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Discrete Funk-Radon transforms based on weighted least squares approximation

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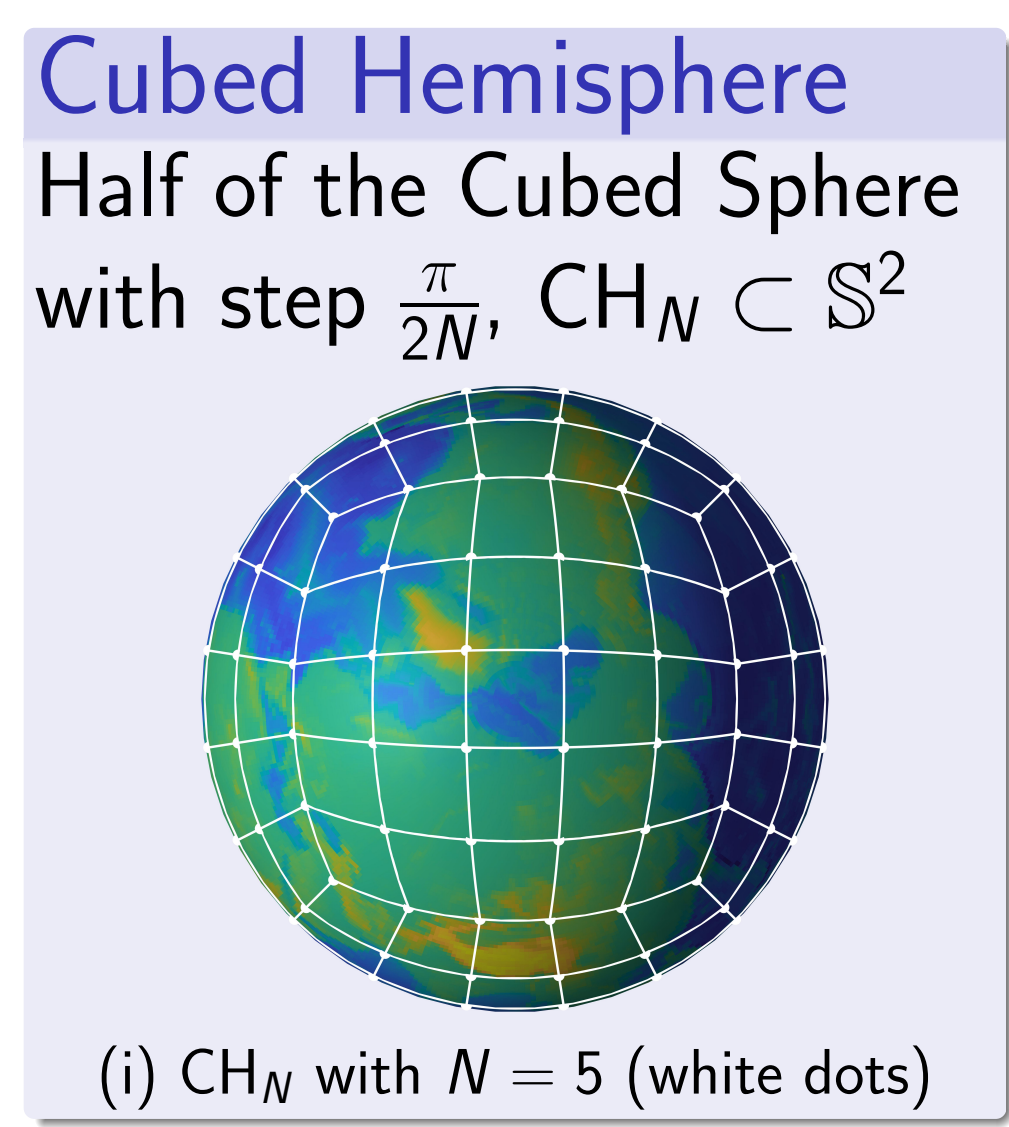


HAL Authorization

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Framework

- Exploratory study about discrete Funk-Radon transforms inspired from [3, 4]
- Features of the studied transforms:
 - the spherical grid is extracted from the Cubed Sphere
 - a spectral method is composed with weighted least squares
 - the weight comes from a quadrature rule
- Numerical study about the effect of the quadrature weights



