

# Technical Report Associated with the Paper ”Directional Critical Load Index: a Distance-to-Instability Metric for Continuum Robots”

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## I. INTRODUCTION

In the present technical report, associated with the paper [1], we provide the expressions of the derivatives of matrices  $\mathbf{H}$ ,  $\mathbf{G}$  with the respect to (w.r.t.) the variables  $f$ . Sec. II recalls the energy-based discretized modelling employed in [1]. This recall is necessary in order to have a self-contained report. Section III provides the geometrico-static model equations expressions when a finite-differences discretized modelling approach is used [2],[3]. Then, Sec. IV derives the expressions of matrices  $\mathbf{H}$ ,  $\mathbf{G}$  while Sec. V computes the derivatives of matrices  $\mathbf{H}$ ,  $\mathbf{G}$  necessary for the distance-to-instability metric computation.

## II. MODELLING

This section describes the energy-based modelling approach of this paper. Sec. II-A presents the *CRs* relevant variables. Then, Sec. II-B,II-C derive the deformation energy and external load energy, respectively. The total *CR* energy and constraints are discussed in Sec. II-D. Finally, the discretization process is introduced to derive the *CR* geometrico-static model in Sec. II-E.

### A. Description of the Continuum Robot

In this report, a *CR* is considered to be made by  $n$  flexible beams, and the *CR* is actuated by  $m$  motors. The variables associated with the motor actions are grouped into the vector  $\mathbf{q}_u \in \mathbb{R}^m$ . Then, a frame  $\mathcal{F}_p$  is rigidly attached to a specific *CR* location, assumed to be the *EE* reference point. The *EE* pose is described by  $\mathbf{q}_p = [\mathbf{p}_p; \boldsymbol{\alpha}_p] \in \mathbb{R}^{n_c}$ , where  $n_c = 3$  for the planar case,  $n_c \geq 6$  for the spatial case,  $\mathbf{p}_p$  describes the *EE* position, and  $\boldsymbol{\alpha}_p$  is a set of orientation parameters that define the rotation matrix  $\mathbf{R}_p$ . Assuming the same number of controlled and actuated variables, the vector  $\mathbf{q}_c \in \mathbb{R}^m$  stacks the controlled variables. Typically,  $\mathbf{q}_c$  is a subset of  $\mathbf{q}_p$ , and the vector  $\mathbf{q}_u$  collects the remaining variables of  $\mathbf{q}_p$  not included in  $\mathbf{q}_c$ .

### B. Deformation Energy

Let us consider a flexible beam as represented in Fig. 1: the beam is assumed to be one of the  $n$  flexible components of the *CR*, and the index  $i = 1, \dots, n$  denotes the  $i$ -th beam. Being  $L_i$  the length of the  $i$ -th beam, the coordinate  $s_i \in [0, L_i]$  parameterizes the beam’s centerline. A frame  $\mathcal{F}_i(s_i)$  is attached at each beam’s cross-section and the pose of the cross-section is defined by  $\mathbf{T}_i(s_i) \in SE(3)$ :

$$\mathbf{T}_i(s_i) = \begin{bmatrix} \mathbf{R}_i(s_i) & \mathbf{p}_i(s_i) \\ \mathbf{0} & 1 \end{bmatrix} \quad (1)$$

where  $\mathbf{R}_i(s_i) \in SO(3)$ ,  $\mathbf{p}_i(s_i) \in \mathbb{R}^3$  represent the rotation matrix and the position of  $\mathcal{F}_i(s_i)$  with respect to (w.r.t.) a fixed frame  $\mathcal{F}_0$ . To get the deformation energy of the beam, let us first evaluate the strain vector  $\boldsymbol{\xi}_i \in \mathbb{R}^6$  at  $s_i$ :

$$\hat{\boldsymbol{\xi}}_i(s_i) = \mathbf{T}_i^{-1}(s_i) \mathbf{T}_i'(s_i) \quad (2)$$

with  $(\cdot)' = \frac{d}{ds}$ ,  $\boldsymbol{\xi}_i = [\mathbf{u}_i, \mathbf{v}_i]$  and  $\hat{\boldsymbol{\xi}}_i \in se(3)$  is defined as:

$$\hat{\boldsymbol{\xi}}_i(s_i) = \begin{bmatrix} \hat{\mathbf{u}}_i(s_i) & \mathbf{v}_i(s_i) \\ \mathbf{0} & 0 \end{bmatrix} \quad (3)$$

where  $\hat{\mathbf{u}}_i \in so(3)$  is the skew-symmetric matrix obtained by  $\mathbf{u}_i$ . Moreover,  $\mathbf{u}_i \in \mathbb{R}^3$  represents bending and torsion of the beam, and  $\mathbf{v}_i \in \mathbb{R}^3$  describes shear and extensibility. Assuming linear isotropic elasticity, the deformation energy of the beam is given by [4]:

$$V_{ei} = \frac{1}{2} \int_0^{L_i} (\boldsymbol{\xi}_i(s_i) - \boldsymbol{\xi}_i^*(s_i))^T \mathbf{K}_i (\boldsymbol{\xi}_i(s_i) - \boldsymbol{\xi}_i^*(s_i)) ds \quad (4)$$

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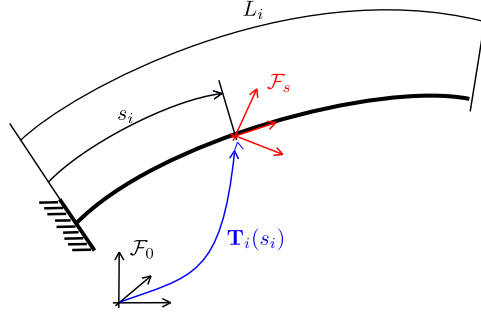


Fig. 1: Continuous parametrization of a flexible beam.

where  $(\cdot)^*$  denotes the undeformed strain configuration, and  $\mathbf{K}_i \in \mathbb{R}^{6 \times 6}$  is the material stiffness matrix. Frequently,  $\mathbf{K}_i = \text{diag}(EI_x, EI_y, GI_z, GA, GA, EA)$  where  $E$  is the Young's modulus,  $G$  is the shear modulus,  $I_x, I_y, I_z$  are the principal inertia moments of the cross-section, and  $A$  is the cross-section area.

### C. External Loads Energy

Let us compute the potential energy due to external *conservative* loads. First, the potential energy due to a distributed force  $\mathbf{f}_{di}$  applied at the  $i$ -th beam is given by:

$$V_{di} = - \int_0^{L_i} \mathbf{f}_{di}(s_i)^T \mathbf{p}_i(s_i) ds \quad (5)$$

Then, concentrated loads<sup>1</sup> are considered. The contribution of a concentrated force  $\mathbf{f}$  applied at  $\mathbf{p}_{app}$  is:

$$V_f = -\mathbf{f}^T \mathbf{p}_{app} \quad (6)$$

where  $\mathbf{p}_{app}$  is the application point of  $\mathbf{f}$ . For instance, when the force is applied to the  $EE$ ,  $\mathbf{p}_{app} = \mathbf{p}_p$  while, if  $\mathbf{f}$  is applied at the coordinate  $s_{app}$  of the  $i$ -th beam,  $\mathbf{p}_{app} = \mathbf{p}_i(s_{app})$ .

For the following derivation, it is convenient to introduce  $f$  and  $\mathbf{d}_f$ :  $f$  represents the magnitude of  $\mathbf{f}$  while  $\mathbf{d}_f$  represents the orientation of segment and its direction in terms of start and end point. Thus, Eq. (6) becomes:

$$V_f = -\mathbf{f}^T \mathbf{p}_{app} = -f \mathbf{d}_f^T \mathbf{p}_{app} = -fh \quad (7)$$

with  $h = \mathbf{d}_f^T \mathbf{p}_{app}$ .

