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Théo Gherdaoui. Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term. 2024. hal-04640822v2

HAL Id: hal-04640822

<https://hal.science/hal-04640822v2>

Preprint submitted on 25 Oct 2024

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Small-Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term

Théo Gherdaoui*

October 25, 2024

Abstract

The goal of this article is to contribute to a better understanding of the relations between the exact controllability of nonlinear PDEs and the control theory for ODEs based on Lie brackets, through a study of the Schrödinger PDE with bilinear control.

We focus on the small-time local controllability (STLC) around an equilibrium, when the linearized system is not controllable. We study the second order term in the Taylor expansion of the state, with respect to the control.

For scalar-input ODEs, quadratic terms never recover controllability: they induce signed drifts in the dynamics (see [10]). Thus proving STLC requires to go at least to the third order. Similar results were proved for the bilinear Schrödinger PDE with scalar-input controls in [16].

In this article, we study the case of multi-input systems. We clarify among the quadratic Lie brackets, those that allow to recover STLC: they are bilinear with respect to two different controls. For ODEs, our result is a consequence of Sussmann's sufficient condition $S(\theta)$ (when focused on quadratic terms), but we propose a new proof, designed to prepare an easier transfer to PDEs. This proof relies on a representation formula of the state inspired by the Magnus formula. By adapting it, we prove a new STLC result for the bilinear Schrödinger PDE.

Keywords: Bilinear Schrödinger equation, infinite-dimensional systems, small-time local exact controllability, power series expansion.

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1 Introduction

1.1 Model and problem

In this article, we study the following Schrödinger equation

$$\begin{cases} i\partial_t\psi(t, x) = -\partial_x^2\psi(t, x) - (u(t)\mu_1(x) + v(t)\mu_2(x))\psi(t, x), & t \in (0, T), x \in (0, 1), \\ \psi(t, 0) = \psi(t, 1) = 0, & t \in (0, T), \\ \psi(0, x) = \psi_0(x), & x \in (0, 1), \end{cases} \quad (1.1)$$

where $\mu_1, \mu_2 : (0, 1) \rightarrow \mathbb{R}$, $T > 0$, $u, v : (0, T) \rightarrow \mathbb{R}$, $\psi : (0, T) \times (0, 1) \rightarrow \mathbb{C}$ and $\|\psi_0\|_{L^2} = 1$. When well-defined, the solution is denoted ψ . When required, we will write $\psi(\cdot; (u, v), \psi_0)$ to refer to this solution to emphasize its dependence on the different parameters.

This equation describes the evolution of the wave function ψ of a quantum particle, in an 1D infinite square potential well $(0, 1)$, subjected to two electric fields with amplitudes $u(t)$ and $v(t)$. The functions μ_1 and μ_2 , called "dipolar moments", model the interaction between the particle's wave function ψ and the two electric fields u, v .

This is multi-input nonlinear control system:

- the state is the wave function $\psi : (0, T) \rightarrow \mathcal{S}$, where \mathcal{S} denotes the $L^2(0, 1)$ sphere,
- the control is $(u, v) : (0, T) \rightarrow \mathbb{R}^2$, it acts bilinearly on the state, through the term $(u(t)\mu_1(x) + v(t)\mu_2(x))\psi(t, x)$, this is a reason why we speak of "bilinear Schrödinger equation".

The ground state is the particular trajectory $\psi_1(t, x) := \varphi_1(x)e^{-i\lambda_1 t}$, with $(u, v) = 0$ where $\varphi_1(x) := \sqrt{2}\sin(\pi x)$ and $\lambda_1 := \pi^2$. We are interested in the local controllability of the nonlinear system (1.1), around the ground state, i.e. realizing small motions around the ground state with small controls: given a maximal size $\varepsilon > 0$ for the controls, does there exist $T > 0$ and $\delta > 0$ such that, for any target ψ_f close enough to the ground state, i.e. $\|\psi_f - \psi_1(T)\| < \delta$, there exist controls $(u, v) : (0, T) \rightarrow \mathbb{R}^2$ such that $\|(u, v)\| < \varepsilon$ and $\psi(T; (u, v), \varphi_1) = \psi_f$. The different norms used are of great importance and will be precised later.

More precisely, we are interested in the **small-time** local controllability around the ground state i.e. realizing, in arbitrary small times $T > 0$, small motions around the ground state with small controls: given $T, \varepsilon > 0$, does there exist $\delta > 0$ such that, for any target ψ_f such that $\|\psi_f - \psi_1(T)\| < \delta$, there exist controls $(u, v) : (0, T) \rightarrow \mathbb{R}^2$ such that $\|(u, v)\| < \varepsilon$ and $\psi(T; (u, v), \varphi_1) = \psi_f$. It corresponds to the local surjectivity around 0 of the nonlinear end-point map $\Theta_T : (u, v) \mapsto \psi(T; (u, v), \varphi_1)$.

Any positive answer to the STLC problem may be thought of as a nonlinear local open mapping theorem, which underlines the deepness and intricacy of this problem, when the inverse mapping theorem (or linear test, see [28, Section 3.1]) cannot be used.

Small-time controllability has particularly relevant physical implications, both from a fundamental viewpoint and for technological applications. Indeed, quantum systems, once engineered, suffer of very short lifespan before decaying (e.g., through spontaneous photon emissions) and loosing their non-classical properties (such as superposition). Hence, the capability of controlling them in a minimal time is also an important challenge in physics.

1.2 The controllable linearized case: constrained functional framework

Under appropriate assumptions on the dipolar moments μ_1, μ_2 the small-time local controllability of (1.1) can be proved by applying the linear test. One then obtains the following statement, that can easily be proved by adapting with two controls the strategy developed in [7, 17] with one control. This statement requires the introduction of the following notations: we define the operator

$$A := -\partial_x^2 \text{ with domain } D(A) = H^2(0, 1) \cap H_0^1(0, 1). \quad (1.2)$$

Except explicit precision, we will work with complex valued functions. We consider the space, $L^2(0, 1)$, endowed with the classical scalar product

$$\forall f, g \in L^2(0, 1), \quad \langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

Classical functional analysis study shows that the eigenvalues and eigenvectors are

$$\lambda_j = (j\pi)^2, \quad \varphi_j := \sqrt{2}\sin(j\pi \cdot), \quad j \geq 1. \quad (1.3)$$

The family $(\varphi_j)_{j \geq 1}$ is an orthonormal basis of $L^2(0, 1)$. We define

$$\psi_j(t, x) := \varphi_j(x)e^{-i\lambda_j t}, \quad (t, x) \in (0, T) \times (0, 1), \quad j \geq 1.$$

These functions are the solutions to the free (with controls $(u, v) \equiv 0$) Schrödinger equation (1.1), with initial data φ_j at time $t = 0$. They are called eigenstates. When $j = 1$, ψ_1 is called the fundamental state, or ground state. We finally define the spaces $H_{(0)}^s(0, 1) = D(A^{s/2})$, equipped with the norm

$$\|\varphi\|_{H_{(0)}^s} := \left(\sum_{j=1}^{+\infty} |j^s \langle \varphi, \varphi_j \rangle|^2 \right)^{1/2}.$$

We consider, for $k \in \mathbb{N}$, $H^k((0, T), \mathbb{R})$, the real Sobolev space, equipped with the usual $H^k(0, T)$ norm and $H_0^k(0, T)$ the adherence of $C_c^\infty(0, T)$ with this norm.

Theorem 1.1. *Let $p, m \in \mathbb{N}$ and $\mu_1, \mu_2 \in H^{2(p+m)+3}(0, 1)$ be such that $\mu_\ell^{(2k+1)}(0) = \mu_\ell^{(2k+1)}(1) = 0$ for $k = 0, \dots, p-1$, $\ell = 1, 2$ and*

$$\exists C > 0, \quad \forall j \in \mathbb{N}^*, \quad |\langle \mu_1 \varphi_1, \varphi_j \rangle| + |\langle \mu_2 \varphi_1, \varphi_j \rangle| \geq \frac{C}{j^{2p+3}}. \quad (1.4)$$

Then the Schrödinger equation (1.1) is $H_0^m((0, T), \mathbb{R})$ -small-time locally controllable in $H_{(0)}^{2(p+m)+3}(0, 1)$: for every $T, \varepsilon > 0$, there exists $\delta > 0$ such that for every $\psi_f \in \mathcal{S} \cap H_{(0)}^{2(p+m)+3}(0, 1)$ with $\|\psi_f - \psi_1(T)\|_{H^{2(p+m)+3}} < \delta$, there exists $(u, v) \in H_0^m((0, T), \mathbb{R})^2$ such that $\|(u, v)\|_{H^m} \leq \varepsilon$ and $\psi(T; (u, v), \varphi_1) = \psi_f$.

In this statement, there are two parameters:

- p is the number of odd derivatives of μ_ℓ that vanish at the boundary, $\ell \in \{1, 2\}$,
- m corresponds to the regularity expected on the controls: $u, v \in H_0^m((0, T), \mathbb{R})$.

The property

$$\forall j \in \mathbb{N}^*, \quad |\langle \mu_1 \varphi_1, \varphi_j \rangle| + |\langle \mu_2 \varphi_1, \varphi_j \rangle| \neq 0 \quad (1.5)$$

is necessary for the controllability of the linearized system around the ground state. The stronger assumption (1.4) guarantees that the linearized system is $H_0^m((0, T), \mathbb{R})$ -small-time controllable in $H^{2(p+m)+3}(0, 1)$. Then, a smoothing effect allows to prove that the end-point map is of class \mathcal{C}^1 between the following spaces

$$\Theta_T : (u, v) \in H_0^m((0, T), \mathbb{R})^2 \mapsto \psi(T; (u, v), \varphi_1) \in H_{(0)}^{2(m+p)+3}(0, 1) \cap \mathcal{S}.$$

Thus, by applying the inverse mapping theorem, we obtain the local controllability of the nonlinear system (1.1).

The choice of functional spaces is determined by the controllability of the linearized system: once p, m are fixed and (1.4) is verified, then we must work with $\psi(T) \in H_{(0)}^{2(p+m)+3}(0, 1)$.

Note that the assumption (1.4) prevents the $\mu_\ell^{(2p+1)}$ from vanishing at the boundary because, if μ_ℓ satisfies the hypotheses of Theorem 1.1, then integration by parts give: for all $k, j \in \mathbb{N}^*$, for all $\ell \in \{1, 2\}$,

$$\langle \mu_\ell \varphi_j, \varphi_k \rangle = \frac{4(-1)^p(p+1)k}{j^{2p+3}\pi^{2p+2}} \left((-1)^{j+k} \mu_\ell^{(2p+1)}(1) - \mu_\ell^{(2p+1)}(0) \right) + \underset{j \rightarrow +\infty}{o} \left(\frac{1}{j^{2p+3}} \right). \quad (1.6)$$

1.3 Our case study and its interpretation in terms of Lie brackets

In this article, we study the case where the linearized system around the ground state is not controllable, so we consider an integer $K \geq 2$ and we assume

$$(\mathbf{H})_{\text{lin}, K, 1} : \langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0.$$

In this case, the linearized system is not controllable: it misses one complex direction: $\langle \psi(t), \varphi_K \rangle \in \mathbb{C}$, i.e. the first order term in the Taylor expansion at $(0, 0)$ of the map $(u, v) \mapsto \langle \psi(t; (u, v), \varphi_1), \varphi_K \rangle$ vanishes. We assume that this is the only direction lost, i.e. all other directions are controllable on the linearized system, with the usual functional framework, so we consider integers $p, m \in \mathbb{N}$ and we assume

$$(\mathbf{H})_{\text{reg}} : \mu_\ell \in H^{2(p+m)+3}((0, 1), \mathbb{R}), \quad \text{with } \mu_\ell^{(2k+1)}|_{\{0,1\}} = 0 \text{ for } 0 \leq k \leq p-1, \quad \ell \in \{1, 2\}.$$

$$(\mathbf{H})_{\text{lin}, K, 2} : \text{There exists } C > 0 \text{ such that } \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| \left(\langle \mu_\ell \varphi_1, \varphi_j \rangle \right)_{1 \leq \ell \leq 2} \right\| \geq \frac{C}{j^{2p+3}}.$$

To control the complex direction $\langle \psi(t), \varphi_K \rangle$ of the nonlinear system (1.1), we will use a power series expansion of the order 2, i.e. the second order term in the Taylor expansion at $(0, 0)$ of the map $(u, v) \mapsto \langle \psi(t; (u, v), \varphi_1), \varphi_K \rangle$. At this step of the proof, we consider an integer $n \in \mathbb{N}^*$ and we assume that the following assumptions hold:

(H)_{quad,K,1}: $\forall k \in \llbracket 1, \lfloor \frac{n+1}{2} \rrbracket \rrbracket$,

$$A_k^1 := (-1)^{k-1} \sum_{j=1}^{+\infty} \left(\lambda_j - \frac{\lambda_1 + \lambda_K}{2} \right) (\lambda_K - \lambda_j)^{k-1} (\lambda_j - \lambda_1)^{k-1} c_j = 0.$$

(H)_{quad,K,2}: $\forall k \in \llbracket 1, \lfloor \frac{n+1}{2} \rrbracket \rrbracket$,

$$A_k^2 := (-1)^{k-1} \sum_{j=1}^{+\infty} \left(\lambda_j - \frac{\lambda_1 + \lambda_K}{2} \right) (\lambda_K - \lambda_j)^{k-1} (\lambda_j - \lambda_1)^{k-1} \tilde{c}_j = 0.$$

(H)_{quad,K,3}: $\forall k \in \llbracket 1, n-1 \rrbracket$,

$$\gamma_k := \sum_{j=1}^{+\infty} \left((\lambda_K - \lambda_j)^{\lfloor \frac{k+1}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{k}{2} \rfloor} d_j - (\lambda_K - \lambda_j)^{\lfloor \frac{k}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{k+1}{2} \rfloor} \tilde{d}_j \right) = 0.$$

(H)_{quad,K,4}:

$$\gamma_n := \sum_{j=1}^{+\infty} \left((\lambda_K - \lambda_j)^{\lfloor \frac{n+1}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{n}{2} \rfloor} d_j - (\lambda_K - \lambda_j)^{\lfloor \frac{n}{2} \rfloor} (\lambda_j - \lambda_1)^{\lfloor \frac{n+1}{2} \rfloor} \tilde{d}_j \right) \neq 0,$$

with the notations: for all $j \geq 1$,

$$\begin{aligned} c_j &:= \langle \mu_1 \varphi_1, \varphi_j \rangle \langle \mu_1 \varphi_j, \varphi_K \rangle, & d_j &:= \langle \mu_2 \varphi_1, \varphi_j \rangle \langle \mu_1 \varphi_j, \varphi_K \rangle, \\ \tilde{c}_j &:= \langle \mu_2 \varphi_1, \varphi_j \rangle \langle \mu_2 \varphi_j, \varphi_K \rangle, & \tilde{d}_j &:= \langle \mu_1 \varphi_1, \varphi_j \rangle \langle \mu_2 \varphi_j, \varphi_K \rangle. \end{aligned}$$

Remark 1.2. *The asymptotic behavior (1.6) and the hypothesis $n \leq 2p + 2$ ensure that all the series written in (H)_{quad,K,1}, (H)_{quad,K,2}, (H)_{quad,K,3}, and (H)_{quad,K,4} converge absolutely. Indeed, for all $\eta \in \mathbb{N}$,*

$$\sum_{j=1}^{+\infty} |c_j| j^{4\eta}, \sum_{j=1}^{+\infty} |\tilde{c}_j| j^{4\eta}, \sum_{j=1}^{+\infty} |d_j| j^{4\eta}, \sum_{j=1}^{+\infty} |\tilde{d}_j| j^{4\eta} \leq C \sum_{j=1}^{+\infty} \frac{1}{j^{4p+6-4\eta}},$$

which converges if $p+1 \geq \eta$. In particular, if $\lfloor \frac{n}{2} \rfloor \leq p$, all the series defined previously, and $A_{\lfloor \frac{n}{2} \rfloor + 1}^1, A_{\lfloor \frac{n}{2} \rfloor + 1}^2, \gamma_{n+1}$ converge also (technical assumption).

The existence of such functions μ_1, μ_2 is proved in Appendix A.1.

For A and B two operators, we define by induction on $k \in \mathbb{N}$ the operator $\text{ad}_A^k(B)$ as: $\text{ad}_A^0(B) = B$ and $\text{ad}_A^{k+1}(B) = \text{ad}_A^k(B)A - A\text{ad}_A^k(B)$. Under appropriate relations between the parameter n and p , the assumptions (H)_{quad,K,1}, (H)_{quad,K,2}, (H)_{quad,K,3} and (H)_{quad,K,4} can be interpreted in terms of Lie brackets:

$$\forall k = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor, \quad 2A_k^1 = (-1)^k \left\langle [\text{ad}_A^{k-1}(B_1), \text{ad}_A^k(B_1)] \varphi_1, \varphi_K \right\rangle, \quad (1.7)$$

$$\forall k = 1, \dots, \left\lfloor \frac{n+1}{2} \right\rfloor, \quad 2A_k^2 = (-1)^k \left\langle [\text{ad}_A^{k-1}(B_2), \text{ad}_A^k(B_2)] \varphi_1, \varphi_K \right\rangle, \quad (1.8)$$

$$\forall k = 1, \dots, n, \quad \gamma_k = (-1)^k \left\langle [\text{ad}_A^{\lfloor \frac{k+1}{2} \rfloor}(B_1), \text{ad}_A^{\lfloor \frac{k}{2} \rfloor}(B_2)] \varphi_1, \varphi_K \right\rangle, \quad (1.9)$$

where A is defined in (1.2), and B_ℓ is the multiplication operator by μ_ℓ in $L^2(0, 1)$. We refer to Propositions A.6, A.7, A.9 and A.10 in appendix for a precise proof.

For the single-input bilinear Schrödinger equation

$$\begin{cases} i\partial_t \psi(t, x) = -\partial_x^2 \psi(t, x) - u(t) \mu_\ell(x) \psi(t, x), & (t, x) \in (0, T) \times (0, 1), \\ \psi(t, 0) = \psi(t, 1) = 0, & t \in (0, T), \\ \psi(0, x) = \psi_0(x), & x \in (0, 1), \end{cases} \quad (1.10)$$

the Lie bracket $2A_k^\ell$ is known to be an obstruction to small-time local controllability, in an appropriate functional setting (see [18]). The Lie brackets $(\gamma_k)_{1 \leq k \leq n}$ are specific to multi-input system (1.1), our positive controllability result relies on γ_n . We need the following definition: for $T > 0$, and $u \in L^1((0, T), \mathbb{R})$, one defines the iterated primitives u_n vanishing at $t = 0$ by induction as

$$u_0 = u, \quad \text{and} \quad \forall n \in \mathbb{N}, \forall t \in [0, T], \quad u_{n+1}(t) = \int_0^T u_n(t) dt.$$

Then, heuristically, it is as if the following terms were included in the quadratic development of the solution:

$$-i \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} A_k^1 \int_0^T u_k^2(t) dt - i \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} A_k^2 \int_0^T v_k^2(t) dt + \sum_{k=1}^n i^k \gamma_k \int_0^T u_{\lfloor \frac{k}{2} \rfloor + 1}(t) v_{\lfloor \frac{k+1}{2} \rfloor}(t) dt.$$

The cancellation assumptions $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ reduce this sum to the simpler expression:

$$i^n \gamma_n \int_0^T u_{\lfloor \frac{n}{2} \rfloor + 1}(t) v_{\lfloor \frac{n+1}{2} \rfloor}(t) dt, \quad (1.11)$$

and they allow use to take advantage of the term $\int_0^T u_{\lfloor \frac{n}{2} \rfloor + 1}(t) v_{\lfloor \frac{n+1}{2} \rfloor}(t) dt$. For this strategy to work, it is thus necessary to consider $K \geq 2$. Indeed, the assumption $(\mathbf{H})_{\text{lin}, \mathbf{K}=1, 2}$ leads to $\max(A_k^1, A_k^2) > 0$. Thus, at least one of the assumptions $(\mathbf{H})_{\text{quad}, 1, 1}$, $(\mathbf{H})_{\text{quad}, 1, 2}$ is not verified.

Therefore, contrary to the single-input case [18], here, in the multi-input case, the quadratic term alone can recover small-time local controllability.

1.4 Main result and proof strategy

Our main result is the following one, i.e. the same conclusion as in Theorem 1.1, when replacing the assumption (1.4) related to the controllability of the linearized system, by the assumptions $(\mathbf{H})_{\text{lin}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$ related to a power series expansion of order two.

Theorem 1.3. *Let $p, m, K \in \mathbb{N}$, $n \in \mathbb{N}^*$ with $K \geq 2$ and $\lfloor \frac{n}{2} \rfloor \leq p$, μ_1, μ_2 satisfying $(\mathbf{H})_{\text{reg}}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$ and $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$. Then the bilinear Schrödinger equation (1.1) is $H_0^m((0, T), \mathbb{R})$ -small-time locally controllable in $H_{(0)}^{2(p+m)+3}(0, 1)$.*

Remark 1.4. *In particular, if μ_1 and μ_2 are $C^\infty([0, 1], \mathbb{R})$, then, for all $m \in \mathbb{N}$, the bilinear Schrödinger equation (1.1) is H_0^m -STLC, with targets in $H_{(0)}^{2(p+m)+3}(0, 1)$, when the condition $(\mathbf{H})_{\text{reg}}$ is satisfied.*

The assumption $(\mathbf{H})_{\text{lin}, \mathbf{K}, 2}$ and the linear test prove that $\mathbb{P}_{\mathcal{H}} \psi(t)$ is $H_0^m((0, T), \mathbb{R})$ -small-time locally controllable in $H_{(0)}^{2(p+m)+3}(0, 1)$ (see Theorem 3.8), where

$$\mathcal{H} := \overline{\text{Span}_{\mathbb{C}}(\varphi_j, j \in \mathbb{N}^* \setminus \{K\})}, \quad (1.12)$$

and the orthogonal projection on \mathcal{H} , which is given by

$$\mathbb{P}_{\mathcal{H}} : \begin{array}{l} L^2(0, 1) \rightarrow \mathcal{H} \\ \psi \mapsto \psi - \langle \psi, \varphi_K \rangle \varphi_K \end{array}$$

Then, the proof of Theorem 1.3 is divided in two steps: given a time $T > 0$, a target ψ_f and $z \in \mathbb{C}$,

- we find $T_1 < T$ and controls (u, v) such that $\langle \psi(T_1; (u, v), \varphi_1), \psi_K(T_1) \rangle \sim z$,

- we apply on (T_1, T) a control such that $\mathbb{P}_{\mathcal{H}}\psi(T) = \mathbb{P}_{\mathcal{H}}\psi_f$,
- we prove that, despite the second step, $\langle \psi(T; (u, v), \varphi_1), \psi_K(T) \rangle$ stays close to z , so that a fixed point argument allows to find $z = z(\psi_f)$ so that $\langle \psi(T; (u, v), \varphi_1), \psi_K(T) \rangle = \langle \psi_f, \psi_K(T) \rangle$.

Finally $\psi(T; (u, v), \varphi_1) = \psi_f$.

For the step 1, we examine the quadratic term of $(u, v) \mapsto \langle \psi(t; (u, v), \varphi_1), \psi_K(t) \rangle$: it contains terms of the form (1.11).

1.5 State of the art

1.5.1 Topological obstructions to exact controllability

In [1], Ball, Marsden and Slemrod proved obstructions to local exact controllability of linear PDEs with bilinear controls. For instance, if the multiplicative operators μ_ℓ are bounded on $H_{(0)}^s(0, 1)$, then system (1.1) is not exactly controllable in $\mathcal{S} \cap H_{(0)}^s(0, 1)$, with controls $u, v \in L_{loc}^p(\mathbb{R}, \mathbb{R})$ and $p > 1$. The fundamental reason behind is that, under these assumptions, the reachable set has empty interior in $H_{(0)}^s(0, 1)$. The case of L_{loc}^1 -controls ($p = 1$) was incorporated in [19] and extensions to nonlinear equations were proved in [25, 26]. After the important work [1], different notions of controllability were studied for system (1.10) such as

- *exact controllability* in more regular spaces, on which μ_ℓ do not define bounded operators: note that in Theorem 1.1, for $m = 0$, μ_ℓ do not define bounded operators on $H_{(0)}^{2p+3}(0, 1)$ because $\mu_\ell^{(2p+1)}|_{\{0,1\}} \neq 0$ (see (1.6), because of assumption (1.4)).
- *approximate controllability*.

