi_J^r(\mathbf{ib}) = \varphi_{\text{pr}(\mathbf{b})}^{-1}(\pi_{I_1}^r(\mathbf{i})) \pi_J^r(\mathbf{b})$ .*

**Proposition 3.3.** *For  $I, J \subseteq S$  and  $\mathbf{b} \in B$  be such that  $\text{pr}(\mathbf{b})$  is an  $I$ -ribbon- $J$ , we have  $\pi_I(\mathbf{b}) = \varphi_{\text{pr}(\mathbf{b})}(\pi_J(\mathbf{b}^{-1}))^{-1}$ , in particular*

$$\pi_I(\mathbf{b}) = 1 \iff \pi_J(\mathbf{b}^{-1}) = 1.$$

*Proof.* We have  $1 = \pi_I(1) = \pi_I(\mathbf{b}\mathbf{b}^{-1}) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_J(\mathbf{b}^{-1}))$ , where the last equality is by Proposition 2.6(8), whence the proposition.  $\square$

**Proposition 3.4.** *Let  $I, J \subseteq S$  and  $\mathbf{b}$  in  $B$  be such that  $\text{pr}(\mathbf{b})$  is reduced- $J$ . Let  $J_1 = I^{\text{pr}(\mathbf{b})} \cap J$ ; then  $\text{pr}(\mathbf{t}_I(\mathbf{b}))$  is a ribbon- $J_1$  and for any  $\mathbf{j} \in B_J$  one has*

$$\pi_I(\mathbf{bj}) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{t}_I(\mathbf{b}))}(\pi_{J_1}(\mathbf{j})) = \pi_I(\mathbf{b}) \pi_{I_1}(\mathbf{t}_I(\mathbf{b})\mathbf{j})$$

where  $I_1 = \text{pr}(\mathbf{t}_I(\mathbf{b}))J_1 = I^{\text{pr}(\boldsymbol{\pi}_I(\mathbf{b}))} \cap \text{pr}(\mathbf{t}_I(\mathbf{b}))J$ . In particular, when  $\boldsymbol{\pi}_I(\mathbf{b}) = 1$  we have  $\boldsymbol{\pi}_I(\mathbf{b}\mathbf{j}) = \boldsymbol{\pi}_{I_1}(\mathbf{b}\mathbf{j})$ .

*Proof.* For any  $s \in J_1$  we have  $\text{pr}(\mathbf{b})s \in I$ . Writing  $\text{pr}(\mathbf{b}) = \text{pr}(\boldsymbol{\pi}_I(\mathbf{b}))\text{pr}(\mathbf{t}_I(\mathbf{b}))$  we deduce that  $\text{pr}(\mathbf{t}_I(\mathbf{b}))s$  lies in  $W_I$ , hence in  $I$  by Lemma 1.1. Since  $\mathbf{t}_I(\mathbf{b})$  is  $I$ -reduced- $J$  we deduce that it is an  $I_1$ -ribbon- $J_1$ . Writing  $\boldsymbol{\pi}_I(\mathbf{b}\mathbf{j}) = \boldsymbol{\pi}_I(\mathbf{b})\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b})\mathbf{j})$ , we can apply Proposition 2.6(9) with  $\mathbf{t}_I(\mathbf{b})$  for  $\mathbf{b}$ , whence  $\boldsymbol{\pi}_I(\mathbf{b}\mathbf{j}) = \boldsymbol{\pi}_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{t}_I(\mathbf{b}))}(\boldsymbol{\pi}_{J_1}(\mathbf{j}))$ . The equality  $\varphi_{\text{pr}(\mathbf{t}_I(\mathbf{b}))}(\boldsymbol{\pi}_{J_1}(\mathbf{j})) = \boldsymbol{\pi}_{I_1}(\mathbf{t}_I(\mathbf{b})\mathbf{j})$  comes from Proposition 2.6(9) with  $\mathbf{t}_I(\mathbf{b})$  for  $\mathbf{b}$  and  $I_1$  for  $I$ , taking in account the fact that  $\boldsymbol{\pi}_{I_1}(\mathbf{t}_I(\mathbf{b})) = 1$  by Corollary 1.9.  $\square$

The right-counterpart of Proposition 3.4 is

**Proposition 3.5.** *Let  $I, J \subseteq S$  and  $\mathbf{b}$  in  $B$  be such that  $\text{pr}(\mathbf{b})$  is  $I$ -reduced. Let  $I_1 = I \cap \text{pr}(\mathbf{b})J$ ; then for any  $\mathbf{i} \in B_I$  we have  $\boldsymbol{\pi}_J^r(\mathbf{i}\mathbf{b}) = \boldsymbol{\pi}_{J_1}^r(\mathbf{i}\mathbf{t}_I^r(\mathbf{b}))\boldsymbol{\pi}_J^r(\mathbf{b})$ , where  $J_1 = I_1^{\text{pr}(\mathbf{t}_I^r(\mathbf{b}))} = I^{\text{pr}(\mathbf{t}_I^r(\mathbf{b}))} \cap \text{pr}(\boldsymbol{\pi}_I^r(\mathbf{b}))J$ . In particular, when  $\boldsymbol{\pi}_J^r(\mathbf{b}) = 1$  we have  $\boldsymbol{\pi}_J^r(\mathbf{i}\mathbf{b}) = \boldsymbol{\pi}_{J_1}^r(\mathbf{i}\mathbf{b})$ .*

**Proposition 3.6.** *Let  $I, J \subseteq S$  and  $\mathbf{b} \in B$ ; set  $\mathbf{b}_1 = \mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b}))$ . Then*

- (1)  $\boldsymbol{\pi}_I(\mathbf{b}_1) = \boldsymbol{\pi}_J^r(\mathbf{b}_1) = 1$ . In particular  $\text{pr}(\mathbf{b}_1)$  is  $I$ -reduced- $J$ .
- (2) Let  $I_1 = I \cap \text{pr}(\mathbf{b}_1)J$  and  $J_1 = I^{\text{pr}(\mathbf{b}_1)} \cap J$ ; then

$$\boldsymbol{\pi}_J^r(\mathbf{b}) = \varphi_{\text{pr}(\mathbf{b}_1)}^{-1}(\boldsymbol{\pi}_{I_1}^r(\boldsymbol{\pi}_I(\mathbf{b})))\boldsymbol{\pi}_{J_1}^r(\mathbf{t}_I(\mathbf{b})).$$

*Proof.* First,  $\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b})) = 1$  and  $\text{pr}(\mathbf{t}_I(\mathbf{b}))$  is  $I$ -reduced by Lemma 2.9. For a similar reason,  $\text{pr}(\mathbf{b}_1)$  is reduced- $J$ , hence it is  $I$ -reduced- $J$  since it has to left-divide (be smaller for the weak order than)  $\text{pr}(\mathbf{t}_I(\mathbf{b}))$  in  $W$ . By definition of  $\mathbf{b}_1$  we have  $\boldsymbol{\pi}_J^r(\mathbf{b}_1) = 1$ . Since  $J_1 \subseteq J$  one has  $\boldsymbol{\pi}_{J_1}^r(\mathbf{b}_1) = \boldsymbol{\pi}_{J_1}^r(\boldsymbol{\pi}_J^r(\mathbf{b}_1)) = 1$ . Applying Lemma 3.1 we get  $\boldsymbol{\pi}_{J_1}(\mathbf{b}_1^{-1}) = 1$  and by Propositions 3.3 and 3.4, we get that  $\boldsymbol{\pi}_{I_1}(\mathbf{b}_1) = 1$ . Since  $\text{pr}(\mathbf{b}_1)$  is  $I$ -reduced- $J$ , applying again Lemma 2.9, we get  $\text{pr}(\mathbf{t}_I(\mathbf{b}_1)) = \text{pr}(\mathbf{b}_1)$  and  $\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b}_1)) = 1$ . By Proposition 3.4 we deduce that  $\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b}_1)\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b}))) = \boldsymbol{\pi}_{I_1}(\mathbf{t}_I(\mathbf{b}_1)\boldsymbol{\pi}_{J_1}^r(\mathbf{t}_I(\mathbf{b})))$ , and therefore belongs to  $B_{I_1}$ . But now  $1 = \boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b})) = \boldsymbol{\pi}_I(\boldsymbol{\pi}_I(\mathbf{b}_1)\mathbf{t}_I(\mathbf{b}_1)\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b}))) = \boldsymbol{\pi}_I(\mathbf{b}_1)\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b}_1)\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b})))$ . So  $\boldsymbol{\pi}_I(\mathbf{b}_1)$  is equal to  $(\boldsymbol{\pi}_I(\mathbf{t}_I(\mathbf{b}_1)\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b}))))^{-1}$  and belongs to  $B_{I_1}$  too. This imposes  $\boldsymbol{\pi}_I(\mathbf{b}_1) = \boldsymbol{\pi}_{I_1}(\boldsymbol{\pi}_I(\mathbf{b}_1)) = \boldsymbol{\pi}_{I_1}(\mathbf{b}_1) = 1$ , the second equality by Corollary 1.9. This concludes the proof of (1).