### D. Continuum Robot Energy and Constraints

The  $CR$  total potential energy is obtained as the sum of the deformation energy and external load contributions, that is:

$$V_{tot} = \sum_{i=1}^n (V_{bi} + V_{di}) + V_f \quad (8)$$

Geometric constraints frequently have to be considered in  $CR$ s. For instance, the closure-loop geometric constraints of  $CPR$ s involve leg and rigid-platform variables [3]. In serial  $CR$ s, when a set of independent variables  $\mathbf{q}_p$  is used to represent the  $EE$  position, geometric constraints are introduced as well. Additionally, when using quaternions to represent orientations, unitary constraints must be enforced. Without loss of generality, the constraints are represented by

$$\Phi = \mathbf{0} \quad (9)$$

where a vector  $\Phi \in \mathbb{R}^{n_\Phi}$  is introduced to stack all the constraints in homogeneous form.  $CR$ s passive joints, (such as connections between the rigid platform to the passive beams in continuum parallel robots), can be modelled. For instance, revolute joints, spherical joints, cylindrical joints, and fixed joints can be modelled. Without loss of generality, the constraint of the  $i$ -th beam  $\Phi_i \in \mathbb{R}^{n_{\Phi_i}}$  can be represented by:

$$\Phi_i = \mathbf{C}_i \begin{bmatrix} (\mathbf{R}_p^T \mathbf{R}_i(L_i) - \mathbf{R}_i^T(L_i) \mathbf{R}_p)^{\vee} \\ \mathbf{p}_i(L_i) - (\mathbf{p}_p + \mathbf{R}_p \mathbf{p}_{fi}) \end{bmatrix} \quad (10)$$

where  $\mathbf{p}_p, \mathbf{R}_p$  are the position and the orientation matrix of  $\mathcal{F}_p$  w.r.t.  $\mathcal{F}_0$ ,  $\mathbf{p}_{fi} \in \mathbb{R}^3$  is a vector pointing from the  $i$ -th joint position to  $\mathcal{F}_p$  (thus constant w.r.t.  $\mathcal{F}_p$ ), respectively. The vector  $\mathbf{p}_i(L_i)$  is the position of the  $i$ -th beams at the coordinates  $L_i$  and  $\mathbf{R}_i(L_i)$  is the orientation matrix at  $L_i$  of the same beam. The superscript  $(\cdot)^\vee$  indicates the extraction of the three independent components of its argument, assumed to be a skew-symmetric matrix. Matrix  $\mathbf{C}_i \in \mathbb{R}^{n_{\Phi_i} \times 6}$  is named joint matrix, and Table I summarizes how  $\mathbf{C}_i$  is structured for different kinds of passive platform joints.

<sup>1</sup>Three-dimensional pure moments, which are non-conservative, are assumed not to appear.

Name	Fixed	Spherical	Revolute Local $z$	Cylindrical Local $z$
$n^\circ DoF$	0	3	1	2
<b>C</b>	$\mathbf{I}_6$	$[\mathbf{0}_{3 \times 3} \quad \mathbf{I}_3]$	$\begin{bmatrix} \mathbf{I}_2 & \mathbf{0}_{2 \times 4} \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_3 \end{bmatrix}$	$\begin{bmatrix} \mathbf{0}_2 & \mathbf{0}_{2 \times 1} & \mathbf{I}_{2 \times 2} & \mathbf{0}_{2 \times 1} \\ \mathbf{I}_2 & \mathbf{0}_{2 \times 1} & \mathbf{0}_2 & \mathbf{0}_{2 \times 1} \end{bmatrix}$

TABLE I: Values of  $\mathbf{C}$  for different platform constraints. The number of  $DoF$  is indicated with  $n^\circ DoF$ ,  $\mathbf{I}_k$  is the identify matrix of dimension  $k$ , and  $\mathbf{0}_{k \times h}$  denotes a matrix of dimension  $k \times h$  full of zeros. Revolute and Cylindrical pairs are assumed to be aligned with the  $z$  local axis.

### E. Discretization and Geometrico-Static Modelling

$CRs$  equilibrium configurations are associated with critical points of  $V_{tot}$ , which depends on  $\mathbf{q}_a, \mathbf{q}_p$ , and a set of continuous strain-functions  $\xi_i$ . Discretization of the potential energy equations through a finite number of variables brings simple but effective  $CRs$  analysis, in particular when dealing with equilibrium stability assessment. When employing a discretization technique,  $V_{tot}$  (defined in Eq. (8)) is discretized with a finite set of variables  $\mathbf{q}_{ei} \in \mathbb{R}^{N_{fi}}$ , with  $N_{fi}$  being the number of discretization variables. Then, being  $N_e = \sum_{i=1}^n N_{fi}$ , the vector  $\mathbf{q}_e = [\mathbf{q}_{e1}, \dots, \mathbf{q}_{en}] \in \mathbb{R}^{N_e}$  stacks all the discretization variables. For later convenience, the vectors  $\mathbf{q}_d = [\mathbf{q}_u, \mathbf{q}_e]$ , and  $\mathbf{x} = [\mathbf{q}_d, \mathbf{q}_c]$  are introduced. The  $CRs$  configuration is thus defined by  $\mathbf{q}_a, \mathbf{x}$ : the total potential energy and the constraints become a discrete function of  $\mathbf{q}_a, \mathbf{x}$ , that is:

$$V_{tot} = V_{tot}(\mathbf{q}_a, \mathbf{x}); \quad \Phi = \Phi(\mathbf{q}_a, \mathbf{x}) \quad (11)$$

In the static case and for a fixed value of  $\mathbf{q}_a$ , feasible  $CRs$  configurations are determined by the minimum of the total potential energy  $V_{tot}$  [3]. However, constraints  $\Phi$  should be enforced, and first-order Lagrange conditions determine critical points of  $V_{tot}$  [5]. Assuming  $\nabla_{\mathbf{x}} \Phi$  full rank,  $\mathbf{x}$  is associated with a critical point of  $V_{tot}$  if there exist a set of Lagrange multipliers  $\lambda \in \mathbb{R}^{n_\Phi}$  such as [5]:

$$\begin{cases} \nabla_{\mathbf{x}} \mathcal{L} = \mathbf{0} \\ \nabla_{\lambda} \mathcal{L} = \mathbf{0} \end{cases} \quad (12)$$

where  $\mathcal{L}$  is the Lagrangian function defined as:

$$\mathcal{L} = V_{tot} + \Phi^T \lambda \quad (13)$$

Equations (12) form the geometrico-static model of a  $CR$ , and it is an undermined set of  $n_c + N_e + n_\Phi$  equations in  $m + n_c + N_e + n_\Phi$  unknowns.

### III. GEOMETRICO-STATIC-MODEL EQUATIONS

This section derives the expressions of Eq.(12) when a finite-differences modelling approach is used. Let us first expand Eq.(12) as follows:

$$\begin{cases} \nabla_{\mathbf{x}} \mathcal{L} = \mathbf{0} \\ \nabla_{\lambda} \mathcal{L} = \mathbf{0} \end{cases} \rightarrow \begin{cases} \nabla_{\mathbf{q}_e} V_{tot} + \nabla_{\mathbf{q}_e} (\lambda^T \Phi) = \mathbf{0} \\ \nabla_{\mathbf{q}_p} V_{tot} + \nabla_{\mathbf{q}_p} (\lambda^T \Phi) = \mathbf{0} \\ \Phi = \mathbf{0} \end{cases} \quad (14)$$

To derive the expression of these equations, let us consider the deformation energy of the  $i$ -th beam of Eq. (4) in the case shear and extensibility are neglected, here reported for clarity:

$$V_{ei} = \frac{1}{2} \int_0^{L_i} (\mathbf{u}_i(s) - \mathbf{u}_i^*(s))^T \mathbf{K} (\mathbf{u}_i(s) - \mathbf{u}_i^*(s)) ds \quad (15)$$

When a finite-differences approach is employed, it is convenient to parametrize the orientation of each cross-section by using unit quaternions  $\mathbf{h}_i \in \mathbb{R}^4$ ,  $\mathbf{h}_i = h_{i1} + h_{i2}\mathbf{e}_1 + h_{i3}\mathbf{e}_2 + h_{i4}\mathbf{e}_3$ ,  $\mathbf{e}_1 = [1, 0, 0]$ ,  $\mathbf{e}_2 = [0, 1, 0]$ ,  $\mathbf{e}_3 = [0, 0, 1]$ . In this case, the  $i$ -th beams curvature  $\mathbf{u}_i(s) \in \mathbb{R}^3$  is computed as follows:

$$u_{ik}(s) = 2\mathbf{h}_i^T \mathbf{B}_k^T \mathbf{h}'_i; \quad k = 1, 2, 3 \quad (16)$$

with the matrices  $\mathbf{B}_k$  structured as follows:

$$\mathbf{B}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{B}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (17)$$

