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Thomas Colas, Julien Grain, Greg Kaplanek, Vincent Vennin. In-in formalism for the entropy of quantum fields in curved spacetimes. *Journal of Cosmology and Astroparticle Physics*, 2024, 08, pp.047. 10.1088/1475-7516/2024/08/047 . hal-04639767

**HAL Id: hal-04639767**

**<https://hal.science/hal-04639767v1>**

Submitted on 9 Oct 2024

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# In-in formalism for the entropy of quantum fields in curved spacetimes

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**Abstract.** We show how to compute the purity and entanglement entropy for quantum fields in a systematic perturbative expansion. To that end, we generalize the in-in formalism to non-unitary dynamics (*i.e.* accounting for the presence of an environment) and to the calculation of quantum information measures, which are not observables in the usual sense. This allows us to reduce the problem to one involving standard correlation functions, and to organize their computation in a diagrammatic expansion for which we construct the corresponding Feynman rules. As an illustration, we apply the formalism to a cosmological setting inspired by the effective field theory of inflation. We find that at late times, non-linear loop corrections share the same time behavior as the linear contribution, and only yield a slight redressing of the purity. In particular, when the environment is heavy compared to the Hubble scale, the phenomenon of recoherence previously encountered is robust to the class of non-linear extensions considered. Bridging the gap between perturbative quantum field theory and open quantum systems paves the way to a better understanding of renormalization and resummation in open effective field theories. It also enables a more systematic exploration of quantum information properties in field theoretic settings.

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## 1 Introduction

The interplay between quantum information theory and quantum field theory has raised an ever-increasing interest in recent years [1–5]. Being able to describe and quantify quantum correlations for relativistic fields is of particular importance in cosmology, where all structures are expected to arise from the gravitational amplification of vacuum quantum fluctuations during an era of accelerated expansion called inflation. How those fluctuations acquire classical properties remains an open issue [6–14], together with the possibility of genuine signatures

of the quantum origin of cosmic structures [15–24]. A thorough investigation of these aspects requires a better characterization of the entanglement acquired by quantum fields in accelerating space-times [25–35], as well as an understanding of the role played by quantum decoherence in cosmology [36–58].

Decoherence [59–61] is the process by which, when a system couples to an environment that remains observationally inaccessible, the reduced state of the system transits from a pure state to a statistical mixture where quantum coherence between the pointer states is lost. When a quantum field is coupled to an environment, it is understood as an open system undergoing non-unitary evolution. Describing this evolution necessitates the development of open quantum field theory [62–66]. These tools (influence functionals, master equations and their stochastic unravelling, *etc.*) have been applied to cosmological or black hole backgrounds in various works, see *e.g.* Refs. [64–96]. They have been used to efficiently resum secular late-time effects in a number of cases [97–102], which makes them particularly well-suited to the description of inflationary fluctuations.

In practice however, these methods often rely on approximation schemes (*e.g.* Markovianity) that substantially differ from the traditional perturbative framework employed in asymptotically non-flat space-times, known as the in-in formalism. In this setup, a systematic expansion in the system-environment coupling is performed, which allows one to rely on the whole suite of Quantum Field Theory (QFT) perturbative tools, such as Feynman diagrams, renormalization methods, *etc.* So far the in-in formalism has been developed for unitary theories mostly [103] (see however Ref. [95]), hence it cannot be directly employed in open QFTs. In particular, quantum purity, which measures how much a quantum field decoheres, cannot be computed using existing in-in results.

The goal of the present work is to bridge this gap, and show how quantum information measures of an open quantum field can be computed in the in-in approach. This has several advantages. First, it allows one to rely on the perturbative QFT tools mentioned above, which greatly help in the treatment of non-linearities and the associated divergences. Second, being expressed in the language of cosmological correlators, it makes it possible to relate quantum information properties to summary statistics of the system. Third, in order to better understand resummation in open QFTs (and what exactly is being resummed), perturbative evolution should be understood as a first step.

There are two main obstacles when dealing with quantum purity or other entropy measures in the in-in framework. First, as already mentioned, in-in methods have been developed in the context of unitary theories, whereas the loss of purity is inherently a non-unitary phenomenon. Second, the in-in framework is constructed to deliver correlation functions, or more generally the expectation value of observables. However, because purity is not an observable<sup>1</sup>, extending the formalism becomes necessary.

In practice, the reduced state of a system  $\mathcal{S}$  is obtained by partially tracing the full density matrix  $\hat{\rho}$  over the Hilbert space of the environment  $\mathcal{E}$ ,

$$\hat{\rho}_{\text{red}} \equiv \text{Tr}_{\mathcal{E}} (\hat{\rho}). \quad (1.1)$$

In the cosmological context, the system is usually comprised of a set of observable degrees of freedom, *e.g.* the curvature perturbation  $\zeta$  on large scales, while the environment can encompass the set of unobservable modes of  $\zeta$  concealed behind the de-Sitter horizon [34, 54, 75], separate

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<sup>1</sup>Purity cannot be expressed as the expectation value of a Hermitian function of the phase-space operators and as such it would require a full state tomography to be assessed [62].

regions in physical space [31], or other matter fields (as we later consider in this work as an application). The computation of correlation functions of system operators relies on the reduced density matrix only, and since  $\hat{\rho}_{\text{red}}$  is Hermitian and trace normalized it can always be decomposed as

$$\hat{\rho}_{\text{red}} = \sum_j p_j |\psi_j\rangle\langle\psi_j| \quad (1.2)$$

where  $\{|\psi_j\rangle\}$  are pointer states and  $p_j$  are non-negative coefficients that sum up to 1. If the system is in a pure state, all  $p_j$  coefficients vanish except one, and  $\hat{\rho}_{\text{red}} = \hat{\rho}_{\text{red}}^2$  is a projector. Otherwise,  $\hat{\rho}_{\text{red}}$  describes a statistical mixture. This arises when the system becomes entangled with the environment, and one possible measure of that entanglement is the quantum purity [104]

$$\gamma \equiv \text{Tr}_{\mathcal{S}} (\hat{\rho}_{\text{red}}^2). \quad (1.3)$$

If the system's Hilbert space has finite dimension  $D$ , then  $1/D \leq \gamma \leq 1$ . Pure states have  $\gamma = 1$ , and maximal decoherence corresponds to  $\gamma \rightarrow 1/D$ , which simply means  $\gamma \rightarrow 0$  in the case of a quantum field. Let us stress that purity is not only a relevant measure of the system's decoherence, it also captures the importance of non-unitary effects and thus signals the need to go beyond unitary treatments. Indeed, if  $\hat{\rho}_{\text{red}}$  evolves unitarily,

$$\frac{d\hat{\rho}_{\text{red}}}{dt} = -i [\hat{H}, \hat{\rho}_{\text{red}}] \quad (1.4)$$

where  $\hat{H}$  is a self-adjoint Hamiltonian, then the purity is conserved by cyclicity of the trace (and likewise for any other entropy tracer). Deviations of the purity from 1 can thus be seen as a measure of departure from Eq. (1.4).

**Summary of the main results:** In this paragraph we provide a summary of the main results. While not all relevant quantities and concepts are fully defined here – detailed definitions will be provided later – the goal is to give a first flavour of the results derived in this work. A more formal summary is presented in the concluding section.

Consider an interaction Hamiltonian of the form

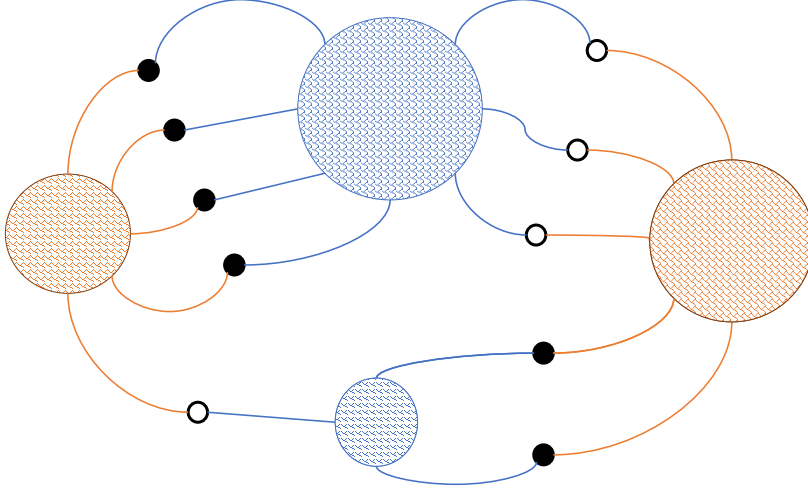
$$g\hat{H}_{\text{int}}(\eta) = g \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hat{\mathcal{O}}_{\mathbf{k}}^{\mathcal{S}}(\eta) \otimes \hat{\mathcal{O}}_{-\mathbf{k}}^{\mathcal{E}}(\eta) \quad (1.5)$$

made of (possibly non-linear) system and environment operators  $\hat{\mathcal{O}}_{\mathcal{S}}$  and  $\hat{\mathcal{O}}_{\mathcal{E}}$  and controlled by a coupling constant  $g$ . The Hamiltonian is expanded in Fourier space where the  $\mathbf{k}$  and  $-\mathbf{k}$  arguments merely result from the isotropy of the background, and  $\eta$  labels time. We perform a systematic expansion of the purity in powers of the interaction Hamiltonian,

$$\gamma = \sum_{n=0}^{\infty} g^n \gamma^{(n)}. \quad (1.6)$$

Assuming that the system is initially in a pure state, one has  $\gamma^{(0)} = 1$ , and the first non-unitary correction reads

$$\gamma^{(2)}(\eta) = -2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2). \quad (1.7)$$



**Figure 1.** Example of a diagram appearing in the computation of  $\gamma^{(n)}$ . Blue and orange blobs represent unequal-time correlators of the system and the environment respectively. To each conversion vertex  $\circ, \bullet$  is associated a power of the coupling constant  $g$ , an integral over time and over three-momenta, and the insertion of  $\widehat{\mathcal{O}}_{\mathbf{k}_i}^S(\eta_i) \otimes \widehat{\mathcal{O}}_{-\mathbf{k}_i}^\mathcal{E}(\eta_i)$ .

It involves the unequal-time, centred, two-point functions

$$\bar{\mathcal{K}}_{S/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \equiv \left\langle \widetilde{\mathcal{O}}_{\pm\mathbf{k}_1}^{S/\mathcal{E}}(\eta_1) \widetilde{\mathcal{O}}_{\pm\mathbf{k}_2}^{S/\mathcal{E}}(\eta_2) \right\rangle - \left\langle \widetilde{\mathcal{O}}_{\pm\mathbf{k}_1}^{S/\mathcal{E}}(\eta_1) \right\rangle \left\langle \widetilde{\mathcal{O}}_{\pm\mathbf{k}_2}^{S/\mathcal{E}}(\eta_2) \right\rangle \quad (1.8)$$

that encode the system and environment statistics in the free theory.

If  $\widehat{\mathcal{O}}_S$  is linear in the phase-space fields of the system, the above reduces to

$$\gamma^{(2)}(\eta) = -4 \int_{\mathbf{k} \in \mathbb{R}^{3+}} d^3\mathbf{k} \det^{(2)} \mathbf{Cov}(k, \eta), \quad (1.9)$$

where  $\det^{(2)} \mathbf{Cov}(k, \eta)$  is the determinant of the covariance matrix at second order in  $\widehat{H}_{\text{int}}$ . We will show that this formula remains correct even when  $\widehat{\mathcal{O}}_S$  is quadratic in the phase-space variables (up to an additional factor 1/2 in the right-hand side), in spite of the system being in a non-Gaussian state in that case. This is because, as we will find, the perturbation to the determinant of the covariance matrix can be related to that of the expectation value of the particle number operator. The expression (1.9) is also independent of the form of  $\widehat{\mathcal{O}}_\mathcal{E}$ , hence it holds true even in the presence of highly non-linear environment operators.

The formula (1.7) can be generalized to arbitrary order  $n$ , where we will find  $n$ -nested time integrals involving products of unequal-time correlators of both the system and the environment. Their calculation can be assisted with a set of Feynman rules to compute diagrams representing the various terms appearing in  $\gamma^{(n)}$ . An example of such diagrams is given in Fig. 1, the detailed meaning of which will become clearer below.

We will then apply these techniques to a model inspired by the EFT of Inflation (EFToI) construction [105], in which the pseudo-Goldstone mode  $\pi_c$  interacts with a hidden scalar sector  $\sigma$  through [106–110]

$$S_{\text{int}} = - \int d\eta d^3\mathbf{x} g \left[ a^3 \pi_c' \sigma - \frac{1}{\Lambda_1} a^2 (\partial_i \pi_c)^2 \sigma - \frac{1}{\Lambda_2} a^2 \pi_c'^2 \sigma \right], \quad (1.10)$$

where  $a$  is the scale factor and  $g$ ,  $\Lambda_1$  and  $\Lambda_2$  are Wilson coefficients. At super-Hubble scales (*i.e.* in the late-time limit) we will find that the purity of a given mode  $k$  is given by

$$\gamma_{\text{NL}}^{(2)}(k, \eta) = \gamma_{\text{lin}}^{(2)}(k, \eta) \left( 1 - \frac{1}{(2\pi)^9} \frac{3 \Delta_\zeta^2}{8 c_s^4} \right) \quad (1.11)$$

with  $c_s$  the speed of sound and  $\Delta_\zeta^2 \sim 10^{-9}$  the amplitude of the primordial power spectrum. The second term in the square brackets corresponds to non-linear contributions (*i.e.* loop corrections), which are found to scale in time in the exact same way as the linear contribution  $\gamma_{\text{lin}}^{(2)}$ , yielding only a slight dressing of the late-time purity. In particular, if the environmental field  $\sigma$  is heavy compared to the Hubble mass, we will find that the system recoheres at late time, and that the non-linear interactions considered in this work do not jeopardize the recoherence process. In the opposite limit where  $\sigma$  is light, we will find that  $\gamma_{\text{lin}}^{(2)}$  scales as  $a^2$ , and decoherence is very efficient.

**Outline:** The article is organized as follow. In Sec. 2, we develop the in-in formalism to compute quantum information properties of cosmological fields. In Sec. 2.1, we present the perturbative expansion underlying the in-in treatment. In Sec. 2.2, we apply it to the computation of the system’s purity. We introduce the concept of spectral purity in Sec. 2.3, to deal with the case where non-linear interactions make different Fourier modes couple, and the reduced density matrix does not factorize in Fourier space. We consider the particular case of Gaussian systems in Sec. 2.4, where we show that the relation between the purity and the covariance matrix also holds for quadratic interactions at leading order. After commenting on the role of resummation in Sec. 2.5, we derive Feynman diagrammatic rules in Sec. 2.6. Lastly, in Sec. 2.7 we take a brief detour to discuss other measures of mixedness, specifically the entanglement entropy. We then apply these techniques to a model of phenomenological interest in Sec. 3, which provides a non-linear extension of the setup considered in Ref. [52]. In Sec. 3.1, we motivate the model from an EFT perspective and discuss its regime of validity. In Sec. 3.2, we compute the second-order perturbative purity and study different cases of interest. In particular, we find that recoherence occurs if the environment is heavy compared to the Hubble scale, and that this conclusion is not altered by the presence of non-linearities. We discuss various prospects this work opens in Sec. 4 and present our main conclusions in Sec. 5, which is followed by a few appendices to which some technical details are deferred.

## 2 Methods

Quantum purity is a measure of decoherence, or “mixedness”, of a given [possibly reduced, see Eq. (1.1)] quantum state  $\hat{\rho}_{\text{red}}$ . It is not an observable in the sense of the definition given in Eq. (1.3), since it is clear that it cannot be written as  $\text{Tr}_S[\hat{\rho}_{\text{red}} \hat{O}]$  for some Hermitian operator  $\hat{O}$ . This is why the standard tools of the in-in formalism cannot be used directly to compute the purity. However, they can be extended, which is the goal of this section.

### 2.1 In-in formalism

The *in-in formalism* [111–113] is a well-established technique aiming at computing perturbatively the expectation value of a given equal-time quantum operator  $\langle \hat{Q}(\eta) \rangle$  where

$$\hat{Q}(\eta) \equiv \hat{O}_1(\eta, \mathbf{x}_1) \hat{O}_2(\eta, \mathbf{x}_2) \cdots \hat{O}_n(\eta, \mathbf{x}_n). \quad (2.1)$$

The operators  $\widehat{\mathcal{O}}_i(\eta, \mathbf{x}_i)$  are constructed locally out of the field operators appearing in the Lagrangian, and the expectation value is taken with respect to the initial Bunch-Davies vacuum state  $|\text{BD}\rangle$ . Given an interaction Hamiltonian  $g\widehat{H}_{\text{int}}$ , the expectation value  $\langle\widehat{Q}(\eta)\rangle$  can be written in the interaction picture

$$\langle\widehat{Q}(\eta)\rangle = \langle\text{BD}|\left[\overline{\mathcal{T}}e^{ig\int_{-\infty(1-i\epsilon)}^{\eta}d\eta'\tilde{H}_{\text{int}}(\eta')}\right]\tilde{Q}(\eta)\left[\mathcal{T}e^{-ig\int_{-\infty(1+i\epsilon)}^{\eta}d\eta'\tilde{H}_{\text{int}}(\eta')}\right]|\text{BD}\rangle \quad (2.2)$$

where  $\mathcal{T}$  and  $\overline{\mathcal{T}}$  represent time and anti-time ordering respectively and tildes denote quantities in the interaction picture, which we introduce properly below in Eqs. (2.8) and (2.10). The  $i\epsilon$  deformation is here to ensure the projection of the adiabatic vacuum of the interacting theory onto the vacuum of the free theory in the asymptotic past, see Refs. [114–116] for in-depth discussions. In practice, the correlation functions of the theory are computed perturbatively, at a given order in  $\widehat{H}_{\text{int}}$ . Defining  $\langle\widehat{Q}(\eta)\rangle^{(n)}$  the  $n^{\text{th}}$  order of the expansion, one has

$$\begin{aligned} \langle\widehat{Q}(\eta)\rangle^{(n)} &= (ig)^n \int_{-\infty}^{\eta} d\eta_n \int_{-\infty}^{\eta_n} d\eta_{n-1} \cdots \int_{-\infty}^{\eta_2} d\eta_1 \\ &\langle\text{BD}|\left[\tilde{H}_{\text{int}}(\eta_1), \left[\tilde{H}_{\text{int}}(\eta_2), \cdots \left[\tilde{H}_{\text{int}}(\eta_n), \tilde{Q}(\eta)\right] \cdots \right]\right]|\text{BD}\rangle \end{aligned} \quad (2.3)$$

where the appropriate  $i\epsilon$  prescription must be used depending on which branch of Eq. (2.2) provides the time integration. This approach has been used in a variety of problems in primordial cosmology to compute cosmological correlators of scalar and tensor perturbations.

**Derivation:** Let us review the derivation of the above expressions. In a unitary setting, the wavefunction  $|\Psi\rangle$  and the density matrix  $\widehat{\rho} = |\Psi\rangle\langle\Psi|$  respectively obey the Schrödinger equation and the Liouville-von-Neumann equation,

$$\frac{d|\Psi(\eta)\rangle}{d\eta} = -i\widehat{H}(\eta)|\Psi(\eta)\rangle \quad \text{and} \quad \frac{d\widehat{\rho}(\eta)}{d\eta} = -i\left[\widehat{H}(\eta), \widehat{\rho}(\eta)\right], \quad (2.4)$$

where  $\widehat{H}(\eta)$  is the Hamiltonian. These equations can be solved formally as

$$|\Psi(\eta)\rangle = \widehat{\mathcal{U}}(\eta, -\infty)|\text{BD}\rangle \quad \text{and} \quad \widehat{\rho}(\eta) = \widehat{\mathcal{U}}(\eta, -\infty)|\text{BD}\rangle\langle\text{BD}|\widehat{\mathcal{U}}^\dagger(\eta, -\infty), \quad (2.5)$$

where  $|\text{BD}\rangle$  is the initial Bunch-Davies vacuum state and we have introduced the evolution operator

$$\widehat{\mathcal{U}}(\eta, -\infty) = \mathcal{T} \exp\left[-i\int_{-\infty}^{\eta}d\eta'\widehat{H}(\eta')\right]. \quad (2.6)$$

This is the so-called Schrödinger picture, where the state (*i.e.*  $|\Psi\rangle$ ) and  $\widehat{\rho}$ ) evolve with  $\widehat{\mathcal{U}}$  and observables  $\widehat{\mathcal{O}}$  evolve only through their explicit dependence on time, if any.

We then divide the Hamiltonian into a free part and an interaction part,

$$\widehat{H}(\eta) = \widehat{H}_0(\eta) + g\widehat{H}_{\text{int}}(\eta) \quad (2.7)$$

and introduce the free evolution operator  $\widehat{\mathcal{U}}_0$ , defined as in Eq. (2.6) where  $\widehat{H}$  is replaced by  $\widehat{H}_0$ . It is often convenient to work in the so-called interaction picture to perform perturbative expansions. In this picture, quantum states evolve with the interaction Hamiltonian  $g\widehat{H}_{\text{int}}$  and



operators evolve with the free Hamiltonian  $\widehat{H}_0$ . The link between the Schrödinger and the interaction picture is given by

$$\widetilde{\rho}(\eta) = \widehat{\mathcal{U}}_0^\dagger(\eta, -\infty) \widehat{\rho}(\eta) \widehat{\mathcal{U}}_0(\eta, -\infty), \quad (2.8)$$

where tildes denote quantities evaluated in the interaction picture. From Eq. (2.4) it is easy to show that the state evolves according to

$$\frac{d\widetilde{\rho}}{d\eta} = -ig \left[ \widetilde{H}_{\text{int}}(\eta), \widetilde{\rho}(\eta) \right], \quad (2.9)$$

where

$$\widetilde{H}_{\text{int}}(\eta) = \widehat{\mathcal{U}}_0^\dagger(\eta, -\infty) \widehat{H}_{\text{int}}(\eta) \widehat{\mathcal{U}}_0(\eta, -\infty) \quad (2.10)$$

and we have used that the evolution operator is Hermitian, *i.e.*  $\widehat{\mathcal{U}}_0 \widehat{\mathcal{U}}_0^\dagger = \widehat{\mathcal{U}}_0^\dagger \widehat{\mathcal{U}}_0 = \text{Id}$ , since  $\widehat{H}_0 = \widehat{H}_0^\dagger$ . One can then introduce a perturbative scheme, where

$$\widetilde{\rho} = \sum_{n=0}^{\infty} g^n \widetilde{\rho}^{(n)} \quad (2.11)$$

with  $\widetilde{\rho}^{(0)} = |\text{BD}\rangle \langle \text{BD}|$ . By inserting Eq. (2.11) into Eq. (2.9) and identifying terms of the same order in  $g$ , one finds

$$\frac{d\widetilde{\rho}^{(n)}}{d\eta} = -ig \left[ \widetilde{H}_{\text{int}}(\eta), \widetilde{\rho}^{(n-1)}(\eta) \right]. \quad (2.12)$$

This can be solved iteratively as

$$\begin{aligned} \widetilde{\rho}^{(n)}(\eta) = & (-i)^n \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \cdots \int_{-\infty}^{\eta_{n-1}} d\eta_n \\ & \left[ \widetilde{H}_{\text{int}}(\eta_1), \left[ \widetilde{H}_{\text{int}}(\eta_2), \cdots \left[ \widetilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \cdots \right] \right]. \end{aligned} \quad (2.13)$$

One can use this expression to compute  $\langle \widehat{Q}(\eta) \rangle^{(n)}$  from Eq. (2.3) since expectation values are computed in the interaction picture via

$$\langle \widehat{Q}(\eta) \rangle = \text{Tr} \left[ \widetilde{Q}(\eta) \widetilde{\rho}(\eta) \right]. \quad (2.14)$$

Inserting Eq. (2.13) into this expression and making iterative use of the relation  $\text{Tr}(\widetilde{A}[\widetilde{B}, \widetilde{C}]) = \text{Tr}([\widetilde{A}, \widetilde{B}]\widetilde{C})$  (a mere consequence of the trace cyclicity) one recovers Eq. (2.3).

## 2.2 Perturbative purity

Let us now partition the physical setup into two subsystems,  $\mathcal{S}$  (system) and  $\mathcal{E}$  (environment), and consider the reduced density matrix  $\widehat{\rho}_{\text{red}} = \text{Tr}_{\mathcal{E}}(\widehat{\rho})$ , see Eq. (1.1), and its purity parameter  $\gamma = \text{Tr}_{\mathcal{S}}(\widehat{\rho}_{\text{red}}^2)$ , see Eq. (1.3). Our goal is to compute this quantity in the perturbative scheme introduced above. We assume that the free Hamiltonian does not entangle the system and the environment, *i.e.* that it is of the form<sup>2</sup>

$$\widehat{H}_0 = \widehat{H}_{\mathcal{S}} \otimes \text{Id}_{\mathcal{E}} + \text{Id}_{\mathcal{S}} \otimes \widehat{H}_{\mathcal{E}}. \quad (2.15)$$

<sup>2</sup>Note that this amounts to treating linear mixing between system and environment as an interaction, which is indeed commonly done in the context of primordial cosmology, see *e.g.* Ref. [113].

In this case the free evolution operator factorizes as  $\widehat{U}_0 = \widehat{U}_{0,S} \otimes \widehat{U}_{0,E}$ , and using partial trace cyclicity this implies that

$$\widetilde{\rho}_{\text{red}}(\eta) = \widehat{U}_{0,S}^\dagger(\eta, -\infty) \widehat{\rho}_{\text{red}}(\eta) \widehat{U}_{0,S}(\eta, -\infty). \quad (2.16)$$

As a consequence, Eq. (1.3) leads to

$$\gamma = \text{Tr}_S (\widetilde{\rho}_{\text{red}}^2) \quad (2.17)$$

and the purity can be computed directly in the interaction picture. In general, one can expand

$$\gamma = \sum_{n=0}^{\infty} g^n \gamma^{(n)} \quad \text{with} \quad \gamma^{(n)} = \sum_{m=0}^n \text{Tr}_S \left[ \widetilde{\rho}_{\text{red}}^{(m)} \widetilde{\rho}_{\text{red}}^{(n-m)} \right] \quad (2.18)$$

and

$$\widetilde{\rho}_{\text{red}}^{(n)} \equiv \text{Tr}_E \left[ \widetilde{\rho}^{(n)} \right]. \quad (2.19)$$

### Leading orders

Let us compute the first few terms in this expansion. At order  $\mathcal{O}(g^0)$ , one finds  $\widetilde{\rho}_{\text{red}}^{(0)} = \text{Tr}_E [|\text{BD}\rangle \langle \text{BD}|]$ . Since both the system and the environment start out in their vacuum states, there is no initial cross-correlation and the state is initially separable,

$$|\text{BD}\rangle \langle \text{BD}| = |\text{BD}\rangle \langle \text{BD}|_S \otimes |\text{BD}\rangle \langle \text{BD}|_E \quad (2.20)$$

hence  $\gamma^{(0)} = 1$ . At order  $\mathcal{O}(g)$ , Eq. (2.13) reduces to

$$\widetilde{\rho}^{(1)}(\eta) = -i \int_{-\infty}^{\eta} d\eta_1 \left[ \widetilde{H}_{\text{int}}(\eta_1), |\text{BD}\rangle \langle \text{BD}| \right]. \quad (2.21)$$

For explicitness, assuming local interactions, we decompose the interaction Hamiltonian in the tensorial basis of local operators

$$\widehat{H}_{\text{int}}(\eta) = \int d^3\mathbf{x} \widehat{\mathcal{O}}^S(\eta, \mathbf{x}) \otimes \widehat{\mathcal{O}}^E(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \widehat{\mathcal{O}}_{\mathbf{k}}^S(\eta) \otimes \widehat{\mathcal{O}}_{-\mathbf{k}}^E(\eta), \quad (2.22)$$

where  $\widehat{\mathcal{O}}^S(\eta, \mathbf{x})$  and  $\widehat{\mathcal{O}}^E(\eta, \mathbf{x})$  act on the Hilbert space of the system and the environment respectively. If the operators  $\widehat{\mathcal{O}}^S$  and  $\widehat{\mathcal{O}}^E$  are linear in the phase-space variables, the interaction only couples modes of opposite wavevectors. However, if they involve non-linear combinations of the phase-space variables,  $\widehat{\mathcal{O}}_{\mathbf{k}}^S$  and  $\widehat{\mathcal{O}}_{-\mathbf{k}}^E$  are given by convolution products in Fourier space and all Fourier modes couple.

