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THE HOMOLOGY OF ADDITIVE FUNCTORS IN PRIME CHARACTERISTIC

AURÉLIEN DJAMENT AND ANTOINE TOUZÉ

ABSTRACT. We compute certain Ext and Tor groups in the category of all functors from an \mathbb{F}_p -linear additive category \mathcal{A} to vector spaces in terms of Ext and Tor computed in the full subcategory of additive functors from \mathcal{A} to vector spaces. We thus obtain group homology computations for general linear groups.

RÉSUMÉ. Nous calculons certains groupes d'extensions et de torsion dans la catégorie de tous les foncteurs depuis une catégorie additive \mathbb{F}_p -linéaire vers les espaces vectoriels, en termes d'Ext et de Tor calculés dans la sous-catégorie pleine des foncteurs additifs de \mathcal{A} vers les espaces vectoriels. Nous obtenons ainsi des calculs d'homologie des groupes linéaires.

1. INTRODUCTION

Let ρ and π be two additive functors from a small (or svelte) additive category \mathcal{A} to the category $\mathcal{V}_{\mathbb{k}}$ of \mathbb{k} -vector spaces. There are two natural contexts in which one may wish to compute the extensions $\text{Ext}^*(\pi, \rho)$. Firstly, we can consider π and ρ as objects of the abelian category $\mathbf{Add}(\mathcal{A}, \mathbb{k})$ of additive functors from \mathcal{A} to \mathbb{k} -vector spaces and natural transformations. Secondly, we can consider π and ρ as objects of the abelian category $\mathcal{F}(\mathcal{A}, \mathbb{k})$ of *all functors* from \mathcal{A} to \mathbb{k} -vector spaces and natural transformations. Since $\mathbf{Add}(\mathcal{A}, \mathbb{k})$ is a full abelian subcategory of $\mathcal{F}(\mathcal{A}, \mathbb{k})$, there is a canonical map of graded vector spaces:

$$\Phi : \text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) .$$

Note that when $\mathcal{A} = \mathbf{P}_R$, the category of finitely generated projective right modules over a ring R , every additive functor is isomorphic to $t_M : P \mapsto P \otimes_R M$ for some unique (R, \mathbb{k}) -bimodule M , and there is a graded isomorphism:

$$\text{Ext}_{\mathbf{Add}(\mathbf{P}_R, \mathbb{k})}^*(t_M, t_N) \simeq \text{Ext}_{R \otimes_{\mathbb{Z}} \mathbb{k}}^*(M, N) .$$

Thus the source of Φ be computed by standard homological algebra in module categories. In sharp contrast, $\mathcal{F}(\mathbf{P}_R, \mathbb{k})$ is not equivalent to a module category if $R \neq 0$, which makes the target of Φ a bit more mysterious; in some cases it is related to topological Hochschild homology [19].

If \mathbb{k} is a field of characteristic zero, then Φ is an isomorphism by [2, Thm 1.2]. This is no longer the case if \mathbb{k} is a field of positive characteristic. For example, if \mathbb{k} is a finite field and if $\mathcal{A} = \mathbf{P}_{\mathbb{k}}$ then the ring $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{k}$ is semi-simple. Hence the source

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of Φ is zero in positive degrees, whereas the fundamental computation of [8, Thm 0.1] shows that the target of Φ may be nonzero in infinitely many positive degrees. Our first result is:

Theorem 1. *Let \mathbb{k} be a perfect field of positive characteristic p , let I denote the embedding $\mathbf{P}_{\mathbb{k}} \hookrightarrow \mathcal{V}_{\mathbb{k}}$, and let E^* denote the graded \mathbb{k} -algebra $\text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I)$ with Yoneda composition. There is an isomorphism of graded \mathbb{k} -algebras*

$$E^* \simeq \mathbb{k}[e_1, e_2, e_3, \dots] / \langle e_1^p, e_2^p, e_3^p, \dots \rangle,$$

where each class e_i has cohomological degree $2p^{i-1}$. Moreover, if the small (or svelte) additive category \mathcal{A} is \mathbb{F}_p -linear, then for all additive functors π and ρ there is a graded isomorphism:

$$\Psi : \text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \otimes_{\mathbb{k}} E^* \simeq \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho).$$

If \mathbb{k} is a finite field, the computation of $\text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I)$ given in this theorem is nothing but [8, Thm 0.1]. Although technically different, theorem 1 is similar in spirit to the main theorems of [11, 12] (see also [17, Cor 4.2]) which compare Hochschild homology and topological Hochschild homology of smooth \mathbb{F}_p -algebras.

The isomorphism Ψ is given by a simple formula. Namely, every functor $\pi : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$ yields an exact restriction functor $\pi^* : \mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k}) \rightarrow \mathcal{F}(\mathcal{A}, \mathbb{k})$, which sends F to the composition $\pi^*F = \overline{F} \circ \pi$, where $\overline{F} : \mathcal{V}_{\mathbb{k}} \rightarrow \mathcal{V}_{\mathbb{k}}$ denotes the left Kan extension of F to the category of all vector spaces, that is, $\overline{F}(v) = \text{colim } F(u)$, with colimit taken over the filtered poset of finite dimensional subspaces u of v , ordered by inclusion. We therefore have a graded map:

$$\pi^* : \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(F, G) \rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi^*F, \pi^*G).$$

Now the isomorphism of theorem 1 is defined by

$$\Psi(e \otimes e') = \Phi(e) \circ \pi^*e'$$

where the symbol " \circ " stands for the Yoneda product of extensions. We observe that Ψ is well-defined even if \mathcal{A} is not \mathbb{F}_p -linear or if \mathbb{k} is not perfect; however the calculations of [15] show that Ψ may fail to be an isomorphism if \mathcal{A} is not \mathbb{F}_p -linear; and remark 6.2 shows that the isomorphism may fail if \mathbb{k} is not perfect.

Comparison of Tor. Theorem 1 can be dualized. If \mathbb{k} is an arbitrary field and \mathcal{A} an arbitrary svelte additive category, there is a tensor product over \mathcal{A} [13, IX.6]:

$$\otimes_{\mathcal{A}} : \mathcal{F}(\mathcal{A}^{\text{op}}, \mathbb{k}) \times \mathcal{F}(\mathcal{A}, \mathbb{k}) \rightarrow \mathcal{V}_{\mathbb{k}},$$

with left derived functors denoted by $\text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F, G)$. One can also consider the restriction of the tensor product to the full subcategories of additive functors, whose left derived functors yield an "additive Tor" denoted by $\text{Tor}^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(F, G)$.

Note that if $\mathcal{A} = \mathbf{P}_R$, the additive Tor can be computed in terms of modules. Indeed, every contravariant functor from \mathcal{A} to vector spaces has the form $h_M : P \mapsto \text{Hom}_R(P, M)$ for some (\mathbb{k}, R) -bimodule M and there is a graded isomorphism:

$$(1) \quad \text{Tor}^{\mathbf{P}_{\mathbb{k}} \otimes_{\mathbb{Z}} \mathbb{k}}(h_M, t_N) \simeq \text{Tor}_*^{R \otimes_{\mathbb{Z}} \mathbb{k}}(M, N).$$

If \mathbb{k} has characteristic zero, then $\text{Tor}^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(\pi, \rho)$ is isomorphic to $\text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi, \rho)$ for all additive functors π and ρ , but this is not the case in positive characteristic. In fact, we can dualize theorem 1 to obtain the following result.

Corollary 2. *Let \mathbb{k} be a perfect field of positive characteristic and let \mathcal{A} be a svelte additive \mathbb{F}_p -linear category. Let T_* denote the graded vector space equal to \mathbb{k} in every even degree, and to zero in every odd degree. There is a graded isomorphism, natural with respect to the additive functors π and ρ :*

$$\mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi, \rho) \simeq \mathrm{Tor}_*^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(\pi, \rho) \otimes_{\mathbb{k}} T_* .$$

Tor between non additive functors. In practice, corollary 2 should be thought as a computation of Tor in the category of all functors, based on the simpler computation of Tor in the category of additive functors.