To prove (2) we decompose  $\mathbf{b}$  as  $\boldsymbol{\pi}_I(\mathbf{b})\mathbf{b}_1\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b}))$ . Applying  $\boldsymbol{\pi}_J^r$ , we get  $\boldsymbol{\pi}_J^r(\mathbf{b}) = \boldsymbol{\pi}_J^r(\boldsymbol{\pi}_I(\mathbf{b})\mathbf{b}_1)\boldsymbol{\pi}_J^r(\mathbf{t}_I(\mathbf{b}))$  (since a term in  $B_J$  factors out on the right in  $\boldsymbol{\pi}_J^r$ ) and then we apply Proposition 3.2 with  $\boldsymbol{\pi}_I(\mathbf{b})$  for  $\mathbf{b}'$  and  $\mathbf{b}_1$  for  $\mathbf{b}$ .  $\square$

**Corollary 3.7.** *Let  $I, J \subseteq S$  and  $\mathbf{b} \in B$  be such that  $\text{pr}(\mathbf{b})$  is a  $I$ -ribbon- $J$ ; then  $\boldsymbol{\pi}_I(\mathbf{b}) = \varphi_{\text{pr}(\mathbf{b})}(\boldsymbol{\pi}_J^r(\mathbf{b}))$ .*

*Proof.* By Proposition 3.3, we have  $\boldsymbol{\pi}_I(\mathbf{b}) = \varphi_{\text{pr}(\mathbf{b})}(\boldsymbol{\pi}_J(\mathbf{b}^{-1}))^{-1} = \varphi_{\text{pr}(\mathbf{b})}(\boldsymbol{\pi}_J^r(\mathbf{b})^{-1})^{-1}$ , the last equality by Proposition 3.1.  $\square$

In general  $\mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b})) \neq \mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b}))$ . For instance in the braid group of type  $A_2$ , with  $\mathbf{S} = \{\mathbf{s}, \mathbf{t}\}$ , let  $I = \{\mathbf{s}\} = J$  and  $\mathbf{b} = \mathbf{st}^2\mathbf{s}$ . We have  $\mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b})) = \mathbf{s}^{-1}\mathbf{t}^2\mathbf{s}$  and  $\mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b})) = \mathbf{st}^2\mathbf{s}^{-1}$  which differ since  $\mathbf{s}^2\mathbf{t}^2 \neq \mathbf{t}^2\mathbf{s}^2$ . In particular the double-coset  $B_I\mathbf{b}B_J$  generally does not contain a unique element  $\mathbf{b}_0$  such that  $\boldsymbol{\pi}_I(\mathbf{b}_0) = \boldsymbol{\pi}_J(\mathbf{b}_0) = 1$ .

**Proposition 3.8.** *Let  $I, J \subseteq S$ . Let  $\mathbf{b}_0, \mathbf{b}_1$  be in  $B$  such that  $\pi_I(\mathbf{b}_0) = \pi_J^r(\mathbf{b}_0) = \pi_I(\mathbf{b}_1) = \pi_J^r(\mathbf{b}_1) = 1$ . Let  $I_1 = I \cap \text{pr}(\mathbf{b}_0)J$  and  $J_1 = I^{\text{pr}(\mathbf{b}_0)} \cap J$ ; then the following are equivalent:*

- (1)  $B_I \mathbf{b}_0 B_J = B_I \mathbf{b}_1 B_J$ .
- (2)  $B_{I_1} \mathbf{b}_0 B_{J_1} = B_{I_1} \mathbf{b}_1 B_{J_1}$ .

Furthermore for any  $\mathbf{i} \in B_I$  and  $\mathbf{j} \in B_J$  such that  $\mathbf{i}\mathbf{b}_0 = \mathbf{b}_1\mathbf{j}$  we have  $\mathbf{i} \in B_{I_1}$ ,  $\mathbf{j} \in B_{J_1}$  and  $\mathbf{i} = \varphi_{\text{pr}(\mathbf{b}_0)}(\mathbf{j})$ .

*Proof.* It is clear that (2) $\Rightarrow$ (1). Assume  $B_I \mathbf{b}_0 B_J = B_I \mathbf{b}_1 B_J$ . Let  $\mathbf{i}$  in  $B_I$  and  $\mathbf{j}$  in  $B_J$  be such that  $\mathbf{i}\mathbf{b}_0 = \mathbf{b}_1\mathbf{j}$ . We have  $\mathbf{j} = \pi_J^r(\mathbf{b}_1\mathbf{j}) = \pi_J^r(\mathbf{i}\mathbf{b}_0) = \varphi_{\text{pr}(\mathbf{b}_0)}^{-1}(\pi_{I_1}^r(\mathbf{i}))$ , the last equality by the counterpart of Proposition 2.6(9). In particular,  $\mathbf{j}$  lies in  $B_{J_1}$ . Symmetrically  $\mathbf{i} = \pi_I(\mathbf{i}\mathbf{b}_0) = \pi_I(\mathbf{b}_1\mathbf{j}) = \varphi_{\text{pr}(\mathbf{b}_1)}(\pi_{I^{\text{pr}(\mathbf{b}_1)} \cap J}(\mathbf{j})) \in B_{I \cap \text{pr}(\mathbf{b}_1)J}$ , the last equality by Proposition 2.6(9). Since  $\text{pr}(\mathbf{b}_0)$  and  $\text{pr}(\mathbf{b}_1)$  are  $I$ -reduced- $J$  elements in the same double coset of  $W_I \backslash W / W_J$  they are equal. Thus  $I \cap \text{pr}(\mathbf{b}_1)J = I_1$  and  $\mathbf{i}$  is in  $B_{I_1}$ .  $\square$

**Corollary 3.9.** *Let  $I, J \subseteq S$  and  $\mathbf{b}_0 \in B$  be such that  $\pi_I(\mathbf{b}_0) = \pi_J^r(\mathbf{b}_0) = 1$ . Then the following are equivalent*

- (1)  $\text{pr}(\mathbf{b}_0 \mathbf{J} \mathbf{b}_0^{-1} \cap \mathbf{I}) = \text{pr}(\mathbf{b}_0)J \cap I$ .
- (2) for every  $\mathbf{b}$  in  $B_I \mathbf{b}_0 B_J$ , if  $\pi_I(\mathbf{b}) = \pi_J^r(\mathbf{b}) = 1$  then  $\mathbf{b} = \mathbf{b}_0$ .
- (3) for every  $\mathbf{b}$  in  $B_I \mathbf{b}_0 B_J$ , one has  $\mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b})) = \mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b})) = \mathbf{b}_0$ .

*Proof.* Set  $w = \text{pr}(\mathbf{b}_0)$ ,  $J_1 = I^w \cap J$  and  $I_1 = {}^w J \cap I$ . Assume (1) holds. Let  $\mathbf{b}$  in  $B$  be such that  $\pi_I(\mathbf{b}) = \pi_J^r(\mathbf{b}) = 1$  with  $B_I \mathbf{b}_0 B_J = B_I \mathbf{b} B_J$ . By proposition 3.8(2) we get  $\mathbf{i}\mathbf{b}_0 = \mathbf{b}\mathbf{j}$  for some  $\mathbf{i} \in B_{I_1}$  and  $\mathbf{j} \in B_{J_1}$  such that  $\mathbf{i} = \varphi_w(\mathbf{j})$ . But by (1),  $\mathbf{b}_0$  conjugates  $\mathbf{J}_1$  to  $\mathbf{I}_1$  thus the conjugation by  $\mathbf{b}_0$  in  $B$  induces the one-to-one map  $\varphi_w$  from  $B_{J_1}$  to  $B_{I_1}$ . So  $\mathbf{b}_0^{-1}\mathbf{i}\mathbf{b}_0 = \mathbf{j}$  thus  $\mathbf{b}_0 = \mathbf{b}$ . Thus (2) holds. Now, (2) implies (3). Indeed, by Proposition 3.6(1) the element  $\mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b}))$  verifies the assumption for  $\mathbf{b}$  in (2), and by symmetry the same holds for  $\mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b}))$ . Finally assume (3). Clearly  $\text{pr}(\mathbf{b}_0 \mathbf{J} \mathbf{b}_0^{-1} \cap \mathbf{I}) \subseteq wJw^{-1} \cap I$  always holds. Let  $s$  in  $I_1$  and  $s'$  in  $J_1$  be such that  $ws'w^{-1} = s$ . Set  $\mathbf{b}_1 = \mathbf{s}^{-1}\mathbf{b}_0\mathbf{s}'$ . We have  $\pi_I(\mathbf{b}_1) = \mathbf{s}^{-1}\pi_I(\mathbf{b}_0\mathbf{s}')$  and by Proposition 1.11 we have  $\pi_I(\mathbf{b}_0\mathbf{s}') = \mathbf{s}$  since  $t_I(\text{pr}(\mathbf{b}_0)) = w$ . Thus  $\pi_I(\mathbf{b}_1) = 1$ . Similarly  $\pi_J^r(\mathbf{b}_1) = 1$  and therefore  $\mathbf{b}_1 = \mathbf{t}_I(\mathbf{t}_J^r(\mathbf{b}_1)) = \mathbf{b}_0$ , the last equality by (3). Hence,  $\mathbf{b}_0 = \mathbf{s}^{-1}\mathbf{b}_0\mathbf{s}'$ . and  $s$  belongs to  $\text{pr}(\mathbf{b}_0 \mathbf{J} \mathbf{b}_0^{-1} \cap \mathbf{I})$ .  $\square$

