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Hyperbolic relaxation of the viscous Cahn–Hilliard equation with a symport term for biological applications

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We consider the hyperbolic relaxation of the viscous Cahn–Hilliard equation with a symport term. This equation is characterized by the presence of the additional inertial term $\tau_D \phi_{tt}$ that accounts for the relaxation of the diffusion flux. We suppose that τ_D is dominated by the viscosity coefficient δ . Endowing the equation with Dirichlet boundary conditions, we show that it generates a dissipative dynamical system acting on a certain phase-space, depending on τ_D . This system is shown to possess a global attractor that is upper semicontinuous at $\tau_D = \delta = 0$. Then, we construct a family of exponential attractors $\mathfrak{F}_{\tau_D, \delta}$ which is a robust perturbation of an exponential attractor of the Cahn–Hilliard equation; namely, the symmetric Hausdorff distance between $\mathfrak{F}_{\tau_D, \delta}$ and $\mathfrak{F}_{0,0}$ tends to 0 as (τ_D, δ) tends to $(0, 0)$ in an explicitly controlled way. Finally, we present numerical simulations of the time evolution of weak solutions as a function of parameters.

KEYWORDS

Cahn–Hilliard equations, global attractors, numerical stability, robust exponential attractors, simulations, symport term

MSC CLASSIFICATION

35B40, 35B41, 35M99, 37L25, 65M12, 65M60

1 | INTRODUCTION

The classical Cahn–Hilliard (CH) equation was proposed in earlier studies [1, 2] in order to describe phase separation processes in binary alloys. During the last decades, it has been analyzed in depth from the mathematical and numerical points of view. It would simply be impossible to cite all papers related to the mathematical and numerical analysis of this equation; we refer the interested reader to, for example, Miranville [3]. The CH equation (or some of its variants) has been used in many other applications, such as dealloying in corrosion processes (see Erlebacher et al. [4]), population dynamics (see Cohen & Murray [5]), tumor growth (see Miranville [3]), bacterial films (see Klapper & Dockery [6]), thin films (see Oron et al. [7]), chemistry (see Verdasca & Dewel [8]), and image processing (see Miranville [3]). Recently, researchers have devoted tremendous attention on the relaxed (CH) system, that is, the viscous CH system and its perturbed form with a hyperbolic relaxation effect. Both systems have been well-studied theoretically in papers which are mainly focused on the well-posedness and the global attractor, exponential attractors, and so on (see previous research [9–13]). Formally, the governing equation of the system is slightly different from the CH equation by incorporating two extra terms, including a strong damping (called “viscosity”) term and a hyperbolic relaxation term (called “inertial”). The viscous effects were first proposed by Novick-Cohen [14] in order to introduce additional smoothing effects and can be

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viewed as a singular limit of the phase field equations for phase transitions. The hyperbolic relaxation term was proposed by Galenko and Lebedev [18], Galenko and Jou [17], Criado-Sancho et al. [15], and Galenko [16], in order to describe strongly nonequilibrium decompositions generated by rapid solidification under super cooling into the spinodal region occurring in certain materials (e.g., glasses).

We will give a brief description of the relaxed model equations that we will study in this paper, together with a symport term. We suppose that a binary alloy is cooled down enough to undergo a phase separation process. Naming $\phi = \phi(x, t)$ the relative concentration of one component of the system and $J = J(x, t)$ the diffusion flux, the balance law for concentration gives

$$\phi_t = -\nabla \cdot J. \quad (1.1)$$

In order to describe the evolution for ϕ , we need to introduce a constitutive assumption on J . We shall assume that

$$\tau_D J_t + J = -\nabla \left(\frac{\delta E}{\delta \phi} + \delta \phi_t \right), \quad (1.2)$$

where $\tau_D \geq 0$ is a relaxation parameter, $\delta \geq 0$ is a viscosity parameter, and $E(\phi)$ is the total free energy that takes the form

$$E(\phi) = \int_{\Omega} \left(\frac{\epsilon^2}{2} |\nabla \phi|^2 + F(\phi) \right) dx,$$

ϵ being a positive constant that measures the interfacial width and $F(\phi)$ being a double-well potential accounting for the presence of two components (e.g., $F(s) = (s^2 - 1)^2$).

By combining (1.1) with (1.2) and setting $\epsilon = 1$ for the sake of simplicity, we obtain

$$\tau_D \phi_{tt} + \phi_t = \Delta \mu, \quad (1.3)$$

$$\mu = -\Delta \phi + f(\phi) + \delta \phi_t, \quad (1.4)$$

where $f(\phi) = F'(\phi)$.

Systems (1.3) and (1.4) are usually endowed with the following boundary and initial conditions on the order parameter:

$$\phi(x, t) = \Delta \phi(x, t) = 0, \quad \text{on } \partial \Omega \times (0, +\infty), \quad (1.5)$$

$$\phi(0, t) = \phi_0 \quad \text{and} \quad \tau_D \phi_t(0, t) = \tau_D \phi_1, \quad t > 0,$$

where $\partial \Omega$ is the boundary of the domain Ω occupied by the system.

We actually consider in this paper the following problem

$$\tau_D \phi_{tt} + \phi_t + \Delta(\Delta \phi - f(\phi) + \delta \phi_t) = g(\phi), \quad (1.6)$$

where g is called symport term, typically defined by

$$g : \mathbb{R} \rightarrow \mathbb{R}, s \mapsto \frac{-ks}{k' + |s|}.$$

From a biological point of view, the nonlinear term g in this model stands for lactate transport from capillary, where k is the maximum transport rate and k' is the Michaelis–Menten constant (see earlier works [19–21]). Li et al. [19] consider the CH equation featuring such a source term g . They successfully establish the existence of a biologically relevant local in time solution, under (unfortunately rather restrictive) conditions on the parameters for global validity. Furthermore, they demonstrate additional regularity on the solutions, enabling them to establish a clear separation from the pure states in one and two spatial dimensions.

This article is organized as follows. In Sections 2 and 3, we introduce the problem and establish important notations for operators and spaces. In Section 4, we prove the existence, uniqueness, and regularity of solutions, and in Section 5, we obtain the existence of absorbing sets for the semigroups in the corresponding phase-spaces, as well as in corresponding

higher-order phase-spaces in Section 6. Section 7 presents the proof of the theorem regarding the existence of a robust family of exponential attractors. In the final section, Section 8, we provide numerical simulations.

2 | SETTING OF THE PROBLEM

Let us consider the following initial and boundary value problem in a bounded and regular domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$:

$$\tau_D \phi_{tt} + \phi_t - \Delta(-\Delta\phi + f(\phi) + \delta\phi_t) = g(\phi), \quad \text{in } \Omega \times (0, +\infty), \quad (2.1)$$

$$\phi(0) = \phi_0 \text{ and } \tau_D \phi_t(0) = \tau_D \phi_1, \quad \text{in } \Omega, \quad (2.2)$$

$$\phi = \Delta\phi = 0 \text{ on } \partial\Omega \times (0, \infty). \quad (2.3)$$

The case $\tau_D > 0$ and $\delta = 0$ seems out of reach in dimensions 2 and 3, where the standard energy estimates fail to hold (the problems arise concerning the uniqueness result). This obstacle is bypassed in one dimension, taking advantage of the continuous embedding $H^1(\Omega) \rightarrow C(\bar{\Omega})$ (see previous research [9, 10, 12, 13]). On account of this remark, we need to require that τ_D is dominated by δ . Thus, we make the assumption

$$\tau_D, \delta \in [0, 1] \text{ and } \delta \geq \mu \tau_D \quad \forall \mu \in]0, 1]. \quad (2.4)$$

The assumptions on the nonlinear term f are

$$f(0) = 0 \text{ and } f \in C^2(\mathbb{R}), \quad (2.5)$$

f'' is locally Lipschitz continuous, with

$$|f''(s)| \leq c(1 + |s|) \quad \forall s \in \mathbb{R}, \text{ for some } c \geq 0, \quad (2.6)$$

and

$$\liminf_{|s| \rightarrow \infty} \frac{f(s)}{s} \geq -\lambda_1, \quad (2.7)$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$.

The symport term g satisfies

$$g \in C^1(\mathbb{R}), \quad (2.8)$$

$$g \text{ is bounded}, \quad (2.9)$$

$$g \text{ is Lipschitz continuous}. \quad (2.10)$$

3 | NOTATION AND FUNCTION SPACES

Let us set $H := L^2(\Omega)$ and denote by (\cdot, \cdot) the scalar product both in H and in $H \times H$ and by $\|\cdot\|$ the induced norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic real Banach space X . Next, we set $V := H_0^1(\Omega)$, so that $V' = H^{-1}(\Omega)$ is the topological dual of V . The space V is endowed with the scalar product

$$((\phi, v)) := \int_{\Omega} \nabla\phi \cdot \nabla v dx,$$

and the related norm.

We also denote by $A : D(A) \rightarrow H$ the Laplace operator with homogeneous Dirichlet boundary conditions. It is well-known that A is a strictly positive operator with $D(A) = H^2(\Omega) \cap V$, so that we can define, for $s \in \mathbb{R}$, its powers $A^s : D(A^s) \rightarrow H$. Moreover, we introduce the scale of Hilbert spaces

$$\mathbf{V}_s^0(\tau_D) := D(A^{\frac{s+1}{2}}) \times D(A^{\frac{s-1}{2}}). \quad (3.1)$$

The spaces \mathbf{V}_s^0 are naturally endowed with the graph norm,

$$\|(\phi, v)\|_{\mathbf{V}_s^0(\tau_D)}^2 := \|A^{\frac{s+1}{2}} \phi\|^2 + \tau_D \|A^{\frac{s-1}{2}} v\|^2.$$

For further purposes, we also need to consider a different scale of Hilbert spaces, namely,

$$\mathbf{H}_s = \mathbf{H}_s(\tau_D, \delta) = \begin{cases} D(A^{\frac{s+1}{2}}) \times D(A^{\frac{s-2}{2}}), & \text{if } \tau_D > 0, \\ D(A^{\frac{s+1}{2}}) \times D(A^{\frac{s-2}{2}}), & \text{if } \tau_D = 0 \text{ and } \delta > 0, \\ D(A^{\frac{s+1}{2}}) \times D(A^{\frac{s-3}{2}}) & \text{if } \tau_D = \delta = 0, \end{cases}$$

with norm

$$\|(\phi, v)\|_{\mathbf{H}_s}^2 = \|A^{\frac{s+1}{2}} \phi\|^2 + \tau_D \|A^{\frac{s-1}{2}} v\|^2 + \delta \|A^{\frac{s-2}{2}} v\|^2 + \|A^{\frac{s-3}{2}} v\|^2.$$

Observe that the injections $\mathbf{V}_{s_1}^0 \hookrightarrow \mathbf{V}_{s_2}^0$ and $\mathbf{H}_{s_1} \hookrightarrow \mathbf{H}_{s_2}$ are compact whenever $s_1 > s_2$.

Remark 3.1. It is important to note that alternative boundary conditions could be taken into consideration. For example, we can impose the following:

$$\frac{\partial \phi}{\partial n} = \frac{\partial \Delta \phi}{\partial n} = 0, \quad \text{on } \partial \Omega,$$

where $\frac{\partial}{\partial n}$ represents the outward normal derivative to $\partial \Omega$.

In this scenario, by integrating (2.1) over Ω and applying the divergence theorem, we can easily derive the following identity:

$$\frac{d}{dt} [\tau_D \langle \phi_t \rangle + \langle \phi \rangle] = \langle g(\phi) \rangle, \quad t \geq 0,$$

where $\langle \cdot \rangle = \frac{1}{\text{Vol}(\Omega)} \int_{\Omega} \cdot \, dx$.

However, in order to demonstrate the dissipative nature of the dynamical system, we must introduce an additional constraint into the phase-space. These specific issues will be addressed in a forthcoming paper.

The abstract version of (2.1) and (2.2) reads

$$\tau_D \frac{d^2 \phi}{dt^2} + \frac{d\phi}{dt} + A(A\phi + f(\phi) + \delta \phi_t) = g(\phi). \quad (3.2)$$

By (2.7), F satisfies

$$F(s) \geq -\frac{\eta}{2} s^2 - c_\eta \quad (3.3)$$

for some $\eta < \lambda_1$ and some $c_\eta \geq 0$.

Lemma 3.2. *We assume that (2.4)–(2.9) hold. There exists $R_0 \geq 0$ such that, given any $R_1 \geq 0$, there is $t_0 = t_0(R_1)$ with the following property: for all $(\phi_0, \phi_1) \in \mathbf{V}_0^0$ with*

$$\|(\phi_0, \phi_1)\|_{\mathbf{V}_0^0} \leq R_1,$$

there holds

$$\|(\phi, \phi_t)\|_{\mathbf{V}_0^0} \leq R_0, \quad \forall t \geq t_0. \quad (3.4)$$

Moreover,

$$\sup_{t \geq 0} \|(\phi, \phi_t)\|_{V_0^0} + \int_0^\infty \|\phi_t(y)\|^2 dy \leq C_0 \quad (3.5)$$

for some $C_0 = C_0(R_1) \geq 0$.

Proof. Multiplying (3.2) by $\alpha A^{-1}\phi + A^{-1}\phi_t$ and integrating over Ω , we obtain

$$\frac{d}{dt} Y(t) + (2 - 2\alpha\tau_D) \|A^{-\frac{1}{2}}\phi_t\|^2 + 2\alpha \|A^{\frac{1}{2}}\phi\|^2 + 2\delta \|\phi_t\|^2 + 2\alpha(f(\phi), \phi) = 2\alpha(g(\phi), A^{-1}\phi) + 2(g(\phi), A^{-1}\phi_t), \quad (3.6)$$

where

$$Y(t) = 2\alpha\tau_D(\phi_t, A^{-1}\phi) + \tau_D \|A^{-\frac{1}{2}}\phi_t\|^2 + \alpha \|A^{-\frac{1}{2}}\phi\|^2 + \|A^{\frac{1}{2}}\phi\|^2 + 2\delta \|\phi\|^2 + 2(F(\phi), 1). \quad (3.7)$$

Thanks to (3.3) and Poincaré's inequality, we deduce that

$$2\alpha(f(\phi), \phi) \geq -\alpha(1 - \eta) \|A^{\frac{1}{2}}\phi\|^2 - \alpha c_\eta$$

and

$$2(F(\phi), 1) \geq -(1 - \eta) \|A^{\frac{1}{2}}\phi\|^2 - c_\eta$$

for some $\eta < \lambda_1$.

Thanks to the Cauchy–Schwarz inequality, we have

$$2\alpha\tau_D(\phi_t, A^{-1}\phi) \leq \frac{\tau_D}{2} \|A^{-\frac{1}{2}}\phi_t\|^2 + \frac{\alpha}{2} \|A^{-\frac{1}{2}}\phi\|^2.$$

Combining the above inequalities, we can rewrite (3.7) as

$$c_1(\tau_D \|A^{-\frac{1}{2}}\phi_t\|^2 + \|A^{\frac{1}{2}}\phi\|^2 + \delta \|\phi\|^2) - c_\eta \leq Y(t) \leq c_2(\tau_D \|A^{-\frac{1}{2}}\phi_t\|^2 + \|A^{\frac{1}{2}}\phi\|^2 + \delta \|\phi\|^2) + c_\eta. \quad (3.8)$$

Owing to (2.9) and Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} 2\alpha|(g(\phi), A^{-1}\phi)| &\leq \alpha c \|A^{-1}\phi\| \\ &\leq c \|\phi\|^2 + \alpha c \end{aligned}$$

and

$$\begin{aligned} 2|(g(\phi), A^{-1}\phi_t)| &\leq c \|A^{-1}\phi_t\| \\ &\leq \frac{1}{4} \|A^{-\frac{1}{2}}\phi_t\|^2 + \alpha c. \end{aligned}$$

Inserting the above estimates into (3.6), we deduce, for $\frac{7}{4} - 2\alpha\tau_D > 0$, that

$$\frac{d}{dt} Y(t) + \gamma Y(t) + \delta \|\phi_t\|^2 \leq c, \quad \gamma > 0, \quad c \geq 0. \quad (3.9)$$

Applying Gronwall's lemma to (3.9) (see, e.g., Grasselli et al. [22]), we have

$$\|(\phi, \phi_t)\|_{V_0^0} + \delta \int_0^t \|\phi_t(y)\|^2 dy \leq e^{-\gamma t} \|(\phi_0, \phi_1)\|_{V_0^0} + c', \quad \gamma > 0, \quad c' \geq 0, \quad t \geq 0. \quad (3.10)$$

By making t tend to infinity, we find the results. \square

4 | EXISTENCE, UNIQUENESS, AND REGULARITY OF SOLUTIONS

The estimates below allow us to state the following theorem.

Theorem 4.1. *Assume that (2.4)–(2.9) and (2.10) are verified. Then for $(\phi_0, \phi_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$, systems (2.1)–(2.3) have a unique solution (ϕ, ϕ_t) such that*

$$\phi \in L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega)), \quad (4.1)$$

$$\phi_t \in L^\infty(0, T; L^2(\Omega)), \quad (4.2)$$

$$\delta\phi_t \in L^\infty(0, T; D(A^{-\frac{1}{2}})) \cap L^2(0, T; D(A^{\frac{1}{2}})), \quad (4.3)$$

$$\phi_{tt} \in L^2(0, T; D(A^{-1})). \quad (4.4)$$

Proof. **1) Existence.** The proof of existence is based on the standard Galerkin method and the following estimates.