Even spherical harmonics
 Even spherical harmonics with degree $\leq 2N - 2$

$\mathcal{Y}_{2N-2}^{ev} = \text{Span}\{Y_{2n}^m, |m| \leq 2n \leq 2N - 2\}$

- $f \in \mathcal{Y}_{2N-2}^{ev}$ admits a unique expansion
 $f = \sum_{|m| \leq 2n \leq 2N-2} \hat{f}_{2n}^m Y_{2n}^m$
- Fourier-like coefficients:
 $\hat{f} = [\hat{f}_{2n}^m]_{|m| \leq 2n \leq 2N-2} \in \mathbb{R}^{N(2N-1)}$
- Vandermonde matrix of \mathcal{Y}_{2N-2}^{ev} on CH_N
 $A = [Y_{2n}^m(x)]_{\substack{x \in CH_N \\ |m| \leq 2n \leq 2N-2}} \in \mathbb{R}^{(3N^2+1) \times N(2N-1)}$

Spherical Funk transform

$\mathcal{F}f(\xi) = \frac{1}{2\pi} \int_{x \cdot \xi = 0} f(x) ds, \xi \in \mathbb{S}^2, f: \mathbb{S}^2 \rightarrow \mathbb{R}$

- $\mathcal{F}: \mathcal{Y}_{2N-2}^{ev} \rightarrow \mathcal{Y}_{2N-2}^{ev}$ is isomorphic, with diagonal matrix
 $\Lambda = \text{diag}[(-1)^{n-3-5 \dots (2n-1)} I_{4n+1}, 0 \leq 2n \leq 2N - 2].$

Weighted least squares

$l_\omega(y) = \arg \inf_{f \in \mathcal{Y}_{2N-2}^{ev}} \sum_{x \in CH_N} 2\omega(x) |f(x) - y(x)|^2, y = [y(x)]_{x \in CH_N}$

- $\omega: CH_N \rightarrow (0, \infty)$ is the weight from an even spherical quadrature rule.
- The corresponding normal equation is
 $A^T \Omega A \hat{f} = A^T \Omega y, \text{ with } \Omega = \text{diag}(2\omega(x), x \in CH_N).$
- The degree $2N - 2$ is a numerical guarantee that the problem is well-conditioned if ω is uniform [1].

