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## NONNEGATIVE POLYNOMIALS AND MOMENT PROBLEMS ON ALGEBRAIC CURVES

LORENZO BALDI<sup>\*</sup>, GRIGORIY BLEKHERMAN<sup>†</sup>, AND RAINER SINN<sup>‡</sup>

**ABSTRACT.** The cone of nonnegative polynomials is of fundamental importance in real algebraic geometry, but its facial structure is understood in very few cases. We initiate a systematic study of the facial structure of the cone of nonnegative polynomials  $P$  on a smooth real projective curve  $X$ . We show that there is a duality between its faces and totally real effective divisors on  $X$ . This allows us to fully describe the face lattice in case  $X$  has genus 1. The dual cone  $P^\vee$  is known as the moment cone, and it plays an important role in real analysis. We compute the Carathéodory number of  $P^\vee$  for an elliptic normal curve  $X$ , which measures the complexity of quadrature rules of measures supported on  $X$ . This number can also be interpreted as a maximal typical Waring rank with nonnegative coefficients. Interestingly, the topology of the real locus of  $X$  influences the Carathéodory number of  $P^\vee$ . We apply our results to truncated moment problems on affine cubic curves, where we deduce sharp bounds on the flat extension degree.

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## 1. INTRODUCTION

Providing representation for real nonnegative polynomials is a fundamental challenge in real algebraic geometry, dating back to the works of Hilbert [Hil88] and Artin [Art27] on the existence of sums of squares representations. Over the last century, these results, now called Positivstellensätze, have been extensively studied and generalized, see e.g. [Mar08]. However, the geometric properties of the set of nonnegative polynomials are less understood. More specifically, the polynomials of degree  $2d$  nonnegative on a variety  $X$  form a convex cone, which we denote  $P_{X,2d}$ , but its facial structure is fully known only in very few cases

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[Sch21, Kun14]: degree  $2d$  univariate polynomials (or equivalently, quadratic polynomials on the rational normal curve), quadratic polynomials on  $\mathbb{R}^n$  and quartic polynomials on  $\mathbb{R}^2$ .

We initiate a systematic study of the facial structure of the cone of nonnegative polynomials on a real projective curve  $X$ . We show that there is a nice duality between the faces of the cone of nonnegative polynomials and totally real divisors on  $X$ . For elliptic normal curves, which is the natural next step from the known case of the rational normal curve, we provide a complete description of the face lattice of  $P_{X,2d}$ . Interestingly, our description is independent of the topology of the real locus of the elliptic curve.

While the cone of nonnegative polynomials is a central object in real algebraic geometry, its dual convex cone  $P_{X,2d}^\vee$  plays an important role in real analysis for the study of truncated moment problems. In the analysis literature,  $P_{X,2d}^\vee$  is called the moment cone. It corresponds to the convex cone of linear functionals, acting on polynomials of degree  $2d$ , that can be written as integration with respect to a measure supported on  $X$ . We leverage our understanding of the cone of nonnegative polynomials on elliptic normal curves to find the Carathéodory number of the moment cone, improving results of [dK21]. The Carathéodory number is an important parameter for the complexity of quadrature formulas [RS18], and can also be interpreted as the maximal rank in a real symmetric tensor decomposition with nonnegative coefficients. Interestingly, we find that the Carathéodory number depends on the topology of the real locus: smooth totally real genus one curves can have either one or two connected components, and the Carathéodory number is different in these two cases.

Finally, we leverage our projective study to investigate the truncated moment problem on smooth affine planar cubic curves. These truncated moment problems (for specific rational curves), have been studied e.g. in [Fia11, Zal23]. We provide sharp degree bounds for flat extension, improving the known results. The flat extension bounds apply to all smooth cubic curves, and depend on the real topology of  $X$  and the number of points that  $X$  has at infinity. Based on these results we can claim that the truncated moment problem on smooth cubic planar curves is now fairly well-understood.

**1.1. Main results.** In this section, we state our main contributions. For a detailed comparison with the existing literature, we refer the reader to Section 1.2.

For the study of nonnegative polynomials, it is more convenient to work with homogeneous polynomials (forms) on projective varieties. This is the approach we will take until we discuss moment problems, which are traditionally considered on affine varieties. Let  $X \subset \mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$  be a real projective curve, and denote the real locus of  $X$  by  $X(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R})$ . We work with totally real curves, i.e. curves such that  $X(\mathbb{R})$  is Zariski dense in  $X$ . Our main object of study is the cone of nonnegative forms on  $X$  of degree  $2d$ . By replacing  $X$  with the  $d$ -th Veronese embedding  $v_{n,d}(X)$  we may restrict ourselves to analyzing nonnegative quadratic polynomials on real curves, and thus we define

$$P_{X,2} := \left\{ q \mid \deg q = 2 \text{ and for all } A \in X(\mathbb{R}), q(A) \geq 0 \right\}$$

to be the convex cone of quadratic forms nonnegative on  $X(\mathbb{R})$ . It is known that  $P_{X,2}$  is a closed, pointed convex cone in the real vector space  $R_2$  of quadratic forms on  $X$ . When the context is clear we will also simply use  $P$  to denote  $P_{X,2}$ .

The cone  $P_{X,2}$  has one zero dimensional face: the origin. One-dimensional faces of  $P_{X,2}$  are its extreme rays. To any positive dimensional face  $F$  we associate a unique totally real effective divisor  $\text{div}(F)$ , which we call the *face divisor* of  $F$ , as follows. The face divisor is equal to half the real part of the divisor  $\text{div}q$ , for any quadric  $q$  in the relative interior of  $F$ . In the converse direction, to any totally real effective divisor  $D$  we associate the face  $F_D$  consisting of all quadratic forms  $q$  such that  $\text{div}q \geq 2D$ . The above allows us to define a two maps  $\Phi$  and  $\Psi$  between faces  $F \subset P$  and totally real effective divisors on  $X$ :  $\Phi$  which sends a positive dimensional face of  $P_{X,2}$  to its face divisor  $\text{div}(F)$ , and the map  $\Psi$  which sends a totally real effective divisor  $D$  to its associated face  $F_D$ . We show that the maps  $\Phi$  and  $\Psi$  form a *Galois connection* between the positive dimensional faces of  $P_{X,2}$  and the partially ordered set of totally real effective divisors in the image of  $\Phi$ , i.e. the poset of face divisors.

**Theorem A** (see Theorem 3.2.5 and Theorem 3.2.6). *Let  $(\mathcal{F} \setminus \{\{0\}\}, \subset)$  be the poset of positive-dimensional faces of  $P_{X,2}$ , ordered by inclusion. Then  $\Phi$  and  $\Psi$  form a Galois connection between  $(\mathcal{F} \setminus \{\{0\}\}, \subset)$  and the face divisors  $(\text{Im } \Phi, \leq)$ . This means that, for all positive dimensional faces  $F \in \mathcal{F} \setminus \{\{0\}\}$  and face divisors  $D \in \text{Im } \Phi$ :*

$$\text{div}(F) \leq D \iff F \supseteq F_D$$

Moreover, we have  $(\Psi \circ \Phi)(F) = F_{\text{div}(F)} = F$  for all  $F \in \mathcal{F} \setminus \{0\}$ .