In section 8, we explain how to combine corollary 2 with standard techniques of computations in functor categories to compute Tor between certain non-additive functors. We give an elementary illustration of this in the next example. Let V be a finite-dimensional \mathbb{k} -linear representation of \mathfrak{S}_d , and let F_V be the endofunctor of vector spaces defined by $F_V(v) = v^{\otimes d} \otimes_{\mathfrak{S}_d} V$, where \mathfrak{S}_d acts on $v^{\otimes d}$ by permuting the factors of the tensor product. We extend F_V to an endofunctor \tilde{F}_V of graded vector spaces by keeping the same formula, but by letting \mathfrak{S}_d act on the tensor product of graded vector spaces with a Koszul sign.

Example 3. Let \mathbb{k} be a perfect field of positive characteristic p and let \mathcal{A} be an \mathbb{F}_p -linear additive category. Assume that d is invertible in \mathbb{k} . For all additive functors π and ρ , there is a graded isomorphism of vector spaces

$$\mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi^* S^d, \rho^* F_V) \simeq \tilde{F}_V(\mathrm{Tor}_*^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(\pi, \rho) \otimes_{\mathbb{k}} T_*) .$$

If $d = 1$, this example gives back corollary 2, while the functors $\pi^* S^d$ and $\rho^* F_V$ are not additive if $d \geq 2$: they are polynomial of degree d in the sense of Eilenberg and Mac Lane [6]. Additional and more general Tor-computations are given in section 8. However, our computations typically require that the characteristic of \mathbb{k} is big enough. Computations in small characteristic are much more difficult, we treat them in [3].

Application to the homology of general linear groups. We finish this introduction by recalling one of the main motivations of the present work, namely the connection with the homology of general linear groups.

Every functor $F : \mathbf{P}_R^{\mathrm{op}} \rightarrow \mathcal{V}_{\mathbb{k}}$ determines a sequence of nested right \mathbb{k} -linear representations $F(R^n)$ of $\mathrm{GL}_n(R)$, in which an element g of the general linear group acts as $F(g)$ on the vector space $F(R^n)$. These actions assemble into a right \mathbb{k} -linear action of $\mathrm{GL}_{\infty}(R) = \bigcup_{n \geq 1} \mathrm{GL}_n(R)$ on $F_{\infty} = \bigcup_{n \geq 1} F(R^n)$. Similarly, every functor $G : \mathbf{P}_R \rightarrow \mathcal{V}_{\mathbb{k}}$ determines a left \mathbb{k} -linear representation G_{∞} of $\mathrm{GL}_{\infty}(R)$.

It follows from [1] that when the functors F and G are polynomial in the sense of Eilenberg and Mac Lane, there is a graded isomorphism :

$$(2) \quad H_*(\mathrm{GL}_{\infty}(R), F_{\infty} \otimes_{\mathbb{k}} G_{\infty}) \simeq H_*(\mathrm{GL}_{\infty}(R), \mathbb{k}) \otimes_{\mathbb{k}} \mathrm{Tor}_*^{\mathbb{k}[\mathbf{P}_R]}(F, G) .$$

If R has finite Bass stable rank, then the left hand-side is known [5, 23, 20] to be isomorphic to $H_*(\mathrm{GL}_n(R), F(R^n) \otimes_{\mathbb{k}} G(R^n))$ in low degrees, which makes (2) also interesting for the homology of general linear groups of finite rank.

Typical examples of polynomial functors are the functors $\pi^* S^d$ and $\rho^* F_V$ of example 3, with $\pi = h_M$ and $\rho = t_N$. The corresponding representations of $\mathrm{GL}_{\infty}(R)$ are concretely described by given by $(h_M^* S^d)_{\infty}$ and $(t_N^* F_V)_{\infty}$ can be described as

$$(h_M^* S^d)_{\infty} = S^d(M^{\infty}) , \quad (t_N^* F_V)_{\infty} = F_V(N^{\infty}) = (N^{\infty})^{\otimes d} \otimes_{\mathfrak{S}_d} V ,$$

where N^∞ and M^∞ are the standard representations of $\mathrm{GL}_\infty(R)$ associated to N and M . In other words, $N^\infty = \bigoplus_{k>0} N_k$ is a countable direct sum of copies of N , and a matrix $[r_{ij}]$ of $\mathrm{GL}_\infty(R)$ acts as the endomorphism of N^∞ whose component $N_j \rightarrow N_i$ is left multiplication by r_{ij} ; similarly, $M^\infty = \bigoplus_{k>0} M_k$ is a countable direct sum of copies of M , and a matrix $[r_{ij}]$ of $\mathrm{GL}_\infty(R)$ acts in a transpose way, i.e. as the endomorphism of M^∞ whose component $M_j \rightarrow M_i$ is right multiplication by r_{ji} . Combining example 3 and isomorphism (1) gives a fairly concrete computation of the Tor-term appearing in the right hand-side of (2).

Example 4. Let \mathbb{k} be a perfect field of positive characteristic p , let R be a ring of characteristic p , let V be a \mathfrak{S}_d -module with $d < p$ and let M and N be a right, resp. left, $R \otimes_{\mathbb{Z}} \mathbb{k}$ -module. There is a graded isomorphism, where T_* is a graded vector space which has dimension one in even degrees and dimension 0 in odd degrees:

$$H_*(\mathrm{GL}_\infty(R), S^d(M^\infty) \otimes_{\mathbb{k}} F_V(N^\infty)) \simeq H_*(\mathrm{GL}_\infty(R), \mathbb{k}) \otimes_{\mathbb{k}} \tilde{F}_V(\mathrm{Tor}_*^{R \otimes_{\mathbb{Z}} \mathbb{k}}(M, N) \otimes_{\mathbb{k}} T_*).$$

Further examples of computations can be obtained by combining isomorphism (1), corollary 2 and the computations of section 8.

2. NOTATIONS

Throughout the article, \mathbb{k} is a field, and \mathcal{A} is an additive category [13, VIII.2] which is svelte (i.e. \mathcal{A} has a *set* of isomorphism classes of objects).

We denote by $\mathcal{V}_{\mathbb{k}}$ the category of \mathbb{k} -vector spaces and \mathbb{k} -linear maps, and if R is a ring, we denote by \mathbf{P}_R the category of finitely projective right R -modules and R -linear maps. Thus $\mathbf{P}_{\mathbb{k}}$ is the category of finite-dimensional \mathbb{k} -vector spaces.

We let $\mathcal{F}(\mathcal{A}, \mathbb{k})$ be the category of all functors from \mathcal{A} to $\mathcal{V}_{\mathbb{k}}$ and natural transformations. (Note that there is a *set* of natural transformations between two functors because \mathcal{A} is svelte.) This is a bicomplete abelian category with enough injectives and projectives. We let $\mathbf{Add}(\mathcal{A}, \mathbb{k})$ be the full subcategory of $\mathcal{F}(\mathcal{A}, \mathbb{k})$ on the additive functors. This category is stable under limits and colimits, and also has enough projectives and injectives. We refer the reader to [14] and [18] for an introduction to homological algebra in these functor categories.

We let $\mathcal{P}_{\mathbb{k}}$ be the category of strict polynomial functors of bounded degree over \mathbb{k} as in [9]. Although this category does not appear in the statement of theorem 1 it will play a role in its proof. Strict polynomial functors over \mathbb{k} are highly structured functors $F : \mathbf{P}_{\mathbb{k}} \rightarrow \mathcal{V}_{\mathbb{k}}$, and there is an exact forgetful functor $\mathcal{P}_{\mathbb{k}} \rightarrow \mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})$ which forgets the "strict polynomial structure". If \mathbb{k} is an infinite field, this forgetful functor is fully faithful.

3. \aleph -ADDITIVE ENVELOPES

Our proof of theorem 1 will rely on a variation of the classical notion of the additive envelope of a category.