**Remark 3.10.** *Since one has always  $\text{pr}(\mathbf{b}_0 \mathbf{J} \mathbf{b}_0^{-1} \cap \mathbf{I}) \subseteq \text{pr}(\mathbf{b}_0)J \cap I$ , the three items of Corollary 3.9 are true in the particular case  $\text{pr}(\mathbf{b}_0)J \cap I = \emptyset$ .*

**Lemma 3.11.** *Let  $I, J$  be subsets of  $S$  and let  $\mathbf{w} \in \mathbf{W}$  be the lift of an  $I$ -ribbon- $J$ ; Then  ${}^{\mathbf{w}}B_J = B_I$ .*

*Proof.* For  $j \in J$  we have  $wj = iw$  for some  $i \in I$ . Since  $w$  is  $I$ -reduced- $J$ , the lift to  $\mathbf{W}$  of  $wj$  is  $\mathbf{w}\mathbf{j}$  and the lift of  $iw$  is  $\mathbf{i}\mathbf{w}$ , thus  $\mathbf{w}\mathbf{j} = \mathbf{i}\mathbf{w}$  and  ${}^{\mathbf{w}}\mathbf{j}$  is in  $\mathbf{I}$ .  $\square$

The following proposition shows various reductions of the problem of the intersection of two parabolic subgroups  ${}^{\mathbf{b}}B_J$  and  $B_I$ . (1) shows that we can assume  $\pi_I(\mathbf{b}) = \pi_J^r(\mathbf{b}) = 1$ , (2) shows that we can also assume that  $\text{pr}(\mathbf{b})$  is an  $I$ -ribbon- $J$ , (3) shows that we can also assume that  $\mathbf{b}$  is pure and  $I = J$  (see Proposition 0.4) and (4) shows that if  $I$  is finite, then we can assume that  $B_J$  is a minimal parabolic subgroup containing  $B_J \cap {}^{\mathbf{b}}B_J$ .

**Proposition 3.12.** *Let  $I, J$  be subsets of  $S$  and  $\mathbf{b} \in B$ .*

- (1) Let  $\mathbf{b}_0 = \mathbf{t}_J^r(\mathbf{t}_I(\mathbf{b}))$ . Then  $\pi_I(\mathbf{b}_0) = \pi_J^r(\mathbf{b}_0) = 1$ , and  ${}^{\mathbf{b}}B_J \cap B_I = \pi_I(\mathbf{b})({}^{\mathbf{b}_0}B_J \cap B_I)$
- (2) Assume  $\pi_I(\mathbf{b}) = \pi_J^r(\mathbf{b}) = 1$ . Then  ${}^{\mathbf{b}}B_J \cap B_I = {}^{\mathbf{b}}B_{J_1} \cap B_{I_1}$ , where  $I_1 = I \cap \text{pr}(\mathbf{b})J$  and  $J_1 = I^{\text{pr}(\mathbf{b})} \cap J$  (so that  $\text{pr}(\mathbf{b})$  is an  $I_1$ -ribbon- $J_1$ ).
- (3) Assume that  $\pi_I(\mathbf{b}) = 1$  and that  $\text{pr}(\mathbf{b})$  is an  $I$ -ribbon- $J$ . Then if  $\mathbf{w} \in \mathbf{W}$  lifts  $\text{pr}(\mathbf{b})$  and  $\mathbf{p}$  is the (pure) element such that  $\mathbf{b} = \mathbf{p}\mathbf{w}$ , we have  $\pi_I(\mathbf{p}) = 1$  and  $B_I \cap {}^{\mathbf{b}}B_J = B_I \cap {}^{\mathbf{p}}B_I = C_{B_I}(\mathbf{p})$ .
- (4) For  $I$  and  $\mathbf{p}$  as in item (3); assume  $I$  finite, then the intersection  $B_I \cap {}^{\mathbf{p}}B_I$  is conjugate in  $B_I$  to  $B_J \cap {}^{\mathbf{p}'}B_J$  for some  $J \subseteq I$  and  $\mathbf{p}' \in P$  such that  $\pi_J(\mathbf{p}') = 1$  and such that  $B_J$  is a minimal parabolic subgroup containing  $B_J \cap {}^{\mathbf{p}'}B_J$ .

*Proof.* (1) is clear from Proposition 3.6(1).

Let us prove (2). Let  $\mathbf{j} \in B_J \cap B_I^{\mathbf{b}}$  and  $\mathbf{i} \in B_I$  be such that  $\mathbf{b}\mathbf{j} = \mathbf{i}\mathbf{b}$ . Since  $\text{pr}(\mathbf{b})$  is  $I$ -reduced- $J$  (see Proposition 3.6(1)) we can apply Proposition 2.6(9), which gives  $\pi_I(\mathbf{b}\mathbf{j}) = \pi_I(\mathbf{b})\varphi_{\text{pr}(\mathbf{b})}(\pi_{J_1}(\mathbf{j})) = \varphi_{\text{pr}(\mathbf{b})}(\pi_{J_1}(\mathbf{j}))$ . On the other hand,  $\pi_I(\mathbf{i}\mathbf{b}) = \mathbf{i}\pi_I(\mathbf{b}) = \mathbf{i}$ . So  $\mathbf{i} \in B_{I_1}$ .

Similarly we may apply Proposition 3.2 to get  $\mathbf{j} = \pi_J^r(\mathbf{b}\mathbf{j}) = \pi_J^r(\mathbf{i}\mathbf{b}) = \varphi_{\text{pr}(\mathbf{b})}^{-1}(\pi_{I_1}^r(\mathbf{i}))$ , which proves that  $\mathbf{j} \in B_{J_1}$ .

We prove now (3). We first prove that  $\pi_I(\mathbf{p}) = 1$ . By Proposition 2.6(7) we have  $1 = \pi_I(\mathbf{b}) = \pi_I(\mathbf{p})\pi_I(\mathbf{w})$ . By Proposition 2.6(5) we have  $\pi_I(\mathbf{w}) = 1$ , hence  $\pi_I(\mathbf{p}) = 1$ .

Now we have  $B_I \cap {}^{\mathbf{b}}B_J = B_I \cap {}^{\mathbf{p}\mathbf{w}}B_J = B_I \cap {}^{\mathbf{p}}B_I$ , the last equality by Lemma 3.11. If  $\mathbf{i}$  is in  $B_I \cap {}^{\mathbf{p}}B_I$  we have  $\mathbf{p}\mathbf{i} = \mathbf{i}'\mathbf{p}$  for some  $\mathbf{i}' \in B_I$ . Hence  $\pi_I(\mathbf{p}\mathbf{i}) = \pi_I(\mathbf{p})\pi_I(\mathbf{i}) = \pi_I(\mathbf{i}) = \mathbf{i}$  and  $\pi_I(\mathbf{i}'\mathbf{p}) = \mathbf{i}'\pi_I(\mathbf{p}) = \mathbf{i}'$ , so that  $\mathbf{i} = \mathbf{i}'$  is in  $C_{B_I}(\mathbf{p})$ .

We prove (4). Let  ${}^{\mathbf{i}}B_J$  with  $J \subseteq I$  and  $\mathbf{i} \in B_I$  be any parabolic subgroup of  $B_I$  containing  $B_I \cap {}^{\mathbf{p}}B_I$ . Then  ${}^{\mathbf{p}\mathbf{i}}B_J$  also contains  $B_I \cap {}^{\mathbf{p}}B_I$  since this intersection is centralised by  $\mathbf{p}$ . We have thus  ${}^{\mathbf{i}}B_J \cap {}^{\mathbf{p}\mathbf{i}}B_J \supseteq B_I \cap {}^{\mathbf{p}}B_I$ , hence equality  ${}^{\mathbf{i}}B_J \cap {}^{\mathbf{p}\mathbf{i}}B_J = B_I \cap {}^{\mathbf{p}}B_I$ , that is  $B_I \cap {}^{\mathbf{p}}B_I$  is conjugate in  $B_I$  to  $B_J \cap {}^{\mathbf{p}'}B_J$  where  $\mathbf{p}' = \mathbf{i}^{-1}\mathbf{p}\mathbf{i}$ . We have  $\pi_I(\mathbf{p}') = \mathbf{i}^{-1}\pi_I(\mathbf{p})\mathbf{i} = 1$ . If we take  ${}^{\mathbf{i}}B_J$  minimal containing  $B_I \cap {}^{\mathbf{p}}B_I$  we get (4); such a parabolic exists since  $I$  is finite.  $\square$

Note that if we assume that the intersection of two parabolic subgroups is a parabolic subgroup, then in (4) above  $B_J = B_J \cap {}^{\mathbf{p}'}B_J$  and by the proof of (3) we have  $\mathbf{p}' \in C_P(B_J)$ .

#### 4. MINIMAL LENGTH ELEMENTS IN A CONJUGACY CLASS

In this section, we first recall a result on elements of  $B^+$  (Proposition 4.5) which is an important step in proving that in spherical type Artin groups there is a unique minimal parabolic subgroup containing a given element (see [7, Introduction of section 6]). We then show that a conjecture (Conjecture 4.14) which generalises Proposition 4.5 to all conjugacy classes is equivalent in all Artin groups to the existence of a unique minimal parabolic subgroup containing a given element. We conclude with a partial result (Proposition 4.17) on conjugacy.