The study of the face lattice of  $P$  thus reduces to understanding the following crucial question: *which totally real effective divisors are face divisors of some face  $F$  of  $P$ ?* For the case of the rational normal curve  $X = \nu_{1,d}(\mathbb{P}^1) \subset \mathbb{P}^d$  it is known (see Example 3.1.1) that any real effective divisor of degree at most  $d$  is a face divisor, i.e. there are no constraints on face divisors, except for the natural restriction on the degree. We will show that the situation is only slightly more complicated in genus 1, i.e., for elliptic normal curves  $X \subset \mathbb{P}^n$ . In particular, for rational and elliptic normal curves the poset of divisors associated to faces of  $P$  is a lower set in the poset of all divisors. That means that if  $D' \leq D$  for a face divisor  $D$ , then  $D'$  is also a face divisor: in this case, we can extend the Galois connection in Theorem A from the face divisors to all totally real effective divisors. As we show in Example 3.2.8, the set of face divisor is not always a lower set in  $\text{Div}(X)$ : therefore we ask the following question.

**Question.** *For which projective curves  $X$  is the set of face divisors a lower set in the poset  $\text{Div}_{\geq 0}(X)$ ?*

To study the case of elliptic normal curves, we connect extreme rays of  $P$  defined by nonnegative quadrics with the maximal number of real zeroes to positive two-torsion points in the Jacobian of  $X$ , see Definition 3.4.4. In particular, we have the following asymptotic result, which holds in arbitrary genus.

**Theorem B** (see Corollary 3.4.10). *Let  $X \subset \mathbb{P}^n$  be a totally real smooth irreducible curve of genus  $g$ . For all sufficiently large  $d$ , there are  $2^g$  distinct families of extreme rays of  $P_{X,2d}$  defined by forms with the maximal number of real zeroes, which are parametrized by the positive 2-torsion points  $J(X)_2^+ \cong (\mathbb{Z}/2\mathbb{Z})^g$ .*

In Theorem B we study only extreme rays defined by forms with the maximal number of real zeros. However, it is shown in Example 3.3.3 that not all the extreme rays are always of this form. It is therefore natural to ask the following question.

**Question.** *Describe all extreme rays of  $P_{X,2d}$  for sufficiently large  $d$ .*

We now investigate in detail the genus one case. For the special case of plane cubics, we describe in Section 4.1 the facial structure of the cone of nonnegative quadratic forms by elementary methods, providing explicit Artin-type nonnegativity certificates. More generally, the previous results allow us to completely describe the facial structure of  $P_{X,2d}$  for elliptic normal curves. Since the Veronese embedding of an elliptic normal curve is again an elliptic normal curve, also in this case it is sufficient to study the cone  $P_{X,2}$  on nonnegative quadratic forms.

**Theorem C** (see Theorem 4.2.3). *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve with embedding given by the divisor  $E$ . If we denote  $\{O, T\} = J(\mathbb{R})_2^+$  the positive 2-torsion points, then all the proper faces of  $P_{X,2}$  can be described as follows:*

- (i)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $1 \leq \deg D \leq n$ . In this case, we have  $\dim F_D = 2(n+1 - \deg D)$ .
- (ii)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $n+1$  such that  $[D - E] = O \in J(\mathbb{R})_2^+$ . In this case,  $F_D = \mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$  and  $q = \ell^2$  is the square of a linear form.
- (iii)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $n+1$  such that  $[D - E] = T \in J(\mathbb{R})_2^+$ . In this case,  $F_D = \mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$ , and  $q$  is not a sum of squares.

It is clear that on an elliptic normal curve  $X \subset \mathbb{P}^n$  any face divisor  $F_D$  has degree at most  $n+1$ , since  $\deg X = n+1$ . Theorem C shows that up to degree  $n$  any totally real effective divisor  $D$  is a face divisor. On the other hand, if  $D$  has degree  $n+1$  then  $D$  is a face divisor if and only if the points of  $D$  sum (in the group law of  $X$ ) to a positive 2-torsion point. Theorem C also shows that the set of face divisors is a lower set in the poset of totally real effective divisors on  $X$ , as in the case of  $\mathbb{P}^1$ .

Now we move to the study of the dual cone  $P_{X,2}^\vee$ , which consists of linear functionals in the dual space to quadrics on  $X$ , that can be expressed as integration with respect to a measure supported on  $X(\mathbb{R})$ . Equivalently,  $P^\vee$  is the convex hull of the set of point evaluations on the points of  $X(\mathbb{R})$ , restricted to quadrics on  $X$ . The Carathéodory number  $\mathcal{C}_{X,2}$  of  $P_{X,2}^\vee$  is the minimal natural number  $\mathcal{C}_{X,2}$  such that any linear functional  $L \in P_{X,2}^\vee$  is conic sum of at most  $\mathcal{C}_{X,2}$  point evaluations. This important problem has been studied from different perspectives, and it is equivalent to determining the minimal number of nodes in a quadrature

rule, or the maximal rank in a symmetric tensor decomposition with nonnegative coefficients. We refer the reader to Section 1.2 for a detailed comparison with the existing literature.

In particular, the study of the nonnegative cone on elliptic normal curves allows us to determine the Carathéodory numbers, which remarkably depend on the topology of the real locus of the curve.

**Theorem D** (see Theorem 5.1.4). *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve.*

- (i) *If  $X(\mathbb{R})$  is connected, then  $C_{X,2} = \deg X = n + 1$ .*
- (ii) *If  $X(\mathbb{R})$  is disconnected, then  $C_{X,2} = \deg X + 1 = n + 2$ .*

The proof of Theorem D is inspired by Hilbert’s proof that every nonnegative ternary quartic is a sum of at most three squares (see [BPSV19] for a modern exposition). It is natural to ask if the same proof technique can be applied to curves of higher genus as well.

**Question.** *Is the Carathéodory number determined by the topology of  $X(\mathbb{R})$  for curves of arbitrary genus and sufficiently high degree?*

We now describe our results on the truncated moment problem. We switch from the projective setting to the affine one, and focus on the planar case, which is the most studied in the literature. Let then  $X$  be an affine real cubic plane curve, whose projectivization is smooth. In this affine setting, the moment cone  $P_{X, \leq 2d}^\vee$  is not necessarily closed, and it is equal to the projective moment cone minus the evaluations at points at infinity.

The *moment problem* consists of determining, for a linear functional  $L$ , whether  $L \in P_{X, \leq 2d}^\vee$  or not. The most effective and studied criterion to solve this problem is the *flat extension* criterion, which can be described as follows. Given a linear functional  $L$  acting on polynomials on degree  $\leq 2d$ , we say that  $\tilde{L}$ , acting on polynomials of degree  $\leq 2d + 2$ , is a flat extension of  $L$  if:

- (i)  $L$  is the restriction of  $\tilde{L}$  to polynomials of degree  $\leq 2d$ ; and
- (ii) the bilinear forms  $(p, q) \mapsto L(pq)$  and  $(p, q) \mapsto \tilde{L}(pq)$  (defined respectively on polynomials of degree  $\leq d$  and  $\leq d + 1$ ) have the same rank.