Let \aleph be a regular cardinal and let k be a commutative ring. A k -linear category is \aleph -additive if it has all \aleph -direct sums, that is, if all direct sums indexed by sets of cardinality less than \aleph exist. A k -linear functor between two such categories is \aleph -additive if it preserves \aleph -direct sums.

Definition 3.1. Given a k -linear category \mathcal{K} with null object, we let \mathcal{K}^{\aleph} be the category whose objects are the families of objects of \mathcal{K} indexed by sets of cardinality less or equal to \aleph . Such an object is denoted by a formal direct sum $\bigoplus_{i \in \mathcal{I}} x_i$. The

morphisms $f : \bigoplus_{j \in \mathcal{J}} x_j \rightarrow \bigoplus_{i \in \mathcal{I}} y_i$ are the ‘matrices’ $[f_{ij}]_{(i,j) \in \mathcal{I} \times \mathcal{J}}$ with entries $f_{ij} \in \mathcal{K}(x_j, y_i)$ and such that for all j_0 only a finite number of morphisms f_{ij_0} are nonzero. The composition of morphisms is given by matrix multiplication.

We identify \mathcal{K} with the full subcategory of \mathcal{K}^{\aleph} on the formal direct sums with only one object, and we let $\iota : \mathcal{K} \hookrightarrow \mathcal{K}^{\aleph}$ be the inclusion. The category \mathcal{K}^{\aleph} is called the \aleph -additive envelope of \mathcal{K} .

The definition of morphisms in \mathcal{K}^{\aleph} shows that the formal direct sum $\bigoplus_{i \in \mathcal{I}} x_i$ is the categorical coproduct of the x_i in \mathcal{K}^{\aleph} , and also the categorical product if \mathcal{I} is finite, which justifies the direct sum notation. Point (2) of the next elementary proposition justifies the name given to \mathcal{K}^{\aleph} .

Proposition 3.2. *The category \mathcal{K}^{\aleph} is k -linear and \aleph -additive. Moreover:*

- (1) *An object x of \mathcal{K}^{\aleph} is isomorphic to a finite direct sum of objects of \mathcal{K} if and only if $\mathcal{K}^{\aleph}(x, -) : \mathcal{K}^{\aleph} \rightarrow k\text{-Mod}$ is \aleph -additive,*
- (2) *Every k -linear functor $F : \mathcal{K} \rightarrow \mathcal{L}$ with \aleph -additive codomain extends to a unique (up to isomorphism) \aleph -additive functor $F^{\aleph} : \mathcal{K}^{\aleph} \rightarrow \mathcal{L}$.*
- (3) *If \mathcal{K} is svelte, then \mathcal{K}^{\aleph} is svelte.*

Proof. (1) The objects of \mathcal{K} (hence their finite direct sums) are \aleph -additive by the definition of morphisms in \mathcal{K}^{\aleph} . Conversely, if $x = \bigoplus x_i$ is an object such that $\mathcal{K}^{\aleph}(x, -)$ is \aleph -additive, the isomorphism $\mathcal{K}^{\aleph}(x, \bigoplus x_i) \simeq \bigoplus \mathcal{K}^{\aleph}(x, x_i)$ shows that id_x factors through a finite direct sum of the x_i , hence x is isomorphic to a finite direct sum of objects of \mathcal{K} .

(2) For all objects $x = \bigoplus x_i$ we choose a direct sum $\bigoplus F(x_i)$ in \mathcal{L} . The assignment $F^{\aleph}(x) = \bigoplus F(x_i)$ defines an \aleph -additive functor such that $F^{\aleph} \circ \iota = F$. Uniqueness follows from the fact that given any pair of \aleph -additive functors $F', G' : \mathcal{K}^{\aleph} \rightarrow \mathcal{L}$, every natural transformation θ between their restrictions to \mathcal{K} extends uniquely into a natural transformation $\theta' : F' \rightarrow G'$. To be more specific, for all formal direct sum $\bigoplus x_i$, the map $\theta'_{\bigoplus x_i}$ is the composition

$$F'(\bigoplus x_i) \xleftarrow{\simeq} \bigoplus F'(x_i) \xrightarrow{\bigoplus \theta_{x_i}} \bigoplus G'(x_i) \xrightarrow{\simeq} G'(\bigoplus x_i).$$

(3) The set of iso classes of \mathcal{K}^{\aleph} has cardinal less or equal to that of \mathcal{K} and \aleph . \square

For example, using the universal property (2), $\mathbf{P}_{\mathbb{k}}^{\aleph}$ is easily seen to be equivalent to the category of \mathbb{k} -vector spaces of dimension less than \aleph .

We now come to the usefulness of additive envelopes when computing Ext in functor categories. We are interested in the case $k = \mathbb{k}$ is a field and $\mathcal{K} = \mathcal{A}$.

Let $F : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$ be a functor, possibly non-additive. If $x = \bigoplus_{i \in \mathcal{I}} x_i$ is an object of \mathcal{A}^{\aleph} , then the left Kan extension of F along $\iota : \mathcal{A} \rightarrow \mathcal{A}^{\aleph}$ can be computed by

$$\text{Lan}_{\iota} F(x) = \text{colim}_{y \in \mathcal{F}} F(y),$$

where \mathcal{F} denotes the poset of finite direct sums $\bigoplus_{i \in \mathcal{J}} x_i$, ordered by canonical inclusions. (Indeed \mathcal{F} is cofinal in the comma category $\mathcal{K} \downarrow x$.) This implies that if F is additive then $\text{Lan}_{\iota} F$ is isomorphic to F^{\aleph} , hence additive. Since \mathcal{F} is filtered, this also implies that $\text{Lan}_{\iota} : \mathcal{F}(\mathcal{A}, \mathbb{k}) \rightarrow \mathcal{F}(\mathcal{A}^{\aleph}, \mathbb{k})$ is exact. The following result follows by standard homological algebra (see e.g. [18, Lm 1.4(v)]).

Proposition 3.3. *For all functors $F : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$ and $G : \mathcal{A}^{\aleph} \rightarrow \mathcal{V}_{\mathbb{k}}$, restriction along $\iota : \mathcal{A} \hookrightarrow \mathcal{A}^{\aleph}$ yields a graded isomorphism:*

$$\mathrm{Ext}_{\mathcal{F}(\mathcal{A}^{\aleph}, \mathbb{k})}^*(\mathrm{Lan}_{\iota} F, G) \simeq \mathrm{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(F, G \circ \iota) .$$

Moreover, if F and G are additive then a similar isomorphism also holds for Ext computed in the categories of additive functors.

Next proposition reveals the advantage of working with the larger category \mathcal{A}^{\aleph} .

Proposition 3.4. *Assume that for all pairs (a, b) of objects of \mathcal{A} , $\mathcal{A}(a, b)$ is an \mathbb{F}_p -vector space of cardinal less than \aleph . Then for all a in \mathcal{A} , the functor $\mathcal{A}^{\aleph}(a, -) : \mathcal{A}^{\aleph} \rightarrow \mathbf{P}_{\mathbb{F}_p}^{\aleph}$ has a left adjoint $a \otimes -$. Moreover, for all functors $F : \mathbf{P}_{\mathbb{F}_p} \rightarrow \mathcal{V}_{\mathbb{k}}$ and $G : \mathcal{A}^{\aleph} \rightarrow \mathcal{V}_{\mathbb{k}}$, there is a graded isomorphism:*

$$\mathrm{Ext}_{\mathcal{F}(\mathcal{A}^{\aleph}, \mathbb{k})}^*(F \circ \mathcal{A}^{\aleph}(a, -), G) \simeq \mathrm{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})}^*(F, G \circ (a \otimes -)) .$$

which is induced by precomposition by $a \otimes -$ and restriction along the unit of adjunction $x \mapsto \mathcal{A}(a, a \otimes x)$. If F and G are additive then a similar isomorphism also holds for Ext computed in the categories of additive functors.