**Positive elements.** The goal of this subsection is to introduce the concepts and results on elements of  $B^+$  needed, ending with a proof of Proposition 4.5.

For  $I \subseteq S$  and  $\mathbf{b} \in B^+$  we define  $H_I(\mathbf{b})$  as the longest prefix of  $\mathbf{b}$  in  $B_I^+$ ; we say that an element  $\mathbf{g} \in B^+$  is  $I$ -reduced if  $H_I(\mathbf{g}) = 1$ . We say that an element  $\mathbf{g} \in B^+$

is an  $I$ -ribbon if it is  $I$ -reduced and  $\mathbf{I}^{\mathbf{g}} \subseteq B^+$ . Since  $\ell_{\mathbb{Z}}$  is constant on a conjugacy class, this is seen to be equivalent to  $\mathbf{I}^{\mathbf{g}} \subseteq \mathbf{S}$ . For  $\mathbf{b} \in B^+$  we define the support  $\text{supp}(\mathbf{b})$  as the smallest subset  $I \subseteq S$  such that  $\mathbf{b} \in B_I^+$ . This is well defined since the two sides of a braid relation involve the same elements of  $\mathbf{S}$ . Note that the notions of  $I$ -reduced element and of  $I$ -ribbon in  $B^+$  are coherent with the corresponding definitions in  $W$  introduced in Section 1: the element  $\mathbf{w}$  in  $\mathbf{W}$  is  $I$ -reduced (*resp.* a  $I$ -ribbon) if and only if  $\text{pr}(\mathbf{w})$  is. This is not true if  $\mathbf{w} \notin \mathbf{W}$ . For instance  $\mathbf{s}^2$  is not  $\{s\}$ -reduced but  $\text{pr}(\mathbf{s})$  is, and is even an  $\{s\}$ -ribbon. If  $\mathbf{s}$  and  $\mathbf{t}$  do not commute, then  $\mathbf{s}^2\mathbf{t}$  is  $\{t\}$ -reduced but  $\text{pr}(\mathbf{s}^2\mathbf{t}) = t$  is not.

Recall that the Garside normal form has been introduced in Section 2. For  $\mathbf{b} \in B^+$  we denote by  $\text{head}(\mathbf{b})$  the first term of the Garside normal form of  $\mathbf{b}$  and define  $\text{tail}(\mathbf{b}) = \text{head}(\mathbf{b})^{-1}\mathbf{b}$ .

- Lemma 4.1.** (1) For  $\mathbf{b}, \mathbf{g} \in B^+$ , if  $\mathbf{b}^{\mathbf{g}} \in B^+$  then  $\mathbf{b}^{\text{head}(\mathbf{g})} \in B^+$   
(2) If  $\mathbf{g} \in B^+$  is an  $I$ -ribbon, then  $\text{head}(\mathbf{g})$  is an  $I$ -ribbon.  
(3) If  $\mathbf{g} \in B^+$  is an  $I$ -ribbon, then  $\text{pr}(\mathbf{g})$  is an  $I$ -ribbon.

*Proof.* For (1), the assumption  $\mathbf{b}^{\mathbf{g}} \in B^+$  is equivalent to  $\mathbf{g}$  left-dividing  $\mathbf{b}\mathbf{g}$  from which it follows that  $\text{head}(\mathbf{g})$  left-divides  $\text{head}(\mathbf{b}\mathbf{g}) = \text{head}(\mathbf{b}\text{head}(\mathbf{g}))$ , in particular  $\text{head}(\mathbf{g})$  left-divides  $\mathbf{b}\text{head}(\mathbf{g})$ , that is  $\mathbf{b}^{\text{head}(\mathbf{g})} \in B^+$ .

For (2), by (1) we get  $\mathbf{I}^{\text{head}(\mathbf{g})} \subseteq B^+$ . And clearly if  $\mathbf{g}$  is  $I$ -reduced then  $\text{head}(\mathbf{g})$  also.

Finally (3) is clear if  $\mathbf{g} \in \mathbf{W}$  and the general case follows from (2) by induction on the number of terms in the Garside normal form of  $\mathbf{g}$ .  $\square$

**Lemma 4.2.** Let  $\mathbf{g} \in B^+$  be an  $I$ -ribbon and let  $\mathbf{h} \in B^+$ . Then  $H_I(\mathbf{g}\mathbf{h})^{\mathbf{g}} = H_{I^{\text{pr}(\mathbf{g})}}(\mathbf{h})$  and  $H_I(\mathbf{g}\mathbf{h})^{-1}\mathbf{g}\mathbf{h} = \mathbf{g}H_{I^{\text{pr}(\mathbf{g})}}(\mathbf{h})^{-1}\mathbf{h}$ .

*Proof.* Let  $\mathbf{s} \in \mathbf{I}$  and set  $\mathbf{s}' = \mathbf{s}^{\mathbf{g}}$ . Both formulae clearly follow if we show that it is equivalent that  $\mathbf{s}$  left-divides  $\mathbf{g}\mathbf{h}$  or that  $\mathbf{s}'$  left-divides  $\mathbf{h}$ .

Now from  $\mathbf{s}\mathbf{g} = \mathbf{g}\mathbf{s}'$  it follows that the right-lcm of  $\mathbf{s}$  and  $\mathbf{g}$  divides  $\mathbf{s}\mathbf{g}$ , thus it must be equal to  $\mathbf{s}\mathbf{g}$ , because  $\ell_{\mathbb{Z}}(\text{right-lcm}(\mathbf{s}, \mathbf{g}))$  is at least  $\ell_{\mathbb{Z}}(\mathbf{g}) + 1 = \ell_{\mathbb{Z}}(\mathbf{s}\mathbf{g})$ . Thus  $\mathbf{s}$  left-dividing  $\mathbf{g}\mathbf{h}$  is equivalent to  $\mathbf{s}\mathbf{g}$  left-dividing  $\mathbf{g}\mathbf{h}$  or equivalently to  $\mathbf{g}\mathbf{s}'$  left-dividing  $\mathbf{g}\mathbf{h}$ , which is finally equivalent to  $\mathbf{s}'$  left-dividing  $\mathbf{h}$ .  $\square$

**Proposition 4.3.** Let  $\mathbf{g}, \mathbf{b} \in B^+$  be such that  $\mathbf{g}$  is  $\text{supp}(\mathbf{b})$ -reduced and such that  $\mathbf{b}^{\mathbf{g}} \in B^+$ . Then  $\mathbf{g}$  is a  $\text{supp}(\mathbf{b})$ -ribbon and we have  $\text{supp}(\mathbf{b}^{\mathbf{g}}) = \text{supp}(\mathbf{b})^{\text{pr}(\mathbf{g})}$ .

*Proof.* We first show that we can reduce to the case where  $\mathbf{g}$  is simple, by arguing by induction on the number of terms of the Garside normal form of  $\mathbf{g}$ .

By Lemma 4.1(1) we have  $\mathbf{b}^{\text{head}(\mathbf{g})} \in B^+$ ; and for  $\mathbf{g}$  to be  $\text{supp}(\mathbf{b})$ -reduced, we certainly need  $\text{head}(\mathbf{g})$  to be  $\text{supp}(\mathbf{b})$ -reduced.

If we know the theorem in the case where  $\mathbf{g}$  is simple, it follows that the set  $\mathbf{J} = \mathbf{I}^{\text{head}(\mathbf{g})}$ , where  $I = \text{supp}(\mathbf{b})$ , is a subset of  $\mathbf{S}$ ; and thus  $\mathbf{b}^{\text{head}(\mathbf{g})}$  has support  $J$ . We also have that  $\text{tail}(\mathbf{g})$  is  $J$ -reduced by Lemma 4.2 applied with  $\text{head}(\mathbf{g})$  for  $\mathbf{g}$  and  $\text{tail}(\mathbf{g})$  for  $\mathbf{h}$ . Since  $(\mathbf{b}^{\text{head}(\mathbf{g})})^{\text{tail}(\mathbf{g})} \in B^+$ , we conclude by induction on the number of terms of the normal form of  $\mathbf{g}$  that  $\mathbf{J}^{\text{tail}(\mathbf{g})} \subseteq \mathbf{S}$ , whence the result since  $\mathbf{J}^{\text{tail}(\mathbf{g})} = \mathbf{I}^{\mathbf{g}}$ .

We now show the theorem when  $\mathbf{g}$  is simple. We write the condition  $\mathbf{b}^{\mathbf{g}} \in B^+$  as  $\mathbf{b}\mathbf{g} = \mathbf{g}\mathbf{u}$  with  $\mathbf{u} \in B^+$ .

We proceed by an induction on  $\ell_{\mathbb{Z}}(\mathbf{b})$ . For  $\mathbf{s} \in \mathbf{S}$  dividing  $\mathbf{b}$ , we write  $\mathbf{b} = \mathbf{s}\mathbf{b}'$ . We have that  $\mathbf{s}$  left-divides  $\mathbf{b}\mathbf{g} = \mathbf{g}\mathbf{u}$ .