Thus, inserting Eq. (16) into Eq. (15), and by assuming  $\mathbf{u}_i^* = \mathbf{0}$  for simplicity, the following expression of  $V_{ei}$  is obtained:

$$V_{ei} = \frac{1}{2} \int_0^{L_i} \left( \sum_{k=1}^3 K_k (2\mathbf{h}_i^T \mathbf{B}_k \mathbf{h}'_i)^2 \right) ds \quad (18)$$

The finite-difference approximation is introduced by first discretizing the rod into  $N_{elt}$  elements of equal length  $L_{ei} = L/N_{elt}$ , with the orientation of the  $j$ -th element of the  $i$ -th beam being defined by  $\mathbf{h}_{ij}$ . The vector  $\mathbf{q}_{ei}$  collects the  $N_{elt}$  quaternions  $\mathbf{h}_{ij}$  of the  $i$ -th beam. The expression of  $V_{ei}$  becomes:

$$V_{ei} = \sum_{i=1}^{N_{elt}} V_{eij}; \quad V_{eij} = \frac{1}{2} \int_0^{L_{ei}} \left( \sum_{k=1}^3 K_k (2\mathbf{h}_{ij}^T \mathbf{B}_k \mathbf{h}'_{ij})^2 \right) ds \quad (19)$$

The expression of  $\mathbf{h}'_{ij}$  is then approximated by the use of a first-order backward finite difference approximation:

$$\mathbf{h}'_{ij} \simeq \frac{\mathbf{h}_{ij} - \mathbf{h}_{ij-1}}{L_{ei}} \quad (20)$$

Inserting Eq. (20) into Eq. (19) and integrating in  $s$  results in the following expression of  $V_{eij}$ :

$$V_{eij} = \sum_{k=1}^3 \frac{K_k}{L_e} \left( 2\mathbf{h}_{ij}^T \mathbf{B}_k \frac{\mathbf{h}_{ij} - \mathbf{h}_{ij-1}}{L_e} \right)^2 \quad (21)$$

Once the term  $V_{ei}$  is obtained, to compute  $\mathcal{L}$ , it is necessary to evaluate the influence of distributed loads  $V_{di}$  (Eq. (5)). First, the position of each beam's element  $\mathbf{p}_{ij}$  is recovered by the use of the following formula:

$$\mathbf{p}_i(s) = \mathbf{p}_{ij} + x\mathbf{R}_{ij}\mathbf{e}_3; \quad x = s - jL_{ei} \quad (22)$$

with  $\mathbf{R}_{ij}$  the orientation matrix of the  $j$ -th element, recovered by the knowledge of  $\mathbf{h}_{ij}$ .  $\mathbf{p}_0, \mathbf{h}_0$ , that are the base position and orientation of the beam, are usually computed from  $q_{ai}$ , depending on the employed actuator. To compute  $V_{di}$ , let us recall its expression of Eq. (5) for clarity:

$$V_{di} = - \int_0^L \mathbf{f}_d^T \mathbf{p}_i ds \quad (23)$$

By discretizing the beam into  $N_{elt}$  equal elements, inserting Eq. (22) into Eq. (23), and integrating in  $s$ , the following expression is obtained:

$$V_{di} = \sum_{j=1}^{N_{elt}} V_{dij} \quad V_{dij} = -\mathbf{f}_d^T (\mathbf{p}_{ij} + L_e \mathbf{R}_{ij} \mathbf{e}_3) \quad (24)$$

At this stage, the derivation of the geometrico-static model equations starts. Being  $\Phi_i$  the constrain vector of the  $i$ -th beam, we assume that only fixed constraints appears. This case is selected as it is the most general, and others kind of joints may be considered as subcases of the fixed-joint case. Being  $\mathbf{p}_{ij}, \mathbf{R}_{ij}$  the values of the position and orientation of the last element of the  $i$ -th beam computed from Eq. (22), the expression of  $\Phi_i$  is computed as follows:

$$\Phi_i = \mathbf{C}_i \left[ \begin{array}{c} (\mathbf{R}_p^T \mathbf{R}_{ij} - \mathbf{R}_{ij} \mathbf{R}_p) \\ \mathbf{p}_{ij} - (\mathbf{p}_p + \mathbf{R}_p \mathbf{p}_{fi}) \end{array} \right] \quad j = N_{elt} \quad (25)$$

Let us derive  $\nabla_{\mathbf{q}_{ei}} \mathcal{L}$ , by considering separately the contribution of the deformation energy, distributed loads energy, and geometrical constraints:

$$\nabla_{\mathbf{q}_{ei}} \mathcal{L} = \nabla_{\mathbf{q}_{ei}} V_{ei} + \nabla_{\mathbf{q}_{ei}} V_{di} + \nabla_{\mathbf{q}_{ei}} (\Phi_i^T \lambda_i) \quad (26)$$

For each term, the expression of  $\nabla_{\mathbf{h}_{ij}}(\cdot)$ , that is the  $j$ -th component of  $\nabla_{\mathbf{q}_{ei}}(\cdot)$ , is derived. Let us expand the expression of  $V_{ei}$  as follows to illustrate where  $\mathbf{h}_{ij}$  appears in  $V_{ei}$ :

$$V_{ei} = \sum_{j=1}^{N_{elt}} \sum_{k=1}^3 \frac{K_k}{L_{ei}} \left( \dots + \left( 2\mathbf{h}_{ij}^T \mathbf{B}_k \frac{\mathbf{h}_{ij} - \mathbf{h}_{ij-1}}{L_e} \right)^2 + \left( 2\mathbf{h}_{ij+1}^T \mathbf{B}_k \frac{\mathbf{h}_{ij+1} - \mathbf{h}_{ij}}{L_e} \right)^2 + \dots \right) \quad (27)$$

The term  $\mathbf{h}_{ij}$ , as a cause of the finite-difference approximation, appears in the term  $j$  and  $j+1$ . Thus,  $\nabla_{\mathbf{h}_{ij}} V_{ei}$  is structured as the sum of two terms:

$$\nabla_{\mathbf{h}_{ij}} V_{ei} = \mathbf{a}_{ij} + \mathbf{b}_i \quad (28)$$

and the terms  $\mathbf{a}_{ij}, \mathbf{b}_{ik}, A_{ijk}$  can be computed as follows:

$$\mathbf{a}_{ij} = +2 \sum_{k=1}^3 \frac{K_k}{L_{ei}} \mathbf{B}_k \mathbf{h}_{ij} A_{ijk}, \quad \mathbf{a}_{ij} \in \mathbb{R}^4 \quad (29)$$

$$\mathbf{b}_{ij} = -2 \sum_{k=1}^3 \frac{K_k}{L_{ei}} \mathbf{B}_k \mathbf{h}_{i(j+1)} A_{ijk}, \quad \mathbf{b}_{ij} \in \mathbb{R}^4 \quad (30)$$

$$A_{ijk} = -2\mathbf{h}_{i(j+1)}^T \mathbf{B}_k^T \mathbf{h}_{ij}, \quad A_{ijk} \in \mathbb{R} \quad (31)$$

Then, the term  $\nabla_{\mathbf{h}_{ij}} V_{di}$  is calculated by differentiating Eq.(24) w.r.t.  $\mathbf{h}_{ij}$ :

$$\nabla_{\mathbf{h}_{ij}} V_{di} = \sum_{r=1}^{N_{elt}} \mathbf{f}_d^T (\nabla_{\mathbf{h}_{ij}} \mathbf{p}_{ir} + L_{ei} \nabla_{\mathbf{h}_{ij}} (\mathbf{R}_{ir} \mathbf{e}_3)) \quad (32)$$

with the terms  $\nabla_{\mathbf{h}_{ij}} \mathbf{p}_{ir}$  and  $\nabla_{\mathbf{h}_{ij}} (\mathbf{R}_{ir} \mathbf{e}_3)$  compute as follows:

$$\nabla_{\mathbf{h}_{ij}} \mathbf{p}_{ir} = L_{ei} \mathbf{D}_{3ij}; \quad j \leq r; \quad \nabla_{\mathbf{h}_{ij}} \mathbf{p}_{ir} = \mathbf{0}; \quad j > r \quad (33)$$

$$\nabla_{\mathbf{h}_{ij}} (\mathbf{R}_{ir} \mathbf{e}_3) = \mathbf{D}_{3ij}; \quad j = r; \quad \nabla_{\mathbf{h}_{ij}} (\mathbf{R}_{ir} \mathbf{e}_3) = \mathbf{0}; \quad j \neq r \quad (34)$$