Proof. Let v be a \mathbb{F}_p -vector space with basis $(b_i)_{i \in I}$. We let $v \otimes x := \bigoplus_{i \in I} x_i$ where each x_i denotes a copy of x . The \mathbb{F}_p -linear map $v \mapsto \mathcal{A}^{\aleph}(x, v \otimes x)$, sending b_i to the canonical inclusion of x as the factor x_i of $v \otimes x$ is initial in $v \downarrow \mathcal{A}^{\aleph}(x, -)$, and the existence of the adjoint follows from [21, Lm 4.6.1]. The Ext -isomorphism is then given by [18, Lm 1.5]. \square

4. RECOLLECTIONS OF FROBENIUS TWISTS

Assume that the field \mathbb{k} is perfect, of positive characteristic p . For all integers r , the r -th Frobenius twist of a vector space v is the vector space $v^{(r)}$ which equals v as an abelian group, with action of \mathbb{k} given by $\lambda \cdot x := \lambda^{p^{-r}} x$. This construction, which may also be seen as *tensoring* with \mathbb{k} with action twisted by $\lambda \mapsto \lambda^{p^r}$, is natural with respect to the vector space v . If $r \geq 0$ then the r -th Frobenius twist is the underlying functor of a strict polynomial functor $I^{(r)}$.

We denote by E_r^* the graded \mathbb{k} -algebra of self-extensions of $I^{(r)}$ in strict polynomial functors. This algebra is computed in [9, Thm 4.10] (with a slightly different notation for the generators):

$$E_r^* = \mathrm{Ext}_{\mathcal{P}_{\mathbb{k}}}^*(I^{(r)}, I^{(r)}) \simeq \mathbb{k}[e_1, \dots, e_r] / \langle e_1^p = e_2^p = \dots = e_r^p = 0 \rangle ,$$

where each e_i is a class of cohomological degree $2p^{i-1}$. In particular, as a graded vector space, E_r^* has dimension one in degree $2i$ for $0 \leq i < p^r$, and it is zero in the other degrees. Precomposition by the Frobenius twist $I^{(1)}$ yields a morphism of graded \mathbb{k} -algebras: $E_r^* \rightarrow E_{r+1}^*$. It is shown in [9, Cor 4.9] that this map is injective. For dimension reasons, it is an isomorphism in degrees less than $2p^r$. We denote by E_{∞}^* the colimit of the E_r^* . Thus, as graded \mathbb{k} -algebras we have

$$E_{\infty}^* \simeq \mathbb{k}[e_i, i \geq 1] / \langle e_i^p = 0 \text{ for all } i \geq 1 \rangle$$

and the canonical map $E_r^* \rightarrow E_{\infty}^*$ identifies E_r^* with the subalgebra of E_{∞}^* generated by the classes e_1, \dots, e_r .

For all $r \geq 0$, there is a morphism of graded \mathbb{k} -algebras $\alpha_r^{\mathbb{k}} : E_r^* \rightarrow E^*$ defined as the composition of the Ext -map induced by the forgetful functor $\mathcal{P}_{\mathbb{k}} \rightarrow \mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})$

and precomposition by the $(-r)$ -th Frobenius twist (here we need that \mathbb{k} is perfect!):

$$\alpha_r^{\mathbb{k}} : E_r^* = \text{Ext}_{\mathcal{P}_{\mathbb{k}}}^*(I^{(r)}, I^{(r)}) \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I^{(r)}, I^{(r)}) \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I) = E^* .$$

Passing to the colimit yields a morphism of graded \mathbb{k} -algebras $\alpha_{\infty}^{\mathbb{k}} : E_{\infty}^* \rightarrow E^*$. The following result is a special case of [7, Thm 3.10], which can be proved directly by comparing the computations of [8] and [9].

Theorem 4.1 ([8, 9]). *If $\mathbb{k} = \mathbb{F}_p$, the map $\alpha_{\infty}^{\mathbb{F}_p}$ is an isomorphism. Equivalently, for all integers r , the maps $\alpha_r^{\mathbb{F}_p}$ are isomorphisms in degrees less than $2p^r$.*

5. A FIRST Ext-COMPUTATION IN $\mathcal{F}(\mathcal{A}, \mathbb{k})$

In this section, \mathbb{k} is a perfect field of positive characteristic p . Given an additive functor $\pi : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$ and $r \in \mathbb{N} \cup \{+\infty\}$, we denote by π_r^* the morphism of graded \mathbb{k} -algebras defined as the composition:

$$\pi_r^* : E_r^* \xrightarrow{\alpha_r} E^* = \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I) \xrightarrow{\pi^*} \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, \pi)$$

where α_r is recalled in section 4 and π^* is induced by evaluation on π . The goal of this section is to prove the following theorem, which is a preliminary version of theorem 1.

Theorem 5.1. *Let \mathbb{k} be a perfect field of positive characteristic p . Assume that \mathcal{A} is \mathbb{F}_p -linear. For all additive functors $\pi, \rho : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$, the following map is an isomorphism of graded \mathbb{k} -vector spaces:*

$$\Psi_{\infty} : \begin{array}{ccc} \text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \otimes_{\mathbb{k}} E_{\infty}^* & \rightarrow & \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \\ e \otimes e' & \mapsto & \Phi(e) \circ \pi_{\infty}^*(e') \end{array} .$$

We shall prove theorem 5.1 in several steps.

Lemma 5.2 (First step). *Theorem 5.1 holds if $\mathcal{A} = \mathbf{P}_{\mathbb{F}_p}$.*

Proof. There is an equivalence of categories $\mathcal{V}_{\mathbb{k}} \simeq \mathbf{Add}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})$ which sends a \mathbb{k} -vector space u to the additive functor $v \mapsto v \otimes_{\mathbb{F}_p} u$. Thus the additive functor π and ρ are direct sums of copies of the additive functor $t : v \mapsto v \otimes_{\mathbb{F}_p} \mathbb{k}$. Hence by additivity of Ext , Φ , π_{∞}^* , and by degreewise finiteness of E_{∞}^* , the proof reduces to the case $\pi = \rho = t$. In this case we have

$$\text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(t, t) = \text{Ext}_{\mathbb{k}}^*(\mathbb{k}, \mathbb{k}) = \text{Hom}_{\mathbb{k}}(\mathbb{k}, \mathbb{k}) \simeq \mathbb{k} .$$

Thus the proof reduces further to showing that $t_{\infty}^* : E_{\infty}^* \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})}^*(t, t)$ is an isomorphism. The latter is equivalent to showing that for all positive integers r , t_r^* is an isomorphism in degree less than $2p^r$.

For this purpose, we consider the commutative square

$$\begin{array}{ccc} \text{Ext}_{\mathcal{P}_{\mathbb{k}}}^i(I^{(r)}, I^{(r)}) & \xrightarrow{t_r^*} & \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})}^i(t, t) \\ \simeq \uparrow & & \simeq \uparrow \\ k \otimes_{\mathbb{F}_p} \text{Ext}_{\mathcal{P}_{\mathbb{F}_p}}^i(I^{(r)}, I^{(r)}) & \xrightarrow{\mathbb{k} \otimes \alpha_r^{\mathbb{F}_p}} & k \otimes_{\mathbb{F}_p} \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{F}_p)}^i(I, I) \end{array}$$

where the vertical isomorphism on the left is the base change isomorphism for strict polynomial functors [22, 2.7] and the vertical isomorphism on the right is induced by tensorization with \mathbb{k} (the latter is an iso since I is fp_{∞} by [8, Prop 10.1]). As

recalled in section 4 $\alpha_r^{\mathbb{F}_p}$ is an isomorphism in degrees less than $2p^r$, hence t_r^* is an isomorphism in degrees less than $2p^r$. \square