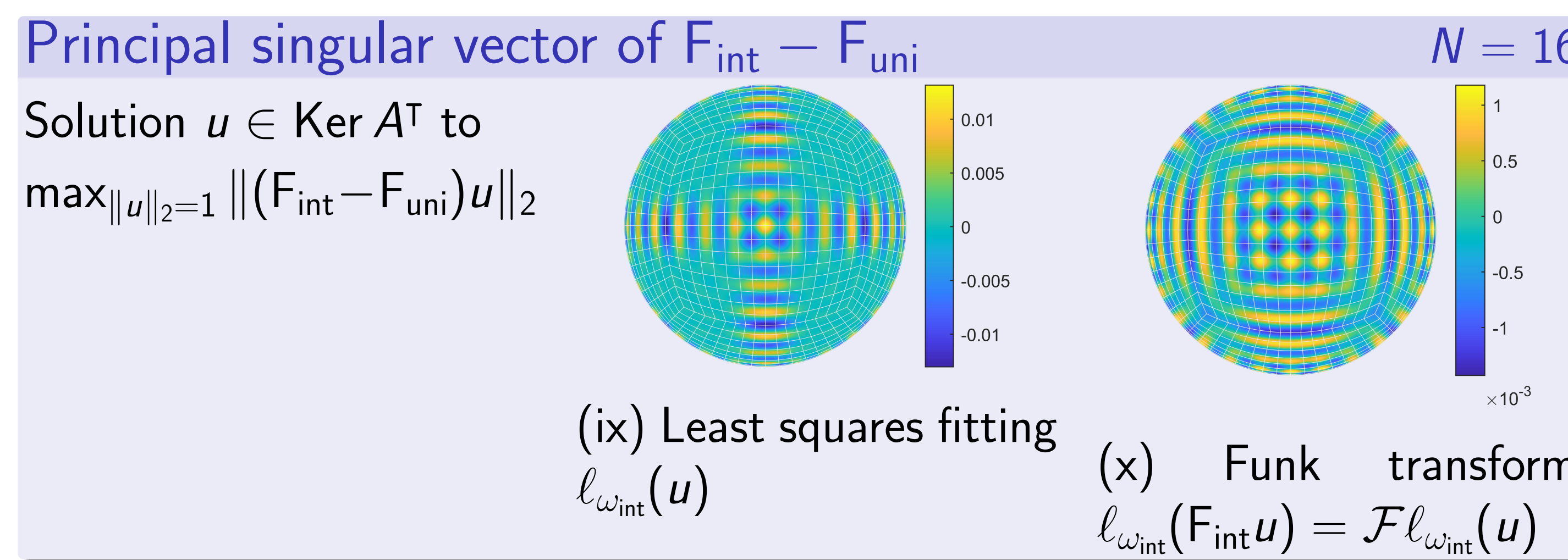
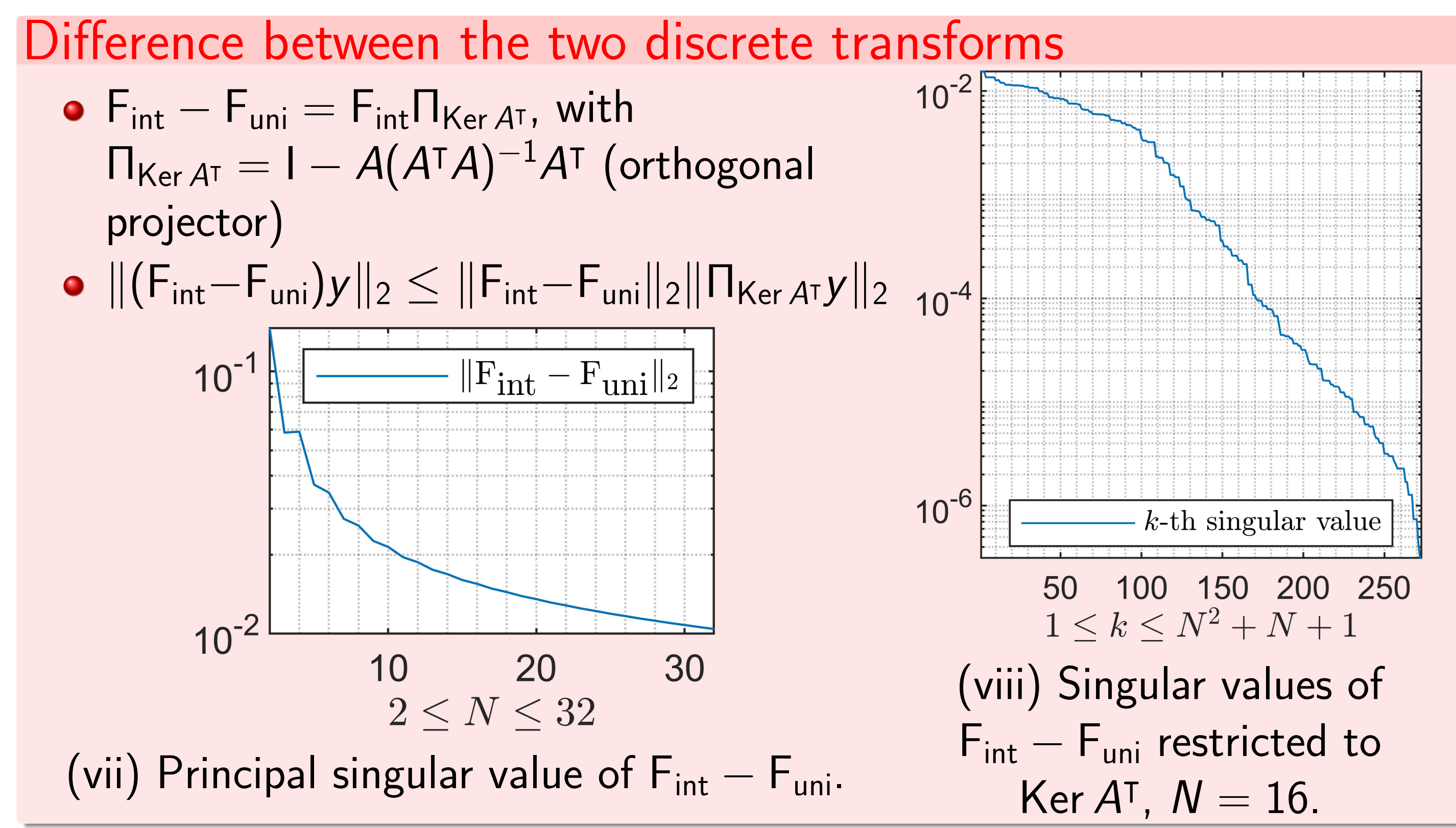