To go on we need the following lemma.

**Lemma 4.4.** *Let  $\mathbf{g} \in B^+$  be simple and let  $\mathbf{s} \in \mathbf{S}$ ,  $\mathbf{u} \in B^+$  be such that  $\mathbf{s}$  does not divide  $\mathbf{g}$  but  $\mathbf{s}$  divides  $\mathbf{gu}$ . Then we have  $\mathbf{u} = \mathbf{u}'\mathbf{t}\mathbf{u}''$  where  $\mathbf{t} \in \mathbf{S}$ ,  $\mathbf{u}', \mathbf{u}'' \in B^+$  are such that  $\mathbf{sgu}' = \mathbf{gu}'\mathbf{t}$  is simple.*

*Proof.* First note that  $\mathbf{s}$  left-divides  $\text{head}(\mathbf{gu})$ . Let  $\mathbf{gu}_1 = \text{head}(\mathbf{gu})$ . Since in  $W$  we have  $\ell_S(\mathbf{s}\mathbf{g}\mathbf{u}_1) < \ell_S(\mathbf{g}\mathbf{u}_1)$  and  $\ell_S(\mathbf{s}\mathbf{g}) > \ell_S(\mathbf{g})$  (in other words the reflection  $\mathbf{s}$  is in the left descent set of  $\mathbf{g}\mathbf{u}_1$  but not in that of  $\mathbf{g}$ ), we have by the exchange lemma  $\mathbf{g}\mathbf{u}_1 = \mathbf{g}\mathbf{u}'\mathbf{t}\mathbf{u}_2$  where  $\mathbf{t} \in S$  and  $\mathbf{s}\mathbf{g}\mathbf{u}' = \mathbf{g}\mathbf{u}'\mathbf{t}$ . Lifting back to  $B^+$  we get the lemma with  $\mathbf{u}'' = \mathbf{u}_2 \text{tail}(\mathbf{gu})$ .  $\square$

We now resume the proof of the proposition. Since  $\mathbf{s}$  does not divide  $\mathbf{g}$ , the lemma gives  $\mathbf{u} = \mathbf{u}'\mathbf{t}\mathbf{u}''$  with  $\mathbf{t} \in \mathbf{S}$  and  $\mathbf{sgu}' = \mathbf{gu}'\mathbf{t} \in \mathbf{W}$ . From  $\mathbf{b}\mathbf{g} = \mathbf{s}\mathbf{b}'\mathbf{g} = \mathbf{gu} = \mathbf{gu}'\mathbf{t}\mathbf{u}'' = \mathbf{sgu}'\mathbf{u}''$  we deduce  $\mathbf{b}'\mathbf{g} = \mathbf{gu}'\mathbf{u}''$ . Thus  $\mathbf{b}', \mathbf{g}$  satisfy the assumptions of the theorem, thus by induction on  $\ell_{\mathbb{Z}}(\mathbf{b})$  we have  $\mathbf{K}^{\mathbf{g}} \subseteq \mathbf{S}$  where  $K = \text{supp}(\mathbf{b}')$ . We still have to prove that  $\mathbf{s}^{\mathbf{g}} \in \mathbf{S}$ . This is already proven unless  $\mathbf{s} \notin \mathbf{K}$ , which we assume now.

We write  $\mathbf{b}'\mathbf{g} = \mathbf{g}\mathbf{b}''$  for some  $\mathbf{b}'' \in B$ . Since  $\mathbf{s}\mathbf{g}\mathbf{b}'' = \mathbf{gu} = \mathbf{gu}'\mathbf{t}\mathbf{u}'' = \mathbf{sgu}'\mathbf{u}''$ , we have  $\mathbf{b}'' = \mathbf{u}'\mathbf{u}'' \in B^+$ . Now  $\mathbf{g}\mathbf{y}\mathbf{g}^{-1} \in \mathbf{K}$ , for any  $\mathbf{y} \in \text{supp}(\mathbf{b}'')$  hence the element  $\mathbf{v} = \mathbf{g}\mathbf{u}'\mathbf{g}^{-1}$  is in  $B_K^+$ . From  $\mathbf{sgu}' = \mathbf{gu}'\mathbf{t}$  we get  $\mathbf{sv}\mathbf{g} = \mathbf{v}\mathbf{g}\mathbf{t} \in \mathbf{W}$ , which we write as  $(\mathbf{v}^{-1}\mathbf{sv})\mathbf{g} = \mathbf{g}\mathbf{t}$ . Since  $\mathbf{g}$  is  $\text{supp}(\mathbf{b})$ -reduced, its image in  $W$  is also and in  $W$  we have  $1 + \ell_S(\mathbf{g}) = \ell_S(\mathbf{g}\mathbf{t}) = \ell_S(\mathbf{v}^{-1}\mathbf{sv}\mathbf{g}) = \ell_S(\mathbf{v}^{-1}\mathbf{sv}) + \ell_S(\mathbf{g})$ , the last equality since  $\mathbf{v}^{-1}\mathbf{sv}$  is in  $W_{\text{supp}(\mathbf{b})}$ , whence  $\ell_S(\mathbf{v}^{-1}\mathbf{sv}) = 1$ . Now  $\mathbf{v}$  and  $\mathbf{sv}$  are simple since  $\mathbf{sv}\mathbf{g} = \mathbf{sgu}'$ , which is simple. Thus  $\mathbf{v}$  divides  $\mathbf{sv}$  because  $l(\mathbf{v}) + l(\mathbf{v}^{-1}\mathbf{sv}) = l(\mathbf{sv})$  which implies that  $\mathbf{v}\mathbf{x} = \mathbf{sv}$  where  $\mathbf{x} \in \mathbf{S}$  lifts  $\mathbf{v}^{-1}\mathbf{sv} \in S$ . It follows that  $\mathbf{v}^{-1}\mathbf{sv} \in \mathbf{S}$ . But  $\text{supp}(\mathbf{v}) \subset \mathbf{K}$  so that  $\mathbf{s} \notin \text{supp}(\mathbf{v})$ , hence  $\mathbf{v}^{-1}\mathbf{sv} \in \mathbf{S}$  implies  $\mathbf{v}^{-1}\mathbf{sv} = \mathbf{s}$ , thus  $\mathbf{s}^{\mathbf{g}} = \mathbf{t}$ .  $\square$

**Proposition 4.5.** *Let  $I \subseteq S$  and let  $\mathbf{p} \in B^+$  be a pure element such that  $H_I(\mathbf{p}) = 1$ ; if  $\mathbf{p}$  commutes with an element in  $B^+$  of support  $I$  then  $\mathbf{p}$  centralises  $B_I$ .*

*Proof.* Let  $\mathbf{i} \in B^+$  have support  $I$ . By Proposition 4.3 conjugating by  $\mathbf{p}$  maps  $\mathbf{I}$  into  $\mathbf{S}$ . Since  $\mathbf{p}$  is pure applying pr to the equality  $\mathbf{p}\mathbf{s} = \mathbf{s}'\mathbf{p}$  with  $\mathbf{s} \in \mathbf{I}$ ,  $\mathbf{s}' \in \mathbf{S}$  we get  $\text{pr}(\mathbf{s}) = \text{pr}(\mathbf{s}')$  hence  $\mathbf{s} = \mathbf{s}'$ .  $\square$

**General elements.** Recall that in the introduction we have defined  $\ell_{\mathbf{S}^{\pm 1}} : B \rightarrow \mathbb{N}$  to be the length function on  $B$  with respect to the generating set  $\mathbf{S}^{\pm 1}$ . For  $\mathbf{b} \in B$ , it is equal to  $\ell(\mathbf{b})$ , the length of a minimal word  $b \in (\mathbf{S}^{\pm 1})^*$  representing  $\mathbf{b}$ .

**Definition 4.6.** *For a word  $b \in (\mathbf{S}^{\pm 1})^*$ , we call  $\text{supp}(b)$  the minimal subset  $I \subseteq S$  such that  $b \in (\mathbf{I}^{\pm 1})^*$ .*

The following proposition, which enables the definition of support for all elements, is the convexity theorem of Charney and Paris [5, Theorem 1.2].

**Proposition 4.7.** *For  $\mathbf{b} \in B$ , the set  $\text{supp}(b)$  is independent of the word  $b$  of shortest length representing  $\mathbf{b}$ . We call it support of  $\mathbf{b}$ , denoted by  $\text{supp}(\mathbf{b})$ . For any word  $b'$  representing  $\mathbf{b}$  we have  $\text{supp}(b') \supseteq \text{supp}(\mathbf{b})$ .*

Note that the definition of support for elements of  $B$  is compatible with that given above Lemma 4.1 for elements of  $B^+$ .

*Proof.* Let  $b$  be a word of minimal length for  $\mathbf{b}$  and let  $J$  be the support of a word  $b'$  for  $\mathbf{b}$ . Since  $\hat{\pi}_J(b)$  represents  $\mathbf{b}$ , by minimality we have  $\ell(\hat{\pi}_J(b)) = \ell(b)$ , whence by Remark 1.3(5),  $\text{supp}(b) \subseteq J$ , which proves the proposition.  $\square$

We denote by  $\text{Conj}(\mathbf{b})$  the conjugacy class of  $\mathbf{b}$  in  $B$ , or  $\text{Conj}_B(\mathbf{b})$  when we want to specify  $B$ .