Lemma 5.3 (Second step). *Let \mathcal{A} be an arbitrary svelte, additive \mathbb{F}_p -linear category. Then theorem 5.1 holds if $\pi = \mathbb{k} \otimes_{\mathbb{F}_p} \mathcal{A}(a, -)$ for some $a \in \mathcal{A}$.*

Proof. Let \aleph be a cardinal larger than the cardinal of $\mathcal{A}(x, y)$ for all x and y , and let \mathcal{A}^\aleph be the \aleph -additive envelope of \mathcal{A} , as in definition 3.1. Let $\pi' = k \otimes_{\mathbb{Z}} \mathcal{A}^\aleph(a, -)$ and let $\rho' : \mathcal{A}^\aleph \rightarrow k\text{-Mod}$ be an arbitrary additive extension of ρ . Then π' is \aleph -additive, hence it is the left Kan extension of π to \mathcal{A}^\aleph , hence by proposition 3.3, lemma 5.3 reduces to proving that

$$\Psi_\infty : \text{Ext}_{\mathbf{Add}(\mathcal{A}^\aleph, \mathbb{k})}^*(\pi', \rho') \otimes_{\mathbb{k}} E_\infty^* \rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}^\aleph, \mathbb{k})}^*(\pi', \rho')$$

is an isomorphism. Now let $t' : \mathbf{P}_{\mathbb{F}_p}^\aleph \rightarrow \mathcal{V}_{\mathbb{k}}$ be given by $t'(v) = \mathbb{k} \otimes_{\mathbb{F}_p} v$. Then proposition 3.4 provides a commutative square with vertical isomorphisms:

$$\begin{array}{ccc} \text{Ext}_{\mathbf{Add}(\mathcal{A}^\aleph, \mathbb{k})}^*(\pi', \rho') \otimes_{\mathbb{k}} E_\infty^* & \xrightarrow{\Psi_\infty} & \text{Ext}_{\mathcal{F}(\mathcal{A}^\aleph, \mathbb{k})}^*(\pi', \rho') \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Ext}_{\mathbf{Add}(\mathbf{P}_{\mathbb{F}_p}^\aleph, \mathbb{k})}^*(t', \rho' \circ (a \otimes -)) \otimes_{\mathbb{k}} E_\infty^* & \xrightarrow{\Psi_\infty} & \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}^\aleph, \mathbb{k})}^*(t', \rho' \circ (a \otimes -)) \end{array} .$$

Thus, in order to prove lemma 5.3, it suffices to prove that the lower ψ_∞ in this square is an isomorphism. By proposition 3.3 again, this is equivalent to showing that

$$\Psi_\infty : \text{Ext}_{\mathbf{Add}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})}^*(t, \rho_a) \otimes_{\mathbb{k}} E_\infty^* \rightarrow \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{F}_p}, \mathbb{k})}^*(t, \rho_a)$$

is an isomorphism, where t and ρ_a denote the restrictions of t' and $\rho' \circ (a \otimes -)$ to $\mathbf{P}_{\mathbb{F}_p}$. But this latter Ψ_∞ is an isomorphism by lemma 5.2. \square

We can now conclude the proof of theorem 5.1 by a spectral sequence argument.

End of the proof of theorem 5.1. By lemma 5.3, Ψ_∞ is an isomorphism for $\pi = k \otimes_{\mathbb{Z}} \mathcal{A}(a, -)$ for all objects a of \mathcal{A} . Now every projective object of $\mathbf{Add}(\mathcal{A}, \mathbb{k})$ is a direct summand of a direct sum of such functors; as Ext sends arbitrary direct sums at the source to products, which are here both exact, and by degreewise finiteness of E_∞^* , Ψ_∞ is an isomorphism for all projectives π of $\mathbf{Add}(\mathcal{A}, \mathbb{k})$.

Now let π be an arbitrary object of $\mathbf{Add}(\mathcal{A}, \mathbb{k})$, let P be a projective resolution of π in $\mathbf{Add}(\mathcal{A}, \mathbb{k})$, and let Q be an injective resolution of ρ in $\mathcal{F}(\mathcal{A}, \mathbb{k})$. We consider the bicomplexes:

$$C^{p,q} = \text{Hom}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}(P_p, \rho) \otimes_{\mathbb{k}} E_\infty^q, \quad D^{p,q} = \text{Hom}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}(P_p, Q^q).$$

Here we consider E_∞^* as a complex with zero differential, hence the second differential of C is zero. We have two associated spectral sequences:

$$\begin{aligned} E_1^{p,q}(C) &= \text{Hom}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}(P_p, \rho) \otimes_{\mathbb{k}} E_\infty^q \Rightarrow (\text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}(\pi, \rho) \otimes_{\mathbb{k}} E_\infty)^{p+q}, \\ E_1^{p,q}(D) &= \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^q(P_p, \rho) \Rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^{p+q}(\pi, \rho). \end{aligned}$$

For all even integers q , we choose a cycle $z(q)$ representing $e_\rho(q)$ in the complex $\text{Hom}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}(\pi, Q)$. Then the morphism of bicomplexes $\chi^{p,q} : C^{p,q} \rightarrow D^{p,q}$ such that

$\chi^{p,q}(f \otimes e_\infty(q)) = z(q) \circ f$ induces a morphism of spectral sequences $E(\chi)$. By construction, the morphisms

$$\begin{aligned} E_1^{p,*}(\chi) &: \text{Hom}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}(P_p, \rho) \otimes_{\mathbb{k}} E_\infty^* \rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(P_p, \rho) \\ \text{Tot}(\chi) &: \text{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \otimes_{\mathbb{k}} E_\infty^* \rightarrow \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, \rho) \end{aligned}$$

are both equal to Ψ_∞ . Thus $E_1(\chi)$ is an isomorphism, which implies that $\text{Tot}(\chi)$ is an isomorphism. \square

6. PROOF OF THEOREM 1

The following lemma gives the first part of theorem 1, granting the computation of E_∞^* recalled in section 4.

Lemma 6.1. *The map $\alpha_\infty^{\mathbb{k}} : E_\infty^* \rightarrow E^* = \text{Ext}_{\mathcal{F}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I)$ is an isomorphism for all perfect fields \mathbb{k} of positive characteristic.*

Proof. We use a Hochschild homology computation. The enveloping ring $\mathbb{k} \otimes_{\mathbb{Z}} \mathbb{k} = \mathbb{k} \otimes_{\mathbb{F}_p} \mathbb{k}$ and every field extension of \mathbb{F}_p is a filtered colimit of \mathbb{F}_p -subalgebras which are smooth and essentially of finite type. Thus $\text{HH}_*(\mathbb{k})$ is an exterior algebra over the \mathbb{k} -vector space of Kähler forms $\Omega^1(\mathbb{k}/\mathbb{F}_p)$ by the Hochschild-Kostant-Rosenberg theorem [10, Cor 2.13]. Since \mathbb{k} is perfect, $\Omega^1(\mathbb{k}/\mathbb{F}_p)$ is zero, so that $\text{HH}_i(\mathbb{k}) = \mathbb{k}$ for $i = 0$, and zero otherwise. By duality, this implies that $\text{HH}^i(\mathbb{k}) = \mathbb{k}$ for $i = 0$, and zero otherwise. Now the equivalence of categories

$$\begin{array}{ccc} \mathbb{k} \otimes_{\mathbb{F}_p} \mathbb{k}\text{-Mod} & \simeq & \mathbf{Add}(\mathbf{P}_{\mathbb{k}}, \mathbb{k}) \\ M & \mapsto & [v \mapsto v \otimes_{\mathbb{k}} M] \end{array}$$

sends the (\mathbb{k}, \mathbb{k}) -bimodule \mathbb{k} to I , hence the graded vector space

$$\text{Ext}_{\mathbf{Add}(\mathbf{P}_{\mathbb{k}}, \mathbb{k})}^*(I, I) \simeq \text{Ext}_{\mathbb{k} \otimes_{\mathbb{F}_p} \mathbb{k}}^*(\mathbb{k}, \mathbb{k}) \simeq \text{HH}^*(\mathbb{k})$$

equals zero in positive degree, and \mathbb{k} in degree zero. Thus, theorem 5.1 with $\mathcal{A} = \mathbf{P}_{\mathbb{k}}$ and $\pi = \rho = I$ implies that $I^* = \alpha_\infty^{\mathbb{k}} : E_\infty^* \rightarrow E^*$ is an isomorphism. \square