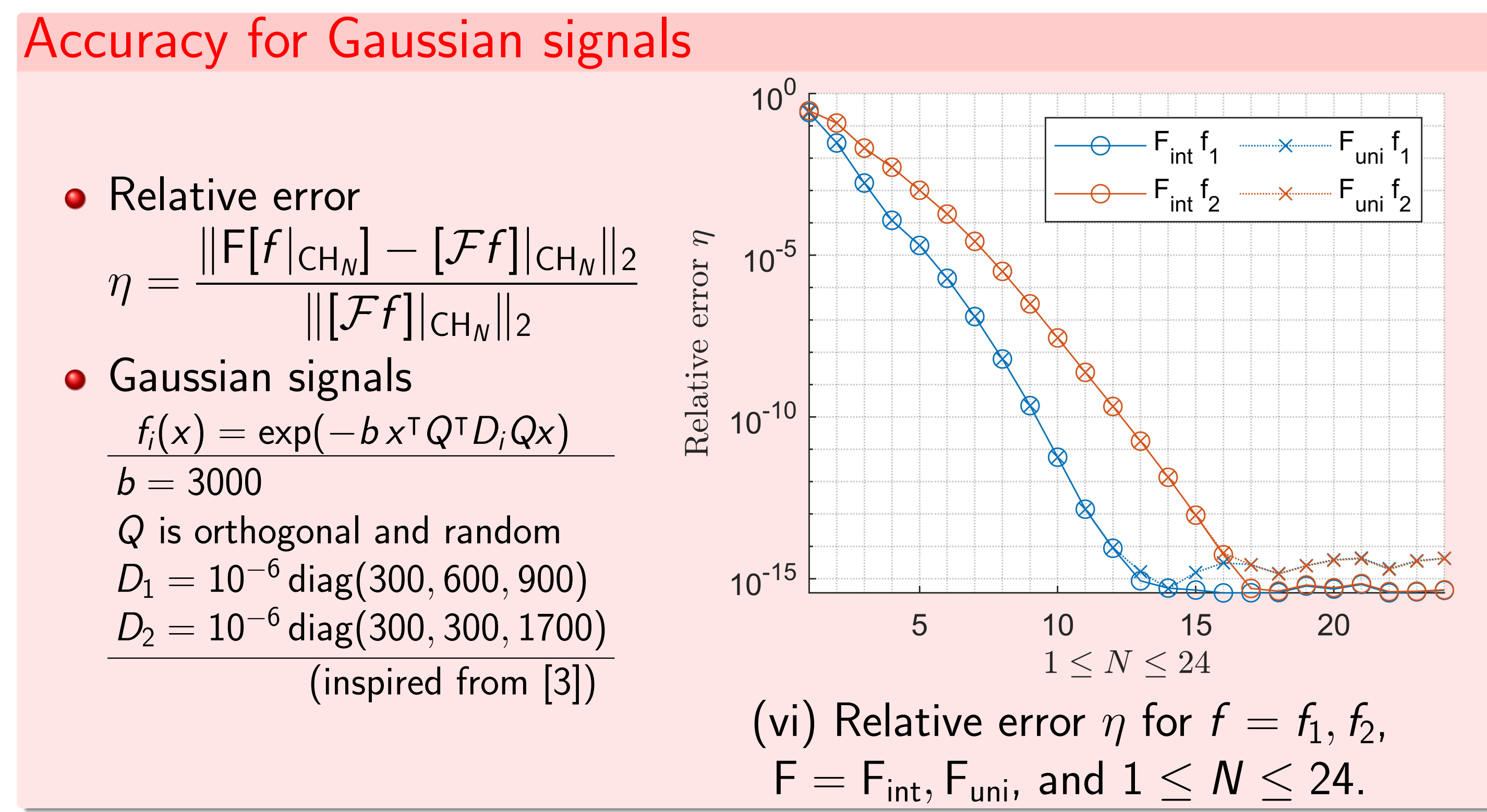