1.5.2 Exact controllability in more regular spaces, by linear test

For system (1.10), local exact controllability was first proved in [2, 4] with Nash-Moser techniques, to deal with an apparent derivative loss problem, and then in [7] with a classical inverse mapping theorem, thanks to a regularizing effect. By grafting other ingredients onto this core strategy, global (resp. local) exact controllability in regular spaces was proved for different models in [40, 42] (resp. [16]).

1.5.3 Single-input systems: quadratic obstructions to STLC

In [27], Coron denied the L^∞ -small-time local controllability result for system (1.10), with a particular dipolar moment μ_ℓ , thanks to a drift. In [13], Beauchard and Morancey gave general assumptions on μ_ℓ to deny L^∞ -STLC of (1.10). In [18], Bournissou proved that this drift also occurs with small control in $W^{-1,\infty}$. Quadratic terms have also been used to create obstructions to the controllability of other single-input systems, for example in [10] for ODEs, in [37] for the Burgers' equation, [11] for the heat equation, [30] for KdV, and [43] for KdV-Neumann.

With the exception of a non-physical PDE designed for - see [11, Section 5 and 6], for single-input systems, the quadratic terms generally do not recover small-time controllability.

1.5.4 Single-input systems: power series expansion of order 2 or 3

Nevertheless, a power series expansion of order 2 allows us to recover a direction lost **in large time**: this strategy is used in [13] for (1.10). This method is also used for other equations, such as KdV, in [22]. If the order 2 vanishes, a power series of order 3 can be used to recover the STLC, for example in [29], for KdV. If the order 2 doesn't cancel out, but the term of order 3 is strong enough, this expansion can also give the STLC (see [16], for equation (1.10)).

1.5.5 Large time and small-time approximate controllability

The first results of global approximate controllability of bilinear Schrödinger equations were obtained in large time (see [14, 23, 34, 41, 44]). For particular systems, a large time is indeed necessary for the approximate controllability (see [5, 6]).

Small-time approximate controllability between eigenstates for Schrödinger equations on the torus is proved by Duca and Nersesyan in [31], by means of an infinite-dimensional geometric control approach (saturation argument). Related results have been subsequently established in [15, 24, 33].

1.5.6 Multi-input ODE: Lie brackets technics for a multi-input ODE

In Section 2, we propose a toy model: a finite-dimensional control-affine system. Thus, we give a new proof of a particular case of the Sussmann's $S(\theta)$ condition (see [45]). This proof relies on a new representation formula of the state (inspired by the Magnus formula), in terms of Lie brackets. We use the same quadratic brackets as in [20]. We will discuss these articles in more detail later in the paper.

1.6 Small-time continuously approximately reachable vector

In this section, we recall a result from [16] that will be a key tool in this article.

Definition 1.5 (Concatenation). *Let $0 < T_1 < T_2$, $u : [0, T_1] \rightarrow \mathbb{R}$ and $\tilde{u} : [0, T_2] \rightarrow \mathbb{R}$. One defines the function $u\#\tilde{u} : [0, T_1 + T_2] \rightarrow \mathbb{R}$ as*

$$u\#\tilde{u} := u\mathbb{1}_{(0, T_1)} + \tilde{u}(\cdot - T_1)\mathbb{1}_{(T_1, T_1 + T_2)}.$$

Let $(X, \|\cdot\|_X)$ be a Banach space over \mathbb{R} . Let $(E_T, \|\cdot\|_{E_T})$ be a family of normed vector spaces of functions defined on $[0, T]$ for $T > 0$. Assume that for all $T_1, T_2 > 0$, for all $u \in E_{T_1}$, $\tilde{u} \in E_{T_2}$, $u\#\tilde{u} \in E_{T_1 + T_2}$ and the following inequality holds

$$\|u\#\tilde{u}\|_{E_{T_1 + T_2}} \leq \|u\|_{E_{T_1}} + \|\tilde{u}\|_{E_{T_2}}.$$

Finally, let $(\mathcal{F}_T)_{T > 0}$ be a family of functions from $X \times E_T$ to X for $T > 0$. The goal is to prove that the map \mathcal{F}_T is locally onto. To do this, we use the following definition; this is an equivalent of the notion of tangent vector, but in infinite dimension.

Definition 1.6 (Small-time continuously approximately reachable vector). *A vector $\xi \in X$ is called a small-time continuously approximately reachable vector if there exists a continuous map $\Xi : [0, +\infty[\rightarrow X$ with $\Xi(0) = \xi$ such that for all $T > 0$, there exist $C, \rho, s > 0$ and a continuous map $z \in (-\rho, \rho) \mapsto u_z \in E_T$ such that,*

$$\forall z \in (-\rho, \rho), \quad \|\mathcal{F}_T(0, u_z) - z\Xi(T)\|_X \leq C|z|^{1+s} \quad \text{with} \quad \|u_z\|_{E_T} \leq C|z|^s.$$

Theorem 1.7. *Assume the following hypotheses hold.*

(A₁) *For all $T > 0$, $\mathcal{F}_T : X \times E_T \rightarrow X$ is of class \mathcal{C}^2 on a neighborhood of $(0, 0)$ with $\mathcal{F}_T(0, 0) = 0$.*

(A₂) *For all $x \in X$, $T \in \mathbb{R}_+ \mapsto d\mathcal{F}_T(0, 0) \cdot (x, 0) \in X$ can be continuously extended at zero with $d\mathcal{F}_0(0, 0) \cdot (x, 0) = x$.*

(A₃) *For all $T_1, T_2 > 0$, for all $x \in X$, for all $u \in E_{T_1}$ and $v \in E_{T_2}$,*

$$\mathcal{F}_{T_1 + T_2}(x, u\#v) = \mathcal{F}_{T_2}(\mathcal{F}_{T_1}(x, u), v).$$

(A₄) *The space $H := \text{Im}(d\mathcal{F}_T(0, 0) \cdot (0, \cdot))$ doesn't depend on time, is closed and of finite codimension n .*

(A₅) *There exists \mathcal{M} , a supplementary of H that admits a basis $(\xi_i)_{1 \leq i \leq n}$ of small-time continuously approximately reachable vectors.*

Then, for all $T > 0$, \mathcal{F}_T is locally onto from zero: for all $\eta > 0$, there exists $\delta > 0$ such that for all $x_f \in X$ with $\|x_f\|_X < \delta$, there exists $u \in E_T$ with $\|u\|_{E_T} < \eta$ such that

$$\mathcal{F}_T(0, u) = x_f.$$

Remark 1.8. Using a change of function argument, this theorem is also true around another equilibrium than $(0, 0)$.

1.7 Plan of the paper

The paper is organized as follow: in section 2, we present a proof of a theorem of STLC thanks to quadratic terms for control-affine system in finite dimension, as a toy-model for the bilinear Schrödinger equation. In section 3, we present classical properties about the bilinear Schrödinger equation. Finally, in the section 4, we give the proof of the main theorem of this article. Some elements of proof are developed in the appendix.

2 The finite-dimensional case

2.1 Framework and notations

One considers the control-affine system

$$x'(t) = f_0(x(t)) + u(t)f_1(x(t)) + v(t)f_2(x(t)), \quad (2.1)$$

where the state $x(t) \in \mathbb{R}^d$, the controls are scalar-input functions $u(t), v(t) \in \mathbb{R}$, and f_0, f_1 and f_2 are analytic vector fields on a neighborhood of 0 in \mathbb{R}^d , such that $f_0(0) = 0$. The last hypothesis ensures that 0 is an equilibrium of the free control-affine system (i.e. with $(u, v) \equiv 0$).

For each $T > T_1 > 0$, $u, v \in L^1((T_1, T), \mathbb{R})$, there exists a unique maximal mild solution to (2.1) with initial data p at time T_1 , which we will denote by $x(\cdot; (u, v), p)$. We will sometimes note $x(\cdot; (u, v), p, T_1)$. As we are interested in small-time, and small controls, the solution is well-defined up to time T . The following concepts were introduced by Beauchard and Marbach in [10], in 2018.

Definition 2.1 ($W_0^{m, \infty}$ -STLC). *Let $m \in \mathbb{N}$. We say that system (2.1) is $W_0^{m, \infty}$ -STLC when, for every $T, \varepsilon > 0$, there exists $\delta > 0$, such that, for every $x_f \in B(0, \delta)$, there exist $u, v \in W_0^{m, \infty}((0, T), \mathbb{R})$ with $\|(u, v)\|_{W^{m, \infty}} \leq \varepsilon$ and $x(T; (u, v), 0) = x_f$.*

Definition 2.2 (Smooth-STLC). *The system (2.1) is smooth-STLC if it is $W_0^{m, \infty}$ -STLC, for every $m \in \mathbb{N}$.*

We use the definitions and notations of Beauchard and Marbach in [12]. Let $X := \{X_0, X_1, X_2\}$ be a set of three non-commutative indeterminates.

Definition 2.3 (Free algebra). *We consider $\mathcal{A}(X)$ the free algebra generated by X over the field \mathbb{R} , i.e. the unital associative algebra of polynomials of the indeterminates X_0, X_1 and X_2 .*

Definition 2.4 (Free Lie algebra). *For $a, b \in \mathcal{A}(X)$, one defines the Lie bracket of a and b as $[a, b] := ab - ba$, also called $ad_a(b)$. One defines by induction on $n \in \mathbb{N}$, $ad_a^{n+1}(b) = [a, ad_a^n(b)]$. This operation is anti-symmetric and satisfies the Jacobi identity: for all $a, b, c \in \mathcal{A}(X)$,*

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0.$$

Let $\mathcal{L}(X)$ be the free Lie algebra generated by X over the field \mathbb{R} , i.e. the smallest linear subspace of $\mathcal{A}(X)$ containing X and stable by the Lie bracket $[\cdot, \cdot]$.

Definition 2.5 (Iterated brackets). Let $\text{Br}(X)$ be the free magma over X , i.e. the set of iterated brackets of elements of X , defined by induction as: $X_0, X_1, X_2 \in \text{Br}(X)$ and if $a, b \in \text{Br}(X)$, then the ordered pair (a, b) belongs to $\text{Br}(X)$.

There is a natural evaluation mapping E from $\text{Br}(X)$ to $\mathcal{L}(X)$ defined by induction by $E(X_i) := X_i$ for $i = 0, 1, 2$ and $E((a, b)) := [E(a), E(b)]$.

Definition 2.6 (Length and homogeneous layers within $\mathcal{L}(X)$). For $b \in \text{Br}(X)$, $|b|$ denotes the length of b . For $i \in \llbracket 0, 2 \rrbracket$, for $b \in \text{Br}(X)$, $n_i(b)$ denotes the number of occurrences of the indeterminate X_i in b . We will use the notation: $n(b) = n_1(b) + n_2(b) = |b| - n_0(b)$. For $i \in \mathbb{N}$, $S_i(X)$ is the vector subspace of $\mathcal{L}(X)$ defined by

$$S_i(X) := \text{Span}\{E(b), b \in \text{Br}(X), n(b) = i\}.$$

Definition 2.7 (Left and right factors). For $b \in \text{Br}(X)$ with $|b| > 1$, b can be written in a unique way as $b = (b_1, b_2)$, with $b_1, b_2 \in \text{Br}(X)$. We use the notations $\lambda(b) = b_1$ and $\mu(b) = b_2$, which define maps $\lambda, \mu : \text{Br}(X) \setminus X \rightarrow \text{Br}(X)$.

Definition 2.8 (Bracket integration $b0^\nu$). For $b \in \text{Br}(X)$ and $\nu \in \mathbb{N}$, we use the unconventional short-hand $b0^\nu$ to denote the right-iterated bracket $(\dots(b, X_0), \dots, X_0)$, where X_0 appears ν times.

Definition 2.9 (Lie bracket of vector fields). Let $f, g : \Omega \rightarrow \mathbb{R}^d$ be two smooth vector fields on an open subset Ω of \mathbb{R}^d . One defines

$$[f, g] : x \in \Omega \mapsto Dg_x f(x) - Df_x g(x). \quad (2.2)$$

Definition 2.10 (Evaluated Lie bracket). Let f_0, f_1, f_2 be $\mathcal{C}^\infty(\Omega, \mathbb{R}^d)$ vector fields on an open subset Ω of \mathbb{R}^d and $f = \{f_0, f_1, f_2\}$.

For $b \in \mathcal{L}(X)$, we define $f_b := \Lambda(b)$, where $\Lambda : \mathcal{L}(X) \rightarrow \mathcal{C}^\infty(\Omega, \mathbb{R}^d)$ is the unique homomorphism of Lie algebras such that $\Lambda(X_i) = f_i$, for $i \in \llbracket 0, 2 \rrbracket$.

We will write f_b instead of $f_{E(b)}$ when $b \in \text{Br}(X)$. Finally, for $\mathcal{N} \subset \text{Br}(X)$, we use the notation

$$\mathcal{N}(f)(0) := \text{Span}\{f_b(0), b \in \mathcal{N}\} \subseteq \mathbb{R}^d. \quad (2.3)$$

2.2 An adapted basis of the free Lie algebra

Definition 2.11 (Hall set). A Hall set is a subset \mathcal{B} of $\text{Br}(X)$ endowed with a total order $<$ such that

- $X \subset \mathcal{B}$,
- for all $b_1, b_2 \in \text{Br}(X)$, $(b_1, b_2) \in \mathcal{B}$ iff $b_1, b_2 \in \mathcal{B}$, $b_1 < b_2$ and either $b_2 \in X$ or $\lambda(b_2) \leq b_1$,
- for all $b_1, b_2 \in \mathcal{B}$ such that $(b_1, b_2) \in \mathcal{B}$ then $b_1 < (b_1, b_2)$.

Theorem 2.12 (Viennot, [36]). Let $\mathcal{B} \subset \text{Br}(X)$ be a Hall set. Then $E(\mathcal{B})$ is a basis of $\mathcal{L}(X)$.

For a Hall set \mathcal{B} and $A \subset \mathbb{N}$, we denote by \mathcal{B}_A :

$$\mathcal{B}_A := \{b \in \mathcal{B}, n(b) \in A\}.$$

When A is a singleton, we write \mathcal{B}_N instead of $\mathcal{B}_{\{N\}}$.

The definition of a Hall set is also an algorithm for its construction. Indeed, the subsets \mathcal{B}_N of a Hall set \mathcal{B} can be constructed by induction on N . One may start, for example, with $\mathcal{B}_0 = \{X_0\}$ and $\mathcal{B}_1 = \{X_1 0^{\nu_1}, X_2 0^{\nu_2}, \nu_1, \nu_2 \in \mathbb{N}\}$ with the following order

$$\forall k \in \mathbb{N}, \quad X_1 0^k < X_2 0^k < X_1 0^{k+1} < X_2 0^{k+1} < \dots < X_0.$$

which is compatible with the three axioms above. For $N \geq 2$, to find all Hall elements $b \in \mathcal{B}_N$ given $\mathcal{B}_{\llbracket 1, N-1 \rrbracket}$, one adds first all (a, b) with $a \in \mathcal{B}_{N-1}$, $b \in X$ and $a < b$. Then for each bracket $b = (b_1, b_2) \in \mathcal{B}_{\llbracket 1, N-1 \rrbracket}$, one adds all the (a, b) with $a \in \mathcal{B}_{N-n(b)}$ and $b_1 \leq a < b$. Finally, one inserts the newly generated elements of \mathcal{B}_N into an ordering, maintaining the condition that $a < (a, b)$. With this construction, one obtains the following statement.

Proposition 2.13. *There exists a Hall set \mathcal{B} such that X_0 is maximal,*

$$\mathcal{B}_1 = \{M_j^1 := X_1 0^j, \quad M_j^2 := X_2 0^j; \quad j \in \mathbb{N}\}, \quad (2.4)$$

and $\mathcal{B}_2 = \mathcal{B}_{2,good} \cup \mathcal{B}_{2,bad}$ with

$$\mathcal{B}_{2,bad} = \{W_{j,l}^1 := (M_{j-1}^1, M_j^1) 0^l, \quad W_{j,l}^2 := (M_{j-1}^2, M_j^2) 0^l; \quad j \in \mathbb{N}^*, l \in \mathbb{N}\}, \quad (2.5)$$

and

$$\mathcal{B}_{2,good} = \left\{ C_{j,l} := (-1)^j \left(M_{\lfloor \frac{j+1}{2} \rfloor}^1, M_{\lfloor \frac{j}{2} \rfloor}^2 \right) 0^l; \quad j, l \in \mathbb{N} \right\}. \quad (2.6)$$

We used the following notation.

Notation. For $b = (b_1, b_2) \in \text{Br}(X)$ and $j \in \mathbb{N}$, we use the notation $(-1)^j b$ as $(-1)^j b = (b_2, b_1)$ if j is odd and $(-1)^j b = b$ if j is even.

Remark 2.14. When $l = 0$, we will write W_j^1, W_j^2, C_j instead of $W_{j,0}^1, W_{j,0}^2$ and $C_{j,0}$.

2.3 A quadratic sufficient condition for STLC

The purpose of this section is to prove the following theorem.

Theorem 2.15. *Let $d, L \geq 1$, f_0, f_1, f_2 be analytic vector fields on a neighborhood of 0 in \mathbb{R}^d , such that $f_0(0) = 0$. Assume that $r := \dim(S_1(f)(0)) < d$ and that there exist brackets $b_{r+1}, \dots, b_d \in \mathcal{B}_{2,good}$ such that $\max_{i \in \llbracket r+1, d \rrbracket} |b_i| = L$ and*

$$S_1(f)(0) \oplus \text{Span}(f_{b_{r+1}}(0), \dots, f_{b_d}(0)) = \mathbb{R}^d. \quad (2.7)$$

One assumes that

$$\text{for all } b \in \mathcal{B}_{2,bad} \text{ such that } |b| \leq L, \quad f_b(0) \in S_1(f)(0). \quad (2.8)$$

Then, the control-affine system (2.1) is smooth-STLC.

Remark 2.16. The case $r = d$ corresponds to the linear test, see for instance [28], or [10, Theorem 1] for the smooth-STLC case.

A very simple system for which the theorem applies is the following one

$$\begin{cases} x_1' &= u \\ x_2' &= x_1 v \end{cases}.$$

Indeed, some computations on Lie brackets give: $f_{M_0^1}(0) = e_1$ and $f_{C_0}(0) = e_2$. Thus, the LARC (2.7) is verified. Moreover, $\{b \in \mathcal{B}_{2,bad}, |b| \leq |C_0|\} = \emptyset$. Then, the condition (2.8) is verified. Thus, the system is smooth-STLC.

Theorem 2.15 gives a sufficient condition for smooth-STLC; this condition is not necessary: indeed, consider the system

$$\begin{cases} x_1' &= u \\ x_2' &= v \\ x_3' &= x_1^2 + x_2^2 + \alpha x_1 x_2 \end{cases},$$

with $\alpha \in \mathbb{R}^*$. One can prove that $S_1(f)(0) = \text{Span}(e_1, e_2)$, and $f_{C_1}(0) = \alpha e_3$. Then, the LARC (2.7) is verified. Moreover, $f_{W_1^1}(0) = 2e_3 \notin S_1(f)(0)$, and $|W_1^1| = |C_1|$. Then, the hypothesis (2.8) is not verified. However, if $|\alpha| > 2$, one can prove that the system is smooth-STLC.

Theorem 2.15 is a corollary of Sussmann's $S(\theta)$ -condition (see Appendix A.3.1). We propose another proof of Theorem 2.15, relying on a representation formula of the state,

presented in Propositions 2.19 and 2.20. The advantage of this alternative proof strategy is that it can be adapted to the infinite dimensional framework of PDEs: the representation formula does not hold in its full generality for the state of the Schrödinger PDE, but its leading terms can be extracted from an expansion of the solution (see Proposition 4.1), so that we can conclude with a similar proof strategy.

In [20], the author is interested in particular linear and quadratic ODEs systems of the form $\begin{cases} x' = Ax + Bu, \\ w'_i = {}^t x D_i x, \end{cases} \quad 1 \leq i \leq r$, with D_i , well-chosen symmetric matrices. He proves a small-time global controllability result. For that, Brockett shows that the matrices D_i are associated with good brackets, see [20, Theorem 3.4]. The good quadratic Lie brackets involved are elements shaped as (M_ν^1, X_2) , $\nu \in \mathbb{N}$. They are linear combination of elements of our base of $\mathcal{B}_{2,good}$.

2.4 Some tools for the proof

Definition 2.17 (Coordinates of the second kind). *Let $\mathcal{B} \subset \text{Br}(X)$ be a Hall set. The coordinates of the second kind associated with \mathcal{B} is the unique family $(\xi_b)_{b \in \mathcal{B}}$ of functionals $\mathbb{R}_+ \times L_{\text{loc}}^1(\mathbb{R}_+; \mathbb{R})^2 \rightarrow \mathbb{R}$ defined by induction as: for every $t > 0$ and $u, v \in L^1((0, t), \mathbb{R})$,*

- $\xi_{X_0}(t) := t$, $\xi_{X_1}(t, u) := \int_0^t u(s) ds = u_1(t)$ and $\xi_{X_2}(t, v) := \int_0^t v(s) ds = v_1(t)$,
- for $b \in \mathcal{B} \setminus X$, there exists a unique couple (b_1, b_2) of elements of \mathcal{B} such that $b_1 < b_2$ and a unique maximal integer $m \geq 1$ with $b = \text{ad}_{b_1}^m(b_2)$, and then

$$\xi_b(t, (u, v)) := \frac{1}{m!} \int_0^t \xi_{b_1}^m(s, (u, v)) \dot{\xi}_{b_2}(s, (u, v)) ds. \quad (2.9)$$

With this definition, one immediately obtains the following expressions.

Proposition 2.18 (Coordinates of the second kind). *The following equalities hold.*

1. For every $j \in \mathbb{N}$, for every $t > 0$,

$$\xi_{M_j^1}(t, (u, v)) = u_{j+1}(t), \quad \xi_{M_j^2}(t, (u, v)) = v_{j+1}(t). \quad (2.10)$$

2. For every $j \in \mathbb{N}^*$, $l \in \mathbb{N}$, for every $t > 0$,

$$\begin{aligned} \xi_{W_{j,l}^1}(t, (u, v)) &= \frac{1}{2} \int_0^t \frac{(t-s)^l}{l!} u_j^2(s) ds, \\ \xi_{W_{j,l}^2}(t, (u, v)) &= \frac{1}{2} \int_0^t \frac{(t-s)^l}{l!} v_j^2(s) ds. \end{aligned} \quad (2.11)$$

3. For every $j, l \in \mathbb{N}$, for every $t > 0$,

$$\xi_{C_{j,l}}(t, (u, v)) = \int_0^t \frac{(t-s)^l}{l!} u_{\lfloor \frac{j}{2} \rfloor + 1}(s) v_{\lfloor \frac{j+1}{2} \rfloor}(s) ds. \quad (2.12)$$

Proof. The first two points are proved by Beauchard and Marbach in [12, Proposition 3.7]. For the last one, the definition by induction on ξ and some computations lead to the result. \square

Proposition 2.19 (Magnus formula). *Let $M \in \mathbb{N}$, $\delta, T > 0$, $f_0, f_1, f_2 : B(0, 5\delta) \rightarrow \mathbb{R}^d$ be analytic vector fields with $T \|f_0\|_{C^0} \leq \delta$. There exist $\gamma, C > 0$ such that, for every $u, v \in L^1((0, T), \mathbb{R}^d)$ with $\|u\|_{L^1}, \|v\|_{L^1} \leq \gamma$, $p \in \mathcal{B}(0, \delta)$ and $t \in [0, \gamma]$,*

$$\left\| x(t; (u, v), p, 0) - e^{\mathcal{Z}_M(t; f, (u, v))} e^{t f_0} p \right\| \leq C \|(u, v)\|_{L^1(0, t)}^{M+1}, \quad (2.13)$$

with

$$\mathcal{Z}_M(t; f, (u, v))(0) = \sum_{b \in \mathcal{B}_{[1, M]}} \eta_b(t, (u, v)) f_b(0),$$

where the series is absolutely convergent, and the following explicit expressions hold for every $t > 0$ and $u, v \in L^1(0, t)$:

1. if $b \in \mathcal{B}_1$, then $\eta_b(t, (u, v)) = \xi_b(t, (u, v))$,
2. if $b \in \mathcal{B}_{2, \text{good}}$, then

$$\eta_b(t, (u, v)) = \xi_b(t, (u, v)) + \sum_{j+k=n_0(b)} \gamma_{j,k}^b u_{j+1}(t) v_{k+1}(t), \quad (2.14)$$

where $\gamma_{j,k}^b \in \mathbb{R}$.