End of the proof of theorem 1. We observe that $\Psi_\infty = \Psi \circ (\text{id} \otimes \alpha_\infty^{\mathbb{k}})$, where Ψ_∞ is the isomorphism of theorem 5.1 and Ψ is the map of theorem 1. Since $\text{id} \otimes \alpha_\infty^{\mathbb{k}}$ is an isomorphism by lemma 6.1, we conclude that Ψ is an isomorphism too. \square

Remark 6.2. The proof of lemma 6.1 shows that for $\mathcal{A} = \mathbf{P}_{\mathbb{k}}$ and $\pi = \rho = I$ the source of Ψ is equal to $\text{HH}^*(\mathbb{k}) \otimes_{\mathbb{k}} E^*$ while the target of Ψ is equal to E^* . If \mathbb{k} is not perfect then $\text{HH}^*(\mathbb{k})$ may be non-trivial so that Ψ may fail to be an isomorphism.

7. DUALIZATION OF THEOREM 1

Given a vector space V and a functor $F : \mathcal{A}^{\text{op}} \rightarrow \mathcal{V}_{\mathbb{k}}$, we let $D_V F : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$ be the functor such that $D_V F(a) = \text{Hom}_{\mathbb{k}}(F(a), V)$. Then the tensor product over \mathcal{A} is characterized by the isomorphism, natural with respect to F , G and V :

$$\text{Hom}_{\mathbb{k}}(F \otimes_{\mathcal{A}} G, V) \simeq \text{Hom}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}(G, D_V F).$$

By deriving this isomorphism, we obtain isomorphisms, natural with respect to F , G , V , in which $\text{Hom}_{\mathbb{k}}(-, V)$ should be interpreted in the sense of a restricted dual (i.e. applied degreewise):

$$\text{Hom}_{\mathbb{k}}(\text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F, G), V) \simeq \text{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(G, D_V F),$$

There is a similar isomorphism involving Tor and Ext between additive functors. These isomorphisms allow us to prove corollary 2.

Proof of corollary 2. We have a chain of graded isomorphisms, natural with respect to π , ρ and V , in which the third isomorphism is provided by theorem 1:

$$\begin{aligned} \mathrm{Hom}_{\mathbb{k}}(\mathrm{Tor}_*^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(\pi, \rho) \otimes_{\mathbb{k}} T_*, V) &\simeq \mathrm{Hom}_{\mathbb{k}}(\mathrm{Tor}_*^{\mathbb{k} \otimes_{\mathbb{Z}} \mathcal{A}}(\pi, \rho), V) \otimes_{\mathbb{k}} E^* \\ &\simeq \mathrm{Ext}_{\mathbf{Add}(\mathcal{A}, \mathbb{k})}^*(\pi, D_V \rho) \otimes_{\mathbb{k}} E^* \\ &\simeq \mathrm{Ext}_{\mathcal{F}(\mathcal{A}, \mathbb{k})}^*(\pi, D_V \rho) \\ &\simeq \mathrm{Hom}_{\mathbb{k}}(\mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi, \rho), V) . \end{aligned}$$

Hence the result follows from the Yoneda lemma. \square

8. SOME TOR-COMPUTATIONS BETWEEN NON-ADDITIVE FUNCTORS

Additive functors are building blocks for interesting non-additive functors. In this section, we explain how to compute Tor between these non-additive functors from the Tor between their additive building blocks. The results of the section should not be surprising for experts, though they cannot be found in the literature. Combined with our theorem 1, these results yield example 3 of the introduction, and many other computations.

In this section \mathcal{A} and \mathcal{B} are arbitrary svelte additive categories, and \mathbb{k} is an arbitrary field. Unadorned tensor products are taken over \mathbb{k} . We first give Tor-versions of some well-known properties of Ext, see e.g. [7, proof of Thm 1.7] or [18]. Given a functor $\phi : \mathcal{A} \rightarrow \mathcal{B}$, we have a canonical restriction map

$$\mathrm{res}^{\phi} : \mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F \circ \phi, G \circ \phi) \rightarrow \mathrm{Tor}_*^{\mathbb{k}[\mathcal{B}]}(F, G) .$$

The following lemma is a straightforward check.

Lemma 8.1. *Let $\phi : \mathcal{A} \rightleftarrows \mathcal{B} : \psi$ be an adjoint pair, and let $u : \mathrm{id} \rightarrow \psi \circ \phi$ and $e : \phi \circ \psi \rightarrow \mathrm{id}$ be the unit and the counit of an adjunction. Then the following composition is an isomorphism, whose inverse is induced by $F(e)$ and res^{ψ} :*

$$\mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F \circ \phi, G) \xrightarrow{G(u)_*} \mathrm{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F \circ \phi, G \circ \psi \circ \phi) \xrightarrow{\mathrm{res}^{\phi}} \mathrm{Tor}_*^{\mathbb{k}[\mathcal{B}]}(F, G \circ \psi) .$$

In the sequel, lemma 8.1 will be applied to the d -th iterated diagonal functor $D : \mathcal{A} \rightarrow \mathcal{A}^d$, $D(a) = (a, \dots, a)$, and the d -th iterated direct sum functor $\Pi : \mathcal{A}^d \rightarrow \mathcal{A}$, $\Pi(a_1, \dots, a_d) = a_1 \oplus \dots \oplus a_d$, which are adjoint to each other on both sides, giving isomorphisms:

$$\begin{aligned} \mathrm{Tor}_*^{k[\mathcal{A}^d]}(F \circ \Pi, G) &\simeq \mathrm{Tor}_*^{k[\mathcal{A}]}(F, G \circ D) , \\ \mathrm{Tor}_*^{k[\mathcal{A}]}(F' \circ D, G') &\simeq \mathrm{Tor}_*^{k[\mathcal{A}^d]}(F', G' \circ \Pi) . \end{aligned}$$

The second property is a Künneth isomorphism. We let $F_1 \boxtimes \dots \boxtimes F_d : \mathcal{B}^d \rightarrow \mathcal{V}_{\mathbb{k}}$ denote the external tensor product of functors $F_i : \mathcal{B} \rightarrow \mathcal{V}_{\mathbb{k}}$, defined by:

$$(F_1 \boxtimes \dots \boxtimes F_d)(b_1, \dots, b_d) = F_1(b_1) \otimes \dots \otimes F_d(b_d) .$$

Lemma 8.2. *For all functors $F_i : \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{V}_{\mathbb{k}}$ and $G_i : \mathcal{A} \rightarrow \mathcal{V}_{\mathbb{k}}$, with $1 \leq i \leq d$, there is a Künneth isomorphism:*

$$\kappa : \mathrm{Tor}_*^{k[\mathcal{A}^d]}(F_1 \boxtimes \dots \boxtimes F_d, G_1 \boxtimes \dots \boxtimes G_d) \simeq \bigotimes_{1 \leq i \leq d} \mathrm{Tor}_*^{k[\mathcal{A}]}(F_i, G_i) .$$

Proof. The tensor product $- \otimes_{\mathcal{B}} G : \mathcal{F}(\mathcal{B}^{\text{op}}, \mathbb{k}) \rightarrow \mathcal{V}_{\mathbb{k}}$ is the unique cocontinuous functor, such that there is a "Yoneda isomorphism" $P_b \otimes_{\mathcal{B}} G \simeq G(b)$, natural with respect to b and G , where P_b is a standard projective: $P_b = \mathbb{k}[\mathcal{B}(-, b)]$. The external tensor product $P_{a_1} \boxtimes \cdots \boxtimes P_{a_d}$ is isomorphic to $P_{(a_1, \dots, a_d)}$ hence the Yoneda isomorphisms yield an isomorphism, natural with respect to functors G_i and the standard projectives F_i :

$$(F_1 \boxtimes \cdots \boxtimes F_d) \otimes_{\mathcal{A}^d} (G_1 \boxtimes \cdots \boxtimes G_d) \simeq \bigotimes_{1 \leq i \leq d} G_i(a_i) \simeq \bigotimes_{1 \leq i \leq d} F_i \otimes_{\mathcal{A}} G_i .$$