The matrix  $\mathbf{D}_{3ij}$  (and the matrices  $\mathbf{D}_{1ij}, \mathbf{D}_{2ij}$  that introduced for later convenience), are structured as follows:

$$\mathbf{D}_{1ij} = 2 \begin{bmatrix} +h_{1ij} & +h_{2ij} & -h_{3ij} & -h_{4ij} \\ +h_{4ij} & +h_{3ij} & +h_{2ij} & +h_{1ij} \\ -h_{3ij} & +h_{4ij} & -h_{1ij} & +h_{2ij} \end{bmatrix} \quad (35)$$

$$\mathbf{D}_{2ij} = 2 \begin{bmatrix} -h_{4ij} & +h_{3ij} & +h_{2ij} & -h_{1ij} \\ +h_{1ij} & -h_{2ij} & +h_{3ij} & -h_{4ij} \\ +h_{2ij} & +h_{1ij} & +h_{4ij} & +h_{3ij} \end{bmatrix} \quad (36)$$

$$\mathbf{D}_{3ij} = 2 \begin{bmatrix} +h_{3ij} & +h_{4ij} & +h_{1ij} & +h_{2ij} \\ -h_{2ij} & -h_{1ij} & +h_{4ij} & +h_{3ij} \\ +h_{1ij} & -h_{2ij} & -h_{3ij} & +h_{4ij} \end{bmatrix} \quad (37)$$

Finally, to compute  $\nabla_{\mathbf{q}_{ei}} (\Phi_i^T \lambda_i)$ , the  $j$ -th component  $\nabla_{\mathbf{h}_{ij}} (\Phi_i^T \lambda_i)$  is calculated. For the following derivation, it is convenient to simplify the expression of the orientation constraints. Being  $\mathbf{d}_{p1}, \mathbf{d}_{p2}, \mathbf{d}_{p3}$  the columns of the platform rotation matrix such as  $\mathbf{R}_p = [\mathbf{d}_{p1}, \mathbf{d}_{p2}, \mathbf{d}_{p3}]$ , and  $\mathbf{d}_{i1}, \mathbf{d}_{i2}, \mathbf{d}_{i3}$  the columns of the  $i$ -th beam rotation matrix at  $s = L$  ( $\mathbf{R}_{ij} = [\mathbf{d}_{i1}, \mathbf{d}_{i2}, \mathbf{d}_{i3}], j = N_{elt}$ ), the following equivalent expression of the orientation constraint is obtained:

$$(\mathbf{R}_p^T \mathbf{R}_i(L_i) - \mathbf{R}_i^T(L_i) \mathbf{R}_p)^\vee = \begin{cases} \mathbf{d}_{p1}^T \mathbf{d}_2 - \mathbf{d}_{p2}^T \mathbf{d}_1 \\ \mathbf{d}_{p1}^T \mathbf{d}_3 - \mathbf{d}_{p3}^T \mathbf{d}_1 \\ \mathbf{d}_{p2}^T \mathbf{d}_3 - \mathbf{d}_{p3}^T \mathbf{d}_2 \end{cases} \quad (38)$$

Thus,  $\lambda_i^T \Phi_i$  is explicitly computed as follows:

$$\lambda_i^T \Phi_i = [\lambda_{4i}, \lambda_{5i}, \lambda_{6i}] (\mathbf{p}_i(L_i) - (\mathbf{p}_p + \mathbf{R}_p \mathbf{p}_{fi})) + \lambda_{1i} (\mathbf{d}_{p1}^T \mathbf{d}_2 - \mathbf{d}_{p2}^T \mathbf{d}_1) + \lambda_{2i} (\mathbf{d}_{p1}^T \mathbf{d}_3 - \mathbf{d}_{p3}^T \mathbf{d}_1) + \lambda_{3i} (\mathbf{d}_{p2}^T \mathbf{d}_3 - \mathbf{d}_{p3}^T \mathbf{d}_2) \quad (39)$$

Deriving Eq. (39) w.r.t.  $\mathbf{h}_{ij}$  results in two terms:

$$\nabla_{\mathbf{h}_{ij}} (\lambda_i^T \Phi_i) = \mathbf{c}_{ij} + \mathbf{d}_{ij} \quad (40)$$

where  $\mathbf{c}_{ij} \in \mathbb{R}^4$ :

$$\mathbf{c}_{ij} = L_{ei} [\lambda_{4i}, \lambda_{5i}, \lambda_{6i}] \mathbf{D}_{3ij} \quad (41)$$

Instead, the term  $\mathbf{d}_{ij} \in \mathbb{R}^4$  is non-null only for  $j = N_{elt}$ , and it is structured as follows:

$$\mathbf{d}_{ij} = \lambda_{1i} (\mathbf{d}_{p1}^T \mathbf{D}_{2ij} - \mathbf{d}_{p2}^T \mathbf{D}_{1ij}) + \lambda_{2i} (\mathbf{d}_{p1}^T \mathbf{D}_{3ij} - \mathbf{d}_{p3}^T \mathbf{D}_{1ij}) + \lambda_{3i} (\mathbf{d}_{p2}^T \mathbf{D}_{3ij} - \mathbf{d}_{p3}^T \mathbf{D}_{2ij}), \quad j = N_{elt} \quad (42)$$

Finally, the last required term for the geometrico-static model derivation is  $\nabla_{\mathbf{q}_p} \mathcal{L}$ , that can be computed as:

$$\nabla_{\mathbf{q}_p} \mathcal{L} = \nabla_{\mathbf{q}_p} V_{tot} + \nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i) \quad (43)$$

where  $\nabla_{\mathbf{q}_p} V = -[\mathbf{0}_{3 \times 1}; \mathbf{I}_3]$  and, being  $\mathbf{q}_p = [\mathbf{p}_p, \alpha] \in \mathbb{R}^{n_c}$  with  $\mathbf{p}_p \in \mathbb{R}^3$  the platform position and  $\alpha \in \mathbb{R}^{n_c-3}$  the platform orientation parameters, the term  $\nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i)$  is structured as follows:

$$\nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i) = -[\lambda_{4i}, \lambda_{5i}, \lambda_{6i}] [\mathbf{I}_3 \quad \mathbf{m}_{p1}] + [\mathbf{0}_{3 \times 1} \quad \mathbf{m}_{p2}] \quad (44)$$

the vector  $\mathbf{m}_{p1} \in \mathbb{R}^{n_c-3}$  is:

$$\mathbf{m}_{p1} = \frac{\partial}{\partial \alpha} (\mathbf{R}_p \mathbf{p}_{fi}) = p_{fi1} \frac{\partial \mathbf{d}_{p1}}{\partial \alpha} + p_{fi2} \frac{\partial \mathbf{d}_{p2}}{\partial \alpha} + p_{fi3} \frac{\partial \mathbf{d}_{p3}}{\partial \alpha} \quad (45)$$

with  $\partial \mathbf{d}_{pk} / \partial \alpha$  to be computed in relation to the selected platform orientation parametrization. The expression of  $\mathbf{m}_{p2} \in \mathbb{R}^{n_c-3}$  depends on the platform orientation parametrization as well, and it is structured as:

$$\mathbf{m}_{p2} = \lambda_{1i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \mathbf{d}_{2ij} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \alpha} \mathbf{d}_{1ij} \right) + \lambda_{2i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \mathbf{d}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \mathbf{d}_{1ij} \right) + \lambda_{3i} \left( \frac{\partial \mathbf{d}_{p2}^T}{\partial \alpha} \mathbf{d}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \mathbf{d}_{2ij} \right), \quad j = N_{elt} \quad (46)$$

with  $\partial \mathbf{d}_{pk}^T / \partial \alpha$  to be computed in relation to the selected platform orientation parametrization.