From this proposition, one can deduce the following one, thanks to Lemma A.15.

Corollary 2.20 (Magnus formula 2). *Let $M \in \mathbb{N}^*$, $\delta, T > 0$, $f_0, f_1, f_2 : B(0, 2\delta) \rightarrow \mathbb{R}^d$ be analytic vector fields with $f_0(0) = 0$ and $T \|f_0\|_\infty \leq \delta$. For $u, v \in L^1((0, T), \mathbb{R})$, as $\|(u, v)\|_{L^1} \rightarrow 0$,*

$$x(t; (u, v), 0) = \mathcal{Z}_M(t; f, (u, v))(0) + \mathcal{O}\left(\|(u, v)\|_{L^1(0,t)}^{M+1} + \|x(t; (u, v))\|^{1+\frac{1}{M}}\right). \quad (2.15)$$

These representation formulas rely on [9]. The path for obtaining it is presented in appendix.

Remark 2.21 (Homogeneity property). *Let $\bar{u}, \bar{v} \in L^1(0, 1)$, $\lambda, \mu \in \mathbb{R}$, $T > 0$, $u : t \in (0, T) \mapsto \lambda \bar{u}(\frac{t}{T})$ and $v : t \in (0, T) \mapsto \mu \bar{v}(\frac{t}{T})$. Then, for all $b \in \mathcal{B}$,*

$$\eta_b(T, (u, v)) = \lambda^{n_1(b)} \mu^{n_2(b)} T^{|b|} \eta_b(1, (\bar{u}, \bar{v})). \quad (2.16)$$

Lemma 2.22. *For every $M \in \mathbb{N}^*$, there exists $C_M > 0$ such that, for every $T \geq 0$, $u, v \in L^1((0, T), \mathbb{R})$, $b \in \mathcal{B}$ with $n(b) \leq M$ and $t \in [0, T]$,*

$$|\eta_b(t, (u, v))| \leq \frac{C_M}{|b|!} t^{n_0(b)} \|(u, v)\|_{L^1(0,t)}^{n(b)}. \quad (2.17)$$

Proof. This lemma is proved in [9, Proposition 52]. \square

2.5 A quadratic moment problem

To prove Theorem 2.15, we solve quadratic moment problems.

Proposition 2.23 (Moment problems). *Let B be a finite subset of $\mathcal{B}_1 \cup \mathcal{B}_{2, \text{good}}$ and $b_0 \in B \cap \mathcal{B}_{2, \text{good}}$. There exist $u, v \in C_c^\infty((0, 1), \mathbb{R})$ such that,*

$$\text{for every } b \in B, \quad \eta_b(1, (u, v)) = \delta_{b, b_0}. \quad (2.18)$$

Proof. One may assume that $B = \{M_\nu^1, M_\nu^2, C_{\mu, \nu}, (\mu, \nu) \in \llbracket 0, \mu^* \rrbracket \times \llbracket 0, \nu^* \rrbracket\}$. One considers $N = \mu^* + \nu^* + 1$. We are looking for u and v in the form of $u = \phi^{(N)}$ and $v = \psi^{(N)}$, with $\phi, \psi \in C_c^\infty((0, 1), \mathbb{R})$. Then, for every $\mu \in \llbracket 0, \mu^* \rrbracket$, $\xi_{M_\mu^1}(1, (u, v)) = u_{\mu+1}(1) = 0$. One recalls that for $b \in \mathcal{B}_1$, $\eta_b = \xi_b$. Thus, (2.18) is already verified for $b \in B \cap \mathcal{B}_1$.

The equation (2.14) gives: for all $b \in \mathcal{B}_{2, \text{good}} \cap B$, $\eta_b(1, (u, v)) = \xi_b(1, (u, v))$. Moreover, using (2.12), one obtains for all $(\mu, \nu) \in \llbracket 0, \mu^* \rrbracket \times \llbracket 0, \nu^* \rrbracket$,

$$\eta_{C_{\mu, \nu}}(1, (u, v)) = \int_0^1 \frac{(1-t)^\nu}{\nu!} u_{\lfloor \frac{\mu}{2} \rfloor + 1}(t) v_{\lfloor \frac{\mu+1}{2} \rfloor}(t) dt.$$

Using integration by parts, one has

$$\eta_{C_{\mu, \nu}}(1, (u, v)) = (-1)^{N - \lfloor \frac{\mu+1}{2} \rfloor} \int_0^1 \psi(t) \left(\frac{(1-\cdot)^\nu}{\nu!} \phi^{(N - \lfloor \frac{\mu}{2} \rfloor - 1)} \right)^{(N - \lfloor \frac{\mu+1}{2} \rfloor)}(t) dt.$$

Using the Leibniz formula, one gets: $\eta_{C_{\mu,\nu}}(1, (u, v)) = \int_0^1 \psi(t) f_{\mu,\nu}(t) dt$, with

$$f_{\mu,\nu} : t \in (0, 1) \mapsto \sum_{k=0}^{\min(N - \lfloor \frac{\mu+1}{2} \rfloor, \nu)} \binom{N - \lfloor \frac{\mu+1}{2} \rfloor}{k} \frac{(-1)^{k+N - \lfloor \frac{\mu+1}{2} \rfloor}}{(\nu - k)!} (1-t)^{\nu-k} \phi^{(2N-1-\mu-k)}(t).$$

Step 1: There exists $\phi \in C_c^\infty((0, 1), \mathbb{R})$ s.t. the family $\mathcal{F} := (f_{\mu,\nu})_{\substack{0 \leq \mu \leq \mu^ \\ 0 \leq \nu \leq \nu^*}}$ is linearly independent on $(0, 1)$. Let $m \geq \nu^* + 2$ and $\chi \in C_c^\infty((0, 1), \mathbb{R})$ such that $\chi \equiv 1$ on $(\frac{1}{4}, \frac{3}{4})$. One defines $\phi : t \mapsto e^{t^m} \chi(t) \in C_c^\infty((0, 1), \mathbb{R})$. By induction, one can prove that: for all $l \in \mathbb{N}$, there exists $P_l \in \mathbb{R}[X]$ with $\deg(P_l) = l(m-1)$ such that, for all $t \in (\frac{1}{4}, \frac{3}{4})$, $\phi^{(l)}(t) = e^{t^m} P_l(t)$. Then,*

$$\forall (\mu, \nu) \in \llbracket 0, \mu^* \rrbracket \times \llbracket 0, \nu^* \rrbracket, \quad \deg \left(f_{\mu,\nu}(t) e^{-t^m} \Big|_{(\frac{1}{4}, \frac{3}{4})} \right) = \nu + (m-1)(2N-1-\mu). \quad (2.19)$$

Thanks to the choice of m , all the degrees (2.19) are different. Then, $(t \mapsto f_{\mu,\nu}(t) e^{-t^m})_{\substack{0 \leq \mu \leq \mu^* \\ 0 \leq \nu \leq \nu^*}}$ is linearly independent on $(0, 1)$. Consequently, one obtains the result for \mathcal{F} .

Step 2: Construction of u and v . One chooses $u = \phi^{(N)}$, with the function ϕ obtained at step 1. By construction of ϕ , there exists $\psi \in C_c^\infty((0, 1), \mathbb{R})$, solution to the following moment problems

$$\text{for every } (\mu, \nu) \in \llbracket 0, \mu^* \rrbracket \times \llbracket 0, \nu^* \rrbracket, \quad \int_0^1 \psi(t) f_{\mu,\nu}(t) dt = \delta_{C_{\mu,\nu}, b_0}.$$

We conclude with $v := \psi^{(N)}$. □

2.6 Tangent vector

Let $m \geq 0$, we prove that the system is $W_0^{m,\infty}$ -STLC. Taking into account the assumption (2.7), we can consider \mathbb{P} , the linear projection on $\text{Span}(f_{b_i}(0))_{r+1 \leq i \leq d}$ parallel to $S_1(f)(0)$. The purpose of this section is to prove:

Proposition 2.24. *Let $j \in \llbracket r+1, d \rrbracket$. The vector $f_{b_j}(0)$ is small-time $W_0^{m,\infty}$ -continuously approximately reachable vector associated with vector variations $e^{\frac{1}{2}H_0} f_{b_j}(0)$.*

The proof is divided in two steps. In the first one, we prove that the system can move in the direction lost at the linear order $f_{b_j}(0)$, in projection.

Proposition 2.25. *Let $j \in \llbracket r+1, d \rrbracket$. Let $q_j := \lfloor \frac{n_0(b_j)}{2} \rfloor$, $s_j := \frac{1}{4(|b_j|+m)}$, $\alpha_j = \frac{3}{8} + \frac{m}{8(|b_j|+m)}$ and $\delta_j := \alpha_j + (q_j + 2)s_j$. For all $T_1 > 0$, there exist $C, \rho > 0$, and a continuous map $z \in \mathbb{R} \mapsto (u_z, v_z) \in W_0^{m,\infty}(0, T_1)^2$ such that*

$$\forall z \in (-\rho, \rho), \quad \|\mathbb{P}(x(T_1; (u_z, v_z), 0, 0)) - z f_{b_j}(0)\| \leq C|z|^{1+s_j}. \quad (2.20)$$

The size of the controls is given by: for all $k \geq 0$, for all $r \in [1, +\infty]$, there exists $C > 0$, such that

$$\forall z \in (-\rho, \rho), \quad \|u_z\|_{W^{k,r}}, \|v_z\|_{W^{k,r}} \leq C|z|^{\alpha_j + s_j(\frac{1}{r} - k)}. \quad (2.21)$$

Finally, for all $z \in (-\rho, \rho)$,

$$\|x(T_1; (u_z, v_z), 0, 0)\| \leq C|z|^{\delta_j}. \quad (2.22)$$

Note that $\delta_j > \frac{1}{2}$.

Proof. Let $T_1 > 0$ and $\rho = T_1^{\frac{1}{s_j}}$. One defines the brackets set $B = \{M_j^1, M_j^2, j \in \llbracket 0, q_j \rrbracket\} \cup \{b \in \mathcal{B}_{2,good}, |b| \leq L\}$, with $L := \max_{i \in \llbracket r+1, d \rrbracket} |b_i|$. Thanks to Proposition 2.23, we consider $\bar{u}, \bar{v} \in C_c^\infty(0, 1)$ such that $\eta_b(1, (\bar{u}, \bar{v})) = \delta_{b, b_j}$, for all $b \in B$. We define, for $z \in \mathbb{R}^*$,

$$u_z, v_z : t \in [0, T_1] \mapsto \text{sgn}(z) |z|^{\alpha_j} \bar{u} \left(\frac{|z|^{s_j} - T_1 + t}{|z|^{s_j}} \right), |z|^{\alpha_j} \bar{v} \left(\frac{|z|^{s_j} - T_1 + t}{|z|^{s_j}} \right).$$

For all $z \in (-\rho, \rho)$, one has

$$\text{Supp} \left((u_z)_{q_j+1} \right), \text{Supp} \left((v_z)_{q_j+1} \right) \subset (T_1 - |z|^{s_j}, T_1) \subset (0, T_1). \quad (2.23)$$

For all $k \geq 0$, $r \in [1, +\infty[$, using the Poincaré's inequality, the support condition (2.23), and a change of variables, we get for all $z \in (-\rho, \rho) \setminus \{0\}$,

$$\|u_z\|_{W^{k,r}}^r \leq C \int_0^1 \left| |z|^{\alpha_j - ks_j} \bar{u}^{(k)}(\sigma) \right|^r |z|^{s_j} d\sigma \leq C |z|^{(\alpha_j - ks_j)r + s_j} \left\| \bar{u}^{(k)} \right\|_{L^r}^r.$$

The inequality with $r = +\infty$ is proved in the same way and we get (2.21).

We prove (2.22); we estimate the linear term. Note that, for all $z \in (-\rho, \rho)$, for all $i \in \{1, 2\}$, for all $k \in \mathbb{N}$, the homogeneity property (2.16) gives

$$\begin{aligned} \eta_{M_k^i}(T_1, (u_z, v_z)) &= \eta_{M_k^i} \left(|z|^{s_j}, \left(\text{sgn}(z) |z|^{\alpha_j} \bar{u} \left(\frac{\cdot}{|z|^{s_j}} \right), |z|^{\alpha_j} \bar{v} \left(\frac{\cdot}{|z|^{s_j}} \right) \right) \right) \\ &= \text{sgn}(z)^{\delta_{1,i}} |z|^{\alpha_j + |M_k^i| s_j} \eta_{M_k^i}(1, (\bar{u}, \bar{v})). \end{aligned} \quad (2.24)$$

Consequently, the definition of (\bar{u}, \bar{v}) and (2.24) gives: for all $z \in (-\rho, \rho)$, for all $i \in \{1, 2\}$, for all $k \in \llbracket 0, q_j \rrbracket$, $\eta_{M_k^i}(T_1, (u_z, v_z)) = 0$ and,

$$\mathcal{Z}_1(T_1; f, (u_z, v_z))(0) = \sum_{i=1}^2 \sum_{k=q_j+1}^{+\infty} \eta_{M_k^i}(T_1, (u_z, v_z)) f_{M_k^i}(0) = \mathcal{O} \left(|z|^{\alpha_j + (q_j+2)s_j} \right). \quad (2.25)$$

The Magnus formula (2.15) with $M = 1$ leads to

$$x(T_1; (u_z, v_z), 0, 0) = \mathcal{Z}_1(T_1; f, (u_z, v_z))(0) + \mathcal{O} \left(\|(u_z, v_z)\|_{L^1}^2 + \|x(T_1; (u_z, v_z), 0, 0)\|^2 \right).$$

Using the size of the controls given by (2.21) with $(k, r) = (0, 1)$ and (2.25), one has:

$$x(T_1; (u_z, v_z), 0, 0) = \mathcal{O} \left(|z|^{\delta_j} + |z|^{2\alpha_j + 2s_j} + \|x(T_1; (u_z, v_z), 0, 0)\|^2 \right).$$

As $\frac{1}{2} < \delta_j \leq 2\alpha_j + 2s_j$, one gets (2.22).

Now, we prove (2.20). By definition,

$$\mathcal{Z}_2(T_1; f, (u_z, v_z))(0) = \mathcal{Z}_1(T_1; f, (u_z, v_z))(0) + \sum_{b \in \mathcal{B}_2} \eta_b(T_1, (u_z, v_z)) f_b(0) = \mathcal{O} \left(|z|^{\delta_j} \right), \quad (2.26)$$

using (2.25), the estimates (2.17) and (2.21) with $(k, r) = (0, 1)$. Then, using the map \mathbb{P} , the hypothesis (2.8) on the short brackets of $\mathcal{B}_{2,bad}$ and the homogeneity property (2.16), we obtain

$$\mathbb{P} \left(\mathcal{Z}_2(T_1; f, (u_z, v_z))(0) \right) = \sum_{\substack{b \in \{\mathcal{B}_{2,good}, |\mathbf{b}| \leq L\} \\ \cup \{\mathcal{B}_{2,bad}, |\mathbf{b}| > L\}}} \text{sgn}(z)^{n_1(b)} |z|^{2\alpha_j + |\mathbf{b}| s_j} \eta_b(1, (\bar{u}, \bar{v})) \mathbb{P} \left(f_b(0) \right).$$

Using the definition of (\bar{u}, \bar{v}) and the fact that $2\alpha_j + |b_j| s_j = 1$, one gets

$$\mathbb{P} \left(\mathcal{Z}_2(T_1; f, (u_z, v_z))(0) \right) = z f_{b_j}(0) + \mathcal{O} \left(|z|^{1+s_j} \right). \quad (2.27)$$

Then, using the Magnus representation formula (2.13) with $M = 2$, Lemma A.15, and the projection \mathbb{P} , one has

$$\begin{aligned} \mathbb{P} \left(x(T_1; (u_z, v_z), 0, 0) \right) &= \mathbb{P} \left(\mathcal{Z}_2(T_1; f, (u_z, v_z))(0) \right) + \\ &\mathcal{O} \left(\|\mathcal{Z}_2(T_1; f, (u_z, v_z))(0)\|_{\infty} \|D\mathcal{Z}_2(T_1; f, (u_z, v_z))\|_{\infty} + \|(u_z, v_z)\|_{L^1}^3 \right). \end{aligned} \quad (2.28)$$

Finally, note that

$$DZ_2(T_1; f, (u_z, v_z)) = \sum_{b \in \mathcal{B}_{[1,2]}} \eta_b(T_1, (u_z, v_z)) Df_b.$$

Then, with the same steps than for (2.26), one gets

$$\|DZ_2(T_1; f, (u_z, v_z))\|_\infty = \mathcal{O}(|z|^{\delta_j}). \quad (2.29)$$

Finally, using (2.26), (2.27), (2.29) and (2.21) with $(k, r) = (0, 1)$ in (2.28), one obtains (2.20), as $1 + s_j \leq 2\delta_j \leq 3\alpha_j + 3s_j$. \square

One defines $H_0 = Df_0(0)$. In the second step of the proof, we correct the linear part of the solution.

Proposition 2.26. *Let $j \in \llbracket r+1, d \rrbracket$. For all $T_1 + 1 > T > T_1 > 0$, there exist $C, \rho > 0$ and a continuous map $z \mapsto (U_z, V_z) \in W_0^{m, \infty}(0, T)^2$ such that*

$$\forall z \in (-\rho, \rho), \quad \left\| x(T; (U_z, V_z), 0, 0) - ze^{(T-T_1)H_0} f_{b_j}(0) \right\| \leq C|z|^{1+s_j}, \quad (2.30)$$

with the following estimate on the family of controls

$$\|U_z\|_{W^{m, \infty}}, \|V_z\|_{W^{m, \infty}} \leq C|z|^{\frac{1}{4}}. \quad (2.31)$$

Proof. Let $0 < T_1 < T < T_1 + 1$, one defines, for $z \in \mathbb{R}$,

$$U_z, V_z := u_z \mathbb{1}_{[0, T_1]} + \tilde{u}_z \mathbb{1}_{[T_1, T]}, v_z \mathbb{1}_{[0, T_1]} + \tilde{v}_z \mathbb{1}_{[T_1, T]},$$

where $u_z, v_z \in W_0^{m, \infty}(0, T_1)$ are the controls defined in the previous proposition, and $\tilde{u}_z, \tilde{v}_z \in W_0^{m, \infty}(T_1, T)$ are the controls that correct the linear part of the solution, i.e. the controls that drive the solution from $x(T_1; (u_z, v_z), 0, 0)$ to $\mathbb{P}_{S_1(f)(0)}(ze^{(T-T_1)H_0} f_{b_j}(0))$ in projection on $S_1(f)(0)$, that is to say

$$\mathbb{P}_{S_1(f)(0)}(x(T; (\tilde{u}_z, \tilde{v}_z), x(T_1; (u_z, v_z), 0, 0), T_1)) = \mathbb{P}_{S_1(f)(0)}(ze^{(T-T_1)H_0} f_{b_j}(0)).$$

Note that $\mathbb{P}_{S_1(f)(0)} = I - \mathbb{P}$. Then,

$$\|(\tilde{u}_z, \tilde{v}_z)\|_{W^{m, \infty}} \leq C \left(\|x(T_1; (u_z, v_z), 0, 0)\| + |z| \left\| e^{(T-T_1)H_0} f_{b_j}(0) \right\| \right) \leq C|z|^{\delta_j}, \quad (2.32)$$

the last estimate is given by the (2.22). Moreover, the control estimate (2.21) with $(k, r) = (m, \infty)$ gives

$$\|(u_z, v_z)\|_{W^{m, \infty}} \leq C|z|^{\alpha_j - ms_j} \leq C|z|^{\frac{1}{4}}. \quad (2.33)$$

The equations (2.32) and (2.33) give (2.31), as $\delta_j > \frac{1}{2}$. Furthermore, by construction,

$$\begin{aligned} \left\| x(T; (U_z, V_z), 0, 0) - ze^{(T-T_1)H_0} f_{b_j}(0) \right\| &= \\ & \left\| \mathbb{P} \left(x(T; (U_z, V_z), 0, 0) - ze^{(T-T_1)H_0} f_{b_j}(0) \right) \right\|. \end{aligned} \quad (2.34)$$

We apply Lemma A.19 with $p = x(T_1; (u_z, v_z), 0, 0)$ and $(u, v) = (\tilde{u}_z, \tilde{v}_z)$ to obtain

$$\begin{aligned} x(T; (U_z, V_z), 0, 0) &= x(T; (0, 0), x(T_1; (u_z, v_z), 0, 0), T_1) + x(T; (\tilde{u}_z, \tilde{v}_z), 0, T_1) \\ &+ \mathcal{O} \left(\|x(T_1; (u_z, v_z), 0, 0)\| \|(\tilde{u}_z, \tilde{v}_z)\|_{L^\infty} + \|x(T_1; (u_z, v_z), 0, 0)\|^2 \right). \end{aligned} \quad (2.35)$$

The equations (2.22), (2.32), (2.34), and (2.35) give:

$$\begin{aligned} \left\| x(T; (U_z, V_z), 0, 0) - ze^{(T-T_1)H_0} f_{b_j}(0) \right\| &\leq \left\| \mathbb{P}(x(T; (\tilde{u}_z, \tilde{v}_z), 0, T_1)) \right\| + \\ & \left\| \mathbb{P} \left(x(T; (0, 0), x(T_1; (u_z, v_z), 0, 0), T_1) - ze^{(T-T_1)H_0} f_{b_j}(0) \right) \right\| + \mathcal{O}(|z|^{2\delta_j}). \end{aligned} \quad (2.36)$$

We designate the second term of the right-hand side of (2.36) by A , and Lemma A.18 gives with $p = x(T_1; (u_z, v_z), 0, 0)$

$$A = \left\| \mathbb{P} \left(e^{(T-T_1)H_0} x(T_1; (u_z, v_z), 0, 0) - z e^{(T-T_1)H_0} f_{b_j}(0) \right) \right\| + \mathcal{O} \left(\|x(T_1; (u_z, v_z), 0, 0)\|^2 \right).$$

For all $b \in \mathcal{B}_1$, $H_0 f_b(0) = Df_0(0) f_b(0) - Df_b(0) f_0(0) = f_{b0}(0)$, with $b0 \in \mathcal{B}_1$. Then, $S_1(f)(0)$ is stable by $e^{(T-T_1)H_0}$. Consequently,

$$e^{(T-T_1)H_0} (I - \mathbb{P}) (x(T_1; (u_z, v_z), 0, 0)) \in S_1(f)(0) = \ker(\mathbb{P}),$$

and, with (2.22), we obtain

$$\begin{aligned} A &= \left\| \mathbb{P} \left(e^{(T-T_1)H_0} \mathbb{P} (x(T_1; (u_z, v_z), 0, 0)) - z e^{(T-T_1)H_0} f_{b_j}(0) \right) \right\| + \mathcal{O}(|z|^{2\delta_j}), \\ A &= \mathcal{O} \left(\left\| \mathbb{P} (x(T_1; (u_z, v_z), 0, 0)) - z f_{b_j}(0) \right\| + |z|^{2\delta_j} \right). \end{aligned}$$

Finally, the equation (2.20) gives

$$A = \mathcal{O}(|z|^{1+s_j}), \quad (2.37)$$

as $1 + s_j \leq 2\delta_j$. To obtain (2.30), we finally estimate the first term of the right-hand side of (2.36), let's say B . Using once again the Magnus formula (2.13) with $M = 2$, Lemma A.15, and the projection \mathbb{P} , one has

$$\begin{aligned} \mathbb{P} (x(T; (\tilde{u}_z, \tilde{v}_z), 0, T_1)) &= \mathbb{P} (\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z))(0)) + \\ &\mathcal{O} \left(\|\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z))(0)\|_\infty \|D\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z))\|_\infty + \|(\tilde{u}_z, \tilde{v}_z)\|_{L^1}^3 \right). \end{aligned} \quad (2.38)$$

Again, (2.17) and (2.32) gives

$$\begin{aligned} \|\mathbb{P} (\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z)))\|_\infty &= \mathcal{O} \left(\|(\tilde{u}_z, \tilde{v}_z)\|_{L^1}^2 \right) = \mathcal{O}(|z|^{2\delta_j}), \\ \|\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z))\|_\infty, \|D\mathcal{Z}_2(T; f, (\tilde{u}_z, \tilde{v}_z))\|_\infty &= \mathcal{O}(\|(\tilde{u}_z, \tilde{v}_z)\|_{L^1}) = \mathcal{O}(|z|^{\delta_j}). \end{aligned}$$

Therefore, thanks to the estimates (2.38), one has:

$$B = \mathcal{O} \left(|z|^{2\delta_j} + \|(\tilde{u}_z, \tilde{v}_z)\|_{L^1}^3 \right) = \mathcal{O}(|z|^{1+s_j}), \quad (2.39)$$

as $1 + s_j \leq 2\delta_j \leq 3\alpha_j + 3s_j$. To conclude, the equations (2.36), (2.37) and (2.39) lead to the (2.30). \square

We are now in a position to write the proof of the main result of this section.