The result follows from this isomorphism by taking projective resolutions. \square

Now we define some non-additive functors of interest. We fix additive functors π_1, \dots, π_d from \mathcal{A}^{op} to $\mathcal{V}_{\mathbb{k}}$ and we consider the (non-additive if $d > 1$) functor:

$$\begin{aligned} \pi^{\otimes} := \pi_1 \otimes \cdots \otimes \pi_d : \mathcal{A}^{\text{op}} &\rightarrow \mathcal{V}_{\mathbb{k}} \\ a &\mapsto \pi_1(a) \otimes \cdots \otimes \pi_n(a) . \end{aligned}$$

We also fix covariant additive functors ρ_1, \dots, ρ_e we set $\rho^{\otimes} := \rho_1 \otimes \cdots \otimes \rho_e$, and we consider the graded vector space

$$\mathbb{T}_* = \text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi^{\otimes}, \rho^{\otimes}) .$$

The next two propositions compute this graded vector space. The first one is a variant of Pirashvili's vanishing lemma [18, Cor 2.13], [16].

Proposition 8.3. *If $d \neq e$, then $\mathbb{T}_* = 0$.*

Proof. Every (covariant or contravariant) functor F from \mathcal{A} to $\mathcal{V}_{\mathbb{k}}$ splits as a direct sum of a constant functor with value $F(0)$ and a "reduced" functor rF , that is, a functor such that $(rF)(0) = 0$. Reduced functors have projective resolutions by direct sums of reduced standard projectives, and the tensor product of a reduced functor with a constant functor is zero, hence we obtain:

(*) *Assume either that F is reduced and G is constant, or that G is reduced and F is constant. Then $\text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(F, G) = 0$.*

Now assume for example that $d > e$. Let $\pi^{\boxtimes} := \pi_1 \boxtimes \cdots \boxtimes \pi_d$. Then $\pi^{\otimes} = \pi^{\boxtimes} \circ D$, hence lemma 8.1 yields an isomorphism $\mathbb{T}_* \simeq \text{Tor}_*^{\mathbb{k}[\mathcal{A}^d]}(\pi^{\boxtimes}, \rho^{\otimes} \circ \Pi)$. By additivity of the ρ_i , the functor $\rho^{\otimes} \circ \Pi$ is isomorphic to a direct sum of external tensor products $F_1 \boxtimes \cdots \boxtimes F_d$ and since $d > e$ at least one F_i is constant. Thus, the result follows from the Künneth isomorphism of lemma 8.2 together with the cancellation property (*). \square

From now on, we assume that $d = e$. Every permutation $\sigma \in \mathfrak{S}_d$ defines a morphism, still denoted by σ , from the tensor product π^{\otimes} to the swapped tensor product $\pi^{\otimes} \sigma := \pi_{\sigma(1)} \otimes \cdots \otimes \pi_{\sigma(d)}$. To be more specific, σ is given by

$$\begin{aligned} \sigma : \pi_1(a) \otimes \cdots \otimes \pi_d(a) &\rightarrow \pi_{\sigma(1)}(a) \otimes \cdots \otimes \pi_{\sigma(d)}(a) \\ x_1 \otimes \cdots \otimes x_d &\mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(d)} . \end{aligned}$$

We set

$$\mathbb{T}_*^{\sigma} := \bigotimes_{1 \leq i \leq d} \text{Tor}_*^{\mathbb{k}[\mathcal{A}]}(\pi_{\sigma(i)}, \rho_i) ,$$

and we denote by Υ the graded \mathbb{k} -linear map defined as the following composition (where the first map is induced the maps $\sigma : \pi^{\otimes} \rightarrow \pi^{\otimes} \sigma$, and a notation like $\pi^{\boxtimes} \sigma$

stands for the external tensor product $\pi_{\sigma(1)} \boxtimes \cdots \boxtimes \pi_{\sigma(d)}$ and the last isomorphism is the Künneth isomorphism of lemma 8.2):

$$\Upsilon : \mathbb{T}_* \xrightarrow{\prod \sigma_*} \bigoplus_{\sigma \in \mathfrak{S}_d} \mathrm{Tor}_*^{\mathbb{k}[A]}(\pi^\otimes \sigma, \rho^\otimes) \xrightarrow{\oplus \mathrm{res}^D} \bigoplus_{\sigma \in \mathfrak{S}_d} \mathrm{Tor}_*^{\mathbb{k}[A^d]}(\pi^\boxtimes \sigma, \rho^\boxtimes) \simeq \bigoplus_{\sigma \in \mathfrak{S}_d} \mathbb{T}_*^\sigma.$$

The next proposition is our computation of \mathbb{T}_* when $d = e$.

Proposition 8.4. *The map Υ is an isomorphism.*

Proof. Since the π_i are additive, there is a decomposition

$$\pi^\otimes(a_1 \oplus \cdots \oplus a_d) = \bigoplus_{1 \leq i_1, \dots, i_d \leq d} \pi_1(a_{i_1}) \otimes \cdots \otimes \pi_d(a_{i_d}).$$

For all $\sigma \in \mathfrak{S}_d$ we let $f_\sigma : \pi^\otimes(a_1 \oplus \cdots \oplus a_d) \rightarrow \pi_{\sigma(1)}(a_1) \otimes \cdots \otimes \pi_{\sigma(d)}(a_d)$ be the map which is zero on all summands of $\pi^\otimes(a_1 \oplus \cdots \oplus a_d)$, except on the summand such that $i_k = \sigma^{-1}(k)$ for all k , where it is defined by $f_\sigma(x_1 \otimes \cdots \otimes x_d) = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(d)}$. The maps f_σ are the components of an epimorphism

$$f : \pi^\otimes \circ \Pi \rightarrow \bigoplus_{\sigma \in \mathfrak{S}_d} \pi^\boxtimes \sigma,$$

which satisfies the following properties.

- (1) The kernel of f is a direct summand of $\pi^\otimes \circ \Pi$, and it is isomorphic to a direct sum of external tensor products $F_1 \boxtimes \cdots \boxtimes F_d$ in which at least one of the F_i is constant.
- (2) The map $\prod \sigma : \pi^\otimes \rightarrow \bigoplus_{\sigma \in \mathfrak{S}_d} \pi^\otimes \sigma$ is equal to the following composition, in which $u : \mathrm{id} \rightarrow \Pi \circ D$ is the unit of an adjunction between D and Π :

$$\pi^\otimes \xrightarrow{\pi^\otimes(u)} \pi^\otimes \circ \Pi \circ D \xrightarrow{f(D)} \bigoplus_{\sigma \in \mathfrak{S}_d} \pi^\boxtimes \sigma \circ D = \bigoplus_{\sigma \in \mathfrak{S}_d} \pi^\otimes \sigma.$$

Property (2) implies that there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Tor}_*^{\mathbb{k}[A]}(\pi^\otimes \circ \Pi \circ D, \rho^\otimes) & \xrightarrow{\mathrm{res}^D} & \mathrm{Tor}_*^{\mathbb{k}[A^d]}(\pi^\otimes \circ \Pi, \rho^\boxtimes) \\ \pi^\otimes(u)_* \nearrow & \downarrow f(D)_* & \downarrow f_* \\ \mathbb{T}_* \xrightarrow{\prod \sigma} \mathrm{Tor}_*^{\mathbb{k}[A]}(\bigoplus \pi^\otimes \sigma, \rho^\otimes) & \xrightarrow{\mathrm{res}^D} & \mathrm{Tor}_*^{\mathbb{k}[A^d]}(\bigoplus \pi^\boxtimes \sigma, \rho^\boxtimes) \end{array}$$

and property (1) implies (by the same reasoning as in the proof of proposition 8.3) that f_* is an isomorphism. Now the composition of res^D and $\pi^\otimes(u)_*$ is an isomorphism by lemma 8.1, hence the bottom row of the diagram is an isomorphism. The result follows. \square