The flat extension criterion states that if  $L$  has a flat extension, then  $L \in P_{X, \leq 2d}^\vee$ .

Leveraging our projective study of the Carathéodory numbers, we show that for plane cubics the flat extension criterion is not always a necessary condition for  $L \in P_{X, \leq 2d}^\vee$ . To find an effective necessary condition, we introduce the notion of *almost flat extension* of  $L$ . This is an extension  $\tilde{L}$  of  $L$  to degree  $\leq 2d + 4$ , where the rank of the associate bilinear forms is allowed to grow by one, instead of remaining constant (see Definition 6.1.2 for a precise definition).

With the use of almost flat extensions, we can solve the moment problem for affine plane cubics .

**Theorem E** (see Proposition 6.2.1 and Theorems 6.2.2 and 6.2.4). *Let  $X(\mathbb{R}) \subset \mathbb{R}^2$  be the affine real locus of a totally real plane cubic, whose projectivization  $\overline{X}$  is smooth, and let  $L$  be a linear functional acting on  $\mathbb{R}[x, y]_{\leq 2d} / \mathcal{I}(X)_{\leq 2d}$ . Then  $L$  is a moment linear functional if and only if  $L$  admits an almost flat extension. Moreover:*

- (i) *if  $X(\mathbb{R})$  is connected and the projective closure  $\overline{X}(\mathbb{R})$  contains only one point at infinity, then  $L$  is a moment linear functional if and only if  $L$  admits a flat extension;*
- (ii) *if  $X(\mathbb{R})$  has two connected components, then  $L$  is a moment linear functional if and only if  $L$  admits an almost flat extension, and the almost flat extension condition cannot be replaced by flat extension.*

To the best of our knowledge, this is the first solution of the moment problem for irreducible, nonrational curves. We refer to Section 1.2 for a detailed comparison with the existing literature.

**1.2. Related works.** The first part of the manuscript deals with the geometrical properties of the convex cone of nonnegative polynomials, and in particular on the set of extreme rays (see Theorem A and Theorem B). The first results in this direction were obtained in [CL77], where they provide several examples of extreme rays (that are not SoS) of the nonnegative cone for the case  $X = \mathbb{P}^n$ . In particular, they highlight the role of the zeroes of nonnegative polynomials, to determine if they define extreme rays or not. We refer to the survey [Rez00] for more details on the developments in the twentieth century, and to [Sch21] for the description of Hilbert’s cases and other results of local nature.

Beyond such results, the case of ternary sextics was studied recently in [KS18]. In particular, the authors characterize the sets of nine points in the projective plane that admit a nonnegative, extreme ternary sextic vanishing precisely on them. These results are only partial, and before Theorem C a complete description of the extreme rays and the faces of the nonnegative cone was known to be possible only for Hilbert’s cases.

Results related to our study of nonnegative quadratic forms have been discussed in [Vin93] and [Kum19] for nonnegative differential forms. In these works the properties of the real Jacobian variety are exploited, in analogy to Theorem B and Theorem C.

We now focus on the study of the dual convex cones  $P_{X,2}^\vee$ , and in particular of their Carathéodory numbers. Exact results concerning Carathéodory numbers are rare in the literature. The rational univariate case is e.g. completely solved in [Sch17, Cor. 9.16]. To the best of our knowledge Theorem C is the first complete description for non-rational curves, and the first time where the topology of the real supporting set is shown to play a key role in the analysis of Carathéodory numbers.

Asymptotic estimates for Carathéodory numbers on (affine) curves have been recently studied in [RS18]. There, the point of view is the one of minimal nodes necessary for quadrature rules. Other asymptotic estimates for compact affine curves can be found in [dK21, Th. 4.8]. Theorem C shows that, in the genus one case, the topology of the real locus distinguishes between the two possibilities left open from [dK21, Th. 4.8]. Other estimates on the Carathéodory numbers for higher dimensional cases have been investigated in [dS18, dK21, GHK23].

We can also interpret the Carathéodory number as the rank of a real symmetric tensor decomposition with nonnegative coefficients, using forms supported on the Veronese embedding of the curve (we refer to Section 5.2 for more details). The similarities and differences between the complex case, real case, and real case with nonnegative coefficients have been investigated in [QCL16, BS16, Ang17, ABC18]. Finally, let us mention that our results on Carathéodory numbers rely on the analysis of convex hulls of (Veronese embeddings of) curves. The existence of semidefinite representations for such convex hulls was studied in [Hen11, Sch11, Sch18].

We conclude by studying the moment problem in the case of affine plane cubic curves in Theorem E. This problem has been studied in [Fia11] for the curve  $y = x^3$ , in [Zal23] for curves of the form  $y = q(x)$  and  $yx^\ell = 1$ , and in [Zal22] for the union of parallel lines. Other results for curves defined by sparse equations have been studied in [Zal21]. All these works deal with (unions of) rational curves, and the authors are able to characterize membership in the moment cone with the flat extension condition (partial studies for genus one curves can be found in [Bha20]). The flat extension condition is not necessary in our genus one case, and to overcome this problem we introduce the notion of almost flat extension. Moreover, we remark that these results apply for linear functionals  $L \in P_{X,\leq 2d}^\vee$  for large enough  $d$ , while Theorem E applies for any  $d \geq 1$ .

**1.3. Structure of the manuscript.** The next sections are structured as follows. In Section 2 we summarize the necessary preliminaries on real algebraic curves (Section 2.1), convex geometry (Section 2.2) and on the duality between nonnegative forms and measures (Section 2.3).

In Section 3, we start by presenting the facial structure of  $P_{\mathbb{P}^1,2d}$  (Section 3.1). To generalize the description to arbitrary curves, we show in Section 3.2 how to move in the general case from the faces of the cone of nonnegative quadrics to (totally real, effective) divisors and, vice versa, from divisors to faces. We then turn our attention to the dimension of the faces in Section 3.3. We finally describe the extreme rays arising from nonnegative quadrics with the maximal number of real zeroes in Section 3.4.

In Section 4, we investigate the genus one case, and in particular plane cubics (Section 4.1) and elliptic normal curves (Section 4.2), showing a complete description of the facial structure of the nonnegative cone.

In Section 5 we move to the dual setting, i.e. to the study of Carathéodory numbers for elliptic normal curves. In Section 5.1 we determine exactly the Carathéodory numbers in both the connected and disconnected case, while in Section 5.2 we interpret our results in terms of Waring decompositions.

Section 6 is dedicated to the study of flat extension properties for affine plane cubic curves. In Section 6.1 we introduce the concept of almost flat extension, and in Section 6.2 we prove that whether flat extension and almost flat extension are necessary and sufficient to solve the moment problem depends on the topology of the real locus and on the number of real points at infinity.

