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## Paradigms of linear algebra

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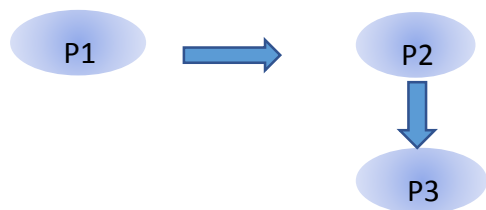
*This study is part of the ECOS-ANID C22H03 project, which involves four universities in Chile and France: UV, UPLA, PUCV (Chile) and UPCité (France). The aim of this project is to characterise and analyse the personal mathematical work of future teachers of mathematics in linear algebra at the beginning of university. It also aims to develop open tasks with automatic feedback on digital media in the field of linear algebra. We present here the constitution of the theoretical framework, more particularly the paradigms, and the analysis and implementation of a task that encourages students' work in different paradigms of linear algebra.*

*Keywords: paradigm, linear algebra, mathematical work, task-design, automatic feedback*

### INTRODUCTION

The theory of Mathematical Work Spaces (MWS) takes into account epistemological and cognitive aspects of work by crossing them with semiotic, discursive and instrumental dimensions for the specific study of mathematical work in an educational environment (Kuzniak et al., 2022). The notion of paradigm is central to the theory. It is used to characterise the MWS of a mathematical domain. To identify paradigms, we consider the historical, epistemological and curricular aspects of the mathematical domain in question. Paradigms help the researcher to analyse mathematical work and ensure that work processes are consistent.

Paradigms have already been worked on for geometry, analysis and probability. For each domain, we have identified three paradigms (P1, P2, P3) with an idea of horizontal and vertical mathematisation in three stages (Treffers, 1978). Horizontal mathematisation refers mainly to the formalisation and unification of objects and procedures from paradigm 1 within paradigm 2, while vertical mathematisation refers to the generalisation of objects and procedures within paradigm 3.



**Figure 1: Schema of paradigms within horizontal and vertical mathematisation**

In this paper, we focus on linear algebra (LA). We performed an historical, epistemological and curricular review to highlight the three paradigms we have

identified. We experiment a task with some students with an online assessment platform. The task aims to favour the student's flexibility between paradigms.

## **HISTORICAL, EPISTEMOLOGICAL AND CURRICULUM APPROACH**

For this study, we based ourselves essentially on the work of Dorier (1997). To simplify we distinguish 3 periods in order to align with the 3 paradigms we will define later.

### **First period: linear systems and geometry**

Systems of linear numerical equations played an important role in providing initial problems and concepts. Techniques for solving systems of linear equations are initially based on eliminations or substitutions (Eastern and Western civilisations). In 1750, Euler composed linear equations as sums of two equations, which was the first step towards interpreting them as mathematical objects that could be operated on. But determinants, along with minors and Cramer's method (1850), are the predominant method for solving them. For instance, in 1879, Frobenius defined dependence, linear independence and the notion of rank with minors. The algebraisation of geometry, with Descartes (1637) and Fermat's methods (1643), brings geometry and algebra closer together. But this rapprochement is not achieved directly between points and  $n$ -uples, which could be interpreted as a definite advance towards AL, but rather between geometric curves and algebraic equations. Although algebraic methods demonstrate the simplicity of linear geometry, they are still insufficient for the emergence of a new theory.

### **Second period: unified theory of linear algebra for systems and geometry**

The theory is initiated with the work of Grassmann (1844), who developed a theory with results equivalent to the vector spaces theory, such as the properties of operations, linear combinations and dimension. But his work was discarded by the mathematician until his death. In the 2nd half of the 19th century, a kind of unified theory of finite-dimensional linear algebra emerged that could be considered as a theory of  $\mathbf{R}^n$  spaces. This theory, which takes into account the initially implicit duality between systems of linear equations and their solution with  $n$ -uples, see the development of the first concepts of AL, in particular linear dependence and rank. However, the linearity of  $\mathbf{R}^n$  in itself is not primitive; the studies of linear systems are predominant. Determinants and matrices are still amalgamated, because matrices have to be considered as algebraic objects in order to go further in the conceptualisation. Hamilton's theory of quaternions played an important role in the development of what will become matrix algebra.