We rewrite (3.2) in the equivalent form

$$\tau_D\phi_{tt} + \phi_t + A(A\phi + f(\phi) + \delta\phi_t) - g(\phi) = 0. \quad (4.5)$$

Multiplying (4.5) by $\phi_t + \alpha\phi + \frac{1}{4}A^{-2}\phi_{tt}$ and integrating over Ω , we obtain

$$\begin{aligned} \frac{d}{dt}H(t) + \left(\frac{3}{2} - 2\alpha\tau_D\right) \|\phi_t\|^2 + 2\alpha\|A\phi\|^2 + 2\delta\|A^{\frac{1}{2}}\phi_t\|^2 + \frac{\tau_D}{2}\|A^{-1}\phi_{tt}\|^2 &= \frac{1}{2}(\phi_t f'(\phi), A^{-1}\phi_t) \\ - 2(Af(\phi), \phi_t) - 2\alpha(Af(\phi), \phi) + 2(g(\phi), \phi_t) + 2\alpha(g(\phi), \phi) + \frac{1}{2}(g(\phi), A^{-2}\phi_{tt}), \end{aligned} \quad (4.6)$$

where

$$H(t) = \tau_D\|\phi_t\|^2 + 2\alpha\tau_D(\phi, \phi_t) + \alpha\|\phi\|^2 + \|A\phi\|^2 + \frac{1}{2}\|A^{-1}\phi_t\|^2 + \frac{1}{2}(\phi, \phi_t) + \frac{\delta}{2}\|A^{-\frac{1}{2}}\phi_t\|^2 + \frac{1}{2}(f(\phi), A^{-1}\phi_t). \quad (4.7)$$

Thus, for $\alpha \in (0, \frac{1}{2\tau_D})$, we deduce that

$$\begin{aligned} H(t) &\geq c_3(\tau_D\|\phi_t\|^2 + \|A\phi\|^2 + \delta\|A^{-\frac{1}{2}}\phi_t\|^2 + \|A^{-1}\phi_t\|^2), \quad c_3 > 0, \\ H(t) &\leq c_4(\tau_D\|\phi_t\|^2 + \|A\phi\|^2 + \delta\|A^{-\frac{1}{2}}\phi_t\|^2 + \|A^{-1}\phi_t\|^2) + c_5, \quad c_4 > 0, \quad c_5 \geq 0. \end{aligned} \quad (4.8)$$

Owing to (2.9), (3.5), and Young's, Poincaré's, and proper interpolation inequalities, we find

$$\begin{aligned} 2\alpha|(A^{\frac{1}{2}}\phi f'(\phi), A^{\frac{1}{2}}\phi)| &\leq \alpha\|A^{\frac{1}{2}}\phi\|^2\|f'(\phi)\|_{L^\infty(\Omega)} \\ &\leq \alpha c(1 + \|\phi\|_{L^\infty(\Omega)}^2) \\ &\leq \frac{\alpha}{4}\|A\phi\|^2 + c_6 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}|(\phi_t f'(\phi), A^{-1}\phi_t)| &\leq c\|f'(\phi)\|_{L^\infty(\Omega)}\|\phi_t\|\|A^{-\frac{1}{2}}\phi_t\| \\ &\leq c(1 + \|\phi\|_{L^\infty(\Omega)}^2)\|\phi_t\| \\ &\leq \frac{\alpha}{4}\|A\phi\|^2 + \frac{1}{4}\|\phi_t\|^2 + c_7. \end{aligned}$$

Moreover,

$$\begin{aligned} 2|(g(\phi), \phi_t)| &\leq \|g(\phi)\| \|\phi_t\| \\ &\leq c \|\phi_t\| \\ &\leq \frac{1}{4} \|\phi_t\|^2 + c_8 \end{aligned}$$

and

$$\begin{aligned} 2\alpha|(g(\phi), \phi)| &\leq \alpha \|g(\phi)\| \|\phi\| \\ &\leq \alpha c \|\phi\| \\ &\leq \frac{\alpha}{4} \|A\phi\|^2 + c_9. \end{aligned}$$

Besides,

$$\begin{aligned} \frac{1}{2}|(g(\phi), A^{-2}\phi_t)| &\leq \|g(\phi)\| \|A^{-2}\phi_t\| \\ &\leq c \|A^{-1}\phi_t\| \\ &\leq \frac{\tau_D}{4} \|A^{-1}\phi_t\|^2 + c_{10}. \end{aligned}$$

Combining the above estimates, we can rewrite (4.6) as

$$\frac{d}{dt}H(t) + c_{11}H(t) + \delta \|A^{\frac{1}{2}}\phi_t\|^2 + \tau_D \|A^{-1}\phi_t\|^2 \leq c_{12}. \quad (4.9)$$

Applying Gronwall's lemma to (4.9), we obtain

$$H(t) + \int_0^T (\delta \|A^{\frac{1}{2}}\phi_t(s)\|^2 + \tau_D \|A^{-1}\phi_t(s)\|^2) ds \leq H(0)e^{-c_{11}t} + \frac{c_{12}}{c_{11}}, \quad c_{11} > 0, \quad t \geq 0. \quad (4.10)$$

Consider a spectral basis, $(\omega_i)_{i \geq 1}$ of $-\Delta$, with Dirichlet boundary conditions, associated with the eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots$ which forms an orthonormal basis in $L^2(\Omega)$ and an orthogonal one in $H_0^1(\Omega)$. We set $V_n = \text{Span}\{\omega_1, \dots, \omega_n\}$ and

$$\phi_n = \sum_{i=1}^n \phi_{i,n} \omega_i.$$

Then we consider the following approximating problem, written in functional form,

$$\tau_D \frac{d^2 \phi_n}{dt^2} + \frac{d\phi_n}{dt} + A \left(A\phi_n + f(\phi_n) + \delta \frac{d\phi_n}{dt} \right) = g(\phi_n), \quad (4.11)$$

$$\phi_n(0) = \phi_{n,0}, \quad \tau_D \frac{d}{dt} \phi_n(0) = \tau_D \phi_{n,1}. \quad (4.12)$$

Replacing ϕ by ϕ_n in estimate (4.10) and thanks to (4.8), we conclude that

- is bounded in $L^\infty(0, T; H^2(\Omega) \cap H_0^1(\Omega))$,
- is bounded in $L^\infty(0, T; L^2(\Omega))$,
- is bounded in $L^\infty(0, T; D(A^{-\frac{1}{2}})) \cap L^2(0, T; D(A^{\frac{1}{2}}))$,
- is bounded in $L^2(0, T; D(A^{-1}))$.

Finally, the passage to the limit is standard and is based on classical Aubin–Lions compactness results (see, e.g., Miranville [3]), which proves the first part of the Theorem.

2) Uniqueness. Consider $(\phi^1, \frac{d\phi^1}{dt})$ and $(\phi^2, \frac{d\phi^2}{dt})$ two solutions of systems (2.1) and (2.2) with respective initial conditions $(\phi_0^1, \frac{d\phi_0^1}{dt})$ and $(\phi_0^2, \frac{d\phi_0^2}{dt})$ in $H^2(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega)$. We set

$$\left(\phi, \frac{d\phi}{dt} \right) = \left(\phi^1 - \phi^2, \frac{d\phi^1}{dt} - \frac{d\phi^2}{dt} \right).$$

Systems (2.1) and (2.2) become

$$\tau_D \phi_{tt} + \phi_t + \delta A \phi_t + A^2 \phi + A(f(\phi^1) - f(\phi^2)) = g(\phi^1) - g(\phi^2) \quad (4.13)$$

and

$$\phi(0) = \phi_0^1 - \phi_0^2, \tau_D \phi_t(0) = \tau_D \phi_1^1 - \tau_D \phi_1^2. \quad (4.14)$$

Multiplying (4.13) by ϕ_t and integrating over Ω , we have

$$\frac{d}{dt} Y(t) + 2\|\phi_t\|^2 + 2\delta \|A^{\frac{1}{2}} \phi_t\|^2 = -2(A(f(\phi^1) - f(\phi^2)), \phi_t) + 2(g(\phi^1) - g(\phi^2), \phi_t), \quad (4.15)$$

where

$$Y(t) = \tau_D \|\phi_t\|^2 + \|A\phi\|^2.$$

Owing to (2.5), (2.6), (3.3), (2.10), and Young's, Poincaré's, and proper interpolation inequalities, we deduce that

$$\begin{aligned} -2|(A(f(\phi^1) - f(\phi^2)), \phi_t)| &\leq \|A(f(\phi^1) - f(\phi^2))\| \|\phi_t\| \\ &\leq (\|A\phi^2(f'(\phi^1) - f'(\phi^2))\| + \| |A^{\frac{1}{2}} \phi^2|^2 (f''(\phi^1) - f''(\phi^2)) \| \\ &\quad + \|A\phi f'(\phi^1)\| + \| |A^{\frac{1}{2}} \phi|^2 (f''(\phi^1)) \|) \|\phi_t\| \\ &\leq c(\|A\phi^2\| \|\phi\| \int_0^1 f''(s\phi^1 + (1-s)\phi^2) ds + \| |A^{\frac{1}{2}} \phi^2|^2 \| \|f''(\phi^1) - f''(\phi^2)\| \\ &\quad + \|A\phi\| \|f'(\phi^1)\| + \| |A^{\frac{1}{2}} \phi|^2 \| \|f''(\phi^1)\|) \|\phi_t\| \\ &\leq c(\|A\phi^2\| \|\phi\|_{L^\infty(\Omega)}^2 + \| |A^{\frac{1}{2}} \phi^2|^2 \| \|\phi\|_{L^\infty(\Omega)} + \|A\phi\| (1 + \|\phi^1\|_{L^\infty(\Omega)}^2) \\ &\quad + \| |A^{\frac{1}{2}} \phi|^2 \| \|\phi^2\|_{L^\infty(\Omega)}) \|\phi_t\| \\ &\leq (\text{thanks to (4.10) and Agmon's inequality}) \\ &\leq c\|A\phi\| \|\phi_t\| \\ &\leq c\|A\phi\|^2 + \|\phi_t\|^2 \end{aligned}$$

and

$$\begin{aligned} |(g(\phi^1) - g(\phi^2), \phi_t)| &\leq \|g(\phi^1) - g(\phi^2)\| \|\phi_t\| \\ &\leq c\|\phi\| \|\phi_t\| \\ &\leq c\|A\phi\|^2 + \|\phi_t\|^2. \end{aligned}$$

Combining the above estimates, we deduce from (4.15) that

$$\frac{d}{dt} Y(t) + \delta \| |A^{\frac{1}{2}} \phi_t|^2 \| \leq c_{15} Y(t), \quad c_{15} > 0, t \geq 0. \quad (4.16)$$

Applying Gronwall's lemma to (4.16), we obtain

$$\|(\phi, \phi_t)\|_{\mathbf{V}_1^0} \leq e^{c_{15}t} \|(\phi_0, \phi_1)\|_{\mathbf{V}_1^0}, \quad c_{15} > 0, t \geq 0. \quad (4.17)$$

This yields the uniqueness, as well the continuous dependence with respect to the initial data. \square

Let $S_{\tau_D, \delta}^0(t)$ be the semigroup associated with (2.1) and (2.2) and defined on the phase-space \mathbf{V}_1^0 .

By means of the semigroup $S_{\tau_D, \delta}^0(t)$, we can now construct a one-parameter family of operators

$$S_{\tau_D, \delta}^1(t) : \mathbf{H}_1 \rightarrow \mathbf{H}_1$$

in the following manner:

$$S_{\tau_D, \delta}^1(t)(\phi_0, \phi_1) = \begin{cases} S_{\tau_D, \delta}^0(t)(\phi_0, \phi_1), & \text{if } \tau_D > 0, \\ (S_{0, \delta}^0(t)\phi_0, L_\delta(S_{0, \delta}^0(t)\phi_0)) & \text{if } \tau_D = 0, \end{cases}$$

where

$$L_\delta(\phi) = -(\mathbf{I} + \delta A)^{-1}[A(A\phi + f(\phi)) - g(\phi)]. \tag{4.18}$$

Observe that, when $\tau_D = 0$, we read directly from the parabolic equation the equalities

$$\phi(t) = S_{0,\delta}^0(t)\phi_0 \text{ and } \phi_t(t) = L_\delta(S_{0,\delta}^0(t)\phi_0).$$

Therefore,

$$S_{\tau_D,\delta}^1(t)(\phi_0, \phi_1) = (\phi(t), \phi_t(t)),$$

both for the parabolic and the hyperbolic cases. It should be noted that, when $\tau_D = 0$, the initial datum ϕ_1 is somehow fictitious, for it does not play any role in the dynamics of the system.

Remark 4.2. Obviously, when $\tau_D > 0$, $S_{\tau_D,\delta}^1(t)$ fulfills the semigroup properties on the space \mathbf{H}_1 . This is false when $\tau_D = 0$. Indeed, in this case, $S_{0,\delta}^1(t)$ maps the space \mathbf{H}_1 into the manifold $M_1(\delta)$ of \mathbf{H}_1 given by

$$M_1(\delta) = \{(\phi, \phi_t) \in \mathbf{H}_1(0, \delta) : \phi_t = L_\delta(\phi)\}.$$

In particular, $S_{0,\delta}^1(0)$ is the projection of \mathbf{H}_1 onto $M_1(\delta)$. However, it is easily seen that the restriction of $S_{0,\delta}^1(t)$ on $M_1(\delta)$ is again a semigroup.

The above remark suggests the way to construct the proper phase-space for $S_{0,\delta}^1(t)$. Hence, we define

$$\mathbf{V}_s^1(\tau_D, \delta) = \begin{cases} \mathbf{H}_s(\tau_D, \delta), & \text{if } \tau_D > 0, \\ M_s(\delta) & \text{if } \tau_D = 0, \end{cases}$$

where

$$M_s(\delta) = \{(\phi, \phi_t) \in \mathbf{H}_s(0, \delta) : \phi_t = L_\delta(\phi)\}.$$

It is immediate to check that \mathbf{V}_s^1 is a closed subset of \mathbf{H}_s . Thus, \mathbf{V}_s^1 is a complete metric space, with the distance inherited by the norm of \mathbf{H}_s . Abusing the notation, we will write

$$\|(\phi, \phi_t)\|_{\mathbf{V}_s^1} \text{ to mean } \|(\phi, \phi_t)\|_{\mathbf{H}_s},$$

for all $(\phi, \phi_t) \in \mathbf{V}_s^1$.

Lemma 4.3. For any $R \geq 0$, there exists $K = K(R) \geq 1$ such that, for any two initial data $\phi_1, \phi_2 \in \mathbf{V}_1^\ell$ with $\|\phi_i\|_{\mathbf{V}_1^\ell} \leq R$ (here and in the sequel, $\ell = 0, 1$), there holds

$$\|S_{\tau_D,\delta}^\ell(t)\phi_1 - S_{\tau_D,\delta}^\ell(t)\phi_2\|_{\mathbf{V}_1^\ell} \leq Ke^{kt}\|\phi_1 - \phi_2\|_{\mathbf{V}_1^\ell},$$

where

$$\phi_1 = (\phi_{0,1}, \phi_{1,1}) \text{ and } \phi_2 = (\phi_{0,2}, \phi_{1,2}).$$

Proof. In this proof, the constant $c \geq 0$ depends on R . Firstly, we consider the case $\ell = 0$. Given two solutions ϕ^1 and ϕ^2 corresponding to different initial data $\phi_1, \phi_2 \in \mathbf{V}_1^0$, the difference $\bar{\phi} = \phi^1 - \phi^2$ fulfills the equation

$$\tau_D \bar{\phi}_{tt} + \bar{\phi}_t + \delta A \bar{\phi}_t + A^2 \bar{\phi} + A(f(\phi^1) - f(\phi^2)) = g(\phi^1) - g(\phi^2). \tag{4.19}$$

Multiplying (4.19) by $\bar{\phi}_t$ over Ω , we obtain

$$\frac{d}{dt}(\tau_D \|\bar{\phi}_t\|^2 + \|A\bar{\phi}\|^2) + 2\|\bar{\phi}_t\|^2 + 2\delta\|A^{\frac{1}{2}}\bar{\phi}\|^2 = -2(A(f(\phi^1) - f(\phi^2)), \bar{\phi}_t) + 2(g(\phi^1) - g(\phi^2), \bar{\phi}_t). \tag{4.20}$$

Owing to (2.6), (2.7), (2.10), and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned}
 -2|(A(f(\phi^1) - f(\phi^2)), \bar{\phi}_t)| &\leq \|A(f(\phi^1) - f(\phi^2))\| \|\bar{\phi}_t\| \\
 &\leq \|A\phi^2(f'(\phi^1) - f'(\phi^2)) + |A^{\frac{1}{2}}\phi^2|^2(f''(\phi^1) - f''(\phi^2)) \\
 &\quad + A\bar{\phi}f'(\phi^1) + |A^{\frac{1}{2}}\phi|^2(f''(\phi^1))\| \|\bar{\phi}_t\| \\
 &\leq c(\|A\phi^2\| \|\bar{\phi}\| \int_0^1 f''(s\phi^1 + (1-s)\phi^2) ds + \| |A^{\frac{1}{2}}\phi^2|^2 \| \|f''(\phi^1) - f''(\phi^2)\| \\
 &\quad + \|A\bar{\phi}\| \|f'(\phi^1)\| + \| |A^{\frac{1}{2}}\bar{\phi}|^2 \| \|f''(\phi^1)\|) \|\bar{\phi}_t\| \\
 &\leq c(\|A\phi^2\| \|\bar{\phi}\|_{L^\infty(\Omega)}^2 + \| |A^{\frac{1}{2}}\phi^2|^2 \| \|\bar{\phi}\|_{L^\infty(\Omega)} + \|A\bar{\phi}\| (1 + \|\phi^1\|_{L^\infty(\Omega)}^2 \\
 &\quad + \| |A^{\frac{1}{2}}\bar{\phi}|^2 \| \|\phi^2\|_{L^\infty(\Omega)}) \|\bar{\phi}_t\| \\
 &\leq (\text{thanks to (4.10) and Agmon's inequality}) \\
 &\leq \|A\bar{\phi}\| \|\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\|^2 + \frac{1}{2}\|\bar{\phi}_t\|^2
 \end{aligned}$$

and

$$\begin{aligned}
 2|(g(\phi^1) - g(\phi^2), \bar{\phi}_t)| &\leq \|(g(\phi^1) - g(\phi^2))\| \|\bar{\phi}_t\| \\
 &\leq c\|\phi^1 - \phi^2\| \|\bar{\phi}_t\| \\
 &\leq c\|\bar{\phi}\| \|\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\| \|\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\|^2 + \frac{1}{2}\|\bar{\phi}_t\|^2.
 \end{aligned}$$