Proof of Proposition 2.24. One considers $T > 0$, and $T_1 := \frac{T}{2}$. Then, the previous proposition gives the result. \square

2.7 Proof of Theorem 2.15

Now, we can write the proof of Theorem 2.15.

Proof of Theorem 2.15. We use Theorem 1.7. More precisely, we consider $X = \mathbb{R}^d$ and $E_T = W_0^{m, \infty}(0, T)^2$. Moreover, for all $T > 0$,

$$\mathcal{F}_T : (0, (u, v)) \mapsto x(T; (u, v), 0).$$

We have to check all the assumptions to obtain the controllability result.

(A₁) This is well-known that the end-point map is regular around the equilibrium.

(A₂) The differential at $(0, (0, 0))$ is given $d\mathcal{F}_T(0, (0, 0))(x_0, (\bar{u}, \bar{v})) = Y(T)$ where Y is the solution to the linearized system

$$\begin{cases} Y'(t) &= Df_0(0)Y(t) + \bar{u}(t)f_1(0) + \bar{v}(t)f_2(0) \\ Y(0) &= x_0 \end{cases}.$$

Then, for all $x_0 \in \mathbb{R}^d$, $T \mapsto d\mathcal{F}_T(0, (0, 0))(x_0, (0, 0)) = e^{TDf_0(0)}x_0$ is continuous on \mathbb{R} , and $d\mathcal{F}_0(0, (0, 0))(x_0, (0, 0)) = x_0$.

(A₃) This point is linked to the semi-group property of the solution.

(A₄) By hypothesis, $H = \text{Im}(\mathcal{F}_T(0, 0) \cdot (0, \cdot))$ is a closed subspace (finite dimension), this is the reachable set of the linearized system around 0. This space doesn't depend on T , and its codimension is finite.

(A₅) Finally, Proposition 2.24 shows that, for all $j \in \llbracket r+1, d \rrbracket$, $f_{b_j}(0)$ is a small-time continuously approximately reachable vector. Then, as a supplementary space of H is given by $\text{Span}\left(\left(f_{b_j}(0)\right)_{r+1 \leq j \leq d}\right)$, this condition is verified.

By Theorem 1.7, the multi-input control-affine system (2.1) is $W_0^{m, \infty}$ -STLC. We obtain the desired result. \square

3 Preliminaries on the Schrödinger equation

In all the document, we will note $\omega_j := \lambda_j - \lambda_1$, and $\nu_j := \lambda_K - \lambda_j$, with $j \geq 1$, $K \geq 2$. For $\varphi \in L^2(0, 1)$, $t \in \mathbb{R}$, one defines

$$e^{-iAt}\varphi = \sum_{k=1}^{+\infty} \langle \varphi, \varphi_k \rangle e^{-i\lambda_k t} \varphi_k.$$

One recalls that λ_j, φ_j are defined in (1.3). For the rest of this paper, $n \geq 1$, $p, m \geq 0$, $K \geq 2$ are fixed integers. Let us discuss about the well-posedness of (1.1).

3.1 Well-posedness

Theorem 3.1 (Well-posedness). *Let $T > 0$, μ_1, μ_2 satisfying $(\mathbf{H})_{\text{reg}}$, $u, v \in H_0^m((0, T), \mathbb{R})$, $\psi_0 \in H_{(0)}^{2(p+m)+3}(0, 1)$, and $f \in H_0^m\left((0, T), H^{2p+3} \cap H_{(0)}^{2p+1}(0, 1)\right)$. There exists a unique weak solution to the following equation*

$$\begin{cases} i\partial_t \psi = -\partial_x^2 \psi - (u\mu_1 + v\mu_2)\psi - f, & (0, T) \times (0, 1), \\ \psi(\cdot, 0) = \psi(\cdot, 1) = 0, & (0, T), \\ \psi(0, \cdot) = \psi_0, & (0, 1), \end{cases} \quad (3.1)$$

i.e. a function $\psi \in C^m\left([0, T], H_{(0)}^{2p+3}(0, 1)\right)$ such that the following equality holds in $H_{(0)}^{2p+3}$ for every $t \in [0, T]$

$$\psi(t) = e^{-iAt}\psi_0 + i \int_0^t e^{-iA(t-s)}((u(s)\mu_1 + v(s)\mu_2)\psi(s) + f(s))ds. \quad (3.2)$$

Moreover, $\psi(T) \in H_{(0)}^{2(p+m)+3}$ and the following estimates hold: for every $R > 0$, there exists $C = C(R, \mu_1, \mu_2, T) > 0$ such that, if $\|u\|_{H^m}, \|v\|_{H^m} < R$,

$$\|\psi\|_{C^m([0, T], H_{(0)}^{2p+3})} \leq C \left(\|\psi_0\|_{H_{(0)}^{2(p+m)+3}} + \|f\|_{H^m((0, T), H^{2p+3})} \right), \quad (3.3)$$

$$\|\psi(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C \left(\|\psi_0\|_{H_{(0)}^{2(p+m)+3}} + \|f\|_{H^m((0, T), H^{2p+3})} \right). \quad (3.4)$$

See [17, Theorem 2.1] for details about the proof.

Lemma 3.2. *Let $k \in \mathbb{N}$ and $\mu \in W^{2k, \infty}(0, 1)$. Then, μ is a bounded operator on $H_{(0)}^{2k}(0, 1)$ iff all the derivatives of μ of odd order less than or equal to $2k - 3$ vanish at the boundary.*

Proof. For all $\varphi \in H_{(0)}^{2k}(0, 1)$, for all $l \in \{0, \dots, k - 1\}$, one has, thanks to Leibniz formula

$$(\mu\varphi)^{(2l)} = \sum_{j=0}^l \binom{2l}{2j} \mu^{(2j)} \varphi^{(2(l-j))} + \sum_{j=0}^{l-1} \binom{2l}{2j+1} \mu^{(2j+1)} \varphi^{(2(l-j)-1)}.$$

The first sum vanishes at $x = 0, 1$ because $\varphi^{(2(l-j))}|_{\{0,1\}} = 0$. If $\mu^{(2j+1)}|_{\{0,1\}} = 0$ for $2j+1 \leq 2l-1 \leq 2k-3$, the second sum is 0 at $x = 0, 1$. Conversely, let $l \in \{0, \dots, k-2\}$. Considering $\varphi_l \in H_{(0)}^{2k}(0, 1)$ satisfying $\varphi_l^{(2(k-j)-3)}(0) = \delta_{j,l}$ for $0 \leq j \leq k-2$, we obtain the result. \square

Remark 3.3. *Let μ be a function satisfying $(\mathbf{H}_{\text{reg}})$. Then, $\varphi \mapsto \mu\varphi$ is a continuous mapping from $H^{2p+3} \cap H_{(0)}^{2p+1}$ to $H^{2p+3} \cap H_{(0)}^{2p+1}$. However, $\mu^{(2p+1)}(0), \mu^{(2p+1)}(1)$ a priori don't vanish, so this application doesn't preserve $H_{(0)}^{2p+3}(0, 1)$. This problem is circumvented by (3.3) and (3.4).*

The following statement gives the dependency of the solution with respect to the initial condition. This is the adaptation of Lemma A.19, with the Schrödinger equation.

Proposition 3.4. *Let $T > 0$, μ_1, μ_2 satisfying $(\mathbf{H}_{\text{reg}})$, and $\psi_0 \in H_{(0)}^{2(p+m)+3}(0, 1)$. For all $R > 0$, there exists $C = C(T, \mu_1, \mu_2, R) > 0$ such that for all $u, v \in H_0^m(0, T)$ with $\|u\|_{H^m}, \|v\|_{H^m} < R$, one has*

$$\begin{aligned} \|\psi(T; (u, v), \psi_0 + \varphi_1) - \psi(T; (u, v), \varphi_1) - \psi(T; (0, 0), \psi_0)\|_{H_{(0)}^{2(p+m)+3}} \\ \leq C \|(u, v)\|_{H^m} \|\psi_0\|_{H_{(0)}^{2(p+m)+3}}. \end{aligned} \quad (3.5)$$

Proof. One defines for $T > 0$,

$$\Lambda : t \in [0, T] \mapsto \psi(t; (u, v), \psi_0 + \varphi_1) - \psi(t; (u, v), \varphi_1) - \psi(t; (0, 0), \psi_0).$$

The function Λ is the solution to the following bilinear Schrödinger equation

$$\begin{cases} i\partial_t \Lambda = -\partial_x^2 \Lambda - (u\mu_1 + v\mu_2)\Lambda - (u\mu_1 + v\mu_2)\psi(\cdot; (0, 0), \psi_0), & (0, T) \times (0, 1), \\ \Lambda(\cdot, 0) = \Lambda(\cdot, 1) = 0, & (0, T), \\ \Lambda(0, \cdot) = 0, & (0, 1). \end{cases}$$

Then, the inequality (3.4) applied to this equation gives

$$\|\Lambda(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C \|(u\mu_1 + v\mu_2)\psi(\cdot; (0, 0), \psi_0)\|_{H^m((0, T), H^{2p+3})}. \quad (3.6)$$

When $m \geq 1$, the Sobolev space $H^m(0, T)$ has an algebra structure, and one obtains the following inequality

$$\|\Lambda(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C \|(u, v)\|_{H^m} \max_{i \in [1, 2]} \|\mu_i \psi(\cdot; (0, 0), \psi_0)\|_{H^m((0, T), H^{2p+3})}.$$

Once again, we use the algebra structure of the Sobolev space $H^{2p+3}(0, 1)$ to obtain

$$\|\Lambda(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C \max_{i \in [1, 2]} \|\mu_i\|_{H^{2p+3}} \|(u, v)\|_{H^m} \|\psi(\cdot; (0, 0), \psi_0)\|_{H^m((0, T), H_{(0)}^{2p+3})}.$$

Noticing that $\psi(t; (0, 0), \psi_0) = e^{-iAt}\psi_0$,

$$\|\psi(\cdot; (0, 0), \psi_0)\|_{H^m((0, T), H_{(0)}^{2p+3})} \leq C \|\psi_0\|_{H_{(0)}^{2(p+m)+3}}.$$

This leads to the conclusion. Finally, let's consider the case where $m = 0$. The following estimation holds

$$\|u\mu_1\psi(\cdot; (0, 0), \psi_0)\|_{L^2((0, T), H^{2p+3})}^2 = \int_0^T |u(t)|^2 \|\mu_1\psi(t; (0, 0), \psi_0)\|_{H^{2p+3}}^2 dt.$$

Using the algebra structure of $H^{2p+3}(0, 1)$,

$$\|u\mu_1\psi(\cdot; (0, 0), \psi_0)\|_{L^2((0, T), H^{2p+3})}^2 \leq C \|\mu_1\|_{H^{2p+3}}^2 \int_0^T |u(t)|^2 \|\psi(t; (0, 0), \psi_0)\|_{H_{(0)}^{2p+3}}^2 dt.$$

The same estimate can be obtained with the term in $\mu_2 v$. Thus, the equation (3.6) gives

$$\|\Lambda(T)\|_{H_{(0)}^{2p+3}} \leq C \max_{i \in \llbracket 1, 2 \rrbracket} \|\mu_i\|_{H^{2p+3}} \|(u, v)\|_{L^2} \|\psi(\cdot; (0, 0), \psi_0)\|_{L^\infty((0, T), H_{(0)}^{2p+3})}.$$

The inequality (3.3) (with $m = 0$) gives the conclusion. \square

3.2 Expansion of the solution

We want to make an asymptotic development of the solution ψ , with small controls. Let $u, v \in H_0^m(0, T)$ be two fixed controls. The first-order term $\Psi \in \mathcal{C}^m([0, T], H_{(0)}^{2p+3}(0, 1))$ is the solution to the linearized system of (3.1) around the free trajectory $(\psi_1, (u, v) \equiv 0)$, i.e.

$$\begin{cases} i\partial_t \Psi = -\partial_x^2 \Psi - (u\mu_1 + v\mu_2)\psi_1, & (0, T) \times (0, 1), \\ \Psi(\cdot, 0) = \Psi(\cdot, 1) = 0, & (0, T), \\ \Psi(0, \cdot) = 0, & (0, 1). \end{cases} \quad (3.7)$$

Using (3.2), the solution is given by: $\forall t \in [0, T]$,

$$\Psi(t) = i \sum_{j=1}^{+\infty} \left(\langle \mu_1 \varphi_1, \varphi_j \rangle \int_0^t u(s) e^{i\omega_j s} ds + \langle \mu_2 \varphi_1, \varphi_j \rangle \int_0^t v(s) e^{i\omega_j s} ds \right) \psi_j(t). \quad (3.8)$$

Definition 3.5 (Weak norms of Sobolev spaces). *For any integer $k \in \mathbb{N}^*$, the negative $H^{-k}(0, T)$ space is endowed with the norm: $\forall u \in H^{-k}(0, T)$,*

$$\|u\|_{H^{-k}(0, T)} := |u_1(T)| + \|u_k\|_{L^2(0, T)}.$$

Proposition 3.6 (Weak estimates). *Let $T > 0$, $k, p \in \mathbb{N}$ be such that $k \leq p$, μ_1, μ_2 be two functions satisfying $(\mathbf{H})_{\text{reg}}$ and $u, v \in H_0^m(0, T)$. For every $R > 0$, there exists a constant $C = C(T, \mu_1, \mu_2, R) > 0$ such that, if $\|u\|_{H^m}, \|v\|_{H^m} < R$ and $u_2(T) = \dots = u_{k+1}(T) = 0$, $v_1(T) = \dots = v_{k+1}(T) = 0$, the following estimates hold: for all $l = -(k+1), \dots, m$,*

$$\|(\psi(T; (u, v), \varphi_1) - \psi_1 - \Psi)(T)\|_{H_{(0)}^{2(p+1)+3}} \leq C \|(u, v)\|_{H^l} \|(u, v)\|_{H^m}. \quad (3.9)$$

Proof. This point is proved in [18, Proposition 4.5]. \square

In the framework of this article, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$ implies $\langle \Psi(T), \varphi_K \rangle \equiv 0$, for every controls (u, v) . To determine the evolution of ψ in this direction, we refine the approximation, and extend the development of the solution to the quadratic term. The second-order term $\xi \in \mathcal{C}^m([0, T], H_{(0)}^{2p+3}(0, 1))$ is the solution to the following system

$$\begin{cases} i\partial_t \xi = -\partial_x^2 \xi - (u\mu_1 + v\mu_2)\Psi, & (0, T) \times (0, 1), \\ \xi(\cdot, 0) = \xi(\cdot, 1) = 0, & (0, T), \\ \xi(0, \cdot) = 0, & (0, 1). \end{cases} \quad (3.10)$$

We will sometimes note $\xi(\cdot; (u, v))$ for more precision. The idea is that $\psi(T; (u, v), \varphi_1) \simeq \psi_1(T) + \Psi(T) + \xi(T)$. Thus,

$$\langle \psi(T; (u, v), \varphi_1), \psi_K(T) \rangle \simeq 0 + 0 + \langle \xi(T), \psi_K(T) \rangle,$$

the first term being 0 thanks to orthogonality ($K \neq 1$), and the second one by hypothesis $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$. For $1 \leq \alpha, \beta \leq 2$, one defines

$$h_{\alpha, \beta} : (t, s) \in [0, T]^2 \mapsto - \sum_{j=1}^{+\infty} \langle \mu_\alpha \varphi_1, \varphi_j \rangle \langle \varphi_j, \mu_\beta \varphi_K \rangle e^{i\nu_j t + i\omega_j s}. \quad (3.11)$$

We finally use the notation, for $u, v \in L^2((0, T), \mathbb{R})$,

$$\begin{aligned} \mathcal{F}_T^1(u) &:= \int_0^T u(t) \left(\int_0^t h_{1,1}(t, s) u(s) ds \right) dt, \\ \mathcal{F}_T^2(v) &:= \int_0^T v(t) \left(\int_0^t h_{2,2}(t, s) v(s) ds \right) dt, \end{aligned} \quad (3.12)$$

$$\mathcal{G}_T(u, v) := \int_0^T u(t) \left(\int_0^t h_{2,1}(t, s) v(s) ds \right) dt + \int_0^T v(t) \left(\int_0^t h_{1,2}(t, s) u(s) ds \right) dt. \quad (3.13)$$

The formulas (3.2) and (3.8) lead to

$$\langle \xi(T), \psi_K(T) \rangle = \mathcal{F}_T^1(u) + \mathcal{G}_T(u, v) + \mathcal{F}_T^2(v). \quad (3.14)$$

Lemma 3.7. *Assume that μ_1, μ_2 satisfy $(\mathbf{H})_{\text{reg}}$. Then, as $\|(u, v)\|_{L^2} \rightarrow 0$,*

$$\|\psi(\cdot; (u, v), \varphi_1) - \psi_1 - \Psi - \xi\|_{L^\infty((0, T), H_{(0)}^{2p+3})} = \mathcal{O}\left(\|u\|_{L^2(0, T)}^3 + \|v\|_{L^2(0, T)}^3\right). \quad (3.15)$$

Proof. First step: linear remainder. One defines $\Lambda : t \in [0, T] \mapsto \psi(t; (u, v), \varphi_1) - \psi_1(t)$. The function Λ is solution to the following PDE

$$\begin{cases} i\partial_t \Lambda = -\partial_x^2 \Lambda - (u\mu_1 + v\mu_2)\Lambda - (u\mu_1 + v\mu_2)\psi_1, & (0, T) \times (0, 1), \\ \Lambda(\cdot, 0) = \Lambda(\cdot, 1) = 0, & (0, T), \\ \Lambda(0, \cdot) = 0, & (0, 1). \end{cases}$$

As $\psi_1(t) \in H_{(0)}^{2p+3}(0, 1)$, and μ_1, μ_2 verify $(\mathbf{H})_{\text{reg}}$, Remark 3.3 ensures that $(u(t)\mu_1 + v(t)\mu_2)\psi_1(t) \in H^{2p+3} \cap H_{(0)}^{2p+1}(0, 1)$. Then, we can use the regularization inequality given by (3.3) with $m = 0$ to obtain

$$\|\Lambda\|_{L^\infty((0, T), H_{(0)}^{2p+3})} \leq C \|(u\mu_1 + v\mu_2)\psi_1\|_{L^2((0, T), H^{2p+3})}. \quad (3.16)$$

Note that

$$\|u\mu_1\psi_1\|_{L^2((0, T), H^{2p+3})}^2 = \int_0^T |u(t)|^2 \|\mu_1\varphi_1 e^{-i\lambda_1 t}\|_{H^{2p+3}}^2 dt = \|\mu_1\varphi_1\|_{H^{2p+3}}^2 \|u\|_{L^2(0, T)}^2.$$

We can obtain the same estimate for $\|v\mu_2\psi_1\|_{L^2((0, T), H^{2p+3})}$. Then, thanks to (3.16), one gets

$$\|\psi(\cdot; (u, v), \varphi_1) - \psi_1\|_{L^\infty((0, T), H_{(0)}^{2p+3})} = \mathcal{O}\left(\|u\|_{L^2(0, T)} + \|v\|_{L^2(0, T)}\right).$$

Second step: quadratic remainder. Using the same strategy, one defines $\Lambda : t \in [0, T] \mapsto \psi(t; (u, v), \varphi_1) - \psi_1(t) - \Psi(t)$. The function Λ is solution to the following PDE

$$\begin{cases} i\partial_t \Lambda = -\partial_x^2 \Lambda - (u\mu_1 + v\mu_2)(\psi(\cdot; (u, v), \varphi_1) - \psi_1), & (0, T) \times (0, 1), \\ \Lambda(\cdot, 0) = \Lambda(\cdot, 1) = 0, & (0, T), \\ \Lambda(0, \cdot) = 0, & (0, 1). \end{cases}$$

By the same way, one can use the estimate (3.3) with $m = 0$ in this equation to obtain

$$\|\Lambda\|_{L^\infty((0, T), H_{(0)}^{2p+3})} \leq C \|(u\mu_1 + v\mu_2)(\psi(\cdot; (u, v), \varphi_1) - \psi_1)\|_{L^2((0, T), H^{2p+3})}.$$

Computing this norm, and using the algebra structure of the Sobolev space $H^{2p+3}(0, 1)$,

$$\begin{aligned} & \|u\mu_1(\psi(\cdot; (u, v), \varphi_1) - \psi_1)\|_{L^2((0, T), H^{2p+3})} \\ & \leq C \|\psi(\cdot; (u, v), \varphi_1) - \psi_1\|_{L^\infty((0, T), H_{(0)}^{2p+3})} \|\mu_1\|_{H^{2p+3}} \|u\|_{L^2(0, T)}. \end{aligned} \quad (3.17)$$

Then, using the inequality proved in the first step in (3.17), we obtain

$$\|\psi(\cdot; (u, v), \varphi_1) - \psi_1 - \Psi\|_{L^\infty((0, T), H_{(0)}^{2p+3})} = \mathcal{O}\left(\|u\|_{L^2(0, T)}^2 + \|v\|_{L^2(0, T)}^2\right).$$

Third step: cubic remainder. We conclude with the same arguments. \square

3.3 Control in projection

The following result is adapted from a result proved by Bournissou in [17]. This theorem gives the controllability in projection on \mathcal{H} , defined in (1.12). More precisely, the statement is the following.