#### IV. MATRICES COMPUTATION

This Section derives the expression of the matrices required for the distance-to-instability metric computation. In particular, matrices  $\mathbf{H}$  and  $\mathbf{G}$  are necessary, and their expression is the following:

$$\mathbf{H} = \begin{bmatrix} \nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_e} \mathcal{L}) & \nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_p} \mathcal{L}) \\ \nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_e} \mathcal{L}) & \nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_p} \mathcal{L}) \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} \nabla_{\lambda} (\nabla_{\mathbf{q}_e} \mathcal{L}) \\ \nabla_{\lambda} (\nabla_{\mathbf{q}_p} \mathcal{L}) \end{bmatrix} \quad (47)$$

The next sections derives each required term separately .

##### A. Computation of $\nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_e} \mathcal{L})$

Let us start by computing the derivatives of  $\nabla_{\mathbf{q}_e} \mathcal{L}$  w.r.t.  $\mathbf{q}_{ei}$ : this term is obtained by considering  $V_{ei}, V_{di}$  and  $\lambda_i^T \Phi_i$  separately. The derivative of  $\nabla_{\mathbf{q}_{ei}} V_{ei}$  w.r.t.  $\mathbf{q}_{ei}$  is computed according to Eq. (28) as a diagonal block matrix:

$$\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{ei}) = \begin{bmatrix} \mathbf{E}_{i1} & \mathbf{F}_{i2}^T & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{i2} & \mathbf{E}_{i2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{E}_{iN_{elt}-1} & \mathbf{F}_{iN_{elt}}^T \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{F}_{iN_{elt}} & \mathbf{E}_{iN_{elt}} \end{bmatrix} \quad (48)$$

in which:

$$\mathbf{E}_{ij} = \mathbf{M}_{ij} + \mathbf{N}_{ij+1} \quad (49)$$

$$\mathbf{F}_{ij} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} ((\mathbf{B}_k \mathbf{h}_{ij})(\mathbf{B}_k \mathbf{h}_{ij+1})^T - \mathbf{B}_k A_{ijk}/2) \quad (50)$$

$$\mathbf{M}_{ij} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} (\mathbf{B}_k \mathbf{h}_{ij})(\mathbf{B}_k^T \mathbf{h}_{ij})^T \quad (51)$$

$$\mathbf{N}_{ij} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} (\mathbf{B}_k \mathbf{h}_{ij+1})(\mathbf{B}_k \mathbf{h}_{ij+1})^T \quad (52)$$

Let us consider now  $\nabla_{\mathbf{q}_{ei}} V_{di}$ . To compute derivative of  $\nabla_{\mathbf{h}_{ij}} V_{di}$  w.r.t.  $\mathbf{h}_{ij}$ , it is first convenient to insert  $\mathbf{f}_d = [f_{dx}, f_{dy}, f_{dz}]$  inside Eq. (32), and to obtain the following expression:

$$\nabla_{\mathbf{h}_{ij}} V_{di} = \sum_{r=1}^{N_{elt}} \nabla_{\mathbf{h}_{ij}} (\mathbf{f}_d^T \mathbf{p}_{ir}) + L_{ei} \nabla_{\mathbf{h}_{ij}} (\mathbf{f}_d^T \mathbf{R}_{ir} \mathbf{e}_3) \quad (53)$$

Then, it is convenient to define the following operator that transforms the generic vector  $\mathbf{z} = [z_1, z_2, z_3] \in \mathbb{R}^3$  into a  $4 \times 4$  matrix structured as follows:

$$\tilde{\mathbf{z}} = \begin{bmatrix} +z_3 & -z_2 & +z_1 & 0 \\ -z_2 & -z_3 & 0 & +z_1 \\ +z_1 & 0 & -z_3 & +z_2 \\ 0 & +z_1 & +z_2 & +z_3 \end{bmatrix} \quad (54)$$

The derivative of  $\nabla_{\mathbf{h}_{ij}} V_{di}$  w.r.t.  $\mathbf{h}_{ij}$  is expressed as follows:

$$\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{h}_{ij}} (\mathbf{f}_d^T \mathbf{p}_{ir})) = 4\tilde{\mathbf{f}}_d \quad (55)$$

Then, let us consider  $\nabla_{\mathbf{q}_{ei}} \lambda_i^T \Phi_i$  of Eq.(40). First, let us compute the derivative of Eq.(40) w.r.t.  $\mathbf{h}_{ij}$ , that is:

$$\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{h}_{ij}} (\lambda_i^T \Phi_i)) = \nabla_{\mathbf{h}_i} \mathbf{c}_i + \nabla_{\mathbf{h}_i} \mathbf{d}_i \quad (56)$$

Considering Eq.(41), and by introducing  $\lambda_{4:6i} = [\lambda_{4i}, \lambda_{5i}, \lambda_{6i}]$ , the expression  $\nabla_{\mathbf{h}_i} \mathbf{c}_i$  is obtained as:

$$\nabla_{\mathbf{h}_i} \mathbf{c}_{ij} = 2L_{ei} \widetilde{\lambda_{4:6i}} \quad (57)$$

The term  $\nabla_{\mathbf{h}_{ij}} \mathbf{d}_{ij}$  is non-null only for  $j = N_{elt}$ , and it is structured as follows:

$$\nabla_{\mathbf{h}_{ij}} \mathbf{d}_{ij} = \lambda_{1i} (\widetilde{\mathbf{d}_{p1}} - \widetilde{\mathbf{d}_{p2}}) + \lambda_{2i} (\widetilde{\mathbf{d}_{p1}} - \widetilde{\mathbf{d}_{p3}}) + \lambda_{3i} (\widetilde{\mathbf{d}_{p2}} - \widetilde{\mathbf{d}_{p3}}), \quad j = N_{elt} \quad (58)$$

### B. Computation of $\nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_p} \mathcal{L})$

By inspecting the terms of  $\nabla_{\mathbf{q}_p} \mathcal{L}$ , the only term involving  $\mathbf{q}_e$  is  $(\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)$ . The derivative of  $\nabla_{\mathbf{q}_p} \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i$  w.r.t.  $\mathbf{h}_{ij}$  is non-null only for  $j = N_{elt}$ , and it computed as follows:

$$\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{q}_p} \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i) = \begin{bmatrix} \mathbf{0}_{3 \times 4} \\ \nabla_{\mathbf{h}_{ij}} \mathbf{m}_{p2} \end{bmatrix} \quad (59)$$

in which:

$$\begin{aligned} \nabla_{\mathbf{h}_{ij}} \mathbf{m}_{p2} = & \lambda_{1i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{2ij} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{1ij} \right) + \lambda_{2i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{1ij} \right) + \\ & + \lambda_{3i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{2ij} \right), \quad j = N_{elt} \end{aligned} \quad (60)$$

Since derivatives w.r.t.  $\mathbf{q}_p, \mathbf{q}_e$  are commutative,  $\nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_p} \mathcal{L}) = \nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_e} \mathcal{L})^T$ .