Combining the above estimates, we have

$$\frac{d}{dt} \|(\bar{\phi}, \bar{\phi}_t)\|_{\mathbf{V}_1^0} + \|\bar{\phi}_t\|^2 \leq c\|(\bar{\phi}, \bar{\phi}_t)\|_{\mathbf{V}_1^0}. \quad (4.21)$$

Applying Gronwall's lemma to (4.21) gives

$$\|(\bar{\phi}, \bar{\phi}_t)\|_{\mathbf{V}_1^0}^2 + \int_0^t \|\bar{\phi}_t(y)\|^2 dy \leq e^{ct} \|(\bar{\phi}(0), \bar{\phi}_t(0))\|_{\mathbf{V}_1^0}^2, \quad c > 0, t \geq 0. \quad (4.22)$$

To treat the case $\ell = 1$, we test (4.19) by $A^{-2}\bar{\phi}_{tt}$ to obtain

$$\begin{aligned}
 \frac{d}{dt} H(t) - 2\|\bar{\phi}_t\|^2 + 2\tau_D \|A^{-1}\bar{\phi}_{tt}\|^2 &= 2((f'(\phi^1) - f'(\phi^2))\phi_t^2, A^{-1}\bar{\phi}_t) + 2(f'(\phi^1)\bar{\phi}_t, A^{-1}\bar{\phi}_t) \\
 &\quad + 2(g(\phi^1) - g(\phi^2), A^{-2}\bar{\phi}_{tt}),
 \end{aligned} \quad (4.23)$$

where

$$H(t) = \|A^{-1}\bar{\phi}_t\|^2 + 2(\bar{\phi}, \bar{\phi}_t) + \delta \| |A^{-\frac{1}{2}}\bar{\phi}_t|^2 \| + 2(f(\phi^1) - f(\phi^2), A^{-1}\bar{\phi}_t)$$

is such that

$$c_{16}(\|A^{-1}\bar{\phi}_t\|^2 + \delta \| |A^{-\frac{1}{2}}\bar{\phi}_t|^2 \| - \|A\bar{\phi}\|^2) \leq H(t) \leq c_{17}(\|A^{-1}\bar{\phi}_t\|^2 + \delta \| |A^{-\frac{1}{2}}\bar{\phi}_t|^2 \| + \|A\bar{\phi}\|^2).$$

Due to (2.6), (2.7), (2.10), and Young's, Poincaré's, and proper interpolation inequalities, we deduce that

$$\begin{aligned}
 2|(f'(\phi^1)\bar{\phi}_t, A^{-1}\bar{\phi}_t)| &\leq \|f'(\phi^1)\|_{L^\infty(\Omega)} \|\bar{\phi}_t\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|\bar{\phi}_t\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|\bar{\phi}_t\|^2
 \end{aligned}$$

and

$$\begin{aligned}
 2|(f'(\phi^1) - f'(\phi^2))\phi_t^2, A^{-1}\bar{\phi}_t| &\leq c\|f'(\phi^1) - f'(\phi^2)\| \|\phi_t^2\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|\bar{\phi}\| \int_0^1 f''(s\phi^1 + (1-s)\phi^2) ds \|\phi_t^2\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|\bar{\phi}\| \|\phi_t^2\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\| \|\phi_t^2\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\|^2 \|\phi_t^2\|^2 + \|A^{-1}\bar{\phi}_t\|^2.
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 2|(g(\phi^1) - g(\phi^2), A^{-2}\bar{\phi}_t)| &\leq \|g(\phi^1) - g(\phi^2)\| \|A^{-2}\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\| \|A^{-1}\bar{\phi}_t\| \\
 &\leq c\|A\bar{\phi}\|^2 + \tau_D \|A^{-1}\bar{\phi}_t\|^2.
 \end{aligned}$$

Combining the above inequalities, we can rewrite (4.23) as

$$\frac{d}{dt}H(t) + \tau_D \|A^{-1}\bar{\phi}_t\|^2 \leq c'H(t) + c\|\bar{\phi}_t\|^2 + c\|A\bar{\phi}\|^2 \|\phi_t^2\|^2 + c\|A\bar{\phi}\|^2. \tag{4.24}$$

Thanks to (4.22), (3.5), and Gronwall's lemma, we conclude that

$$\|(\bar{\phi}, \bar{\phi}_t)\|_{V_1} \leq ce^{c't} \|(\bar{\phi}(0), \bar{\phi}_t(0))\|_{V_1}, \quad c > 0, \quad t \geq 0. \tag{4.25}$$

Estimates (4.25) and (4.22) give the result of the lemma. □

5 | DISSIPATIVITY

Our first goal is to prove the existence of a bounded absorbing set for the semigroup $S_{\tau_D, \delta}^\ell(t)$ on the space-phase V_1^ℓ . To do so, we state the following Lemma.

Lemma 5.1. *Given $R \geq 0$ and $\nu \geq 0$, there exist $K_1 = K_1(R) \geq 0$, $K_0 = K_0(\nu) \geq 0$, and $e_0 > 0$ such that whenever*

$$\|(\phi_0, \phi_1)\|_{V_0} \leq \nu \text{ and } \|(\phi_0, \phi_1)\|_{V_1} \leq R,$$

there holds

$$\|S_{\tau_D, \delta}^\ell(t)(\phi_0, \phi_1)\|_{V_1} \leq K_1 e^{-e_0 t} + K_0. \tag{5.1}$$

Proof. Multiplying (3.2) by $\phi_t + \frac{\mu}{2}A\phi + \frac{1}{4}A^{-2}\phi_t$ (with $\mu > 0$ small enough), we have

$$\begin{aligned}
 \frac{d}{dt}Y(t) + (2\delta - \mu\tau_D)\|A^{\frac{1}{2}}\phi_t\|^2 + \frac{3}{2}\|\phi_t\|^2 + \mu\|A^{\frac{3}{2}}\phi\|^2 + \frac{\tau_D}{2}\|A^{-1}\phi_t\|^2 &= -\mu(Af(\phi), A\phi) \\
 + \frac{1}{2}(\phi_t f'(\phi), A^{-1}\phi_t) - 2(\phi_t f'(\phi), A\phi) + 2(g(\phi), \phi_t) + \mu(g(\phi), A\phi) + \frac{1}{2}(g(\phi), A^{-2}\phi_t), &
 \end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
 Y(t) = \tau_D\|\phi_t\|^2 + \mu\tau_D(A\phi, \phi_t) + \frac{\mu}{2}\|A^{\frac{1}{2}}\phi\|^2 + \frac{1}{2}(\phi, \phi_t) + \frac{\mu\delta}{2}\|A\phi\|^2 + \|A\phi\|^2 + \frac{1}{4}\|A^{-1}\phi_t\|^2 \\
 + \frac{\delta}{2}\|A^{-\frac{1}{2}}\phi_t\|^2 + \frac{1}{2}(f(\phi), A^{-1}\phi_t) + 2(f(\phi), A\phi). &
 \end{aligned} \tag{5.3}$$

Thanks to (2.6), (2.7), (3.5), and Young's, Poincaré's, and proper interpolation inequalities, we deduce that

$$\begin{aligned} \frac{1}{2}|(f(\phi), A^{-1}\phi_t)| &\leq \|\phi\| \|f'(\phi)\|_{L^\infty(\Omega)} \|A^{-1}\phi_t\| \\ &\leq c(1 + \|\phi\|_{L^\infty(\Omega)}^2) \|A^{-1}\phi_t\| \\ &\leq c(1 + \|A\phi\|) \|A^{-\frac{1}{2}}\phi_t\| \\ &\leq c\|A\phi\|^2 + c\|A^{-\frac{1}{2}}\phi_t\|^2 + c \end{aligned}$$

and

$$\begin{aligned} |(A^{-1}\phi_t, A\phi)| &\leq \|A^{-1}\phi_t\| \|A\phi\| \\ &\leq \|A^{-\frac{1}{2}}\phi_t\| \|A\phi\| \\ &\leq c\|A\phi\|^2 + \frac{1}{4}\|A^{-\frac{1}{2}}\phi_t\|^2. \end{aligned}$$

Besides,

$$\begin{aligned} \mu\tau_D|(A\phi, \phi_t)| &\leq \mu\tau_D\|A\phi\| \|\phi_t\| \\ &\leq \delta\|A\phi\| \|A^{\frac{1}{2}}\phi_t\| \\ &\leq c\|A\phi\|^2 + \delta\|A^{\frac{1}{2}}\phi_t\|^2. \end{aligned}$$

Thus,

$$c_{18}\|S_{\tau_D, \delta}^\ell(t)(\phi_0, \phi_1)\|_{V_1^\ell}^2 \leq Y(t) \leq c_{19}\|S_{\tau_D, \delta}^\ell(t)(\phi_0, \phi_1)\|_{V_1^\ell}^2 + c, \quad c_{39} > 0, \quad c_{38} > 0, \quad c \geq 0.$$

Similarly,

$$\begin{aligned} \frac{1}{2}|(\phi_t f'(\phi), A^{-1}\phi_t)| &\leq \|\phi_t\| \|f'(\phi)\|_{L^\infty(\Omega)} \|A^{-1}\phi_t\| \\ &\leq c(1 + \|\phi\|_{L^\infty(\Omega)}^2) \|\phi_t\| \|A^{-1}\phi_t\| \\ &\leq c(1 + \|A\phi\|) \|\phi_t\| \|\phi_t\| \\ &\leq \frac{1}{4}\|\phi_t\|^2 + c\|\phi_t\|^2 \|A\phi\|^2 \end{aligned}$$

and

$$\begin{aligned} -\mu|(Af(\phi), A\phi)| &\leq \mu\|A^{\frac{1}{2}}\phi\| \|f'(\phi)\|_{L^\infty(\Omega)} \|A^{\frac{3}{2}}\phi\| \\ &\leq \mu c(1 + \|A\phi\|) \|A^{\frac{3}{2}}\phi\| \\ &\leq \frac{\mu}{4}\|A^{\frac{3}{2}}\phi\|^2 + c. \end{aligned}$$

Then (2.9) becomes, thanks to Young's and Poincaré's inequalities,

$$\begin{aligned} \mu|(g(\phi), A\phi)| &\leq c\mu\|g(\phi)\| \|A\phi\| \\ &\leq c\mu\|A^{\frac{3}{2}}\phi\| \\ &\leq \frac{\mu}{4}\|A^{\frac{3}{2}}\phi\|^2 + c. \end{aligned}$$

and

$$\begin{aligned} 2|(g(\phi), \phi_t)| &\leq \|g(\phi)\| \|\phi_t\| \\ &\leq c\|\phi_t\| \\ &\leq \frac{1}{4}\|\phi_t\|^2 + c. \end{aligned}$$

Besides,

$$\begin{aligned} \frac{1}{2}|(g(\phi), A^{-2}\phi_{tt})| &\leq \|g(\phi)\| \|A^{-2}\phi_{tt}\| \\ &\leq c \|A^{-1}\phi_{tt}\| \\ &\leq \frac{\tau_D}{4} \|A^{-1}\phi_{tt}\|^2 + c. \end{aligned}$$

Now, combining the above estimates, we can rewrite (5.2) as follows:

$$\frac{d}{dt}Y(t) + e_0Y(t) + \frac{\mu}{2}\|A^{\frac{3}{2}}\phi\|^2 + \|\phi_t\|^2 + \delta\|A^{\frac{1}{2}}\phi_t\|^2 + \frac{\tau_D}{4}\|A^{-1}\phi_{tt}\|^2 \leq c + \|\phi_t\|^2Y(t), \quad (5.4)$$

for some $e_0 > 0$.

Using (3.5), we are allowed to apply Lemma 3.1 of [23] which entails

$$Y(t) \leq cY(0)e^{-e_0t} + c', \quad c > 0, e_0 > 0, c' \geq 0, t \geq 0. \quad (5.5)$$

Then, we state the following result. □

Theorem 5.2. *The semigroup $S_{\tau_D, \delta}^{\ell}(t)$ associated with (2.1) and (2.2) possesses a bounded absorbing set \mathbf{B}_0^{ℓ} in \mathbf{V}_1^{ℓ} such that, for every bounded set $B \subset \mathbf{V}_1^{\ell}$, there exists $t_0 = t_0(B) \geq 0$ such that $t \geq t_0$ implies*

$$S_{\tau_D, \delta}^{\ell}(t)B \subset \mathbf{B}_0^{\ell} \forall t \geq t_0;$$

obviously, $\mathbf{B}_0^0 = \mathbf{B}_0^0(\tau_D)$ and $\mathbf{B}_0^1 = \mathbf{B}_0^1(\tau_D, \delta)$.

Proof. Let $\|(\phi_0, \phi_1)\|_{\mathbf{V}_1^{\ell}} \leq R$. In particular $\|(\phi_0, \phi_1)\|_{\mathbf{V}_0^0} \leq \frac{R}{\sqrt{\lambda_1}}$. Thanks to Lemma 3.2, we obtain

$$\|(\phi, \phi_t)\|_{\mathbf{V}_0^0} \leq R_0t \geq t_0 = t_0 \left(\frac{R}{\sqrt{\lambda_1}} \right).$$

Applying Lemma 5.1, we can see that

$$\|(\phi(t_0), \phi_t(t_0))\|_{\mathbf{V}_1^{\ell}} \leq \bar{R} = K_1(R)e^{-e_0t_0} + K_0 \left(\frac{R}{\sqrt{\lambda_1}} \right).$$

Thus, for $t \geq t_0$ and Lemma 5.1 again, we conclude that

$$\|(\phi(t), \phi_t(t))\|_{\mathbf{V}_1^{\ell}} \leq K_1(\bar{R})e^{-e_0t}e^{e_0t_0} + K_0(R_0), \quad e_0 > 0,$$

and the result follows. □

5.1 | Existence of the global attractor

The global attractor is the smallest compact set of the phase space that is invariant by the semigroup $S_{\tau_D, \delta}^{\ell}(t)$ and attracts all bounded sets of initial data.

Theorem 5.3. *Under the hypotheses of Theorem 4.1, the semigroups $S_{\tau_D, \delta}^{\ell}(t)$ ($\ell = 0, 1$) defined on the phase space $\mathbf{V}_1^{\ell}(\tau_D, \delta)$ have a global attractor $A_{\tau_D, \delta}^{\ell}$ in $\mathbf{V}_2^{\ell}(\tau_D, \delta)$.*

Proof. We decompose the semigroup $S_{\tau_D, \delta}^{\ell}(t)$ into the sum of two parts, where the first one tends to 0 when t tends to infinity, while the second one is asymptotically compact in $\mathbf{V}_2^{\ell}(\tau_D, \delta)$,

$$S_{\tau_D, \delta}^{\ell}(t) = S_{\tau_D, \delta}^{1, \ell}(t) + S_{\tau_D, \delta}^{2, \ell}(t) \geq 0.$$

Consider the following decomposition

$$(\phi, \phi_t) = (w, w_t) + (v, v_t), \quad (5.6)$$

where (v, v_t) is the solution of

$$\tau_D v_{tt} + v_t + A^2 v + \delta A v_t = 0 \quad (5.7)$$

and

$$v(0) = \phi_0, \tau_D v_t(0) = \tau_D \phi_1, \quad (5.8)$$

associated with $S_{\tau_D, \delta}^{1, \ell}(t)$, and (w, w_t) is solution of

$$\tau_D w_{tt} + w_t + A^2 w + \delta A w_t + A f(\phi) = g(\phi) \quad (5.9)$$

and

$$w(0) = 0, \tau_D w_t(0) = 0, \quad (5.10)$$

associated with $S_{\tau_D, \delta}^{2, \ell}(t)$.

Firstly, we consider the case $\ell = 0$. We multiply (5.7) by $\alpha v + v_t$ and get

$$\frac{d}{dt} H(t) + (2 - 2\alpha\tau_D) \|v_t\|^2 + 2\alpha \|Av\|^2 + \delta \|A^{\frac{1}{2}} v_t\|^2 = 0, \quad (5.11)$$

where

$$H(t) = 2\alpha\tau_D (v_t, v) + \alpha \|v\|^2 + \|Av\|^2 + \alpha\delta \|A^{\frac{1}{2}} v\|^2.$$

Thus, for $\alpha \in (0, \frac{1}{2\tau_D})$, we obtain

$$c_{19}(\tau_D \|v_t\|^2 + \|Av\|^2 + \|A^{\frac{1}{2}} v_t\|^2) \leq H(t) \leq c_{20}(\tau_D \|v_t\|^2 + \|Av\|^2 + \|A^{\frac{1}{2}} v_t\|^2).$$

Combining the above estimates, we can rewrite (5.11) as follows:

$$\frac{d}{dt} H(t) + cH(t) + \delta \|A^{\frac{1}{2}} v_t\|^2 \leq 0, \quad c > 0. \quad (5.12)$$

Applying Gronwall's lemma to (5.12), we obtain

$$H(t) + \delta \int_0^t \|A^{\frac{1}{2}} v_t(s)\|^2 ds \leq H(0)e^{-ct}, \quad c > 0, \quad t \geq 0, \quad (5.13)$$

and the first part of the theorem is proved.