Recalling that  $\widetilde{\rho}^{(0)} = |\text{BD}\rangle \langle \text{BD}|_S \otimes |\text{BD}\rangle \langle \text{BD}|_E$ , this gives rise to

$$\begin{aligned} \widetilde{\rho}^{(1)}(\eta) = -i \int_{-\infty}^{\eta} d\eta_1 \int \frac{d^3\mathbf{k}}{(2\pi)^3} & \left[ \widetilde{\mathcal{O}}_{\mathbf{k}}^S(\eta_1) |\text{BD}\rangle \langle \text{BD}|_S \otimes \widetilde{\mathcal{O}}_{-\mathbf{k}}^E(\eta_1) |\text{BD}\rangle \langle \text{BD}|_E \right. \\ & \left. - |\text{BD}\rangle \langle \text{BD}|_S \widetilde{\mathcal{O}}_{\mathbf{k}}^S(\eta_1) \otimes |\text{BD}\rangle \langle \text{BD}|_E \widetilde{\mathcal{O}}_{-\mathbf{k}}^E(\eta_1) \right]. \end{aligned} \quad (2.23)$$

As a consequence, tracing over the environment Hilbert space, we obtain

$$\widetilde{\rho}_{\text{red}}^{(1)}(\eta) = -i \int_{-\infty}^{\eta} d\eta_1 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[ \widetilde{\mathcal{O}}_{\mathbf{k}}^S(\eta_1), |\text{BD}\rangle \langle \text{BD}|_S \right] \mathcal{K}_E^{(1)}(\mathbf{k}, \eta_1) \quad (2.24)$$

where

$$\mathcal{K}_{S/\mathcal{E}}^{(1)}(\mathbf{k}, \eta_1) \equiv \text{Tr}_{S/\mathcal{E}} \left[ \tilde{\mathcal{O}}_{\pm\mathbf{k}}^{S/\mathcal{E}}(\eta_1) |\text{BD}\rangle \langle \text{BD}|_{S/\mathcal{E}} \right] = \langle \text{BD}| \tilde{\mathcal{O}}_{\pm\mathbf{k}}^{S/\mathcal{E}}(\eta_1) |\text{BD}\rangle_{S/\mathcal{E}}. \quad (2.25)$$

In this expression,  $\mathcal{K}_S^{(1)}(\mathbf{k}, \eta_1)$ , which is introduced for later convenience, is the vacuum expectation value of  $\tilde{\mathcal{O}}_{\mathbf{k}}^S(\eta_1)$ , while  $\mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}, \eta_1)$  is the vacuum expectation value of  $\tilde{\mathcal{O}}_{-\mathbf{k}}^{\mathcal{E}}(\eta_1)$ . Since  $\tilde{\mathcal{O}}^S(\mathbf{x})$  is a Hermitian operator, one has  $\tilde{\mathcal{O}}_{-\mathbf{k}}^S = (\tilde{\mathcal{O}}_{\mathbf{k}}^S)^\dagger$ . If the free, linear evolution is isotropic, the free evolution operator  $\hat{U}_0(\eta_1, -\infty)$  only depends on  $k = |\mathbf{k}|$ , hence one also has  $\tilde{\mathcal{O}}_{-\mathbf{k}}^S = (\tilde{\mathcal{O}}_{\mathbf{k}}^S)^\dagger$  and

$$\mathcal{K}_S^{(1)}(-\mathbf{k}, \eta_1) = \left[ \mathcal{K}_S^{(1)}(\mathbf{k}, \eta_1) \right]^*, \quad (2.26)$$

and similarly for  $\tilde{\mathcal{O}}_{-\mathbf{k}}^{\mathcal{E}}(\eta_1)$ . Following Eq. (2.18), this gives rise to

$$\gamma^{(1)} = -2i \int_{-\infty}^{\eta} d\eta_1 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \text{Tr}_S \left\{ |\text{BD}\rangle \langle \text{BD}|_S \left[ \tilde{\mathcal{O}}_{\mathbf{k}}^S(\eta_1), |\text{BD}\rangle \langle \text{BD}|_S \right] \right\} \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}, \eta_1) = 0 \quad (2.27)$$

since the trace operator is cyclic. We conclude that decoherence does not proceed at order  $g$ , a well-known fact indeed, see *e.g.* Ref. [62]. This can be understood as follows. At order  $g$ , the modification to the quantum state of the system,  $\tilde{\rho}_{\text{red}}^{(1)}$ , only involves the one-point function of the operators  $\tilde{\mathcal{O}}^{S/\mathcal{E}}$  appearing in the interaction, see Eq. (2.25). Upon performing the replacement

$$\tilde{\mathcal{O}}^{S/\mathcal{E}} = \mathcal{K}_{S/\mathcal{E}}^{(1)} \text{Id}_{S/\mathcal{E}} + \delta\tilde{\mathcal{O}}^{S/\mathcal{E}} \quad (2.28)$$

in the Hamiltonian, and expressing its interacting part  $\hat{H}_{\text{int}}$  in terms of the centred  $\delta\tilde{\mathcal{O}}^{S/\mathcal{E}}$  operators, one obtains a setup in which the free-evolution terms  $\hat{H}_S$  and  $\hat{H}_{\mathcal{E}}$  are modified but the one-point functions  $\mathcal{K}_{S/\mathcal{E}}^{(1)}$  vanish, hence  $\tilde{\rho}_{\text{red}}^{(1)} = 0$ . Such a replacement leaves the definition of the system unchanged, hence its quantum state remains indeed pure at order  $g$ .

Let us carry on the expansion and consider the order  $\mathcal{O}(g^2)$ . Expanding the commutators and tracing over the environment Hilbert space, Eq. (2.13) gives rise to

$$\begin{aligned} \tilde{\rho}_{\text{red}}^{(2)}(\eta) = & - \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3} \\ & \left\{ \left[ \tilde{\mathcal{O}}_{\mathbf{k}_1}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{k}_2}^S(\eta_2) |\text{BD}\rangle \langle \text{BD}|_S - \tilde{\mathcal{O}}_{\mathbf{k}_2}^S(\eta_2) |\text{BD}\rangle \langle \text{BD}|_S \tilde{\mathcal{O}}_{\mathbf{k}_1}^S(\eta_1) \right] \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \right. \\ & \left. - \left[ \tilde{\mathcal{O}}_{\mathbf{k}_1}^S(\eta_1) |\text{BD}\rangle \langle \text{BD}|_S \tilde{\mathcal{O}}_{\mathbf{k}_2}^S(\eta_2) - |\text{BD}\rangle \langle \text{BD}|_S \tilde{\mathcal{O}}_{\mathbf{k}_2}^S(\eta_2) \tilde{\mathcal{O}}_{\mathbf{k}_1}^S(\eta_1) \right] \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right\} \end{aligned} \quad (2.29)$$

where, similarly to Eq. (2.25), we introduce

$$\mathcal{K}_{S/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \equiv \text{Tr}_{S/\mathcal{E}} \left[ \tilde{\mathcal{O}}_{\pm\mathbf{k}_1}^{S/\mathcal{E}}(\eta_1) \tilde{\mathcal{O}}_{\pm\mathbf{k}_2}^{S/\mathcal{E}}(\eta_2) |\text{BD}\rangle \langle \text{BD}|_{S/\mathcal{E}} \right] \quad (2.30)$$

$$= \langle \text{BD}| \tilde{\mathcal{O}}_{\pm\mathbf{k}_1}^{S/\mathcal{E}}(\eta_1) \tilde{\mathcal{O}}_{\pm\mathbf{k}_2}^{S/\mathcal{E}}(\eta_2) |\text{BD}\rangle_{S/\mathcal{E}}. \quad (2.31)$$

Note that, since  $\tilde{\mathcal{O}}_{-\mathbf{k}}^{S/\mathcal{E}} = (\tilde{\mathcal{O}}_{\mathbf{k}}^{S/\mathcal{E}})^\dagger$  as mentioned above, one has

$$\mathcal{K}_{S/\mathcal{E}}^{(2)}(-\mathbf{k}_2, -\mathbf{k}_1, \eta_2, \eta_1) = \left[ \mathcal{K}_{S/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \right]^*. \quad (2.32)$$

Therefore, upon changing integration variables  $\mathbf{k}_1 \rightarrow -\mathbf{k}_1$  and  $\mathbf{k}_2 \rightarrow -\mathbf{k}_2$ , the third line of Eq. (2.29) is nothing but the Hermitian conjugate of the second line, which is consistent with the fact that the reduced density matrix is Hermitian. In order to derive the second-order perturbative purity  $\gamma^{(2)}$ , from Eq. (2.18) we have to compute two terms. The first one is

$$\text{Tr}_{\mathcal{S}} \left[ \left( \tilde{\rho}_{\text{red}}^{(1)} \right)^2 \right] = \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \left\{ \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2) \right. \\ \left. \left[ \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) + \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right] \right\} \quad (2.33)$$

where we defined the centred two-point functions

$$\bar{\mathcal{K}}_{\mathcal{S}/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \equiv \mathcal{K}_{\mathcal{S}/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) - \mathcal{K}_{\mathcal{S}/\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{S}/\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2). \quad (2.34)$$

When deriving the above expressions, we have used that  $(|\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}})^2 = |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}}$ , i.e. the fact that the system is initially placed in a pure state, as well as

$$\text{Tr}_{\mathcal{S}} \left[ \tilde{\mathcal{O}}_{\mathbf{k}_1}^{\mathcal{S}}(\eta_1) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \tilde{\mathcal{O}}_{\mathbf{k}_2}^{\mathcal{S}}(\eta_2) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \right] = \mathcal{K}_{\mathcal{S}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{S}}^{(1)}(\mathbf{k}_2, \eta_2). \quad (2.35)$$

The second term we have to compute is given by

$$\text{Tr}_{\mathcal{S}} \left[ \tilde{\rho}_{\text{red}}^{(0)} \tilde{\rho}_{\text{red}}^{(2)} \right] = - \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \quad (2.36) \\ \left[ \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) + \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right].$$

The integrand is invariant under the  $1 \leftrightarrow 2$  label exchange, hence it can be written as an un-nested time integral by replacing  $\int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \rightarrow \frac{1}{2} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2$ . Making further use of Eqs. (2.26) and (2.32), the above results lead to

**Second-order perturbative purity:**

$$\gamma^{(2)} = -2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2). \quad (2.37)$$

At this order, the purity involves two unequal-time correlators, one that describes the state the system in the free theory and one that describes the environment. If these correlators feature non-singular behavior in the coincident time limit, Eq. (2.37) implies that  $\gamma$  always decreases initially. Indeed, denoting the initial time by  $\eta_{\text{in}}$  (although in practice we let  $\eta_{\text{in}} = -\infty$ ) one has

$$\left. \frac{d\gamma^{(2)}}{d\eta} \right|_{\eta=\eta_{\text{in}}} = 0 \quad (2.38)$$

and

$$\left. \frac{d^2 \gamma^{(2)}}{d\eta^2} \right|_{\eta=\eta_{\text{in}}} = -4g^2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \Re \left[ \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_{\text{in}}, \eta_{\text{in}}) \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_{\text{in}}, \eta_{\text{in}}) \right]. \quad (2.39)$$

In this expression,  $\bar{\mathcal{K}}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_{\text{in}}, \eta_{\text{in}})$  and  $\bar{\mathcal{K}}_\mathcal{E}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_{\text{in}}, \eta_{\text{in}})$  are nothing but the initial power spectra of the centred (and possibly composite)  $\hat{\mathcal{O}}^S$  and  $\hat{\mathcal{O}}^\mathcal{E}$  operators. Once integrated over all Fourier modes, the right-hand-side of Eq. (2.39) can be shown to be non-negative. This implies that, close enough to the initial time,  $\gamma \simeq 1 - \frac{1}{2}\gamma''(\eta_{\text{in}})(\eta - \eta_{\text{in}})^2$  necessarily decreases, and is below one. The fact that purity remains below one indicates that at order  $g^2$ , perturbation theory provides a *complete-positive and trace-preserving* (CPTP) dynamical map [62]. If  $\gamma < 0$  is found at some later time, this simply indicates the breakdown of perturbation theory, but, under the above assumptions, there is always a finite time interval around  $\eta_{\text{in}}$  where it is valid. This expression also implies that Eq. (2.39) can be seen as the squared decay rate of purity initially, hence it provides a relevant time scale for the evolution of purity, which is simply made of the product of the power spectra of the system and environment operators involved in the interaction. For linear interactions, the correlators  $\bar{\mathcal{K}}_S^{(2)}$  and  $\bar{\mathcal{K}}_\mathcal{E}^{(2)}$  are non singular in the time-coincident limit and the above considerations hold. However, as stressed in Ref. [102], they do not apply in general when non-linear interactions are at play.

Let us also note that under the form (2.37), the purity is symmetric under exchanging the system and the environment, hence it is the same for both subsystems. This holds at all orders,<sup>3</sup> and comes from the fact that purity is directly related to entropy measures such as linear entropy, the entanglement entropy [104] (see Sec. 2.7), or the Rényi-2 entropy [117] as we shall now see. Since it quantifies the amount of information shared between the system and the environment, it must indeed be a symmetric quantity [104].

### 2.3 Rényi-2 entropy

In the case of linear interactions, the system remains in a Gaussian state whose Fourier modes are uncoupled, and the reduced density matrix can be factorized into

$$\tilde{\rho}_{\text{red}} = \bigotimes_{\mathbf{k} \in \mathbb{R}^{3+}} \tilde{\rho}_{\text{red}}(\mathbf{k}). \quad (2.40)$$

In this expression, the tensorial product is taken over  $\mathbb{R}^{3+}$  to avoid double counting of the degrees of freedom. For each Fourier mode  $\mathbf{k}$ , one can compute  $\text{Tr}[\tilde{\rho}_{\text{red}}^2(\mathbf{k})]$  and assign a purity per mode  $\mathbf{k}$ . This is relevant in cosmology, where a particular emphasis is often given to how much decoherence proceeds at a given scale. However, if non-linearities are present, this factorization cannot be performed anymore, and the purity cannot be computed mode by mode. This is why we introduce and propose the concept of “spectral purity” in this section, to quantify decoherence *at a given scale* even in the presence of mode coupling.

A first remark is that purity is a multiplicative (rather than additive) quantity. Indeed, if  $\mathcal{A}$  and  $\mathcal{B}$  denote two Hilbert spaces in which  $\tilde{\rho}_A$  and  $\tilde{\rho}_B$  lie, the purity of the whole state  $\tilde{\rho} = \tilde{\rho}_A \otimes \tilde{\rho}_B$  is given by  $\text{Tr}(\tilde{\rho}^2) = \text{Tr}(\tilde{\rho}_A^2 \otimes \tilde{\rho}_B^2) = \text{Tr}_\mathcal{A}(\tilde{\rho}_A^2) \text{Tr}_\mathcal{B}(\tilde{\rho}_B^2)$ , hence one obtains the product of the individual purities. In order to deal with additive quantities instead, it is convenient to

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<sup>3</sup>This can be shown using the Schmidt decomposition of the pure state describing the whole system-environment setup,  $|\Psi\rangle = \sum_i \lambda_i |i\rangle_\mathcal{S} |i\rangle_\mathcal{E}$ , where  $\{|i\rangle_\mathcal{S}\}$  and  $\{|i\rangle_\mathcal{E}\}$  are orthonormal sets of states for the system and the environment respectively. This is a mere consequence of the singular value decomposition (the sum may be discrete or continuous). From this expression one can easily compute  $\hat{\rho}_{\text{red}} = \text{Tr}_\mathcal{E}(\hat{\rho}) = \sum_i \lambda_i^2 |i\rangle_\mathcal{S} \langle i|_\mathcal{S}$ , hence the purity of the system reads  $\gamma = \text{Tr}_\mathcal{S}(\hat{\rho}_{\text{red}}^2) = \sum_i \lambda_i^4$ , and a similar calculation gives the same result for the purity of the environment.

introduce the logarithm of the purity, which is nothing but the *Rényi-2 entropy* [104, 117–119]<sup>4</sup>

$$S_2(\widehat{\rho}_{\text{red}}) \equiv -\ln(\gamma). \quad (2.42)$$

This quantity is naturally additive. In the absence of mode coupling, when the factorization (2.40) is valid, one can define its Fourier version as

$$\delta(\mathbf{0}) \ln \text{Tr} [\widetilde{\rho}_{\text{red}}(\mathbf{k})\widetilde{\rho}_{\text{red}}(\mathbf{k}')] = -S_2(k)\delta(\mathbf{k} - \mathbf{k}'). \quad (2.43)$$

The presence of the volume factor  $\delta(\mathbf{0})$  arises when taking the continuum limit in Fourier space [54, 91]. In this expression, the Dirac distribution arises from the fact that if  $\mathbf{k} \neq \mathbf{k}'$  then  $\widetilde{\rho}_{\text{red}}(\mathbf{k})$  and  $\widetilde{\rho}_{\text{red}}(\mathbf{k}')$  live in different Hilbert spaces, hence  $\text{Tr} [\widetilde{\rho}_{\text{red}}(\mathbf{k})\widetilde{\rho}_{\text{red}}(\mathbf{k}')] = \text{Tr} [\widetilde{\rho}_{\text{red}}(\mathbf{k})]\text{Tr}[\widetilde{\rho}_{\text{red}}(\mathbf{k}')] = 1$ . Combining the above expressions, one obtains, in the factorized state (2.40)

$$\begin{aligned} S_2(\widetilde{\rho}_{\text{red}}) &= -\ln \prod_{\mathbf{k} \in \mathbb{R}^{3+}} \text{Tr} [\widetilde{\rho}_{\text{red}}^2(\mathbf{k})] = -\delta(\mathbf{0}) \int_{\mathbf{k} \in \mathbb{R}^{3+}} d^3\mathbf{k} \ln \text{Tr} [\widetilde{\rho}_{\text{red}}^2(\mathbf{k})] \\ &= \delta(\mathbf{0}) \int_{\mathbf{k} \in \mathbb{R}^{3+}} d^3\mathbf{k} S_2(k). \end{aligned} \quad (2.44)$$

Even though Eq. (2.44) was derived in the factorized state (2.40), it is always the case that  $S_2$  can be cast in the form

$$S_2(\widetilde{\rho}_{\text{red}}) = \delta(\mathbf{0}) \int_{\mathbf{k} \in \mathbb{R}^{3+}} d^3\mathbf{k} S_2(k), \quad (2.45)$$

even in the presence of mode coupling, as we will see below. Our strategy is therefore to express the purity as an integral over Fourier space, and to identify the integrand with the one in the above expression to read off the spectral purity  $S_2(k)$ . In this way, the spectral purity can be defined for any type of interaction. In practice, it is convenient to introduce the reduced spectral purity  $\mathcal{S}_2(k) = 2\pi k^3 S_2(k)$ , such that  $S_2(\widetilde{\rho}_{\text{red}}) = \delta(\mathbf{0}) \int d \ln k \mathcal{S}_2(k)$ . Scale invariance of the reduced spectral purity thus corresponds to volume-law entropy production.

At order  $g^2$  in the perturbative expansion, the purity is given by Eq. (2.37). In this expression, since the background is homogeneous and isotropic, the system and environment unequal-time two-point functions  $\bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2)$  and  $\bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2)$  must be of the form

$$\bar{\mathcal{K}}_{\mathcal{S}/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \equiv \bar{\mathcal{K}}_{\mathcal{S}/\mathcal{E}}^{(2)}(k_1, \eta_1, \eta_2)\delta(\mathbf{k}_1 + \mathbf{k}_2), \quad (2.46)$$

which defines  $\bar{\mathcal{K}}_{\mathcal{S}/\mathcal{E}}^{(2)}(k_1, \eta_1, \eta_2)$ . One can thus rewrite Eq. (2.37) as

$$\gamma^{(2)} = -2(2\pi)^{-3} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(k, \eta_1, \eta_2)\bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta_1, \eta_2)\delta(\mathbf{0}). \quad (2.47)$$

At order  $g^2$ , one has  $S_2(\widetilde{\rho}_{\text{red}}) = -g^2\gamma^{(2)}$ , and by identification with Eq. (2.45) one obtains for the spectral purity

$$S_2^{(2)}(k) = 4(2\pi)^{-6} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(k, \eta_1, \eta_2)\bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta_1, \eta_2). \quad (2.48)$$

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<sup>4</sup>This quantity is called “entropy” in the sense that it provides a measure of the correlations between two subsystems  $A$  and  $B$ , through the *Rényi-2 mutual information* [13, 117–120]

$$\mathcal{I}_2(A, B) = S_2(\widehat{\rho}_A) + S_2(\widehat{\rho}_B) - S_2(\widehat{\rho}_{AB}), \quad (2.41)$$

which is the relative Rényi-2 entropy between the full state  $\widehat{\rho}_{AB}$  and its lowest-order factorized form  $\widehat{\rho}_A \otimes \widehat{\rho}_B$ .

## 2.4 Gaussian case

As mentioned above, when interactions are linear the system remains in a Gaussian state that factorizes according to Eq. (2.40), where each  $\tilde{\rho}_{\text{red}}(\mathbf{k})$  describes a Gaussian state. Gaussian states are fully described by their covariance matrix [104, 120, 121]

$$\delta(\mathbf{0})\mathbf{Cov}(k, \eta) \equiv \frac{1}{2}\text{Tr}_S \left[ \left\{ \tilde{\mathbf{z}}(\mathbf{k}, \eta), \tilde{\mathbf{z}}^\dagger(\mathbf{k}, \eta) \right\} \tilde{\rho}_{\text{red}}(\eta) \right] \quad (2.49)$$

where  $\tilde{\mathbf{z}}(\mathbf{k}, \eta)$  is a vector gathering all phase-space coordinates of the Fourier mode  $\mathbf{k}$  for the system. In particular,  $\text{Tr}[\tilde{\rho}_{\text{red}}^2(\mathbf{k})]$  is given by the determinant of the covariance matrix,

$$\ln \text{Tr}[\tilde{\rho}_{\text{red}}^2(\mathbf{k})] = \ln \left[ \frac{1}{4 \det \mathbf{Cov}(k, \eta)} \right], \quad (2.50)$$

where for explicitness we have assumed that the system is made of a single scalar degree of freedom, hence  $\tilde{\mathbf{z}}(\mathbf{k}, \eta)$  has two entries. By comparison with Eq. (2.43), this leads to

$$S_2(k) = \ln [4 \det \mathbf{Cov}(k, \eta)]. \quad (2.51)$$

When non-linear interactions are at play, the system is not in a Gaussian state and the above expression does not apply. In that case, the purity is not only sensitive to the two-point functions but to all higher-order correlation functions. However, in Appendix A, we show that Eq. (2.51) remains valid at order  $g^2$  for interactions that are quadratic in the system's phase-space variables, with an additional factor 1/2. In other words, we show that, for linear *and* quadratic interactions,

$$S_2^{(2)}(k) = \frac{4}{n} \det^{(2)} \mathbf{Cov}(k, \eta), \quad (2.52)$$

where  $n = 1$  for linear interactions (the result then coincides with Eq. (2.51) when expanded at order  $g^2$ ) and  $n = 2$  for quadratic interactions. The fact that the perturbed purity is still entirely determined by the determinant of the covariance matrix even for quadratic interactions is rather remarkable, since as mentioned above non-linear interactions probe all correlation functions in principle. In the context of cosmological perturbation theory, the dominant interaction terms are cubic [122], hence they are either linear or quadratic in the system's variables, and the above result implies that the power spectra of the system are enough to determine its perturbed purity.

## 2.5 Partial resummation

For Gaussian states, Eq. (2.51) is an exact statement and it is therefore valid to all orders in perturbation theory. This is of substantial practical use, since perturbation theory generically breaks down at late times. Indeed, the in-in formalism expands unitary time evolution operators (schematically) as  $e^{-i(H_0 + gH_{\text{int}})t} \simeq e^{-iH_0 t} (1 - igH_{\text{int}}t + \dots)$ , which is a good approximation for the true dynamics only at relatively early times where  $gt$  is small compared to timescales set by the interaction. Secular growth does not occur in computations of scattering amplitudes since interactions are taken to be turned off in the asymptotic past and future (where momentum eigenstates are taken to be free) – however, in virtually any other setting where the interactions stay turned on, secular growth is a persistent issue. This is the case, for example, in the calculation of correlators in flat-space [101, 123, 124], cosmological space-times [97–100, 125–135], or even black-hole backgrounds [136].

In general, at second-order in perturbation theory, one finds that  $\det \mathbf{Cov} \sim 1/4 + g^2 t^m$  (for some power  $m$ ). The perturbative treatment is therefore a priori valid only for early times  $g^2 t^m \ll 1$ . The exact relation (2.50) provides a simple yet powerful way to resum late-time breakdowns. It leads to  $\gamma \sim g^{-2} t^{-m}$ , which remains reliable even at late times where  $g^2 t^m \gg 1$ . It is the non-perturbative nature of Eq. (2.51) that allows one to trust this approximation out to much later times, since the dynamics have been resummed to all orders in  $g^2 t^m$ . This type of a resummation was used in Refs. [52, 91, 100, 102] for the calculation of  $\gamma$ . In Sec. 3, we illustrate the use of such a resummation (see Fig. 14), and otherwise leave more detailed investigations (for instance, involving master equations) of late-time resummations for future work.

## 2.6 Higher orders

Beyond the lowest order correction to the purity of the system, it might be of interest to derive higher-order corrections, for instance in cosmological applications to relate the evolution of the purity to higher-order statistics (bispectrum, trispectrum and so on). We now explain how the above formalism can be extended to higher orders. This will lead us to diagrammatic rules for the calculation of the purity, allowing for systematic expansions similar to those commonly employed in (unitary) perturbative QFT.

### 2.6.1 Kraus representation

At order  $n$ , the integrand of Eq. (2.13) involves the interaction Hamiltonian evaluated at  $n$  different times  $\eta_1, \dots, \eta_n$ . Let us introduce the  $\tilde{\mathcal{K}}^{(i)}$  vectors, which contain all products of  $i$  interaction Hamiltonians, where the indices of the time arguments are ordered (which corresponds to anti-ordering the time arguments, since in Eq. (2.13),  $\eta_1 > \eta_2 > \dots > \eta_n$ ). More explicitly,

$$\tilde{\mathcal{K}}^{(0)} \equiv [\mathbb{I}] \quad (2.53)$$

$$\tilde{\mathcal{K}}^{(1)} \equiv [\tilde{H}_{\text{int}}(\eta_1), \tilde{H}_{\text{int}}(\eta_2), \dots, \tilde{H}_{\text{int}}(\eta_n)] \quad (2.54)$$

$$\tilde{\mathcal{K}}^{(2)} \equiv [\tilde{H}_{\text{int}}(\eta_1)\tilde{H}_{\text{int}}(\eta_2), \dots, \tilde{H}_{\text{int}}(\eta_i)\tilde{H}_{\text{int}}(\eta_j), \dots] \quad \text{with } i < j \quad (2.55)$$

$$\tilde{\mathcal{K}}^{(3)} \equiv [\tilde{H}_{\text{int}}(\eta_1)\tilde{H}_{\text{int}}(\eta_2)\tilde{H}_{\text{int}}(\eta_3), \dots, \tilde{H}_{\text{int}}(\eta_i)\tilde{H}_{\text{int}}(\eta_j)\tilde{H}_{\text{int}}(\eta_k), \dots] \quad \text{with } i < j < k \quad (2.56)$$

...

$$\tilde{\mathcal{K}}^{(n)} \equiv [\tilde{H}_{\text{int}}(\eta_1)\tilde{H}_{\text{int}}(\eta_2)\dots\tilde{H}_{\text{int}}(\eta_n)] . \quad (2.57)$$

Each  $\tilde{\mathcal{K}}^{(i)}$  vector contains  $\binom{n}{i} = n!/i!(n-i)!$  elements, which corresponds to choosing  $i$  instants  $\eta_j$  amongst the  $n$  ones. When expanding the commutator in Eq. (2.13), one finds

$$\left[ \tilde{H}_{\text{int}}(\eta_1), \left[ \tilde{H}_{\text{int}}(\eta_2), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \tilde{\mathcal{K}}_{\binom{n}{i}-j+1}^{(n-i)} \right)^\dagger . \quad (2.58)$$

This can be shown by induction and the detailed proof, which fixes the ordering of the elements in each vector  $\tilde{\mathcal{K}}^{(i)}$ , is given in Appendix B. As a consequence, Eq. (2.13) can be written as



$$\tilde{\rho}^{(n)}(\eta) = (-i)^n \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \cdots \int_{-\infty}^{\eta_{n-1}} d\eta_n \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \tilde{\mathcal{K}}_{\binom{n}{i}-j+1}^{(n-i)} \right)^\dagger. \quad (2.59)$$

Note that taking  $i \rightarrow n - i$  and  $j \rightarrow \binom{n}{i} - j + 1$  gives the Hermitian conjugate of the above expression, which is consistent with the fact that the density matrix is Hermitian at every order. This also means that, in practice, it is enough to compute the terms for  $i$  up to  $\lfloor \frac{n}{2} \rfloor$  (the floor of  $n/2$ ) and take the Hermitian part of the result.