### **Third period: toward a formal theory**

At the end of the 19th century (and for 40 years), two different approaches are taken: the axiomatisation of AL and the infinite dimension. Peano (1888) proposes the first axiomatisation of AL, referring to Grassmann, with an implicit reference to dimension (if we have a base with  $n$  elements, can we find a smaller generating family?). Pincherle (1901) studies problems in analysis and infinite (countable) dimension. He introduces

the notions of rank of families of vectors, vector subspaces, hyperplanes and changes of basis, and he justifies that the vector space of numerical sequences is of infinite dimension. But his axiomatic theory seemed complex enough to deal with problems involving analytic functions. Burali-Forti and Marcolongo (1912) defines an intrinsic calculus with coordinate-free points and vectors. Weyl (1918) shows that the study of linear systems led to the same axiomatisation as geometry. Dorier says that the desire for axiomatisation is a posture that is not limited to a few applications. Finally, the structures of algebra, including AL, emerge from several contributions in algebra.

### **A few points on the curricular approach**

Traces of these three eras can be found in the curricula of the four universities of the project, with two different introductions to linear algebra, only the first being closer to the historical genesis:

- Linear systems, pivot, geometry in dimensions 2 and 3 THEN the vector space structure with matrices and linear applications, reductions of endomorphisms (PUCV and UPC);
- Matrices and applications to systems, without geometry in dimensions 2 and 3, PLUS vector space structure, linear applications and back to matrices, reductions of endomorphisms (UV and UPLA).

We therefore define three paradigms of linear algebra, based on the historical genesis and consistent with study programmes in universities.

### **PARADIGMS OF LINEAR ALGEBRA**

Roughly speaking, AL1 corresponds to the first period of AL, AL2 to the unified theory of  $\mathbf{R}^n$  at the end of the 19th century (second period) and AL3 to the axiomatisation of the 20th century.

#### **AL1: Paradigm 1**

This paradigm proposes algorithmic work on linear systems and on the geometry of the plane and space (dimensions 2 and 3) with coordinates. The notion of linearity is central but algorithmisation is favoured (work is mostly on instrumental dimension) with calculations without reference to the algebraic structures. The objects are treated with few links between them. In particular, the difference between affine space and vector space is not observed: it is the same structure (it is not distinguished from the Euclidean structure). Algorithmic processes serve to validate the work (discursive dimension of the work). One of the main features of working in AL1 is that there is a direct, or usual, representation of the objects we are working with: objects of plane and spatial geometry as well as algebraic (linear) equations (semiotic dimension of work). Matrices can appear as a semiotic rewriting of systems, which limits the number of signs used by focusing on the row-column position. The systems are solved using Gauss's pivot method, with operations on the rows or columns, or Cramer's method with the calculation of the determinant.

## **AL2: Paradigm 2**

This paradigm proposes working in  $\mathbf{R}^n$  with a reinterpretation of linear systems in terms of matrix products, based on matrix algebra. Matrix algebra appears but algorithmisation is always present (instrumental dimension of work). Some algebraic structures are made explicit (in particular those of matrices and  $\mathbf{R}^n$ ) and objects are treated with greater connections between them, with reference to the usual and implicit structures of  $\mathbf{R}^n$  (discursive dimension of work takes a greater part). Systems of linear equations are reinterpreted using the product of matrices. Planes and lines are interpreted as special cases of vector subspaces, as are  $\mathbf{R}^2$  and  $\mathbf{R}^3$ . In particular, we are no longer limited to dimension 3 and we go beyond the obstacle of dimension 4. The notion of linear application between  $\mathbf{R}^n$  spaces is central (especially for validation), with the notions of rank, kernel and image, and we work explicitly with the bases of  $\mathbf{R}^n$ , mainly the canonical base, but also with other bases. There is a possibility for change of base and conjugation, in particular for the reduction of matrices (diagonalization, trigonalization) using eigenvectors and eigenvalues (possibly with a use of  $\mathbf{C}$ ) with algorithmic work.