Now, multiplying (5.9) by $\alpha Aw + Aw_t$, we find

$$\begin{aligned} \frac{d}{dt} Y(t) + (2 - 2\alpha\tau_D) \|A^{\frac{1}{2}} w_t\|^2 + 2\alpha \|A^{\frac{3}{2}} w\|^2 + 2\delta \|Aw_t\|^2 &= -2\alpha (A f(\phi), Aw) \\ &+ 2(A(\phi_t f'(\phi)), Aw) + 2\alpha (g(\phi), Aw) + 2(g(\phi), Aw_t), \end{aligned} \quad (5.14)$$

where

$$Y(t) = 2\alpha (w_t, Aw) + \tau_D \|A^{\frac{1}{2}} w_t\|^2 + \alpha \|A^{\frac{1}{2}} w\|^2 + 2\|A^{\frac{3}{2}} w\|^2 + \alpha\delta \|Aw\|^2.$$

Let $\alpha > 0$ such that $\alpha \in (0, \frac{1}{2\tau_D})$. We have

$$c_{21}(\tau_D)||A^{\frac{1}{2}}w_t||^2 + ||Aw||^2 + ||A^{\frac{3}{2}}w||^2 \leq Y(t) \leq c_{22}(\tau_D)||A^{\frac{1}{2}}w_t||^2 + ||Aw||^2 + ||A^{\frac{3}{2}}w||^2, \quad c_{21} > 0, \quad c_{22} > 0.$$

Exploiting (2.6), (2.7), (3.5), (6.1), and Young's, Poincaré's, and proper interpolation inequalities, we find

$$-2\alpha|(Af(\phi), Aw)| \leq \alpha||A^{\frac{1}{2}}\phi|||f'(\phi)||_{L^\infty(\Omega)}||A^{\frac{3}{2}}w|| \leq \frac{\alpha}{2}||A^{\frac{3}{2}}w||^2 + c$$

and

$$2|(A(\phi_t f'(\phi)), Aw)| \leq ||A^{\frac{1}{2}}(\phi_t f'(\phi))|||A^{\frac{3}{2}}w|| \leq \frac{\alpha}{4}||A^{\frac{3}{2}}w||^2 + c.$$

Owing to (2.9) and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned} 2|\alpha(g(\phi), Aw)| &\leq \alpha||g(\phi)|||Aw|| \\ &\leq \alpha c||A^{\frac{3}{2}}w|| \\ &\leq \frac{\alpha}{4}||A^{\frac{3}{2}}w||^2 + c \end{aligned}$$

and

$$\begin{aligned} 2|(g(\phi), Aw_t)| &\leq ||g(\phi)|||Aw_t|| \\ &\leq c||Aw_t|| \\ &\leq \delta||Aw_t||^2 + c. \end{aligned}$$

Combining the above estimates, we have

$$\frac{d}{dt}Y(t) + (2 - 2\alpha\tau_D)||A^{\frac{1}{2}}w_t||^2 + \alpha||A^{\frac{3}{2}}w||^2 + \delta||Aw_t||^2 \leq c. \tag{5.15}$$

Therefore, for $2 - 2\alpha\tau_D > 0$, we deduce that

$$\frac{d}{dt}Y(t) + c'Y(t) \leq c, \quad c' > 0, \quad c \geq 0. \tag{5.16}$$

Applying Gronwall's lemma to (5.16), we conclude that

$$|(w, w_t)||_{V_2^0(\tau_D)} \leq e^{-c't} |(w(0), w_t(0))||_{V_2^0(\tau_D)} + c'', \quad c' > 0, \quad c'' \geq 0. \tag{5.17}$$

The estimates (5.13) and (5.17) give the results of the theorem.

To treat the case $\ell = 1$, we multiply (5.7) by $\frac{1}{2}A^{-2}v_u$ to get

$$\frac{d}{dt} (||A^{-1}v_t||^2 + 2(v, v_t) + \delta||A^{-\frac{1}{2}}v_t||^2) - 2||v_t||^2 + 2\tau_D||A^{-1}v_u||^2 = 0. \tag{5.18}$$

The sum of (5.18) and (5.11) gives

$$\frac{d}{dt} (|(v, v_t)||_{V_1^1} + |(v, v_t)||_{V_1^1} + \tau_D||A^{-1}v_u||^2 + \delta||A^{\frac{1}{2}}v_t||^2) \leq 0. \tag{5.19}$$

Applying Gronwall's lemma to (5.19), we obtain the first part of the Theorem.

Now, multiplying (5.9) by $A^{-1}w_{tt}$, we obtain

$$\begin{aligned} \frac{d}{dt} \left((Aw, w_t) + \frac{1}{2} \|A^{-\frac{1}{2}}w_t\|^2 + \frac{\delta}{2} \|w_t\|^2 \right) - \|A^{\frac{1}{2}}w_t\|^2 + \tau_D \|A^{-\frac{1}{2}}w_{tt}\|^2 = -(A^{\frac{1}{2}}\phi f'(\phi), A^{-\frac{1}{2}}w_{tt}) \\ + (g(\phi), A^{-1}w_{tt}). \end{aligned} \quad (5.20)$$

Summing (5.20) and (5.15), we obtain

$$\frac{d}{dt} \|(w, w_t)\|_{\mathbf{V}_2^\ell} + c \|(w, w_t)\|_{\mathbf{V}_2^\ell} + \delta \|Aw_t\|^2 + \frac{\tau_D}{2} \|A^{-\frac{1}{2}}w_{tt}\|^2 \leq c', \quad c > 0, \quad c' \geq 0. \quad (5.21)$$

Applying Gronwall's lemma to (5.21), we conclude that

$$\|(w, w_t)\|_{\mathbf{V}_2^\ell} \leq e^{-ct} \|(w(0), w_t(0))\|_{\mathbf{V}_2^\ell} + c'', \quad c > 0, \quad c'' \geq 0. \quad (5.22)$$

The estimates (5.19) and (5.22) give the results of the theorem in the case $\ell = 1$. \square

Corollary 5.4. *The global attractor $\mathcal{A}_{\tau_D, \delta}^\ell$ is bounded in \mathbf{V}_2^ℓ , with a bound that is independent of τ_D and δ .*

For an idea of the proof, see earlier studies [12, 13].

6 | DISSIPATIVITY IN HIGHER-ORDER SPACES

Our goal in this section is to show that for f and g satisfying suitable compatibility conditions, the semigroup $S_{\tau_D, \delta}^\ell(t)$ defined on the space \mathbf{V}_n^ℓ (for $1 \leq n \leq 4$) has an absorbing and closed set $\mathbf{B}_n^\ell(\tau_D, \delta)$ in \mathbf{V}_n^ℓ . We assume that

$$f \in C^{n+1}(\mathbf{R}) \text{ if } n = 3, 4, \quad f''(0) = 0 \quad (6.1)$$

and

$$g^{(n-1)} \text{ is bounded.} \quad (6.2)$$

Theorem 6.1. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Then there exists $R_n \geq 0$ such that the closed ball $\mathbf{B}_n^\ell(\tau_D, \delta)$ of \mathbf{V}_n^ℓ centered at zero of radius R_n is an absorbing set for $S_{\tau_D, \delta}^\ell(t)$ in \mathbf{V}_n^ℓ . That is, for every bounded set $B \subset \mathbf{V}_n^\ell$, there exists $t_n = t_n(B) \geq 0$ such that*

$$S_{\tau_D, \delta}^\ell(t)B \subset \mathbf{B}_n^\ell(\tau_D, \delta), \quad \forall t \geq t_n,$$

where

$$\mathbf{B}_n^0 = \mathbf{B}_n^0(\tau_D) \text{ and } \mathbf{B}_n^1 = \mathbf{B}_n^1(\tau_D, \delta).$$

Theorem 6.2. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Given any bounded set $B \subset \mathbf{V}_n^\ell$, there exists $C_n = C_n(B) \geq 0$ such that*

$$\sup_{z_0 \in B} \|S_{\tau_D, \delta}^\ell(t)z_0\|_{\mathbf{V}_m^\ell} + \sup_{z_0 \in B} \int_t^{t+1} \|A^{\frac{n}{2}}\phi_t(y)\|^2 dy \leq C_n, \quad \forall t \geq 0, \quad (6.3)$$

where $z_0 = (\phi(0), \phi_t(0))$.

The proof of Theorems 6.1 and 6.2 follows directly from Lemma 6.3. Indeed, one can proceed exactly as in the proofs of Theorem 5.2 and Lemma 5.1. The rest follows from the next lemma (in particular, for Theorem 6.2, one has to integrate formula (6.10) below over $(t, t+1)$).

Lemma 6.3. *Let the hypotheses of Theorem 6.1 hold. Given $r_n \geq 0$, $r_{n-1} \geq 0$, there are $K_{n-1} = K_{n-1}(r_{n-1}) \geq 0$, $K_n = K_n(r_n) \geq 0$ and $v_n > 0$ such that if*

$$\|z_0\|_{V_{n-1}^c} \leq r_{n-1} \text{ and } \|z_0\|_{V_n^c} \leq r_n,$$

then

$$\|S_{\tau_D, \delta}^{\ell}(t)z_0\|_{V_n^c} \leq K_n e^{-v_n t} + K_{n-1}, \quad t \geq 0. \quad (6.4)$$

Proof. Throughout this proof, c may depend on r_{n-1} . We exploit an inductive argument on $n \geq 1$. So, we assume that

$$\sup_{\|z_0\|_{V_{n-1}^c} \leq r_{n-1}} \left(\|A^{\frac{n+1}{2}} \phi\|^2 + \|A^{\frac{n-1}{2}} \phi_t\|^2 \right) \leq c. \quad (6.5)$$

Notice that, for $n = 1$, (6.5) holds thanks to estimate (4.9). In particular, the assumptions on f yield

$$\sup_{t \geq 0} \sum_{j=1}^{n+1} \|f^{(j)}(\phi)\|_{L^\infty(\Omega)} \leq c \text{ and } f(\phi) \in D(A^{\frac{n}{2}}). \quad (6.6)$$

Consequently,

$$\|A^{\frac{n}{2}} f(\phi)\| \leq c \text{ and } \|A^{\frac{n}{2}}(f'(\phi)\phi_t)\| \leq c \|A^{\frac{n}{2}} \phi_t\|. \quad (6.7)$$

Multiplying (3.2) by $\alpha A^n \phi + A^n \phi_t + \frac{1}{2} A^{n-2} \phi_{tt}$, we have

$$\begin{aligned} \frac{d}{dt} Y_n + (1 - 2\alpha\tau_D) \|A^{\frac{n}{2}} \phi_t\|^2 + 2\alpha \|A^{\frac{n+2}{2}} \phi\|^2 + 2\delta \|A^{\frac{n+1}{2}} \phi_t\|^2 + \tau_D \|A^{\frac{n-2}{2}} \phi_{tt}\|^2 &= (g(\phi), A^{n-2} \phi_{tt}) \\ -2\alpha (A f(\phi), A^n \phi) + 2(A^{\frac{n}{2}}(f'(\phi)\phi_t), A^{\frac{n+2}{2}} \phi) + (A^{\frac{n}{2}}(f'(\phi)\phi_t), A^{\frac{n-2}{2}} \phi_t) &+ 2\alpha (g(\phi), A^n \phi) + 2(g(\phi), A^n t), \end{aligned} \quad (6.8)$$

where

$$\begin{aligned} Y_n(t) &= 2\alpha\tau_D (A^{\frac{n}{2}} \phi_t, A^{\frac{n}{2}} \phi) + \tau_D \|A^{\frac{n}{2}} \phi_t\|^2 + \alpha \|A^{\frac{n+2}{2}} \phi\|^2 + \frac{1}{2} \|A^{\frac{n-2}{2}} \phi_t\|^2 + \|A^{\frac{n+2}{2}} \phi\|^2 + (A^{\frac{n}{2}} \phi_t, A^{\frac{n}{2}} \phi) \\ &+ \alpha\delta \|A^{\frac{n+1}{2}} \phi\|^2 + \delta \|A^{\frac{n-1}{2}} \phi_t\|^2 + 2(A^{\frac{n}{2}} f(\phi), A^{\frac{n+2}{2}} \phi) + 2(A^{\frac{n}{2}} f(\phi), A^{\frac{n-2}{2}} \phi_t). \end{aligned}$$

Hence, for $\alpha \in \left(0, \frac{1}{2\tau_D}\right)$, we obtain

$$\begin{aligned} Y_n &\geq c_{23} (\tau_D \|A^{\frac{n}{2}} \phi_t\|^2 + \|A^{\frac{n+2}{2}} \phi\|^2 + \delta \|A^{\frac{n-1}{2}} \phi_t\|^2 + \|A^{\frac{n-2}{2}} \phi_t\|^2), \quad c_{23} > 0, \\ Y_n &\leq c_{24} (\tau_D \|A^{\frac{n}{2}} \phi_t\|^2 + \|A^{\frac{n+2}{2}} \phi\|^2 + \delta \|A^{\frac{n-1}{2}} \phi_t\|^2 + \|A^{\frac{n-2}{2}} \phi_t\|^2), \quad c_{24} > 0. \end{aligned}$$

Owing to (6.6), (6.7), and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned} 2|(A^{\frac{n}{2}}(f'(\phi)\phi_t), A^{\frac{n+2}{2}} \phi)| &\leq c \|A^{\frac{n}{2}}(f'(\phi)\phi_t)\| \|A^{\frac{n+2}{2}} \phi\| \\ &\leq c \|A^{\frac{n}{2}} \phi_t\| \|A^{\frac{n+2}{2}} \phi\| \\ &\leq \alpha \|A^{\frac{n+2}{2}} \phi\|^2 + \frac{1}{4} \|A^{\frac{n}{2}} \phi_t\|^2 \end{aligned}$$

and

$$\begin{aligned} -2\alpha |(A^{\frac{n}{2}} f(\phi), A^{\frac{n+2}{2}} \phi)| &\leq \alpha \|A^{\frac{n}{2}} f(\phi)\| \|A^{\frac{n+2}{2}} \phi\| \\ &\leq \frac{\alpha}{4} \|A^{\frac{n+2}{2}} \phi\|^2 + c. \end{aligned}$$

Using (6.2), Young's and Poincaré's inequalities imply

$$\begin{aligned} 2\alpha|(g(\phi), A^n\phi)| &\leq \alpha\|A^{\frac{n-2}{2}}g(\phi)\|\|A^{\frac{n+2}{2}}\phi\| \\ &\leq \alpha c\|A^{\frac{n+2}{2}}\phi\| \\ &\leq \frac{\alpha}{4}\|A^{\frac{n+2}{2}}\phi\|^2 + c \end{aligned}$$

and

$$\begin{aligned} 2|(g(\phi), A^n\phi_t)| &\leq \|A^{\frac{n}{2}}g(\phi)\|\|A^{\frac{n}{2}}\phi_t\| \\ &\leq c\|A^{\frac{n}{2}}\phi_t\| \\ &\leq \frac{1}{4}\|A^{\frac{n}{2}}\phi_t\|^2 + c. \end{aligned}$$

Inserting the above estimates into (6.8), we infer that

$$\frac{d}{dt}Y_n + \left(\frac{1}{2} - 2\alpha\tau_D\right)\|A^{\frac{n}{2}}\phi_t\|^2 + \frac{\alpha}{2}\|A^{\frac{n+2}{2}}\phi\|^2 + 2\delta\|A^{\frac{n+1}{2}}\phi_t\|^2 + \frac{\tau_D}{2}\|A^{\frac{n-2}{2}}\phi_{tt}\|^2 \leq c, \quad (6.9)$$

where, for $\frac{1}{2} - 2\alpha\tau_D > 0$, we have

$$\frac{d}{dt}Y_n + \nu_n Y_n + c'\|A^{\frac{n}{2}}\phi_t\|^2 + 2\delta\|A^{\frac{n+1}{2}}\phi_t\|^2 + \frac{\tau_D}{2}\|A^{\frac{n-2}{2}}\phi_{tt}\|^2 \leq c, \quad \nu_n > 0, \quad c \geq 0. \quad (6.10)$$

Applying Gronwall's lemma to (6.10), we obtain the result. \square

Proposition 6.4. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Then, there exist $\omega_n > 0$ and an increasing function $J_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\text{dist}_{\mathbf{V}^\ell}(S_{\tau_D, \delta}^\ell(t)B, B_n^\ell) \leq J_n(R)e^{-\omega_n t}, \quad t \geq 0, \quad (6.11)$$

for all $B \subset \mathbf{V}_1^\ell$ with $\sup_{z \in B} \|z\|_{\mathbf{V}_1^\ell} \leq R$, where $\text{dist}_{\mathbf{V}^\ell}$ denotes the standard Hausdorff semidistance in \mathbf{V}_1^ℓ .

The proof of Proposition 6.4 follows directly from the following two lemmas.

Lemma 6.5 (see, e.g., Fabrie et al. [24]). *Let $S(t)$ be a strongly continuous semigroup on a metric space (\mathbf{V}, d) . Let $B_0, B_1, B_2 \subset \mathbf{V}$ be such that*

$$\text{dist}_{\mathbf{V}}(S(t)B_0, B_1) \leq M_1 e^{-\alpha_1 t}, \quad \text{dist}_{\mathbf{V}}(S(t)B_1, B_2) \leq M_2 e^{-\alpha_2 t},$$

for some $\alpha_1, \alpha_2 > 0$ and $M_1, M_2 \geq 0$.

Assume also that for all $z_1, z_2 \in \cup_{t \geq 0} S(t)B_j$ ($j = 0, 1, 2$), there holds

$$d(S(t)z_1, S(t)z_2) \leq M_0 e^{\alpha_0 t} d(z_1, z_2),$$

for some $\alpha_0 \geq 0$ and some $M_0 \geq 0$. Then it follows that

$$\text{dist}_{\mathbf{V}}(S(t)B_0, B_2) \leq M e^{-\alpha t}, \quad (6.12)$$

where $\alpha = \frac{\alpha_1 \alpha_2}{\alpha_0 + \alpha_1 + \alpha_2}$ and $M = M_0 M_1 + M_2$.

Lemma 6.6. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Then there holds*

$$\text{dist}_{\mathbf{V}^\ell}(S_{\tau_D, \delta}^\ell(t)B_{n-1}^\ell, B_n^\ell) \leq L_n e^{-\omega t}, \quad \forall t \geq 0, \quad (6.13)$$

for some $L_n \geq 0$ and some $\omega > 0$.