Theorem 3.8. *Let $p, m, k \geq 0$, $K \geq 2$, such that $k \leq p$, μ_1, μ_2 functions satisfying $(\mathbf{H})_{\text{reg}}$, and $(\mathbf{H})_{\text{lin}, K, 2}$. Then, the bilinear Schrödinger equation (1.1) is H_0^m -STLC in projection on \mathcal{H} around the ground state in $H_{(0)}^{2(p+m)+3}$. More precisely, for any $T_1 < T$, there exists $\delta > 0$, $C > 0$, and a C^1 map $\Gamma_{T_1, T} : \mathcal{V}_{T_1} \times \mathcal{V}_T \rightarrow H_0^m((T_1, T), \mathbb{R})^2$, where*

$$\begin{aligned} \mathcal{V}_{T_1} & := \left\{ \psi_0 \in \mathcal{S} \cap H_{(0)}^{2(p+m)+3}(0, 1), \quad \|\psi_0 - \psi_1(T_1)\|_{H_{(0)}^{2(p+m)+3}} < \delta \right\} \\ \mathcal{V}_T & := \left\{ \psi_f \in \mathcal{H} \cap H_{(0)}^{2(p+m)+3}(0, 1), \quad \|\psi_f - \psi_1(T)\|_{H_{(0)}^{2(p+m)+3}} < \delta \right\} \end{aligned}$$

such that, $\Gamma_{T_1, T}(\psi_1(T_1), \psi_1(T)) = (0, 0)$ and for every $\psi_0, \psi_f \in \mathcal{V}_{T_1} \times \mathcal{V}_T$, the solution to (1.1) on (T_1, T) with initial data ψ_0 in $t = T_1$, and controls $u, v := \Gamma_{T_1, T}(\psi_0, \psi_f)$ satisfies

$$\mathbb{P}_{\mathcal{H}}(\psi(T; \Gamma_{T_1, T}(\psi_0, \psi_f), \psi_0)) = \psi_f,$$

with the following boundary conditions

$$u_2(T) = \dots = u_{k+1}(T) = 0, \quad \text{and} \quad v_2(T) \dots = v_{k+1}(T) = 0.$$

Finally, for all $l \in \{-(k+1), \dots, m\}$,

$$\|u\|_{H^l} \leq C \left(\|\psi_0 - \psi_1(T_1)\|_{H_{(0)}^{2(p+l)+3}} + \|\psi_f - \psi_1(T)\|_{H_{(0)}^{2(p+l)+3}} \right). \quad (3.18)$$

Moreover, if $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$, one can ensure $v_1(T) = 0$.

Remark 3.9. *The hypothesis $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$ allows us to solve the moment problems (given by the linear test) in the direction φ_1 with the control u . Indeed, since $\omega_1 = 0$,*

$$\langle \Psi(T), \psi_1(T) \rangle = i \langle \mu_1 \varphi_1, \varphi_1 \rangle u_1(T) + i \langle \mu_2 \varphi_1, \varphi_1 \rangle v_1(T).$$

Thus, one can impose the border condition $v_1(T) = 0$. This property is also verified by the nonlinear system thanks to the iteration on the Banach fixed-point theorem.

4 Proof of the main theorem

The proof of the main theorem is divided in the following steps:

1. In the first step, we use the quadratic expansion of the solution in the direction $\psi_K(T)$ (given by Proposition 4.1) to move along the real direction lost $i^n \varphi_K$.

2. In the second step, we correct linearly the system in the other directions \mathcal{H} , thanks to Theorem 3.8. Then, we examine the impact of this second step on the correction made in the first one. A priori, the second step can destroy the work of the first, and we use weak estimates to prove that this is not the case.
3. We use a trick introduced by Kawski in [35], and executed by Bournissou in [16] to move along the other real direction lost $i^{n+1}\varphi_K$.
4. We apply Theorem 1.7 (based on the Brouwer fixed-point theorem) to obtain STLC.

In order to extract the leading terms of the dynamic of the system (as in finite dimension, with the Magnus-type representation formula), we manipulate the expression given by (3.14). This is the purpose of the following subsection.

4.1 Asymptotic estimates on the quadratic term of the solution

Notation. For $T > 0$, $j \geq 1$, and $u, v \in L^2(0, T)$, we use the notation

$$I_T^j(u, v) := \int_0^T u(t) e^{i\nu_j t} \left(\int_0^t v(s) e^{i\omega_j s} ds \right) dt, \quad \mathcal{I}_T(u, v) := \int_0^T u(t) v(t) e^{i\omega_K t} dt.$$

We define $q := \lfloor \frac{n}{2} \rfloor$. As a remainder, the sequence $(c_j)_{j \geq 1}$ is defined in Section 1.3 as $c_j = \langle \mu_1 \varphi_1, \varphi_j \rangle \langle \mu_1 \varphi_j, \varphi_K \rangle$. The goal of this subsection is to prove the following proposition.

Proposition 4.1. *Let $u, v \in L^2((0, T), \mathbb{R})$ be such that $u_i(T) = 0$ for $1 \leq i \leq q+1$, and $v_i(T) = 0$ for $1 \leq i \leq q+1$. Furthermore, assume that $(\mathbf{H})_{\text{reg}}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ hold. Then,*

$$\begin{aligned} \langle \xi(T), \psi_K(T) \rangle &= -i A_{q+1}^1 \int_0^T u_{q+1}^2(t) e^{i\omega_K t} dt + i^n \gamma_n \int_0^T u_{q+1}(t) v_{n-q}(t) e^{i\omega_K t} dt \\ &\quad - i A_{q+1}^2 \int_0^T v_{q+1}^2(t) e^{i\omega_K t} dt + R_{T,1}^{q+1}(u) + \rho_T^{q+1}(u, v) + R_{T,2}^{q+1}(v). \end{aligned} \quad (4.1)$$

with

$$\begin{aligned} R_{T,1}^{q+1}(u) &:= (-1)^q \sum_{j=1}^{+\infty} c_j \nu_j^{q+1} \omega_j^{q+1} I_T^j(u_{q+1}, u_{q+1}), \\ R_{T,2}^{q+1}(v) &:= (-1)^q \sum_{j=1}^{+\infty} \tilde{c}_j \nu_j^{q+1} \omega_j^{q+1} I_T^j(v_{q+1}, v_{q+1}), \\ \rho_T^{q+1}(u, v) &:= (-i)^{n-1} \left(\sum_{j=1}^{+\infty} d_j \nu_j^{q+1} \omega_j^{n-q} I_T^j(u_{q+1}, v_{n-q}) + \sum_{j=1}^{+\infty} \tilde{d}_j \nu_j^{n-q} \omega_j^{q+1} I_T^j(v_{n-q}, u_{q+1}) \right). \end{aligned}$$

The following lemma is strongly inspired by [18, Proposition 5.1].

Lemma 4.2. *Let $T > 0$, $0 \leq l \leq q$ and assume that $u \in L^2((0, T), \mathbb{R})$ such that $u_i(T) = 0$ for $2 \leq i \leq l+1$. If $(\mathbf{H})_{\text{reg}}$ holds, then*

$$\begin{aligned} \mathcal{F}_T^1(u) &= -u_1(T) \sum_{j=1}^{+\infty} c_j e^{i\nu_j T} \left(\int_0^T u(s) e^{i\omega_j s} ds \right) + \frac{u_1(T)^2}{2} e^{i\omega_K T} \sum_{j=1}^{+\infty} c_j \\ &\quad - i \sum_{k=1}^{l+1} A_k^1 \int_0^T u_k^2(t) e^{i\omega_K t} dt + R_{T,1}^{l+1}(u). \end{aligned} \quad (4.2)$$

Proof. We prove the statement by a finite induction on $l \in \llbracket 0, q \rrbracket$. If the formula is true for a fixed $l \leq q-1$, then, for all $u \in L^2((0, 1), \mathbb{R})$ such that $u_i(T) = 0$ for $2 \leq i \leq l+2$, an integration by parts gives

$$(-1)^l R_{T,1}^{l+1}(u) = A + B, \quad (4.3)$$

with

$$A = -i \sum_{j=1}^{+\infty} c_j \nu_j^{l+2} \omega_j^{l+1} I_T^j(u_{l+2}, u_{l+1}), \quad B := - \sum_{j=1}^{+\infty} c_j \nu_j^{l+1} \omega_j^{l+1} \int_0^T u_{l+2}(t) u_{l+1}(t) e^{i\omega_\kappa t} dt.$$

Then, with integration by parts, we obtain

$$A = -i \sum_{j=1}^{+\infty} c_j \nu_j^{l+2} \omega_j^{l+1} \int_0^T u_{l+2}^2(t) e^{i\omega_\kappa t} dt + (-1)^l R_{T,1}^{l+2}(u). \quad (4.4)$$

$$B = \frac{i\omega_K}{2} \sum_{j=1}^{+\infty} c_j \nu_j^{l+1} \omega_j^{l+1} \int_0^T u_{l+2}^2(t) e^{i\omega_\kappa t} dt. \quad (4.5)$$

Using (4.3), (4.4), (4.5) and the induction hypothesis, we conclude. The initialization is proved by the same manipulations, but there are boundary terms. The starting point is the equality given by (3.11) and (3.12),

$$\mathcal{F}_T^1(u) = - \sum_{j=1}^{+\infty} c_j I_T^j(u, u).$$

All series converge thanks to the hypothesis $\lfloor \frac{n}{2} \rfloor \leq p$ (see Remark 1.2). \square

One recalls that, for any integer $k \in \mathbb{N}^*$, the negative $H^{-k}(0, T)$ space is endowed with the norm: $\forall u \in H^{-k}(0, T)$, $\|u\|_{H^{-k}(0, T)} := |u_1(T)| + \|u_k\|_{L^2(0, T)}$. Using the expression of Lemma 4.2, one can obtain the following estimate.

Corollary 4.3. *Let $T > 0$, and assume that $u \in L^2((0, T), \mathbb{R})$ such that $u_i(T) = 0$ for $2 \leq i \leq q+1$. If $(\mathbf{H})_{\text{reg}}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$ hold, then*

$$|\mathcal{F}_T^1(u)| = \mathcal{O} \left(|u_1(T)|^2 + \|u_{q+1}\|_{L^2}^2 \right) = \mathcal{O} \left(\|u\|_{H^{-(q+1)}}^2 \right). \quad (4.6)$$

Proof. Using Lemma 4.2 with $l = q$, we obtain

$$\begin{aligned} \mathcal{F}_T^1(u) &= -u_1(T) \sum_{j=1}^{+\infty} c_j e^{i\nu_j T} \left(\int_0^T u(s) e^{i\omega_j s} ds \right) + \frac{u_1(T)^2}{2} e^{i\omega_\kappa T} \sum_{j=1}^{+\infty} c_j \\ &\quad - iA_{q+1}^1 \int_0^T u_{q+1}^2(t) e^{i\omega_\kappa t} dt + R_{T,1}^{q+1}(u). \end{aligned}$$

Note that $A_{q+1}^1 = 0$ when n is odd, but not necessarily when n is even. We can manipulate the first term: using integration by parts,

$$\sum_{j=1}^{+\infty} c_j e^{i\nu_j T} \left(\int_0^T u(s) e^{i\omega_j s} ds \right) = \sum_{j=1}^{+\infty} c_j e^{i\nu_j T} \left(u_1(T) e^{i\omega_j T} + (-i\omega_j)^{q+1} \int_0^T u_{q+1}(s) e^{i\omega_j s} ds \right).$$

Finally, the term $R_{T,1}^{q+1}(u)$ is estimated by $\|u\|_{H^{-(q+1)}}^2$ because, for all $j \geq 1$,

$$\left| I_T^j(u_{q+1}, u_{q+1}) \right| \leq \|u_{q+1}\|_{L^1}^2.$$

This formula leads to the result. \square

Now, we can follow the same approach with the crossed terms. More precisely,

Lemma 4.4. *Let $T > 0$ and assume that $u, v \in L^2((0, T), \mathbb{R})$ such that $u_i(T) = 0$ for $2 \leq i \leq q+1$ and $v_i(T) = 0$ for $1 \leq i \leq q+1$. If $(\mathbf{H})_{\text{reg}}$ holds, then*

$$\begin{aligned} \mathcal{G}_T(u, v) &= \sum_{k=1}^q (-1)^k \gamma_{2k} \mathcal{I}_T(u_{k+1}, v_k) + i \sum_{k=0}^{n-q-1} (-1)^k \gamma_{2k+1} \mathcal{I}_T(u_{k+1}, v_{k+1}) \\ &\quad - u_1(T) \sum_{j=1}^{+\infty} d_j e^{i\nu_j T} \left(\int_0^T v(s) e^{i\omega_j s} ds \right) + \rho_T^{q+1}(u, v). \end{aligned} \quad (4.7)$$

Proof. We deal with the case where n is odd. Thanks to (3.11) and (3.13), we obtain: $\mathcal{G}_T(u, v) = -(A + B)$ with

$$A = \sum_{j=1}^{+\infty} d_j I_T^j(u, v), \quad B = \sum_{j=1}^{+\infty} \tilde{d}_j I_T^j(v, u).$$

First, we prove by a (finite) induction on $0 \leq l \leq q$ that: for all $u, v \in L^2((0, T), \mathbb{R})$ such that $u_i(T) = 0$ for $2 \leq i \leq l+1$,

$$\begin{aligned} A &= u_1(T) \sum_{j=1}^{+\infty} d_j e^{i\nu_j T} \left(\int_0^T v(s) e^{i\omega_j s} ds \right) + (-1)^{l+1} \sum_{j=1}^{+\infty} d_j \nu_j^{l+1} \omega_j^{l+1} I_T^j(u_{l+1}, v_{l+1}) + \\ &\quad \sum_{k=0}^l (-1)^{k+1} \sum_{j=1}^{+\infty} d_j \nu_j^k \omega_j^k \mathcal{I}_T(u_{k+1}, v_k) + i \sum_{k=0}^l (-1)^{k+1} \sum_{j=1}^{+\infty} d_j \nu_j^{k+1} \omega_j^k \mathcal{I}_T(u_{k+1}, v_{k+1}). \end{aligned} \quad (4.8)$$

For $l = 0$, with an integration by parts on A , we get

$$A = u_1(T) \sum_{j=1}^{+\infty} d_j e^{i\nu_j T} \left(\int_0^T v(s) e^{i\omega_j s} ds \right) - \sum_{j=1}^{+\infty} d_j \mathcal{I}_T(u_1, v) - i \sum_{j=1}^{+\infty} d_j \nu_j I_T^j(u_1, v).$$

Another integration on v by parts gives

$$\begin{aligned} A &= u_1(T) \sum_{j=1}^{+\infty} d_j e^{i\nu_j T} \left(\int_0^T v(s) e^{i\omega_j s} ds \right) - \sum_{j=1}^{+\infty} d_j \nu_j \omega_j I_T^j(u_1, v_1) + \\ &\quad - \sum_{j=1}^{+\infty} d_j \mathcal{I}_T(u_1, v) - i \sum_{j=1}^{+\infty} d_j \nu_j \mathcal{I}_T(u_1, v_1). \end{aligned}$$

For the induction step, it suffices to do two integration by parts in the formula given by the induction hypothesis. Similarly, we prove that for $0 \leq l \leq q$, for all $u, v \in L^2((0, T), \mathbb{R})$ such that $v_i(T) = 0$ for $1 \leq i \leq l+1$,

$$\begin{aligned} B &= \sum_{k=0}^l (-1)^k \left(\sum_{j=1}^{+\infty} \tilde{d}_j \nu_j^k \omega_j^k \mathcal{I}_T(u_{k+1}, v_k) + i \sum_{j=1}^{+\infty} \tilde{d}_j \nu_j^k \omega_j^{k+1} \mathcal{I}_T(u_{k+1}, v_{k+1}) \right) + \\ &\quad (-1)^{l+1} \sum_{j=1}^{+\infty} \tilde{d}_j \nu_j^{l+1} \omega_j^{l+1} I_T^j(v_{l+1}, u_{l+1}). \end{aligned} \quad (4.9)$$

Using $\mathcal{G}_T(u, v) = -(A + B)$, (4.8) and (4.9) with $l = q$, we conclude. To obtain the result in the even case, we manipulate the expression in the same way, with v_q instead of v_{q+1} . Once again, all the series converge thanks to the hypothesis $\lfloor \frac{n}{2} \rfloor \leq p$ (see Remark 1.2). \square

Using the expression given by Lemma 4.4, we get the following estimate.

Corollary 4.5. *Let $T > 0$, $u, v \in L^2((0, T), \mathbb{R})$ be such that, $u_i(T) = 0$ for $2 \leq i \leq q+1$, $v_i(T) = 0$ for $1 \leq i \leq q+1$. If $(\mathbf{H})_{\text{reg}}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ hold,*

$$|\mathcal{G}_T(u, v)| = \mathcal{O}(\|u\|_{H^{-(q+1)}} \|v\|_{H^{-(n-q)}}). \quad (4.10)$$

Proof. We deal with the case where n is odd, so $n = 2q + 1$. First, using Lemma 4.4 and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, we obtain

$$\mathcal{G}_T(u, v) = -u_1(T) \sum_{j=1}^{+\infty} d_j e^{i\nu_j T} (-i\omega_j)^{q+1} \left(\int_0^T v_{q+1}(s) e^{i\omega_j s} ds \right) + \rho_T^{q+1}(u, v) + i(-1)^q \gamma_n \mathcal{I}_T(u_{q+1}, v_{q+1}).$$

This equality leads to the result, by definition of the norm in a negative Sobolev space, as

$$\left| \rho_T^{q+1}(u, v) \right| \leq C \|u\|_{H^{-(q+1)}} \|v\|_{H^{-(q+1)}}, \quad |\mathcal{I}_T(u_{q+1}, v_{q+1})| \leq C \|u\|_{H^{-(q+1)}} \|v\|_{H^{-(q+1)}}.$$

We can prove this statement in the same way when n is even. \square

Proof of Proposition 4.1. Once again, we deal with the case where n is odd. Then, using the hypothesis $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, Lemma 4.2 with $l = q$ gives

$$\mathcal{F}_T^1(u) = R_{T,1}^{q+1}(u). \quad (4.11)$$

Similarly, the hypothesis $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$ leads to

$$\mathcal{F}_T^2(v) = R_{T,2}^{q+1}(v). \quad (4.12)$$

Using Lemma 4.4 and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, one has

$$\mathcal{G}_T(u, v) = i(-1)^q \gamma_n \int_0^T u_{q+1}(t) v_{q+1}(t) e^{i\omega_K t} dt + \rho_T^{q+1}(u, v). \quad (4.13)$$

Using the equations (3.12), (3.13), (3.14), (4.11), (4.12), and (4.13), we obtain (4.1). \square

4.2 A concatenation lemma

For all the rest of the document, we assume that $(\mathbf{H})_{\text{reg}}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$ hold. The goal of the following lemma is to examine the interaction between the first and the second step.

Lemma 4.6 (A composition lemma). *Assume that $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$. Let $0 < T_1 < T$, $u, v \in L^2(0, T_1)$ be such that $u_i(T_1) = v_i(T_1) = 0$ for $1 \leq i \leq q + 1$. Let $(\tilde{u}, \tilde{v}) = \Gamma_{T_1, T}(\psi(T_1; (u, v), \varphi_1), \psi_1(T))$, where Γ is defined in Theorem 3.8. Finally, one defines $U = u \mathbb{1}_{(0, T_1)} + \tilde{u} \mathbb{1}_{(T_1, T)}$ and $V = v \mathbb{1}_{(0, T_1)} + \tilde{v} \mathbb{1}_{(T_1, T)}$. Then,*

$$\begin{aligned} \langle \xi(T; (U, V)), \psi_K(T) \rangle - \langle \xi(T_1; (u, v)), \psi_K(T_1) \rangle &= \mathcal{O} \left(\|(\tilde{u}, \tilde{v})\|_{H^{-(q+1)}}^2 \right. \\ &\quad \left. + \|(u, v)\|_{H^{-(q+1)}} \|(\tilde{u}, \tilde{v})\|_{H^{-(q+1)}} + \|\tilde{u}\|_{H^{-(q+1)}} \|\tilde{v}\|_{H^{-(n-q)}} \right). \end{aligned} \quad (4.14)$$

Proof. Using the formula (3.14), we get

$$\langle \xi(T; (U, V)), \psi_K(T) \rangle - \langle \xi(T_1; (u, v)), \psi_K(T_1) \rangle = \langle \xi(T; (\tilde{u}, \tilde{v})), \psi_K(T) \rangle + G_{T_1, T}(U, V), \quad (4.15)$$

where $G_{T_1, T}$ is the bilinear form given by

$$\begin{aligned} G_{T_1, T}(U, V) &= \int_{T_1}^T \tilde{u}(t) \left(\int_0^{T_1} h_{1,1}(t, s) u(s) ds + \int_0^{T_1} h_{2,1}(t, s) v(s) ds \right) dt \\ &\quad + \int_{T_1}^T \tilde{v}(t) \left(\int_0^{T_1} h_{1,2}(t, s) u(s) ds + \int_0^{T_1} h_{2,2}(t, s) v(s) ds \right) dt. \end{aligned} \quad (4.16)$$

Each term of (4.16) can be written as a sum of product of two integrals, for example

$$\int_{T_1}^T \tilde{u}(t) \left(\int_0^{T_1} h_{1,1}(t, s) u(s) ds \right) dt = - \sum_{j=1}^{+\infty} c_j \left(\int_{T_1}^T \tilde{u}(t) e^{i\nu_j t} dt \right) \left(\int_0^{T_1} u(s) e^{i\omega_j s} ds \right).$$

Using integration par parts, one can write the expression as

$$- \sum_{j=1}^{+\infty} c_j \left(\tilde{u}_1(T) e^{i\nu_j T} + (-i\nu_j)^{q+1} \int_{T_1}^T \tilde{u}_{q+1}(t) e^{i\nu_j t} dt \right) (-i\omega_j)^{q+1} \int_0^{T_1} u_{q+1}(s) e^{i\omega_j s} ds.$$

By Cauchy–Schwarz’s inequality,

$$\left| \int_{T_1}^T \tilde{u}(t) \left(\int_0^{T_1} h_{1,1}(t, s) u(s) ds \right) dt \right| \leq C \| (u, v) \|_{H^{-(q+1)}} \| (\tilde{u}, \tilde{v}) \|_{H^{-(q+1)}}.$$

We obtain the same estimation with the other terms of (4.16). Thus,

$$G_{T_1, T}(U, V) = \mathcal{O}(\| (u, v) \|_{H^{-(q+1)}} \| (\tilde{u}, \tilde{v}) \|_{H^{-(q+1)}}). \quad (4.17)$$

Moreover,

$$\langle \xi(T; (\tilde{u}, \tilde{v})), \psi_K(T) \rangle = \mathcal{F}_T^1(\tilde{u}) + \mathcal{F}_T^2(\tilde{v}) + \mathcal{G}_T(\tilde{u}, \tilde{v}). \quad (4.18)$$

Using (4.15), (4.17), (4.18), (4.6), and (4.10) (thanks to the boundary conditions given by the STLC in projection Theorem 3.8), we obtain the result. The assumption $\lfloor \frac{n}{2} \rfloor \leq p$ gives the convergence of all the series written (see Remark 1.2). \square

4.3 Motion along φ_K and $i\varphi_K$

The next proposition implements step 1 of the proof strategy explained at the beginning of Section 4.

Proposition 4.7. *We denote $s_n := \frac{1}{4(n+m+2)}$ and $\alpha_n := \frac{3}{8} + \frac{m}{8(n+m+2)}$. For all $T_1 > 0$, there exist $C, \rho > 0$ and a continuous map $z \mapsto (u_z, v_z)$ from \mathbb{R} to $H_0^m(0, T_1)^2$ such that*

$$\forall z \in (-\rho, \rho), \quad |\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle - i^n z| \leq C |z|^{1+s_n}. \quad (4.19)$$

The size of the controls is given by: for all $k \in \mathbb{Z}_{\geq -(q+1)}$, for all $r \in [1, +\infty]$, there exists $C > 0$ such that

$$\forall z \in (-\rho, \rho), \quad \|u_z\|_{W^{k,r}}, \|v_z\|_{W^{k,r}} \leq C |z|^{\alpha_n + s_n(\frac{1}{r} - k)}. \quad (4.20)$$

Finally, for all $\varepsilon \in (0, \frac{3}{4})$, there exists $C > 0$ so that for all $z \in (-\rho, \rho)$,

$$\| \psi(T_1; (u_z, v_z), \varphi_1) - \psi_1(T_1) \|_{H_{(0)}^{2(p+l)+3}} \leq C |z|^{\tau_l}, \quad l = -(q+1), \dots, m, \quad (4.21)$$

with $\tau_l := \alpha_n + s_n(\frac{3}{4} - \varepsilon - l) > \frac{1}{4}$. Note that $\tau_{-(q+1)} > \frac{1}{2}$ if $\varepsilon < \frac{1}{4}$.