## **AL3: Paradigm 3**

This paradigm proposes work based on the algebraic structure of vector spaces, in finite dimension or not (discursive dimension of work). It is characterised by the fact that the algebraic structures are now explicit. They serve for validation of the work and there is a strong relationship between the mathematical objects involved. As a result, there is a greater capacity for abstraction and generalisation of the basic notions of linear algebra. In this paradigm, the notions of dimension and basis of a vector space are explicit and important. The work is done with different vector spaces, without being limited either to  $\mathbf{R}^n$  or to finite dimension. The semiotic treatments of these objects are essentially algebraic (no more link with geometry) and their representation depends on the space considered. A vector can now designate a matrix, a function, a sequence or a formal series, among other things. The objects are abstract. Their generality goes far beyond the framework of  $\mathbf{R}^n$  spaces.

## **A TASK TO PROMOTE THE FLEXIBILITY BETWEEN PARADIGMS**

This project aims to study the mathematical work of trainee teachers when they interact with an online assessment platform that performs automatic feedback (Gaona et al., 2022; Gaona & Menares, 2021). Tasks plays a central role in this work. We distinguish open-ended tasks as ones that allow for different solutions.

Starting from our analysis of research into linear algebra, we particularly note the tasks proposed by Taylor et al. (2008) (Figure 2) which allow the transformations of a figure in the plane with matrices to be worked on from different points of view. The task can be approached using a trial-and-error strategy or with a system of linear equations (mostly in paradigm AL1) or taking into account the matrix as a linear application (paradigm AL2).

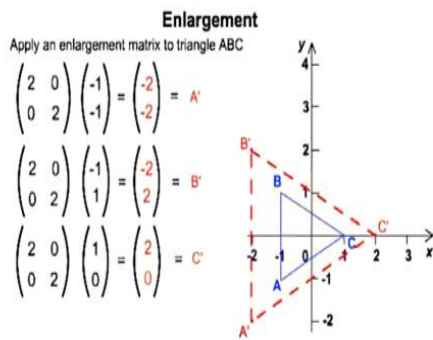


Fig. 6. Animation for illustrating how a matrix can be used to enlarge a simple 2D shape consisting of vectors.

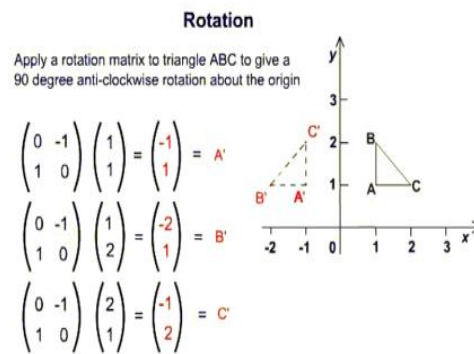


Fig. 8. Animation for illustrating how a matrix can be used to rotate a simple 2D shape consisting of vectors.

## Figure 2. Tasks extracted from the article by Taylor and al. (2008)

In the same way, the tasks that were designed in the project were designed to allow for a flexibility between the different paradigms. Two of these tasks adapted from the one Taylor and al. are shown in Figure 3 (task 1 and 2).

Task 1: Find a  $2 \times 2$  matrix A such that when multiplied by the vectors joining the origin with the vertices of the triangle, it becomes a triangle which is in the second quadrant.