### C. Computation of $\nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_p} \mathcal{L})$

To compute  $\nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_p} \mathcal{L})$ , let us consider  $V_{tot}$  and  $\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i$  separately. Since  $\nabla_{\mathbf{q}_p} V_{tot}$  is constant, its derivative w.r.t.  $\mathbf{q}_p$  is null. Instead, the derivative of  $\nabla_{\mathbf{q}_p} (\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)$  w.r.t.  $\mathbf{q}_p$  is structured as follows:

$$\nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_p} (\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)) = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{m}_{p3} \end{bmatrix} \quad (61)$$

where the matrix  $\mathbf{m}_{p3} \in \mathbb{R}^{3 \times 3}$  is defined as:

$$\mathbf{m}_{p3} = \begin{bmatrix} \mathbf{m}_{p31} \\ \mathbf{m}_{p32} \\ \mathbf{m}_{p33} \end{bmatrix} \quad (62)$$

and the vectors  $\mathbf{m}_{p3i}$  are obtained by differentiating Eq. (44) w.r.t.  $\alpha_i$ . In particular,  $\mathbf{m}_{p3i}$  is structured as follows:

$$\mathbf{m}_{p3i} = -[\lambda_{4i}, \lambda_{5i}, \lambda_{6i}] \frac{\partial \mathbf{m}_{p1}}{\partial \alpha_i} + \frac{\partial \mathbf{m}_{p2}}{\partial \alpha_i} \quad (63)$$

where:

$$\frac{\partial \mathbf{m}_{p1}}{\partial \alpha_i} = p_{fi1} \frac{\partial^2 \mathbf{d}_{p1}}{\partial \boldsymbol{\alpha} \partial \alpha_i} + p_{fi2} \frac{\partial^2 \mathbf{d}_{p2}}{\partial \boldsymbol{\alpha} \partial \alpha_i} + p_{fi3} \frac{\partial^2 \mathbf{d}_{p3}}{\partial \boldsymbol{\alpha} \partial \alpha_i} \quad (64)$$

$$\frac{\partial \mathbf{m}_{p2}}{\partial \alpha_i} = \lambda_{1i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{2ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{1ij} \right) + \lambda_{2i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{1ij} \right) + \lambda_{3i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha} \alpha_i} \mathbf{d}_{2ij} \right), \quad j = N_{elt} \quad (65)$$

with the derivatives of the platform rotation matrix columns to be computed in relation to the selected platform orientation parametrization.

### D. Computation of $\nabla_{\boldsymbol{\lambda}_i} (\nabla_{\mathbf{q}_e} \mathcal{L})$

Let us calculate  $\nabla_{\boldsymbol{\lambda}_i} (\nabla_{\mathbf{q}_e} \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)$ . Derivating  $\mathbf{c}_{ij}$  w.r.t.  $\boldsymbol{\lambda}_i$  results in:

$$\nabla_{\boldsymbol{\lambda}_i} \mathbf{c}_{ij} = L_{ei} \begin{bmatrix} \mathbf{0}_{3 \times 4} \\ \mathbf{D}_{3ij} \end{bmatrix} \quad (66)$$

while, by differentiating  $\mathbf{d}_{ij}$  w.r.t.  $\boldsymbol{\lambda}_i$ , the following expression is obtained:

$$\nabla_{\boldsymbol{\lambda}_i} \mathbf{d}_{ij} = \begin{bmatrix} \mathbf{d}_{p1}^T \mathbf{D}_{2ij} - \mathbf{d}_{p2}^T \mathbf{D}_{1ij} \\ \mathbf{d}_{p1}^T \mathbf{D}_{3ij} - \mathbf{d}_{p3}^T \mathbf{D}_{1ij} \\ \mathbf{d}_{p2}^T \mathbf{D}_{3ij} - \mathbf{d}_{p3}^T \mathbf{D}_{2ij} \\ \mathbf{0}_{3 \times 4} \end{bmatrix}, \quad j = N_{elt} \quad (67)$$

### E. Computation of $\nabla_{\lambda} (\nabla_{\mathbf{q}_p} \mathcal{L})$

Let us compute the derivatives of  $\nabla_{\mathbf{q}_p} \mathcal{L}$ . In particular,  $\nabla_{\mathbf{q}_p} V_{tot}$  is constant, and its first derivative w.r.t.  $\lambda_i$  vanishes. Instead, let us consider  $\nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i)$  of Eq.(44), where a differentiation w.r.t.  $\lambda$  results in:

$$\nabla_{\lambda_i} \left( \nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i) \right) = \begin{bmatrix} \mathbf{0}_{3 \times 3} & -\mathbf{I}_{3 \times 3} \\ \nabla_{\lambda_{4:6i}} \mathbf{m}_{p2} & \mathbf{0}_{3 \times 3} \end{bmatrix} \quad (68)$$

where  $\nabla_{\lambda_{4:6i}} \mathbf{m}_{p2} \in \mathbb{R}^{3 \times 3}$  is structured as follows:

$$\nabla_{\lambda_{4:6i}} \mathbf{m}_{p2} = \begin{bmatrix} \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \mathbf{d}_{2ij} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \alpha} \mathbf{d}_{1ij}, & \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \mathbf{d}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \mathbf{d}_{1ij}, & \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \mathbf{d}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \mathbf{d}_{2ij} \end{bmatrix} \quad (69)$$

with the derivatives of the platform rotation matrix columns to be computed in relation to the selected orientation parametrization.

## V. COMPUTATION OF MATRICES DERIVATIVES

The goal of this Section is to compute the derivatives of  $\mathbf{H}, \mathbf{G}$ . In particular, we seek to compute:

$$\sum_{i=1}^{n_y} \frac{\partial \mathbf{H}}{\partial y_i} \frac{\partial y_i}{\partial f} \quad (70)$$

$$\sum_{i=1}^{n_y} \frac{\partial \mathbf{G}}{\partial y_i} \frac{\partial y_i}{\partial f} \quad (71)$$

where  $\mathbf{y} = [\mathbf{q}_a, \mathbf{q}_e, \mathbf{q}_p, \lambda]$  and  $n_y$  is the dimension of  $\mathbf{y}$ . Since  $\mathbf{H}, \mathbf{G}$  are structured as illustrated in Eq.(47), the derivatives of each term in  $\mathbf{H}, \mathbf{G}$  are computed separately in the following. The values of the terms  $\partial y_i / \partial f$  are assumed to be known.

$$\mathbf{H} = \begin{bmatrix} \nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_e} \mathcal{L}) & \nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_p} \mathcal{L}) \\ \nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_e} \mathcal{L}) & \nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_p} \mathcal{L}) \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} \nabla_{\lambda} (\nabla_{\mathbf{q}_e} \mathcal{L}) \\ \nabla_{\lambda} (\nabla_{\mathbf{q}_p} \mathcal{L}) \end{bmatrix} \quad (72)$$

### A. Computation of $\partial (\nabla_{\mathbf{q}_e} (\nabla_{\mathbf{q}_e} \mathcal{L})) / \partial f$

Let us first consider  $\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} \mathcal{L})$  which is structured as:

$$\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} \mathcal{L}) = \nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{ei}) + (\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{di})) + \left( \nabla_{\mathbf{q}_{ei}} \left( \nabla_{\mathbf{q}_{ei}} (\lambda_i^T \Phi) \right) \right) \quad (73)$$

The derivative of  $\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{ei})$  w.r.t.  $f$  is obtained by derivating Eq. (48):

$$\frac{\partial}{\partial f} (\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{ei})) = \begin{bmatrix} \frac{\partial}{\partial f} (\mathbf{E}_{i1}) & \frac{\partial}{\partial f} (\mathbf{F}_{i2}^T) & \cdots & \mathbf{0} & \mathbf{0} \\ \frac{\partial}{\partial f} (\mathbf{F}_{i2}) & \frac{\partial}{\partial f} (\mathbf{E}_{i2}) & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{\partial}{\partial f} (\mathbf{E}_{iN_{elt}-1}) & \frac{\partial}{\partial f} (\mathbf{F}_{iN_{elt}}^T) \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{\partial}{\partial f} (\mathbf{F}_{iN_{elt}}) & \frac{\partial}{\partial f} (\mathbf{E}_{iN_{elt}}) \end{bmatrix} \quad (74)$$

in which:

$$\frac{\partial \mathbf{E}_{ij}}{\partial f} = \frac{\partial \mathbf{M}_{ij}}{\partial f} + \frac{\partial \mathbf{N}_{ij+1}}{\partial f} \quad (75)$$

$$\frac{\partial \mathbf{F}_{ij}}{\partial f} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} \left( (\mathbf{B}_k \frac{\partial \mathbf{h}_{ij}}{\partial f}) (\mathbf{B}_k \mathbf{h}_{ij+1})^T + (\mathbf{B}_k \mathbf{h}_{ij}) (\mathbf{B}_k \frac{\partial \mathbf{h}_{ij+1}}{\partial f})^T - \mathbf{B}_k \frac{1}{2} \frac{\partial A_{ijk}}{\partial f} \right) \quad (76)$$