Proof. Let $T_1 > 0$ and $\rho = T_1^{\frac{1}{s_n}}$. Thanks to $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$, we considers $\bar{u}, \bar{v} \in C_c^\infty(\mathbb{R}, \mathbb{R})^2$ such that $\text{Supp}(\bar{u}), \text{Supp}(\bar{v}) \subset (0, 1)$ and $\int_0^1 \bar{u}(t) \bar{v}^{(2q+1-n)}(t) dt = \frac{1}{\gamma_n}$. Then, we define for $z \in \mathbb{R}^*$,

$$u_z, v_z : t \in [0, T_1] \mapsto \text{sgn}(z) |z|^{\alpha_n} \bar{u}^{(q+1)} \left(\frac{t}{|z|^{s_n}} \right), |z|^{\alpha_n} \bar{v}^{(q+1)} \left(\frac{t}{|z|^{s_n}} \right).$$

Then, for all $z \in (-\rho, \rho)$, one has

$$\text{Supp} \left((u_z)_{q+1} \right), \text{Supp} \left((v_z)_{q+1} \right) \subset (0, |z|^{s_n}) \subset (0, T_1). \quad (4.22)$$

By definition,

$$\text{for all } z \in (-\rho, \rho), \text{ for all } i \in \llbracket 1, q+1 \rrbracket, \quad (u_z)_i(T_1) = (v_z)_i(T_1) = 0. \quad (4.23)$$

Then, we can compute the norm of the controls. For all $k \in \mathbb{Z}_{\geq -(q+1)}$, $r \in [1, +\infty[$, using, the notation $u^{(k)} = u_{-k}$ if $k < 0$ and the Poincaré's inequality, we get for all $z \in (-\rho, \rho) \setminus \{0\}$,

$$\|u_z\|_{W^{k,r}}^r \leq C \left\| u_z^{(k)} \right\|_{L^r}^r = C \int_0^1 \left| |z|^{\alpha_n - ks_n} \bar{u}^{(q+1+k)} \left(\frac{t}{|z|^{s_n}} \right) \right|^r dt.$$

Then, using a change of variables, one obtains

$$\|u_z\|_{W^{k,r}} \leq C |z|^{(\alpha_n - ks_n) + \frac{sn}{r}} \left\| \bar{u}^{(q+1+k)} \right\|_{L^r}.$$

The inequality with $r = +\infty$ is similarly proved. We prove the inequality for v_z in the same way to get (4.20). Then, the mapping can be extended to 0. The continuity is given by the previous inequality. We use the expansion (4.1), the support condition (4.22) and (4.23) to obtain, for every $z \in (-\rho, \rho)$,

$$\begin{aligned} \langle \xi(T_1), \psi_K(T_1) \rangle &= i^n \gamma_n \int_0^{|z|^{s_n}} (u_z)_{q+1}(t) (v_z)_{n-q}(t) e^{i\omega_K t} dt + \rho_{T_1}^{q+1}(u_z, v_z) + R_{T_1,1}^{q+1}(u_z) \\ &\quad + R_{T_1,2}^{q+1}(v_z) - i A_{q+1}^1 \int_0^{|z|^{s_n}} (u_z)_{q+1}(t)^2 e^{i\omega_K t} dt - i A_{q+1}^2 \int_0^{|z|^{s_n}} (v_z)_{q+1}(t)^2 e^{i\omega_K t} dt. \end{aligned}$$

Note that, for all $k \in \llbracket 0, q+1 \rrbracket$, one has $(u_z)_k = \text{sgn}(z) |z|^{\alpha_n + ks_n} \bar{u}^{(q+1-k)} \left(\frac{\cdot}{|z|^{s_n}} \right)$ and $(v_z)_k = |z|^{\alpha_n + ks_n} \bar{v}^{(q+1-k)} \left(\frac{\cdot}{|z|^{s_n}} \right)$. Then, with the change of variables $t = |z|^{s_n} \sigma$, one has

$$\begin{aligned} \langle \xi(T_1), \psi_K(T_1) \rangle &= i^n \gamma_n \text{sgn}(z) |z|^{2\alpha_n + (n+2)s_n} \int_0^1 \bar{u}(\sigma) \bar{v}^{(2q+1-n)}(\sigma) e^{i\omega_K |z|^{s_n} \sigma} d\sigma + \rho_{T_1}^{q+1}(u_z, v_z) \\ &\quad + R_{T_1,1}^{q+1}(u_z) + R_{T_1,2}^{q+1}(v_z) - i |z|^{2\alpha_n + (2q+3)s_n} \int_0^1 (A_{q+1}^1 \bar{u}(\sigma)^2 + A_{q+1}^2 \bar{v}(\sigma)^2) e^{i\omega_K |z|^{s_n} \sigma} d\sigma. \end{aligned} \quad (4.24)$$

Note that, if n is even then $n = 2q$. Else, n is odd, then, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$ gives $A_{q+1}^1 = A_{q+1}^2 = 0$. In all the cases, we obtain

$$|z|^{2\alpha_n + (2q+3)s_n} \left| \int_0^1 (A_{q+1}^1 \bar{u}(\sigma)^2 + A_{q+1}^2 \bar{v}(\sigma)^2) e^{i\omega_K |z|^{s_n} \sigma} d\sigma \right| = \mathcal{O} \left(|z|^{2\alpha_n + (n+3)s_n} \right). \quad (4.25)$$

By definition of $R_{T_1,1}^{q+1}$, $\left| R_{T_1,1}^{q+1}(u_z) \right| \leq \sum_{j=1}^{+\infty} \left| c_j \nu_j^{q+1} \omega_j^{q+1} \right| \left| I_{T_1}^j \left((u_z)_{q+1}, (u_z)_{q+1} \right) \right|$. Then, using the explicit formulation of $I_{T_1}^j$ and (4.20) with $(k, r) = (-(q+1), 1)$,

$$\left| R_{T_1,1}^{q+1}(u_z) \right| \leq C \|u_z\|_{W^{-(q+1),1}}^2 \leq C |z|^{2\alpha_n + 2(q+2)s_n} = \mathcal{O} \left(|z|^{2\alpha_n + (n+3)s_n} \right). \quad (4.26)$$

Similarly,

$$\left| R_{T_1,2}^{q+1}(v_z) \right| = \mathcal{O} \left(|z|^{2\alpha_n + (n+3)s_n} \right). \quad (4.27)$$

Finally, one obtains with the same arguments

$$\left| \rho_{T_1}^{q+1}(u_z, v_z) \right| \leq C \|u_z\|_{W^{-(q+1),1}} \|v_z\|_{W^{-(n-q),1}} = \mathcal{O} \left(|z|^{2\alpha_n + (n+3)s_n} \right). \quad (4.28)$$

Using (4.25), (4.26), (4.27), and (4.28) in (4.24), noticing that $2\alpha_n + (n+2)s_n = 1$, and using the expansion $e^{i\omega_K |z|^{s_n} \sigma} = 1 + \mathcal{O}(|z|^{s_n})$, one has

$$\langle \xi(T_1), \psi_K(T_1) \rangle = i^n \gamma_n z \int_0^1 \bar{u}(\sigma) \bar{v}^{(2q+1-n)}(\sigma) d\sigma + \mathcal{O}(|z|^{1+s_n}) = i^n z + \mathcal{O}(|z|^{1+s_n}),$$

by definition of \bar{u}, \bar{v} . Thus, using the error estimates (3.15), and the hypothesis $(\mathbf{H})_{\text{lin}, \mathbf{K}, \mathbf{1}}$, we obtain

$$|\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle - i^n z| = \mathcal{O}\left(|z|^{1+s_n} + \|(u_z, v_z)\|_{L^2}^3\right).$$

Using (4.20) with $(k, r) = (0, 2)$, we obtain (4.19), as $1 + s_n \leq 3\alpha_n + \frac{3}{2}s_n$.

For the estimate (4.21), we use (3.9) with $k = q$ and (4.20) with $(k, r) \in \{(l, 2), (m, 2)\}$ to obtain the existence of $C > 0$ such that, for all $l = -(q+1), \dots, m$,

$$\|\psi(T_1; (u_z, v_z), \varphi_1) - \psi_1(T_1) - \Psi(T_1)\|_{H_{(0)}^{2(p+l)+3}} \leq C|z|^{2\alpha_n + s_n(1-m-l)}. \quad (4.29)$$

Then, we estimate the linear term in weak norms. For all $j \geq 2$, for all $k \geq -(q+1)$, integration by parts give (the same notation as previously is used for $k < 0$) gives

$$\left| \int_0^{T_1} u_z(t) e^{i\omega_j(t-T_1)} dt \right| = \left| \omega_j^{-k} \int_0^{T_1} u_z^{(k)}(t) e^{i\omega_j(t-T_1)} dt \right| \leq C_k \omega_j^{-k} |z|^{\alpha_n + s_n(1-k)}, \quad (4.30)$$

thanks the inequality (4.20), with $(k, r) = (k, 1)$. This inequality is true with $j = 1$ because the left-hand side is zero. Let $k \in \llbracket -(q+1), m \rrbracket$, and $r \in [k, k+1]$. There exists $\theta \in [0, 1]$ such that $r = k + \theta$. Then, using (4.30)

$$\begin{aligned} \left| \int_0^{T_1} u_z(t) e^{i\omega_j(t-T_1)} dt \right| &= \left| \int_0^{T_1} u_z(t) e^{i\omega_j(t-T_1)} dt \right|^{1-\theta} \left| \int_0^{T_1} u_z(t) e^{i\omega_j(t-T_1)} dt \right|^\theta \\ &\leq \left(C_k \omega_j^{-k} |z|^{\alpha_n + s_n(1-k)} \right)^{1-\theta} \left(C_{k+1} \omega_j^{-(k+1)} |z|^{\alpha_n + s_n(1-(k+1))} \right)^\theta \\ &\leq \max_{k \in \llbracket -(q+1), m \rrbracket} (\max(1, C_k) \max(1, C_{k+1})) \omega_j^{-r} |z|^{\alpha_n + s_n(1-r)}. \end{aligned}$$

Then, the inequality (4.30) is true for k real; there exists $C > 0$, uniform in k , such that, for every $j \in \mathbb{N}^*$ and $k \in [-(q+1), m+1]$,

$$\left| \int_0^{T_1} u_z(t) e^{i\omega_j(t-T_1)} dt \right| \leq C \omega_j^{-k} |z|^{\alpha_n + s_n(1-k)}. \quad (4.31)$$

This inequality is also true with v_z . We want to apply this inequality with $k = l$. Nevertheless, the series diverges; we introduce a non-integer perturbation: let $\varepsilon \in (0, 3/4)$ and $l \in \{-(q+1), \dots, m\}$. With $k = \varepsilon + l + \frac{1}{4}$, one obtains

$$\|\Psi(T_1)\|_{H_{(0)}^{2(p+l)+3}} \leq C \left(\sum_{j=1}^{+\infty} \frac{1}{j^{1+4\varepsilon}} \right)^{\frac{1}{2}} |z|^{\tau_l}, \quad (4.32)$$

thanks to the remark 1.2 and the estimate (4.31). Using the equation (4.29) and (4.32), we obtain the desired inequality, because $\tau_l \leq 2\alpha_n + s_n(1-m-l)$. \square

The following statement represents step 2 of the proof strategy explained at the beginning of Section 4.

Proposition 4.8. *Assume that $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$. The vector $i^n \varphi_K$ is a small-time H_0^m -continuously approximately reachable vector associated with vector variations $i^n \psi_K(T)$. More precisely, for all $T > 0$, there exist $C, \rho > 0$ and a continuous map $z \in \mathbb{R} \mapsto (U_z, V_z) \in H_0^m(0, T)^2$ such that*

$$\forall z \in (-\rho, \rho), \quad \|\psi(T; (U_z, V_z), \varphi_1) - \psi_1(T) - i^n z \psi_K(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C|z|^{1+\frac{1}{5}s_n}, \quad (4.33)$$

with the following size estimate on the family of controls

$$\|U_z\|_{H^m}, \|V_z\|_{H^m} \leq C|z|^{\frac{1}{4}}. \quad (4.34)$$

Proof. Let $0 < T_1 < T$, one defines controls that allow to move along $i^n \varphi_K$, and correcting the linear part. More precisely, for $z \in \mathbb{R}$,

$$U_z, V_z := u_z \mathbb{1}_{[0, T_1]} + \tilde{u}_z \mathbb{1}_{[T_1, T]}, v_z \mathbb{1}_{[0, T_1]} + \tilde{v}_z \mathbb{1}_{[T_1, T]},$$

where u_z, v_z are the controls defined by the previous proposition, and \tilde{u}_z, \tilde{v}_z are the controls given by the control in projection Theorem 3.8 with $k = q$, i.e.

$$\tilde{u}_z, \tilde{v}_z = \Gamma_{T_1, T}(\psi(T_1; (u_z, v_z), \varphi_1), \psi_1(T)).$$

Then, for all $z \in (-\rho, \rho)$ (with $\rho = \min(\rho_1, 1)$, where ρ_1 is given by the previous proposition with T_1),

$$\|U_z\|_{H^m}, \|V_z\|_{H^m} \leq \|(u_z, v_z)\|_{H^m} + \|(\tilde{u}_z, \tilde{v}_z)\|_{H^m}.$$

The first term is estimated by (4.20) with $(k, r) = (m, 2)$. For the second one, we use the simultaneous estimates (3.18) and (4.21) to obtain, for all $l \in \llbracket -(q+1), m \rrbracket$,

$$\|(\tilde{u}_z, \tilde{v}_z)\|_{H^l} \leq C \|\psi(T_1; (u_z, v_z), \varphi_1) - \psi_1(T_1)\|_{H_{(0)}^{2(p+l)+3}} \leq C|z|^{\tau_l}. \quad (4.35)$$

Noticing that $\tau_m > \frac{1}{4}$ and $\alpha_n + s_n(\frac{1}{2} - m) > \frac{1}{4}$, one has (4.34). For the motion along $i^n \varphi_K$, by construction

$$\mathbb{P}_{\mathcal{H}}(\psi(T; (U_z, V_z), \varphi_1)) = \psi_1(T) = \mathbb{P}_{\mathcal{H}}(\psi_1(T) + i^n z \psi_K(T)),$$

where \mathcal{H} is defined in (1.12). Thus, we just have to estimate $|\langle \psi(T; (U_z, V_z), \varphi_1), \psi_K(T) \rangle - i^n z|$. Using the triangular inequality,

$$\begin{aligned} |\langle \psi(T; (U_z, V_z), \varphi_1), \psi_K(T) \rangle - i^n z| &\leq |\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle - i^n z| + \\ &|\langle \psi(T; (U_z, V_z), \varphi_1), \psi_K(T) \rangle - \langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle|. \end{aligned} \quad (4.36)$$

The first term is estimated by (4.19). To study the second one, we use the equations (4.14), and (3.15). Then, we get

$$\begin{aligned} |\langle \psi(T; (U_z, V_z), \varphi_1), \psi_K(T) \rangle - \langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle| &= \mathcal{O} \left(\|(\tilde{u}_z, \tilde{v}_z)\|_{H^{-(q+1)}}^2 + \right. \\ &\left. \|(u_z, v_z)\|_{H^{-(q+1)}} \|(\tilde{u}_z, \tilde{v}_z)\|_{H^{-(q+1)}} + \|\tilde{u}_z\|_{H^{-(q+1)}} \|\tilde{v}_z\|_{H^{-(n-q)}} + \|(U_z, V_z)\|_{L^2}^3 \right). \end{aligned} \quad (4.37)$$

We choose $0 < \varepsilon < \frac{1}{48}$. We estimate these terms

1. The error term: using (4.20) with $(k, r) = (0, 2)$ and (4.35) with $l = 0$, and because $\frac{1}{5}s_n \leq \frac{1}{16}$,

$$\begin{aligned} \|(U_z, V_z)\|_{L^2}^3 &\leq C|z|^{3\alpha_n + \frac{3}{2}s_n} + C|z|^{3\tau_0} \leq C|z|^{1 + \frac{1}{5}s_n} + C|z|^{\frac{17}{16} + (\frac{1}{16} - 3\varepsilon s_n) + \frac{9}{4}s_n} \\ &\leq C|z|^{1 + \frac{1}{5}s_n} + C|z|^{\frac{17}{16} + (\frac{1}{16} - 3\varepsilon)} \leq C|z|^{1 + \frac{1}{5}s_n} + C|z|^{\frac{17}{16}} \leq C|z|^{1 + \frac{1}{5}s_n}. \end{aligned}$$

2. Using (4.35) with $l = -(q+1)$, $\|(\tilde{u}_z, \tilde{v}_z)\|_{H^{-(q+1)}}^2 \leq C|z|^{2\tau_{-(q+1)}}$. We obtain the result because

- a. if n is odd, $2\tau_{-(q+1)} = 2\alpha_n + s_n(n + 2 + \frac{1}{2} - 2\varepsilon) = 1 + s_n(\frac{1}{2} - 2\varepsilon) \geq 1 + \frac{1}{5}s_n$.
- b. if n is even, $2\tau_{-(q+1)} = 2\alpha_n + s_n(n + 2 + \frac{3}{2} - 2\varepsilon) \geq 1 + \frac{1}{5}s_n$.

3. Once again, using (4.20) with $(k, r) = (-(q+1), 2)$ and (4.35) with $l = -(q+1)$,

$$\|(u_z, v_z)\|_{H^{-(q+1)}} \|(\tilde{u}_z, \tilde{v}_z)\|_{H^{-(q+1)}} \leq C|z|^{\alpha_n + s_n(\frac{1}{2} + q + 1) + \tau_{-(q+1)}}.$$

With the same arguments, we obtain the result.

4. If n is even, there is a last term: using (4.35) with $l = -q$ and $l = -(q+1)$,

$$\|\tilde{u}_z\|_{H^{-(q+1)}} \|\tilde{v}_z\|_{H^{-q}} \leq C|z|^{\tau_{-(q+1)} + \tau_{-q}}.$$

Furthermore, $\tau_{-(q+1)} + \tau_{-q} = 2\alpha_n + s_n \left(n + 2 + \frac{1}{2} - 2\varepsilon\right) = 1 + s_n \left(\frac{1}{2} - 2\varepsilon\right) \geq 1 + \frac{1}{5}s_n$.

Finally, the continuity of $z \in \mathbb{R} \mapsto (u_z, v_z) \in H_0^m(0, T_1)^2$ is given by the previous proposition, and the continuity of $z \in \mathbb{R} \mapsto (\tilde{u}_z, \tilde{v}_z) \in H_0^m(T_1, T)^2$ results from the regularity of $\Gamma_{T_1, T}$ and the regularity of the Schrödinger equation, with respect to the controls. \square

Now, we have to show that it is possible to move in the direction $i^{n+1}\varphi_K$: this is the step 3 of the proof strategy explained at the beginning of Section 4.

Proposition 4.9. *Assume that $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$. The vector $i^{n+1}\varphi_K$ is a small-time H_0^m -continuously approximately reachable vector associated with vector variations $i^{n+1}\psi_K(T)$. More precisely, there exists $T^* > 0$, such that, for all $T \in (0, T^*)$, there exist $C, \rho > 0$ and a continuous map $z \mapsto (U_z, V_z)$ from \mathbb{R} to $H_0^m(0, T)^2$ such that,*

$$\forall z \in (-\rho, \rho), \quad \|\psi(T; (U_z, V_z), \varphi_1) - \psi_1(T) - i^{n+1}z\psi_K(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C|z|^{1+\frac{1}{5}s_n},$$

with the following size estimate on the family of controls

$$\|U_z\|_{H^m}, \|V_z\|_{H^m} \leq C|z|^{\frac{1}{4}}.$$

Proof. We consider $(u_z, v_z)_{z \in \mathbb{R}}$ the family of controls associated with $i^n \varphi_K$ in Proposition 4.8. First, we show that there exists $C > 0$, for all $(\alpha, \beta) \in \mathbb{R}^2$, small enough,

$$\|\psi(3T; (u_{\alpha, \beta}, v_{\alpha, \beta}), \varphi_1) - \psi_1(3T) - i^n (\beta e^{2i\omega_K T} + \alpha) \psi_K(3T)\|_{H_{(0)}^{2(p+m)+3}} \leq C|(\alpha, \beta)|^{1+\frac{1}{5}s_n}, \quad (4.38)$$

with $u_{\alpha, \beta} = u_\alpha \#_{[0, T]} u_\beta$ and $v_{\alpha, \beta} = v_\alpha \#_{[0, T]} v_\beta$. Indeed, by Proposition 4.8, there exist $C, \rho > 0$ such that, for all $\alpha \in (-\rho, \rho)$,

$$\|\psi(T; (u_\alpha, v_\alpha), \varphi_1) - \psi_1(T) - i^n \alpha \psi_K(T)\|_{H_{(0)}^{2(p+m)+3}} \leq C|\alpha|^{1+\frac{1}{5}s_n}, \quad (4.39)$$

$$\|(u_\alpha, v_\alpha)\|_{H^m} \leq C|\alpha|^{\frac{1}{4}}. \quad (4.40)$$

In the interval $[T, 2T]$, the evolution of the system is free ($(u, v) \equiv 0$), the solution is given by $\psi(2T) = e^{-iA(2T-T)}\psi(T)$. Using that e^{-iAT} is an isometry from $H_{(0)}^{2(p+m)+3}$ to $H_{(0)}^{2(p+m)+3}$,

$$\|\psi(2T; (u_\alpha \#_{[0, T]}, v_\alpha \#_{[0, T]}), \varphi_1) - \psi_1(2T) - i^n \alpha \psi_K(2T)\|_{H_{(0)}^{2(p+m)+3}} \leq C|\alpha|^{1+\frac{1}{5}s_n}. \quad (4.41)$$

Let $\lambda \in \mathbb{C}$, $\psi_0 \in H_{(0)}^{2(p+m)+3}(0, 1)$, and $u, v \in H_0^m((0, T), \mathbb{R})$. Using the uniqueness of the solution to (3.1) (with $f = 0$), one has: $\psi(\cdot; (u, v), \lambda\psi_0) = \lambda\psi(\cdot; (u, v), \psi_0)$. This property, applied with $\lambda := e^{2i\lambda_1 T}$, and the semi-group property of the bilinear Schrödinger equation lead to

$$\psi(3T; (u_{\alpha, \beta}, v_{\alpha, \beta}), \varphi_1) e^{2i\lambda_1 T} = \psi(T; (u_\beta, v_\beta), \psi(2T; (u_\alpha \#_{[0, T]}, v_\alpha \#_{[0, T]}), \varphi_1) e^{2i\lambda_1 T}).$$

Then, the estimate (3.5) with $\psi_0 = \psi(2T; (u_\alpha \#_{[0, T]}, v_\alpha \#_{[0, T]}), \varphi_1) e^{2i\lambda_1 T} - \varphi_1$ gives

$$\begin{aligned} & \|\psi(3T; (u_{\alpha, \beta}, v_{\alpha, \beta}), \varphi_1) e^{2i\lambda_1 T} - \psi(T; (u_\beta, v_\beta), \varphi_1) - \psi(T; (0, 0), \psi_0)\|_{H_{(0)}^{2(p+m)+3}} \\ & \leq C \|(u_\beta, v_\beta)\|_{H^m} \|\psi(2T; (u_\alpha \#_{[0, T]}, v_\alpha \#_{[0, T]}), \varphi_1) - \psi_1(2T)\|_{H_{(0)}^{2(p+m)+3}}. \end{aligned}$$

Using the estimation (4.40) with $\alpha = \beta$, and (4.41), one gets

$$\|\psi(3T; (u_{\alpha, \beta}, v_{\alpha, \beta}), \varphi_1) e^{2i\lambda_1 T} - \psi(T; (u_\beta, v_\beta), \varphi_1) - e^{-iAT}(\psi_0)\|_{H_{(0)}^{2(p+m)+3}} \leq C|\beta|^{\frac{1}{4}}|\alpha|. \quad (4.42)$$

Using (4.41) in (4.42), we obtain

$$\begin{aligned} & \left\| \psi(3T; (u_{\alpha,\beta}, v_{\alpha,\beta}), \varphi_1) - \psi(T; (u_\beta, v_\beta), \varphi_1) e^{-2i\lambda_1 T} - e^{-iAT} (i^n \alpha \psi_K(2T)) \right\|_{H_{(0)}^{2(p+m)+3}} \\ & \leq C \left(|\beta|^{\frac{1}{4}} |\alpha| + |\alpha|^{1+\frac{1}{5}s_n} \right). \end{aligned} \quad (4.43)$$

Finally, as $e^{-iAT}(i^n \alpha \psi_K(2T)) = i^n \alpha \psi_K(3T)$, using (4.39) with β instead of α , we obtain (4.38). Thus, for $T \in \left(0, \frac{\pi}{2\omega_K}\right)$, and $z \in \mathbb{R}$, taking $\beta = \frac{z}{\sin(2\omega_K T)}$ and $\alpha = -\beta \cos(2\omega_K T)$, we conclude. \square

4.4 Conclusion

Now, we can easily write the proof of the main theorem of this article; this is the last step of the proof strategy explained at the beginning of Section 4.