Proof. In this proof, $c \geq 0$ may depend on the radius R_{n-1} of $B_{\tau_D, \delta}^{n-1}$. We need to show that the solution map is the sum of a term exponentially decreasing in the space \mathbf{V}_1^ℓ and a term uniformly bounded in \mathbf{V}_n^ℓ . We consider the decomposition (5.6) of the semigroup. For initial data $(\phi_0, \phi_1) \in B_0^\ell$, we multiply (5.7) by $\alpha v + v_t$ and using Theorem 5.3, we have

$$\|(v(t), v_t(t))\|_{\mathbf{V}_1^\ell} \leq c \|(\phi_0, \phi_1)\|_{\mathbf{V}_1^\ell} e^{-\omega t}, \quad t \geq 0, \quad (6.14)$$

for some $\omega > 0$. Systems (5.7)–(5.8) are therefore exponentially stable in \mathbf{V}_1^ℓ . Now multiplying (5.9) by $\alpha A^n w + A^n w_t$ and following Lemma 6.3, we find

$$\|(w(t), w_t(t))\|_{\mathbf{V}_n^\ell} \leq c, \quad \forall t \geq 0,$$

for some $c \geq 0$ depending on the radius R_{n-1} of $B_{\tau_D, \delta}^{n-1}$. Redefining R_n to be greater than or equal to the above constant c , we reach the desired conclusion. \square

Proof. We observe that, thanks to Lemmas 4.3 and 6.5, the transitivity of attraction applies, so that

$$\text{dist}_{\mathbf{V}_1^\ell}(S_{\tau_D, \delta}^\ell(t)B_0^\ell, B_n^\ell) \leq L'_n e^{-\omega'_n t}, \quad \forall t \geq 0, \quad (6.15)$$

for some $L'_n \geq 0$ and $\omega'_n > 0$. Now, for any $B \subset \mathbf{V}_1^\ell$, with $\sup_{(\phi, \phi_t) \in B} \|(\phi, \phi_t)\|_{\mathbf{V}_1^\ell} \leq R$, Theorem 5.2 yields

$$\text{dist}_{\mathbf{V}_1^\ell}(S_{\tau_D, \delta}^\ell(t)B, B_0^\ell) \leq J_0(R) e^{-\omega_0 t}, \quad t \geq 0, \quad (6.16)$$

for some $\omega_0 > 0$ and some increasing function $J_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. Therefore, by Lemmas 4.3 and 6.5, we find the result.

Corollary 6.7. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. The global attractor $\mathcal{A}_{\tau_D, \delta}$ is bounded in B_n^0 , with a bound that is independent of τ_D and δ .*

Theorem 6.8. *Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Then, the lifting $\mathcal{A}_{0,0}$ of the global attractor is upper semicontinuous at zero with respect to the family $\mathcal{A}_{\tau_D, \delta}$ in the Hausdorff semidistance of \mathbf{V}_1^0 , that is,*

$$\lim_{\delta \rightarrow 0} [\text{dist}_{\mathbf{V}_1^0}(\{\mathcal{A}_{\tau_D, \delta}, \mathcal{A}_{0,0}\})] = 0.$$

Remark 6.9. The result of the theorem is always true if we take $\tau_D \rightarrow 0$ and $\delta \rightarrow \delta_0 > 0$, thanks to (2.4).

The proof of Theorem 6.8 uses the following two lemmas.

Lemma 6.10. *The set A is bounded in \mathbf{V}_3^0 . In addition, when $\omega > 0$, any solution $\phi(t)$ departing from the attractor $\mathcal{A}_{\tau_D, \delta}$ fulfills the inequality*

$$\tau_D \|\phi_{tt}\|^2 \leq Q, \quad \forall t \geq 0, \quad (6.17)$$

for some $Q \geq 0$, that is independent of τ_D and δ .

Proof. Thanks to Corollary 6.7, $\|A^2 \phi(0)\|$ is bounded. We first consider the case $\tau_D = 0$. Since $\mathcal{A}_{0, \delta}$ is bounded in $D(A^3)$, for every fixed $\phi(0) \in \mathcal{A}_{0, \delta}$, the corresponding ϕ satisfies

$$A\phi_t = AL_\delta(\phi).$$

Hence, appealing to the invariance of the global attractor, we get a uniform bound for $\|A\phi_t(0)\|$, which yields the boundedness of $\mathcal{A}_{0, \delta}$ in \mathbf{V}_3^0 .

When $\tau_D > 0$, we introduce the following auxiliary problem, formally obtained by taking the time-derivative of (3.2):

$$\tau_D u_{tt} + u_t + A(Au + f'(\phi)u + \delta u_t) = u g'(\phi) \quad (6.18)$$

and

$$u(0) = \phi_1, u_t(0) = \frac{1}{\tau_D}(-\phi_1 - A(A\phi(0) + f(\phi(0)) + \delta\phi_1) + g(\phi(0))), \quad (6.19)$$

with $(\phi(0), \phi_1) \in \mathcal{A}_{\tau_D, \delta}$ and $\phi_t = u$.

For any $T > 0$, the problem has a unique solution

$$(u, u_t) \in C([0, T], \mathbf{V}_1^0).$$

Multiplying (6.18) by $u_t + \mu Au$ (with $\mu > 0$ enough small), we obtain

$$\begin{aligned} \frac{d}{dt}Y(t) + (2\delta - \mu\tau_D)||A^{\frac{1}{2}}u_t||^2 + 2||u_t||^2 + 2\mu||A^{\frac{3}{2}}u||^2 &= -2(A(uf'(\phi)), u_t) \\ &- 2\mu(A(uf'(\phi)), Au) + 2(ug'(\phi), u_t) + 2\mu(ug'(\phi), Au), \end{aligned} \quad (6.20)$$

where

$$Y(t) = \tau_D||u_t||^2 + 2\mu\tau_D(u_t, Au) + \frac{\mu}{2}||A^{\frac{1}{2}}u||^2 + ||Au||^2 + \delta\mu||Au||^2.$$

Let $\mu > 0$ be such that

$$c_{27}(\tau_D||u_t||^2 + ||Au||^2 + ||A^{\frac{1}{2}}u||^2) \leq Y(t) \leq c_{28}(\tau_D||u_t||^2 + ||Au||^2 + ||A^{\frac{1}{2}}u||^2).$$

Owing to (6.7) and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned} -2|(A(uf'(\phi)), u_t)| &\leq ||A(uf'(\phi))|| ||u_t|| \\ &\leq c||A^{\frac{3}{2}}u|| ||u_t|| \\ &\leq \frac{\mu}{4}||A^{\frac{3}{2}}u||^2 + \frac{1}{2}||u_t||^2 \end{aligned}$$

and

$$\begin{aligned} \mu|(A(uf'(\phi)), Au)| &\leq \mu||A^{\frac{3}{2}}u|| ||A^{\frac{1}{2}}u|| \\ &\leq \frac{\mu}{4}||A^{\frac{3}{2}}u||^2 + c||A^{\frac{1}{2}}u||^2 \\ &\leq \frac{\mu}{4}||A^{\frac{3}{2}}u||^2 + c||A^{-\frac{1}{2}}u||^2. \end{aligned}$$

Thanks to (6.2) and Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} 2|(ug'(\phi), u_t)| &\leq ||ug'(\phi))|| ||u_t|| \\ &\leq \frac{\mu}{4}||A^{\frac{3}{2}}u||^2 + \frac{1}{2}||u_t||^2 \end{aligned}$$

and

$$\begin{aligned} \mu|(ug'(\phi), Au)| &\leq \mu||A^{-\frac{1}{2}}(ug'(\phi))|| ||A^{\frac{3}{2}}u|| \\ &\leq \mu c||A^{-\frac{1}{2}}u|| ||A^{\frac{3}{2}}u|| \\ &\leq c||A^{-\frac{1}{2}}u||^2 + \frac{\mu}{4}||A^{\frac{3}{2}}u||^2. \end{aligned}$$

Combining the above estimates in (6.20), we deduce that

$$\frac{d}{dt}Y(t) + \epsilon Y(t) + \delta||A^{\frac{1}{2}}u_t||^2 + \mu||A^{\frac{3}{2}}u||^2 \leq c||A^{-\frac{1}{2}}u||^2. \quad (6.21)$$

Integrating the estimate (6.21) over $(0, t)$, we conclude that

$$\tau_D \|\phi_t\|^2 + \|A\phi_t\|^2 \leq c \left(1 + \frac{1}{\tau_D} e^{-\epsilon t} \right), \quad \epsilon > 0, c > 0, t \geq 0.$$

Thanks to the invariance of the global attractor, this bound is independent of τ_D . \square

We consider the projection $P : \mathbf{V}_1^0(1) \rightarrow \mathbf{V}_1^0(0)$, defined as

$$P(\phi(0), \phi_1) = \phi(0).$$

We have the following lemma.

Lemma 6.11. *The family $G = \{\hat{\phi} \in C(\mathbb{R}^+, \mathbf{V}_1^0(0))\}$, where*

$$\hat{\phi}(t) = \begin{cases} PS_{\tau_D, \delta}^0(t)(\phi_0, \phi_1), & \text{if } \tau_D > 0, \\ (S_{0, \delta}^0(t)P(\phi_0, \phi_1)), & \text{if } \tau_D = 0, \end{cases}$$

is equicontinuous at zero.

Proof of Theorem 6.8. Assuming the opposite, there would exist sequences τ_{D_n}, δ_n satisfying (2.4) with $\delta_n \rightarrow 0$, $(\phi_n, \phi_{t,n}) = z_n \in A_{\tau_{D_n}, \delta_n}$, and a constant $Q > 0$ such that

$$\text{dist}_{\mathbf{V}_1^0(1)}(z_n, \mathcal{A}_{0,0}) \geq Q.$$

Assume that $\tau_{D_n} > 0$ for all $n \in \mathbb{N}$. Denote $z^n(t) = S_{\tau_{D_n}, \delta_n}^0(t)z_n$, and observe that this function can be extended to a bounded complete trajectory of $z^n(t) = S_{\tau_{D_n}, \delta_n}^0(t)$. Moreover, the set

$$\cup_{t \in \mathbb{R}} \cup_{n \in \mathbb{N}} z^n(t) \subset A$$

is relatively compact in $\mathbf{V}_1^0(1)$. The family Pz^n is equicontinuous from \mathbb{R} to $\mathbf{V}_1^0(0)$, thanks to Lemma 6.11 and to the semigroup properties. Thus, by the Ascoli theorem and a classical diagonalization procedure, we deduce the existence of $\bar{z} \in C(\mathbb{R}, \mathbf{V}_1^0(0))$ such that, up to a subsequence,

$$Pz^n \rightarrow \bar{z} \text{ in } C([-N, N], \mathbf{V}_1^0(0)),$$

for any $N > 0$, and

$$\sup_{t \in \mathbb{R}} \|\bar{z}(t)\|_{\mathbf{V}_1^0(0)} < \infty.$$

Arguing exactly as in Grasselli and Pata [25], we can show that \bar{z} is a bounded complete trajectory of $S_{0,0}^0(t)$. Then, by the characterization of the global attractor as the set of all complete bounded trajectories, we conclude that $\bar{z}(0) \in \mathcal{A}_{0,0}$. Finally, exploiting the boundedness of A in $\mathbf{V}_3^0(1)$, we easily obtain the convergence $z^n \rightarrow \bar{z}(0)$ in $\mathbf{V}_1^0(1)$, as well as the validity of Lemma 6.10, and we obtain a contradiction. \square

7 | ROBUST EXPONENTIAL ATTRACTORS

In this section, we formulate one of the main results of the paper; namely, there exists a robust (in a sense that will be made precise later) family of exponential attractors for the semigroups $S_{\tau_D, \delta}^1(t)$ on the space-phases $\mathbf{V}_1^1(\tau_D, \delta)$ associated to problems (2.1) and (2.3).

We begin with some definitions.

Definition 7.1. Let $S(t)$ be a strongly continuous semigroup on a complete metric space $(\mathbf{V}, d_{\mathbf{V}})$. A compact set $\mathfrak{F} \subset \mathbf{V}$ is called an exponential attractor or inertial set for $S(t)$ if the following conditions hold:

1. \mathfrak{F} is positively invariant for $S(t)$, that is, $S(t)\mathfrak{F} \subset \mathfrak{F}$ for every $t \geq 0$,
2. $\dim_F(\mathfrak{F}, \mathbf{V}) < \infty$, that is, \mathfrak{F} has finite fractal dimension,
3. There exists an increasing function $J : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $k > 0$ such that there holds

$$\text{dist}_{\mathbf{V}}(S(t)B, \mathfrak{F}) \leq J(R)e^{-kt},$$

for any set $B \subset \mathbf{V}$ with $\sup_{z \in B} d_{\mathbf{V}}(z, z_*) \leq R$, where z_* is a selected vector.

Recall that the fractal dimension $\dim_F(K, \mathbf{V})$ of a compact subset $K \subset \mathbf{V}$ is given by

$$\dim_F(K, \mathbf{V}) = \limsup_{\epsilon \rightarrow 0} \frac{\log N_{\epsilon}(K, \mathbf{V})}{\log \frac{1}{\epsilon}},$$

where $N_{\epsilon}(K, \mathbf{V})$ is the minimum number of ϵ -balls of \mathbf{V} necessary to cover K . The quantity $\mathbf{H}_{\epsilon}(K, \mathbf{V}) = \log N_{\epsilon}(K, \mathbf{V})$ is usually referred to as the Kolmogorov ϵ -entropy of K . Notice that the entropy and thus the fractal dimension depend on the underlying space \mathbf{V} .

Remark 7.2. The above definition is actually more general than the one in Miranville [21], since here the basin of exponential attraction is the whole phase-space and not just a compact invariant subset of the phase-space.

Theorem 7.3. Let (6.1) and (6.2) hold for $1 \leq n \leq 4$. Then there is a family $\{\mathfrak{F}_{\tau_D, \delta}\} \subset \mathbf{V}_1^1(\tau_D, \delta)$ of exponential attractor for the semigroups $S_{\tau_D, \delta}^1(t)$ acting on the phase-space \mathbf{V}_1^1 . In addition, the family $\{\mathfrak{F}_{\tau_D, \delta}\}$ satisfies the following properties:

- (1) $\{\mathfrak{F}_{\tau_D, \delta}\}$ is bounded in $\mathbf{V}_4^1(\tau_D, \delta)$, uniformly with respect to τ_D and δ ,
- (2) The rate of exponential attraction k and the function J of definition 7.1 are independent of τ_D and δ ,
- (3) $\dim_F(\mathfrak{F}_{\tau_D, \delta}, \mathbf{V}_1^1)$ is uniformly bounded with respect to τ_D and δ ,
- (4) The symmetric Hausdorff distance between $\mathfrak{F}_{\tau_D, \delta}$ and $\mathfrak{F}_{0,0}$ satisfies

$$\text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}^{\text{sym}}(\mathfrak{F}_{\tau_D, \delta}, \mathfrak{F}_{0,0}) \leq c(\tau_D + \delta)^{\tau},$$

for some $c \geq 0$, $\tau \in (0, 1)$ where denotes the symmetric Hausdorff distance between sets is $\text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}^{\text{sym}}$ defined by

$$\text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}^{\text{sym}}(A, B) := \max(\text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}(A, B), \text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}(B, A)).$$

The constants k , τ , C , and the function J can be computed explicitly.

With reference to definition 7.1, we agree to select $z_* = 0 \in \mathbf{V}_1^1(\tau_D, \delta)$, since $f(0) = 0$. In particular, assertion (4) of the above theorem guarantees that the family $\{\mathfrak{F}_{\tau_D, \delta}\}$ is robust with respect to the (singular) double limit $\tau_D, \delta \rightarrow 0$, provided, of course, that it is taken according to condition (2.4).

Remark 7.4. It is straightforward to realize that, for $\tau_D > 0$, the set $\mathfrak{F}_{\tau_D, \delta}$ is an exponential attractor for the semigroup $S_{\tau_D, \delta}^0(t)$ as well, whereas for $\tau_D = 0$, the projection of $\mathfrak{F}_{0, \delta}$ on the first component is an exponential attractor for $S_{0, \delta}^0(t)$.

The key step in order to prove Theorem 7.3 is the next lemma, whose proof is given in this section of the paper. To this aim, we introduce the sets

$$\mathbf{B} = \mathbf{B}(\tau_D, \delta) = \begin{cases} B_4^1(\tau_D, \delta), & \text{if } \tau_D > 0 \\ \{(\phi(0), L_{\delta}(\phi(0))) \in \mathbf{V}_4^1(0, \delta) : \|A^3\phi(0)\| \leq R_4\}, & \text{if } \tau_D = 0, \end{cases}$$

where R_4 is the radius of the ball B_4^1 of \mathbf{V}_4^1 (recall that R_4 is the same for all τ_D and δ).

It is apparent that $B_4^1 \subset \mathbf{B}$. Besides, when $\tau_D = 0$, if $(\phi(0), L_\delta(\phi(0))) \in \mathbf{B}$, it turns out that

$$\|(\phi(0), L_\delta(\phi(0)))\|_{V_4^1} \leq r \|A^3 \phi(0)\| \leq rR_4,$$

for some $r > 1$. Indeed, this easily follows from the fact that L_δ is (locally Lipschitz) continuous from $D(A^3)$ to $D(A)$, uniformly with respect to δ , and from the bound

$$\|A(\mathbf{I} + \delta A)^{-1}\| \leq \frac{1}{\delta}, \forall \delta > 0.$$

Thus, we conclude that

$$B_4^1 \subset \mathbf{B} \subset rB_4^1,$$

which implies that \mathbf{B} fulfills the very same properties as B_4^1 .