$$\frac{\partial \mathbf{M}_{ij}}{\partial f} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} \left( (\mathbf{B}_k \mathbf{h}_{ij}) (\mathbf{B}_k^T \frac{\partial \mathbf{h}_{ij}}{\partial f})^T + (\mathbf{B}_k \mathbf{h}_{ij}) (\mathbf{B}_k^T \frac{\partial \mathbf{h}_{ij}}{\partial f})^T \right) \quad (77)$$

$$\frac{\partial \mathbf{N}_{ij}}{\partial f} = -4 \sum_{k=1}^3 \frac{K_k}{L_{ei}} \left( (\mathbf{B}_k \frac{\partial \mathbf{h}_{ij+1}}{\partial f}) (\mathbf{B}_k \mathbf{h}_{ij+1})^T + (\mathbf{B}_k \mathbf{h}_{ij+1}) (\mathbf{B}_k \frac{\partial \mathbf{h}_{ij+1}}{\partial f})^T \right) \quad (78)$$

$$\frac{\partial A_{ijk}}{\partial f} = -2 \left( \frac{\partial \mathbf{h}_{i(j+1)}}{\partial f}^T \mathbf{B}_k^T \mathbf{h}_{ij} + \mathbf{h}_{i(j+1)}^T \mathbf{B}_k^T \frac{\partial \mathbf{h}_{ij}}{\partial f} \right) \quad (79)$$



Let us consider the  $j$ -th term of  $\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} V_{di})$ . Since  $\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{h}_{ij}} V_{di})$  of Eq.(55) is constant, if derivative w.r.t.  $f$  is null:

$$\frac{\partial}{\partial f} \nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{h}_{ij}} V_{di}) = \mathbf{0} \quad (80)$$

To compute the  $j$ -th term of  $\nabla_{\mathbf{q}_{ei}} (\nabla_{\mathbf{q}_{ei}} (\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}))$ , let us calculate the derivative w.r.t.  $f$  of Eq.(56):

$$\frac{\partial}{\partial f} \nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{h}_{ij}} (\boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)) = \frac{\partial}{\partial f} (\nabla_{\mathbf{h}_i} \mathbf{c}_{ij}) + \frac{\partial}{\partial f} (\nabla_{\mathbf{h}_i} \mathbf{d}_{ij}) \quad (81)$$

in which:

$$\frac{\partial}{\partial f} (\nabla_{\mathbf{h}_i} \mathbf{c}_{ij}) = 2L_{ei} \left( \frac{\partial \widetilde{\boldsymbol{\lambda}}_{4:6i}}{\partial f} \right) \quad (82)$$

and:

$$\begin{aligned} \frac{\partial}{\partial f} (\nabla_{\mathbf{h}_{ij}} \mathbf{d}_{ij}) &= \frac{\partial \lambda_{1i}}{\partial f} (\widetilde{\mathbf{d}}_{p1} - \widetilde{\mathbf{d}}_{p2}) + \frac{\partial \lambda_{2i}}{\partial f} (\widetilde{\mathbf{d}}_{p1} - \widetilde{\mathbf{d}}_{p3}) + \frac{\partial \lambda_{3i}}{\partial f} (\widetilde{\mathbf{d}}_{p2} - \widetilde{\mathbf{d}}_{p3}) + \\ &+ \lambda_{1i} \left( \frac{\partial \widetilde{\mathbf{d}}_{p1}}{\partial f} - \frac{\partial \widetilde{\mathbf{d}}_{p2}}{\partial f} \right) + \lambda_{2i} \left( \frac{\partial \widetilde{\mathbf{d}}_{p1}}{\partial f} - \frac{\partial \widetilde{\mathbf{d}}_{p3}}{\partial f} \right) + \lambda_{3i} \left( \frac{\partial \widetilde{\mathbf{d}}_{p2}}{\partial f} - \frac{\partial \widetilde{\mathbf{d}}_{p3}}{\partial f} \right), \quad j = N_{elt} \end{aligned} \quad (83)$$

In particular, the derivatives of the rotation matrix columns  $\mathbf{d}_{pi}$  are computed by first calculating  $\partial \mathbf{R}_p / \partial f$  and then extracting the columns:

$$\frac{\partial \mathbf{R}_p}{\partial f} = \sum_{i=1}^{n_e-3} \frac{\partial \mathbf{R}_p}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial f} \quad (84)$$

where  $\alpha_i \ i = 1, \dots, n_e - 3$  is the  $i$ -th parameter of the platform orientation.

### B. Computation of $\partial (\nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_e} \mathcal{L})) \partial f$

The derivative of  $\nabla_{\mathbf{q}_p} (\nabla_{\mathbf{q}_e} \mathcal{L})$  w.r.t.  $f$  is computed by differentiating its elementary term  $\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{q}_p} \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)$  w.r.t.  $f$

$$\frac{\partial}{\partial f} (\nabla_{\mathbf{h}_{ij}} (\nabla_{\mathbf{q}_p} \boldsymbol{\lambda}_i^T \boldsymbol{\Phi}_i)) = \left[ \begin{array}{c} \mathbf{0}_{3 \times 4} \\ \partial (\nabla_{\mathbf{h}_{ij}} \mathbf{m}_{p2}) / \partial f \end{array} \right] \quad (85)$$

in which the term  $\partial (\nabla_{\mathbf{h}_{ij}} \mathbf{m}_{p2}) / \partial f$  is composed as a sum of three terms:

$$\frac{\partial}{\partial f} (\nabla_{\mathbf{h}_{ij}} \mathbf{m}_{p2}) = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \quad (86)$$

where:

$$\begin{aligned} \mathbf{p}_1 &= \frac{\partial \lambda_{1i}}{\partial f} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{2ij} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{1ij} \right) + \frac{\partial \lambda_{2i}}{\partial f} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{1ij} \right) + \\ &+ \frac{\partial \lambda_{3i}}{\partial f} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{3ij} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \mathbf{D}_{2ij} \right), \quad j = N_{elt} \end{aligned} \quad (87)$$

$$\begin{aligned} \mathbf{p}_2 &= \lambda_{1i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{2ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{1ij} \right) + \lambda_{2i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{3ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{1ij} \right) + \\ &+ \lambda_{3i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha} \partial f} \mathbf{D}_{2ij} \right), \quad j = N_{elt} \end{aligned} \quad (88)$$

$$\begin{aligned} \mathbf{p}_3 &= \lambda_{1i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{2ij}}{\partial f} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{1ij}}{\partial f} \right) + \lambda_{2i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{3ij}}{\partial f} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{1ij}}{\partial f} \right) + \\ &+ \lambda_{3i} \left( \frac{\partial \mathbf{d}_{p1}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{3ij}}{\partial f} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \boldsymbol{\alpha}} \frac{\partial \mathbf{D}_{2ij}}{\partial f} \right), \quad j = N_{elt} \end{aligned} \quad (89)$$

where  $\partial \mathbf{D}_{kij} / \partial f, k = 1, 2, 3$  are computed according to Eq.(37) by using  $\partial \mathbf{h}_{ij} / \partial f$ , and the derivatives of the platform rotation matrix columns to be computed in relation to the selected platform orientation parametrization.