We are now in a position where we can trace over the environmental degrees of freedom. For local interactions of the type (2.22), one obtains

$$\begin{aligned} \tilde{\rho}_{\text{red}}^{(n)}(\eta) = & (-i)^n \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \cdots \int_{-\infty}^{\eta_{n-1}} d\eta_n \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \cdots \int \frac{d^3 \mathbf{k}_n}{(2\pi)^3} \\ & \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \mathcal{K}_{\mathcal{E},i,j} \tilde{\mathcal{K}}_{\mathcal{S},j}^{(i)} |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \left( \tilde{\mathcal{K}}_{\mathcal{S},\binom{n}{i}-j+1}^{(n-i)} \right)^\dagger, \end{aligned} \quad (2.60)$$

where

$$\mathcal{K}_{\mathcal{E},i,j} = \langle \text{BD}| \left( \tilde{\mathcal{K}}_{\mathcal{E},\binom{n}{i}-j+1}^{(n-i)} \right)^\dagger \tilde{\mathcal{K}}_{\mathcal{E},j}^{(i)} |\text{BD}\rangle_{\mathcal{E}}. \quad (2.61)$$

Here,  $\tilde{\mathcal{K}}_{\mathcal{S},j}^{(i)}$  are defined as in Eq. (2.53)-(2.57), where each  $\tilde{H}_{\text{int}}(\eta_\ell)$  is replaced with  $\tilde{O}_{\mathbf{k}_\ell}^{\mathcal{S}}(\eta_\ell)$ , with  $\ell = 1 \cdots i$ . Likewise,  $\tilde{\mathcal{K}}_{\mathcal{E},j}^{(i)}$  are defined as in Eqs. (2.53)-(2.57) where each  $\tilde{H}_{\text{int}}(\eta_\ell)$  is replaced with  $\tilde{O}_{-\mathbf{k}_\ell}^{\mathcal{E}}(\eta_\ell)$ . With  $n = 2$ , one recovers Eq. (2.29). With  $n = 3$ , one finds

$$\begin{aligned} \tilde{\rho}_{\text{red}}^{(3)}(\eta) = & i \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int_{-\infty}^{\eta_2} d\eta_3 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_3}{(2\pi)^3} \\ & \left\{ \mathcal{K}_{\mathcal{E}}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \eta_1, \eta_2, \eta_3) \left[ \tilde{O}_{\mathbf{k}_1}^{\mathcal{S}}(\eta_1), \tilde{O}_{\mathbf{k}_2}^{\mathcal{S}}(\eta_2) \tilde{O}_{\mathbf{k}_3}^{\mathcal{S}}(\eta_3) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \right] \right. \\ & \left. - \mathcal{K}_{\mathcal{E}}^{(3)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3; \eta_2, \eta_1, \eta_3) \left[ \tilde{O}_{\mathbf{k}_1}^{\mathcal{S}}(\eta_1), \tilde{O}_{\mathbf{k}_3}^{\mathcal{S}}(\eta_3) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \tilde{O}_{\mathbf{k}_2}^{\mathcal{S}}(\eta_2) \right] \right\} + \text{h.c.} \end{aligned} \quad (2.62)$$

where we have introduced

$$\mathcal{K}_{\mathcal{E}}^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; \eta_1, \eta_2, \eta_3) \equiv \langle \text{BD}| \tilde{O}_{-\mathbf{k}_1}^{\mathcal{E}}(\eta_1) \tilde{O}_{-\mathbf{k}_2}^{\mathcal{E}}(\eta_2) \tilde{O}_{-\mathbf{k}_3}^{\mathcal{E}}(\eta_3) |\text{BD}\rangle_{\mathcal{E}}, \quad (2.63)$$

and higher orders can be computed similarly. When  $n$  gets large, the list of terms to write down becomes long: there are  $2^n$  correlators of the type (2.61), although using trace cyclicity and Hermiticity under index relabelling as mentioned below Eq. (2.59) leaves  $2^n/4$  correlators to compute (hence 1 correlator when  $n = 2$  and 2 correlators when  $n = 3$ , this is consistent with Eqs. (2.29) and (2.62) respectively). The number of terms still grows rapidly, which is why we now provide diagrammatic rules to assist the calculation. These will allow us to organize the computation of  $\tilde{\rho}_{\text{red}}^{(n)}$ , and of the terms  $\text{Tr}_{\mathcal{S}}[\tilde{\rho}_{\text{red}}^{(m)} \tilde{\rho}_{\text{red}}^{(n-m)}]$  that appear in the calculation of  $\gamma^{(n)}$  in Eq. (2.18).

### 2.6.2 Diagrammatic rules

From Eq. (2.60), the Feynman rules for the  $n^{\text{th}}$ -order perturbative purity are given as follows.

**Feynman rules for  $\tilde{\rho}_{\text{red}}^{(n)}$**  (an example is given in Fig. 2):

1. Draw a diagram with one environment *blob* connected to  $n$  environment propagators leading to  $n$  conversion vertices, themselves connected to  $n$  system propagators. This diagram corresponds to a fixed  $i$  and  $j$  in Eq. (2.60), the vertices are labelled by  $\ell$ .

2. To each vertex is associated a

$$(-i) \int_{-\infty}^{\eta} d\eta_{\ell} \int \frac{d^3 \mathbf{k}_{\ell}}{(2\pi)^3}. \quad (2.64)$$

The diagram is time-ordered such that there is an overall

$$\Theta(\eta_{\ell_1} - \eta_{\ell_2}) \quad \forall \ell_1 < \ell_2. \quad (2.65)$$

Amongst the  $n$  vertices,  $i$  vertices are ‘‘coloured’’ with  $\circ$ , and  $n - i$  with  $\bullet$  which contributes a factor  $(-1)$ . To the  $\circ$  vertices are attached  $\tilde{O}_{\mathbf{k}_{\ell}}^S(\eta_{\ell})$  (system propagator) and  $\tilde{O}_{-\mathbf{k}_{\ell}}^{\mathcal{E}}(\eta_{\ell})$  (environment propagator), and to the  $\bullet$  vertices are attached  $[\tilde{O}_{\mathbf{k}_{\ell}}^S(\eta_{\ell})]^{\dagger}$  and  $[\tilde{O}_{-\mathbf{k}_{\ell}}^{\mathcal{E}}(\eta_{\ell})]^{\dagger}$ .

3. The amplitude part of the diagram is controlled by

$$\langle \text{BD} | \tilde{\mathcal{T}}[\bullet_{\mathcal{E}}] \mathcal{T}[\circ_{\mathcal{E}}] | \text{BD} \rangle_{\mathcal{E}} \quad (2.66)$$

where  $\mathcal{T}[\bullet_{\mathcal{E}}]$  corresponds to the time-ordering of the environment operators connected to the  $\bullet$  vertices, and  $\tilde{\mathcal{T}}[\circ_{\mathcal{E}}]$  corresponds to the anti-time-ordering of the environment operators connected to the  $\circ$  vertices.

4. The operator part of the diagram is controlled by

$$\mathcal{T}[\circ_S] | \text{BD} \rangle \langle \text{BD} |_S \tilde{\mathcal{T}}[\bullet_S] \quad (2.67)$$

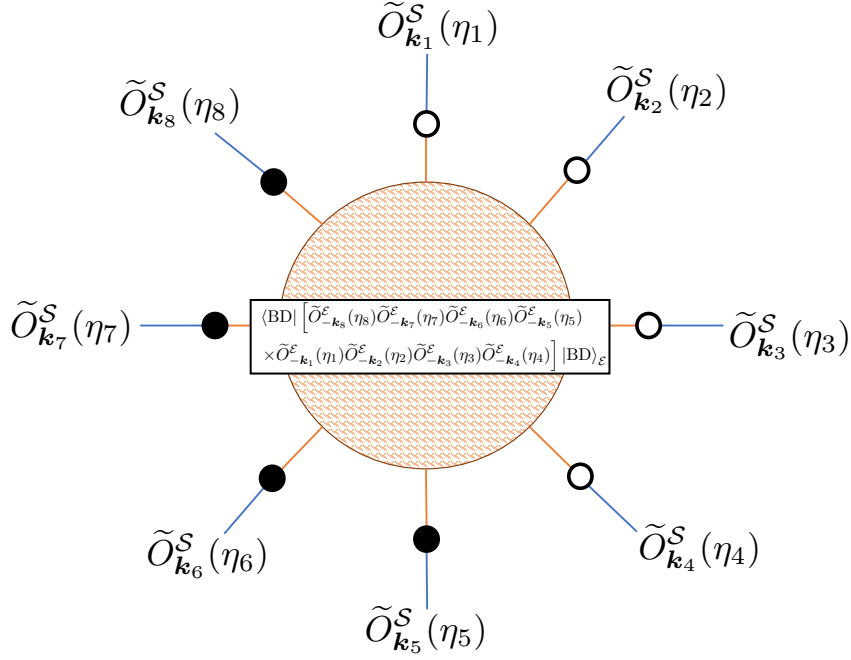
with similar notations.

5. Sum over all possibilities for the number  $i$  of  $\circ$  vertices and their arrangement in the diagram (labeled by  $j$ ).

The diagrams for  $\tilde{\rho}_{\text{red}}^{(n)}$  constitute the building blocks for the ones of  $\gamma^{(n)}$ . The latter follow from Eq. (2.18) and are given as follows.

**Feynman rules for  $\gamma^{(n)}$**  (an example is given in Fig. 3):

1. Draw one of the diagrams appearing in  $\tilde{\rho}_{\text{red}}^{(m)}$  (called *left cluster*) and one of the



**Figure 2.** One of the terms appearing in the computation of  $\tilde{\rho}_{\text{red}}^{(8)}$ . The system is represented in blue and the environment in orange. This diagram has  $n = 8$  vertices, labeled by  $\ell$ , with  $i = 4$  of them being  $\circ$  vertices.

diagrams appearing in  $\tilde{\rho}_{\text{red}}^{(n-m)}$  called *right cluster*) for  $m$  between 0 and  $\lfloor \frac{n}{2} \rfloor$ .

2. Add two system blobs between the clusters: the first one is connected to all  $\bullet$  vertices of the left cluster and all  $\circ$  vertices of the right cluster, while the second one is connected to all  $\circ$  vertices of the left cluster and all  $\bullet$  vertices of the right cluster.
3. Each system blob contributes to

$$\langle \text{BD} | \tilde{\mathcal{T}} [\bullet_S] \mathcal{T} [\circ_S] | \text{BD} \rangle_S, \quad (2.68)$$

which is further multiplied by the amplitude parts of the environment blobs.

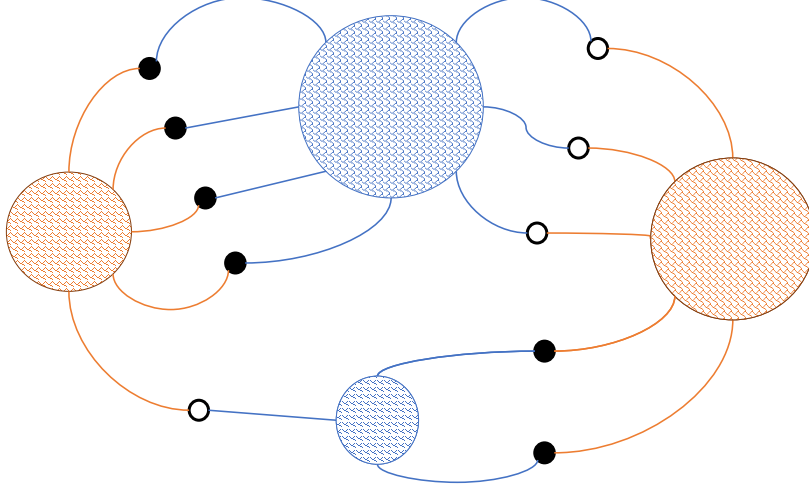
4. Sum over  $m$  from 0 to  $\lfloor \frac{n}{2} \rfloor$ . There is a prefactor 2 if  $m \neq n/2$ .

In order to get more familiar with the above diagrammatic representation, let us employ it to compute perturbative purity up to order  $g^4$ .

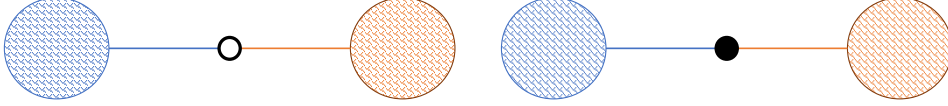
$\gamma^{(1)}$ : At first order, there are only two diagrams, displayed in Fig. 4. Using the Feynman rules derived above, we recover the result given in Eq. (2.27), that is

$$\gamma^{(1)} = -2i \int_{-\infty}^{\eta} d\eta_1 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[ \mathcal{K}_S^{(1)}(\mathbf{k}, \eta_1) \mathcal{K}_\varepsilon^{(1)}(\mathbf{k}, \eta_1) - \mathcal{K}_S^{(1)}(\mathbf{k}, \eta_1) \mathcal{K}_\varepsilon^{(1)}(\mathbf{k}, \eta_1) \right] = 0. \quad (2.69)$$

The first non-trivial contribution thus arises at second order, as already mentioned above.



**Figure 3.** One of the terms appearing in the computation of  $\gamma^{(10)}$  for  $m = 5$ .



**Figure 4.** The two terms appearing in the computation of  $\gamma^{(1)}$ .

$\gamma^{(2)}$ : Second-order purity is made of two kinds of diagrams that come from  $m = 0$  and  $m = 1$ . To simplify the discussions, let us introduce

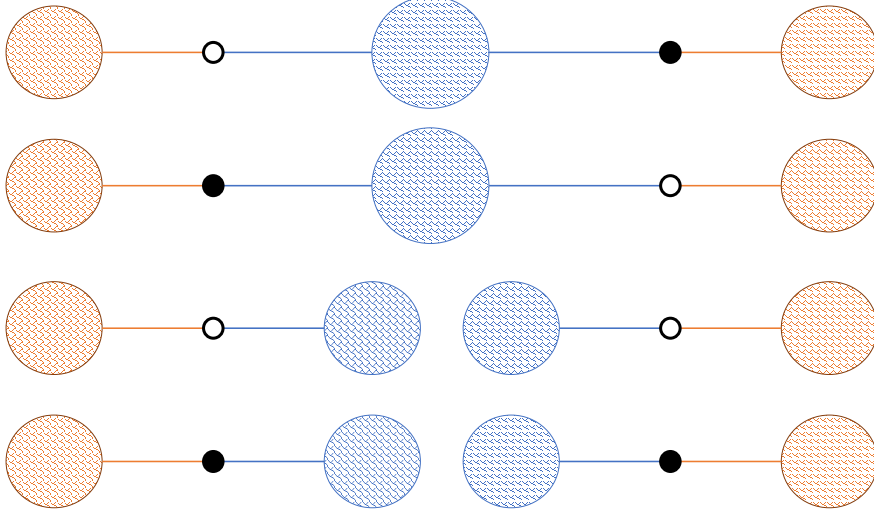
$$\gamma^{(n-m,m)} \equiv 2^{1-\delta_{n,m-n}} \text{Tr}_S \left[ \tilde{\rho}_{\text{red}}^{(m)} \tilde{\rho}_{\text{red}}^{(n-m)} \right]. \quad (2.70)$$

The diagrams for  $\gamma^{(1,1)}$  are given in Fig. 5 and the ones for  $\gamma^{(2,0)}$  in Fig. 6. Let us first consider  $\gamma^{(1,1)}$ . Following the Feynman rules given above, we recover Eq. (2.33), namely

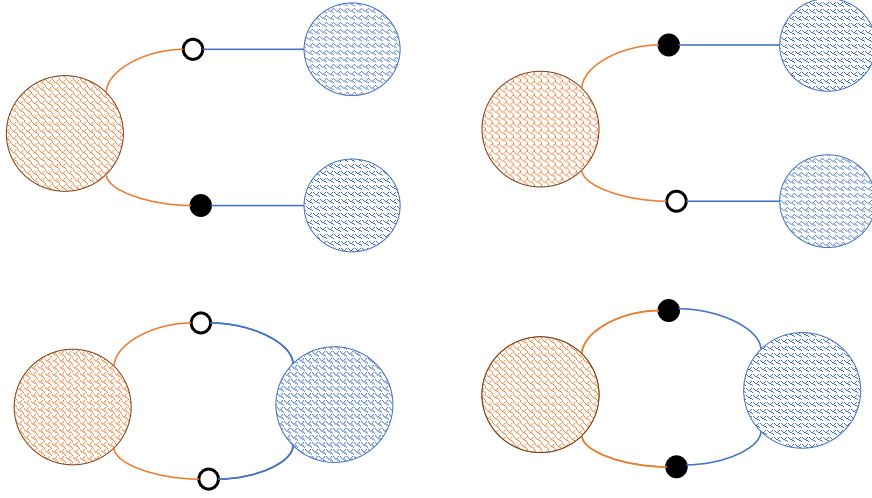
$$\begin{aligned} \gamma^{(1,1)} = & (-i)^2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \\ & \left[ -\mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2) - \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2) \right. \\ & \left. + \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2) + \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_{\mathcal{E}}^{(1)}(\mathbf{k}_2, \eta_2) \right]. \end{aligned} \quad (2.71)$$

Similarly for  $\gamma^{(2,0)}$ , one can recover from Fig. 6 the result of Eq. (2.36), that is

$$\begin{aligned} \gamma^{(2,0)} = & 2(-i)^2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \\ & \left[ -\mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) - \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right. \\ & \left. + \mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) + \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right]. \end{aligned} \quad (2.72)$$



**Figure 5.** Diagrams appearing in the computation of  $\gamma^{(1,1)}$ .



**Figure 6.** Diagrams appearing in the computation of  $\gamma^{(2,0)}$ .

$\gamma^{(3)}$ : We have to compute diagrams for  $\gamma^{(3,0)}$  and  $\gamma^{(2,1)}$ , which are represented in Figs. 7 and 8 respectively. From the diagrammatic representation, one has

$$\begin{aligned} \gamma^{(3,0)} = & 2(-i)^3 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int_{-\infty}^{\eta_2} d\eta_3 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_3}{(2\pi)^3} \\ & \left[ \mathcal{K}_S^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \eta_1, \eta_2, \eta_3) \mathcal{K}_E^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \eta_1, \eta_2, \eta_3) \right. \\ & - \mathcal{K}_S^{(3)}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_1, \eta_3, \eta_2, \eta_1) \mathcal{K}_E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_1, \eta_3, \eta_2, \eta_1) \\ & \left. + \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(2)}(\mathbf{k}_3, \mathbf{k}_2, \eta_3, \eta_2) \mathcal{K}_E^{(3)}(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_1, \eta_3, \eta_2, \eta_1) \right] \end{aligned}$$

$$\begin{aligned}
& + \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_S^{(2)}(\mathbf{k}_3, \mathbf{k}_1, \eta_3, \eta_1) \mathcal{K}_E^{(3)}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{k}_2, \eta_3, \eta_1, \eta_2) \\
& + \mathcal{K}_S^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \mathcal{K}_E^{(3)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3, \eta_2, \eta_1, \eta_3) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_3, \eta_2, \eta_3) \mathcal{K}_E^{(3)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \eta_1, \eta_2, \eta_3) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_3, \eta_1, \eta_3) \mathcal{K}_E^{(3)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3, \eta_2, \eta_1, \eta_3) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \mathcal{K}_E^{(3)}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{k}_2, \eta_3, \eta_1, \eta_2) \Big]. \tag{2.73}
\end{aligned}$$

This coincides with what can be directly obtained from Eq. (2.62). Similarly, we find

$$\begin{aligned}
\gamma^{(2,1)} = & 2(-i)^3 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int_{-\infty}^{\eta_2} d\eta_3 \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_2}{(2\pi)^3} \int \frac{d^3 \mathbf{k}_3}{(2\pi)^3} \\
& \left[ \mathcal{K}_S^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \right. \\
& - \mathcal{K}_S^{(3)}(\mathbf{k}_3, \mathbf{k}_1, \mathbf{k}_2, \eta_3, \eta_1, \eta_2) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_3, \eta_2, \eta_3) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_3, \eta_1, \eta_3) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \\
& + \mathcal{K}_S^{(1)}(\mathbf{k}_1, \eta_1) \mathcal{K}_S^{(2)}(\mathbf{k}_3, \mathbf{k}_2, \eta_3, \eta_2) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \\
& + \mathcal{K}_S^{(1)}(\mathbf{k}_2, \eta_2) \mathcal{K}_S^{(2)}(\mathbf{k}_3, \mathbf{k}_1, \eta_3, \eta_1) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \\
& - \mathcal{K}_S^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_S^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \\
& \left. + \mathcal{K}_S^{(3)}(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3, \eta_2, \eta_1, \eta_3) \mathcal{K}_E^{(1)}(\mathbf{k}_3, \eta_3) \mathcal{K}_E^{(2)}(\mathbf{k}_2, \mathbf{k}_1, \eta_2, \eta_1) \right]. \tag{2.74}
\end{aligned}$$

$\gamma^{(4)}$ : In Figs. 9, 10 and 11, we draw the diagrams needed to compute  $\gamma^{(4)}$ . They correspond to  $\gamma^{(4,0)}$ ,  $\gamma^{(3,1)}$  and  $\gamma^{(2,2)}$  respectively. We do not write the corresponding expressions here, for the sake of brevity, but they can be obtained straightforwardly following similar lines.

Ultimately, the computation boils down to the evaluation of the blobs representing the various unequal-time correlators of the system and the environment in a model of interest. If the theory does not contain any system and environment non-linearities (that is if  $\widehat{H}_S$  and  $\widehat{H}_E$  are quadratic), the expectation values  $\langle \widetilde{\mathcal{T}}[\bullet_S] \mathcal{T}[\circ_S] \rangle_S$  and  $\langle \widetilde{\mathcal{T}}[\bullet_E] \mathcal{T}[\circ_E] \rangle_E$  can readily be obtained by performing Wick contractions.<sup>5</sup> Beyond this simple case, one may wish to further connect the current representation to the standard in-in diagrammatics [113] in order to evaluate the blobs in terms of time and momenta integrals of the system and environment propagators. In this work we will not consider setups with system or environment self-interactions, although we come back to this issue in Sec. 5.

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<sup>5</sup>In this sense, we notice that in the in-in path integral language, the  $\circ$  vertices correspond to field insertions from the + branch of the path integral and the  $\bullet$  vertices to field insertions from the - branch of the path integral (see Refs. [65, 137] for a more in-depth discussion of the relations between operator and path integral formalisms).

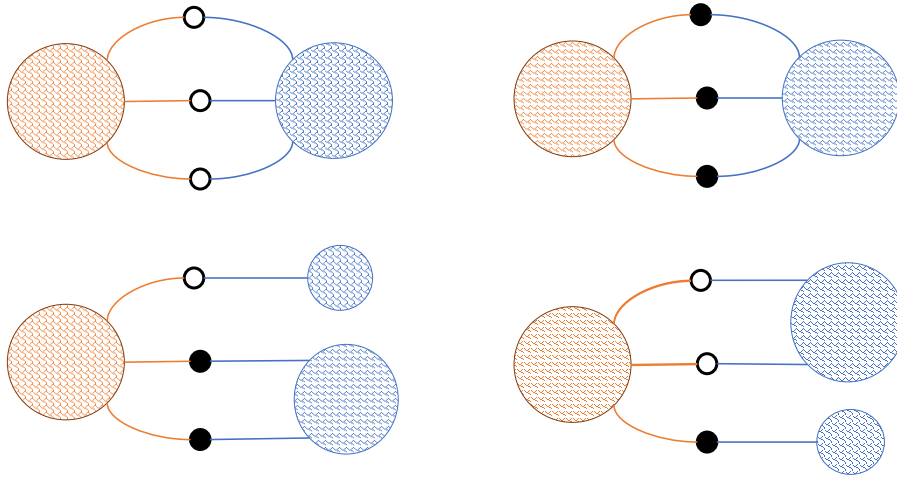


Figure 7. Diagrams appearing in the computation of  $\gamma^{(3,0)}$ .

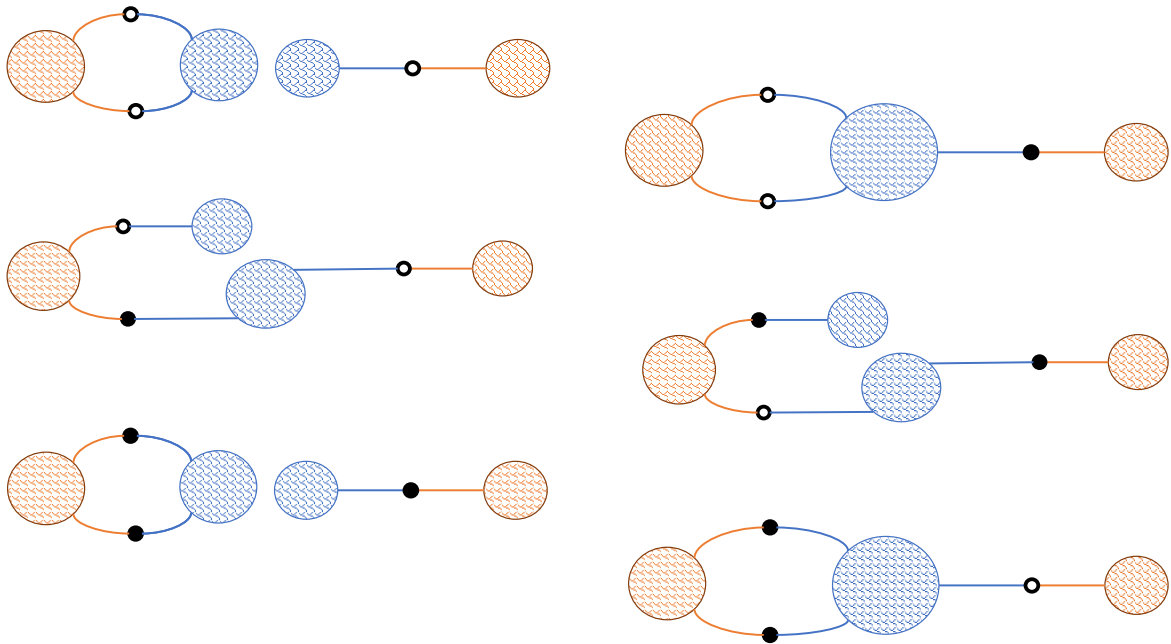


Figure 8. Diagrams appearing in the computation of  $\gamma^{(2,1)}$ .

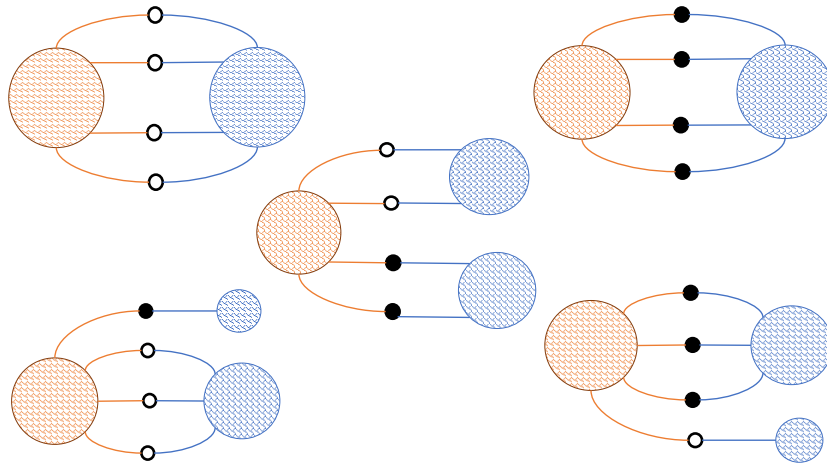


Figure 9. Diagrams appearing in the computation of  $\gamma^{(4,0)}$ .