## 2. PRELIMINARIES

To make the article accessible to different audiences, we summarize in this section the basic definitions and notations about real algebraic curves (Section 2.1), convex geometry (Section 2.2), and finally about nonnegative forms and moment problems (Section 2.3).

**2.1. Real algebraic curves.** We refer to [Man20] for the basics of real algebraic geometry, which we briefly recall in this section. In the following, varieties are always to be understood as *reduced*, (*geometrically*) *irreducible* and *smooth*.

An  $\mathbb{R}$ -variety is a pair  $(X, \sigma)$ , where  $X$  is an (abstract) complex algebraic variety and  $\sigma$  is an anti-regular (or anti-holomorphic) involution on  $X$ , see [Man20, Def. 2.1.12]. We will often omit  $\sigma$  from the notation. The *real locus* of the  $\mathbb{R}$ -variety  $(X, \sigma)$ , denoted  $X(\mathbb{R})$ , is the set of fixed points for  $\sigma$ , i.e.  $X(\mathbb{R}) := \{x \in X : \sigma(x) = x\}$ . We denote by  $\mathbb{C}(X)$  the  $\mathbb{C}$ -algebra of rational functions and by  $\mathbb{R}(X)$  the  $\mathbb{R}$ -algebra of real rational functions on  $X$ . We say that an  $\mathbb{R}$ -variety is *totally real* if  $X(\mathbb{R})$  is Zariski dense in  $X$  (these are called varieties with *enough real points* in [Man20]).

Concretely, we are interested in *real algebraic projective subvarieties*, which we call *real varieties* for short. These are subvarieties  $X \subset \mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$  of the projective space, which are invariant under the natural involution of  $\mathbb{P}^n$ , i.e. the coordinate-wise conjugation. Equivalently, real varieties  $X \subset \mathbb{P}^n$  are given as the zero locus of (finitely many) real  $n$ -variate homogeneous polynomials  $\{p_1, \dots, p_a\} \subset \mathbb{R}[x_0, \dots, x_n] = \mathbb{R}[\mathbf{x}]$ . The real homogeneous coordinate ring is  $\mathbb{R}[X] = \mathbb{R}[\mathbf{x}]/I$ , where  $I = \mathcal{I}(X)$  is the ideal of (real) polynomials vanishing on  $X \subset \mathbb{P}^n$ .  $\mathbb{R}[X]$  is a graded ring, with the grading induced by  $\mathbb{R}[\mathbf{x}] = \mathbb{R}[\mathbb{P}^n]$ . We denote by  $R_k := \mathbb{R}[X]_k = \mathbb{R}[\mathbf{x}]_k/I_k$  the  $k$ -graded part, i.e. the  $\mathbb{R}$ -vector space of homogeneous real polynomials (or *forms*) of degree  $k$  on  $X(\mathbb{R})$ . The field of real rational functions on  $X$  is  $\mathbb{R}(X) = \text{Quot}(\mathbb{R}[U])$ , where  $U \subset X$  is an open real affine variety and  $\mathbb{R}[U]$  is the coordinate ring of  $U$ .

If  $\dim X = 1$  (dimension as a complex algebraic variety), we say that  $(X, \sigma)$  (resp.  $X \subset \mathbb{P}^n$ ) is a  $\mathbb{R}$ -curve (resp. a real projective curve). When  $X$  is totally real, the real locus  $X(\mathbb{R})$  of a  $\mathbb{R}$ -curve is a real differentiable manifold of dimension 1.

A (Weil) *divisor*  $D = \sum_{A \in X} a_A A$  on a curve  $X$  is an element of the free abelian group on the points of  $X$ . This means that a divisor is a formal combination of points  $A \in X$  with integer coefficients  $a_A \in \mathbb{Z}$ , where  $a_A = 0$  for all but finitely many  $A \in X$ . We denote by  $(\text{Div } X, +)$  the abelian group of divisors on  $X$ . A divisor  $D = \sum_{A \in X} a_A A \in \text{Div } X$  is called *effective* if  $a_A \geq 0$  for all  $A$ . The *support* of  $D$ , denoted  $\text{supp } D$ , is the set of  $A \in X$  such that  $a_A \neq 0$ .

We say that a divisor  $D = \sum_{A \in X} a_A A$  on a  $\mathbb{R}$ -curve  $X$  is *real* if it is invariant under  $\sigma$ , i.e. if  $\sum_{A \in X} a_A A = \sum_{A \in X} a_A \sigma(A)$ . We say that  $D = \sum_{A \in X} a_A A$  is *totally real* if the support of  $D$  is contained in  $X(\mathbb{R})$ , i.e. if  $a_A = 0$  for all  $A \in X \setminus X(\mathbb{R})$ . We are particularly interested in totally real effective divisors, i.e. divisors with nonnegative integer coefficients whose support is included in  $X(\mathbb{R})$ .

Given a divisor  $D = \sum_{A \in X} a_A A$ , we write  $D_{\mathbb{R}} = \sum_{A \in X(\mathbb{R})} a_A A$  and  $D_{\mathbb{C}} = \sum_{A \in X \setminus X(\mathbb{R})} a_A A$ . Then  $D$  is totally real if and only if  $D = D_{\mathbb{R}}$ .

**2.2. Convex geometry.** We refer to [Roc70, Bar02] for the basics of convex geometry, which we briefly recall in this section.

Given a finite dimensional vector space  $V$ , we call  $C \subset V$  a *convex cone* if  $\mathbb{R}_{\geq 0} \cdot C \subset C$  and  $C + C \subset C$ . Given  $B \subset V$ , we denote by  $\text{cone}(B)$  the smallest convex cone containing  $B$ , called the *conic hull* of  $B$ . We say that a convex cone  $C$  is *closed* if it is closed as a subset of  $V$  with the Euclidean topology. We say that  $C$  is *pointed* if  $C \cap -C = \{0\}$ . In this paper, we will mostly work with convex cones inside  $R_2 = \mathbb{R}[X]_2$ , the real vector space of quadratic forms restricted to a totally real projective curve  $X \subset \mathbb{P}^n$ .

A *face* of a convex cone  $C$  is a convex subset  $F \subset C$  such that, for all  $a, b \in C$ ,  $a + b \in F$  implies  $a \in F$  and  $b \in F$ . The *dimension* of a face  $F$ , denoted by  $\dim F$ , is the dimension of the smallest vector subspace  $W \subset V$  containing  $F$ , called the *linear span* of  $F$ . Faces of dimension one are called *extreme rays* of  $C$ . A closed pointed convex cone is equal to the conic hull (or convex hull) of its extreme rays, see e.g. [Roc70, Th. 18.5].

The *relative interior* of a face  $F$ , denoted  $\text{ri } F$ , is the interior of  $F$  when regarded as a subset of its linear span, equipped with the Euclidean topology. If, given two faces  $F_1, F_2 \subset C$ , there exists  $a \in \text{ri } F_1 \cap \text{ri } F_2$ , then  $F_1 = F_2$  (see [Roc70, Th. 18.1.2]).