Lemma 7.5. *There exist positive constants Δ_i , $i = 1, 2, \dots$ (independent of τ_D and δ) such that the following hold:*

(L1) *For every $z_0 \in \mathbf{B}(0, 0) \subset V_1^1(\tau_D, \delta)$,*

$$\|z_0\|_{V_1^1(\tau_D, \delta)} \leq \Delta_1 \|z_0\|_{V_1^1(0, 0)} + \Delta_2 \sqrt{\tau_D + \delta}.$$

(L2) *There exists $t^* \geq t_B$ such that the map $S_{\tau_D, \delta}^1 = S_{\tau_D, \delta}^1(t^*)$ can be decomposed into the sum*

$$S_{\tau_D, \delta}^1 = V_{\tau_D, \delta} + W_{\tau_D, \delta},$$

where for every $z_1, z_2 \in \mathbf{B}(\tau_D, \delta)$,

$$\begin{aligned} \|V_{\tau_D, \delta} z_1 - V_{\tau_D, \delta} z_2\|_{V_1^1(\tau_D, \delta)} &\leq \frac{1}{4} \|z_1 - z_2\|_{V_1^1(\tau_D, \delta)}, \\ \|W_{\tau_D, \delta} z_1 - W_{\tau_D, \delta} z_2\|_{V_2^1(\tau_D, \delta)} &\leq \Delta_3 \|z_1 - z_2\|_{V_1^1(\tau_D, \delta)}. \end{aligned}$$

(L3) *There exist (nonlinear) projections $\Pi_{\tau_D, \delta} : \mathbf{B}(\tau_D, \delta) \rightarrow \mathbf{B}(0, 0)$ such that*

$$\Pi_{\tau_D, \delta} \mathbf{B}(\tau_D, \delta) = \mathbf{B}(0, 0)$$

and

$$\|(S_{\tau_D, \delta}^1)^n z_0 - (S_{0, 0}^1)^n \Pi_{\tau_D, \delta} z_0\|_{V_1^1(\tau_D, \delta)} \leq \Delta_4^n (\tau_D + \delta), \forall n \in \mathbb{N},$$

for all $z_0 \in \mathbf{B}(\tau_D, \delta)$.

(L4) *There holds*

$$\|S_{\tau_D, \delta}^1 z_0 - S_{0, 0}^1 z_0\|_{V_1^1(0, 0)} \leq \Delta_5 (\tau_D + \delta) + \Delta_6 \|z_0 - \bar{z}_0\|_{V_1^1(0, 0)},$$

for every $t \in [0, t^*]$ and every $z_0 \in \mathbf{B}(\tau_D, \delta) \subset V_1^1(0, 0)$, $\bar{z}_0 \in \mathbf{B}(0, 0)$.

(L5) *The map*

$$(t, z) \mapsto S_{\tau_D, \delta}^1 z : [0, t^*] \times \mathbf{B}(\tau_D, \delta) \rightarrow V_1^1(\tau_D, \delta)$$

is Lipschitz continuous with a Lipschitz constant that may depend on τ_D and δ . Here $\mathbf{B}(\tau_D, \delta)$ is endowed with the metric topology of $V_1^1(\tau_D, \delta)$. Besides, the Lipschitz constant of the map $z \rightarrow S_{\tau_D, \delta}^1 z$ is independent of τ_D , δ and $t \in [0, t^*]$.

By Lemma 7.5 and the obvious fact that the compact embeddings $H_2(\tau_D, \delta) \hookrightarrow H_1(\tau_D, \delta)$ are uniform, in the sense that the quantity $\mathbf{H}_c(B_1, H_2(\tau_D, \delta))$, where B_1 denotes the unit ball of $H_1(\tau_D, \delta)$, is independent of τ_D and δ , we are in a position to exploit an abstract result due to Fabrie et al. [24]. The conclusion that can be drawn is “almost” the statement of Theorem 7.3. Precisely, we find a family of sets $\{\mathfrak{F}_{\tau_D, \delta}\}$ satisfying conditions (1), (2), and (4) of the theorem, with (2) replaced by the weaker attraction property

$$\text{dist}_{\mathbf{V}_1^1(\tau_D, \delta)}(S_{\tau_D, \delta}^1(t)\mathbf{B}(\tau_D, \delta), \mathfrak{F}_{\tau_D, \delta}) \leq J_* e^{-k_* t},$$

for some $J_* \geq 0$ and $k_* \geq 0$. But it is then readily seen that, thanks to the transitivity of exponential attraction, collecting the above formula, Lemma 4.3, and Theorems 5.3 and 6.4 (with \mathbf{B} in place of B_1^1), we get the desired conclusion (2), which completes the proof of Theorem 7.3.

Proof of Lemma 7.5. In what follows, the constants $c \geq 0$ and $c' \geq 0$ may depend on R_4 .

Proof of (L1). Let $z_0 \in \mathbf{B}(0, 0) \subset \mathbf{V}_1^1(\tau_D, \delta)$. Then,

$$\begin{aligned} \|z_0\|_{\mathbf{V}_1^1(\tau_D, \delta)}^2 &= \|A\phi(0)\|^2 + \tau_D \|\phi_1\|^2 + \delta \|A^{-\frac{1}{2}}\phi_1\|^2 \\ &\leq \|A\phi(0)\|^2 + \tau_D \|\phi_1\|^2 + c\delta \|\phi_1\|^2 \\ &\leq \|z_0\|_{\mathbf{V}_1^1(0, 0)}^2 + c(\tau_D + \delta). \end{aligned}$$

Thus,

$$\|z_0\|_{\mathbf{V}_1^1(\tau_D, \delta)}^2 \leq \Delta_1 \|z_0\|_{\mathbf{V}_1^1(0, 0)}^2 + \Delta_2(\tau_D + \delta).$$

Proof of (L2). Let $z_0 \in \mathbf{B}$ and define the maps $V_{\tau_D, \delta}(t)$ and $W_{\tau_D, \delta}(t)$ as follows:

$$V_{\tau_D, \delta}(t)z_0 = (v(t), v_t(t)), W_{\tau_D, \delta}(t)z_0 = (w(t), w_t(t)),$$

with v and w solutions to (5.7) and (5.8) and (5.9) and (5.10), respectively.

Multiplying (5.7) by $\alpha v + v_t$ and thanks to Gronwall’s lemma, we deduce that

$$\|(v, v_t)\|_{\mathbf{V}_1^1} \leq ce^{-c't} \|(v(0), v_t(0))\|_{\mathbf{V}_1^1}. \tag{7.1}$$

Thus,

$$\|V_{\tau_D, \delta}z_1 - V_{\tau_D, \delta}z_2\|_{\mathbf{V}_1^1(\tau_D, \delta)} \leq \frac{1}{4} \|z_1 - z_2\|_{\mathbf{V}_1^1(\tau_D, \delta)}, \forall z_1, z_2 \in \mathbf{B}$$

for some $c' > 0$.

It is then apparent that we can fix a $t^* \geq t_B$ such that the map $V_{\tau_D, \delta} = V_{\tau_D, \delta}(t^*)$ satisfies the first inequality of (L2).

The corresponding difference $w(t) = w^1(t) + w^2(t)$ fulfills (5.9),

$$\tau_D w_{tt} + w_t + A^2 w + \delta A w_t + A(f(\phi^1) - f(\phi^2)) = g(\phi^1) - g(\phi^2), \tag{7.2}$$

for all $z_1, z_2 \in \mathbf{B}$ fixed initial data.

We multiply (7.2) by $\frac{\mu}{2} A^2 w + A w_t + \frac{1}{8} A^{-1} w_{tt}$ to obtain

$$\begin{aligned} \frac{d}{dt} H(t) + (2\delta - \mu\tau_D) \|A w_t\|^2 + \frac{7}{4} \|A^{\frac{1}{2}} w_t\|^2 + \mu \|A^2 w\|^2 + \frac{\tau_D}{4} \|A^{-\frac{1}{2}} w_{tt}\|^2 &= -\mu(A(f(\phi^1) - f(\phi^2)), A^2 w) \\ + \mu(g(\phi^1) - g(\phi^2), A^2 w) + \frac{1}{4}(g(\phi^1) - g(\phi^2), A^{-1} w_{tt}) + 2(g(\phi^1) - g(\phi^2), A w_t) &\tag{7.3} \\ + 2(\phi_t^1 f'(\phi^1) - \phi_t^2 f'(\phi^2), A^2 w) + \frac{1}{4}(\phi_t^1 f'(\phi^1) - \phi_t^2 f'(\phi^2), w_t), & \end{aligned}$$

where

$$H(t) = \tau_D \|A^{\frac{1}{2}} w_t\|^2 + \mu \tau_D (Aw_t, Aw) + \frac{\mu}{2} \|Aw\|^2 + \|A^{\frac{3}{2}} w\|^2 + \frac{1}{4} (w_t, Aw) + \frac{1}{8} \|A^{-\frac{1}{2}} w_t\|^2 + \frac{\delta}{2} \|wt\|^2 + \frac{\delta \mu}{2} \|A^{\frac{3}{2}} w\|^2 + 2(A(f(\phi^1) - f(\phi^2)), Aw) + \frac{1}{4} (f(\phi^1) - f(\phi^2), w_t)$$

satisfies

$$c_{30} \|(w, w_t)\|_{V_2^1(\tau_D, \delta)}^2 - c \|A\phi\|^2 \leq H(t) \leq c_{31} \|(w, w_t)\|_{V_2^1(\tau_D, \delta)}^2 + c \|A\phi\|^2, \quad c_{30} > 0, \quad c_{31} > 0.$$

Thanks to (2.10), (2.4), and Young's, Hölder's, and Poincaré's inequalities, we have

$$\frac{d}{dt} H(t) + \delta \|Aw_t\|^2 + \frac{5}{4} \|A^{\frac{1}{2}} w_t\|^2 + \frac{\mu}{4} \|A^2 w\|^2 + \frac{\tau_D}{8} \|A^{-\frac{1}{2}} w_{tt}\|^2 \leq c(\|\phi_t\|^2 + \|A\phi\|^2). \tag{7.4}$$

Integrating (7.4) over $(0, t)$ and thanks to Lemma 4.3, we have

$$\|W_{\tau_D, \delta} z_1 - W_{\tau_D, \delta} z_2\|_{V_2^1(\tau_D, \delta)} \leq ce^{ct^*} \|z_1 - z_2\|_{V_1^1(\tau_D, \delta)}, \tag{7.5}$$

where $W_{\tau_D, \delta} = W_{\tau_D, \delta}(t^*)$.

Proof of (L3). Assume that τ_D and δ satisfy (2.4) and $z_0 \in \mathbf{B}$. Set $\Pi_{\tau_D, \delta} = \Pi$, where Π is given by

$$\Pi(\phi_0, \phi_1) = (\phi_0, L_0(\phi_0)),$$

with L_0 as in (4.18) (for $\delta = 0$). By construction, $\Pi : \mathbf{B}(\tau_D, \delta) \rightarrow \mathbf{B}(0, 0)$. In order to show the desired inequality, we exploit the following decomposition. We write the first component $\phi(t)$ of the solution $S_{\tau_D, \delta}^1(t)z_0$ as

$$\phi(t) = \phi^0(t) + \tau_D \hat{\phi} \left(\frac{t}{\tau_D} \right) + \tau_D R(t), \tag{7.6}$$

where ϕ^0 solves the limiting parabolic problem

$$\begin{cases} \phi^0 + A(A\phi^0 + f(\phi^0)) = g(\phi^0), \\ \phi^0(0) = \phi_0. \end{cases}$$

Setting

$$\tau = \frac{t}{\tau_D} \text{ and } \hat{\phi}_0 = \phi_1 - \phi_t^0(0) \in D(A),$$

the boundary layer $\hat{\phi} = \hat{\phi}(\tau)$ is the solution to

$$\begin{cases} \hat{\phi}_{\tau\tau} + \hat{\phi}_{\tau} + \delta A \hat{\phi}_{\tau} = 0, \\ \hat{\phi}_{\tau}(0) = \hat{\phi}_0, \\ \lim_{\tau \rightarrow \infty} \hat{\phi}(\tau) = 0. \end{cases}$$

It is easily seen that $\hat{\phi}(\tau)$ and $\hat{\phi}_{\tau}(\tau)$ can be expressed as

$$\hat{\phi}(\tau) = -(\mathbf{I} + \delta A)^{-1} \mathbf{T}(\tau) \hat{\phi}_0 \tag{7.7}$$

and

$$\hat{\phi}_{\tau}(\tau) = \mathbf{T}(\tau) \hat{\phi}_0, \tag{7.8}$$

where \mathbf{T} is the strongly continuous semigroup on $L^2(\Omega)$ whose infinitesimal generator is $-(\mathbf{I} + \delta A)$.

Notice that $\mathbf{T}(\tau)$ is exponentially stable, and more precisely,

$$\|\mathbf{T}(\tau)\| \leq e^{-\tau}, \forall \tau \geq 0. \quad (7.9)$$

Since $\hat{\phi}_\tau(\tau) = \tau_D \hat{\phi}\left(\frac{t}{\tau_D}\right)$, we get at once the bound

$$\sup_{\tau \geq 0} \|A\hat{\phi}(\tau)\| + \int_0^\infty \|A\hat{\phi}_\tau\left(\frac{y}{\tau_D}\right)\| dy \leq c. \quad (7.10)$$

We start by providing an estimate for the component ϕ^0 .

Lemma 7.6. *There holds*

$$\|A\phi_t^0\|^2 + \|A^{-1}\phi_{tt}^0\|^2 + \int_0^\infty [\|\phi_{tt}^0(y)\|^2 + \|A^{-2}\phi_{ttt}^0(y)\|^2] dy \leq c, \quad (7.11)$$

for any $t \geq 0$.

Proof. The estimate of the term $\|A\phi_t^0\|^2$ is already known, due to the fact that $z_0 \in \mathbf{B}$. We consider the auxiliary problem

$$\begin{cases} \psi_t + A(A\psi + \psi f'(\phi^0)) = \psi g'(\phi^0), \\ \psi(0) = L(\phi_0), \end{cases}$$

where $\phi_t^0 = \psi$. This system admits a unique solution in $C([0, T], D(A))$.

We multiply the first equation of the system by $\psi_t + A\psi$ and obtain

$$\begin{aligned} \frac{d}{dt} (\|A^{\frac{1}{2}}\psi\|^2 + \|A\psi_t\|^2) + 2\|\psi_t\|^2 + 2\|A^{\frac{3}{2}}\psi\|^2 &= -2(A(f'(\phi^0)\psi), \psi_t) \\ &+ 2(A(f'(\phi^0)\psi), A\psi) + 2(\psi g'(\phi^0), \psi_t) + 2(\psi g'(\phi^0), A\psi). \end{aligned} \quad (7.12)$$

Owing to (6.6), (6.7), and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned} -2|(A(f'(\phi^0)\psi), \psi_t)| &\leq \|A(f'(\phi^0)\psi)\| \|\psi_t\| \\ &\leq \|A\psi\| \|\psi_t\| \\ &\leq c\|A^{\frac{3}{2}}\psi\| \|\psi_t\| \\ &\leq \frac{1}{2}\|\psi_t\|^2 + \frac{1}{2}\|A^{\frac{3}{2}}\psi\|^2 \end{aligned}$$

and

$$\begin{aligned} 2|(A(f'(\phi^0)\psi), A\psi)| &\leq \|A^{\frac{1}{2}}(f'(\phi^0)\psi)\| \|A^{\frac{3}{2}}\psi\| \\ &\leq c\|A^{\frac{1}{2}}\psi\| \|A^{\frac{3}{2}}\psi\| \\ &\leq c\|A^{\frac{1}{2}}\psi\|^2 + \frac{1}{2}\|A^{\frac{3}{2}}\psi\|^2. \end{aligned}$$

Using (6.2) and Young's and Poincaré's inequalities, we obtain

$$\begin{aligned} 2|(\psi g'(\phi^0), \psi_t)| &\leq \|\psi g'(\phi^0)\| \|\psi_t\| \\ &\leq c\|A^{\frac{1}{2}}\psi\| \|\psi_t\| \\ &\leq c\|A^{\frac{1}{2}}\psi\|^2 + \frac{1}{2}\|\psi_t\|^2 \end{aligned}$$

and

$$\begin{aligned} 2|(\psi g'(\phi^0), A\psi)| &\leq \|\psi g'(\phi^0)\| \|A\psi\| \\ &\leq c \|A^{\frac{1}{2}}\psi\|^2. \end{aligned}$$

Combining the above estimates in (7.12), we have

$$\frac{d}{dt} (\|A^{\frac{1}{2}}\psi\|^2 + \|A\psi_t\|^2) + \|\psi_t\|^2 + \|A^{\frac{3}{2}}\psi\|^2 \leq c \|A^{\frac{1}{2}}\psi\|^2. \quad (7.13)$$

Integrating the above inequality over $(0, t)$, passing to the limit as $t \rightarrow \infty$, and thanks to (4.9), we get

$$\int_0^\infty [\|A^{\frac{3}{2}}\phi_t^0(y)\|^2 + \|\phi_{tt}^0(y)\|^2] dy \leq c. \quad (7.14)$$

We deduce the integral estimate for ϕ_{tt}^0 . In order to control the two remaining terms, we introduce a further auxiliary problem,

$$\begin{cases} \theta_t + A(A\theta + \theta f'(\phi^0) + f''(\phi^0)|\phi_t^0|^2) = \theta g'(\phi^0) + g''(\phi^0)|\phi_t^0|^2, \\ \theta(0) = -[A(AL_0(\phi_0)) + f'(\phi_0)L_0(\phi_0) - g'(\phi_0)L_0(\phi_0)], \end{cases}$$

where $\theta_t = \psi_{tt}$. This system admits a unique solution in $C([0, T], D(A^{-1}))$.