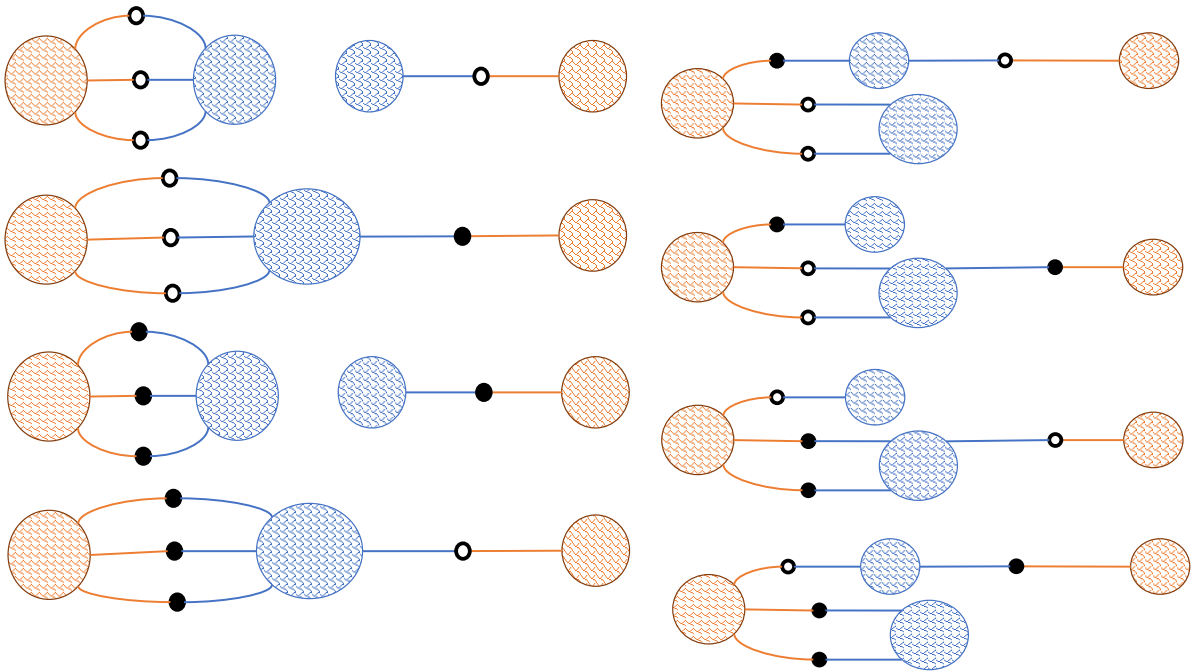
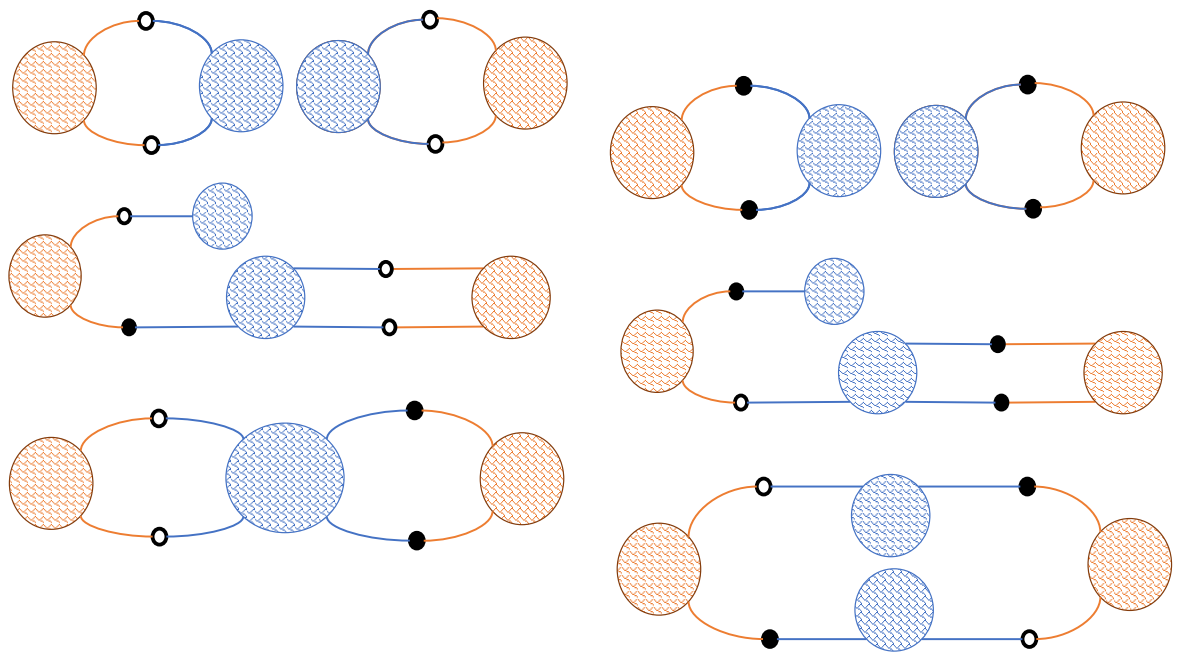


Figure 10. Diagrams appearing in the computation of  $\gamma^{(3,1)}$ .





**Figure 11.** Diagrams appearing in the computation of  $\gamma^{(2,2)}$ .

## 2.7 Other quantum information measures

Although the focus so far has been on computing the purity, we mention here that our formalism can be easily extended to computing other quantum information measures. Let us first mention the Rényi- $N$  entropy [117]

$$S_N \equiv \frac{1}{1-N} \ln [\text{Tr}_{\mathcal{S}} (\hat{\rho}_{\text{red}}^N)]. \quad (2.75)$$

When  $N = 2$ , this reduces to the Rényi-2 entropy given in Eq. (2.42), hence to the purity. The Rényi- $N$  entropy turns out to lend itself well to our formalism, as one must simply take  $N$  product of the reduced density matrix already computed in previous subsections. In particular, using Eq. (2.19) one finds that for integers  $N \geq 3$

$$\begin{aligned} \text{Tr}_{\mathcal{S}} (\hat{\rho}_{\text{red}}^N) = & \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)}] + gN \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)}] \\ & + g^2 \left\{ N \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(2)}] + N \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} (\hat{\rho}_{\text{red}}^{(1)})^2] + \frac{N(N-3)}{2} \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)} \hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)}] \right\} \\ & + \mathcal{O}(g^3), \end{aligned} \quad (2.76)$$

where cyclicity of the trace has been applied and we have used that the initial state is pure, as in Eq. (2.20), so that  $(\hat{\rho}_{\text{red}}^{(0)})^2 = \hat{\rho}_{\text{red}}^{(0)}$ . Note furthermore that  $\text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)}] = 0$  as found in Eq. (2.27), which also implies  $\text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)} \hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(1)}] = 0$ . Finally one also has  $\text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)}] = 1$  and from Eq. (2.23) one can show explicitly that  $\text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} (\hat{\rho}_{\text{red}}^{(1)})^2] = \frac{1}{2} \text{Tr}_{\mathcal{S}} [(\hat{\rho}_{\text{red}}^{(1)})^2]$ , so the above formula reduces to

$$\text{Tr}_{\mathcal{S}} (\hat{\rho}_{\text{red}}^N) = 1 + Ng^2 \left\{ \text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}}^{(0)} \hat{\rho}_{\text{red}}^{(2)}] + \frac{1}{2} \text{Tr}_{\mathcal{S}} [(\hat{\rho}_{\text{red}}^{(1)})^2] \right\} + \mathcal{O}(g^3). \quad (2.77)$$

Since the term involving  $N(N-3)$  dropped out, this is in fact valid for all integers  $N \geq 2$ . One recognizes the combination giving  $\gamma^{(2)}$  in Eq. (2.18), hence we have found that

$$S_N = \frac{N}{N-1} \frac{1-\gamma}{2} + \mathcal{O}(g^3), \quad (2.78)$$

which is obviously consistent with Eq. (2.42) when  $N = 2$ .

Another quantity of interest is the von-Neumann entanglement entropy of the system, defined as

$$S_{\text{EE}} = -\text{Tr} (\hat{\rho}_{\text{red}} \ln \hat{\rho}_{\text{red}}). \quad (2.79)$$

This is the quantum analog of the so-called Shannon entropy in classical-information theory, and it encapsulates the information inherent to the possible outcomes of a measurement on a quantum system. It is central to the computation of many other key quantum information metrics. Perturbation theory of von-Neumann entropies has been considered in various contexts before [2, 4, 138–146], and we now connect our formalism to these works. Formally, one can show that  $S_{\text{EE}} = \lim_{N \rightarrow 1} S_N$ , hence the entanglement entropy can be obtained from analytic continuation of the Rényi entropy, using what is known as the replica trick [147, 148]. In the present framework, this cannot be done from Eq. (2.77) directly, since it is valid for  $N \geq 2$  only.

However once can expand the entanglement entropy around a pure state for which  $\hat{\rho}_{\text{red}}^2 \simeq \hat{\rho}_{\text{red}}$  and close to which the Mercator series can be employed

$$\text{Tr}_{\mathcal{S}} [\hat{\rho}_{\text{red}} \ln (\hat{\rho}_{\text{red}})] = -\text{Tr}_{\mathcal{S}} \left[ \hat{\rho}_{\text{red}} \sum_{n=1}^{\infty} \frac{(1 - \hat{\rho}_{\text{red}})^n}{n} \right]. \quad (2.80)$$

At leading order, one finds  $\text{Tr}_{\mathcal{S}}[\widehat{\rho}_{\text{red}} \ln(\widehat{\rho}_{\text{red}})] \simeq \text{Tr}_{\mathcal{S}}[\widehat{\rho}_{\text{red}}^2] - 1$ , hence

$$S_{\text{EE}} = 1 - \gamma + \mathcal{O}(g^3). \quad (2.81)$$

In fact, this relation is the reason why the Rényi-2 entropy is sometimes called the “linear entropy”: it corresponds to a linearized version of the von-Neumann entropy where only the first term in the Mercator series is kept. At higher order, the linear entropy and the von-Neumann entropy are different, but the above framework still allows one to express the von-Neumann entropy in terms of the diagrams involved in the linear entropy.

### 3 Application

In this section, we apply the techniques developed above to a model of phenomenological interest for primordial cosmology.

#### 3.1 EFT-inspired model

Based on the Effective Field Theory of Inflation (EFToI) [105, 149–151] and quasi-single field inflation/the gelaton scenario [106–110, 152–157], we illustrate our method on a subset of possible interactions between the curvature perturbation  $\zeta$  (the system) and a massive scalar degree of freedom  $\sigma$  (the environment). We describe the model below.

Perturbing about a quasi de-Sitter background and always working to leading order in the slow-roll parameter  $\epsilon_1 := -\dot{H}/H^2 \ll 1$ , where  $H = \dot{a}/a$  is the Hubble expansion rate and  $a$  is the scale factor, the first ingredient in the considered model is the pseudo-Goldstone boson  $\pi$  which is related to the curvature perturbation  $\zeta$  as  $\zeta = -H\pi$  at leading order in cosmological perturbation theory. The formalism describes inflation as a symmetry-breaking phenomenon in which  $\pi$  non-linearly realizes the broken time-diffeomorphism invariance due to the presence of the inflaton field. Being an effective field theory, the Goldstone boson  $\pi$  determines the low-energy spectrum of the theory with higher-order effects parametrized by operators compatible with the remaining spatial diffeomorphism symmetry, and systematically organized in a derivative expansion. We work in the decoupling regime<sup>6</sup> in which the physics of the Goldstone boson decouples from the other metric fluctuations so that  $\pi$  is the only scalar degree of freedom considered. Its action takes the form [105]

$$S_{\mathcal{S}} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} M_{\text{Pl}}^2 R + M_{\text{Pl}}^2 H^2 (2\epsilon_1 - 3) - M_{\text{Pl}}^2 H^2 \epsilon_1 \delta g^{00} + \frac{M_2^4}{2!} (\delta g^{00})^2 + \frac{M_3^4}{3!} (\delta g^{00})^3 + \dots \right]. \quad (3.1)$$

The scales  $M_2$  and  $M_3$  are in general time-dependent<sup>7</sup>, and  $\pi$  is related to the time-time component of the metric fluctuation  $\delta g^{00}$  via

$$\delta g^{00} = -2\dot{\pi} - \dot{\pi}^2 + \frac{(\partial_i \pi)^2}{a^2} \quad (3.2)$$

where  $(\partial_i \pi)^2 = \delta^{ij} \partial_i \pi \partial_j \pi$ . Functions of the background in Eq. (3.1) are evaluated at time  $t + \pi$  such that, expanding in  $\pi$ , we observe a cancellation of the terms linear in  $\pi$  (tadpoles). This cancellation takes place “on-shell”, *i.e.* when the Friedmann equations hold [105]. The action

<sup>6</sup>This is valid at energies larger than  $E_{\text{mix}} \sim M_2^2/M_{\text{Pl}}$ , see Ref. [105].

<sup>7</sup>In setups where  $H$  and  $\dot{H}$  do not vary significantly over a Hubble time, it is natural to assume that  $M_2$  and  $M_3$  do not vary much either, and treat these coefficients as constants.

(3.1) neglects terms  $\mathcal{O}[(\delta g^{00})^4]$  and others involving the extrinsic curvature (and so neglects the two transverse modes of the graviton), which are indeed subdominant in the decoupling regime mentioned above. Higher-order derivatives of the fluctuations are also dropped.

The second ingredient to the model considered is an additional massive field  $\sigma$ , minimally-coupled, with an action [108–110]

$$S_{\mathcal{E}} = \int d^4x \sqrt{-g} \left( -\frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} m^2 \sigma^2 + \dots \right). \quad (3.3)$$

Here we neglect cubic self-interactions of  $\sigma$  (6 operators in total) and higher. The interactions between  $\pi$  and  $\sigma$  are then fixed by the EFToI symmetries [152] – we choose to examine only a subset of the possible interactions which are linear in  $\sigma$  so that

$$S_{\text{int}} = \int d^4x \sqrt{-g} \left[ \widetilde{M}_1^3 \delta g^{00} \sigma + \widetilde{M}_3^3 (\delta g^{00})^2 \sigma + \dots \right] \quad (3.4)$$

and neglect the other possible interactions for simplicity (which means ignoring 9 other operators like  $\sim \delta g^{00} \sigma^2$ ). Note that phenomenological properties of this model have been studied in Refs. [109, 110, 156–158]. In this work, we treat  $\pi$  as the “system” and  $\sigma$  as the “environment”.

Expanding Eq. (3.1) in terms of  $\pi$ , the system free action (3.1) simplifies to (working in terms of conformal time  $\eta$  and enforcing background evolution to cancel tadpoles)

$$S_S = \int d\eta d^3\mathbf{x} a^2 \left[ \frac{1}{2} \pi_c'^2 - \frac{1}{2} c_s^2 (\partial_i \pi_c)^2 \right] \quad (3.5)$$

where cubic self interactions of  $\pi$  have been neglected, and we have introduced the canonically normalized field

$$\pi_c \equiv \frac{\sqrt{2\epsilon_1} H M_{\text{Pl}}}{c_s} \pi \quad \text{with} \quad c_s^2 \equiv \left( 1 + \frac{2M_2^4}{M_{\text{Pl}}^2 H^2 \epsilon_1} \right)^{-1}. \quad (3.6)$$

In this expression, we used the speed of sound  $c_s$ , forbidding superluminal propagation,  $c_s^2 < 1$ , by enforcing  $M_2^4 > 0$ . The environment free action (3.3) is simply

$$S_{\mathcal{E}} = \int d\eta d^3\mathbf{x} a^2 \left[ \frac{1}{2} \sigma'^2 - \frac{1}{2} (\partial_i \sigma)^2 - \frac{1}{2} m^2 a^2 \sigma^2 \right] \quad (3.7)$$

and the interaction action, expanded to cubic order in the field operators, has the form

$$S_{\text{int}} = - \int d\eta d^3\mathbf{x} g \left[ a^3 \pi_c' \sigma - \frac{1}{\Lambda_1} a^2 (\partial_i \pi_c)^2 \sigma - \frac{1}{\Lambda_2} a^2 \pi_c'^2 \sigma \right], \quad (3.8)$$

with the definitions

$$g \equiv \frac{2c_s \widetilde{M}_1^3}{\sqrt{2\epsilon_1} H M_{\text{Pl}}}, \quad \frac{1}{\Lambda_1} \equiv \frac{c_s}{2\sqrt{2\epsilon_1} H M_{\text{Pl}}} \quad \text{and} \quad \frac{1}{\Lambda_2} \equiv \frac{c_s}{2\sqrt{2\epsilon_1} H M_{\text{Pl}}} \left( \frac{4\widetilde{M}_3^3}{\widetilde{M}_1^3} - 1 \right). \quad (3.9)$$

One can see that  $S_{\text{int}}$  is controlled by an overall dimensionful coupling  $g$  and contains three terms. On the one hand, the first two terms in Eq. (3.8) are both generated by the operator  $\widetilde{M}_1^3 \delta g^{00} \sigma$  upon using Eq. (3.2). On the other hand, the term  $\pi_c'^2 \sigma$  is generated by both  $\widetilde{M}_1^3 \delta g^{00} \sigma$  and  $\widetilde{M}_3^3 (\delta g^{00})^2 \sigma$ , which is why the scale  $\Lambda_1$  is not always fixed in the same way as  $\Lambda_2$ . Note that

in the case where the term  $\widetilde{M}_3^3(\delta g^{00})^2\sigma$  in Eq. (3.4) can be neglected, the relation  $\Lambda_2 = -\Lambda_1$  holds, and this will play a key role in dealing with some of the divergences we encounter below [we return to this point below Eq. (3.62) later on].

In what follows, it will also be convenient to parametrize our results in terms of the amplitude of the reduced power spectrum of curvature perturbations [159–161],

$$\Delta_\zeta^2 \equiv \frac{H^2}{8\pi^2\epsilon_1 M_{\text{Pl}}^2 c_s} \simeq 2.2 \times 10^{-9} \quad (3.10)$$

since this combination will appear in various expressions. The numerical value is quoted from measurements of the CMB temperature anisotropies [162].

### 3.1.1 Canonical variables and Hamiltonian

We rescale the field variables according to

$$v_\pi = a\pi_c \quad \text{and} \quad v_\sigma = a\sigma, \quad (3.11)$$

in order for their kinetic term to be canonically normalized. By further adding the total derivative  $\frac{1}{2}\frac{d}{d\eta}\left(\frac{a'}{a}v_\pi^2 + \frac{a'}{a}v_\sigma^2\right)$  to the Lagrangian density  $\mathcal{L}$  in  $S = \int d\eta d^3\mathbf{x} \mathcal{L}$  (see Refs. [24, 163] for recent discussions on this topic), the action  $S = S_S + S_E + S_{\text{int}}$  is given by

$$S_S = \int d\eta d^3\mathbf{x} \frac{1}{2} \left[ (v'_\pi)^2 + \frac{a''}{a}v_\pi^2 - c_s^2(\partial_i v_\pi)^2 \right], \quad (3.12)$$

$$S_E = \int d\eta d^3\mathbf{x} \frac{1}{2} \left[ (v'_\sigma)^2 + \left(\frac{a''}{a} - a^2 m^2\right)v_\sigma^2 - (\partial_i v_\sigma)^2 \right], \quad (3.13)$$

$$S_{\text{int}} = -g \int d\eta d^3\mathbf{x} \left[ av_\sigma \left(v'_\pi - \frac{a'}{a}v_\pi\right) - \frac{v_\sigma}{a\Lambda_1}(\partial_i v_\pi)^2 - \frac{v_\sigma}{a\Lambda_2} \left(v'_\pi - \frac{a'}{a}v_\pi\right)^2 \right]. \quad (3.14)$$

The canonical momenta are determined from

$$p_\pi = \frac{\partial\mathcal{L}}{\partial v'_\pi} = v'_\pi \left(1 + \frac{2g}{a\Lambda_2}v_\sigma\right) - gav_\sigma - \frac{2ga'}{a^2\Lambda_2}v_\pi v_\sigma, \quad (3.15)$$

$$p_\sigma = \frac{\partial\mathcal{L}}{\partial v'_\sigma} = v'_\sigma, \quad (3.16)$$

and the Hamiltonian is then obtained by performing a Legendre transform

$$\begin{aligned} H &= \int d^3\mathbf{x} [p_\pi v'_\pi + p_\sigma v'_\sigma - \mathcal{L}] \\ &= \int d^3\mathbf{x} \left\{ \frac{1}{2} \left[ p_\pi^2 + c_s^2(\partial_i v_\pi)^2 - \frac{a''}{a}v_\pi^2 \right] + \frac{1}{2} \left[ p_\sigma^2 + (\partial_i v_\sigma)^2 + \left(a^2 m^2 - \frac{a''}{a}\right)v_\sigma^2 \right] \right. \\ &\quad + agv_\sigma \left(p_\pi - \frac{a'}{a}v_\pi\right) - \frac{gv_\sigma}{a\Lambda_1}(\partial_i v_\pi)^2 - \frac{gv_\sigma}{a\Lambda_2} \left(p_\pi - \frac{a'}{a}v_\pi\right)^2 \\ &\quad \left. + \frac{g^2 v_\sigma^2}{2a^2\Lambda_2^2} \frac{1}{1 + \frac{2g}{a\Lambda_2}v_\sigma} \left(a^2\Lambda_2 - 2p_\pi + 2\frac{a'}{a}v_\pi\right)^2 \right\}. \end{aligned} \quad (3.18)$$

In this expression, the terms in the first line are quadratic and correspond to the “free” Hamiltonian, *i.e.* they leave the  $\pi$  and  $\sigma$  sectors uncoupled. The terms in the second line belong to

the interaction Hamiltonian – they are either quadratic or cubic, and arise at order  $g$ . When expanding the terms in the third line at quadratic order in the fields, one finds  $a^2 g^2 v_\sigma^2/2$ , which is non interacting. It can thus be appended to the first line, which corresponds to renormalizing the mass of the  $\sigma$  field according to  $m^2 \rightarrow m^2 + g^2$ . All other terms of the third line are interacting terms, they are cubic or higher order in the fields, and they arise at order  $g^2$  and above. At leading order in  $g$  they can thus be neglected (the validity of this truncation is further discussed in Sec. 3.1.3), and the Hamiltonian  $H = H_S + H_E + gH_{\text{int}}$  becomes

$$H_S = \int d^3\mathbf{x} \frac{1}{2} \left[ p_\pi^2 + c_s^2 (\partial_i v_\pi)^2 - \frac{a''}{a} v_\pi^2 \right], \quad (3.19)$$

$$H_E = \int d^3\mathbf{x} \frac{1}{2} \left\{ p_\sigma^2 + (\partial_i v_\sigma)^2 + \left[ a^2 (m^2 + g^2) - \frac{a''}{a} \right] v_\sigma^2 \right\}, \quad (3.20)$$

$$gH_{\text{int}} = \int d^3\mathbf{x} g \left[ a v_\sigma \left( p_\pi - \frac{a'}{a} v_\pi \right) - \frac{v_\sigma}{a\Lambda_1} (\partial_i v_\pi)^2 - \frac{v_\sigma}{a\Lambda_2} \left( p_\pi - \frac{a'}{a} v_\pi \right)^2 \right]. \quad (3.21)$$

### 3.1.2 Mode functions

In the interaction picture, the field operators evolve in the free theory and are thus most usefully expressed in momentum space by Fourier expanding

$$\tilde{v}_i(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{v}_i(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{and} \quad \tilde{p}_i(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{p}_i(\mathbf{k}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (3.22)$$

for each field  $i = \pi, \sigma$ , where  $\tilde{v}_i(\mathbf{k}, \eta)$  and  $\tilde{p}_i(\mathbf{k}, \eta)$  can be further expanded in creation and annihilation operators

$$\tilde{v}_i(\mathbf{k}, \eta) = v_i(k, \eta) \hat{a}_{\mathbf{k}}^i + v_i^*(k, \eta) \hat{a}_{-\mathbf{k}}^{i\dagger} \quad \text{and} \quad \tilde{p}_i(\mathbf{k}, \eta) = p_i(k, \eta) \hat{a}_{\mathbf{k}}^i + p_i^*(k, \eta) \hat{a}_{-\mathbf{k}}^{i\dagger}. \quad (3.23)$$

The creation and annihilation operators  $\hat{a}_{\mathbf{k}}^{i\dagger}$  and  $\hat{a}_{\mathbf{k}}^i$  obey the canonical commutation relations

$$\left[ \hat{a}_{\mathbf{k}}^i, \hat{a}_{\mathbf{q}}^{j\dagger} \right] = \delta(\mathbf{k} - \mathbf{q}) \delta_{i,j}, \quad (3.24)$$

which leads to the Wronskian conditions

$$v_i(k, \eta) p_i^*(k, \eta) - v_i^*(k, \eta) p_i(k, \eta) = i \quad (3.25)$$

for the mode functions. Heisenberg's equation yields the classical equations of motion for the mode functions, i.e.

$$v'_\pi = p_\pi, \quad p'_\pi = \left( c_s^2 k^2 - \frac{a''}{a} \right) v_\pi, \quad (3.26)$$

$$v'_\sigma = p_\sigma, \quad p'_\sigma = \left( k^2 + a^2 m^2 - \frac{a''}{a} \right) v_\sigma, \quad (3.27)$$

where the  $k$  arguments are omitted for simplicity, since Fourier modes decouple in the free theory. Hamilton's equations can be combined into the so-called Mukhanov-Sasaki equations

$$v''_\pi + \left( c_s^2 k^2 - \frac{2}{\eta^2} \right) v_\pi = 0 \quad \text{and} \quad v''_\sigma + \left( k^2 - \frac{\nu_\sigma^2 - \frac{1}{4}}{\eta^2} \right) v_\sigma = 0, \quad (3.28)$$

where at leading order in slow roll we have replaced  $a = -(H\eta)^{-1}$ , and where we have introduced

$$\nu_\sigma = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \equiv i\mu_\sigma, \quad (3.29)$$

which is a pure imaginary if  $m^2 > \frac{9}{4}H^2$  and real otherwise. By normalizing the mode functions to the Bunch-Davies vacuum [164] in the asymptotic, sub-Hubble past, one obtains<sup>8</sup>

$$v_\pi(k, \eta) = \frac{e^{-ic_s k\eta}}{\sqrt{2c_s k}} \left(1 - \frac{i}{c_s k\eta}\right), \quad (3.30)$$

$$v_\sigma(k, \eta) = \frac{\sqrt{-\pi\eta}}{2} e^{-\frac{\pi}{2}\mu_\sigma + i\frac{\pi}{4}} H_{i\mu_\sigma}^{(1)}(-k\eta), \quad (3.31)$$

where  $H_\nu^{(1)}$  is the Hankel function of the first kind and of order  $\nu$ . The mode functions of the momentum operators read

$$p_\pi(k, \eta) = -i\sqrt{\frac{c_s k}{2}} e^{-ic_s k\eta} \left(1 - \frac{i}{c_s k\eta} - \frac{1}{c_s^2 k^2 \eta^2}\right), \quad (3.32)$$

$$p_\sigma(k, \eta) = -\frac{1}{2}\sqrt{\frac{\pi}{-\eta}} e^{-\frac{\pi}{2}\mu_\sigma + i\frac{\pi}{4}} \left[\left(i\mu_\sigma + \frac{1}{2}\right) H_{i\mu_\sigma}^{(1)}(-k\eta) + k\eta H_{i\mu_\sigma+1}^{(1)}(-k\eta)\right]. \quad (3.33)$$

One can check that Eqs. (3.30)-(3.33) obey the Wronskian condition given above.

### 3.1.3 Early-time breakdown of the $g$ expansion

Above we have restricted the Hamiltonian to terms that are linear in  $g$ , and it is important to assess the regime of validity of this truncation. Upon inspection of Eq. (3.18) it is clear that the terms we have neglected are controlled by  $gv_\sigma/(a\Lambda)$ , hence our results only apply when the condition

$$\left|\frac{gv_\sigma}{a\Lambda}\right| \ll 1 \quad (3.34)$$

is satisfied. The validity of this condition can be qualitatively assessed as follows. In real space,  $\langle \tilde{v}_\sigma(\mathbf{x}) \rangle = 0$  and  $\langle \tilde{v}_\sigma^2(\mathbf{x}) \rangle = \int d \ln k \frac{k^3}{2\pi^2} v_\sigma^2(k)$ . The contribution from a given scale  $k$  to  $v_\sigma$  can thus be estimated as  $k^{3/2}|v_\sigma(k)|$ , which is of order  $k$  on sub-Hubble scales and of order  $aH$  on super-Hubble scales if  $\sigma$  is massless (if  $\sigma$  is massive, the super-Hubble behavior is further suppressed, hence the following estimate is conservative). At super-Hubble scales, the condition (3.34) thus requires that

$$g \ll \frac{\sqrt{\epsilon_1} M_{\text{Pl}}}{c_s} \quad (3.35)$$

upon using Eq. (3.9), where we further assume  $\widetilde{M}_3 \lesssim \widetilde{M}_1$ . This implies that  $g$  needs to be parametrically suppressed compared to the Planck mass, and that the theory breaks down when  $\epsilon_1 \rightarrow 0$ , which is expected from the EFToI perspective. At sub-Hubble scales, the condition (3.34) reduces to

$$\frac{aH}{k} \gg \frac{gc_s}{\sqrt{\epsilon_1} M_{\text{Pl}}}. \quad (3.36)$$

---

<sup>8</sup>In the asymptotic past,  $k/a \gg H, m, g$ , hence the fields become decoupled and their mode functions evolve on an effectively quasi-static background. This allows one to set both sectors in the free vacuum state of Minkowski space-times. We stress that decoupling at small scales is necessary for the Bunch-Davies prescription to be applied [121, 165].

This implies that our expansion breaks down at small distances, and becomes valid only once the scales of interest cross  $gc_s/(\sqrt{\epsilon_1}M_{\text{Pl}})$  times the Hubble radius. It is interesting to notice that Eq. (3.35) requires that the scale below which the expansion breaks down lies well within the Hubble radius. Moreover, if one works with

$$g \lesssim \frac{\sqrt{\epsilon_1}H}{c_s}, \quad (3.37)$$

which is more restrictive than Eq. (3.35), then that critical scale falls below the Planck length. In this case, the theory becomes valid at all super-Planckian distances, to which it should be restricted anyway. The main conclusion is that the effective theory we are working with is expected to break down at early time, hence below we employ it to investigate late-time behavior only.