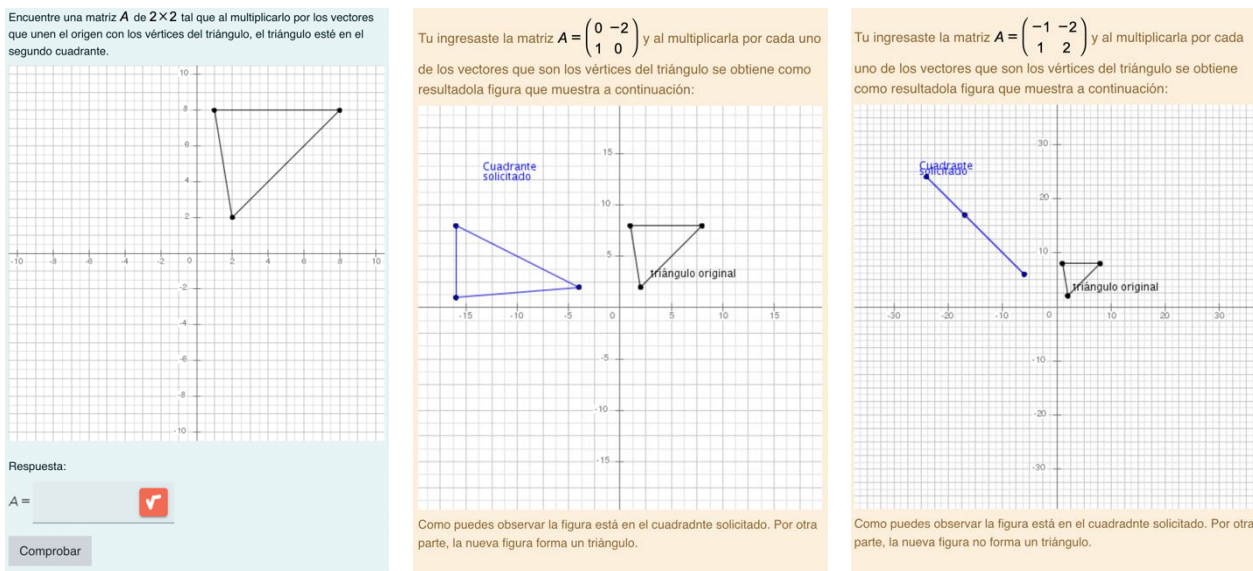
Task 2: Find a  $2 \times 2$  matrix A such that when multiplied by the vectors joining the origin with the vertices of the triangle, it becomes a segment which is in the second quadrant.

In these tasks, the vertices of the given triangle are random with integer coordinates and could be in different quadrants. For the tasks 1 and 2 (Figure 3) they appear in the first quadrant and the answer (triangle or segment) is awaited in the second quadrant. There are other versions of these tasks where a triangle is asked to be moved to a specific quadrant, with non-orthonormal axes or with non-congruent triangles. Due to space limitations, only the version in figure 3 (two initial tasks 1 and 2) will be analysed in the paper. Depending on the answer given by the student, the system provides specific feedback. If the answer is correct, the feedback indicates that the given matrix when multiplied by each of the vectors that are the vertices of the given triangle, results in a triangle in the second quadrant. If the answer is incorrect, the feedback aims to make the student reflect on the resulting triangle and determine the error (Figure 3).

For the tasks 1 and 2, before receiving feedback, we consider four possible strategies: E1) Find the matrix by trial and error, E2) Fix three points in the chosen quadrant and assemble a system of four variables and six equations, E3) Fix two vectors in the chosen quadrant and assemble a system of equations, and E4) Deliver a matrix from knowledge about matrices as operators.

The E1 strategy of trial and error seems to us to be quite expected given the characteristics of the platform. The work is mostly with the semiotic dimension and the AL1 paradigm.

To approach the task with strategy E2, students visually determine the approximate coordinates of the starting point and assemble a system of six equations and four variables, arbitrarily defining the resulting vectors (semiotic dimension of the work). We place mostly this strategy in the AL1 category as it is an algorithmic work (instrumental dimension of the work) on a system of equations without any discursive work involving linear knowledge. However, the system is likely to be inconsistent and its resolution can foster the work in AL2.



**Figure 3. Task 1 constructed in the online assessment system, with the statement (left), with automatic feedback for a correct answer (centre), and a feedback for an incorrect answer (right).**

If they choose strategy E3, that is, if they choose only two arbitrary vectors, they build a system of four variables and four equations and use the fact that the third vector is a linear combination of the two previous ones. Then the work has a greater discursive dimension with theoretical knowledge of linear algebra. The work, in this case, is mostly located in the AL2 paradigm as it works with vectors and the idea of linear dependence. The task 2 is specifically designed to foster this shift of paradigm.