**Proposition 4.8.** *Let  $\mathbf{i}, \mathbf{j} \in B$  be conjugate elements with respective supports  $I$  and  $J$ . Assume  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{j})$  minimal in  $\text{Conj}(\mathbf{i})$ . Let  $\mathbf{g} \in B$  be such that  $\mathbf{ig} = \mathbf{gj}$ ; then  $\text{pr}(\mathbf{t}_I(\mathbf{g}))$  is a  $I'$ -ribbon- $J$  for some  $I' \subseteq I$ . If in addition  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i})$  is minimal in  $\text{Conj}(\mathbf{i})$ , then  $I' = I$ .*

*Proof.* We have  $\pi_I(\mathbf{ig}) = \mathbf{i}\pi_I(\mathbf{g}) = \pi_I(\mathbf{gj})$ . By Proposition 1.11 we have  $\pi_I(\mathbf{gj}) = \pi_I(\mathbf{g})\mathbf{i}'$  for some element  $\mathbf{i}' \in B_I$  such that given a minimal word  $j$  for  $\mathbf{j}$ , the word  $i' = p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j)) \cap \Phi_I))$  is a representative for  $\mathbf{i}'$ . But  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i}') \leq \ell(p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j)) \cap \Phi_I))) \leq \ell(j) = \ell_{\mathbf{S}^{\pm 1}}(\mathbf{j}) \leq \ell_{\mathbf{S}^{\pm 1}}(\mathbf{i}')$ , where the last inequality comes from the minimality of  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{j})$  in its conjugacy class, since  $\mathbf{i}'$  is conjugate to  $\mathbf{j}$ . Thus we have  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i}') = \ell_{\mathbf{S}^{\pm 1}}(\mathbf{j}) = \ell(p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j)) \cap \Phi_I)))$ . This forces all terms of  $\vec{N}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j))$  to be in  $\Phi_I$ . Using that  $\vec{N}$  is an involution, we get  $i' = p^{*-1}(\vec{N}(\vec{N}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j)))) = p^{*-1}({}^{t_I(\text{pr}(\mathbf{g}))}p^*(j))$ . But we know that in the formula for  $\hat{\pi}_I$  the roots to which one applies  $p^{*-1}$  are in  $\Pi_I^{\pm 1}$ . Hence  $t_I(\text{pr}(\mathbf{g})) = \text{pr}(\mathbf{t}_I(\mathbf{g}))$  is an  $I'$ -ribbon- $J$  where  $I'$  is the support of  $i'$  (and  $\mathbf{i}' = \varphi_{t_I(\text{pr}(\mathbf{g}))}(\mathbf{j})$ ).

If  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i})$  is minimal in the conjugacy class of  $\mathbf{i}$ , we may apply the previous result to the equality  $\mathbf{i}'\pi_I(\mathbf{g})^{-1} = \pi_I(\mathbf{g})^{-1}\mathbf{i}$  to conclude that  $t_{I'}(\text{pr}(\pi_I(\mathbf{g})^{-1}))$  is an  $I''$ -ribbon- $I$  for some  $I'' \subseteq I'$ . But this implies that  $t_{I'}(\text{pr}(\pi_I(\mathbf{g})^{-1}))$  is reduced- $I$ ; since this element is in  $W_I$ , this implies  $t_{I'}(\text{pr}(\pi_I(\mathbf{g})^{-1})) = 1$  thus  $I'' = I = I'$ .  $\square$

**Corollary 4.9.** *Let  $\mathbf{i}, \mathbf{j} \in B$  be conjugate elements of respective supports  $I, J$ ; assume that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i})$  and  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{j})$  are equal and minimal in  $\text{Conj}(\mathbf{i})$ . Then*

- (1)  $I$  and  $J$  are  $W$ -conjugate.
- (2) For  $\mathbf{g}$  such that  $\mathbf{ig} = \mathbf{gj}$ , the element  $\mathbf{w} \in \mathbf{W}$  defined by  $\text{pr}(\mathbf{w}) = \text{pr}(\mathbf{t}_I(\mathbf{g}))$  is an  $I$ -ribbon- $J$  such that  $\mathbf{i}^{\pi_I(\mathbf{g})\mathbf{w}} = \mathbf{j}$ .

*Proof.* (1) comes (2). (2) comes from the facts in the proof of Proposition 4.8 that  $I' = I$  and  $\mathbf{i}^{\pi_I(\mathbf{g})} = \mathbf{i}' = \varphi_{t_I(\text{pr}(\mathbf{g}))}(\mathbf{j}) = \varphi_{\text{pr}(\mathbf{w})}(\mathbf{j})$ .  $\square$

**Proposition 4.10.** *For  $I \subseteq S$  let  $\mathbf{i} \in B_I$ . Let  $\mathbf{g} \in B$  be such that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{g}^{-1}\mathbf{ig})$  is minimal in  $\text{Conj}_B(\mathbf{i})$ . Then  $\pi_I(\mathbf{g})^{-1}\mathbf{i}\pi_I(\mathbf{g}) \in \text{Conj}_{B_I}(\mathbf{i})$  has  $\ell_{\mathbf{S}^{\pm 1}}$  minimal in  $\text{Conj}_B(\mathbf{i})$ .*

*Proof.* This is a consequence of the proof of Proposition 4.8 where it is shown that the element  $\mathbf{i}' = \mathbf{i}^{\pi_I(\mathbf{g})}$  is has  $\ell_{\mathbf{S}^{\pm 1}}$  minimal in  $\text{Conj}_B(\mathbf{i})$ .  $\square$

From Proposition 4.10, we immediately deduce

**Corollary 4.11.** *For  $I \subseteq S$  and  $\mathbf{b} \in B$ , either  $\text{Conj}_B(\mathbf{b}) \cap B_I$  is empty or it contains an element  $\mathbf{i}$  such that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i})$  is minimal in  $\text{Conj}_B(\mathbf{b})$ .*

**Proposition 4.12.** *Let  $\mathbf{b} \in B$  be such that  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}(\mathbf{b})$ . Then  $B_{\text{supp}(\mathbf{b})}$  is a minimal parabolic subgroup containing  $\mathbf{b}$ , and any minimal parabolic subgroup containing  $\mathbf{b}$  is of the form  ${}^{\mathbf{p}}B_{\text{supp}(\mathbf{b})}$  for some element  $\mathbf{p}$  in  $C_P(\mathbf{b})$  such that  $\pi_{\text{supp}(\mathbf{b})}(\mathbf{p}) = 1$ .*

*Proof.* We first observe that any minimal parabolic subgroup containing  $\mathbf{b}$  is of the form  ${}^{\mathfrak{g}}B_I$  with  $\ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b}^{\mathfrak{g}}) = \ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b})$  and  $\text{supp}(\mathbf{b}^{\mathfrak{g}}) = I$ . Indeed if  ${}^{\mathfrak{g}}B_I$  is minimal containing  $\mathbf{b}$  then we apply by Proposition 4.10 with  $\mathbf{g}^{-1}$  for  $\mathfrak{g}$  and  $\mathbf{b}^{\mathfrak{g}}$  for  $\mathbf{i}$ . This shows that up to replacing  $\mathfrak{g}$  by  $\mathfrak{g}\pi_I(\mathbf{g}^{-1})$  we may assume that  $\ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b}^{\mathfrak{g}}) = \ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b})$ ; and we certainly have  $\text{supp}(\mathbf{b}^{\mathfrak{g}}) = I$  otherwise the proper parabolic subgroup  $B_{\text{supp}(\mathbf{b}^{\mathfrak{g}})}$  of  $B_I$  would contain  $\mathbf{b}^{\mathfrak{g}}$  and  ${}^{\mathfrak{g}}B_I$  would not be minimal containing  $\mathbf{b}$ .

Let now  ${}^{\mathfrak{g}}B_I$  be any minimal parabolic subgroup containing  $\mathbf{b}$ . We may assume that  $\ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b}^{\mathfrak{g}}) = \ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b})$  and that  $\text{supp}(\mathbf{b}^{\mathfrak{g}}) = I$ . It follows by Corollary 4.9(1) that  $\text{supp}(\mathbf{b})$  is conjugate to  $I$ . So, up to changing  $\mathfrak{g}$ , we could have started with a parabolic subgroup  ${}^{\mathfrak{g}}B_I$  where  $I = \text{supp}(\mathbf{b}) = \text{supp}(\mathbf{b}^{\mathfrak{g}})$ . It also follows from Corollary 4.9(2) that  $\mathbf{b}^{\mathfrak{g}} = \mathbf{b}^{\pi_I(\mathfrak{g})\mathbf{w}}$  where  $\mathbf{w} \in \mathbf{W}$  is an  $I$ -ribbon- $I$  and where  $\mathbf{p} = \mathbf{g}\mathbf{w}^{-1}\pi_I(\mathfrak{g})^{-1}$  is pure. In particular  $B_I^{\pi_I(\mathfrak{g})\mathbf{w}} = B_I$  so  ${}^{\mathfrak{g}}B_I = {}^{\mathbf{p}}B_I$ . We have  $\mathbf{b} \in B_I \cap {}^{\mathbf{p}}B_I = \pi_I(\mathbf{p})(B_I \cap \mathbf{t}_I(\mathbf{p})B_I) = \pi_I(\mathbf{p})C_{B_I}(\mathbf{t}_I(\mathbf{p})) = C_{B_I}(\mathbf{p}\pi_I(\mathbf{p})^{-1})$ , the second equality by Proposition 0.4, thus  $\mathbf{b}$  commutes to  $\mathbf{p}\pi_I(\mathbf{p})^{-1}$ . Now  ${}^{\mathbf{p}\pi_I(\mathbf{p})^{-1}}B_I = {}^{\mathbf{p}}B_I$ . Replacing  $\mathbf{p}$  by  $\mathbf{p}\pi_I(\mathbf{p})^{-1}$  we get that any minimal parabolic subgroup containing  $\mathbf{b}$  is as in the statement.