### 3.2 Perturbative entropies

Let us now determine the in-in evolution of the purity of the system. We use the quantum interaction Hamiltonian (3.21) recast in momentum space as

$$g\tilde{H}_{\text{int}}(\eta) = g \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[ \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}, \eta) + \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}, \eta) \right] \otimes \tilde{v}_\sigma(-\mathbf{k}, \eta) \quad (3.38)$$

where

$$\tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}, \eta) \equiv a(\eta)\tilde{p}_\pi(\mathbf{k}, \eta) - a'(\eta)\tilde{v}_\pi(\mathbf{k}, \eta) \quad (3.39)$$

$$\begin{aligned} \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}, \eta) \equiv & \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k}) \left\{ \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{a(\eta)\Lambda_1} \tilde{v}_\pi(\mathbf{p}_1, \eta) \tilde{v}_\pi(\mathbf{p}_2, \eta) \right. \\ & \left. - \frac{1}{a(\eta)\Lambda_2} \left[ \tilde{p}_\pi(\mathbf{p}_1, \eta) - \frac{a'(\eta)}{a(\eta)}\tilde{v}_\pi(\mathbf{p}_1, \eta) \right] \left[ \tilde{p}_\pi(\mathbf{p}_2, \eta) - \frac{a'(\eta)}{a(\eta)}\tilde{v}_\pi(\mathbf{p}_2, \eta) \right] \right\}. \end{aligned} \quad (3.40)$$

We have written  $g\tilde{H}_{\text{int}}$  in the form of Eq. (2.22) so that the machinery from Sec. 2 can be employed directly. Recall that, according to Eq. (2.48), at leading order in the coupling one has to compute

$$S_2(k) \simeq \frac{4g^2}{(2\pi)^6} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(k, \eta_1, \eta_2) \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta_1, \eta_2) + \mathcal{O}(g^3). \quad (3.41)$$

**Memory kernels** Using the definition (2.46) together with Eqs. (2.31) and (2.34), the two-point function in the environment takes the simple form

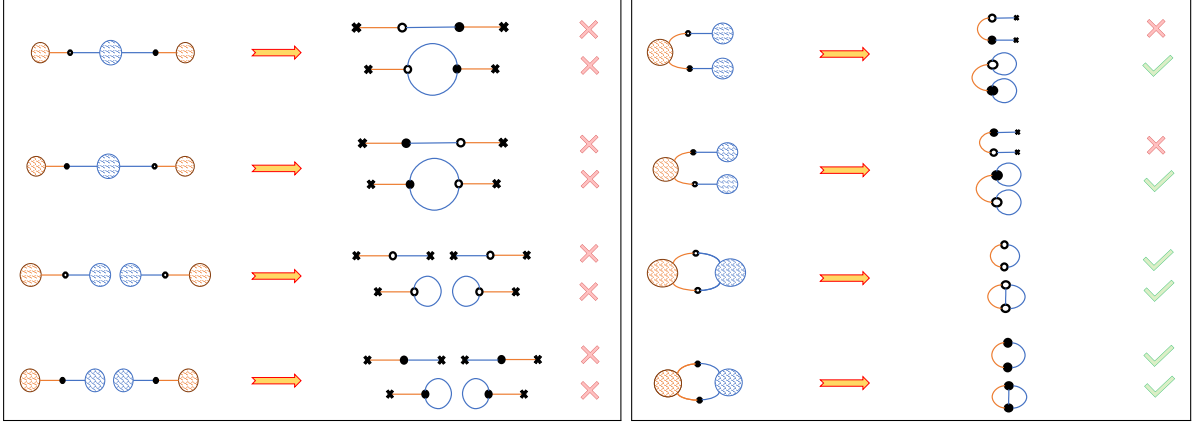
$$\bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta_1, \eta_2) = v_\sigma(k, \eta_1) v_\sigma^*(k, \eta_2), \quad (3.42)$$

whose mass-dependence we take various limits of in the proceeding subsections. For the system, the memory kernel is made of two pieces,

$$\bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(k, \eta_1, \eta_2) = \bar{\mathcal{K}}_{\mathcal{S}, \text{lin}}^{(2)}(k, \eta_1, \eta_2) + \bar{\mathcal{K}}_{\mathcal{S}, \text{NL}}^{(2)}(k, \eta_1, \eta_2), \quad (3.43)$$

which correspond to the two-point functions of  $\tilde{O}_{\text{lin}}^{\mathcal{S}}$  and  $\tilde{O}_{\text{NL}}^{\mathcal{S}}$  respectively. The cross-term between  $\tilde{O}_{\text{lin}}^{\mathcal{S}}$  and  $\tilde{O}_{\text{NL}}^{\mathcal{S}}$  vanishes since it involves cubic powers of field operators, the expectation





**Figure 12.** Diagrams involved in the calculation of  $\gamma^{(1,1)}$  (left panel) and  $\gamma^{(2,0)}$  (right panel), with quadratic and cubic vertices of the form  $\pi\sigma$  and  $\pi^2\sigma$ . Contributions with a cross represent vanishing one-point functions. One can see that  $\gamma^{(1,1)} = 0$ , all contributions being proportional to the environment one-point function. For  $\gamma^{(2,0)}$ , both the quadratic and cubic vertices lead to non-vanishing second-order contributions, which induce departures from a pure state.

value of which vanishes on Gaussian states. A detailed calculation is presented in Appendix C, where it is shown that

$$\bar{\mathcal{K}}_{S,\text{lin}}^{(2)}(k, \eta_1, \eta_2) = \frac{c_s k}{2H^2 \eta_1 \eta_2} e^{-i c_s k (\eta_1 - \eta_2)} \quad (3.44)$$

and

$$\begin{aligned} \bar{\mathcal{K}}_{S,\text{NL}}^{(2)}(k, \eta_1, \eta_2) &= \frac{H^2 \eta_1 \eta_2}{(2\pi)^{12}} \frac{\pi}{32k} \int_k^\infty dP \int_0^k dQ e^{-i c_s P (\eta_1 - \eta_2)} \\ &\times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s \Lambda_1} \left[ 1 - \frac{2i}{c_s (P+Q) \eta_1} \right] \left[ 1 - \frac{2i}{c_s (P-Q) \eta_1} \right] - \frac{c_s (P^2 - Q^2)}{\Lambda_2} \right\} \\ &\times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s \Lambda_1} \left[ 1 + \frac{2i}{c_s (P-Q) \eta_2} \right] \left[ 1 + \frac{2i}{c_s (P+Q) \eta_2} \right] - \frac{c_s (P^2 - Q^2)}{\Lambda_2} \right\}. \end{aligned} \quad (3.45)$$

In practice, after Wick contractions of the field operators, and once conservation of momentum is imposed, the system correlator can be expressed as an integral over two momenta (these are the two momenta flowing through the double loops in the right panel of Fig. 12). The expression (3.45) is what remains after the angular part these two momenta has been integrated away [and after an additional rotation is performed – see Eq. (C.25)].

It is of course possible to perform the two remaining integrals over  $P$  and  $Q$  in Eq. (3.45) before inserting the result in Eq. (3.41). In particular, by examining the  $P \rightarrow \infty$  limit of the integrand, one can show that the dominant contribution in the coincident limit  $\eta_1 \rightarrow \eta_2$  reads

$$\begin{aligned} \bar{\mathcal{K}}_{S,\text{NL}}^{(2)}(k, \eta_1, \eta_2) &= \frac{H^2 \eta_1 \eta_2}{(2\pi)^{12}} \frac{\pi}{32k} \int_k^\infty dP \int_0^k dQ e^{-i c_s P (\eta_1 - \eta_2)} \left[ \left( \frac{1}{c_s \Lambda_1} - \frac{c_s}{\Lambda_2} \right) P^4 + \mathcal{O}(P^3) \right] \\ &\simeq \frac{H^2 \eta_1 \eta_2}{(2\pi)^{12}} \frac{\pi}{32k} \left\{ 24i \frac{c_s}{\Lambda_2} - \frac{1}{c_s \Lambda_1} \right\} \frac{1}{c_s^5 (\eta_1 - \eta_2)^5} + \mathcal{O} \left[ (\eta_1 - \eta_2)^{-4} \right]. \end{aligned} \quad (3.46)$$

This singular coincident limit can be understood as contributing to the UV divergences that later arise, which may be regulated using a short-time cutoff. A more desirable regularization

scheme however can be used if one instead keeps the correlator represented in terms of the momentum integrals as in Eq. (3.45), and then performs the  $P$  and  $Q$  integrals only *after* any time integrals occurring in the linear entropy formula (3.41). This is a useful tactic because the  $P$  and  $Q$  variables cleanly separate the UV and IR behavior of the integrand and therefore, more standard methods of UV regularization from QFT (e.g. dimensional regularization) can be straightforwardly applied (this would also apply to IR divergences if other interactions with less derivatives were studied).

**Spectral purity** The total linear entropy at second-order in the coupling can be represented by the diagrams depicted in Fig. 12. They are obtained by applying the techniques of Sec. 2.6.2 to the model considered here. With notations analogous to Eq. (3.43), the linear entropy receives two contributions,

$$S_2(k) \simeq S_2^{\text{lin}}(k) + S_2^{\text{NL}}(k) + \mathcal{O}(g^3) \quad (3.47)$$

with

$$S_2^i(k) \equiv \frac{4g^2}{(2\pi)^6} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \bar{\mathcal{K}}_{\mathcal{S},i}^{(2)}(k, \eta_1, \eta_2) \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta_1, \eta_2) \quad \text{for } i = \text{lin, NL} . \quad (3.48)$$

Using Eqs. (3.42) and (3.44), the linear contribution reduces to [91, 102]

$$S_2^{\text{lin}}(k) = \frac{g^2 c_s}{(2\pi)^6 H^2} |\mathfrak{L}_k(\eta)|^2 \quad \text{with} \quad \mathfrak{L}(\eta) \equiv \int_{-\infty}^{\eta} d\eta' e^{-ic_s k \eta'} \sqrt{\frac{\pi k}{-2\eta'}} e^{-\frac{\pi}{2}\mu_\sigma + \frac{i\pi}{4}} H_{i\mu_\sigma}^{(1)}(-k\eta') . \quad (3.49)$$

In this form it is obvious that the linear entropy is always non-negative, hence that purity remains below one and that the map is CPTP, see the discussion below Eq. (2.39). Similarly, the non-linear contribution can be expressed as

$$S_2^{\text{NL}}(k) = \frac{g^2 H^2}{(2\pi)^{18}} \frac{\pi}{16k^4} \int_k^\infty dP \int_0^k dQ |\mathfrak{N}_k(P, Q, \eta)|^2 \quad (3.50)$$

with

$$\begin{aligned} \mathfrak{N}_k(P, Q, \eta) &\equiv \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_\sigma + \frac{i\pi}{4}} \int_{-\infty}^{\eta} d\eta' (-k\eta')^{\frac{3}{2}} H_{i\mu}^{(1)}(-k\eta') e^{-ic_s P \eta'} \\ &\times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s \Lambda_1} \left[ 1 - \frac{2i}{c_s (P+Q)\eta'} \right] \left[ 1 - \frac{2i}{c_s (P-Q)\eta'} \right] - \frac{c_s (P^2 - Q^2)}{\Lambda_2} \right\} . \end{aligned} \quad (3.51)$$

In the analysis that follows, we first perform the time integral in Eq. (3.51) before performing the one over  $P$  and  $Q$  in Eq. (3.50), for the reasons mentioned above.

Finally, a quantity of physical interest below is the decoherence *rate*, obtained by differentiating Eq. (2.48) with respect to time,

$$\frac{dS_2(k)}{d\eta} \simeq \frac{8g^2}{(2\pi)^6} \int_{-\infty}^{\eta} d\eta' \Re \left[ \bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(k, \eta, \eta') \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta, \eta') \right] + \mathcal{O}(g^3) . \quad (3.52)$$

This has the advantage of involving a single integral over time, and it can be divided into a linear and a non-linear contribution by time differentiating Eqs. (3.49) and (3.50) respectively. The other advantage of working with the decoherence rate is that, as explained in Sec. 3.1.3,

the EFT treatment of the model we consider is expected to break down at early time. As a consequence, finite contributions to the linear entropy coming from early-time interactions cannot be properly accounted for. Such contributions drop out from the late-time decoherence rate, which may thus be argued to be the proper quantity to compute in the EFT.

In what follows, we compute the spectral purity in three cases for the environment mass: massless ( $m = 0$ ), conformal ( $m = \sqrt{2}H$ ) and heavy ( $m \gg H$ ).

### 3.2.1 Massless environment

Let us first consider the case where the environment is massless,  $m = 0$ , such that  $i\mu_\sigma = 3/2$  and Eq. (3.31) reduces to

$$v_\sigma(k, \eta) = \frac{e^{-ik\eta}}{\sqrt{2k}} \left( 1 - \frac{i}{k\eta} \right). \quad (3.53)$$

#### Linear contribution

The linear contribution to the entropy depends on the integral  $\mathfrak{L}_k(\eta)$  defined in Eq. (3.49), which is evaluated for the massless case in Appendix D and one finds

$$S_2^{\text{lin}}(k) = \frac{g^2 c_s}{(2\pi)^6 H^2} \left| \frac{ie^{-i(1+c_s)k\eta}}{-k\eta} + c_s \{ \text{Ei}[-i(1+c_s)k\eta] - i\pi \} \right|^2. \quad (3.54)$$

Here, Ei is the exponential integral function defined in (D.6). It is convenient to express the above in terms of the number of  $e$ -folds  $N = \ln(a)$  spent by the mode  $k$  outside the Hubble radius. In a de-Sitter universe it is given by

$$N_k = -\log(-k\eta), \quad (3.55)$$

such that  $N_k > 0$  and  $N_k < 0$  correspond to before and after Hubble-crossing, respectively. The early ( $N_k \ll -1$ ) and late-time ( $N_k \gg 1$ ) behavior of the entropy can be obtained by expanding Eq. (3.54) and one finds

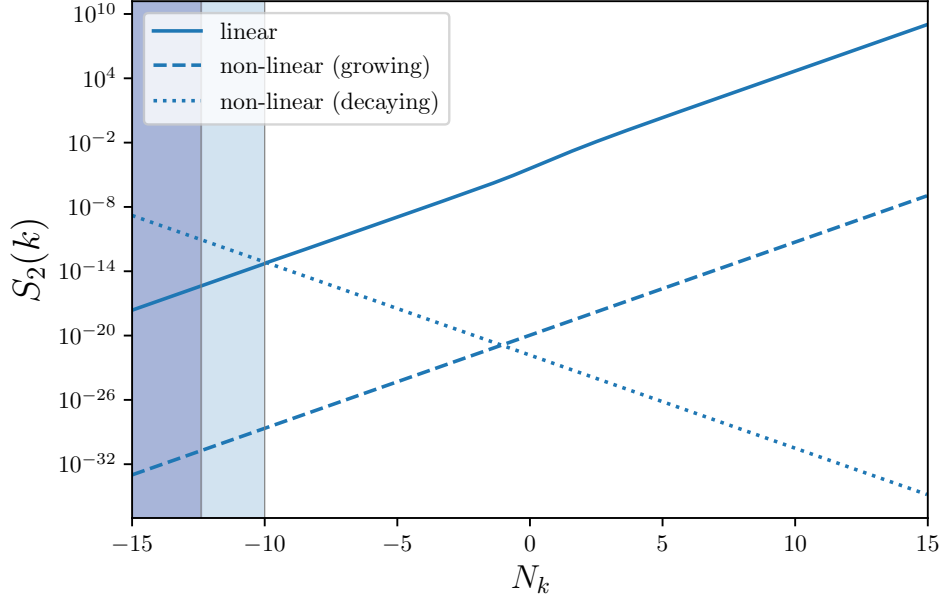
$$S_2^{\text{lin}}(k) \simeq \begin{cases} \frac{c_s g^2}{(2\pi)^6 H^2} \frac{e^{2N_k}}{(1+c_s)^2} & \text{(sub-Hubble)} \\ \frac{c_s g^2}{(2\pi)^6 H^2} e^{2N_k} & \text{(super-Hubble)} \end{cases}. \quad (3.56)$$

The entropy is thus monotonously increasing, at a rate which is the same in both asymptotic regimes, *i.e.*  $S_2^{\text{lin}} \propto a^2$ , with only a slight increase in the overall amplitude at around Hubble crossing. This can be further checked in Fig. 13 where Eq. (3.54) is displayed.

#### Non-linear contribution

Evaluating Eq. (3.51) with Eq. (3.53), in Appendix D we show that

$$\begin{aligned} \mathfrak{N}_k(P, Q, \eta) = & e^{-i(k+c_s P)\eta} \left\{ -\frac{4ik(P^2 + Q^2 - 2k^2)}{c_s^3(P^2 - Q^2)\Lambda_1} \frac{1}{-k\eta} + \frac{c_s(P^2 - Q^2)(2k + c_s P)}{(k + c_s P)^2 \Lambda_2} \right. \\ & - \frac{(P^2 + Q^2 - 2k^2)}{c_s(k + c_s P)\Lambda_1} \left[ \frac{4kP}{c_s(P^2 - Q^2)} + \frac{2k + c_s P}{k + c_s P} \right] \\ & \left. + \frac{i}{c_s(k + c_s P)} \left[ \frac{P^2 + Q^2 - 2k^2}{\Lambda_1} - \frac{c_s^2(P^2 - Q^2)}{\Lambda_2} \right] (-k\eta) \right\}. \end{aligned} \quad (3.57)$$



**Figure 13.** Spectral purity  $S_2(k)$  as a function of the number of  $e$ -folds  $N_k$  from Hubble-exit time, in the case of a massless environment with  $g = 0.01 H$  and  $c_s = 1$ . Loop corrections overtake the tree-order result at early times given by Eq. (3.65), which is displayed with the light-blue shaded region. The EFT also breaks down at early time, see Eq. (3.66), which is displayed in dark blue. At late time, both the linear and non-linear contributions grow like  $a^2$ , the latter being suppressed with respect to the former by  $\Delta_\zeta^2/(4\pi^2 c_s)^4$ .

Inserting this expression into the formula (3.50) one obtains UV divergences of the form  $S_2^{\text{NL}} \propto \int_k^\infty dP P^n$  for  $P \rightarrow \infty$  with  $n = -1, 0, 1, 2$ . These are reminiscent of the singular equal-time limit of the environment correlator [102] as discussed below Eq. (3.46). Here they appear as large-momentum divergences, which makes it possible to employ a variant of dimensional regularization where we take for example<sup>9</sup>

$$\int_k^\infty \frac{dP}{k + c_s P} \rightarrow \int_k^\infty \frac{dP}{k + c_s P} \left(\frac{P}{\mu}\right)^\varepsilon \simeq -\frac{1}{c_s} \left\{ \frac{1}{\varepsilon} - \log \left[ \frac{c_s \mu}{(1 + c_s)k} \right] \right\} + \mathcal{O}(\varepsilon). \quad (3.58)$$

Here we have expanded near  $\varepsilon \simeq 0$  and taken  $\mu > 0$  to be an arbitrary mass scale. Performing the  $P$ - and then  $Q$ -integrals in Eq. (3.50) leads to

$$S_2^{\text{NL}}(k) = \frac{g^2 H^2}{(2\pi)^{18}} \frac{\pi}{16} (\mathcal{C}_1 e^{2N_k} + \mathcal{C}_2 + \mathcal{C}_3 e^{-2N_k}), \quad (3.59)$$

written in terms of the number of  $e$ -folds defined in Eq. (3.55) and with time-independent coefficients given by

$$\begin{aligned} \mathcal{C}_1 &= -\frac{48}{c_s^6 \Lambda_1^2}, \\ \mathcal{C}_2 &= \frac{4}{c_s^3} \left( \frac{9}{c_s^4 \Lambda_1^2} - \frac{1}{\Lambda_2^2} \right) \left\{ \frac{1}{\varepsilon} - \log \left[ \frac{c_s \mu}{(c_s + 1)k} \right] \right\} + \frac{264c_s^6 + 792c_s^5 - 1064c_s^4 - 3975c_s^3 - 1625c_s^2 + 1935c_s + 1185}{45c_s^7 (c_s + 1)^3 \Lambda_1^2} \end{aligned} \quad (3.60)$$

<sup>9</sup>Strictly speaking, this regularization scheme is only tangentially related to dimensional regularization, which would require the use of the mode functions in  $(4 + \varepsilon)$ -dimensional de Sitter space.

$$-\frac{2(96c_s^6+288c_s^5+544c_s^4+855c_s^3+885c_s^2+585c_s+195)}{45c_s^3(c_s+1)^3\Lambda_1\Lambda_2} + \frac{24c_s^6+72c_s^5+216c_s^4+405c_s^3+275c_s^2-45c_s-75}{45c_s^3(c_s+1)^3\Lambda_2^2} \quad (3.61)$$

$$\mathcal{C}_3 = \frac{4}{c_s^3} \left( \frac{1}{c_s^2\Lambda_1} - \frac{1}{\Lambda_2} \right) \left( \frac{c_s^2-3}{3\Lambda_2} + \frac{3-5c_s^2}{3c_s^2\Lambda_1} \right) \left\{ \frac{1}{\varepsilon} - \log \left[ \frac{c_s\mu}{(c_s+1)k} \right] \right\} \\ + \frac{88c_s^4+60c_s^3-80c_s^2-45c_s+15}{15c_s^7(c_s+1)\Lambda_1^2} - \frac{2(32c_s^4+40c_s^3-60c_s^2-45c_s+15)}{15c_s^3(c_s+1)\Lambda_1\Lambda_2} + \frac{8c_s^4+20c_s^3-40c_s^2-45c_s+15}{15c_s^3(c_s+1)\Lambda_2^2} \quad (3.62)$$

with further details of the derivation given in Appendix D.

**UV logarithmic divergences:** The coefficients  $\mathcal{C}_2$  and  $\mathcal{C}_3$  feature UV-divergent terms that we now discuss. As argued below Eq. (3.52), a quantity that is immune to finite non-EFT effects at small scales is the entropy production rate,

$$\frac{dS_2^{\text{NL}}(k)}{dN_k} = \frac{g^2 H^2}{(2\pi)^{18}} \frac{\pi}{16} (\mathcal{C}_1 e^{2N_k} - \mathcal{C}_3 e^{-2N_k}), \quad (3.63)$$

from which  $\mathcal{C}_2$  is absent. At late time indeed, the constant term  $\mathcal{C}_2$  may receive contributions from effects occurring before the EFT became valid, which we cannot properly account for and that may cancel out the divergences in Eq. (3.61). By considering the decoherence rate we are focusing on late-time effects, which are within the reach of the EFT.

For  $\mathcal{C}_3$ , the divergent term is controlled by  $(c_s^2-3)/\Lambda_2 + (3-5c_s^2)/(c_s^2\Lambda_1)$ . As mentioned below Eq. (3.9), in the case where the term controlled by  $\widetilde{M}_3$  can be neglected in the interaction Lagrangian (3.4), the relation  $\Lambda_2 = -\Lambda_1$  holds. Moreover, if the term controlled by  $M_2$  in the system's Lagrangian (3.1) can be neglected too, then  $c_s^2 = 1$ , see Eq. (3.6). Under these conditions the divergent term in  $\mathcal{C}_3$  thus cancels out. Both  $M_2$  and  $\widetilde{M}_3$  control terms involving  $(\delta g^{00})^2 \supset \dot{\pi}^3, \dot{\pi}(\partial_i \pi)^2$ , which are expected to contribute to spectral purity (by coupling a given Fourier mode to all other Fourier modes) but which we have neglected. It is therefore reasonable to assume that these contributions cancel out the divergent term in  $\mathcal{C}_3$ . In other words, only when  $c_s = 1$  and  $\Lambda_2 = -\Lambda_1$  can one consistently discard terms of order  $(\delta g^{00})^2$  in the Lagrangian and we will therefore assume that this is the case in what follows.

Under these conditions  $\mathcal{C}_3$  is finite and given by

$$\mathcal{C}_3 = -\frac{4}{3\Lambda_1^2}. \quad (3.64)$$

The term it controls in Eq. (3.59) is displayed with the dotted line in Fig. 13, while the term controlled by  $\mathcal{C}_1$  is displayed with the dashed line. Let us further discuss the contributions of these terms at early and late time.

**Early-time perturbative breakdown:** At sub-Hubble scales, the non-linear contribution takes over the linear contribution, since

$$\left. \frac{S_2^{\text{NL}}(k)}{S_2^{\text{lin}}(k)} \right|_{\text{sub-Hubble}} = -\frac{\Delta_\zeta^2}{24(2\pi)^9} e^{-4N_k}, \quad (3.65)$$

where we have used Eqs. (3.9) and (3.10). This becomes larger than unity (in absolute value) at early time, when

$$\frac{aH}{k} < \left[ \frac{\Delta_\zeta^2}{24(2\pi)^9} \right]^{1/4} \simeq 5 \times 10^{-5}, \quad (3.66)$$

*i.e.*  $\simeq 10$   $e$ -folds before Hubble crossing. One should recall nonetheless that the EFT treatment breaks down when

$$\frac{aH}{k} < \frac{g}{H} \sqrt{8\pi^2 \Delta_\zeta^2}, \quad (3.67)$$

see Eq. (3.36). The two breaking scales coincide when  $g/H = 0.1$ . Therefore, when  $g$  is parametrically smaller than  $H$ , the EFT becomes valid before the loop corrections we have computed become sub-dominant. In any case, the above discussion confirms that our results should be trusted at late time only, *i.e.* around and after Hubble crossing.

**Late-time decoherence:** At super-Hubble scales, both the linear and the non-linear contributions to spectral purity grow like  $a^2$  and adding them together leads to

$$S_2(k)|_{\text{super-Hubble}} = \frac{c_s}{(2\pi)^6} \frac{g^2}{H^2} \left[ 1 - \frac{1}{(2\pi)^9} \frac{3}{8} \frac{\Delta_\zeta^2}{c_s^4} \right] e^{2N_k}. \quad (3.68)$$

At late time, the entropy thus grows as the universe’s “area”  $a^2$ . The fact that it grows confirms the CPTP nature of the non-unitary evolution, see the discussion below Eq. (2.39). Loop contributions only provide a small correction (as long as  $c_s > 10^{-4}$ ), and decoherence, which we may define as  $S_2(k) > 1$ , occurs when

$$\frac{aH}{k} > (2\pi)^3 \frac{H}{g\sqrt{c_s}}. \quad (3.69)$$

The link between purity and linear entropy is given by Eq. (2.42), which allows one to define an effective purity per Fourier mode

$$\gamma_k = e^{-S_2(k)}. \quad (3.70)$$

This quantity is displayed in Fig. 14, which confirms that decoherence takes place soon after Hubble crossing. One might be concerned that, since  $S_2$  has been computed to order  $g^2$  in perturbation theory, only a linearized version of Eq. (3.70) should be used, namely  $\gamma_k \simeq 1 - S_2(k)$ , for consistency. However, in Ref. [91] it is shown that Eq. (3.70) is able to efficiently resum secular effects due to the late-time growth of the linear entropy. This was shown by comparison with an exact, transport-equation based solution as presented in Ref. [52]. This is why we trust Eq. (3.70) to provide a reliable assessment of decoherence even in the strongly decohered regime.

### 3.2.2 Conformal environment

Let us consider the case where the environmental field  $\sigma$  is conformally coupled to gravity, *i.e.*  $i\mu_\sigma = \frac{1}{2}$ . In this case Eq. (3.31) reduces to  $v_\sigma(k, \eta) = e^{-ik\eta}/\sqrt{2\pi}$  and in Appendix D we show that the linear contribution to the entropy is given by

$$S_2^{\text{lin}} = \frac{g^2 c_s}{(2\pi)^6 H^2} |\text{Ei}[-i(c_s + 1)k\eta] - i\pi|^2 \simeq \begin{cases} \frac{c_s g^2}{(2\pi)^6 H^2} \frac{e^{2N_k}}{(1 + c_s)^2} & (\text{sub-Hubble}) \\ \frac{c_s g^2}{(2\pi)^6 H^2} N_k^2 & (\text{super-Hubble}) \end{cases}. \quad (3.71)$$

We have checked that the agreement between these expressions and the result of the transport-equation method (exact numerical treatment) presented in Refs. [52, 91] is excellent. It is interesting to notice that, at sub-Hubble scale, we get the exact same behavior as in the massless

case, see Eq. (3.56). However, at super-Hubble scales, entropy production is much slower, since it proceeds as  $S_2 \propto \ln^2(a)$  instead of  $S_2 \propto a^2$ .

The non-linear contribution can be computed following similar steps as in the massless case. At early time, before the Fourier mode crosses the Compton length, *i.e.* when  $k/a > m$ , the same result as in the massless-environment case are recovered. At late time, Eq. (3.51) is evaluated by first integrating over  $\eta'$ , then expanding the result in  $-k\eta \ll 1$ , and finally integrate over  $P$  and  $Q$ . This leads to

$$S_2(k)|_{\text{super-Hubble}} \simeq \frac{c_s}{(2\pi)^6} \frac{g^2}{H^2} \left[ 1 - \frac{1}{(2\pi)^9} \frac{3}{8} \frac{\Delta_\zeta^2}{c_s^4} \right] N_k^2. \quad (3.72)$$

As before, the non-linear contribution increases with time at the same rate as the linear one, and is suppressed by  $\Delta_\zeta^2/c_s^4$ . The case of a conformal environment is displayed in Fig. 14 with orange curves.