To approach the task with the E4 strategy, students are expected to use transformations (central and axial symmetries, projections, rotations) with an interpretation of the situation in affine plane geometry and the production of a matrix. This approach can be mostly with visualisation (semiotic work): matrices can be seen as a game of changing the signs of some coordinates showing work in AL1. It can, however, play a role in understanding the linear algebra of quarter-turns and favour a shift of paradigm. The ambiguity in AL1 between affine geometry and the vector plane is unlikely to be resolved by students. However, the matrix work is specific to AL2, so the strategy E4 is classified between AL1 and AL2.

In all these strategies, the use of technology can be present, not only classical software such as Geogebra (among others) but also those with generative AI such as ChatGPT (among others). In the latter case, it is interesting to think about the potential dialogue

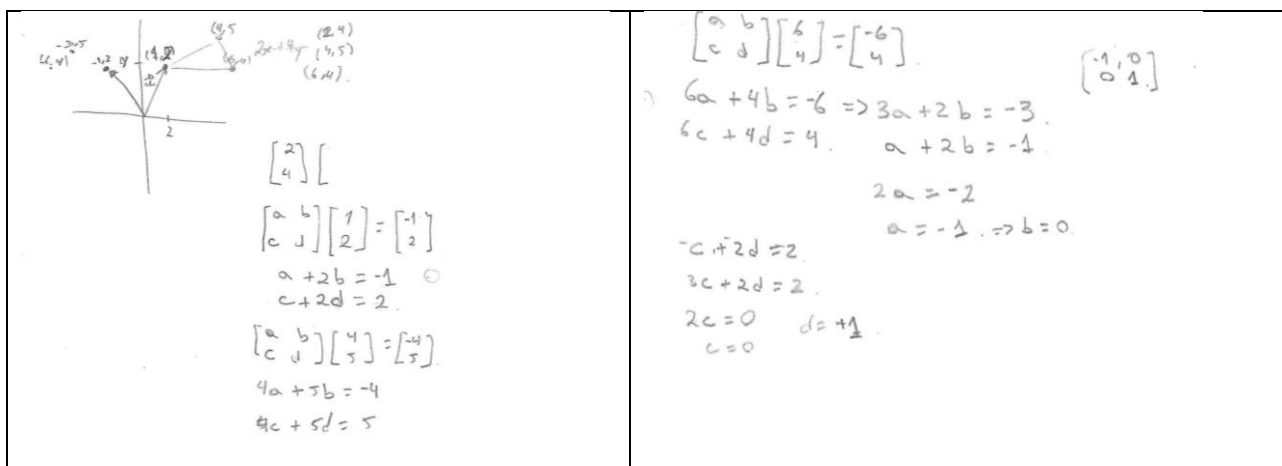
that can be produced between the machines and the students. In our methodology the dialogues between students and ChatGPT are recorded, together with screen recording (unfortunately without audio due to a technical problem), in order to understand students' work and identify the paradigms at stake, especially if they choose the strategy E4.

## OBSERVATION OF TWO GROUPS OF STUDENTS

### Group 1

The session observed lasts 1 hour. The two students of group 1 initially work unsuccessfully on one task 1 and one task 2. They ask ChatGPT questions such as “how can you multiply a vector with a  $2 \times 2$  matrix?” “How do you transform a vector into a matrix?” ChatGPT suggested various off-topic answers and essentially gave the formula for multiplying a  $2 \times 2$  matrix by a vector in the plane.

At 30 min, the students return to a task 1. The proposed triangle is (ABC) where A(1,2); B(3,5) and C(6,4). Mouse movements are observed to drag point A to point (-1,2), then to drag B to point (-3,5). The students return to chatGPT to view the multiplication formula again. This is followed by a long moment on their paper (Figure 4).