To show that these parabolic subgroups are actually minimal it is sufficient to show that  $B_{\text{supp}(\mathbf{b})}$  is a minimal parabolic subgroup containing  $\mathbf{b}$ . By what we have just seen, any minimal parabolic subgroup of  $B_{\text{supp}(\mathbf{b})}$  containing  $\mathbf{b}$  is of the form  ${}^{\mathbf{p}}B_{\text{supp}(\mathbf{b})}$  with  $\mathbf{p} \in P_{\text{supp}(\mathbf{b})}$  and  $\pi_{\text{supp}(\mathbf{b})}(\mathbf{p}) = 1$ , so  $\mathbf{p} = 1$ .  $\square$

Since minimality of parabolic subgroups transfers by conjugation, Proposition 4.12 proves the existence of a minimal parabolic subgroup containing any given element.

If the intersection of two parabolic subgroups is parabolic, there exists a unique minimal parabolic subgroup containing an element. Thus the validity of the following conjecture is supported by the results of [1, 6, 7, 8, 17]

**Conjecture 4.13.** *There exists a unique minimal parabolic subgroup containing a given  $\mathbf{b} \in B$ .*

The following conjecture generalises Proposition 4.5, replacing  $H_I$  by  $\pi_I$ . Note that for a positive element,  $H_I$  is a prefix of  $\pi_I$ .

**Conjecture 4.14.** *Let  $\mathbf{b} \in B$  such that  $\ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}(\mathbf{b})$ . Then any  $\mathbf{p} \in P$  which centralises  $\mathbf{b}$  and is such that  $\pi_{\text{supp}(\mathbf{b})}(\mathbf{p}) = 1$  centralises  $B_{\text{supp}(\mathbf{b})}$ .*

**Remark 4.15.** *The  $\ell_{\mathfrak{S}^{\pm 1}}$ -minimality assumption in Conjecture 4.14 is necessary. In the braid group of  $\mathfrak{S}_4$  with  $\mathfrak{S} = \{\mathbf{s}, \mathbf{t}, \mathbf{u}\}$  take  $\mathbf{b} = \mathbf{s}^{\mathbf{t}}$ ,  $I = \text{supp}(\mathbf{b}) = \{\mathbf{s}, \mathbf{t}\}$ , and  $\mathbf{p} = (\mathbf{u}^2)^{\mathbf{t}}$ . Then  $\pi_I(\mathbf{p}) = \pi_I^r(\mathbf{p}) = 1$  and  $\mathbf{b}^{\mathbf{p}} = \mathbf{b}$  but  $\mathbf{p}$  does not centralise  $\mathbf{s}$  or  $\mathbf{t}$ .*

**Proposition 4.16.** *Conjectures 4.13 and 4.14 are equivalent.*

*Proof.* Assuming Conjecture 4.13, if  $\mathbf{b}$  and  $\mathbf{p}$  are as in 4.14, then by Proposition 4.12  $B_I$  and  ${}^{\mathbf{p}}B_I$ , where  $I = \text{supp}(\mathbf{b})$ , are two minimal parabolic subgroups containing  $\mathbf{b}$ , hence they are equal and by Proposition 0.4  $\mathbf{p}$  centralises  $B_I$ , whence Conjecture 4.14.

We prove the converse. Up to conjugating  $\mathbf{b}$ , we can assume that  $\ell_{\mathfrak{S}^{\pm 1}}(\mathbf{b})$  is minimal in  $\text{Conj}_B(\mathbf{b})$ . Then, by Proposition 4.12, the minimal parabolic subgroups containing  $\mathbf{b}$  are the  ${}^{\mathbf{p}}B_{\text{supp}(\mathbf{b})}$  for  $\mathbf{p}$  in  $C_P(\mathbf{b})$  such that  $\pi_{\text{supp}(\mathbf{b})}(\mathbf{p}) = 1$ . By Conjecture 4.14,  $\mathbf{p}$  centralises  $B_{\text{supp}(\mathbf{b})}$ . We deduce that  $B_{\text{supp}(\mathbf{b})} = {}^{\mathbf{p}}B_{\text{supp}(\mathbf{b})}$  is the only minimal parabolic subgroup containing  $\mathbf{b}$ .  $\square$



Given a Coxeter system  $(W, S)$  and a pair  $(I, J)$  with  $I, J \subseteq S$ , we define the ribbon isomorphism problem in  $W$  for  $(I, J)$  to be the following problem: determine all the bijective maps  $\varphi : J \rightarrow I$  that extend to an isomorphism  $\varphi_w : W_J \rightarrow W_I$  such that  $w$  is a  $I$ -ribbon- $J$  in  $W$ .

**Proposition 4.17.** *Assume  $\Lambda$  is a non-empty set of subsets of  $S$  such that for any  $I$  in  $\Lambda$ , the word problem and the conjugacy problem are solvable in  $B_I$  and the ribbon isomorphism problem is solvable in  $W$  for any pair  $(I, J)$  in  $\Lambda$ . Then the two following problems are solvable.*

- (1) *Given  $I$  in  $\Lambda$  and a word  $b$  on  $\mathbf{I}^{\pm 1}$ , decide whether the word  $b$  represents the unity in  $B$ .*
- (2) *Given two finite subsets  $I, J$  in  $\Lambda$  and words  $i, j$  on  $\mathbf{I}^{\pm 1}$  and  $\mathbf{J}^{\pm 1}$ , respectively, decide whether or not  $i$  and  $j$  represent conjugate elements in  $B$ .*

Moreover, if the solution to the conjugacy problem in each parabolic subgroup  $B_I$  with  $I \in \Lambda$  provides a conjugating element, then the solution of problem (2) provides a conjugating element in  $B$  too.

*Proof.* Solution to problem (1) is clear as  $B_I$  embeds in  $B$  and we have a solution to the word problem in  $B_I$ . Given  $i$  and  $j$  as in (2), denote by  $\mathbf{i}$  and  $\mathbf{j}$  the elements represented by  $i$  and  $j$ , respectively. By the solution to the conjugacy problem in the parabolic subgroup  $B_I$ , by testing for conjugacy the words in  $(\mathbf{I}^{\pm 1})^*$  of length less than or equal to that of a word for  $\mathbf{i}$ , we can find a representative word on  $\mathbf{I}^{\pm 1}$  for each element  $\mathbf{i}' \in B_I$  that is conjugate to  $\mathbf{i}$  and such that  $\ell_{\mathbf{I}^{\pm 1}}(\mathbf{i}')$  is minimal in  $\text{Conj}_{B_I}(\mathbf{i})$ . By Proposition 4.7,  $\ell_{\mathbf{I}^{\pm 1}}(\mathbf{i}') = \ell_{\mathbf{S}^{\pm 1}}(\mathbf{i}')$  and by Proposition 4.10,  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{i}')$  is also minimal in  $\text{Conj}_B(\mathbf{i})$ . Similarly, we can find a representative word on  $\mathbf{J}^{\pm 1}$  for an element  $\mathbf{j}' \in B_J$  that is conjugate to  $\mathbf{j}$  and such that  $\ell_{\mathbf{J}^{\pm 1}}(\mathbf{j}')$  is minimal in  $\text{Conj}_{B_J}(\mathbf{j})$ . Again,  $\ell_{\mathbf{S}^{\pm 1}}(\mathbf{j}')$  is minimal in  $\text{Conj}_B(\mathbf{j})$ . Now  $\mathbf{i}$  and  $\mathbf{j}$  are conjugate if and only if some  $\mathbf{i}'$  and  $\mathbf{j}'$  are. By the proof of Proposition 4.10, some  $\mathbf{i}'$  is conjugate to  $\mathbf{j}'$  if and only if they are conjugate by the lift to  $\mathbf{W}$  of some  $I'$ -ribbon- $J'$  in  $W$  where  $J'$  is the support of  $\mathbf{j}'$  and  $I'$  is the support of  $\mathbf{i}'$ . Since the ribbon isomorphism problem is solvable in  $W$  for any pair  $(I, J)$  in  $\Lambda$ , we are done.  $\square$

When  $S$  is finite the hypothesis concerning the ribbon isomorphism problem is always satisfied as stated in the following result. As a consequence, we get Proposition 0.7.