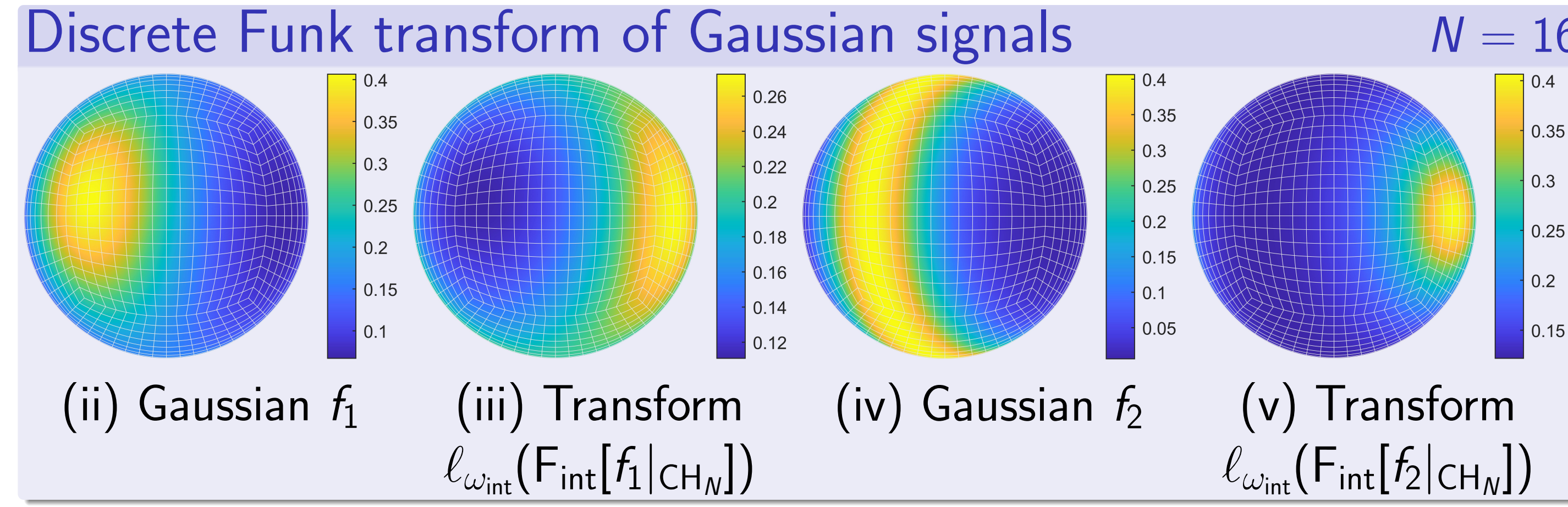
Discrete Funk transform on the Cubed Hemisphere