Multiplying the first equation by $A^{-4}\theta$, we obtain

$$\begin{aligned} \frac{d}{dt} \|A^{-1}\theta\|^2 + 2\|A^{-2}\theta_t\|^2 &= -2(A^{-1}(f'(\phi^0)\theta), A^{-2}\theta_t) - 2(A^{-1}(f''(\phi^0)|\phi_t^0|^2), A^{-2}\theta_t) \\ &\quad + 2(g'(\phi^0)\theta, A^{-4}\theta_t) + 2(g''(\phi^0)|\phi_t^0|^2), A^{-4}\theta_t). \end{aligned} \quad (7.15)$$

Thanks to (6.6), (6.7), and Young's, Poincaré's, and proper interpolation inequalities, we obtain

$$\begin{aligned} -2|(A^{-1}(f'(\phi^0)\theta), A^{-2}\theta_t)| &\leq \|A^{-1}(f'(\phi^0)\theta)\| \|A^{-2}\theta_t\| \\ &\leq c\|\theta\| \|A^{-2}\theta_t\| \\ &\leq c\|\theta\|^2 + \frac{1}{4}\|A^{-2}\theta_t\|^2 \end{aligned}$$

and

$$\begin{aligned} -2|(A^{-1}(f''(\phi^0)|\phi_t^0|^2), A^{-2}\theta_t)| &\leq \|A^{-1}(f''(\phi^0)|\phi_t^0|^2)\| \|A^{-2}\theta_t\| \\ &\leq c\|A^{-1}\phi_t^0\|_{L^4(\Omega)}^4 + \frac{1}{4}\|A^{-2}\theta_t\|^2 \\ &\leq c\|A^{\frac{1}{2}}\phi_t^0\|^4 + \frac{1}{4}\|A^{-2}\theta_t\|^2 \\ &\leq c\|A^{\frac{3}{2}}\phi_t^0\|^2 + \frac{1}{4}\|A^{-2}\theta_t\|^2. \end{aligned}$$

Owing to (6.2) and Young's and Poincaré's inequalities, we have

$$\begin{aligned} 2|(g'(\phi^0)\theta), A^{-4}\theta_t)| &\leq \|(g'(\phi^0)\theta)\| \|A^{-4}\theta_t\| \\ &\leq c\|\theta\| \|A^{-2}\theta_t\| \\ &\leq c\|\theta\|^2 + \frac{1}{4}\|A^{-2}\theta_t\|^2 \end{aligned}$$

and

$$\begin{aligned}
 2|(g''(\phi^0)|\phi_t^0|^2), A^{-4}\theta_t| &\leq \|g''(\phi^0)|\phi_t^0|^2\| \|A^{-4}\theta_t\| \\
 &\leq \| |\phi_t^0|^2 \| \|A^{-2}\theta_t\| \\
 &\leq \|\phi_t^0\|_{L^4(\Omega)}^4 + \frac{1}{2}\|A^{-4}\theta_t\|^2 \\
 &\leq c\|A^{\frac{1}{2}}\phi_t^0\|^4 + \frac{1}{4}\|A^{-2}\theta_t\|^2 \\
 &\leq c\|A^{\frac{3}{2}}\phi_t^0\|^2 + \frac{1}{4}\|A^{-2}\theta_t\|^2.
 \end{aligned}$$

Combining the above estimates, we can rewrite (7.15) as

$$\frac{d}{dt}\|A^{-1}\theta\|^2 + \|A^{-2}\theta_t\|^2 \leq c(\|A^{\frac{3}{2}}\phi_t^0\|^2 + \|\phi_{tt}^0\|^2). \tag{7.16}$$

Integrating the above differential inequality over $(0, t)$, passing to the limit as $t \rightarrow \infty$, and thanks to (7.14), we get

$$\int_0^\infty (\|\phi_{tt}^0(y)\|^2)dy \leq c. \tag{7.17}$$

Inequalities (7.14) and (7.17) give the result. □

Lemma 7.7. *The remainder $R(t)$ of decomposition (7.6) satisfies the bound*

$$\|(R(t), R_t(t))\|_{V_1} \leq c \left(1 + \frac{\delta}{\tau_D}\right) e^{ct}, \forall t \geq 0. \tag{7.18}$$

Proof. Let

$$\begin{aligned}
 \gamma_0(t) &= \int_0^1 f'(s\phi(t) + (1-s)\phi^0(t))ds, \\
 \gamma_1(t) &= \int_0^1 g'(s\phi(t) + (1-s)\phi^0(t))ds.
 \end{aligned}$$

Notice that

$$\begin{aligned}
 f(\phi) - f(\phi^0) &= \tau_D \gamma_0 \hat{\phi} \left(\frac{t}{\tau_D}\right) + \tau_D \gamma_0 R(t), \\
 g(\phi) - g(\phi^0) &= \tau_D \gamma_1 \hat{\phi} \left(\frac{t}{\tau_D}\right) + \tau_D \gamma_1 R(t).
 \end{aligned}$$

By definition, R fulfills the problem

$$\begin{cases} \tau_D R_{tt} + R_t + \delta A R_t + A^2 R = -\phi_{tt}^0 - \frac{\delta}{\tau_D} A \phi_t^0 - A^2 \hat{\phi} \left(\frac{\cdot}{\tau_D}\right) - A(\gamma_0 \hat{\phi} \left(\frac{\cdot}{\tau_D}\right) - A\gamma_0 R + \gamma_1 \hat{\phi} \left(\frac{\cdot}{\tau_D}\right) + \gamma_1 R, \\ R(0) = (\mathbf{I} + \delta A)^{-1} \widehat{\phi}(0), \\ R_t(0) = 0. \end{cases}$$

Multiplying the first equation of the system by R_t , we deduce that

$$\begin{aligned}
 \frac{d}{dt}H(t) + 2\|R_t\|^2 + 2\delta\|A^{\frac{1}{2}}R_t\|^2 &= -2(\phi_{tt}^0, R_t) - 2\frac{\delta}{\tau_D}(A\phi_t^0, R_t) + 2(A\hat{\phi}_t \left(\frac{\cdot}{\tau_D}\right), AR) \\
 &\quad - 2(A(\gamma_0 \hat{\phi} \left(\frac{\cdot}{\tau_D}\right), R_t) - 2(A\gamma_0 R, R_t) + 2(\gamma_1 \hat{\phi} \left(\frac{\cdot}{\tau_D}\right), R_t) + 2(\gamma_1 R, R),
 \end{aligned} \tag{7.19}$$

where

$$H(t) = \tau_D ||R_t||^2 + ||AR||^2 + 2(A(\gamma_0 \hat{\phi} \left(\frac{\cdot}{\tau_D} \right), AR)$$

satisfies

$$c_{30}(\tau_D ||R_t||^2 + ||AR||^2) - c \leq H(t) \leq c_{31}(\tau_D ||R_t||^2 + ||AR||^2) + c, \quad c_{30} > 0, \quad c_{31} > 0, \quad c \geq 0.$$

Thanks to Young's and Poincaré's inequalities, we get

$$\begin{aligned} -2(\phi_{tt}^0, R_t) - 2\frac{\delta}{\tau_D}(A\phi_t^0, R_t) &\leq ||\phi_{tt}^0|| ||R_t|| + \frac{\delta}{\tau_D} ||A\phi_t^0|| ||R_t|| \\ &\leq c ||\phi_{tt}^0||^2 + \frac{1}{2} ||R_t||^2 + \frac{\delta^2}{\tau_D^2} ||A\phi_t^0||^2 \end{aligned}$$

and

$$\begin{aligned} 2(A\hat{\phi}_t \left(\frac{\cdot}{\tau_D} \right), AR) - 2(A(\gamma_0 \hat{\phi} \left(\frac{\cdot}{\tau_D} \right), R_t) &\leq ||A\hat{\phi}_t \left(\frac{\cdot}{\tau_D} \right)|| ||AR|| + ||A(\gamma_0 \hat{\phi} \left(\frac{\cdot}{\tau_D} \right)|| ||R_t|| \\ &\leq c ||A\hat{\phi}_t \left(\frac{\cdot}{\tau_D} \right)||^2 + c ||AR||^2 + c ||A\hat{\phi} \left(\frac{\cdot}{\tau_D} \right)||^2 + \frac{1}{8} ||R_t||^2. \end{aligned}$$

In the same way,

$$\begin{aligned} -2(A\gamma_0 R, R_t) + 2(\gamma_1 \hat{\phi} \left(\frac{\cdot}{\tau_D} \right), R_t) + 2(\gamma_1 R, R) &\leq ||A\gamma_0 R|| ||R_t|| + ||\gamma_1 \hat{\phi} \left(\frac{\cdot}{\tau_D} \right)|| ||R_t|| + ||\gamma_1 R|| ||R_t|| \\ &\leq c ||AR||^2 + \frac{3}{8} ||R_t||^2 + c ||A\hat{\phi} \left(\frac{\cdot}{\tau_D} \right)||^2. \end{aligned}$$

Combining the above into (7.19), we find

$$\frac{d}{dt} H(t) + ||R_t||^2 + 2\delta ||A^{\frac{1}{2}} R_t||^2 \leq c ||AR||^2 + c ||A\hat{\phi} \left(\frac{\cdot}{\tau_D} \right)||^2 + c ||A\hat{\phi}_t \left(\frac{\cdot}{\tau_D} \right)||^2 + c ||\phi_{tt}^0||^2 + \frac{\delta^2}{\tau_D^2} ||A\phi_t^0||^2. \quad (7.20)$$

Hence, thanks to (7.10) and (7.6), we conclude that

$$||AR||^2 + \tau_D ||R_t||^2 + \int_t^{t+1} (\delta ||A^{\frac{1}{2}} R_t||^2 + ||R_t||^2) dy \leq c \left(1 + \frac{\delta^2}{\tau_D^2} \right) e^{ct} \forall t \geq 0. \quad (7.21)$$

This finishes the first part of the proof.

Multiplying the first equation by $A^{-2}R_{tt}$, we have

$$\begin{aligned} \frac{d}{dt} Y(t) + 2\tau_D ||A^{-1}R_{tt}||^2 &= 2||R_t||^2 - g'(\phi^0)\phi_t^0, A^{-2}R_t + \frac{2\delta}{\tau_D}(\phi_{tt}^0, A^{-1}R_t) + 2(A\hat{\phi}, A^{-1}R_t) \\ &+ \frac{2}{\tau_D}(f'(\phi)(\phi_t - \phi_t^0), A^{-1}R_t) + \frac{2}{\tau_D}(g'(\phi)(\phi_t - \phi_t^0), A^{-2}R_t) + \frac{2}{\tau_D}((f'(\phi) - f'(\phi^0))\phi_t^0, A^{-1}R_t) \\ &+ \frac{2}{\tau_D}((g'(\phi) + 2(A^{-2}\phi_{tt}^0), R_t), \end{aligned} \quad (7.22)$$

where

$$\begin{aligned} Y(t) &= \delta ||A^{-\frac{1}{2}} R_t||^2 + ||A^{-1}R_t||^2 + 2(R_t, R) + 2(A^{-1}\phi_{tt}^0, A^{-1}R_t) + 2(\hat{\phi}, R_t) \\ &+ \frac{2\delta}{\tau_D}(\phi_t^0, A^{-1}R_t) + 2(\gamma_0 \hat{\phi}, A^{-1}R_t) + 2(\gamma_0 R, A^{-1}R_t) - 2(\gamma_2 R, A^{-2}R_t) \end{aligned}$$

satisfies

$$c_{33}(\delta \|A^{-\frac{1}{2}}R_t\|^2 + \|A^{-1}R_t\|^2 + \|AR\|^2) - c \left(1 + \frac{\delta^2}{\tau_D^2}\right) \leq Y(t) \leq c_{34}(\delta \|A^{-\frac{1}{2}}R_t\|^2 + \|A^{-1}R_t\|^2 + \|AR\|^2) + c \left(1 + \frac{\delta^2}{\tau_D^2}\right).$$

Thanks to Young's and Poincaré's inequalities, we find

$$\begin{aligned} & \frac{2}{\tau_D}(f'(\phi)(\phi_t - \phi_t^0, A^{-1}R_t) + \frac{2}{\tau_D}(g'(\phi)(\phi_t - \phi_t^0, A^{-2}R_t) + \frac{2}{\tau_D}((f'(\phi) - f'(\phi^0))\phi_t^0, A^{-1}R_t) \\ & + \frac{2}{\tau_D}((g'(\phi) - g'(\phi^0))\phi_t^0, A^{-2}R_t) \leq c\|A^{-1}R_t\|^2 \\ & + c\|AR\|^2 + c\|R_t\|^2 + c\|A\hat{\phi}\|^2 + c\|\hat{\phi}_t\|^2. \end{aligned}$$

Combining the above estimates into (7.22), we deduce that

$$\frac{d}{dt}Y(t) + 2\tau_D\|A^{-1}R_t\|^2 \leq c\|A^{-1}R_t\|^2 + c\|AR\|^2 + c\|R_t\|^2 + c\|A\hat{\phi}\|^2 + c\|\hat{\phi}_t\|^2. \quad (7.23)$$

Applying Gronwall's lemma, we conclude that

$$\delta\|A^{-\frac{1}{2}}R_t\|^2 + \|A^{-1}R_t\|^2 \leq c \left(1 + \frac{\delta^2}{\tau_D^2}\right) e^{ct}. \quad (7.24)$$

The inequalities (7.21) and (7.24) give the result of the lemma. \square

Lemma 7.8. *There holds*

$$\|S_{\tau_D, \delta}^1(t)z_0 - S_{0,0}^1(t)\Pi z_0\|_{V_1^1(\tau_D, \delta)} \leq c(\delta + \tau_D)e^{ct} + ce^{-\frac{t}{\tau_D}}. \quad (7.25)$$

Proof. From the point of view of decomposition (7.6), we have

$$\|S_{\tau_D, \delta}^1(t)z_0 - S_{0,0}^1(t)\Pi z_0\|_{V_1^1(\tau_D, \delta)} = \tau_D\|(\hat{\phi} + R, \hat{\phi}_t + R_t)\|.$$

Thanks to (7.7), (7.9), (7.10), and Lemma 7.18, we get

$$\|S_{\tau_D, \delta}^1(t)z_0 - S_{0,0}^1(t)\Pi z_0\|_{V_1^1(\tau_D, \delta)} \leq c(\delta + \tau_D)e^{ct} + ce^{-\frac{t}{\tau_D}}. \quad (7.26)$$

This gives the proof of (L3). \square

Proof of (L4). Fix $z_0 \in \mathbf{B}(\tau_D, \delta)$ and $\bar{z}_0 \in \mathbf{B}(0, 0)$. Since $\bar{z}_0 = \Pi z_0$, we get

$$\|S_{\tau_D, \delta}^1(t)z_0 - S_{0,0}^1(t)\Pi z_0\|_{V_1^1(0,0)} \leq \|S_{\tau_D, \delta}^1(t)z_0 - S_{0,0}^1(t)\Pi z_0\|_{V_1^1(\tau_D, \delta)} + \|S_{0,0}^1(t)\Pi z_0 - S_{0,0}^1(t)\Pi \bar{z}_0\|_{V_1^1(0,0)}.$$

Lemmas 4.3 and 7.25 give (L4).

Proof of (L5). For $z_1, z_2 \in \mathbf{B}$ and $t_1, t_2 \in [0, t^*]$, we have

$$\|S_{\tau_D, \delta}^1(t_1)z_1 - S_{\tau_D, \delta}^1(t_1)z_2\|_{V_1^1(\tau_D, \delta)} \leq \|S_{\tau_D, \delta}^1(t_1)z_1 - S_{\tau_D, \delta}^1(t_2)z_2\|_{V_1^1(\tau_D, \delta)} + \|S_{\tau_D, \delta}^1(t_2)z_2 - S_{\tau_D, \delta}^1(t_1)z_2\|_{V_1^1(\tau_D, \delta)}.$$

By taking $t_1 = t_2$ and thanks to Lemma 4.3, we obtain

$$\|S_{\tau_D, \delta}^1(t_1)z_1 - S_{\tau_D, \delta}^1(t_1)z_2\|_{V_1^1(\tau_D, \delta)} \leq ce^{ct^*} \|z_1 - z_2\|_{V_1^1(\tau_D, \delta)}. \quad (7.27)$$

There remains to prove the second part of (L5) for $t_1 \neq t_2$.

Lemma 7.9. *For τ_D and δ satisfies (2.4), there exists $Q_{\tau_D, \delta}$ such that*

$$\sup_{z_0 \in \mathbf{B}} \sup_{t \in [0, t^*]} \|(\phi_t, \phi_{tt})\|_{V_1^1(\tau_D, \delta)} \leq Q_{\tau_D, \delta}. \quad (7.28)$$

Proof. We multiply (6.18) by $u_t + \frac{1}{8}A^{-2}u_{tt}$ to obtain

$$\begin{aligned} \frac{d}{dt}H(t) + \frac{\tau_D}{2}\|A^{-1}u_{tt}\|^2 + \frac{7}{4}\|u_t\|^2 + 2\delta\|A^{\frac{1}{2}}u_t\|^2 &= -2(A(f'(\phi)u), u_t) \\ &+ \frac{1}{4}(A(f'(\phi)u)_t, A^{-2}u_t) + 2(g'(\phi)u, u_t) + \frac{1}{4}(g'(\phi)u, A^{-2}u_{tt}). \end{aligned} \quad (7.29)$$

Owing to (6.7) and Young's and Poincaré's inequalities, we deduce that

$$\begin{aligned} -2(A(f'(\phi)u), u_t) + \frac{1}{4}(A(f'(\phi)u)_t, A^{-2}u_t) &\leq \|A(f'(\phi)u)\| \|u_t\| + \|(A(f'(\phi)u)_t)\| \|A^{-2}u_t\| \\ &\leq c\|Au\|^2 + c\|u\|_{L^4(\Omega)}^4 + \frac{1}{2}\|u_t\|^2 \\ &\leq c\|u\|^2\|Au\|^2 + c\|Au\|^2 + \frac{1}{2}\|u_t\|^2 \\ &\leq c\|Au\|^2 + \frac{1}{2}\|u_t\|^2. \end{aligned}$$

Exploiting (6.2) and Young's and Poincaré's inequalities, we find

$$\begin{aligned} 2(g'(\phi)u, u_t) + \frac{1}{4}(g'(\phi)u, A^{-2}u_{tt}) &\leq \|g'(\phi)u\| \|u_t\| + \|g'(\phi)u\| \|A^{-2}u_{tt}\| \\ &\leq \|u\| \|u_t\| + \|u\| \|A^{-1}u_{tt}\| \\ &\leq \frac{1}{2}\|u_t\|^2 + \frac{\tau_D}{4}\|A^{-1}u_{tt}\|^2 + c. \end{aligned}$$

Thus, (7.29) becomes

$$\frac{d}{dt}H(t) + \|u_t\|^2 + 2\delta\|A^{\frac{1}{2}}u_t\|^2 + \frac{\tau_D}{4}\|A^{-1}u_{tt}\|^2 \leq c(1 + H). \quad (7.30)$$

Integrating estimate (7.30) over $(0, t)$, we find the result of the lemma. \square

On account of Lemma 7.9, we conclude that

$$\begin{aligned} \|S_{\tau_D, \delta}^1(t_1)z_1 - S_{\tau_D, \delta}^1(t_2)z_2\|_{V_1^1(\tau_D, \delta)} &\leq \sup_{z_0 \in \mathbf{B}} \sup_{t \in [0, t^*]} \|(\phi_t, \phi_{tt})\|_{V_1^1(\tau_D, \delta)} |t_1 - t_2| \\ &\leq Q_{\tau_D, \delta} |t_1 - t_2|. \end{aligned}$$

Hence, the proof of (L5). \square

8 | NUMERICAL RESULTS

As far as the numerical simulations of the model is concerned, we rewrite (2.1) in the following equivalent form:

$$(S) \begin{cases} \tau_D \phi_t + \phi_t - \Delta \mu = g(\phi), \\ \mu = -\Delta \phi + f(\phi) + \delta \phi_t, \end{cases}$$

where $\tau_D, \delta \geq 0$ are the relaxation and viscosity parameters.