**Proposition 4.18.** *Assume  $(W, S)$  is a Coxeter system with  $S$  finite. The ribbon isomorphism problem is solvable in  $W$  for any pair  $(I, J)$  with  $I, J \subseteq S$ .*

*Proof.* Let  $I, J \subseteq S$ . Let  $\Lambda$  denote the set of subsets  $I' \subseteq S$  whose Coxeter graph is isomorphic to the Coxeter graph of  $I$ , and let  $s$  be the number of graph automorphisms of the Coxeter graph of  $I$ . Since  $S$  and  $I$  are finite it takes finite time to compute  $\Lambda$  and  $s$ . Any one-to-one map  $\varphi : I' \rightarrow I$  that extends to an isomorphism  $\varphi : W_{I'} \rightarrow W_I$  has to induce a isomorphism of graphs. So the set  $\tilde{\Lambda}$  of pairs  $(I', \varphi)$  such that  $\varphi : I' \rightarrow I$  extends to an isomorphism  $\varphi : W_{I'} \rightarrow W_I$  is finite and its cardinality is bounded by  $|\Lambda| \times s$ . On the other hand, every ribbon can be decomposed into a finite product of elementary ribbons  $(I', v[s, I''], I'')$  (see [4, Proposition 2.3] and the set of elementary ribbons is finite (its cardinality is at most  $|S - I| \times |\Lambda|$ ); the set of associated isomorphisms is easy to determine. Now, consider the finite oriented labelled graph with vertex set  $\tilde{\Lambda}$  and such that there is an edge from  $(I', \varphi')$  to  $(I'', \varphi'')$  labelled by the elementary  $I'$ -ribbon- $I''$   $\tau$

if  $\varphi'' = \varphi' \circ \varphi_\tau$ . Then any  $I$ -ribbon can be read as a path in this graph based at  $(I, \text{Id})$ , and any path without loop has a length of at most  $|\tilde{\Lambda}|$ , which is less than  $|\Lambda| \times s$ . So, computing all the isomorphisms that correspond to a path whose length is no more than  $|\Lambda| \times s$  provides an answer to the ribbon isomorphism problem.  $\square$

## 5. A NEW TOPOLOGICAL VERSION OF THE RETRACTION

In this section we assume that  $S$  is finite, since we use the results of [19]. We did not check whether Van der Lek's results could be extended to infinite  $S$ . According to Lemma 2.14, it is sufficient to define  $\pi_I$  on the pure braid group and on simple braids to define it everywhere. We now give a topological definition of  $\pi_I$  restricted to  $\text{pr}^{-1}(W_I)$ .

Let  $V$  be the reflection representation of  $W$ , and  $V_{\mathbb{C}}$  be the complexified space. The set  $T$  is in bijection with the reflecting hyperplanes of  $W$ . Let  $H_s$  be the hyperplane corresponding to  $s \in T$ . We define  $X = V - \bigcup_{s \in T} H_s$  and similarly  $X_I = V - \bigcup_{s \in T_I} H_s$ . Any element of  $V_{\mathbb{C}}$  is of the form  $v = v_1 + \sqrt{-1}v_2$  where  $v_1, v_2 \in V$ . We write  $v_1 = \Re(v)$  and  $v_2 = \Im(v)$ . Now, following Van der Lek, let  $\mathcal{C}$  be the open Tits cone in  $V$  and let  $Y = \{v \in X \mid \Im(v) \in \mathcal{C}\}$  ([19, II, (3.2)]). We define similarly  $Y_I$  (note that though the Tits cone of  $W_I$  is bigger than  $\mathcal{C}$  its intersection with  $X_I$  is the same). Choosing some  $y_0 \in Y$ , the pure braid group is  $P = \Pi_1(Y, y_0)$  and similarly the pure braid group of  $B_I$  is  $P_I = \Pi_1(Y_I, y_0)$ . There is a natural morphism  $P \rightarrow P_I$  (since  $Y \subseteq Y_I$ ). We will show that this morphism is equal to  $\pi_I$ . More generally, we have

**Lemma 5.1.**  $\Pi_1(Y/W_I, \bar{y}_0) \simeq \text{pr}^{-1}(W_I)$

*Proof.* We have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & P & \longrightarrow & B = \Pi_1(Y/W) & \xrightarrow{\text{pr}} & W & \longrightarrow & 1 \\ & & \parallel & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & P & \longrightarrow & \Pi_1(Y/W_I, \bar{y}_0) & \longrightarrow & W_I & \longrightarrow & 1 \end{array}$$

where each line corresponds to a covering. The vertical map in the middle is induced by the natural map  $Y/W_I \rightarrow Y/W$ ; it is injective since the left one is an isomorphism. This diagram proves the lemma.  $\square$

**Proposition 5.2.** *The natural morphism  $\Pi_1(Y/W_I, \bar{y}_0) \rightarrow \Pi_1(Y_I/W_I, \bar{y}_0)$  coincides with the restriction of  $\pi_I$  to  $\text{pr}^{-1}(W_I)$ .*

*Proof.* We prove that the two morphisms coincide on generators of  $\text{pr}^{-1}(W_I)$ . By Lemma 2.19 this group is generated by  $B_I$  and the elements  $\mathbf{w}\mathbf{s}^2\mathbf{w}^{-1}$  with  $\mathbf{s} \in \mathbf{S}$  and  $\mathbf{w}\mathbf{s} \in \mathbf{W}$  an  $I$ -reduced element. Let  $C$  be the fundamental chamber in  $V$ , that is the subset defined by  $\alpha_s(x) > 0$  for all  $s \in S$  where  $\alpha_s$  is a linear form such that  $H_s = \{x \in V_{\mathbb{C}} \mid \alpha_s(x) = 0\}$ . The chambers are the images of  $C$  under  $W$ . Let  $F_I$  be the facet of  $C$  defined by  $\alpha_s(x) = 0$  for  $s \in I$  and  $\alpha_s(x) > 0$  for  $s \notin I$ . As in [19, II, Definition 4.8], we define  $Y(I)$  as the subset of  $Y$  formed of elements whose imaginary part is in  $F_I$  or in a chamber whose closure contains  $F_I$ . Then as a subgroup of  $\Pi_1(Y/W_I, \bar{y}_0)$ , the group  $B_I$  is the image of  $\Pi_1(Y(I)/W_I, \bar{y}_0)$ , and the natural morphism to  $\Pi_1(Y_I/W_I, \bar{y}_0)$  is an isomorphism since Van der Lek [19, II, Proposition 4.9] shows that the inclusion induces a homotopy equivalence between  $Y(I)$  and  $Y_I$ .

For the elements  $\mathbf{ws}^2\mathbf{w}^{-1}$ , we need to show that their image in  $\Pi_1(Y_I/W_I, \bar{y}_0)$  is trivial. But we claim that the image is already trivial in the pure braid group  $\Pi_1(Y_I, y_0)$ . This can be seen from the description in Van der Lek [19, I, Theorem 2.21 and II, Discussion 3.10] of such paths by galleries. The element  $\mathbf{s}^2$  corresponds to the gallery  $C, sC, C$ . If  $s \notin I$  the chambers  $C$  and  $sC$  are in the same chamber of  $Y_I$  so that the gallery becomes trivial in  $Y_I$ . By conjugation, the same happens to  $\mathbf{ws}^2\mathbf{w}^{-1}$ .  $\square$

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## INDEX

$\underline{\alpha}$ , 6  
 $\alpha_t$ , 6  
 $B$ , 1  
 $B^+$ , 1  
 $B_I$ , 2  
 $\mathbb{C}$ , 15  
 Conj, 22  
 $\Delta, \Delta_I$ , 2  
 $\bar{\Gamma}$ , 15  
 head, 20  
 $H_I$ , 19  
 inv, 10  
 $I$ -reduced, 19  
 $\ell$ , 5  
 $\ell_S$ , 1  
 $\ell_{\mathbf{S}^{\pm 1}}$ , 3  
 $\ell_{\mathbb{Z}}$ , 1  
 $N$ , 8  
 $\mathbf{N}$ , 13  
 $\tilde{N}$ , 6  
 $P$ , 1  
 $p, p^*$ , 6  
 $\Phi, \Phi^+$ , 5  
 $\Phi^*$ , 6  
 $\varphi_w$ , 8  
 $\Phi_I, \Phi_I^+$ , 6  
 $(\pm\Pi)^*$ , 6  
 $\Pi$ , 5  
 $\Pi_I$ , 6  
 $\pi_I$ , 10  
 $\tilde{\pi}_I$ , 5  
 $\pi_I$ , 11  
 $\pi_I^r$ , 16  
 $\hat{\text{pr}}$ , 7  
 $\text{pr}$ , 1  
 $\text{prod}(\underline{\alpha})$ , 6  
 rev, 16  
 $S$ , 1  
 $\mathbf{S}$ , 1  
 $\mathbf{S}^{\pm 1}, (\mathbf{S}^{\pm 1})^*$ , 4  
 $sf(j)$ , 5  
 $s_{\alpha}$ , 6  
 supp, 20  
 $T, T_I$ , 5  
 tail, 20  
 $\mathbf{t}_I$ , 11  
 $t_I$ , 4  
 $t_I^r$ , 16  
 $W$ , 1  
 $W_I$ , 2  
 $\mathbf{W}$ , 1  
 $\mathbf{w}, w$ , 1