### 3.2.3 Heavy environment

Let us finally consider the case where the environment is heavy ( $m \gg H$ ), where for simplicity we additionally assume that  $c_s = 1$  and focus on the entropy production rate. Inserting Eq. (3.31) into Eq. (3.49) leads to [52, 91]

$$\left. \frac{dS_2^{\text{lin}}(k)}{d\eta} \right|_{c_s=1} = -\frac{kg^2}{32\pi^5 H^2} \Re \left[ e^{-\pi\mu} H_{i\mu}^{(1)}(-k\eta) e^{-ik\eta} F_\mu(-k\eta) \right] \quad (3.73)$$

for the linear contribution, where

$$F_\mu(z) \equiv \gamma_{\mu_{\mathcal{F}}}^*(z) g_{\mu_{\mathcal{F}}}(z) + \delta_{\mu_{\mathcal{F}}}^*(z) g_{-\mu_{\mathcal{F}}}(z) \quad (3.74)$$

and we have introduced the notations

$$\gamma_{\mu_{\mathcal{F}}}(z) \equiv \frac{1 + \coth \pi\mu_{\mathcal{F}}}{\Gamma(1 + i\mu_{\mathcal{F}})} \left(\frac{z}{2}\right)^{i\mu_{\mathcal{F}}}, \quad \delta_{\mu_{\mathcal{F}}}(z) \equiv \frac{-1}{\sinh \pi\mu_{\mathcal{F}}} \frac{1}{\Gamma(1 - i\mu_{\mathcal{F}})} \left(\frac{z}{2}\right)^{-i\mu_{\mathcal{F}}} \quad (3.75)$$

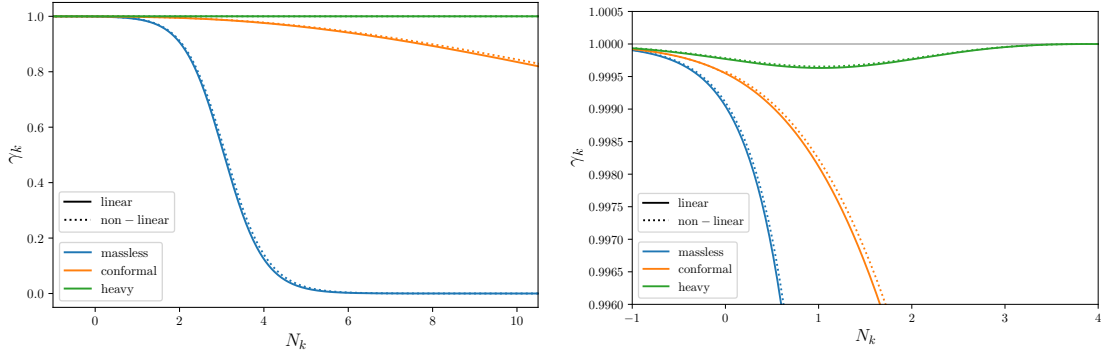
and

$$g_{\mu_{\mathcal{F}}}(z) = \frac{1}{1 - 2i\mu_{\mathcal{F}}} {}_2F_2 \left( \begin{matrix} \frac{1}{2} - i\mu_{\mathcal{F}}, \frac{1}{2} - i\mu_{\mathcal{F}} \\ \frac{3}{2} - i\mu_{\mathcal{F}}, 1 - 2i\mu_{\mathcal{F}} \end{matrix} ; -2iz \right), \quad (3.76)$$

with  ${}_2F_2$  being the (2, 2)-generalized hypergeometric function. We have checked that the agreement of these expressions with the transport-equation method is again excellent. The above expressions can be expanded at early and late time and one finds

$$\left. \frac{dS_2^{\text{lin}}(k)}{dN_k} \right|_{c_s=1} = \begin{cases} \frac{g^2}{2(2\pi)^6 H^2} e^{2N_k} & (\text{sub-Hubble}) \\ -\frac{g^2}{(2\pi)^6 H^2} \frac{4}{\mu(1 + 4\mu^2)} e^{-N_k} & (\text{super-Hubble}) \end{cases}. \quad (3.77)$$

At sub-Hubble scales, one recovers again the same result as in the massless and the conformal case, see Eqs. (3.56) and (3.71) respectively. At super-Hubble scales however, the situation is quite different since the entropy production rate becomes negative, which signals the presence



**Figure 14.** Per-mode purity  $\gamma_k$ , see Eq. (3.70), as a function of the number of  $e$ -folds  $N_k$  spent by a given scale outside the Hubble radius, for  $g = 0.05H$ ,  $c_s = 1$  and  $\Delta_\zeta^2/(4\pi^2 c_s)^4 = 1$ . The value of  $\Delta_\zeta^2$  is unrealistically large, but this is to amplify non-linear corrections that would otherwise not be visible by eye. The solid lines stand for the linear contributions, and the dotted lines correspond to the sum of the linear and non-linear contributions. If the environment is massless (blue curves) or conformally coupled (orange curves) decoherence takes place at late time, while recoherence occurs after a transient phase of slight decoherence if the environment is heavy (green curves). The right panel zooms in on the region  $\gamma_k \simeq 1$ .

of recoherence [52]. In the heavy mass limit  $m \gg H$ ,  $\mu \simeq m/H$ , see Eq. (3.29), and at late time one obtains

$$\gamma_k|_{\text{super-Hubble}} = \gamma_\infty - \frac{g^2}{16\pi^6 H^2} \frac{H^3}{m^3} e^{-N_k}, \quad (3.78)$$

where  $\gamma_\infty$  has been computed in Ref. [91] and is given by

$$\gamma_\infty \simeq 1 - \frac{g^2}{8\pi^4 H^2} e^{-2\pi \frac{m}{H}}. \quad (3.79)$$

For the non-linear contribution, closed-forms expressions cannot be reached, but the relevant integrals can be expanded in the sub- and super-Hubble regimes. At early time, before the Compton length is crossed out, the same results as in the massless and conformal cases are recovered. At late time, we find

$$\left. \frac{dS_2(k)}{dN_k} \right|_{\text{super-Hubble}} \simeq \frac{g^2}{(2\pi)^6 H^2} \frac{4}{\mu(1+4\mu^2)} \left[ 1 - \frac{1}{(2\pi)^9} \frac{3}{8} \frac{\Delta_\zeta^2}{c_s^4} \right] e^{-N_k}, \quad (3.80)$$

where we have reinstated  $c_s$ . Again, the non-linear contribution has the same time dependence as the linear one, and is suppressed by the exact same amount, see Eqs. (3.68) and (3.72). It is thus interesting to notice that the recoherence mechanism identified in Ref. [52] seem robust to loop corrections. Since non-linear interactions mix all Fourier modes, hence substantially enlarge the size of the effective environment compared to linear interactions that only mix Fourier modes of the same wavevector, this may not be obvious a priori and constitutes an important verification. The heavy-environment case is displayed in green in Fig. 14.

## 4 Outlook

We have shown how quantum purity (also known as linear entropy), although not an observable, can be computed perturbatively using in-in methods. We have applied our formalism to a model



of cosmological interest and found that under conditions that we identified, the perturbative calculation is indeed under control, and provides a relevant framework to investigate quantum decoherence of cosmological fluctuations. Let us now mention a few prospects our work opens, both conceptual and phenomenological, before presenting concluding remarks.

#### 4.1 Scaling dimensions

Beyond the specific model we have considered, a generic question of interest consists in identifying the interactions leading to late-time decoherence of the curvature perturbation. Let us consider Eq. (3.52) again, which we reproduce here in the case where the interaction strength can depend on time explicitly

$$\frac{dS_2(k)}{d\eta} \simeq \frac{8}{(2\pi)^6} g(\eta) \int_{-\infty}^{\eta} d\eta' g(\eta') \Re \left[ \bar{\mathcal{K}}_S^{(2)}(k, \eta, \eta') \bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(k, \eta, \eta') \right]. \quad (4.1)$$

Let us assume that  $g$  evolves in time as  $g \propto a^{\Delta_g}$ , where  $\Delta_g$  is the scaling dimension of the coupling. Our goal is to determine the scaling dimension of the decoherence rate, *i.e.* the quantity  $\Delta_s$  such that

$$\frac{dS_2}{d\eta} \propto a^{\Delta_s} \quad \text{at late time.} \quad (4.2)$$

If  $\Delta_s < 0$ , the purity asymptotes a constant at late time, while if  $\Delta_s \geq 0$ , the linear entropy keeps increasing at late time and decoherence takes place.

We parametrize the late-time (or “near-boundary” in the language of Refs. [166, 167]) scaling behavior of the operators appearing in the interaction Hamiltonian  $\tilde{H}_{\text{int}}(\eta)$  with

$$\tilde{\mathcal{O}}_{\mathbf{k}}^{S/\mathcal{E}}(\eta) \rightarrow a^{\Delta_+^{S/\mathcal{E}}} \tilde{\mathcal{O}}_+^{S/\mathcal{E}} \quad (4.3)$$

where  $\tilde{\mathcal{O}}_+^{S/\mathcal{E}}$  are time-independent up to  $\mathcal{O}(a^{-1})$  corrections. Assuming that the late-time behavior of the entropy production is controlled by the late-time scaling of the two-point functions of the system and the environment,

$$\bar{\mathcal{K}}_S^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \rightarrow [a(\eta_1)a(\eta_2)]^{\Delta_+^S}, \quad (4.4)$$

$$\bar{\mathcal{K}}_{\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \rightarrow [a(\eta_1)a(\eta_2)]^{\Delta_+^{\mathcal{E}}}, \quad (4.5)$$

we obtain

**Scaling dimension of entropy production:**

$$\Delta_s = 2\Delta_+^S + 2\Delta_+^{\mathcal{E}} + 2\Delta_g - 1. \quad (4.6)$$

Let us consider a few example of interests. For the model discussed in Sec. 3, the system is described by a massless field, and by expanding Eq. (3.30) at late time one finds  $\pi \propto \text{constant} + \eta^2$ , hence  $\Delta^\pi = 0$  and  $\Delta^{\pi'} = -1$ . The environment consists of a test field with mass  $m_\sigma$ , for which by expanding Eq. (3.31) at late time one finds [166, 167]

$$\Delta_+^{\mathcal{E}} = \Re \left( \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \right) - \frac{3}{2}. \quad (4.7)$$

There are three interaction terms in Eq. (3.8), which come with coupling scaling-dimensions  $\Delta_g^{\pi'\sigma} = 3$ ,  $\Delta_g^{(\partial_i\pi)^2\sigma} = 2$  and  $\Delta_g^{\pi'^2\sigma} = 2$  respectively. With Eq. (4.6) one thus finds

$$\begin{aligned}\Delta_s^{\pi'\sigma} &= \Delta_s^{(\partial_i\pi)^2\sigma} \equiv \Delta_s = 3 + 2\Delta_+^{\mathcal{E}}, \\ \Delta_s^{\pi'^2\sigma} &= -1 + 2\Delta_+^{\mathcal{E}}.\end{aligned}\tag{4.8}$$

Several comments are in order. First, the  $\pi'^2\sigma$  interaction becomes negligible at late time since its scaling dimension is systematically smaller than for the other interaction terms. This is consistent with the results of Sec. 3 where  $\mathcal{C}_1$  depends on  $\Lambda_1$  but not on  $\Lambda_2$ , see Eq. (3.60). Second, the scaling dimension of the non-linear contributions coincide with the one of the linear term, which is again consistent with the results of Sec. 3, see Eqs. (3.68), (3.72) and (3.80). Third, for  $m = 0$  one obtains  $\Delta_s = 3$  which is compatible with Eq. (3.68), for  $m = \sqrt{2}H$  (conformal environment) one obtains  $\Delta_s = 1$  which is compatible with Eq. (3.72), and for  $m > 3H/2$  one finds  $\Delta_s = 0$ , which is again compatible with Eq. (3.80), recalling that  $\partial_\eta = a\partial_N$ .

More than a mere consistency check of the Sec. 3, this shows that the rate at which decoherence or recoherence proceeds at late time can be readily anticipated from scaling arguments, which proves particularly convenient.<sup>10</sup> Another example of the scaling argument is the case where, instead of letting the curvature perturbation interact with a scalar hidden sector, one considers environments made of higher-spin particles, like for instance the tensor modes  $\gamma_{ij}$  of General Relativity [54]. Applying the above procedure to Maldacena's seminal cubic action derived in Ref. [122], we observe that among the tensor-scalar-scalar interaction terms, only  $\gamma_{ij}(\partial_i\zeta)(\partial_j\zeta)$  leads to late-time decoherence of the curvature perturbations with  $\Delta_s = 3$ . Similarly, in the scalar-tensor-tensor sector, the only late-time decohering operator is  $\zeta(\partial_\ell\gamma_{ij})(\partial_\ell\gamma_{ij})$  with  $\Delta_s = 3$ . Both of these contributions are  $\epsilon_1^2$  suppressed and lead to decoherence as found in Ref. [54].<sup>11</sup> Importantly, we note that interactions generating late-time decoherence are not so common.<sup>12</sup> One could also apply similar techniques to spin 1/2 (see e.g. Ref. [168]) and spin 1 (see e.g. Refs. [169–172]) environments, which we leave for future work.

Finally, let us stress that the scaling dimension may not only allow one to determine whether decoherence takes place at late time or not, it may also determine whether quantum features get erased or not. Indeed, in Refs. [13, 120], it was found that due to the competition between decoherence and quantum squeezing at large scales, quantum discord still grows at late time if decoherence proceeds slow enough, i.e.  $\gamma_k \propto a^{-p}$  with  $p < 4$ . This corresponds to a scaling dimension  $\Delta_s < 5$ . Three cases can therefore be encountered:

#### Scaling dimension and quantum-to-classical transition.

- $\Delta_s < 0$ : purity asymptotes a constant (possibly with recoherence) and decoherence does not take place.
- $0 \leq \Delta_s < 5$ : decoherence takes place but quantum signatures remain (quantum

<sup>10</sup>Note that there are cases where the late-time behavior of the purity is not only controlled by the late-time scaling (4.3) and the scaling argument fails. For instance, in Ref. [102], only light environments satisfy the scaling argument but heavy environments do not.

<sup>11</sup>Resumming secular effects can sometimes change the scaling and this is why Ref. [54] rather finds  $\Delta_s = 4$  at late times using a Markovian master equation.

<sup>12</sup>If instead of considering a bipartition in terms of quantum numbers (mass, spin), one considers a bipartition in terms of a momentum cutoff  $k_{UV}$  [54, 75], one could also investigate EFToI self-interactions  $\pi^3$  and  $(\partial_i\pi)^2\pi'$ , which would naively lead to scaling dimensions  $\Delta_s = -6$  and  $\Delta_s = -2$  respectively, hence no late-time decoherence of the long-wavelength modes.

discord still increases).

- $\Delta_s \geq 5$ : decoherence takes place and quantum signatures are erased.

## 4.2 Kraus measurement

We have seen in Sec. 2.6 that the  $n^{\text{th}}$ -order correction to the total density matrix  $\tilde{\rho}^{(n)}$  can be written under the form (2.59). This brings our perturbative framework to a language that is close to the one of *quantum operations* or *dynamical maps* [62], where the evolution of the system is written as

$$\tilde{\rho}^{(n)}(\eta) = \sum_{m \in \mathcal{M}} \Phi_m (|\text{BD}\rangle \langle \text{BD}|). \quad (4.9)$$

Here,  $\mathcal{M}$  is the set of possible outcomes or a generalized measurement. The maps  $\Phi_m$  are completely positive and trace preserving if and only if there exists a countable set of operators  $\Omega_{mk}$  fulfilling

$$\sum_k \Omega_{mk}^\dagger \Omega_{mk} \leq 1, \quad (4.10)$$

such that the maps act on the initial vacuum state according to [173]

$$\Phi_m (|\text{BD}\rangle \langle \text{BD}|) = \sum_k \Omega_{mk} |\text{BD}\rangle \langle \text{BD}| \Omega_{mk}^\dagger. \quad (4.11)$$

From Eq. (2.59), it should be possible to relate the  $\Omega_{mk}$  operators to the ones appearing in the Kraus representation of Sec. 2.6.1, *i.e.* to the system and environment unequal-time correlators. In this way, the CPTP constraint (4.10) should give rise to a set of conditions on these correlators. Those might be interpretable in terms of statistical inequalities that would guarantee the existence of underlying well-defined distributions, hence the CPTP nature of the dynamical map, along the lines of Refs. [174, 175]. This would be similar in spirit to the cosmological bootstrap program (see Ref. [176] for a recent review) where unitarity is used to constrain the structure of the correlators [116, 177–179]. In the absence of unitarity, other symmetries can be used to constrain the effective theory, see Refs. [95, 180], and this picture would certainly benefit from CPTP constraints, the exploration of which is left for future work.

## 5 Conclusions

The extent to which cosmological fluctuations decohere in the early universe is key to characterizing the strength of possible signatures of their quantum origin, as well as investigating the imprint of environmental degrees of freedom in general. However, the use of standard in-in techniques to compute decoherence measures, such as quantum purity, is not straightforward for two reasons. First, in-in methods are typically developed for unitary theories, whereas cosmological fluctuations evolve non-unitarily when coupled to an environment and should therefore be treated as an open quantum system. Second, purity and entanglement entropy cannot be simply expressed as the expectation value of observable operators. Consequently, standard perturbative in-in techniques must be adapted to compute these quantum information measures.

The goal of this work is to develop such a framework. We have shown how, at leading order in a perturbative expansion in the interaction strength, purity can be expressed in terms

of the convolution of unequal time correlators of the system and the environment. For linear interactions, the Fourier modes of the system decouple and they remain in a Gaussian state. The amount by which each Fourier sector decoheres is decided by the determinant of its covariance matrix, hence the two-point functions of the system entirely determine its quantum state, as expected.

In the presence of non-linear interactions, the Fourier modes cannot be treated separately but we have introduced the concept of *spectral purity*, in order to quantify the contribution of each scale to the overall entropy of the field. We have also found that, when the interaction is quadratic in the system’s field variables, the spectral purity is still given by the determinant of the covariance matrix (with an additional factor of  $1/2$ ). Technically, this is because the perturbations of the covariance matrix determinant and the particle number expectation value are related to one another. In practice, this means that even when the system is in a non-Gaussian state, its purity is still determined by its Gaussian moments at leading order, provided the interaction is quadratic in its field variables. In the case where both linear and quadratic interactions are present, non-linearities appear through loop corrections to the power spectra at higher order in perturbation theory.

We have cast our perturbative framework in terms of a diagrammatic expansion presented in Sec. 2.6.2, which involves unequal time correlators represented by *blobs*. One has to combine this diagrammatic approach to the usual in-in techniques [113] to compute the blobs. The benefit of this formulation is however to single out the contributions that are effectively contributing to a change in the purity. If one had used an in-in approach from the get-go, a much larger number of diagrams would have had to be considered. This highlights how our formalism can be used in synergy with the usual in-in techniques.

We then applied our formalism to a model of cosmological interest, where adiabatic perturbations couple to entropic degrees of freedom during inflation. By considering a subset of interactions within the Effective Field Theory of Inflation (EFToI), we were able to explore a non-linear extension of the results presented in Ref. [52].

We have found that the perturbative expansion breaks down at early time, but this is also where the EFT is expected to become invalid. At late times however, our framework can be trusted and it indicates that loop corrections, *i.e.* the effects of the considered non-linear interaction terms, are suppressed by  $\Delta_\zeta^2/c_s^4$ , where  $\Delta_\zeta^2$  is the amplitude of the curvature power spectrum at the scale of interest, and  $c_s$  is the effective speed of sound. When the environment is massless, we show that linear entropy grows as  $a^2$ , leading to rapid decoherence after Hubble crossing. For an environment that is conformally coupled, linear entropy grows as  $\ln^2 a$ , resulting in slower decoherence. Finally when the environment is heavy compared to the Hubble scale, the system experiences a transient phase of limited decoherence before recohering. In this case, the purity asymptotes to a constant at late times, remaining exponentially close to unity.

Note that this phenomenon of recoherence was observed in Ref. [52], but doubts arose in the community whether this was merely a result of considering a bilinear mixing  $\pi'\sigma$  between system and environment, raising concerns about the persistence of recoherence in the presence of non-linear interactions. The heuristic reason is the following. Recoherence is a clear symptom of non-Markovianity, while Markovian behavior is usually expected when the environment is “large enough”. In the case of linear interactions, the system (*i.e.* a given Fourier mode of the curvature perturbation) couples to a single degree of freedom (*i.e.* the same Fourier mode of the isocurvature field), and this is why observing non-Markovian signatures is maybe not so surprising. However, in the presence of non-linear interactions, the system couples to *all* Fourier modes of the environment, which might lead one to expect primarily

Markovian evolution. Nevertheless, we have demonstrated that recoherence persists for the class of non-linear operators considered in this work. This indicates a robustness to loop effects, and our discussion of scaling dimensions reveals that far from being an academic curiosity, recoherence is in fact expected in a large number of cases.

The line of research presented in this work could be extended in various directions. First, the formalism we developed is generic and one may want to apply it to other cosmological and non-cosmological setups (see *e.g.* Refs. [49, 50, 58] in the context of black holes and Ref. [96] for holography). When doing so, one might encounter situations in which non-linear self-interactions of the system and/or the environment are at play. In this case, three perturbative expansions should be implemented simultaneously: one in the self-interaction strength of the system  $g_S$ , one in the self-interaction strength of the environment  $g_E$ , and one in the system-environment coupling  $g$ . If  $g_S, g_E \ll g$  (in the right units), the  $g_S$  and  $g_E$  expansions should be performed first, and the framework presented in this work can still be employed, where the self-interaction terms only give rise to corrections in the system and environment unequal-time correlators. In the opposite regime however, a different expansion scheme might be required.

It may also happen that the relevant system and environment bipartition is not defined in terms of fields but rather in terms of different regions in physical space or different mode intervals of a given field. In that case, although the perturbative expansion should follow a similar structure as the one developed in this work, it might need to be generalised if the interaction Hamiltonian is not of the local form (2.22).

Second, one could consider alternative measures of decoherence and quantum entanglement, such as entanglement entropy or logarithmic negativity. At perturbative order, these measures can be simply interconnected, as discussed in Sec. 2.7, but they may deliver different outcomes upon resummation.

Third, as divergences often appear in perturbative computations [181, 182], the renormalization of open quantum systems in the field theoretic context might also require further investigations (see *e.g.* Refs. [183, 184]). In particular, we have found that a divergent contribution to the entropy arises from early-time interactions in the model we considered [this is the term proportional to  $\mathcal{C}_2$  in Eq. (3.59)]. When considering the entropy production rate this term drops out, hence at late time where the EFT become valid it plays no role, but the entropy rate still involves an integral over time, see Eq. (3.52). How early-time interactions decouple from late-time dynamics is therefore not obvious in general. A better understanding of divergences and renormalization in open quantum systems – potentially informed by recent studies of finite-dimensional or harmonic oscillator open systems, such as those in Refs. [185–187] – may be necessary to determine more clearly whether and how a theory decouples at small scales.

Fourth, one might speculate that just as two-point functions characterize purity even for non-linear quadratic interactions at leading order, higher-order statistics like the bispectrum or trispectrum could potentially characterize purity at higher orders in the coupling or in the presence of higher-order interactions.

Fifth and finally, now that the problem is framed in terms of (equal or unequal-time) correlators, further investigation into the role of symmetries, UV unitarity, and locality in the evolution of the system’s quantum information properties would be valuable. We leave this exploration for future work.

## Acknowledgements:

We thank C. Burgess, C. Duaso Pueyo, R. Holman and E. Pajer for useful discussions. This work is supported by STFC consolidated grant ST/X001113/1, ST/T000791/1, ST/T000694/1, ST/X000664/1 and EP/V048422/1.

## A Perturbative Gaussianity

In the case of linear interactions, the system remains in a Gaussian state, hence it is fully characterized by its covariance matrix

$$\delta(\mathbf{0}) \mathbf{Cov}_{ij}(k, \eta) = \frac{1}{2} \left\langle \tilde{z}_i(\mathbf{k}, \eta) \tilde{z}_j^\dagger(\mathbf{k}, \eta) + \tilde{z}_j(\mathbf{k}, \eta) \tilde{z}_i^\dagger(\mathbf{k}, \eta) \right\rangle, \quad (\text{A.1})$$

see Eq. (2.49). Here,  $\tilde{\mathbf{z}}(\mathbf{k}, \eta)$  is a phase-space vector gathering the phase-space variables describing the  $\mathbf{k}$  Fourier mode of the system. For explicitness, we will consider the case where the system is made of a single scalar degree of freedom denoted  $\pi$ , hence  $\tilde{\mathbf{z}} = [\tilde{v}_\pi(\mathbf{k}, \eta), \tilde{p}_\pi(\mathbf{k}, \eta)]^T$  contains a position operator,  $\tilde{v}_\pi(\mathbf{k}, \eta)$ , and its conjugate momentum,  $\tilde{p}_\pi(\mathbf{k}, \eta)$ . The quantum state for the system factorizes between different Fourier modes,  $\tilde{\rho}_{\text{red}} = \bigotimes_{\mathbf{k} \in \mathbb{R}^{3+}} \tilde{\rho}_{\text{red}}(\mathbf{k})$ , see Sec. 2.3, and the purity for each Fourier mode is fully determined by the determinant of the covariance matrix,

$$S_2(k) = \ln [4 \det \mathbf{Cov}(k, \eta)], \quad (\text{A.2})$$

see Sec. 2.4. At order  $g^2$ , this gives rise to

$$S_2^{(2)}(k) = 4 \det^{(2)} \mathbf{Cov}(k, \eta). \quad (\text{A.3})$$

When interactions are non linear, these considerations do not apply, and the purity receives contribution from all  $n$ -point correlation functions. In this appendix, our goal is to show that at order  $g^2$ , for interactions that are quadratic in the system's phase-space variables, only the two-point functions contribute, hence Eq. (A.3) remains valid (with an additional factor 2) even though the system is not in a Gaussian state anymore.

Without loss of generality, we consider the case where the operators appearing in the interaction Hamiltonian are centred, see the discussion below Eq. (2.27), hence the system is not modified at order  $g$  and the first environmental effects appear at order  $g^2$ . At this order, one has

$$\det^{(2)} \mathbf{Cov} = \mathbf{Cov}_{11}^{(2)} \mathbf{Cov}_{22}^{(0)} + \mathbf{Cov}_{22}^{(2)} \mathbf{Cov}_{11}^{(0)} - 2 \mathbf{Cov}_{12}^{(2)} \mathbf{Cov}_{12}^{(0)}. \quad (\text{A.4})$$

In the free theory, the phase-space operators can be expanded onto creation and annihilation operators by means of the free, Bunch-Davies normalized mode functions  $\mathbf{z}(k, \eta)$ , according to

$$\tilde{\mathbf{z}}(\mathbf{k}, \eta) = \mathbf{z}(k, \eta) \hat{c}_{\mathbf{k}} + \mathbf{z}^*(k, \eta) \hat{c}_{-\mathbf{k}}^\dagger, \quad (\text{A.5})$$

where  $\hat{c}_{\mathbf{k}}$  and  $\hat{c}_{\mathbf{k}}^\dagger$  are the annihilation and creation operators and  $\mathbf{z}(k, \eta) = [v_\pi(k, \eta), p_\pi(k, \eta)]^T$  gathers the mode functions. This implies that  $\mathbf{Cov}^{(0)}(k, \eta)$  contains the free power spectra and can be expressed in terms of the mode functions as

$$\mathbf{Cov}_{ij}^{(0)}(k, \eta) = \frac{1}{2} [z_i(k, \eta) z_j^*(k, \eta) + z_i^*(k, \eta) z_j(k, \eta)]. \quad (\text{A.6})$$

The entries of  $\mathbf{Cov}^{(2)}$  correspond to expectation values computed against  $\tilde{\rho}_{\text{red}}^{(2)}$ . For a generic quantum operator  $\widehat{Q}$ , one has

$$\langle \widehat{Q}(\eta) \rangle^{(2)} \equiv \text{Tr} \left[ \tilde{Q}(\eta) \tilde{\rho}_{\text{red}}^{(2)}(\eta) \right] \quad (\text{A.7})$$

where  $\tilde{\rho}_{\text{red}}^{(2)}(\eta)$  is given in Eq. (2.29) that we reproduce here for clarity

$$\begin{aligned} \tilde{\rho}_{\text{red}}^{(2)}(\eta) = & - \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \\ & \left\{ \left[ \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} - \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \right] \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \right. \\ & \left. - \left[ \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) - |\text{BD}\rangle \langle \text{BD}|_{\mathcal{S}} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \right] \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{q}, \mathbf{p}, \eta_2, \eta_1) \right\}. \end{aligned} \quad (\text{A.8})$$

It leads to

$$\begin{aligned} \langle \widehat{Q}(\eta) \rangle^{(2)} = & - \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \\ & \left\{ \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \langle \left[ \tilde{Q}(\eta), \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \right] \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle \right. \\ & \left. - \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{q}, \mathbf{p}, \eta_2, \eta_1) \langle \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \left[ \tilde{Q}(\eta), \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \right] \rangle \right\} \end{aligned} \quad (\text{A.9})$$

where brackets represent vacuum expectation values in the initial vacuum state.