Proof of Theorem 1.3. We use Theorem 1.7. More precisely, we consider $X = H_{(0)}^{2(p+m)+3}(0, 1)$ and $E_T = H_0^m(0, T)^2$. Moreover, for all $T > 0$,

$$\mathcal{F}_T : (\psi_0, (u, v)) \mapsto \psi(T; (u, v), \psi_0).$$

We have to check all the assumptions to obtain the controllability result.

(A₁) This is known that the end-point map is regular around the equilibrium $(\varphi_1, (0, 0))$ (see [17, Proposition 4.2] for \mathcal{C}^1).

(A₂) This point is given by [17, Proposition 4.2].

(A₃) Using the uniqueness of the solution to (3.1), one can prove the following semi-group property: for all $T_1, T_2 > 0$, $\psi_0 \in H_{(0)}^{2(p+m)+3}(0, 1)$, $u, v \in H_0^m((0, T_1), \mathbb{R})$ and $\tilde{u}, \tilde{v} \in H_0^m((0, T_2), \mathbb{R})$,

$$\psi(T_1 + T_2; (u \# \tilde{u}, v \# \tilde{v}), \psi_0) = \psi(T_2; (\tilde{u}, \tilde{v}), \psi(T_1; (u, v), \psi_0)).$$

(A₄) By hypothesis **(H)**_{lin,K,2} and by Theorem 3.8, $H = \mathcal{H}$ is a closed subspace, this is the reachable set of the linearized system around the ground state. This space doesn't depend on T , and its real codimension in $L^2(0, 1)$ is equal to 2.

(A₅) Finally, as $\langle \mu_1 \varphi_1, \varphi_1 \rangle \neq 0$, Propositions 4.8 and 4.9 show that φ_K and $i\varphi_K$ are small-time continuously approximately reachable vectors. Then, as a supplementary space of H is given by $\text{Span}_{\mathbb{R}}(\varphi_K, i\varphi_K)$, this condition is verified.

By Theorem 1.7, the multi-input bilinear Schrödinger equation (1.1) is H_0^m -Small-Time Locally Controllable around the ground state in $H_{(0)}^{2(p+m)+3}(0, 1)$. \square

A Postponed proofs

A.1 Existence of μ_1, μ_2 verifying the hypotheses

This section is inspired by [18] and [16].

Lemma A.1. *Let $\mu \in \mathcal{C}_c^\infty(0, 1)$. For all $n \in \mathbb{N}$, for all $f \in \mathcal{C}^\infty([0, 1])$,*

$$\text{ad}_A^n(\mu)f = \sum_{k=0}^n \alpha_k^n \mu^{(2n-k)} f^{(k)}, \quad (A.1)$$

where $(\alpha_k^n)_{k \in [0, n]}$ are defined by induction as, $\alpha_0^0 := 1$ and,

$$\forall k \in [1, n], \quad \alpha_k^{n+1} := 2\alpha_{k-1}^n + \alpha_k^n, \quad \alpha_0^{n+1} := \alpha_0^n, \quad \alpha_{n+1}^{n+1} := 2\alpha_n^n.$$

Moreover, $\sum_{k=0}^n \alpha_k^n (-1)^k = (-1)^n$.

This lemma is proved by induction in [18, Proposition A.3, Step 1].

Lemma A.2. *For all $n \in \mathbb{N}$, there exist a constant $C > 0$ and a quadratic form Q_n such that, for every $\mu_1, \mu_2 \in \mathcal{C}_c^\infty(0, 1)$,*

1. *The odd good quadratic brackets are estimated as*

$$\gamma_{2n+1} = -\langle [\underline{\text{ad}}_A^n(\mu_2), \underline{\text{ad}}_A^{n+1}(\mu_1)]\varphi_1, \varphi_K \rangle = 2(-1)^{n+1} \langle \mu_1^{(4n+2)} \mu_2 \varphi_1, \varphi_K \rangle + Q_n(\mu_1, \mu_2),$$

$$\text{with } |Q_n(\mu_1, \mu_2)| \leq \|\mu_1\|_{H^{4n+1}} \|\mu_2\|_{L^2}.$$

2. *The even good quadratic brackets are estimated as*

$$\gamma_{2n} = \langle [\underline{\text{ad}}_A^n(\mu_1), \underline{\text{ad}}_A^n(\mu_2)]\varphi_1, \varphi_K \rangle = 4n(-1)^n \langle \mu_1^{(4n-2)} \mu_2' \varphi_1', \varphi_K \rangle + Q_n(\mu_1, \mu_2),$$

$$\text{with } |Q_n(\mu_1, \mu_2)| \leq \|\mu_1\|_{H^{4n-3}} \|\mu_2\|_{H^1}.$$

Proof. The equalities between the brackets and the series defined by γ_i are proved in Proposition A.10. Let $\mu_1, \mu_2 \in \mathcal{C}_c^\infty(0, 1)$. We start with the odd case: for all $f \in \mathcal{C}^\infty([0, 1])$, $n \geq 0$, using (A.1), one has

$$\begin{aligned} [\underline{\text{ad}}_A^n(\mu_2), \underline{\text{ad}}_A^{n+1}(\mu_1)]f &= \sum_{k=0}^{n+1} \sum_{i=0}^n \sum_{l=0}^i \binom{i}{l} \alpha_k^{n+1} \alpha_i^n \mu_1^{(2(n+1)+i-k-l)} \mu_2^{(2n-i)} f^{(k+l)} \\ &\quad - \sum_{k=0}^n \sum_{i=0}^{n+1} \sum_{l=0}^i \binom{i}{l} \alpha_k^n \alpha_i^{n+1} \mu_2^{(2n+i-k-l)} \mu_1^{(2(n+1)-i)} f^{(k+l)}. \end{aligned} \quad (\text{A.2})$$

Moreover, for all $a, b \in \mathbb{N}$, for all $f \in \mathcal{C}^\infty([0, 1])$, there exists $C > 0$, such that

$$\left| \langle \mu_1^{(a)} \mu_2^{(b)}, f \rangle \right| \leq \|\mu_1\|_{H^{a+b}} \|\mu_2\|_{L^2}.$$

Thus, thanks (A.2), one obtains

$$\begin{aligned} &\left| \langle [\underline{\text{ad}}_A^n(\mu_2), \underline{\text{ad}}_A^{n+1}(\mu_1)]\varphi_1, \varphi_K \rangle - \alpha_0^{n+1} \sum_{i=0}^n \alpha_i^n \langle \mu_1^{(2(n+1)+i)} \mu_2^{(2n-i)} \varphi_1, \varphi_K \rangle \right. \\ &\quad \left. + \alpha_0^n \sum_{i=0}^{n+1} \alpha_i^{n+1} \langle \mu_1^{(2(n+1)-i)} \mu_2^{(2n+i)} \varphi_1, \varphi_K \rangle \right| \leq \|\mu_1\|_{H^{4n+1}} \|\mu_2\|_{L^2}. \end{aligned} \quad (\text{A.3})$$

Moreover, using integration by parts, we get for all $i \in \llbracket 0, n \rrbracket$,

$$\left| \langle \mu_1^{(2(n+1)+i)} \mu_2^{(2n-i)} \varphi_1, \varphi_K \rangle - (-1)^i \langle \mu_1^{(4n+2)} \mu_2 \varphi_1, \varphi_K \rangle \right| \leq \|\mu_1\|_{H^{4n+1}} \|\mu_2\|_{L^2}, \quad (\text{A.4})$$

$$\left| \langle \mu_1^{(2(n+1)-i)} \mu_2^{(2n+i)} \varphi_1, \varphi_K \rangle - (-1)^i \langle \mu_1^{(4n+2)} \mu_2 \varphi_1, \varphi_K \rangle \right| \leq \|\mu_1\|_{H^{4n+1}} \|\mu_2\|_{L^2}. \quad (\text{A.5})$$

Using (A.3), (A.4) and (A.5) one has

$$\begin{aligned} &\left| \gamma_{2n+1} + \left(\alpha_0^{n+1} \sum_{i=0}^n \alpha_i^n (-1)^i - \alpha_0^n \sum_{i=0}^{n+1} \alpha_i^{n+1} (-1)^i \right) \langle \mu_1^{(4n+2)} \mu_2 \varphi_1, \varphi_K \rangle \right| \\ &\quad \leq \|\mu_1\|_{H^{4n+1}} \|\mu_2\|_{L^2}. \end{aligned}$$

Finally, as $\sum_{i=0}^n \alpha_i^n (-1)^i = (-1)^n$, and $\alpha_0^n = 1$, one obtains the result. Let us extract the

leading term of the even good brackets. With same manipulations, one obtains for all $n \geq 1$,

$$\begin{aligned} & \left| \gamma_{2n} - \alpha_0^n \sum_{i=0}^n \alpha_i^n \left\langle \left(\mu_1^{(2n-i)} \mu_2^{(2n+i)} - \mu_2^{(2n-i)} \mu_1^{(2n+i)} \right) \varphi_1, \varphi_K \right\rangle - \right. \\ & \quad \left. \alpha_0^n \sum_{i=1}^n i \alpha_i^n \left\langle \left(\mu_1^{(2n-i)} \mu_2^{(2n+i-1)} - \mu_2^{(2n-i)} \mu_1^{(2n+i-1)} \right) \varphi_1', \varphi_K \right\rangle - \right. \\ & \quad \left. \alpha_1^n \sum_{i=0}^n \alpha_i^n \left\langle \left(\mu_1^{(2n-i)} \mu_2^{(2n+i-1)} - \mu_2^{(2n-i)} \mu_1^{(2n+i-1)} \right) \varphi_1', \varphi_K \right\rangle \right| \leq \|\mu_1\|_{H^{4n-3}} \|\mu_2\|_{H^1}. \end{aligned}$$

Thus, we obtain with same estimates

$$\left| \gamma_{2n} - \left(2\alpha_1^n \sum_{i=0}^n \alpha_i^n (-1)^i \right) \langle \mu_1^{(4n-2)} \mu_2' \varphi_1', \varphi_K \rangle \right| \leq \|\mu_1\|_{H^{4n-3}} \|\mu_2\|_{H^1}.$$

By induction, we obtain $\alpha_1^n = 2n$. This proves the result. \square

Lemma A.3. *Let $K \in \mathbb{N}$, $K \geq 2$ and $\bar{x} \in (0, 1)$ such that $\sin(K\pi\bar{x}) = 0$. There exists $\delta > 0$ such that for every $n \in \mathbb{N}^*$, for every J^+ (resp. J^-) open interval of $(\bar{x}, \bar{x} + \delta)$ (resp. $(\bar{x} - \delta, \bar{x})$), there exist $\mu_{1,+}^n, \mu_{2,+}^n$ (resp. $\mu_{1,-}^n, \mu_{2,-}^n$) in $\mathcal{C}_c^\infty(0, 1)$ supported on J^+ (resp. J^-) such that:*

$$\langle \mu_{1,\pm}^n \varphi_1, \varphi_K \rangle = \langle \mu_{2,\pm}^n \varphi_1, \varphi_K \rangle = \gamma_1(\mu_{1,\pm}^n, \mu_{2,\pm}^n) = \cdots = \gamma_{n-1}(\mu_{1,\pm}^n, \mu_{2,\pm}^n) = 0,$$

and

$$\gamma_n(\mu_{1,\pm}^n, \mu_{2,\pm}^n) = \pm 1.$$

The proof of this lemma is the same as the proof of [18, Theorem A.4], using Lemma A.2 instead of [18, Proposition A.3].

The proof of the following theorem is mainly the same given by Bournissou in [16, Theorem A.2].

Theorem A.4. *Let $n \geq 1$, $K \geq 2$, $p, m \geq 0$, such that $\lfloor \frac{n}{2} \rfloor \leq p$. There exist μ_1, μ_2 satisfying $(\mathbf{H})_{\text{reg}}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{lin}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$.*

In order to prove this theorem, we reformulate some hypotheses. More precisely, we will note

$$\text{Supp}(\mu_1), \text{Supp}(\mu_2) \subset [0, 1), \quad (\text{A.6})$$

$$\forall j \in \mathbb{N}^* \setminus \{K\}, \quad \langle \mu_1 \varphi_1, \varphi_j \rangle \langle \mu_2 \varphi_1, \varphi_j \rangle \neq 0, \quad (\text{A.7})$$

$$\mu_1^{(2p+1)}(0) \mu_2^{(2p+1)}(0) \neq 0. \quad (\text{A.8})$$

Note that, if $\mu_1, \mu_2 \in H^{2(p+m)+3} \cap H_0^{2p+1}((0, 1), \mathbb{R})$ are two functions, one has

$$(\text{A.6}), (\text{A.7}), (\text{A.8}) \Rightarrow (\mathbf{H})_{\text{reg}}, (\mathbf{H})_{\text{lin}, \mathbf{K}, 2}.$$

Consequently, we prove the existence of $\mu_1, \mu_2 \in H^{2(p+m)+3} \cap H_0^{2p+1}((0, 1), \mathbb{R})$ verifying $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$, (A.6), (A.7), and (A.8).

Idea of proof. Let $K \in \mathbb{N}$, $K \geq 2$ and $\bar{x} \in (0, 1)$ such that $\varphi_K(\bar{x}) = 0$. As $\varphi_1 > 0$ on $(0, 1)$ and $\varphi_K'(\bar{x}) > 0$, one may assume the existence of $\delta > 0$ such that $\varphi_1 \varphi_K > 0$ on $(\bar{x}, \bar{x} + \delta)$, and $\varphi_1 \varphi_K < 0$ on $(\bar{x} - \delta, \bar{x})$. Let $\eta \in (0, \bar{x} - \delta)$ be such that $\varphi_1 \varphi_K \neq 0$ on $(0, \eta)$.

Step 1: We prove the existence of functions $\mu_1, \mu_2 \in H^{2(p+m)+3} \cap H_0^{2p+1}((0, 1), \mathbb{R})$ satisfying $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$, (A.6), (A.7) and (A.8). We use the same method than Bournissou in [16]; we consider

$$\mathcal{E} := \left\{ \mu_1, \mu_2 \in H^{2(p+m)+3}(0, 1), \mu_1 \equiv \mu_2 \equiv 0 \text{ on } \left[\frac{\eta}{2}, 1 \right], \text{ satisfying } (\mathbf{H})_{\text{lin}, \mathbf{K}, 1} \right\} \cap H_0^{2p+1}(0, 1),$$

$$\mathcal{U} := \{\mu_1, \mu_2 \in \mathcal{E}, \quad \mu_1, \mu_2 \text{ satisfy } (\mathbf{H})_{\text{quad}, \mathbf{K}, 4}, (\text{A.7}) \text{ and } (\text{A.8})\}.$$

The set \mathcal{E} is not empty. The purpose of the step is to prove that \mathcal{U} is not empty. For that, we use the Baire theorem to prove that \mathcal{U} is dense in \mathcal{E} .

Step 2: We prove the existence of functions $\mu_1, \mu_2 \in H^{2(p+m)+3} \cap H_0^{2p+1}((0, 1), \mathbb{R})$ satisfying $(\mathbf{H})_{\text{lin}, \mathbf{K}, 1}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$, (A.6), (A.7) and (A.8). We divide the support of μ_1 and μ_2 , in two intervals: $\text{Supp}(\mu_1) = I \cup I_1$, and $\text{Supp}(\mu_2) = I \cup I_2$, with $I_1 \cap I_2 = \emptyset$. We considers two functions μ_1, μ_2 as given in the previous step. If $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ is not true, we construct, as in [16], a perturbation, supported on I , satisfying $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$, without any consequence on the other properties.

Step 3: *Conclusion* we prove the theorem: we need to take into account the hypotheses $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 2}$. For example, if $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$ is not true, we construct, as in [16], a perturbation, supported on I_1 , satisfying $(\mathbf{H})_{\text{quad}, \mathbf{K}, 1}$, without any consequence on the other properties (in particular, the properties $(\mathbf{H})_{\text{quad}, \mathbf{K}, 3}$ and $(\mathbf{H})_{\text{quad}, \mathbf{K}, 4}$ are not affected, because $\mu_2 \equiv 0$ on I_1 (and $\mu_1 \equiv 0$ on I_2)). \square

A.2 Lie brackets on the Schrödinger equation

A.2.1 Definition of Lie brackets in infinite dimension

Lemma A.5. *Let $k \in \mathbb{N}$, $\mu_1 \in W^{2k, \infty}(0, 1)$ with all its derivatives of odd order less than or equal to $2k - 3$ that vanish at the boundary. Then, for all $\varphi \in H_{(0)}^{2k}(0, 1)$, $\text{ad}_A^k(B_1)\varphi$ is well-defined in $L^2(0, 1)$. Moreover,*

$$\text{ad}_A^k(B_1)\varphi = \sum_{l=0}^k \alpha_l^k \mu_1^{(2k-l)} \varphi^{(l)},$$

with the definitions of the scalar $(\alpha_l^k)_{0 \leq l \leq k}$ given in (A.1).

Proof. One proves this lemma by induction on k . The case $k = 0$ is immediate, because B_1 is an operator on $L^2(0, 1)$. Assume that the result is true for a fixed k , and, one considers $\mu_1 \in W^{2k+2, \infty}(0, 1)$ such that $\mu_1^{(2l+1)}|_{\{0,1\}} = 0$ for $0 \leq l \leq k - 1$. Let $\varphi \in H_{(0)}^{2k+2}(0, 1)$. Then,

1. $A\varphi \in H_{(0)}^{2k}(0, 1)$, so $\text{ad}_A^k(B_1)(A\varphi)$ is well-defined, by the induction hypothesis.
2. We now prove that $\text{ad}_A^k(B_1)\varphi \in H_{(0)}^2(0, 1)$. The induction hypothesis gives: $\text{ad}_A^k(B_1)\varphi = \sum_{l=0}^k \alpha_l^k \mu_1^{(2k-l)} \varphi^{(l)}$. Then, one has $\text{ad}_A^k(B_1) \in H^2(0, 1)$. Moreover, for all $l \in \llbracket 0, k \rrbracket$,
 - a. If l is odd, $l = 2i + 1$, and $\mu_1^{(2k-2i-1)} \varphi^{(l)}|_{\{0,1\}} = 0$ because $2k - 2i - 1 \leq 2k - 1$ so $\mu_1^{(2k-2i-1)}|_{\{0,1\}} = 0$.
 - b. Otherwise, l is even, $\mu_1^{(2k-l)} \varphi^{(l)}|_{\{0,1\}} = 0$ because $l \leq k$ and $\varphi \in H_{(0)}^{2k+2}(0, 1)$.

Then, the bracket is well-defined. The proposition is true by induction on k . \square

Proposition A.6. *Let $k \in \mathbb{N}^*$, $\mu_1 \in W^{4k-2, \infty}(0, 1)$ with all its derivatives of odd order less than or equal to $4k - 5$ that vanish at the boundary. Then, for all $\varphi \in H_{(0)}^{3k-1}(0, 1)$ the operator $[\text{ad}_A^{k-1}(B_1), \text{ad}_A^k(B_1)]\varphi$ is well-defined in $L^2(0, 1)$.*

Proof. Let $\varphi \in H_{(0)}^{3k-1}(0, 1)$. Then $\varphi \in H_{(0)}^{2k}(0, 1)$ so $\text{ad}_A^k(B_1)\varphi$ and $\text{ad}_A^{k-1}(B_1)\varphi$ are well-defined by Proposition A.5 and one has to prove that

1. $\underline{\text{ad}}_A^{k-1}(B_1)\varphi \in H_{(0)}^{2k}(0,1)$. One recalls: $\underline{\text{ad}}_A^{k-1}(B_1)\varphi = \sum_{l=0}^{k-1} \alpha_l^{k-1} \mu_1^{(2k-l-2)} \varphi^{(l)}$. According to the regularity of φ and μ_1 , this expansion gives $\underline{\text{ad}}_A^{k-1}(B_1)\varphi \in H^{2k}(0,1)$. Then, using Leibniz formula, for all $j \in \llbracket 0, k-1 \rrbracket$, $l \in \llbracket 0, k-1 \rrbracket$,

$$\left(\mu_1^{(2k-l-2)} \varphi^{(l)} \right)^{(2j)} = \sum_{i=0}^{2j} \binom{2j}{i} \mu_1^{(2k-l-2+i)} \varphi^{(l+2j-i)}.$$

Each term of this sum vanish at $x = 0, 1$. Indeed,

- a. If i and l have the same parity, $l+2j-i$ is even and $l+2j-i \leq k-1+2(k-1) = 3k-3$ so $\varphi^{(l+2j-i)}|_{\{0,1\}} = 0$.
- b. Otherwise, $2k-l-2+i$ is odd and $2k-l-2+i \leq 2k-2+2j-1 \leq 4k-5$ so $\mu_1^{(2k-l-2+i)}|_{\{0,1\}} = 0$.

2. $\underline{\text{ad}}_A^k(B_1)\varphi \in H_{(0)}^{2k-2}(0,1)$. This is proven in the same way as point 1.

Consequently, the operator $\left[\underline{\text{ad}}_A^{k-1}(B_1), \underline{\text{ad}}_A^k(B_1) \right]$ is well-defined. \square

Then, if $2\lfloor \frac{n+1}{2} \rfloor - 3 \leq p-1$, Proposition A.6 ensures that the brackets in (1.7) and (1.8) are well-defined.

Proposition A.7. *Let $k \in \mathbb{N}$, $\mu_1, \mu_2 \in W^{2k, \infty}(0,1)$ with all their derivatives of odd order less than or equal to $2k-3$ that vanish at the boundary. Then, for all $\varphi \in H_{(0)}^{k+\lfloor \frac{k+1}{2} \rfloor}(0,1)$, $\left[\underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(B_1), \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(B_2) \right] \varphi$ is well-defined in $L^2(0,1)$.*

This proposition can be proved in the same way as the previous one. Similarly, if $n-2 \leq p-1$, the brackets in (1.9) are well-defined, thanks to Proposition A.7.