Given a real vector space  $V$ , we denote by  $V^* := \text{hom}_{\mathbb{R}}(V, \mathbb{R})$  the dual space of  $\mathbb{R}$ -linear functions. If  $C \subset V$  is a convex cone,  $C^\vee := \{L \in V^* : L(V) \subset \mathbb{R}_{\geq 0}\}$  is the *dual cone* to  $C$ . A face  $F \subset C$  is called *exposed* if  $F = L^{-1}(0) \cap C$  for some  $L \in C^\vee$ .

*Conic duality* states that, given a *closed* convex cone  $C$ , we have  $(C^\vee)^\vee = C$ .

**2.3. Nonnegative forms and the moment problem.** We refer to [BPT12, BSV16, Sch17] for the study of cones of nonnegative forms and their dual cones.

In the following, we consider a totally real projective subvariety  $X \subset \mathbb{P}^n$ . If  $k = 2d$  is even, every  $q \in R_{2d} = \mathbb{R}[X]_{2d}$  has a well-defined sign on every  $A \in X(\mathbb{R})$ . We write  $q(A) \geq 0$  (resp.  $q(A) \leq 0$ ) if the sign of  $q$  at  $A$  is 0 or positive (resp. negative). If the form  $q$  is such that  $q(A) \geq 0$  for all  $A \in X(\mathbb{R})$ , we say that  $q$  is *nonnegative*. We define:

$$P_{X,2d} := \{q \in R_{2d} \mid \forall A \in X(\mathbb{R}) \ q(A) \geq 0\} = \{q \in R_{2d} \mid q \text{ is nonnegative}\}$$

$$\Sigma_{X,2d} := \left\{ \sum_{i=1}^r p_i^2 \in R_{2d} \mid r \in \mathbb{N}, p_1, \dots, p_r \in R_d \right\}$$

The elements of  $\Sigma_{X,2d}$  are called *sums of squares* forms. Clearly, every sum of squares is nonnegative, i.e.  $\Sigma_{X,2d} \subset P_{X,2d}$ . When clear from the context, we will write  $P := P_{X,2d}$  and  $\Sigma := \Sigma_{X,2d}$ .  $P$  and  $\Sigma$  are full dimensional, closed, pointed convex cones in the vector space  $R_{2d}$ , see [BSV16].

The dual convex cone  $P^\vee$  can be characterized as follows. Denote by  $\widehat{X}(\mathbb{R}) \subset \mathbb{R}^{n+1}$  the affine cone over  $X(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R})$ , and by  $\mathbb{S}^n$  the unit sphere in  $\mathbb{R}^{n+1}$ . For every  $x \in \widehat{X}(\mathbb{R})$ , denote by  $\text{ev}_x \in R_{2d}^*$  the point evaluation at  $x$ , i.e.  $\text{ev}_x(q) := q(x)$  for  $q \in R_{2d}$ . Then:

$$P^\vee = P_{X,2d}^\vee = \text{cone}(\text{ev}_A : A \in \widehat{X}(\mathbb{R}))$$

see [BSV16]. We will often replace  $\widehat{X}(\mathbb{R})$  by  $X(\mathbb{R})$ , writing  $\text{ev}_A$  for  $A \in X(\mathbb{R})$ , identifying  $A$  with any of its affine representatives in  $\widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$ . The *moment problem* is the problem of determining, given  $L \in R_{2d}^*$ , whether  $L \in P^\vee$  or  $L \notin P^\vee$ .

If  $L \in P^\vee$ , we define the *Carathéodory number* of  $L$  as:

$$C_{X,2d}(L) := \min \left\{ r \in \mathbb{N} \mid \exists x_1, \dots, x_r \in \widehat{X}(\mathbb{R}) \text{ s.t. } L \in \text{cone}(\text{ev}_{x_1}, \dots, \text{ev}_{x_r}) \right\}$$

and the *Carathéodory number* of  $P^\vee$  as

$$C_{X,2d} := \max \left\{ C_{X,2d}(L) : L \in P^\vee \right\}$$

The main goal of this article is to study in detail the convex cones  $P_{X,2d}, P_{X,2d}^\vee$ , to determine the Carathéodory number  $C_{X,2d}$  when  $X \subset \mathbb{P}^n$  is a totally real elliptic normal curve, and finally to apply these results to solve the moment problem on affine plane cubics.

### 3. THE CONVEX CONE OF NONNEGATIVE FORMS ON PROJECTIVE CURVES

In the following,  $X \subset \mathbb{P}^n$  is a smooth, irreducible, totally real projective curve. We want to study the convex cone  $P_{X,2d} \subset R_{2d} = \mathbb{R}[X]_{2d}$  of nonnegative forms on  $X(\mathbb{R})$  of degree  $2d$ . We can restrict to the case  $d = 1$ , since  $\mathbb{R}[X]_{2d} \cong \mathbb{R}[\nu_{n,d}(X)]_{2d}$ , where  $\nu_{n,d}(X)$  denotes the  $d$ -th Veronese embedding of  $X \subset \mathbb{P}^n$ .

We then fix  $P = P_{X,2}$ . Our goal is to understand the relationship between faces  $F \subset P$  and totally real effective divisors  $D \in \text{Div } X$ . Recall that, given any divisor  $D \in \text{Div}(X)$ , we can uniquely write  $D = D_{\mathbb{R}} + D_{\mathbb{C}}$ , where  $\text{supp } D_{\mathbb{R}} \subset X(\mathbb{R})$  and  $\text{supp } D_{\mathbb{C}} \subset X \setminus X(\mathbb{R})$ , and a divisor  $D$  is totally real if and only if  $D = D_{\mathbb{R}}$ .

**3.1. Bivariate forms.** Before developing the general theory, we describe the special case of  $\mathbb{P}^1$ . For a bivariate form  $q \in \mathbb{R}[x_0, x_1]_{2d}$ , we denote  $\text{div } q$  its divisor of zeroes and write  $\text{div } q = (\text{div } q)_{\mathbb{R}} + (\text{div } q)_{\mathbb{C}}$ , where the support of  $(\text{div } q)_{\mathbb{R}}$  and  $(\text{div } q)_{\mathbb{C}}$  is included in  $\mathbb{P}^1(\mathbb{R})$  and  $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{R})$  respectively.

**Example 3.1.1** (see also [Sch21, Prop. 1.4.4]). Let  $P = P_{\mathbb{P}^1,2d}$  be the convex cone of nonnegative bivariate homogeneous polynomials, or bivariate forms, of degree  $2d$ . Every nonnegative  $q \in P$  has all zeroes with even multiplicity on  $\mathbb{P}^1(\mathbb{R})$  (otherwise  $q$  changes sign). We therefore have  $(\text{div } q)_{\mathbb{R}} = 2D$ , for some totally



real effective divisor  $D$  of degree  $\leq d$ . We can determine the faces of  $P$  by specifying real zeroes on  $\mathbb{P}^1$ , as follows: given a totally real effective divisor  $D$  of degree  $\leq d$ , we define

$$F_D := \{q \in P \mid \text{div} q \geq 2D\}$$

These are faces of  $P$ . By divisibility properties of univariate polynomials, every divisor  $D$  imposes  $2 \deg D$  conditions in the  $2d + 1$  dimensional space of bivariate forms  $\mathbb{R}[x_0, x_1]_{2d}$ . More precisely, we have  $\dim F_D = 2(d - \deg D) + 1$ , as we show in the following.