For further computations, we assume that we are given tuples $\mathbf{d} = (d_1, \dots, d_n)$ and $\mathbf{e} = (e_1, \dots, e_m)$ such that $d_1 + \cdots + d_n = d = e_1 + \cdots + e_m$, and that π^\otimes and ρ^\otimes have the following specific form:

$$\pi^\otimes = \pi_1^{\otimes d_1} \otimes \cdots \otimes \pi_n^{\otimes d_n}, \quad \rho^\otimes = \rho_1^{\otimes e_1} \otimes \cdots \otimes \rho_m^{\otimes e_m}.$$

If we denote by $\mathfrak{S}_{\mathbf{d}}$ the Young subgroup $\mathfrak{S}_{d_1} \times \cdots \times \mathfrak{S}_{d_n} \subset \mathfrak{S}_d$, then every $\sigma \in \mathfrak{S}_{\mathbf{d}}$ induces an endomorphism of π^\otimes , and this defines a right action of $\mathfrak{S}_{\mathbf{d}}$ on π^\otimes . Similarly, there is a right action of the Young subgroup $\mathfrak{S}_{\mathbf{e}}$ on ρ^\otimes , hence \mathbb{T}_* is a right $\mathfrak{S}_{\mathbf{d}} \times \mathfrak{S}_{\mathbf{e}}$ -module. The next lemma is a straightforward check.

Lemma 8.5. *If $(\tau, \mu) \in \mathfrak{S}_d \times \mathfrak{S}_e$ and if $x = x_1 \otimes \cdots \otimes x_d \in \mathbb{T}_*^\sigma$, then we let*

$$x \cdot (\tau, \mu) = \epsilon x_{\mu(1)} \otimes \cdots \otimes x_{\mu(d)} \in \mathbb{T}_*^{\tau^{-1}\sigma\mu},$$

where ϵ is the usual Koszul sign determined by μ and the degrees of the x_i , i.e. the sign such that $x_1 \cdots x_d = \epsilon x_{\mu(1)} \cdots x_{\mu(d)}$ in the free graded commutative algebra on x_1, \dots, x_d . This defines a right $\mathfrak{S}_d \times \mathfrak{S}_e$ -action on $\bigoplus_{\sigma \in \mathfrak{S}_d} \mathbb{T}_*^\sigma$ which makes Υ a morphism of $\mathfrak{S}_d \times \mathfrak{S}_e$ -modules.

Now let U be a left \mathfrak{S}_d -module. Then, instead of π^\otimes we can consider the functor $\pi^\otimes \otimes_{\mathfrak{S}_d} U$. Similarly if V is a left \mathfrak{S}_e -module, we consider the functor $\rho^\otimes \otimes_{\mathfrak{S}_e} V$. The following lemma gives another Tor-computation, granted our computation of the right $\mathfrak{S}_d \times \mathfrak{S}_e$ -module \mathbb{T}_* . Note that by Maschke's theorem, the projectivity hypothesis on U and V is automatically satisfied if the integers d_i and e_j are all invertible in \mathbb{k} (thus if \mathbb{k} has characteristic zero, or big prime characteristic).

Proposition 8.6. *Assume that U and V are projective as modules over the symmetric group. There is a graded isomorphism, natural with respect to U and V :*

$$\mathrm{Tor}_*^{\mathbb{k}[A]}(\pi^\otimes \otimes_{\mathfrak{S}_d} U, \rho^\otimes \otimes_{\mathfrak{S}_e} V) \simeq \mathbb{T}_* \otimes_{\mathfrak{S}_d \times \mathfrak{S}_e} (U \otimes V).$$

Proof. The formula is obviously true if U and V are direct sums of copies of the group rings $\mathbb{k}[\mathfrak{S}_d]$ and $\mathbb{k}[\mathfrak{S}_e]$ respectively. The general result follows from the fact that U and V are direct summands of such direct sums. \square

Remark 8.7 (The relevance of $\pi^\otimes \otimes_{\mathfrak{S}_d} U$). Let U_i be a left representation of \mathfrak{S}_{d_i} , and let $F_i : \mathbf{P}_\mathbb{k} \rightarrow \mathcal{V}_\mathbb{k}$ denote the functor such that $F_i(v) = v^{\otimes d_i} \otimes_{\mathfrak{S}_{d_i}} U_i$, where the right action of \mathfrak{S}_{d_i} on $v^{\otimes d_i}$ is given by permuting the factors of the tensor product. Then if $U = \bigotimes_{1 \leq i \leq n} U_i$ we have

$$\pi^\otimes \otimes_{\mathfrak{S}_d} U \simeq \pi_1^* F_1 \otimes \cdots \otimes \pi_n^* F_n.$$

Assume for simplicity that \mathbb{k} is algebraically closed. Then it follows from [4, Thm 4] that if the N_i are projective and simple \mathfrak{S}_{e_i} -modules, and if the additive functors π_1, \dots, π_m are pairwise non-isomorphic, simple, with finite-dimensional values, then $\pi^\otimes \otimes_{\mathfrak{S}_d} U$ is a simple functor. If \mathbb{k} has characteristic zero, such simple functors exhaust all the simple polynomial functors with finite-dimensional values (and all the simple functors with finite-dimensional values if in addition \mathcal{A} is \mathbb{Q} -linear by [4, Thm 3 and Cor 4.15]). If \mathbb{k} has positive characteristic, this does not exhaust all the simple polynomial functors with finite-dimensional values, but provides nonetheless many examples of simple functors.

If $\pi^\otimes = \pi^{\otimes d}$ for some additive functor π , then the description of the source of Υ simplifies a bit. Namely, we have an isomorphism of graded $\mathfrak{S}_d \times \mathfrak{S}_e$ -modules:

$$(3) \quad \bigoplus_{\sigma \in \mathfrak{S}_d} \mathbf{T}_*^\sigma \simeq \mathbb{k}[\mathfrak{S}_d] \otimes \mathbf{T}_*^e$$

where the group ring $\mathbb{k}[\mathfrak{S}_d]$ is placed in degree zero, with action of $\mathfrak{S}_d \times \mathfrak{S}_e$ given by $\sigma \cdot (\tau, \mu) = \tau^{-1}\sigma\mu$ and \mathbf{T}_*^e denotes the tensor product

$$\mathbf{T}_*^e := \mathrm{Tor}_*^{\mathbb{k}[A]}(\pi, \rho_1)^{\otimes e_1} \otimes \cdots \otimes \mathrm{Tor}_*^{\mathbb{k}[A]}(\pi, \rho_m)^{\otimes e_m}$$

with action of $\mathfrak{S}_d \times \mathfrak{S}_e$ action given by $(x_1 \otimes \cdots \otimes x_d) \cdot (\tau, \mu) = \epsilon x_{\mu(1)} \otimes \cdots \otimes x_{\mu(d)}$ with the usual Koszul sign ϵ as in lemma 8.5.

If U is the trivial \mathfrak{S}_d -module of dimension one then

$$\mathbb{T}_* \otimes_{\mathfrak{S}_d \times \mathfrak{S}_e} U \otimes V \simeq (\mathbb{T}_* \otimes_{\mathfrak{S}_d} U) \otimes_{\mathfrak{S}_e} V \simeq \mathbf{T}_*^e \otimes_{\mathfrak{S}_e} V .$$

Hence proposition 8.6 gives the computation of example 3. Further explicit computations could be obtained with other choices of U .

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