A.2.2 Computation of Lie brackets for the Schrödinger equation

Lemma A.8. *Let $k \in \mathbb{N}$ and $\mu \in W^{2k, \infty}(0,1)$ be such that $\mu^{(2l+1)}|_{\{0,1\}} = 0$ for $0 \leq l \leq k-2$. Then, for all $p, q \in \mathbb{N}^*$,*

$$\left\langle \underline{\text{ad}}_A^k(\mu)\varphi_p, \varphi_q \right\rangle = (\lambda_p - \lambda_q)^k \langle \mu\varphi_p, \varphi_q \rangle.$$

Proof. If $k = 0$, the result is true. Then, assume that $k \geq 1$. As $\varphi_p, \varphi_q \in H_{(0)}^{2k}(0,1)$, one has

$$\left\langle \underline{\text{ad}}_A^k(\mu)\varphi_p, \varphi_q \right\rangle = \left\langle [\underline{\text{ad}}_A^{k-1}(\mu), A]\varphi_p, \varphi_q \right\rangle = \left\langle (\lambda_p I_d - A)\underline{\text{ad}}_A^{k-1}(\mu)\varphi_p, \varphi_q \right\rangle$$

Using the symmetry property of A ,

$$\left\langle \underline{\text{ad}}_A^k(\mu)\varphi_p, \varphi_q \right\rangle = \left\langle \underline{\text{ad}}_A^{k-1}(\mu)\varphi_p, (\lambda_p I_d - A)\varphi_q \right\rangle = (\lambda_p - \lambda_q) \left\langle \underline{\text{ad}}_A^{k-1}(\mu)\varphi_p, \varphi_q \right\rangle.$$

An induction gives the result. \square

Proposition A.9. *Let $k \in \mathbb{N}^*$ be a fixed integer, $K \geq 2$ and $\mu_1 \in W^{4k-2, \infty}(0,1)$ be such that, for $0 \leq l \leq 2k-3$ $\mu_1^{(2l+1)}|_{\{0,1\}} = 0$. Then,*

$$\left\langle [\underline{\text{ad}}_A^{k-1}(\mu_1), \underline{\text{ad}}_A^k(\mu_1)]\varphi_1, \varphi_K \right\rangle = 2(-1)^k A_k^1.$$

Proof. By definition,

$$2(-1)^k A_k^1 = \sum_{j=1}^{+\infty} c_j (\lambda_j - \lambda_1)^{k-1} (\lambda_K - \lambda_j)^k - \sum_{j=1}^{+\infty} c_j (\lambda_j - \lambda_1)^k (\lambda_K - \lambda_j)^{k-1}.$$

Using Lemma A.8, we obtain

$$\sum_{j=1}^{+\infty} \left(\left\langle \underline{\text{ad}}_A^{k-1}(\mu_1)\varphi_j, \varphi_1 \right\rangle \left\langle \underline{\text{ad}}_A^k(\mu_1)\varphi_K, \varphi_j \right\rangle - \left\langle \underline{\text{ad}}_A^k(\mu_1)\varphi_j, \varphi_1 \right\rangle \left\langle \underline{\text{ad}}_A^{k-1}(\mu_1)\varphi_K, \varphi_j \right\rangle \right).$$

Using the symmetry of the operators, we get

$$2(-1)^k A_k^1 = (-1)^{k-1} \left\langle \underline{\text{ad}}_A^{k-1}(\mu_1)\varphi_1, \underline{\text{ad}}_A^k(\mu_1)\varphi_K \right\rangle + (-1)^{k-1} \left\langle \underline{\text{ad}}_A^k(\mu_1)\varphi_1, \underline{\text{ad}}_A^{k-1}(\mu_1)\varphi_K \right\rangle.$$

Then,

$$2(-1)^k A_k^1 = - \left\langle \underline{\text{ad}}_A^k(\mu_1) \left(\underline{\text{ad}}_A^{k-1}(\mu_1)\varphi_1 \right), \varphi_K \right\rangle + \left\langle \underline{\text{ad}}_A^{k-1}(\mu_1) \left(\underline{\text{ad}}_A^k(\mu_1)\varphi_1 \right), \varphi_K \right\rangle.$$

Finally,

$$2(-1)^k A_k^1 = \left\langle \left[\underline{\text{ad}}_A^{k-1}(\mu_1), \underline{\text{ad}}_A^k(\mu_1) \right] \varphi_1, \varphi_K \right\rangle.$$

□

Proposition A.10. *Let $k \in \mathbb{N}$, $K \geq 2$, and $\mu_1, \mu_2 \in W^{2k, \infty}(0, 1)$ such that $\mu_i^{(2l+1)}|_{\{0,1\}} = 0$ for $0 \leq l \leq k-2$ and $1 \leq i \leq 2$. Then,*

$$\left\langle \left[\underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1), \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2) \right] \varphi_1, \varphi_K \right\rangle = (-1)^k \gamma_k.$$

Proof. Using Lemma A.8, we notice that

$$\begin{aligned} \gamma_k = \sum_{j=1}^{+\infty} \left(\left\langle \underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1)\varphi_K, \varphi_j \right\rangle \left\langle \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2)\varphi_j, \varphi_1 \right\rangle - \right. \\ \left. \left\langle \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2)\varphi_K, \varphi_j \right\rangle \left\langle \underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1)\varphi_j, \varphi_1 \right\rangle \right) \end{aligned}$$

As $(\varphi_j)_{j \geq 1}$ is an orthonormal basis of L^2 , and thanks to the symmetry/skew-symmetry of the operator,

$$\begin{aligned} \gamma_k = (-1)^{\lfloor \frac{k}{2} \rfloor} \left\langle \underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1)\varphi_K, \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2)\varphi_1 \right\rangle - \\ (-1)^{\lfloor \frac{k+1}{2} \rfloor} \left\langle \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2)\varphi_K, \underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1)\varphi_1 \right\rangle. \end{aligned}$$

Once again, using the symmetry/skew-symmetry of the operator,

$$\gamma_k = (-1)^{\lfloor \frac{k}{2} \rfloor + \lfloor \frac{k+1}{2} \rfloor} \left\langle \left[\underline{\text{ad}}_A^{\lfloor \frac{k+1}{2} \rfloor}(\mu_1), \underline{\text{ad}}_A^{\lfloor \frac{k}{2} \rfloor}(\mu_2) \right] \varphi_1, \varphi_K \right\rangle.$$

This equality completes the proof. □

A.3 Some elements of proof in finite dimension

A.3.1 A proof of Theorem 2.15 via Sussmann's $S(\theta)$ -condition

In this section, we present a proof of Theorem 2.15 relying on Sussmann's $S(\theta)$ -sufficient condition (see [45, Theorem 7.3]), recalled in Proposition A.12 below (see [28, Theorem 3.29]), for which we need the following definition.

Definition A.11. *The map $\sigma : \text{Br}(X) \mapsto \mathcal{L}(X)$ is defined by $\sigma(b) = E(b) + \pi(E(b))$, where $\pi : \mathcal{L}(X) \mapsto \mathcal{L}(X)$ is the unique morphism of Lie algebra such that $\pi(X_0) = X_0$, $\pi(X_1) = X_2$ and $\pi(X_2) = X_1$.*

For instance, if $b = (X_1, (X_1, X_0))$ then $\sigma(b) = [X_1, [X_1, X_0]] + [X_2, [X_2, X_0]]$.

Proposition A.12 (Sussmann's $S(\theta)$ -condition). *Let f_0, f_1, f_2 be analytic vector fields on a neighborhood of 0 in \mathbb{R}^d that satisfy the Lie algebra rank condition, $\text{Lie}(f_0, f_1, f_2)(0) = \mathbb{R}^d$. If there exists $\theta \in [0, 1]$ such that, for every $\mathbf{b} \in \text{Br}(X)$ with $n_0(\mathbf{b})$ odd and both $n_1(\mathbf{b})$ and $n_2(\mathbf{b})$ even, we have*

$$f_{\sigma(\mathbf{b})}(0) \in \text{Span}\{f_b(0); b \in \text{Br}(X), n(b) + \theta n_0(b) < n(\mathbf{b}) + \theta n_0(\mathbf{b})\}, \quad (\text{A.9})$$

then the system $\dot{x} = f_0(x) + uf_1(x) + vf_2(x)$ is L^∞ -STLC.

Proof of Theorem 2.15 thanks to Proposition A.12. We prove by induction on $m \in \mathbb{N}$ that for every $d \in \mathbb{N}^*$, the assumptions of Theorem 2.15 imply $W^{m, \infty}$ -STLC.

Initialization for $m = 0$: we prove L^∞ -STLC by applying Proposition A.12. First (2.7) gives the Lie algebra rank condition,

$$\text{Span}(f_{b_1}(0), \dots, f_{b_d}(0)) = \mathbb{R}^d,$$

with $b_1, \dots, b_r \in \mathcal{B}_1$ such that $\text{Span}(f_{b_1}(0), \dots, f_{b_r}(0)) = S_1(f)(0)$ and $b_{r+1}, \dots, b_d \in \mathcal{B}_{2, \text{good}}$. One considers

$$0 < \theta \leq \frac{1}{\max_{i \in \llbracket 1, d \rrbracket} |b_i|}.$$

Let $L = \max_{i \in \llbracket r+1, d \rrbracket} |b_i|$. Let $\mathbf{b} \in \text{Br}(X)$ be such that $n_0(\mathbf{b})$ is odd and both $n_1(\mathbf{b})$ and $n_2(\mathbf{b})$ are even. Then,

1. Either, $n(\mathbf{b}) \geq 4$ and then (2.7) gives the compensation because, for all $i \in \llbracket 1, d \rrbracket$,

$$n(b_i) + \theta n_0(b_i) \leq 2 + \theta \left(\max_{i \in \llbracket 1, d \rrbracket} |b_i| - 1 \right) < 3 < 4 + \theta n_0(\mathbf{b}) \leq n(\mathbf{b}) + \theta n_0(\mathbf{b}).$$

2. Or, $n(\mathbf{b}) = 2$ i.e. $(n_1(\mathbf{b}), n_2(\mathbf{b})) \in \{(2, 0), (0, 2)\}$ i.e. $\sigma(\mathbf{b}) \in \text{Span}\{\mathcal{B}_{2, \text{bad}}\}$. Thus:
 - a. If $|\mathbf{b}| \leq L$, (2.8) gives, for all $i \in \llbracket 1, r \rrbracket$,

$$n(b_i) + \theta n_0(b_i) = 1 + \theta n_0(b_i) < 2 \leq n(\mathbf{b}) + \theta n_0(\mathbf{b}).$$

- b. If $|\mathbf{b}| > L$, the LARC (2.7) gives once again the compensation because,

$$\begin{aligned} n(b_i) + \theta n_0(b_i) &\leq 1 + \theta (|b_i| - 1) < 2 + \theta n_0(\mathbf{b}) = n(\mathbf{b}) + \theta n_0(\mathbf{b}), & \text{for } i \in \llbracket 1, r \rrbracket, \\ n(b_i) + \theta n_0(b_i) &\leq 2 + \theta (L - 2) < 2 + \theta (L - 1) \leq n(\mathbf{b}) + \theta n_0(\mathbf{b}), & \text{for } i \in \llbracket r + 1, d \rrbracket. \end{aligned}$$

Theses points prove (A.9).

Heredity. We assume the result is proved up to some $m \in \mathbb{N}$. We consider the extended system

$$\begin{cases} \dot{x} = f_0(x) + uf_1(x) + vf_2(x) \\ \dot{u} = w \\ \dot{v} = \omega \end{cases}$$

with state $Y = (x, u, v) \in \mathbb{R}^{d+2}$ and control (w, ω) , i.e. $\frac{dY}{dt} = F_0(Y) + wF_1(Y) + \omega F_2(Y)$ with:

$$F_0(x, u, v) = \begin{pmatrix} f_0(x) + uf_1(x) + vf_2(x) \\ 0 \\ 0 \end{pmatrix}, \quad F_1(x, u, v) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad F_2(x, u, v) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then, for every $i \in \llbracket 1, 2 \rrbracket$,

$$F_{M_i^1}(Y) = [F_i, F_0](Y) = \begin{pmatrix} f_i(x) \\ 0 \\ 0 \end{pmatrix}.$$

Thus, when computing $F_{M_j^i}$ for $j \geq 2$, the derivatives with respect to u and v never come into play, and we obtain

$$\forall j \geq 1, \quad F_{M_j^i}(Y) = \begin{pmatrix} \underline{\text{ad}}_{f_0+uf_1+vf_2}^{j-1}(f_i)(x) \\ 0 \\ 0 \end{pmatrix},$$

where $\underline{\text{ad}}$ denotes the right-bracketing: $\underline{\text{ad}}_f(g) = [g, f]$. Thus, for every $i \in \llbracket 1, 2 \rrbracket$ and $l \in \mathbb{N}$, $F_{W_{1,l}^i} = 0$ and for every $j \geq 2$,

$$F_{W_{j,l}^i}(Y) = \begin{pmatrix} \underline{\text{ad}}_{f_0+uf_1+vf_2}^l [\underline{\text{ad}}_{f_0+uf_1+vf_2}^{j-2}(f_i), \underline{\text{ad}}_{f_0+uf_1+vf_2}^{j-1}(f_i)](x) \\ 0 \\ 0 \end{pmatrix}.$$

Moreover, for every $l \in \mathbb{N}$, $F_{C_{0,l}} = F_{C_{1,l}} = 0$ and, for every $j \geq 2$,

$$F_{C_{j,l}}(Y) = \begin{pmatrix} (-1)^j \underline{\text{ad}}_{f_0+uf_1+vf_2}^l [\underline{\text{ad}}_{f_0+uf_1+vf_2}^{\lfloor \frac{j+1}{2} \rfloor - 1}(f_1), \underline{\text{ad}}_{f_0+uf_1+vf_2}^{\lfloor \frac{j}{2} \rfloor - 1}(f_2)](x) \\ 0 \\ 0 \end{pmatrix}.$$

In particular, for every $i \in \llbracket 1, 2 \rrbracket$, $j \geq 1$, $F_{M_j^i}(0) = \begin{pmatrix} f_{M_{j-1}^i}(0) \\ 0 \\ 0 \end{pmatrix}$.

Moreover, for every $i \in \llbracket 1, 2 \rrbracket$, $j \geq 2$, $l \in \mathbb{N}$,

$$F_{W_{j,l}^i}(0) = \begin{pmatrix} f_{W_{j-1,l}^i}(0) \\ 0 \\ 0 \end{pmatrix}, \quad F_{C_{j,l}}(0) = \begin{pmatrix} f_{C_{j-1,l}}(0) \\ 0 \\ 0 \end{pmatrix}.$$

Thus the extended system satisfies the assumptions of Theorem 2.15 with $d+2 \rightarrow d$. By the induction assumption, the extended system is $W^{m,\infty}$ -STLC. By choosing trajectories of the extended system starting and finishing at $u = 0$, one concludes that the initial system $\dot{x} = f_0(x) + uf_1(x) + vf_2(x)$ is $W^{m+1,\infty}$ -STLC. \square

A.3.2 Proof of the representation formula of Proposition 2.20

Definition A.13 (Support). *Let $a \in \mathcal{L}(X)$. For $b \in \mathcal{B}$, a Hall set, we denote by $\langle a, b \rangle$ the coefficient of $E(b)$ in the expansion of a on the basis $E(\mathcal{B})$. We define*

$$\text{supp}(a) := \{b \in \mathcal{B}, \langle a, b \rangle \neq 0\}.$$

If $A \subset \text{Br}(X)$, we let $\text{supp}(A) := \bigcup_{a \in A} \text{supp}(a)$.

Definition A.14. *We consider the (unital associative) algebra $\widehat{\mathcal{A}}(X)$ of formal power series generated by $\mathcal{A}(X)$. An element $a \in \widehat{\mathcal{A}}(X)$ is a sequence $a = (a_n)_{n \in \mathbb{N}}$ written $a = \sum_{n \in \mathbb{N}} a_n$, where $a_n \in \mathcal{A}_n(X)$ with, in particular, $a_0 \in \mathbb{R}$ being its constant term. We also define the Lie algebra of formal Lie series $\widehat{\mathcal{L}}(X)$ as the Lie algebra of formal power series $a \in \widehat{\mathcal{A}}(X)$ for which $a_n \in \mathcal{L}(X)$ for each $n \in \mathbb{N}$.*

We consider the formal differential equation

$$\begin{cases} \dot{y}(t) = y(t)(X_0 + u(t)X_1 + v(t)X_2), \\ x(0) = 1, \end{cases} \quad (\text{A.10})$$

whose solution is a formal series valued function $y : \mathbb{R}_+ \rightarrow \widehat{\mathcal{A}}(X)$ (see [9, section 2.2]). By [9, Section 2.4, Theorem 41], for every $t > 0$, there exists $\mathcal{Z}_\infty(t; X, (u, v)) \in \widehat{\mathcal{L}}(X)$ such that

$$y(t) = \exp(tX_0) \exp(\mathcal{Z}_\infty(t; X, (u, v))). \quad (\text{A.11})$$

If $\mathcal{B} \subset \text{Br}(X)$ is a Hall set, there exists a unique family $(\eta_b)_{b \in \mathcal{B}}$ of maps $\mathbb{R}_+ \times L_{loc}^1(\mathbb{R}_+, \mathbb{R})^2 \rightarrow \mathbb{R}$, called coordinates of the pseudo first kind associated with \mathcal{B} and they satisfy the announced homogeneity properties in (2.16). Moreover, by [9, Section 2.5], for every $t > 0$ and $u, v \in L^1((0, t), \mathbb{R})$,

$$y(t) = \prod_{b \in \mathcal{B}}^{\leftarrow} e^{\xi_b(t, (u, v))b}. \quad (\text{A.12})$$

Let \mathcal{B} be a Hall set as in Proposition 2.13. We deduce from (A.11), (A.12) and the maximality of X_0 in \mathcal{B} that

$$\exp(\mathcal{Z}_\infty(t; X, (u, v))) = \prod_{b \in \mathcal{B} \setminus \{X_0\}}^{\leftarrow} e^{\xi_b(t, (u, v))b}.$$

By applying the multivariate CBHD formula [9, Proposition 34] to this expression, one obtains $\eta_b = \xi_b$ for every $b \in \mathcal{B}_1$ and for every $b \in \mathcal{B}_2$,

$$\eta_b(t, (u, v)) = \xi_b(t, (u, v)) + \frac{1}{2} \sum \delta_{(b_1, b_2), b} \xi_{b_1}(t, (u, v)) \xi_{b_2}(t, (u, v))$$

where the sum is indexed by the set $\{b_1 > b_2 \in \mathcal{B}_1; b \in \text{supp}(b_1, b_2)\}$ and $\delta_{(b_1, b_2), b}$ denotes the coefficient of b in the expansion of (b_1, b_2) on the basis \mathcal{B} . In particular this sum is finite and involves only elements $b_1, b_2 \in \mathcal{B}_1$ such that $n_0(b_1) + n_0(b_2) = n_0(b)$.

The estimate (2.15) is proved in [9, Proposition 161]. The absolute convergence is proved in [9, Proposition 103] and relies on [8, Theorem 1.9].

A.4 Some lemmas about ODEs

Lemma A.15. *Let $\delta > 0$ and $z \in \mathcal{C}^1(B(0, \delta); \mathbb{R}^d)$ be such that $\|z\|_\infty \leq \delta$. We note $x(\cdot, z, 0)$*

the solution to the system: $\begin{cases} x' &= z(x) \\ x(0) &= 0 \end{cases}$. Then,

$$\|x(1; z, 0) - z(0)\| \leq \|z(0)\| \|Dz\|_\infty e^{\|Dz\|_\infty}.$$

Proof. This lemma is proved in [9, Lemma 160]. \square

Lemma A.16. *Let $\delta > 0$, and $f_0, f_1, f_2 : B(0, 2\delta) \rightarrow \mathbb{R}^d$ be real continuous functions, with $f_0(0) = 0$. There exists $r(\delta) > 0$, such that, for all $T_1 < T < T_1 + 1$, $p \in B(0, r(\delta))$, and $u, v \in L^\infty(T_1, T)$ with $\|u\|_{L^\infty}, \|v\|_{L^\infty} < r(\delta)$, one has*

$$\forall t \in [T_1, T], \quad x(T; (u, v), p, T_1) \in B(0, \delta). \quad (\text{A.13})$$

Proof. This result is linked to continuity with respect to the initial condition \square

Corollary A.17. *Under the same assumptions as the previous lemma, there exists $r(\delta) > 0$, $C(\delta) > 0$, such that, for all $T_1 < T < T_1 + 1$, $p \in B(0, r(\delta))$, and $u, v \in L^\infty(T_1, T)$ with $\|u\|_{L^\infty}, \|v\|_{L^\infty} < r(\delta)$, one has*

$$\|x(T; (u, v), p, T_1)\| \leq C(\delta) (\|p\| + \|(u, v)\|_{L^\infty}).$$

Proof. This statement is based on the previous Lemma and the Grönwall inequality. \square

Lemma A.18. *Let $f_0 : B(0, 2\delta) \rightarrow \mathbb{R}^d$, be \mathcal{C}^2 function, for $\delta > 0$, with $f_0(0) = 0$. There exist $C, \varepsilon > 0$ such that, for all $T_1 < T < T_1 + 1$ and $p \in B(0, \varepsilon)$,*

$$\left\| x(T; (0, 0), p, T_1) - e^{(T-T_1)Df_0(0)} p \right\| \leq C \|p\|^2.$$

Proof. By definition, there exists $h > 0$, s.t. for all $x \in B(0, h)$,

$$\|f_0(x) - f_0(0) - Df_0(0)x\| \leq \left(\frac{1}{2} \|D^2 f_0(0)\| + 1 \right) \|x\|^2. \quad (\text{A.14})$$

Let $p \in B(0, \varepsilon)$ with $\varepsilon = \min(r(\delta), r(h))$. Using the Duhamel formula,

$$\begin{aligned} \left\| x(T; (0, 0), p, T_1) - e^{(T-T_1)Df_0(0)} p \right\| &\leq \\ &\int_{T_1}^T \left\| e^{(t-s)Df_0(0)} \right\| \left\| f_0(x(s, (0, 0), p, T_1)) - Df_0(0)x(s, (0, 0), p, T_1) \right\| ds. \end{aligned}$$

Thank to Lemma A.16, one can use the inequality (A.14) and Corollary A.17 gives the result. \square

Lemma A.19. *Let $\delta > 0$, and $f_0, f_1, f_2 : B(0, 2\delta) \rightarrow \mathbb{R}^d$ be \mathcal{C}^2 functions, with $f_0(0) = 0$. There exist $C, \varepsilon > 0$, such that, for all $T_1 < T < T_1 + 1$, $p \in B(0, \varepsilon)$, and $u, v \in L^\infty(T_1, T)$ with $\|u\|_{L^\infty}, \|v\|_{L^\infty} < \varepsilon$, one has*

$$\|x(T; (u, v), p, T_1) - x(T; (u, v), 0, T_1) - x(T; (0, 0), p, T_1)\| \leq C \left(\|p\| \|(u, v)\|_{L^\infty} + \|p\|^2 \right).$$

Proof. Let $p \in B(0, \varepsilon)$, $u, v \in L^\infty(T_1, T)$ for ε small enough. For simplicity, we use the notations: $x := x(\cdot; (u, v), p, T_1)$, $x_{u,v} := x(\cdot; (u, v), 0, T_1)$, and $x_p := x(\cdot; (0, 0), p, T_1)$. Let $z := x - x_{u,v} - x_p$. By definition, z is solution to the following control-affine system

$$\begin{aligned} z' = & (f_0(z + x_{u,v} + x_p) - f_0(x_{u,v} + x_p)) + (f_0(x_{u,v} + x_p) - f_0(x_{u,v}) - f_0(x_p)) + \\ & u(f_1(z + x_{u,v} + x_p) - f_1(x_{u,v})) + v(f_2(z + x_{u,v} + x_p) - f_2(x_{u,v})), \end{aligned}$$

with $z(T_1) = 0$. We estimate each term of this formulation

1. the first one scales like $\|z\|$,
2. the second one is shaped like $\|p\| (\|p\| + \|(u, v)\|_\infty)$,
3. the other terms are like $\|(u, v)\|_\infty (\|z\| + \|p\|)$.

The Grönwall lemma gives the result. \square

Acknowledgement

The author would like to take particular care in thanking Karine Beauchard and Frédéric Marbach for the many discussions that brought this article to life.

The author acknowledges support from grants ANR-20-CE40-0009fh (Project TRE-COS) and ANR-11-LABX-0020 (Labex Lebesgue), as well as from the Fondation Simone et Cino Del Duca – Institut de France.

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