In practice, the determinant in Eq. (A.4) can be obtained from Eq. (A.9) by replacing

$$\begin{aligned} \tilde{Q}(\eta) = & \mathbf{Cov}_{22}^{(0)}(k, \eta) \tilde{v}_{\pi}(\mathbf{k}, \eta) \tilde{v}_{\pi}(-\mathbf{k}, \eta) + \mathbf{Cov}_{11}^{(0)}(k, \eta) \tilde{p}_{\pi}(\mathbf{k}, \eta) \tilde{p}_{\pi}(-\mathbf{k}, \eta) \\ & - \mathbf{Cov}_{12}^{(0)}(k, \eta) [\tilde{v}_{\pi}(\mathbf{k}, \eta) \tilde{p}_{\pi}(-\mathbf{k}, \eta) + \tilde{p}_{\pi}(\mathbf{k}, \eta) \tilde{v}_{\pi}(-\mathbf{k}, \eta)]. \end{aligned} \quad (\text{A.10})$$

Making use of the canonical commutation relations,

$$[\tilde{v}_{\pi}(\mathbf{k}, \eta), \tilde{v}_{\pi}^{\dagger}(\mathbf{k}', \eta)] = [\tilde{p}_{\pi}(\mathbf{k}, \eta), \tilde{p}_{\pi}^{\dagger}(\mathbf{k}', \eta)] = 0, \quad (\text{A.11})$$

$$[\tilde{v}_{\pi}(\mathbf{k}, \eta), \tilde{p}_{\pi}^{\dagger}(\mathbf{k}', \eta)] = i\delta(\mathbf{k} - \mathbf{k}'), \quad (\text{A.12})$$

and recalling that  $\tilde{v}_{\pi}(-\mathbf{k}, \eta) = \tilde{v}_{\pi}^{\dagger}(\mathbf{k}, \eta)$  and  $\tilde{p}_{\pi}(-\mathbf{k}, \eta) = \tilde{p}_{\pi}^{\dagger}(\mathbf{k}, \eta)$ , the above can be rewritten as

$$\begin{aligned} \tilde{Q}(\eta) = & \mathbf{Cov}_{22}^{(0)}(k, \eta) \tilde{v}_{\pi}^{\dagger}(\mathbf{k}, \eta) \tilde{v}_{\pi}(\mathbf{k}, \eta) + \mathbf{Cov}_{11}^{(0)}(k, \eta) \tilde{p}_{\pi}^{\dagger}(\mathbf{k}, \eta) \tilde{p}_{\pi}(\mathbf{k}, \eta) \\ & - \mathbf{Cov}_{12}^{(0)}(k, \eta) \left[ \tilde{p}_{\pi}^{\dagger}(\mathbf{k}, \eta) \tilde{v}_{\pi}(\mathbf{k}, \eta) + \tilde{v}_{\pi}^{\dagger}(\mathbf{k}, \eta) \tilde{p}_{\pi}(\mathbf{k}, \eta) \right]. \end{aligned} \quad (\text{A.13})$$

Let us cast this relationship in matricial form. First, one has

$$\tilde{Q}(\eta) = \frac{1}{4} \tilde{\mathbf{z}}^{\dagger}(\mathbf{k}, \eta) \left[ \mathbf{Cov}^{(0)}(k, \eta) \right]^{-1} \tilde{\mathbf{z}}(\mathbf{k}, \eta), \quad (\text{A.14})$$

where we have used that the inverse of the free covariance matrix can be written as

$$\left[ \mathbf{Cov}^{(0)}(k, \eta) \right]^{-1} = \frac{1}{\det \mathbf{Cov}^{(0)}} \begin{pmatrix} \mathbf{Cov}_{22}^{(0)}(k, \eta) & -\mathbf{Cov}_{12}^{(0)}(k, \eta) \\ -\mathbf{Cov}_{12}^{(0)}(k, \eta) & \mathbf{Cov}_{11}^{(0)}(k, \eta) \end{pmatrix} \quad (\text{A.15})$$

where  $\det \mathbf{Cov}^{(0)} = 1/4$  since in the free theory the system is placed in a pure Gaussian state. Then, we rewrite Eq. (A.5) as

$$\tilde{\mathbf{z}}(\mathbf{k}, \eta) = \mathbf{M}(k, \eta) \cdot \hat{\mathbf{c}}_{\mathbf{k}} \quad \text{where} \quad \mathbf{M}(k, \eta) = \begin{pmatrix} v_{\pi}(k, \eta) & v_{\pi}^*(k, \eta) \\ p_{\pi}(k, \eta) & p_{\pi}^*(k, \eta) \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{c}}_{\mathbf{k}} = \begin{pmatrix} \hat{c}_{\mathbf{k}} \\ \hat{c}_{-\mathbf{k}}^{\dagger} \end{pmatrix}. \quad (\text{A.16})$$

The creation and annihilation operators obey the commutation relations

$$[\hat{c}_{\mathbf{k}}, \hat{c}_{\mathbf{k}'}^{\dagger}] = \delta(\mathbf{k} - \mathbf{k}') \quad \text{and} \quad [\hat{c}_{\mathbf{k}}, \hat{c}_{\mathbf{k}'}] = [\hat{c}_{\mathbf{k}}^{\dagger}, \hat{c}_{\mathbf{k}'}^{\dagger}] = 0 \quad (\text{A.17})$$

hence for Eqs. (A.11) and (A.12) to be satisfied the free mode functions must be such that

$$v_{\pi}(k, \eta)p_{\pi}^*(k, \eta) - v_{\pi}^*(k, \eta)p_{\pi}(k, \eta) = i. \quad (\text{A.18})$$

Using this relation repeatedly, one can readily show that

$$\mathbf{M}^{\text{T}*}(k, \eta) \cdot [\mathbf{Cov}^{(0)}(k, \eta)]^{-1} \cdot \mathbf{M}(k, \eta) = 2 \mathbf{Id}_2, \quad (\text{A.19})$$

where  $\mathbf{Id}_2$  is the  $2 \times 2$  identity matrix. As a consequence,

$$\tilde{Q}(\eta) = \frac{1}{2} \hat{\mathbf{c}}_{\mathbf{k}}^{\dagger} \hat{\mathbf{c}}_{\mathbf{k}} = \frac{1}{2} \left( \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + \hat{c}_{-\mathbf{k}} \hat{c}_{-\mathbf{k}}^{\dagger} \right), \quad (\text{A.20})$$

hence  $\tilde{Q}(\eta)$  is in fact independent of  $\eta$  and is directly related to the particle-number operator. Inserting Eq. (A.20) into Eq. (A.9), one thus obtains

$$\begin{aligned} \det^{(2)} \mathbf{Cov} = & -\frac{1}{2} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \\ & \left\{ \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \left\langle \left[ \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + \hat{c}_{-\mathbf{k}} \hat{c}_{-\mathbf{k}}^{\dagger}, \tilde{O}_{\mathbf{p}}^S(\eta_1) \right] \tilde{O}_{\mathbf{q}}^S(\eta_2) \right\rangle \right. \\ & \left. - \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{q}, \mathbf{p}, \eta_2, \eta_1) \left\langle \tilde{O}_{\mathbf{q}}^S(\eta_2) \left[ \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + \hat{c}_{-\mathbf{k}} \hat{c}_{-\mathbf{k}}^{\dagger}, \tilde{O}_{\mathbf{p}}^S(\eta_1) \right] \right\rangle \right\}. \end{aligned} \quad (\text{A.21})$$

Recalling that  $\mathcal{K}_{S/\mathcal{E}}^{(2)}(-\mathbf{k}_2, -\mathbf{k}_1, \eta_2, \eta_1) = [\mathcal{K}_{S/\mathcal{E}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2)]^*$ , and using the fact that the  $\tilde{O}^S(\mathbf{x})$  operators are Hermitian, hence  $\tilde{O}_{-\mathbf{k}}^S = (\tilde{O}_{\mathbf{k}}^S)^{\dagger}$ , the two terms appearing in the above expression are complex conjugate of each other, hence one can write

$$\begin{aligned} \det^{(2)} \mathbf{Cov} = & -\int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \\ & \Re \left\{ \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \left\langle \left[ \hat{c}_{\mathbf{k}}^{\dagger} \hat{c}_{\mathbf{k}} + \hat{c}_{-\mathbf{k}} \hat{c}_{-\mathbf{k}}^{\dagger}, \tilde{O}_{\mathbf{p}}^S(\eta_1) \right] \tilde{O}_{\mathbf{q}}^S(\eta_2) \right\rangle \right\}. \end{aligned} \quad (\text{A.22})$$

Upon introducing the particle number operator

$$\hat{N} = \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \hat{c}_{\mathbf{k}} \hat{c}_{\mathbf{k}}^{\dagger}, \quad (\text{A.23})$$

the above leads to

$$\int \frac{d^3 \mathbf{k}}{(2\pi)^3} \det^{(2)} \mathbf{Cov} = -2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3}$$



$$\Re \left\{ \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \left\langle \left[ \widehat{N}, \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \right] \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle \right\}. \quad (\text{A.24})$$

Since  $\widehat{N}$  vanishes when acted on the vacuum, this finally gives

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} \det^{(2)} \mathbf{Cov} = 2 \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \Re \left\{ \mathcal{K}_{\mathcal{E}}^{(2)}(\mathbf{p}, \mathbf{q}, \eta_1, \eta_2) \left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widehat{N} \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle \right\}. \quad (\text{A.25})$$

Our next task is to compute the correlator appearing in the above expression. For explicitness, let us assume that  $\widetilde{\mathcal{O}}^S$  is of order  $n$  in the phase-space operators,

$$\widetilde{\mathcal{O}}^S(\mathbf{x}, \eta) = \mathcal{C}_{\alpha_1 \dots \alpha_n}(\eta) \widetilde{z}_{\alpha_1}(\mathbf{x}, \eta) \dots \widetilde{z}_{\alpha_n}(\mathbf{x}, \eta), \quad (\text{A.26})$$

where  $\alpha_i \in \{1, 2\}$  and  $\mathcal{C}_{\alpha_1 \dots \alpha_n}$  are free coefficients, so when Fourier expanding the above expression  $\widetilde{\mathcal{O}}_{\mathbf{p}}^S$  is of the form

$$\widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) = (2\pi)^3 \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \dots \int \frac{d^3\mathbf{p}_n}{(2\pi)^3} \mathcal{C}_{\alpha_1 \dots \alpha_n}(\eta_1) \widetilde{z}_{\alpha_1}(\mathbf{p}_1, \eta_1) \dots \widetilde{z}_{\alpha_n}(\mathbf{p}_n, \eta_1) \delta(\mathbf{p}_1 + \dots + \mathbf{p}_n - \mathbf{p}). \quad (\text{A.27})$$

As mentioned above, we consider the case where  $\langle \widetilde{\mathcal{O}}_{\mathbf{k}}^S(\eta) \rangle = 0$ . This implies that, when  $n$  is even, the mean value is implicitly subtracted from the above expression. Let us now investigate the first values of  $n$  to gain more intuition on the above results.

- $n = 1$

When  $n = 1$ , one has

$$\begin{aligned} \left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widehat{N} \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} C_{\alpha_1}(\eta_1) C_{\alpha_2}(\eta_2) \\ &\left\langle \left[ z_{\alpha_1}(\mathbf{p}, \eta_1) \widehat{c}_{\mathbf{p}} + z_{\alpha_1}^*(\mathbf{p}, \eta_1) \widehat{c}_{-\mathbf{p}}^\dagger \right] \widehat{N}_{\mathbf{k}} \left[ z_{\alpha_2}(\mathbf{q}, \eta_2) \widehat{c}_{\mathbf{q}} + z_{\alpha_2}^*(\mathbf{q}, \eta_2) \widehat{c}_{-\mathbf{q}}^\dagger \right] \right\rangle, \end{aligned} \quad (\text{A.28})$$

where  $\widehat{N}_{\mathbf{k}} = \widehat{c}_{\mathbf{k}} c_{\mathbf{k}}^\dagger$ . The only non-vanishing contribution arises from creating a particle with  $\widehat{c}_{-\mathbf{q}}^\dagger$  and destroying it with  $\widehat{c}_{\mathbf{p}}$ , so one finds

$$\begin{aligned} \left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widehat{N} \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} C_{\alpha_1}(\eta_1) C_{\alpha_2}(\eta_2) z_{\alpha_1}(\mathbf{p}, \eta_1) z_{\alpha_2}^*(\mathbf{q}, \eta_2) \left\langle \widehat{c}_{\mathbf{p}} \widehat{N}_{\mathbf{k}} \widehat{c}_{-\mathbf{q}}^\dagger \right\rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} C_{\alpha_1}(\eta_1) C_{\alpha_2}(\eta_2) z_{\alpha_1}(\mathbf{p}, \eta_1) z_{\alpha_2}^*(\mathbf{q}, \eta_2) \left\langle \widehat{c}_{\mathbf{p}} \widehat{c}_{-\mathbf{q}}^\dagger \right\rangle \delta(\mathbf{k} + \mathbf{q}) \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} C_{\alpha_1}(\eta_1) C_{\alpha_2}(\eta_2) z_{\alpha_1}(\mathbf{p}, \eta_1) z_{\alpha_2}^*(\mathbf{q}, \eta_2) \delta(\mathbf{p} + \mathbf{q}) \delta(\mathbf{k} + \mathbf{q}) \\ &= C_{\alpha_1}(\eta_1) C_{\alpha_2}(\eta_2) z_{\alpha_1}(\mathbf{p}, \eta_1) z_{\alpha_2}^*(\mathbf{q}, \eta_2) \frac{\delta(\mathbf{p} + \mathbf{q})}{(2\pi)^3}. \end{aligned} \quad (\text{A.29})$$

One can readily see that a similar calculation gives the same result for  $\left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle$ , up to the factor  $(2\pi)^{-3}$ , hence

$$\left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widehat{N} \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle = (2\pi)^{-3} \left\langle \widetilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \widetilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \right\rangle = (2\pi)^{-3} \bar{K}_S^{(2)}(\mathbf{p}, \eta_1, \eta_2) \delta(\mathbf{p} + \mathbf{q}), \quad (\text{A.30})$$

see Eqs. (2.31) and (2.46). Inserting this formula into Eq. (A.25), one has

$$\int d^3\mathbf{k} \det^{(2)}\mathbf{Cov} = 2(2\pi)^{-6} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta_1} d\eta_2 \int d^3\mathbf{p} \Re e \left[ \bar{K}_{\mathcal{E}}^{(2)}(p, \eta_1, \eta_2) \bar{K}_{\mathcal{S}}^{(2)}(p, \eta_1, \eta_2) \right], \quad (\text{A.31})$$

where we have used Eq. (2.46) again. By comparison with Eq. (2.48), this can be written as

$$\int d^3\mathbf{k} \det^{(2)}\mathbf{Cov} = \frac{1}{4} \int d^3\mathbf{k} S_2^{(2)}(k), \quad (\text{A.32})$$

hence

$$S_2^{(2)}(k) = 4\det^{(2)}\mathbf{Cov}, \quad (\text{A.33})$$

which is the result announced, see Eq. (A.3). When  $n = 1$ , as explained in Sec. 2.4 the system remains in a Gaussian state and this result is already known, it also holds at all orders in  $g$ . The above derivation is however a consistency check of the validity of the present formalism.

- $n = 2$

When  $n = 2$ , one has to compute

$$\begin{aligned} \langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \hat{N} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} \int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \int \frac{d^3\mathbf{q}_2}{(2\pi)^3} \\ &\quad \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}) \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}) C_{\alpha_1\alpha_2}(\eta_1) C_{\beta_2\beta_2}(\eta_2) \\ &\quad \left\langle \left[ z_{\alpha_1}(p_1, \eta_1) \hat{c}_{\mathbf{p}_1} + z_{\alpha_1}^*(p_1, \eta_1) \hat{c}_{-\mathbf{p}_1}^\dagger \right] \left[ z_{\alpha_2}(p_2, \eta_1) \hat{c}_{\mathbf{p}_2} + z_{\alpha_2}^*(p_2, \eta_1) \hat{c}_{-\mathbf{p}_2}^\dagger \right] \hat{N}_{\mathbf{k}} \right. \\ &\quad \left. \left[ z_{\beta_1}(q_1, \eta_1) \hat{c}_{\mathbf{q}_1} + z_{\beta_1}^*(q_1, \eta_2) \hat{c}_{-\mathbf{q}_1}^\dagger \right] \left[ z_{\beta_2}(q_2, \eta_1) \hat{c}_{\mathbf{q}_2} + z_{\beta_2}^*(q_2, \eta_2) \hat{c}_{-\mathbf{q}_2}^\dagger \right] \right\rangle. \end{aligned} \quad (\text{A.34})$$

Amongst the several terms arising from developing the square brackets, the only one that is not annihilated by the particle-number operator is the one involving  $\hat{c}_{\mathbf{p}_1} \hat{c}_{\mathbf{p}_2} \hat{N}_{\mathbf{k}} \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger$ . Making use of the commutation relations (A.17), one has

$$\left[ \hat{N}_{\mathbf{k}}, \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger \right] = \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger [\delta(\mathbf{k} + \mathbf{q}_1) + \delta(\mathbf{k} + \mathbf{q}_2)]. \quad (\text{A.35})$$

Upon integrating over  $\mathbf{k}$ , one thus finds

$$\begin{aligned} \langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \hat{N} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle &= 2(2\pi)^{-3} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} \int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \int \frac{d^3\mathbf{q}_2}{(2\pi)^3} \\ &\quad \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}) \delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q}) C_{\alpha_1\alpha_2}(\eta_1) C_{\beta_2\beta_2}(\eta_2) \\ &\quad z_{\alpha_1}(p_1, \eta_1) z_{\alpha_2}(p_2, \eta_1) z_{\beta_1}^*(q_1, \eta_2) z_{\beta_2}^*(q_2, \eta_2) \left\langle \hat{c}_{\mathbf{p}_1} \hat{c}_{\mathbf{p}_2} \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger \right\rangle. \end{aligned} \quad (\text{A.36})$$

This coincides with the expression one obtains for  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$ , up to two differences. First, Eq. (A.36) contains an additional factor 2, which is coming from the two terms appearing in the right-hand side of Eq. (A.35). In other words, since two particles are created and annihilated, the presence of the particle-number operator yields a factor 2. Second, in  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$  there are additional terms, for instance the one involving  $\hat{c}_{\mathbf{p}_1} \hat{c}_{-\mathbf{p}_2}^\dagger \hat{c}_{\mathbf{q}_1} \hat{c}_{-\mathbf{q}_2}^\dagger$ , which do not appear in Eq. (A.36) since they are annihilated by the particle-number operator. However, these terms contribute to  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \rangle \langle \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$ , which is subtracted from the

final result, see the remark below Eq. (A.27). From these considerations one concludes that  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \hat{N} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle = 2(2\pi)^{-3} \langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$ . The rest of the calculation is identical to the case  $n = 1$ , the factor 2 being the only difference, and one thus finds

$$2S_2^{(2)}(k) = 4\det^{(2)}\mathbf{Cov}, \quad (\text{A.37})$$

where the factor 2 has been singled out in the left-hand side. Compared to Eq. (A.33), spectral purity is thus half of what it would be if the state was Gaussian, with the same covariance.

- $n \geq 3$

When  $n = 3$ , one has to compute

$$\begin{aligned} \langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \hat{N} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} \int \frac{d^3\mathbf{p}_3}{(2\pi)^3} \int \frac{d^3\mathbf{q}_1}{(2\pi)^3} \int \frac{d^3\mathbf{q}_2}{(2\pi)^3} \int \frac{d^3\mathbf{q}_3}{(2\pi)^3} \\ &\quad \delta(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - \mathbf{p}) \delta(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 - \mathbf{q}) C_{\alpha_1\alpha_2\alpha_3}(\eta_1) C_{\beta_2\beta_2\beta_3}(\eta_2) \\ &\quad \left\langle \left[ z_{\alpha_1}(p_1, \eta_1) \hat{c}_{\mathbf{p}_1} + z_{\alpha_1}^*(p_1, \eta_1) \hat{c}_{-\mathbf{p}_1}^\dagger \right] \left[ z_{\alpha_2}(p_2, \eta_1) \hat{c}_{\mathbf{p}_2} + z_{\alpha_2}^*(p_2, \eta_1) \hat{c}_{-\mathbf{p}_2}^\dagger \right] \right. \\ &\quad \left[ z_{\alpha_3}(p_3, \eta_1) \hat{c}_{\mathbf{p}_3} + z_{\alpha_3}^*(p_3, \eta_1) \hat{c}_{-\mathbf{p}_3}^\dagger \right] \hat{N}_{\mathbf{k}} \left[ z_{\beta_1}(q_1, \eta_1) \hat{c}_{\mathbf{q}_1} + z_{\beta_1}^*(q_1, \eta_2) \hat{c}_{-\mathbf{q}_1}^\dagger \right] \\ &\quad \left. \left[ z_{\beta_2}(q_2, \eta_1) \hat{c}_{\mathbf{q}_2} + z_{\beta_2}^*(q_2, \eta_2) \hat{c}_{-\mathbf{q}_2}^\dagger \right] \left[ z_{\beta_3}(q_3, \eta_1) \hat{c}_{\mathbf{q}_3} + z_{\beta_3}^*(q_3, \eta_2) \hat{c}_{-\mathbf{q}_3}^\dagger \right] \right\rangle. \end{aligned} \quad (\text{A.38})$$

Several terms need to be distinguished. A first term arises from selecting creation operators only in the right-hand side of the particle-number operator. Making use of the commutation relations (A.17), one has

$$\left[ \hat{N}_{\mathbf{k}}, \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger \right] = \hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger [\delta(\mathbf{k} + \mathbf{q}_1) + \delta(\mathbf{k} + \mathbf{q}_2) + \delta(\mathbf{k} + \mathbf{q}_3)]. \quad (\text{A.39})$$

When integrated over  $\mathbf{k}$ , this thus leaves out a factor 3. This is because  $\hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger$  creates a net number of particles equal to 3. A second category of terms arises from selecting two creation operators and one annihilation operator in the right-hand side of the particle-number operator. There are three such terms:  $\hat{c}_{\mathbf{q}_1} \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger$ ,  $\hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{\mathbf{q}_2} \hat{c}_{-\mathbf{q}_3}^\dagger$  and  $\hat{c}_{-\mathbf{q}_1}^\dagger \hat{c}_{-\mathbf{q}_2} \hat{c}_{\mathbf{q}_3}$  (the latter vanishes when acted on the vacuum). For the first term for instance, using the commutation relations (A.17), one finds

$$\left[ \hat{N}_{\mathbf{k}}, \hat{c}_{\mathbf{q}_1} \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger \right] = \hat{c}_{\mathbf{q}_1} \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger [\delta(\mathbf{k} + \mathbf{q}_2) + \delta(\mathbf{k} + \mathbf{q}_3) - \delta(\mathbf{k} + \mathbf{q}_1)]. \quad (\text{A.40})$$

When integrated over  $\mathbf{k}$ , this leaves out a factor 1, in agreement again with the fact that  $\hat{c}_{\mathbf{q}_1} \hat{c}_{-\mathbf{q}_2}^\dagger \hat{c}_{-\mathbf{q}_3}^\dagger$  creates a net number of particles equal to 1. Therefore, these terms come with multiplicities that differ from their counterpart in the correlator  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$ , hence there is no simple relationship between  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \hat{N} \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$  and  $\langle \tilde{\mathcal{O}}_{\mathbf{p}}^S(\eta_1) \tilde{\mathcal{O}}_{\mathbf{q}}^S(\eta_2) \rangle$ . As a consequence, for  $n \geq 3$  the leading-order purity is not only given by  $\det^{(2)}\mathbf{Cov}$ .

## B Unnesting commutators by induction

In this Appendix, we demonstrate how the commutators appearing in Eq. (2.13) can be un-nested as in Eq. (2.58), which we reproduce here for convenience

$$\left[ \tilde{H}_{\text{int}}(\eta_1), \left[ \tilde{H}_{\text{int}}(\eta_2), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i)} \right)^\dagger. \quad (\text{B.1})$$

This identity can be shown by induction as follows. At orders  $n = 0$  and  $n = 1$ , it is trivially satisfied. Let us now assume that it holds at order  $n$ . The commutator between  $\tilde{H}_{\text{int}}(\eta_0)$  and Eq. (B.1) can be written as

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \\ & \left[ \tilde{H}_{\text{int}}(\eta_0)^{(n)} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i)} \right)^\dagger - {}^{(n)} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i)} \right)^\dagger \tilde{H}_{\text{int}}(\eta_0) \right]. \end{aligned} \quad (\text{B.2})$$

In this expression, we have added an upper left index  $n$  to stress that the  $\tilde{\mathcal{K}}_j^{(i)}$  vectors are constructed out of  $\tilde{H}_{\text{int}}(\eta_1), \dots, \tilde{H}_{\text{int}}(\eta_n)$ . If  ${}^{(n+1)} \tilde{\mathcal{K}}_j^{(i)}$  denote the  $\tilde{\mathcal{K}}_j^{(i)}$  vectors constructed out of  $\tilde{H}_{\text{int}}(\eta_0), \dots, \tilde{H}_{\text{int}}(\eta_n)$ , from Eqs. (2.53)-(2.57), the  $\binom{n}{i-1}$  first elements of the  ${}^{(n+1)} \tilde{\mathcal{K}}^{(i)}$  vector contain  $\tilde{H}_{\text{int}}(\eta_0)$  as the first term, and they can thus be obtained by multiplying  $\tilde{H}_{\text{int}}(\eta_0)$  with the  $\binom{n}{i-1}$  elements of the  ${}^{(n)} \tilde{\mathcal{K}}^{(i-1)}$  vector. In other words,

$${}^{(n+1)} \tilde{\mathcal{K}}_j^{(i)} = \tilde{H}_{\text{int}}(\eta_0) {}^{(n)} \tilde{\mathcal{K}}_j^{(i-1)} \quad \text{for } j \leq \binom{n}{i-1}. \quad (\text{B.3})$$

The subsequent elements of the  ${}^{(n+1)} \tilde{\mathcal{K}}^{(i)}$  vector do not contain  $\tilde{H}_{\text{int}}(\eta_0)$ , and they coincide with the  $\binom{n}{i}$  elements of the  ${}^{(n)} \tilde{\mathcal{K}}^{(i)}$  vector, hence

$${}^{(n+1)} \tilde{\mathcal{K}}_{\binom{n}{i-1}+j}^{(i)} = {}^{(n)} \tilde{\mathcal{K}}_j^{(i)} \quad \text{for } j \leq \binom{n}{i}. \quad (\text{B.4})$$

Since  $\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$ , this indeed determines all elements of the  ${}^{(n+1)} \tilde{\mathcal{K}}^{(i)}$  vector, and the two above relations fix the element ordering in Eqs. (2.53)-(2.57). Inserting them into Eq. (B.2) leads to

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \\ & \left[ {}^{(n+1)} \tilde{\mathcal{K}}_j^{(i+1)} |\text{BD}\rangle \langle \text{BD}| \left( {}^{(n+1)} \tilde{\mathcal{K}}_{\binom{n}{i-1}+j}^{(n-i)} \right)^\dagger - {}^{(n+1)} \tilde{\mathcal{K}}_{\binom{n}{i-1}+j}^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( {}^{(n+1)} \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i+1)} \right)^\dagger \right], \end{aligned} \quad (\text{B.5})$$

and noticing that  $\binom{n}{n-i-1} + \binom{n}{i} = \binom{n+1}{i+1}$ , one finds

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \sum_{i=0}^n (-1)^{n-i} \sum_{j=1}^{\binom{n}{i}} \\ & \left[ \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i+1)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i+1} \tilde{\mathcal{K}}_{(i+1)-j+1}^{(n-i)} \right)^\dagger - \binom{n+1}{i-1} \tilde{\mathcal{K}}_{(i-1)+j}^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i+1)} \right)^\dagger \right]. \end{aligned} \quad (\text{B.6})$$

In the first term, let us relabel  $i+1 \rightarrow i$ , and in the second term, let us relabel  $\binom{n}{i-1} + j \rightarrow j$ . This gives

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \\ & \sum_{i=1}^{n+1} (-1)^{n+1-i} \sum_{j=1}^{\binom{n}{i-1}} \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)-j+1}^{(n+1-i)} \right)^\dagger \\ & + \sum_{i=0}^n (-1)^{n+1-i} \sum_{j=\binom{n}{i-1}+1}^{\binom{n}{i-1}+\binom{n}{i}} \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)+\binom{n}{i-1}-j+1}^{(n-i+1)} \right)^\dagger. \end{aligned} \quad (\text{B.7})$$

Using again that  $\binom{n}{i-1} + \binom{n}{i} = \binom{n+1}{i}$ , this reduces to

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \\ & \sum_{i=1}^{n+1} (-1)^{n+1-i} \sum_{j=1}^{\binom{n}{i-1}} \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)-j+1}^{(n+1-i)} \right)^\dagger \\ & + \sum_{i=0}^n (-1)^{n+1-i} \sum_{j=\binom{n}{i-1}+1}^{\binom{n+1}{i}} \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)-j+1}^{(n-i+1)} \right)^\dagger. \end{aligned} \quad (\text{B.8})$$

The first sum does not have a term with  $i=0$ , but its argument vanishes when evaluated at  $i=0$ , hence the sum can be extended to  $i=0 \dots n+1$ . Likewise, the argument of the second sum vanishes when evaluated at  $i=n+1$ , so the two sums can be extended to  $i=0 \dots n+1$ , and one finds

$$\begin{aligned} & \left[ \tilde{H}_{\text{int}}(\eta_0), \left[ \tilde{H}_{\text{int}}(\eta_1), \dots \left[ \tilde{H}_{\text{int}}(\eta_n), |\text{BD}\rangle \langle \text{BD}| \right] \dots \right] \right] = \\ & \sum_{i=0}^{n+1} (-1)^{n+1-i} \sum_{j=1}^{\binom{n+1}{i}} \binom{n+1}{i} \tilde{\mathcal{K}}_j^{(i)} |\text{BD}\rangle \langle \text{BD}| \left( \binom{n+1}{i} \tilde{\mathcal{K}}_{(i)-j+1}^{(n+1-i)} \right)^\dagger, \end{aligned} \quad (\text{B.9})$$

which concludes the proof of Eq. (B.1) by induction.