- If  $D = 0$  is the empty divisor, then  $F_D = P$ , and we want to show that  $\dim P = 2d + 1$ . Therefore we only need to show that  $P$  has nonempty interior. But this is clear, as e.g. the form  $x_0^{2d} + x_1^{2d}$  is strictly positive on  $\mathbb{P}^1(\mathbb{R})$ , and it can be perturbed in any direction while remaining strictly positive.
- If  $D = A$  for some  $A \in \mathbb{P}^1(\mathbb{R})$ , then  $F_D = F_A$  consists on nonnegative forms vanishing on  $A$ . This imposes two conditions on form of degree  $2d$ , and  $F_D$  is full dimensional in its  $2(d - 1) + 1$  linear span given by  $V_D := \{q \in \mathbb{R}[x_0, x_1]_{2d} : \text{mult}_A(q) \geq 2\}$ . For instance, if  $A = (a : b)$ , then the form  $(bx_0 - ax_1)^2(x_0^{2d-2} + x_1^{2d-2})$  lies in the relative interior of  $F_D$ , as it vanishes exactly with multiplicity 2 at  $A$  on all  $\mathbb{P}^1(\mathbb{R})$ :  $\text{div}((bx_0 - ax_1)^2(x_0^{2d-2} + x_1^{2d-2}))_{\mathbb{R}} = 2A$ .
- In general, if  $D$  is a totally real effective divisor with  $\deg D \leq d$ , then  $F_D$  is full dimensional in  $V_D = \{q \in \mathbb{R}[x_0, x_1]_{2d} : \text{div} q \geq 2D\}$ , which is a  $2(d - \deg D) + 1$  dimensional vector space. A point in the relative interior of  $F_D$  can be constructed as follows. If  $D = A_1 + \dots + A_k$  for points  $A_i = (a_i : b_i) \in \mathbb{P}^1(\mathbb{R})$ , set  $f = \prod_{i=1}^k (b_i x_0 - a_i x_1)^2$ . Then  $f(x_0^{2d-2k} + x_1^{2d-2k})$  lies in the relative interior of  $F_D \subset V_D$ .

All the faces of  $P$  are of the form  $F_D$  for some totally real effective divisor  $D$ . Indeed, if  $F \subset P$  is a face, then pick  $q \in \text{ri} F$ , and write  $(\text{div} q)_{\mathbb{R}} = 2D$ . It is then possible to show that  $q \in \text{ri} F_D$ , and therefore  $F = F_D$  from [Roc70, Cor. 18.1.2] (see also Lemma 3.2.1).

In the next sections, we will generalize Example 3.1.1 to arbitrary totally real projective curves  $X \subset \mathbb{P}^n$ .

**3.2. Faces and divisors.** Let  $X \subset \mathbb{P}^n$  be a smooth, irreducible, totally real projective curve. For  $0 \neq q \in \mathbb{C}[X]$  we consider the divisor of  $q$ , see e.g. [Sha13, p. 152], denoted  $\text{div} q$ . This divisor encodes the intersection points (with multiplicity) of  $X$  inside  $\mathbb{P}^n$  with the hypersurface defined by any  $Q \in \mathbb{C}[\mathbf{x}]$  such that  $Q + I(X) = q$  in  $\mathbb{C}[X]$ :  $\text{div} q = \sum_{A \in X} \text{mult}_A(Q)A$ . We have  $\deg(\text{div} q) = \sum_{A \in X} \text{mult}_A(Q) = \deg q \cdot \deg X$ .

We now consider  $q \in P \subset R_2 = \mathbb{R}[X]_2$ . Since  $q$  does not change sign on  $X(\mathbb{R})$ , it does not change sign on any connected component of  $X(\mathbb{R})$ . Therefore, if  $q$  has a zero at  $A \in X(\mathbb{R})$ , the multiplicity of  $q$  at  $A$  is even. This implies that  $(\text{div} q)_{\mathbb{R}} = 2D$  for some effective totally real divisor  $D$ .

We now show that any non-zero face  $F \subset P$  determines a unique totally real effective divisor  $D$  on  $X \subset \mathbb{P}^n$ .

**Lemma 3.2.1.** *Let  $\{0\} \neq F \subset P$  be a face. Then there exists a unique totally real effective divisor  $D \subset \text{Div}(X)$  such that, for all  $q$  in the relative interior of  $F$ ,  $(\text{div} q)_{\mathbb{R}} = 2D$ .*

*Proof.* For  $A \in X(\mathbb{R})$ , let  $\mu_A = \min\{\text{mult}_A(q) : q \in F\}/2$ . Note that, since  $F \neq \emptyset$ , we have  $\mu_A \neq 0$  only for finitely many  $A \in X(\mathbb{R})$  (any quadratic form  $q \in \mathbb{C}[X]$ ,  $q \neq 0$ , only vanishes at finitely many points on  $X$ ). We show that, for all  $q \in F$  and  $A \in X(\mathbb{R})$ ,  $\text{mult}_A(q) > 2\mu_A$  implies  $q \notin \text{ri} F$ . So we define the totally real effective divisor  $D := \sum_{A \in X(\mathbb{R})} \mu_A A$ . The following argument then shows  $(\text{div} q)_{\mathbb{R}} = 2D$  for all  $q \in \text{ri} F$  as claimed.

Consider the vector space  $V_A = \{q \in R_2 : \text{mult}_A(q) \geq 2\mu_A\}$ . By definition,  $F \subset V_A$ . We construct a linear functional  $L : V_A \rightarrow \mathbb{R}$  as follows. Consider a local parameter  $t$  at  $A \in X(\mathbb{R})$ , and expand  $q \in V_A$  locally around  $A$  so that  $q = a_q t^{2\mu_A} + \dots$ , where the dots represent higher order terms in  $t$ . We define  $L(q) = a_q$ . Since every  $q \in F$  is nonnegative on  $X(\mathbb{R})$  we conclude  $L(q) \geq 0$  for all  $q \in F$ , i.e.  $L \in F^\vee \subset V_A^*$ . By definition of  $\mu_A$ , there exists a form  $g \in F$  with  $\text{mult}_A(g) = 2\mu_A$ , which implies  $L(g) > 0$ . So  $L$  defines a supporting hyperplane to  $F$  containing any  $q \in F$  with  $\text{mult}_A(q) > 2\mu_A$ , showing  $q \notin \text{ri} F$ .  $\square$

Since the divisor  $(\text{div} q)_{\mathbb{R}}$  is equal for all  $q \in \text{ri} F$  by Lemma 3.2.1 we can make the following definition.

**Definition 3.2.2.** Given a face  $\{\{0\}\} \neq F \subset P$ , we define the *face divisor* of  $F$  as the unique totally real divisor  $\text{div}(F)$  such that  $(\text{div} q)_{\mathbb{R}} = 2\text{div}(F)$  for all  $q \in \text{ri} F$ .