We consider a P_1 finite element approach and a semi-implicit Euler time discretization. Let T_h be the triangulation of $\bar{\Omega}$ and $V_h = \{v_h \in C^0(\bar{\Omega}); v_h = 0 \text{ on } \partial\bar{\Omega}, v_h|_K \in P_1, \forall K \in T_h\}$ be the finite element space. We denote by $\Delta t > 0$ the time step. The variational formulation of the semidiscretized model is as follows.

Let $\phi^0, \phi^1 \in V_h$ be given and for $n = 1, 2, \dots$ find $\phi^{n+1}, \mu^{n+1} \in V_h$ such that

$$\left(\tau_D \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\Delta t^2}, \chi \right) + \left(\frac{\phi^{n+1} - \phi^n}{\Delta t}, \chi \right) + (\nabla \mu^{n+1}, \nabla \chi) = (g(\phi^n), \chi), \quad (8.1)$$

$$(\mu^{n+1}, \psi) = (\nabla \phi^{n+1}, \nabla \psi) + (f(\phi^n), \psi) + \left(\delta \frac{\phi^{n+1} - \phi^n}{\Delta t}, \psi \right), \quad (8.2)$$

for all $\chi, \psi \in V_h$ (note that μ^0 and μ^1 are not needed). This scheme is linear. In the next section, we give a condition on the time step Δt which is sufficient for the stability when $g \equiv 0$.

8.1 | Stability by the energy method in the case $g \equiv 0$.

We assume that $g \equiv 0$. In the case $\delta = 0$, a condition on the time step was obtained by the energy method for closely related schemes in earlier studies [26, 27]. The proof here is similar, but we obtain a condition on the time step which is interesting for numerical simulations.

Since $g = 0$, problems (2.1)–(2.3) have a Lyapunov function. Indeed, by applying $(-\Delta)^{-1}$ to (2.1) and taking the L^2 -scalar product with ϕ_t , we see that

$$\frac{\tau_D}{2} \frac{d}{dt} \|\phi_t\|_{-1}^2 + \|\phi_t\|_{-1}^2 + \frac{1}{2} \frac{d}{dt} \|\nabla \phi\|^2 + \frac{d}{dt} F(\phi) + \delta \|\phi_t\|^2 = 0, \quad (8.3)$$

where $F(s) = \int_0^s f(t) dt$. That is, we have

$$\frac{d}{dt} E(\phi, \phi_t) + \|\phi_t\|_{-1}^2 + \delta \|\phi_t\|^2 = 0,$$

where

$$E(\phi, u) = \frac{1}{2} \|\phi\|_1^2 + (F(\phi), 1) + \frac{\tau_D}{2} \|u\|_{-1}^2.$$

If Δt is small enough, then schemes (8.1) and (8.2) also have a Lyapunov functional. For the sake of simplicity, we only consider here the time semidiscrete version of (8.1) and (8.2) which reads as follows: let $\phi^0, \phi^1 \in H_0^1(\Omega) = V$ and for $n = 1, 2, \dots$ find $\phi^{n+1}, \mu^{n+1} \in H_0^1(\Omega)$ such that

$$\tau_D \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\Delta t^2} + \frac{\phi^{n+1} - \phi^n}{\Delta t} - \Delta \mu^{n+1} = 0, \quad (8.4)$$

$$\mu^{n+1} = -\Delta \phi^{n+1} + f(\phi^n) + \delta \frac{\phi^{n+1} - \phi^n}{\Delta t}. \quad (8.5)$$

We assume that f satisfies assumptions (2.5)–(2.7) and that

$$f \text{ is Lipschitz continuous on } \mathbb{R}. \quad (8.6)$$

Let (ϕ^n, μ^n) be a sequence of elements in $V \times V$ that satisfies schemes (8.4) and (8.5). We define the following discrete derivative:

$$u^{n+1} := \frac{\phi^{n+1} - \phi^n}{\Delta t}, \quad n \geq 0 \quad (8.7)$$

and the discrete energy

$$E^n := \frac{\tau_D}{2} \|u^n\|_{-1}^2 + \frac{1}{2} \|\phi^n\|_1^2 + (F(\phi^n), 1), \quad n \geq 0. \quad (8.8)$$

Proposition 8.1. *If $\Delta t \leq \max\left(\frac{2\delta}{L}, \frac{8}{L^2}\right)$, then for all $n \geq 1$, we have*

$$E^{n+1} \leq E^n,$$

where $L = \sup_{s \in \mathbb{R}} |f'(s)|$.

Proof. Multiplying (8.4) by $(-\Delta)^{-1}(\phi^{n+1} - \phi^n)$, we have

$$\frac{\tau_D}{\Delta t^2} (\delta\phi^{n+1} - \delta\phi^n, \phi^{n+1} - \phi^n)_{-1} + \frac{1}{\Delta t} \|\phi^{n+1} - \phi^n\|_{-1}^2 + (\mu^{n+1}, \phi^{n+1} - \phi^n) = 0, \quad (8.9)$$

where $\delta\phi^n = \phi^n - \phi^{n-1}$.

Multiplying (8.5) by $(\phi^{n+1} - \phi^n)$, we obtain

$$\frac{1}{2} (\|\phi^{n+1}\|_1^2 - \|\phi^n\|_1^2 + \|\phi^{n+1} - \phi^n\|_1^2) + \frac{\delta}{\Delta t} \|\phi^{n+1} - \phi^n\|^2 - (\mu^{n+1}, \phi^{n+1} - \phi^n) + (f(\phi^n), \phi^{n+1} - \phi^n) = 0, \quad (8.10)$$

where

$$(\nabla\phi^{n+1}, \nabla(\phi^{n+1} - \phi^n)) = \frac{1}{2} \|\phi^{n+1}\|_1^2 - \frac{1}{2} \|\phi^n\|_1^2 + \frac{1}{2} \|\phi^{n+1} - \phi^n\|_1^2.$$

Let us sum (8.9) and (8.10). We deduce that

$$\begin{aligned} & \frac{\tau_D}{2\Delta t^2} (\|\delta\phi^{n+1} - \delta\phi^n\|_{-1}^2 - \|\delta\phi^n\|_{-1}^2 + \|\delta\phi^{n+1}\|_{-1}^2) + \frac{1}{2} \|\phi^{n+1}\|_1^2 - \frac{1}{2} \|\phi^n\|_1^2 \\ & + \frac{1}{2} \|\phi^{n+1} - \phi^n\|_1^2 + \frac{\delta}{\Delta t} \|\phi^{n+1} - \phi^n\|^2 + \frac{1}{\Delta t} \|\phi^{n+1} - \phi^n\|_{-1}^2 = -(f(\phi^n), \phi^{n+1} - \phi^n). \end{aligned} \quad (8.11)$$

From Taylor's inequality, we get

$$(F(\phi^{n+1}) - F(\phi^n), 1) = (f(\phi^n), \phi^{n+1} - \phi^n) + \frac{1}{2} (f'(\xi_{\phi^{n+1}, \phi^n})(\phi^{n+1} - \phi^n)^2, 1),$$

where $\xi_{\phi^n, \phi^{n+1}}(x) \in [\phi^n(x), \phi^{n+1}(x)]$ for a.e. $x \in \Omega$, and for all $n \geq 0$. Thus,

$$\begin{aligned} & \frac{\tau_D}{2\Delta t^2} \|\delta\phi^{n+1} - \delta\phi^n\|_{-1}^2 + E^{n+1} - E^n + \frac{1}{2} \|\phi^{n+1} - \phi^n\|_1^2 + \frac{\delta}{\Delta t} \|\phi^{n+1} - \phi^n\|^2 \\ & + \frac{1}{\Delta t} \|\phi^{n+1} - \phi^n\|_{-1}^2 \leq \frac{L}{2} \|\phi^{n+1} - \phi^n\|^2, \end{aligned} \quad (8.12)$$

where $L = \|f'\|_{L^\infty(\mathbb{R})}$. Thus, if $\frac{\delta}{\Delta t} \geq \frac{L}{2}$, we have $E^{n+1} \leq E^n$.

Thanks to interpolation and Young's inequalities, we also deduce from (8.12) that

$$\begin{aligned} & \frac{\tau_D}{2\Delta t^2} (\|\delta\phi^{n+1} - \delta\phi^n\|_{-1}^2) + E(\phi^{n+1}) - E(\phi^n) + \frac{1}{2} \|\phi^{n+1} - \phi^n\|_1^2 + \frac{\delta}{\Delta t} \|\phi^{n+1} - \phi^n\|^2 \\ & + \frac{1}{\Delta t} \|\phi^{n+1} - \phi^n\|_{-1}^2 \leq \frac{1}{2} \|\phi^{n+1} - \phi^n\|_1^2 + \frac{L^2}{8} \|\phi^{n+1} - \phi^n\|_{-1}^2. \end{aligned} \quad (8.13)$$

This time, if $\frac{1}{\Delta t} \geq \frac{L^2}{8}$, then $E^{n+1} \leq E^n$. This gives the result. \square

Remark 8.2. Using the Lyapunov functional E^n , it is possible to prove, using the energy estimate (8.12), that the sequence (ϕ^n, μ^n) converges to a steady state as $n \rightarrow \infty$ (see Brachet et al. [28] and references therein). When $g \neq 0$, this result seems out of reach, because there is no Lyapunov functional associated to the system. Numerical simulations in the next section indicate that it is still true.

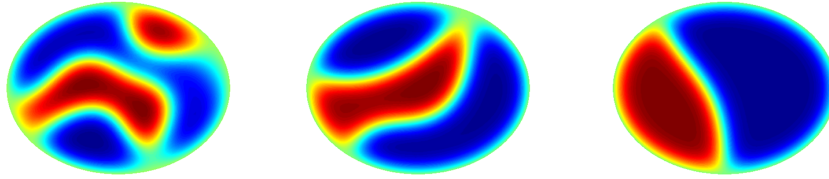


FIGURE 1 Time evolution of the solution at $t = 4.10^{-3}$ (left), $t = 10^{-2}$ (middle), and $t = 29.10^{-3}$ (right) for $\tau_D = \delta = 0$ and $k = 12$. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

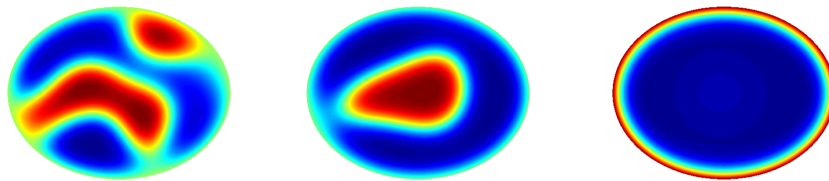


FIGURE 2 Time evolution of the solution at $t = 4.10^{-3}$ (left), $t = 10^{-2}$ (middle), and $t = 29.10^{-3}$ (right) for $\tau_D = 0$, $\delta = 0.02$ and $k = 12$. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

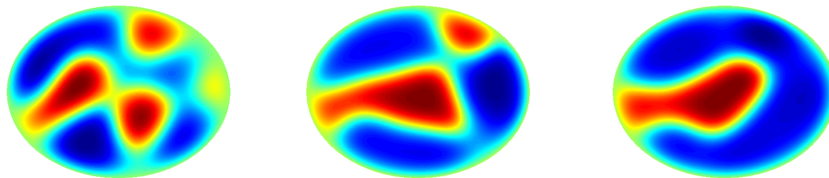


FIGURE 3 Time evolution of the solution at $t = 17.10^{-3}$ (left), $t = 23.10^{-3}$ (middle), and $t = 29.10^{-3}$ (right) for $\tau_D = 0.01$, $\delta = 0.02$, and $k = 12$. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

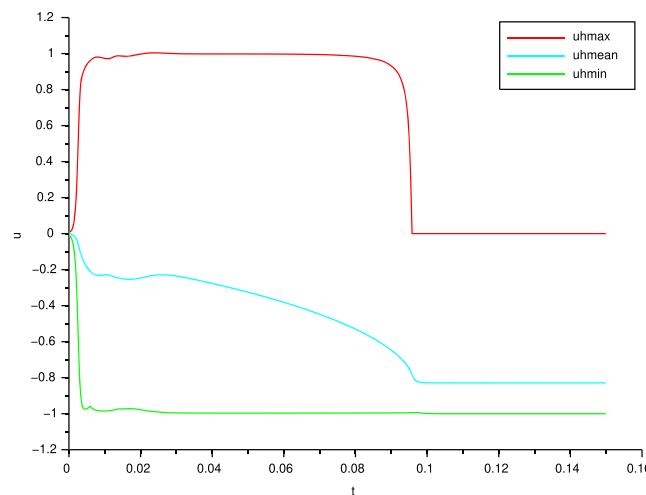


FIGURE 4 Evolution of u_{max} , u_{min} and u_{mean} for 1500 iterations. In this case, we take $k = 12$ and $\tau_D = \delta = 0$, that is, the classical Cahn–Hilliard equation with the symport term. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

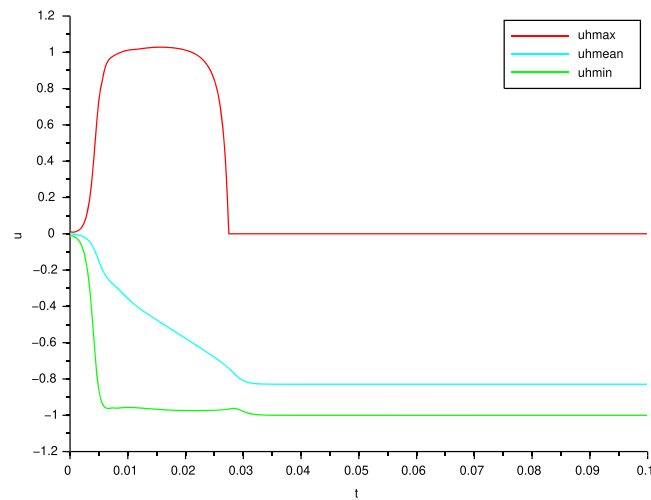


FIGURE 5 Evolution of u_{max} , u_{min} , and u_{mean} for 1000 iterations. In this case, we take $k = 12$, $\tau_D = 0$ and $\delta = 0.02$, that is, the viscous Cahn–Hilliard equation with the sympport term. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

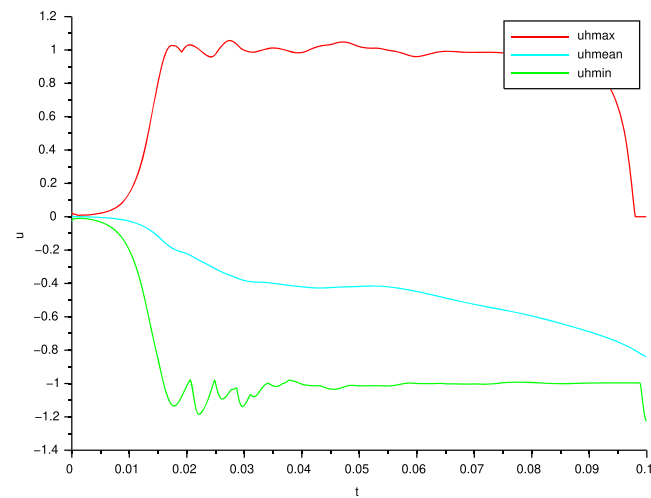


FIGURE 6 Evolution of u_{max} , u_{min} , and u_{mean} for 1000 iterations. In this case, we take $k = 12$, $\tau_D = 0.01$, and $\delta = 0.02$, that is, the hyperbolic relaxation of the viscous Cahn–Hilliard equation with the sympport term. The time step is $\Delta t = 10^{-4}$. [Colour figure can be viewed at wileyonlinelibrary.com]

8.2 | Simulations in two space dimensions

The numerical simulations are performed with the software Freefem++ [29]. In the numerical results presented below, the domain Ω is the ellipse $\{t \in [0, 2\pi]; x = 1.3 \cos(t), y = \sin(t)\}$. The triangulation is obtained by dividing Ω into 6042 triangles and 3122 vertices. We take here $f(s) = \frac{1}{\epsilon^2}(s^3 - s)$ and $g(s) = \frac{-k(s+1)}{2k' + |s+1|}$, where $\epsilon = 0.1$, $k = 12$ and $k' = 1$. In Figures 1–3, we present the evolution of the solution for different times under the influence of the parameters τ_D , δ , and k . Generally, the blue color represents the negative part of the solution and the red color the positive one.

In Figures 4–6, we show the influence of the parameters τ_D , δ , and k on the evolution of the decomposition of phases. We take the initial condition random in the interval $[-0.03, 0.03]$ on a coarse mesh. In the three cases, the solution u of (2.1)–(2.3) evolves from u_0 to a steady state, which takes values close to ± 1 .

AUTHOR CONTRIBUTIONS

Dieunel Dor: Methodology; software; formal analysis; project administration; funding acquisition; writing—original draft; conceptualization; investigation; visualization; validation; data curation; and resources. **Alain Miranville:** Funding acquisition; investigation; conceptualization; validation; visualization; project administration; supervision; formal

analysis; methodology; writing—review and editing; and resources. **Morgan Pierre:** Software; formal analysis; supervision; validation; visualization; methodology; investigation; and funding acquisition.

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CONFLICT OF INTEREST STATEMENT

The authors declare that they have no conflict of interests.

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