### C. Computation of $\partial(\nabla_{\mathbf{q}_p}(\nabla_{\mathbf{q}_p}\mathcal{L}))\partial f$

The derivative of  $\nabla_{\mathbf{q}_p}(\nabla_{\mathbf{q}_p}\mathcal{L})$  w.r.t.  $f$  is structured as follows:

$$\nabla_{\mathbf{q}_p}(\nabla_{\mathbf{q}_p}(\lambda_i^T \Phi_i)) = \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \frac{\partial \mathbf{m}_{p3}}{\partial f} \end{bmatrix} \quad (90)$$

where the matrix  $\partial \mathbf{m}_{p3}/\partial f$  is defined as:

$$\frac{\partial \mathbf{m}_{p3}}{\partial f} = \begin{bmatrix} \partial \mathbf{m}_{p31}/\partial f \\ \partial \mathbf{m}_{p32}/\partial f \\ \partial \mathbf{m}_{p33}/\partial f \end{bmatrix} \quad (91)$$

and the vectors  $\partial \mathbf{m}_{p3i}/\partial f$  is structured as follows:

$$\mathbf{m}_{p3i} = - \left[ \frac{\partial \lambda_{4i}}{\partial f}, \frac{\partial \lambda_{5i}}{\partial f}, \frac{\partial \lambda_{6i}}{\partial f} \right] \frac{\partial \mathbf{m}_{p1}}{\partial \alpha_i} - [\lambda_{4i}, \lambda_{5i}, \lambda_{6i}] \frac{\partial^2 \mathbf{m}_{p1}}{\partial \alpha_i \partial f} + \frac{\partial^2 \mathbf{m}_{p2}}{\partial \alpha_i \partial f} \quad (92)$$

where:

$$\frac{\partial^2 \mathbf{m}_{p1}}{\partial \alpha_i \partial f} = p_{fi1} \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}}{\partial \alpha \partial \alpha_i} \right) + p_{fi2} \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p2}}{\partial \alpha \partial \alpha_i} \right) + p_{fi3} \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p3}}{\partial \alpha \partial \alpha_i} \right) \quad (93)$$

$$\frac{\partial^2 \mathbf{m}_{p2}}{\partial \alpha_i \partial f} = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 \quad (94)$$

$$\begin{aligned} \mathbf{t}_1 = \frac{\partial \lambda_{1i}}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{d}_{2ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \alpha \alpha_i} \mathbf{d}_{1ij} \right) + \frac{\partial \lambda_{2i}}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \mathbf{d}_{1ij} \right) + \\ + \frac{\partial \lambda_{3i}}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \mathbf{d}_{2ij} \right), \quad j = N_{elt} \end{aligned} \quad (95)$$

$$\begin{aligned} \mathbf{t}_2 = \lambda_{1i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{D}_{2ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \alpha \alpha_i} \mathbf{D}_{1ij} \right) \frac{\partial \mathbf{h}_{ij}}{\partial f} + \lambda_{2i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{D}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \mathbf{D}_{1ij} \right) \frac{\partial \mathbf{h}_{ij}}{\partial f} + \\ + \lambda_{3i} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \mathbf{D}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \mathbf{D}_{2ij} \right) \frac{\partial \mathbf{h}_{ij}}{\partial f}, \quad j = N_{elt} \end{aligned} \quad (96)$$

$$\begin{aligned} \mathbf{t}_3 = \lambda_{1i} \left( \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{2ij} - \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{1ij} \right) + \lambda_{2i} \left( \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{3ij} - \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{1ij} \right) + \\ + \lambda_{3i} \left( \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{3ij} - \frac{\partial}{\partial f} \left( \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \alpha_i} \right) \mathbf{d}_{2ij} \right), \quad j = N_{elt} \end{aligned} \quad (97)$$

with the derivatives of the platform rotation matrix columns to be computed in relation to the selected platform orientation parametrization.

### D. Computation of $\partial(\nabla_{\lambda}(\nabla_{\mathbf{q}_e}\mathcal{L}))\partial f$

Let us now compute the derivative of  $\nabla_{\lambda_i}(\nabla_{\mathbf{q}_{ei}}(\lambda_i^T \Phi_i))$  w.r.t.  $f$ . This term is structured as follows:

$$\frac{\partial}{\partial f} \nabla_{\lambda_i}(\nabla_{\mathbf{q}_{ei}}(\lambda_i^T \Phi_i)) = \frac{\partial}{\partial f} (\nabla_{\lambda_i} \mathbf{c}_{ij}) + \frac{\partial}{\partial f} (\nabla_{\lambda_i} \mathbf{d}_{ij}) \quad (98)$$

where:

$$\nabla_{\lambda_i} \mathbf{c}_{ij} = L_{ei} \begin{bmatrix} \mathbf{0}_{3 \times 4} \\ \frac{\partial \mathbf{D}_{3ij}}{\partial f} \end{bmatrix} \quad (99)$$

and:

$$\frac{\partial}{\partial f} \nabla_{\lambda_i} \mathbf{d}_{ij} = \begin{bmatrix} \frac{\partial}{\partial f} (\mathbf{d}_{p1}^T) \mathbf{D}_{2ij} - \frac{\partial}{\partial f} (\mathbf{d}_{p2}^T) \mathbf{D}_{1ij} \\ \frac{\partial}{\partial f} (\mathbf{d}_{p1}^T) \mathbf{D}_{3ij} - \frac{\partial}{\partial f} (\mathbf{d}_{p3}^T) \mathbf{D}_{1ij} \\ \frac{\partial}{\partial f} (\mathbf{d}_{p2}^T) \mathbf{D}_{3ij} - \frac{\partial}{\partial f} (\mathbf{d}_{p3}^T) \mathbf{D}_{2ij} \\ \mathbf{0}_{3 \times 4} \end{bmatrix} + \begin{bmatrix} \mathbf{d}_{p1}^T \frac{\partial}{\partial f} (\mathbf{D}_{2ij}) - \mathbf{d}_{p2}^T \frac{\partial}{\partial f} (\mathbf{D}_{1ij}) \\ \mathbf{d}_{p1}^T \frac{\partial}{\partial f} (\mathbf{D}_{3ij}) - \mathbf{d}_{p3}^T \frac{\partial}{\partial f} (\mathbf{D}_{1ij}) \\ \mathbf{d}_{p2}^T \frac{\partial}{\partial f} (\mathbf{D}_{3ij}) - \mathbf{d}_{p3}^T \frac{\partial}{\partial f} (\mathbf{D}_{2ij}) \\ \mathbf{0}_{3 \times 4} \end{bmatrix}, \quad j = N_{elt} \quad (100)$$

where  $\partial \mathbf{D}_{kij}/\partial f, k = 1, 2, 3$  are computed according to Eq.(37) by using  $\partial \mathbf{h}_{ij}/\partial f$ .

E. Computation of  $\partial (\nabla_{\lambda} (\nabla_{\mathbf{q}_p} \mathcal{L})) \partial f$

Then, the derivative of  $\nabla_{\lambda_i} (\nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i))$  w.r.t.  $f$  is calculated by derivating Eq.(68) w.r.t.  $f$ :

$$\frac{\partial}{\partial f} (\nabla_{\lambda_i} (\nabla_{\mathbf{q}_p} (\lambda_i^T \Phi_i))) = + \begin{bmatrix} \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \frac{\partial}{\partial f} (\nabla_{\lambda_{4:6i}} \mathbf{m}_{p2}) & \mathbf{0}_{3 \times 3} \end{bmatrix} \quad (101)$$

where:

$$\begin{aligned} \frac{\partial}{\partial f} (\nabla_{\lambda_{4:6i}} \mathbf{m}_{p2}) = & \left[ \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \partial f} \mathbf{d}_{2ij} - \frac{\partial^2 \mathbf{d}_{p2}^T}{\partial \alpha \partial f} \mathbf{d}_{1ij}, \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \partial f} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \partial f} \mathbf{d}_{1ij}, \frac{\partial^2 \mathbf{d}_{p1}^T}{\partial \alpha \partial f} \mathbf{d}_{3ij} - \frac{\partial^2 \mathbf{d}_{p3}^T}{\partial \alpha \partial f} \mathbf{d}_{2ij} \right] + \\ & + \left[ \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{2ij}}{\partial f} - \frac{\partial \mathbf{d}_{p2}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{1ij}}{\partial f}, \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{3ij}}{\partial f} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{1ij}}{\partial f}, \frac{\partial \mathbf{d}_{p1}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{3ij}}{\partial f} - \frac{\partial \mathbf{d}_{p3}^T}{\partial \alpha} \frac{\partial \mathbf{d}_{2ij}}{\partial f} \right] \end{aligned} \quad (102)$$

where the term  $\partial^2 \mathbf{d}_{pk}^T / \partial \alpha \partial f$  depends on the specific platform orientation parametrization, and  $\partial \mathbf{d}_{kij} / \partial f$  is obtained as follows:

$$\frac{\partial \mathbf{d}_{kij}}{\partial f} = \mathbf{D}_{kij} \frac{\partial \mathbf{h}_{ij}}{\partial f} \quad k = 1, 2, 3 \quad (103)$$

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