## C System memory kernel

This appendix aims at computing the memory kernel of the system corresponding to the interaction Hamiltonian (3.38), and derive the expressions (3.44) and (3.45).

Using the definition (2.34) together with Eqs. (2.25) and (2.31), one finds

$$\begin{aligned}
\bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) &= \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} + \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&+ \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} + \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&- \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) | \text{BD} \rangle_{\mathcal{E}} \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&- \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) | \text{BD} \rangle_{\mathcal{E}} \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&- \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) | \text{BD} \rangle_{\mathcal{E}} \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&- \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) | \text{BD} \rangle_{\mathcal{E}} \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} .
\end{aligned} \tag{C.1}$$

The operators  $\tilde{O}_{\text{lin}}^{\mathcal{S}}$  and  $\tilde{O}_{\text{NL}}^{\mathcal{S}}$  were introduced in Eqs. (3.39) and (3.40), which we repeat here for convenience

$$\begin{aligned}
\tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}, \eta) &= a(\eta) \tilde{\Pi}_{\pi}(\mathbf{k}, \eta) \tag{C.2} \\
\tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}, \eta) &\equiv \int \frac{d^3 \mathbf{p}_1}{(2\pi)^3} \int \frac{d^3 \mathbf{p}_2}{(2\pi)^3} \frac{\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k})}{a(\eta)} \left[ \frac{\mathbf{p}_1 \cdot \mathbf{p}_2 \tilde{v}_{\pi}(\mathbf{p}_1, \eta) \tilde{v}_{\pi}(\mathbf{p}_2, \eta)}{\Lambda_1} - \frac{\tilde{\Pi}_{\pi}(\mathbf{p}_1, \eta) \tilde{\Pi}_{\pi}(\mathbf{p}_2, \eta)}{\Lambda_2} \right] . \tag{C.3}
\end{aligned}$$

Here we use the shorthand notation

$$\tilde{\Pi}_{\pi}(\mathbf{k}, \eta) \equiv \tilde{p}_{\pi}(\mathbf{k}, \eta) - \frac{a'(\eta)}{a(\eta)} \tilde{v}_{\pi}(\mathbf{k}, \eta) , \tag{C.4}$$

whose mode functions are given by

$$\Pi_{\pi}(k, \eta) \equiv p_{\pi}(k, \eta) - \frac{a'(\eta)}{a(\eta)} \tilde{v}_{\pi}(\mathbf{k}, \eta) = -i \sqrt{\frac{c_{\mathcal{S}} k}{2}} e^{-i c_{\mathcal{S}} k \eta} , \tag{C.5}$$

which follow from Eqs. (3.30) and (3.32). Since  $\tilde{O}_{\text{lin}}^{\mathcal{S}}$  is linear and  $\tilde{O}_{\text{NL}}^{\mathcal{S}}$  is quadratic in the field variables, all vacuum expectation values involving an odd power of fields in Eq. (C.1) vanish, leaving

$$\bar{\mathcal{K}}_{\mathcal{S}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) = \bar{\mathcal{K}}_{\mathcal{S}, \text{lin}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) + \bar{\mathcal{K}}_{\mathcal{S}, \text{NL}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) \tag{C.6}$$

where

$$\bar{\mathcal{K}}_{\mathcal{S}, \text{lin}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) = \langle \text{BD} | \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{lin}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \tag{C.7}$$

and

$$\begin{aligned}
\bar{\mathcal{K}}_{\mathcal{S}, \text{NL}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) &= \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&- \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_1, \eta_1) | \text{BD} \rangle_{\mathcal{E}} \langle \text{BD} | \tilde{O}_{\text{NL}}^{\mathcal{S}}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} .
\end{aligned} \tag{C.8}$$

This also means that all cross-terms between the operators  $\tilde{O}_{\text{lin}}^{\mathcal{S}}$  and  $\tilde{O}_{\text{NL}}^{\mathcal{S}}$  in the system two-point functions vanish, as claimed in the main text.

**Linear contribution** Upon introducing Eqs. (C.2) and (C.5) into Eq. (C.7), one finds that

$$\begin{aligned}
\bar{\mathcal{K}}_{\mathcal{S}, \text{lin}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) &= a(\eta_1) a(\eta_2) \langle \text{BD} | \tilde{\Pi}_{\pi}(\mathbf{k}_1, \eta_1) \tilde{\Pi}_{\pi}(\mathbf{k}_2, \eta_2) | \text{BD} \rangle_{\mathcal{E}} \\
&= a(\eta_1) a(\eta_2) \Pi_{\pi}(k_1, \eta_1) \Pi_{\pi}^*(k_2, \eta_2) \delta(\mathbf{k}_1 + \mathbf{k}_2) \\
&= \frac{c_{\mathcal{S}} k_1}{2H^2 \eta_1 \eta_2} e^{-i c_{\mathcal{S}} k_1 (\eta_1 - \eta_2)} \delta(\mathbf{k}_1 + \mathbf{k}_2) .
\end{aligned} \tag{C.9}$$

When combined with Eq. (2.46), this results in Eq. (3.44) in the main text.

**Non-linear contribution** The computation of  $\bar{\mathcal{K}}_{S,\text{NL}}^{(2)}$  is more involved due to the larger number of field operators contained within the expectation values. Inserting Eq. (C.3) into Eq. (C.8) ones finds

$$\begin{aligned} \bar{\mathcal{K}}_{S,\text{NL}}^{(2)}(\mathbf{k}_1, \mathbf{k}_2, \eta_1, \eta_2) &= \prod_{i,j=1}^2 \int \frac{d^3\mathbf{p}_i}{(2\pi)^3} \int \frac{d^3\mathbf{q}_j}{(2\pi)^3} \frac{\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k}_1)\delta(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{k}_2)}{a(\eta_1)a(\eta_2)} \quad (\text{C.10}) \\ &\times \left\{ \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)(\mathbf{q}_1 \cdot \mathbf{q}_2)}{\Lambda_1^2} [\langle \tilde{v}_\pi(\mathbf{p}_1, \eta_1)\tilde{v}_\pi(\mathbf{p}_2, \eta_1)\tilde{v}_\pi(\mathbf{q}_1, \eta_2)\tilde{v}_\pi(\mathbf{q}_2, \eta_2) \rangle - \langle \tilde{v}_\pi(\mathbf{p}_1, \eta_1)\tilde{v}_\pi(\mathbf{p}_2, \eta_1) \rangle \langle \tilde{v}_\pi(\mathbf{q}_1, \eta_2)\tilde{v}_\pi(\mathbf{q}_2, \eta_2) \rangle] \right. \\ &- \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{\Lambda_1 \Lambda_2} \left[ \langle \tilde{\Pi}_\pi(\mathbf{p}_1, \eta_1)\tilde{\Pi}_\pi(\mathbf{p}_2, \eta_1)\tilde{v}_\pi(\mathbf{q}_1, \eta_2)\tilde{v}_\pi(\mathbf{q}_2, \eta_2) \rangle - \langle \tilde{\Pi}_\pi(\mathbf{p}_1, \eta_1)\tilde{\Pi}_\pi(\mathbf{p}_2, \eta_1) \rangle \langle \tilde{v}_\pi(\mathbf{q}_1, \eta_2)\tilde{v}_\pi(\mathbf{q}_2, \eta_2) \rangle \right] \\ &- \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{\Lambda_1 \Lambda_2} \left[ \langle \tilde{v}_\pi(\mathbf{p}_1, \eta_1)\tilde{v}_\pi(\mathbf{p}_2, \eta_1)\tilde{\Pi}_\pi(\mathbf{q}_1, \eta_2)\tilde{\Pi}_\pi(\mathbf{q}_2, \eta_2) \rangle - \langle \tilde{v}_\pi(\mathbf{p}_1, \eta_1)\tilde{v}_\pi(\mathbf{p}_2, \eta_1) \rangle \langle \tilde{\Pi}_\pi(\mathbf{q}_1, \eta_2)\tilde{\Pi}_\pi(\mathbf{q}_2, \eta_2) \rangle \right] \\ &\left. + \frac{1}{\Lambda_2^2} \left[ \langle \tilde{\Pi}_\pi(\mathbf{p}_1, \eta_1)\tilde{\Pi}_\pi(\mathbf{p}_2, \eta_1)\tilde{\Pi}_\pi(\mathbf{q}_1, \eta_2)\tilde{\Pi}_\pi(\mathbf{q}_2, \eta_2) \rangle - \langle \tilde{\Pi}_\pi(\mathbf{p}_1, \eta_1)\tilde{\Pi}_\pi(\mathbf{p}_2, \eta_1) \rangle \langle \tilde{\Pi}_\pi(\mathbf{q}_1, \eta_2)\tilde{\Pi}_\pi(\mathbf{q}_2, \eta_2) \rangle \right] \right\}, \end{aligned}$$

where we have used the shorthand  $\langle \tilde{\mathcal{O}} \rangle = \langle \text{BD}|\tilde{\mathcal{O}}|\text{BD} \rangle_{\mathcal{E}}$ . Performing the Wick contractions, and extracting a factor of  $\delta(\mathbf{k}_1 + \mathbf{k}_2)$  via Eq. (2.46), leaves

$$\begin{aligned} \bar{\mathcal{K}}_{S,\text{NL}}^{(2)}(k, \eta_1, \eta_2) &= \frac{2}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) \quad (\text{C.11}) \\ &\times \left[ (\mathbf{p} \cdot \mathbf{q})^2 \frac{v_\pi(p, \eta_1)v_\pi(q, \eta_1)v_\pi^*(p, \eta_2)v_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_1^2} + \frac{\Pi_\pi(p, \eta_1)\Pi_\pi(q, \eta_1)\Pi_\pi^*(p, \eta_2)\Pi_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_2^2} \right. \\ &\left. - \mathbf{p} \cdot \mathbf{q} \frac{\Pi_\pi(p, \eta_1)\Pi_\pi(q, \eta_1)v_\pi^*(p, \eta_2)v_\pi^*(q, \eta_2) + v_\pi(p, \eta_1)v_\pi(q, \eta_1)\Pi_\pi^*(p, \eta_2)\Pi_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_1\Lambda_2} \right]. \end{aligned}$$

To proceed we note that the above double-momentum integral consists of terms of the form

$$\mathcal{I}_n[f] := \frac{2}{(2\pi)^6} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(\mathbf{p} + \mathbf{q} - \mathbf{k}) (\mathbf{p} \cdot \mathbf{q})^n f(p, q) \quad (\text{C.12})$$

for three choices of functions  $f = f(p, q)$  depending on  $p$  and  $q$  (and not the angles). Using this compact notation, the correlator can be written in terms of the above integrals for  $n = 0, 1, 2$  such that

$$\begin{aligned} \bar{\mathcal{K}}_{S,\text{NL}}^{(2)}(k, \eta_1, \eta_2) &= \mathcal{I}_2 \left[ \frac{v_\pi(p, \eta_1)v_\pi(q, \eta_1)v_\pi^*(p, \eta_2)v_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_1^2} \right] + \mathcal{I}_0 \left[ \frac{\Pi_\pi(p, \eta_1)\Pi_\pi(q, \eta_1)\Pi_\pi^*(p, \eta_2)\Pi_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_2^2} \right] \\ &- \mathcal{I}_1 \left[ \frac{\Pi_\pi(p, \eta_1)\Pi_\pi(q, \eta_1)v_\pi^*(p, \eta_2)v_\pi^*(q, \eta_2) + v_\pi(p, \eta_1)v_\pi(q, \eta_1)\Pi_\pi^*(p, \eta_2)\Pi_\pi^*(q, \eta_2)}{a(\eta_1)a(\eta_2)\Lambda_1\Lambda_2} \right]. \quad (\text{C.13}) \end{aligned}$$

The next step is to integrate the angles away in Eq. (C.12), which is why we now convert to spherical coordinates  $\mathbf{q} = (q, \theta_q, \varphi_q)$  and  $\mathbf{p} = (p, \theta_p, \varphi_p)$ . With the aim of integrating away  $\mathbf{q}$  in the  $\delta$ -function we note that

$$\delta(\mathbf{q} - \ell) = \frac{\delta(q - \ell)\delta(\theta_q - \theta_\ell)\delta(\varphi_q - \varphi_\ell)}{q^2 |\sin \theta_q|} \quad (\text{C.14})$$

for arbitrary  $\ell$ . Expressing the vector  $\ell \equiv \mathbf{k} - \mathbf{p}$  in spherical coordinates,

$$\ell = \sqrt{p^2 + k^2 - 2kp \cos \theta_p}, \quad \theta_\ell = \cos^{-1} \left( \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right), \quad \varphi_\ell = \varphi_p + \pi, \quad (\text{C.15})$$

turns Eq. (C.12) into

$$\begin{aligned}
I_n[f] &= \frac{2}{(2\pi)^{12}} \int_0^\infty dq q^2 \int_0^\pi d\theta_q \sin \theta_q \int_0^{2\pi} d\varphi_q \int_0^\infty dp p^2 \int_0^\pi d\theta_p \sin \theta_p \int_0^{2\pi} d\varphi_p \\
&\quad \times \frac{1}{q^2 \sin \theta_q} \delta \left( q - \sqrt{p^2 + k^2 - 2kp \cos \theta_p} \right) \delta \left[ \theta_q - \cos^{-1} \left( \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right) \right] \\
&\quad \times \delta [\varphi_q - (\varphi_p + \pi)] \left\{ qp [\sin \theta_q \sin \theta_p \cos(\varphi_q - \varphi_p) + \cos \theta_q \cos \theta_p] \right\}^n f(p, q). \quad (\text{C.16})
\end{aligned}$$

The integration over  $\varphi_q$  simply leads to the replacement of  $\varphi_q$  by  $\varphi_p + \pi$  and the integration over  $\varphi_p$  just yields a factor  $2\pi$ , which leads to

$$\begin{aligned}
I_n[f] &= \frac{2}{(2\pi)^{11}} \int_0^\infty dq \int_0^\pi d\theta_q \int_0^\infty dp p^2 \int_0^\pi d\theta_p \sin \theta_p \delta \left( q - \sqrt{p^2 + k^2 - 2kp \cos \theta_p} \right) \\
&\quad \times \delta \left[ \theta_q - \cos^{-1} \left( \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right) \right] [qp (-\sin \theta_q \sin \theta_p + \cos \theta_q \cos \theta_p)]^n f(p, q). \quad (\text{C.17})
\end{aligned}$$

The next step consists in integrating over  $\theta_q$ . Using the identities

$$\cos \left[ \cos^{-1} \left( \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right) \right] = \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}}, \quad (\text{C.18})$$

$$\sin \left[ \cos^{-1} \left( \frac{k - p \cos \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right) \right] = \frac{p \sin \theta_p}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}}, \quad (\text{C.19})$$

leads to

$$\begin{aligned}
I_n[f] &= \frac{2}{(2\pi)^{11}} \int_0^\infty dq \int_0^\infty dp p^2 \int_0^\pi d\theta_p \sin \theta_p \delta \left( q - \sqrt{p^2 + k^2 - 2kp \cos \theta_p} \right) \\
&\quad \times \left\{ qp \frac{[(k - p \cos \theta_p) \cos \theta_p - p \sin^2 \theta_p]}{\sqrt{p^2 + k^2 - 2kp \cos \theta_p}} \right\}^n f(p, q). \quad (\text{C.20})
\end{aligned}$$

We then switch to  $\mu = \cos \theta_p$  and use the fact that

$$\delta \left( q - \sqrt{p^2 + k^2 - 2pk\mu} \right) = \frac{q}{kp} \delta \left( \mu - \frac{p^2 + k^2 - q^2}{2pk} \right) \quad (\text{C.21})$$

to rewrite

$$I_n[f] = \frac{2}{(2\pi)^{11}k} \int_0^\infty dq \int_0^\infty dp \int_{-1}^1 d\mu \delta \left( \mu - \frac{p^2 + k^2 - q^2}{2pk} \right) \frac{(qp)^{n+1} (k\mu - p)^n}{(p^2 + k^2 - 2kp\mu)^{n/2}} f(p, q). \quad (\text{C.22})$$

After integrating over  $\mu$ , the remaining  $\delta$ -function restricts the integration region to

$$U = \{ |p - k| < q < p + k \} \quad (\text{C.23})$$

which gives

$$I_n[f] = \frac{2}{(2\pi)^{11}k} \int_U dp dq pq \left( \frac{k^2 - p^2 - q^2}{2} \right)^n f(p, q). \quad (\text{C.24})$$



The region  $U$  is inconvenient to integrate over in these variables, which is why we now introduce [54]

$$P = p + q \quad \text{and} \quad Q = p - q. \quad (\text{C.25})$$

This turns Eq. (C.24) into

$$I_n[f] = \frac{1}{4(2\pi)^{11}k} \int_k^\infty dP \int_0^k dQ (P^2 - Q^2) \left( \frac{2k^2 - P^2 - Q^2}{4} \right)^n f(p, q). \quad (\text{C.26})$$

Using this identity allows one to rewrite Eq. (C.13) as

$$\begin{aligned} \bar{\mathcal{K}}_{\mathcal{S},\text{NL}}^{(2)}(k, \eta_1, \eta_2) = & \frac{1}{(2\pi)^{11}} \frac{1}{64k} \int_k^\infty dP \int_0^k dQ (P^2 - Q^2) \\ & \left[ \frac{(P^2 + Q^2 - 2k^2)v_\pi\left(\frac{P+Q}{2}, \eta_1\right)v_\pi\left(\frac{P-Q}{2}, \eta_1\right)}{a(\eta_1)\Lambda_1} + \frac{4\Pi_\pi\left(\frac{P+Q}{2}, \eta_1\right)\Pi_\pi\left(\frac{P-Q}{2}, \eta_1\right)}{a(\eta_1)\Lambda_2} \right] \\ & \left[ \frac{(P^2 + Q^2 - 2k^2)v_\pi^*\left(\frac{P+Q}{2}, \eta_2\right)v_\pi^*\left(\frac{P-Q}{2}, \eta_2\right)}{a(\eta_2)\Lambda_1} + \frac{4\Pi_\pi^*\left(\frac{P+Q}{2}, \eta_2\right)\Pi_\pi^*\left(\frac{P-Q}{2}, \eta_2\right)}{a(\eta_2)\Lambda_2} \right]. \end{aligned} \quad (\text{C.27})$$

With the mode functions (3.30) and (C.5) this reads

$$\begin{aligned} \bar{\mathcal{K}}_{\mathcal{S},\text{NL}}^{(2)}(k, \eta_1, \eta_2) = & \frac{H^2\eta_1\eta_2}{(2\pi)^{12}} \frac{\pi}{32k} \int_k^\infty dP \int_0^k dQ e^{-ic_s P(\eta_1 - \eta_2)} \\ & \times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s\Lambda_1} \left[ 1 - \frac{2i}{c_s(P+Q)\eta_1} \right] \left[ 1 - \frac{2i}{c_s(P-Q)\eta_1} \right] - \frac{c_s(P^2 - Q^2)}{\Lambda_2} \right\} \\ & \times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s\Lambda_1} \left[ 1 + \frac{2i}{c_s(P-Q)\eta_2} \right] \left[ 1 + \frac{2i}{c_s(P+Q)\eta_2} \right] - \frac{c_s(P^2 - Q^2)}{\Lambda_2} \right\}. \end{aligned} \quad (\text{C.28})$$

which coincides with Eq. (3.45) stated the main text.

## D Integrals $\mathfrak{L}_k$ and $\mathfrak{N}_k$

In this appendix we flesh out the mathematical details of the integrals computed in Sec. 3.2, using similar methods to those employed in Ref. [188].

### Linear contribution

In Eq. (3.49),  $S_2^{\text{lin}}$  is expressed in terms of a single time integral, and switching to the dimensionless integration variable

$$z' = -k\eta, \quad (\text{D.1})$$

this integral takes the form

$$\mathfrak{L}_k(\eta) = \sqrt{\frac{\pi}{2}} e^{-\frac{\pi}{2}\mu_\sigma + \frac{i\pi}{4}} \int_{-k\eta}^\infty dz' \frac{e^{ic_s z'} H_{i\mu_\sigma}^{(1)}(z')}{\sqrt{z'}}. \quad (\text{D.2})$$

For the case of a massless environment with  $i\mu_\sigma = \frac{3}{2}$  the integrand simplifies and one finds:

$$\mathfrak{L}_k(\eta)|_{\text{massless}} = \int_{-k\eta}^\infty dz' e^{i(1+c_s)z'} \left( \frac{i}{z'^2} + \frac{1}{z'} \right) \quad (\text{D.3})$$

$$= \frac{ie^{-i(1+c_s)k\eta}}{-k\eta} + c_s \{ \text{Ei}[-i(1+c_s)k\eta] - i\pi \} \quad (\text{D.4})$$

$$\simeq \begin{cases} \frac{i}{-k\eta} + \dots & , -k\eta \ll 1 \\ e^{-i(1+c_s)k\eta} \left[ \frac{i}{(1+c_s)(-k\eta)} + \dots \right] & , -k\eta \gg 1 \end{cases} \quad (\text{D.5})$$

where  $\text{Ei}(z)$  is the exponential integral function defined for  $z \in \mathbb{C} \setminus (-\infty, 0]$  (i.e. with a branch cut along the negative real axis) as

$$\text{Ei}(z) \equiv - \int_{-z}^{\infty} dt \frac{e^{-t}}{t} \quad , \quad (\text{D.6})$$

where the principal value of the integral is taken. In the case of a conformal environment with  $i\mu_\sigma = \frac{1}{2}$  we have:

$$\mathfrak{L}_k(\eta)|_{\text{conformal}} = \int_{-k\eta}^{\infty} dz' \frac{e^{i(1+c_s)z'}}{z'} \quad (\text{D.7})$$

$$= -\text{Ei}[-i(c_s+1)k\eta] + i\pi \quad (\text{D.8})$$

$$\simeq \begin{cases} -\log[-e^{\gamma_E}(c_s+1)k\eta] + \frac{i\pi}{2} + \dots & , -k\eta \ll 1 \\ e^{-i(1+c_s)k\eta} \left[ \frac{i}{(1+c_s)(-k\eta)} + \dots \right] & , -k\eta \gg 1 \end{cases} \quad (\text{D.9})$$

### Non-linear contribution

In order to determine the non-linear contribution to the entropy, we must first compute the integral  $\mathfrak{N}_k(P, Q, \eta)$  defined in Eq. (3.51), and then integrate over  $P$  and  $Q$  using Eq. (3.50) to find  $S_2^{\text{NL}}$ .

We focus on the case of a massless environment, since the calculations for conformal and heavy environments proceed along similar lines. In this case, we use  $i\mu_\sigma = \frac{3}{2}$  and switch the integration variable to  $z' = -k\eta$  as above for simplicity, so that Eq. (3.51) becomes

$$\begin{aligned} \mathfrak{N}_k(P, Q, \eta) &= \frac{1}{k} \int_{-k\eta}^{\infty} dz' e^{i(k+c_s P)\frac{z'}{k}} (i+z') \\ &\times \left\{ \frac{P^2 + Q^2 - 2k^2}{c_s \Lambda_1} \left[ 1 + \frac{2ik}{c_s(P+Q)z'} \right] \left[ 1 - \frac{2i}{c_s(P-Q)z'} \right] - \frac{c_s(P^2 - Q^2)}{\Lambda_2} \right\}. \end{aligned} \quad (\text{D.10})$$

The terms proportional to  $(z')^{-1}$  and  $(z')^{-2}$  converge, meanwhile those proportional to  $(z')^0$  and  $z'$  formally diverge. Understood as a distribution however, such that [54]

$$\int_{-k\eta}^{\infty} dz' e^{i(k+c_s P)\frac{z'}{k}} = e^{-i(k+c_s P)\eta} \frac{ik}{k+c_s P} \quad , \quad (\text{D.11})$$

$$\int_{-k\eta}^{\infty} dz' z' e^{i(k+c_s P)\frac{z'}{k}} = e^{-i(k+c_s P)\eta} \left[ -\frac{k^2}{(k+c_s P)^2} + \frac{ik}{k+c_s P}(-k\eta) \right] \quad , \quad (\text{D.12})$$

we find that  $\mathfrak{N}_k$  evaluates to Eq. (3.57) in the main text. At this stage we use Eq. (3.50) to evaluate the entropy, however we make the change to dimensionless variables

$$a = Q/k \quad \text{and} \quad b = P/k \quad (\text{D.13})$$

so that when used in tandem with (3.57) we find that

$$S_2^{\text{NL}}(k) = \frac{g^2 H^2}{(2\pi)^{18}} \frac{\pi}{16} \left[ \frac{\mathcal{C}_1}{(-k\eta)^2} + \mathcal{C}_2 + \mathcal{C}_3(-k\eta)^2 \right]. \quad (\text{D.14})$$

This is organised in powers of  $-k\eta$ , with time-independent coefficients given by

$$\mathcal{C}_1 = \int_1^\infty db \int_0^1 da \frac{16(a^2 + b^2 - 2)^2}{(b^2 - a^2)^2 c_s^6 \Lambda_1^2} \quad (\text{D.15})$$

$$\mathcal{C}_2 = \int_1^\infty db \int_0^1 da \left\{ \frac{(a^2 + b^2 - 2)^2}{(a^2 - b^2)^2 (bc_s + 1)^4 c_s^4 \Lambda_1^2} \left[ bc_s^3 (b^2 - a^2)^2 (bc_s + 4) + 8bc_s (a^2 + 3b^2) \right. \right. \\ \left. \left. + 8(a^2 + b^2) + 4c_s^2 (a^4 - 2a^2 b^2 + 5b^4) \right] \right. \quad (\text{D.16})$$

$$\left. + \frac{2(a^2 + b^2 - 2) [4(bc_s + 1) - c_s^2 (b^2 - a^2)(bc_s + 2)^2]}{(bc_s + 1)^4 c_s^2 \Lambda_1 \Lambda_2} + \frac{c_s^2 (b^2 - a^2)^2 (bc_s + 2)^2}{(bc_s + 1)^4 \Lambda_2^2} \right\} \\ \mathcal{C}_3 = \int_1^\infty db \int_0^1 da \left[ \frac{(a^2 + b^2 - 2)^2}{(bc_s + 1)^2 c_s^2 \Lambda_1^2} - \frac{2(b^2 - a^2)(a^2 + b^2 - 2)}{(bc_s + 1)^2 \Lambda_1 \Lambda_2} + \frac{c_s^2 (b^2 - a^2)^2}{(bc_s + 1)^2 \Lambda_2^2} \right]. \quad (\text{D.17})$$

We first perform the  $b$  integrals in the above, which contain the UV divergences described in the main text. For example, for  $\mathcal{C}_1$ , taking a partial fraction expansion in  $b$  gives

$$\mathcal{C}_1 = \frac{16}{c_s^6 \Lambda_1^2} \int_1^\infty db \int_0^1 da \left\{ 1 + \frac{a^4 - 1}{a^3} \left( \frac{1}{b - a} - \frac{1}{b + a} \right) + \frac{(a^2 - 1)^2}{a^2} \left[ \frac{1}{(b + a)^2} + \frac{1}{(b - a)^2} \right] \right\}. \quad (\text{D.18})$$

The integrand in the above behaves as  $1 + 4(a^2 - 1)b^{-2} + \mathcal{O}(b^{-3})$  for  $b \gg 1$  and so only the first term must be regulated. This is done in a way that is analogous to Eq. (3.58), taking

$$\int_1^\infty db \rightarrow \int_1^\infty db \left( \frac{bk}{\mu} \right)^\varepsilon \simeq -1 + \mathcal{O}(\varepsilon). \quad (\text{D.19})$$

Notice that the result is independent of the dimensional regulator  $\varepsilon$  here, as is always the case when an integral is power-law divergent in a momentum UV cutoff. The rest of the terms converge giving

$$\mathcal{C}_1 = \frac{16}{c_s^6 \Lambda_1^2} \int_0^1 da \left[ -3 + \frac{2}{a^2} + \frac{2}{a^3} (a^4 - 1) \tanh^{-1}(a) \right] = -\frac{48}{c_s^6 \Lambda_1^2} \quad (\text{D.20})$$

as quoted in Eq. (3.60) of the main text. The computations for Eqs. (3.61) and (3.62) are more cumbersome but follow through similarly, using the regularizations

$$\int_1^\infty db b^n \rightarrow \int_1^\infty db b^n \left( \frac{bk}{\mu} \right)^\varepsilon \simeq -\frac{1}{n+1} + \mathcal{O}(\varepsilon) \quad \text{for } n = 1, 2 \quad (\text{D.21})$$

$$\text{and } \int_1^\infty \frac{db}{b \pm a} \rightarrow \int_1^\infty \frac{db}{b \pm a} \left( \frac{bk}{\mu} \right)^\varepsilon \simeq -\frac{1}{\varepsilon} + \log \left[ \frac{\mu}{(1 \pm a)k} \right] + \mathcal{O}(\varepsilon) \quad (\text{D.22})$$

in addition to Eqs. (3.58) and (D.19).

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