**Figure 4. Extract of students' production of group 1 on one task 1.**

The students write the 3 systems of 2 equations, using the E2 strategy, with the same variables  $a, b, c, d$  as those proposed in ChatGPT's answers. They determine  $a$  and  $b$  from the third system and then  $c$  and  $d$  using the equation  $c+2d=2$  from the first system. This is a method of elimination/substitution that is functional here. This allows them to find good values for  $a, b, c, d$ . Since they are looking for the symmetry matrix with respect to the line  $x=0$  without realising it, and since they have given each point A, B and C its correct image by this symmetry, the system is compatible and the imperfect solution they develop still allows them to find a good matrix. The error in the coordinates of B is transparent. At 39'15, they validate their task 1. At this point, they spent a long time looking at the feedback (2 minutes), whereas previously they had not read the feedback during the unsuccessful trials. When they tried one task 1 again, they



immediately proposed the same matrix  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ , which tends to show that they had developed decontextualised knowledge at this moment.

Students also try task 2. They no longer misread the coordinates by transferring them to their rough draft. There is work for about ten minutes which led to the identification of a generic matrix. They write “*to form general, to find triangles in the 2<sup>nd</sup> quadrant we have  $\begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$ ”, underlining twice the matrix”, which marked the start of work in AL2. This does not allow them to reach a conclusion.*

Finally, they essentially worked in the AL1 paradigm on the E2 strategy. They only started working after an unsuccessful attempt at each of the 2 tasks (first 16 minutes) and after questioning ChatGPT several times. The questions were formulated in technical terms and the ChatGPT responses were at the same level. However these responses helped students to convert matrix product into a system. Working in the AL1 paradigm on the first task enabled success, but only the Sem-Ins dimensions of work was used. Solving the system of 6 equations with 4 unknowns was not a problem insofar as the students constructed a compatible system. There was no need to discuss the system, which ultimately confirmed that the students were working in the AL1 paradigm. Working several time on the same task with the software nevertheless enabled the genericity of the matrix found to be identified but it doesn't permit to develop a discursive work for task 2. The paradigm shift required for task 2 (they had to move towards E3 or E4 strategies) remained too difficult for them, without any right answers for task 2.

## **Group 2**

This group of 3 students quickly explored tasks 1 and 2 through trial and errors. For example, they enter 2 for matrix A in task 1. The system interprets this as 2Id and the feedback therefore focuses on the wrong quadrant. On task 2 they enter  $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ . The result is a segment, but in the same quadrant. This is followed by a question in Google “2×2 matrix to form triangle” and then they click on “how to develop a 2×2 matrix”. After a long moment without any change on the screen, the students retry a question in Google (copying the statement) with an unsuccessful attempt to use ChatGPT. But this leads nowhere.

After a while (17'30), at the start of one task 1, the students enter the matrix  $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$ , which gives a correct answer. So they move on to task 2 (18'30). They quickly enter the matrix  $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  which gives the right quadrant, but not a segment. It is clear that at this point they have understood that this matrix send the triangles to the right quadrant. We also see, at a moment, that the statement for task 1 is copied into ChatGPT. The answer suggests the same matrix and ChatGPT explains the change of

quadrant. This should reinforce students' answers while adding a discursive dimension of their work.

Students do several series. For tasks 1, they always use the same matrix, which works. For tasks 2, on the other hand, they try for instance  $-2$ , interpreted as  $-2Id$ . This gives neither a segment nor the right quadrant. They go to Google with the question "through a matrix, how to find a segment?" but the answers are unconvincing. At one point there were 6'30 of work and the students tries the matrix  $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$  which gives a segment, but not in the right quadrant. There was clearly a discursive dimension that took place during these 6'30, as evidenced by the written productions (Figure 5). In particular, we can see matrices with a row or column of 0s that give segments, as well as the question on the links between obtaining a segment and the determinant of the matrix.