We therefore have a way to move from faces to divisors. We can also move naturally in the other direction, as follows.

**Definition 3.2.3.** Given a totally real effective divisor  $D$ , we define

$$F_D := \{q \in P : \text{div}(q) \geq 2D\}$$

the face of  $P$  consisting of quadratic forms vanishing on  $X(\mathbb{R})$  with at least twice the multiplicity given by  $D$ . We call  $F_D$  the *face associated to the divisor  $D$* .

**Lemma 3.2.4.**  $F_D$  is a face of  $P$ .

*Proof.* Since  $V_D = \{q \in \mathbb{R}[X] : \text{div}(q) \geq 2D\}$  is a linear space, we conclude that  $F_D$  is a convex cone contained in  $P$ . To show that it is a face, pick  $q_1, q_2 \in P$  with  $q_1 + q_2 \in F_D$ . So we have  $\text{mult}_A(q_1 + q_2) \geq \text{mult}_A(2D)$  for all  $A \in X(\mathbb{R})$ . Since  $q_1$  and  $q_2$  are nonnegative on  $X(\mathbb{R})$ , the valuation  $\nu_A(q_1 + q_2)$  is equal to  $\min\{\nu_A(q_1), \nu_A(q_2)\}$  so that we also get  $\text{mult}_A(q_1) \geq \text{mult}_A(2D)$  and  $\text{mult}_A(q_2) \geq \text{mult}_A(2D)$  for all  $A \in X(\mathbb{R})$ . Therefore  $q_1, q_2 \in F_D$ , showing that  $F_D$  is a face.  $\square$

Denote by  $\mathcal{F}$  the set of faces of  $P$  and by  $\text{Div}_{\geq 0}(X(\mathbb{R}))$  the set of totally real effective divisors. We define

$$\begin{aligned} \Phi: \mathcal{F} \setminus \{\{0\}\} &\longrightarrow \text{Div}_{\geq 0}(X(\mathbb{R})) & \Psi: \text{Div}_{\geq 0}(X(\mathbb{R})) \supset \text{Im}(\Phi) &\longrightarrow \mathcal{F} \\ F &\longmapsto \text{div}(F) & D &\longmapsto F_D \end{aligned}$$

$\Phi$  and  $\Psi$  are, by definition, two order-reversing maps between the poset of (nonzero) faces  $\mathcal{F} \setminus \{\{0\}\}$  and the face divisors  $\text{Im}(\Phi) \subset \text{Div}_{\geq 0}(X(\mathbb{R}))$ . Here, we order  $\mathcal{F}$  by inclusion and  $\text{Im}(\Phi)$  by the usual order on  $\text{Div}(X)$  defined as such:  $\sum_{P \in X} a_P P \leq \sum_{P \in X} b_P P$  if  $a_P \leq b_P$  for all  $P \in X$ .

We now show that every face can be realized as the face associated to a totally real effective divisor. In other words, we prove that  $\Psi$  is surjective.

**Theorem 3.2.5.** Let  $X \subset \mathbb{P}^n$  be a totally real curve and  $P = P_{X,2}$  the convex cone of nonnegative quadrics. For any face  $F \subset P$ ,  $F \neq \{0\}$  the following statements hold.

- (i)  $F = F_{\text{div}(F)} = (\Psi \circ \Phi)(F)$ ;
- (ii) The linear span of  $F$  is  $V_{\text{div}(F)} = \{q \in R_2 : \text{div} q \geq 2\text{div}(F)\}$ ;
- (iii)  $\text{ri} F = \{q \in P : (\text{div} q)_{\mathbb{R}} = 2\text{div}(F)\}$ .

*Proof.* Let  $\text{div}(F)$  be the face divisor of  $F$ . For any  $g \in \text{ri} F$ , we have  $(\text{div} g)_{\mathbb{R}} = 2\text{div}(F)$  by definition. We now show that if  $(\text{div} g)_{\mathbb{R}} = 2\text{div}(F)$  for some  $g \in P$  then  $g$  lies in the relative interior of  $F$ .

Consider the vector space  $V = V_{\text{div}(F)} = \{q \in R_2 : (\text{div} q)_{\mathbb{R}} \geq 2\text{div}(F)\}$ . Then we have  $F_{\text{div}(F)} \subset V$  by definition. We now show that any  $g \in P$  with  $(\text{div} g)_{\mathbb{R}} = 2\text{div}(F)$  belongs to the interior of  $F_{\text{div}(F)}$  in  $V$ , which in particular implies that  $V$  is the linear span of  $F_{\text{div}(F)}$ . Since  $(\text{div} g)_{\mathbb{R}} = 2\text{div}(F)$  we have  $g \in F_{\text{div}(F)}$ . We only need to show that for any  $q \in V$  there is an  $\varepsilon > 0$  such that  $g + \varepsilon q \in P$ , i.e., that  $g + \varepsilon q \geq 0$  on  $X(\mathbb{R})$ . This follows from compactness of  $X(\mathbb{R})$ . We switch to the affine cone so that  $g$  and  $q$  are actual functions on  $\widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$ . For every  $A \in \text{supp div}(F)$  let  $x_A \in \widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$  be an affine representative. Since the multiplicity of  $g$  at  $A$  is smaller or equal to the multiplicity of  $q$  at  $A$ , and since  $g$  is nonnegative, there is an open neighborhood  $U_A$  of  $x_A$  in  $\widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$  and an  $\varepsilon > 0$  such that  $g + \varepsilon q \geq 0$  on  $U_A$ . Since  $g(-x) = g(x)$  and  $g$  vanishes only at finitely many points in  $X(\mathbb{R})$ , there exists an  $\varepsilon > 0$  such that  $g + \varepsilon q \geq 0$  on  $U := \bigcup_{A \in \text{supp div}(F)} (U_A \cup -U_A)$ . Next, since all the zeroes of  $g$  belong to  $U$ , we conclude that  $g$  is strictly positive on the compact set  $K := (\widehat{X}(\mathbb{R}) \cap \mathbb{S}^n) \setminus U$ . Since  $q$  is bounded on  $K$  and  $g$  has a strictly positive minimum on  $K$  we get  $g + \varepsilon q > 0$  on  $K$  for sufficiently small  $\varepsilon > 0$ . Overall, we conclude that there is an  $\varepsilon > 0$  such that  $g + \varepsilon q$  is nonnegative on  $X(\mathbb{R})$ , showing that  $g$  is in the relative interior of  $F$ .

This also implies that any interior point of  $F$  is an interior point of  $F_{\text{div}(F)}$ , and thus by [Roc70, Cor. 18.1.2] we have  $F = F_{\text{div}(F)}$ . Moreover, the linear span of  $F$  is  $V$ .