$F = A \Lambda (A^T \Omega A)^{-1} A^T \Omega \in \mathbb{R}^{(3N^2+1) \times (3N^2+1)}$

- $F = [\mathcal{F} \circ l_\omega]|_{CH_N}$, and $l_\omega \circ F = \mathcal{F} \circ l_\omega$
- Exactness property: $F[f]|_{CH_N} = [\mathcal{F}f]|_{CH_N}, f \in \mathcal{Y}_{2N-2}^{ev}$
- Stability estimate: $\|F\|_2^2 \leq \text{cond}(\Omega) \text{cond}(A^T \Omega A)$

Two considered weights

- Uniform weight $\omega(x) = \omega_{uni}(x) = \frac{2\pi}{3N^2+1}$
 - $F = F_{uni}$ coincides with the transform from [1]
- Interpolatory weight $\omega = \omega_{int}$ from [2]
 - $F = F_{int}$ has never been tested



Normal matrix as a perturbation of identity

$A^T \Omega A = I - E$, where E represents integration errors of products $Y_{2n}^m Y_{2n}^{m'}$

N	1	2	3	4	5	6	7	8	9	10	11	12	16	24	32
ω_{uni}	1e-16	1e-01	1e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01	2e-01
ω_{int}	4e-16	5e-16	2e-15	2e-15	5e-03	2e-03	6e-03	4e-03	4e-03	3e-03	3e-03	3e-03	2e-03	1e-03	7e-04

(xi) Spectral radius of $E = I - A^T \Omega A$.

N	1	2	3	4	5	6	7	8	9	10	11	12	16	24	32
ω_{uni}	0.0	13.9	17.8	18.0	18.1	18.1	18.2	18.2	18.2	18.3	18.3	18.3	18.4	18.5	18.6
ω_{int}	0.0	0.0	0.0	0.0	10.9	8.4	12.4	10.5	13.2	11.7	13.7	12.4	13.1	13.3	13.0

(xii) Sparsity score of $E = I - A^T \Omega A$ (%) (with threshold 10^{-14}).

Algebraic expression of optimal quadrature weights, $N \leq 4$

N	x_1	x_2	x_3	$\omega(x_1, x_2, x_3)$
1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{2}$
2	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{70}$
	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{16\pi}{105}$
3	1	0	0	$\frac{4\pi}{21}$
	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{140}$
	$\frac{1}{\sqrt{2+t^2}}$	$\frac{1}{\sqrt{2+t^2}}$	$\frac{1}{\sqrt{2+t^2}}$	$\frac{61\pi}{840} - \frac{3\pi\sqrt{3}}{560}$
4	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{61\pi}{840} + \frac{3\pi\sqrt{3}}{560}$
	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{512\pi}{15015}$
3	$\frac{1}{\sqrt{1+s^2}}$	$\frac{1}{\sqrt{1+s^2}}$	0	$\frac{2048\pi}{45045}$
	1	0	0	$\frac{736\pi}{15015}$

with $s = \sqrt{2} - 1$

(xiii) On $\{CH_N, -CH_N\}$, the octahedral quadrature rule given by ω has the maximum degree of accuracy $4N - 1$ and coincides with ω_{int} . So, $A^T \Omega A = I$.

Conclusion and discussion

- The discrete transforms F_{int} and F_{uni} approximate the Funk transform \mathcal{F} with an interesting accuracy.
- Despite F_{int} and F_{uni} act differently on some patterns, they are very similar.
- Iterative solvers (conjugate gradient, Neumann series,...) converge in a few iterations for $l_{\omega_{int}}$, which motivates the use of suitable weights.
- Similar approaches can be developed for other quadrature rules on other spherical grids.

References

- [1] J.-B. Bellet, A discrete Funk transform on the Cubed Sphere, *Journal of Computational and Applied Mathematics*, 429 (2023).
- [2] J.-B. Bellet, M. Brachet, and J.-P. Croisille, Quadrature and symmetry on the Cubed Sphere, *Journal of Computational and Applied Mathematics*, 409 (2022).
- [3] M. Descoteaux, E. Angelino, S. Fitzgibbons, and R. Deriche, Regularized, Fast, and Robust Analytical Q-Ball Imaging, *Magnetic Resonance in Medicine*, 58 (2007), pp. 497-510.
- [4] C. P. Hess, P. Mukherjee, E. T. Han, D. Xu, and D. B. Vigneron, Q-Ball Reconstruction of Multimodal Fiber Orientations Using The Spherical Harmonic Basis, *Magnetic Resonance in Medicine*, 56 (2006), pp. 104-117.

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