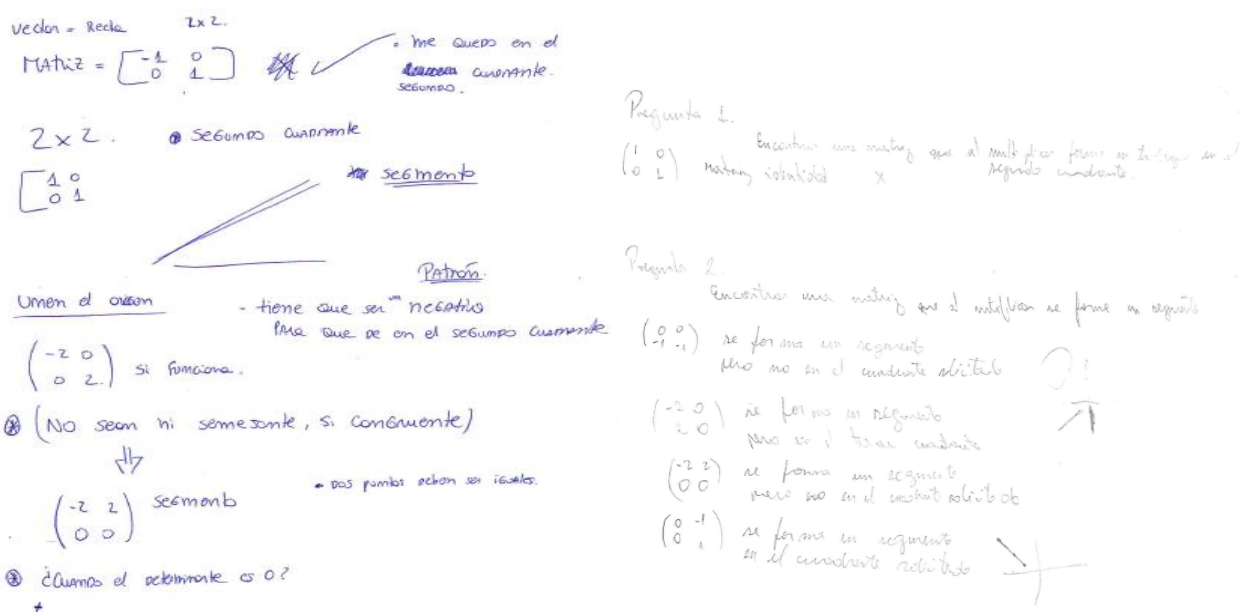


Figure 5. Extract of students' production of group 2 on one task 2.

This group tended to use the E4 strategy, even though it could also be seen some use of the E1 strategy, with trials that were perhaps not much discussed ("just to see"?). Students found two matrices that transform the triangles into the second quadrant. These matrices are supported by the response from ChatGPT, which gives the indication that the abscissa must change sign. Students also answered task 2 correctly with non-invertible matrices consisting of a row or column of zero and a more general question about the determinant. It seems that they had a discursive work, with a search for the genericity of matrices, which enabled them to work in the AL2 paradigm, quickly taking matrices to be an object in their own right which transforms the points of the plane (we can however note the confusion between a number and a 2x2 matrix, which was quickly regulated).

## CONCLUSION

The paradigms make it possible to analyse the syllabuses and question their coherence. On the one hand, in the light of the historical genesis: is it suitable to introduce linear

algebra using the AL2 paradigm when history shows the long genesis during the 19<sup>th</sup> century, initially based on work close to the AL1 paradigm? Secondly, the coherence between the paradigms present in the syllabus and what is expected in the examinations can be interrogated, with paradigm changes that are not made explicit. This contributes to our understanding of education systems.

Of course, and this is the continuation of the project, it is important to analyse the students' work on tasks and refine the analyses in terms of paradigms with the ETM theory. This contribution included an exploration of two pedagogical tasks designed to promote a flexibility and smooth transition between paradigms AL1 and AL2, highlighting the importance of task design in the process of teaching and learning linear algebra.

In parallel, the project has turned to the use of information and communication technologies, in particular online assessment platforms, to study the mathematical work of trainee teachers and the role of technologies. The triangle task, presented as an example, illustrates how students can approach problems using different strategies, highlighting the diversity of possible paradigmatic approaches together with the influence of ChatGPT. However, we show, in this study, the difficulty in working in AL2 paradigm, which seems an important step before working in AL3.

## NOTES

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