Above, we have shown the inclusion  $\text{ri} F_{\text{div}(F)} \supset \{q \in P : (\text{div} q)_{\mathbb{R}} = 2\text{div}(F)\}$ . We now prove the reverse inclusion as well. For  $f \in F_{\text{div}(F)} \setminus \{q \in P : (\text{div} q)_{\mathbb{R}} = 2\text{div}(F)\}$  there exists a real point  $A \in \text{supp div}(F)$  such that  $\text{mult}_A(f) > \text{mult}_A(2\text{div}(F))$ . Consider then  $L: V_A \rightarrow \mathbb{R}$  as in the proof of Lemma 3.2.1. By definition we have  $L(f) = 0$  and  $L(g) > 0$  for all  $g \in \{q \in P : (\text{div} q)_{\mathbb{R}} = 2\text{div}(F)\} \supset \text{ri} F_{\text{div}(F)}$ , concluding the proof.  $\square$

Theorem 3.2.5 allows us to describe the duality between faces and face divisors precisely, as follows.

**Theorem 3.2.6.** *The functions*

$$\begin{aligned} \Phi: \mathcal{F} \setminus \{\{0\}\} &\longrightarrow \text{Div}_{\geq 0}(X(\mathbb{R})) & \Psi: \text{Div}_{\geq 0}(X(\mathbb{R})) \supset \text{Im}(\Phi) &\longrightarrow \mathcal{F} \\ F &\longmapsto \text{div}(F) & D &\longmapsto F_D \end{aligned}$$

form a Galois connection between the positive dimensional faces  $\mathcal{F} \setminus \{\{0\}\}$ , ordered by inclusion, and the set of face divisors  $\text{Im}(\Phi) \subset \text{Div}_{\geq 0}(X(\mathbb{R}))$ . This means that, for all  $F \in \mathcal{F}$  and  $D \in \text{Im} \Phi$  we have

$$\text{div}(F) \leq D \iff F \supseteq F_D.$$

*Proof.* Let  $F$  and  $D$  be such that  $\text{div}(F) \leq D$ . By definition this implies  $F_{\text{div}(F)} \supset F_{\text{div}(F_D)}$ . From Lemma 3.2.4 we deduce that  $F_D$  is a face of  $P$ , while from Theorem 3.2.5  $F_D = F_{\text{div}(F_D)}$  and  $F = F_{\text{div}(F)}$ . Therefore  $F = F_{\text{div}(F)} \supset F_{\text{div}(F_D)} = F_D$ , as we wanted.

For the converse implication, let  $D = \text{div}(\tilde{F}) \in \text{Im} \Phi$  and  $F \supset F_D = F_{\text{div}(\tilde{F})}$ . This implies that  $\text{div}(F) \leq \text{div}(F_{\text{div}(\tilde{F})}) = \text{div}(\tilde{F}) = D$ , concluding the proof.  $\square$

Theorem 3.2.5 implies that  $\Psi$  is surjective. On the other hand,  $\Phi$  is not. Characterizing the image of  $\Phi$  is difficult, and it is naturally connected to existence of nonnegative quadrics, as we now show.

**Lemma 3.2.7.** *Let  $D$  be a totally real effective divisor. The following are equivalent:*

- (i)  $D \in \text{Im}(\Phi)$ , i.e.  $D$  is a face divisor;
- (ii)  $D = \text{div}(F_D)$ ;
- (iii) There exists  $q \in P$  such that  $(\text{div} q)_{\mathbb{R}} = 2D$ .

*Proof.* For (i)  $\implies$  (ii), we have to prove  $\text{div}(F) = \text{div}(F_{\text{div}(F)})$ , which follows from Theorem 3.2.5. The converse implication (ii)  $\implies$  (i) is trivial.

(ii)  $\implies$  (iii) follows by taking  $q \in \text{ri} F_D$  because we have  $(\text{div} q)_{\mathbb{R}} = 2\text{div}(F_D) = 2D$  for such a  $q$ . The final implication (iii)  $\implies$  (ii) follows from the fact that  $(\text{div} q)_{\mathbb{R}} = 2D$  implies  $q \in \text{ri} F_D$ , and thus  $D = \text{div}(F_D)$ .  $\square$

Studying the set  $\text{Im} \Phi$  is challenging. The following example, for instance, shows that  $\text{Im}(\Phi)$  is in general not a lower set in  $\text{Div}_{\geq 0}(X(\mathbb{R}))$ , i.e. for  $D, D' \in \text{Div}_{\geq 0}(X(\mathbb{R}))$ ,  $D \in \text{Im} \Phi$  and  $D' \leq D \not\Rightarrow D' \in \text{Im} \Phi$ .

**Example 3.2.8.** Consider the projective closure  $X \subset \mathbb{P}^2$  of the affine plane quartic defined by

$$2(1 - x_1^2 - 2x_2^2)(1 - 2x_1^2 - x_2^2) - (4x_1^2 - 1)(4x_2^2 - 1) = 0$$

Its real locus consists of four disjoint ovals, symmetric with respect to the  $x_1$  and  $x_2$  axis. Setting  $a = \sqrt{1 - \sqrt{112}/14}$ , we can verify that  $X(\mathbb{R})$  intersects the quadric  $q = a^2 - x_1^2$  in the four points  $(\pm a, \pm \sqrt{1 - 3\sqrt{7}/14})$  with multiplicity 2, and that  $q$  is nonnegative on  $X(\mathbb{R})$ . If we denote these four points by  $A_1, \dots, A_4$  and we define  $D = A_1 + \dots + A_4$ , we therefore have that  $\text{div} q = (\text{div} q)_{\mathbb{R}} = 2D$ , and thus  $D \in \text{Im} \Phi$ .

We now set  $D' = A_1 + A_2 + A_3$ . To show that  $\text{Im} \Phi$  is not a lower set in the totally real effective divisors, we prove that  $D' \notin \text{Im} \Phi$ . Equivalently, we want to show that there does not exist  $q' \in P$  s.t.  $(\text{div} q')_{\mathbb{R}} = 2D'$ . Let  $q' \in R_2$  be a quadric such that  $\text{div} q' \geq 2D$ . It follows from the Cayley-Bacharach theorem (e.g. [EGH96, Th. CB7], applied to the hypersurfaces  $X$  and  $q = 0$  with  $\Gamma' = 2D'$  and  $\Gamma'' = 2P_4$ ) that  $q'$  vanishes with multiplicity 2 at  $P_4$ , which implies that  $\text{div} q' \geq (\text{div} q')_{\mathbb{R}} \geq 2D$ . Therefore, every nonnegative quadric vanishing at  $P_1, P_2, P_3$  vanishes also at  $P_4$ , showing that  $D' = P_1 + P_2 + P_3 \notin \text{Im} \Phi$ . Thus  $\text{Im} \Phi$  is not a lower set.

**3.3. Dimension of faces.** In this section, we use the Riemann-Roch theorem (see e.g. [Sha13]) to compute dimensions of faces of the cone of nonnegative quadrics on  $X$ . The Riemann-Roch Theorem computes the dimension of the Riemann-Roch spaces

$$\mathcal{L}(D) = \{f \in \mathbb{C}(X) \mid D + \text{div}(f) \geq 0\}$$

as a  $\mathbb{C}$ -vector space. Writing  $\ell(D)$  for  $\dim_{\mathbb{C}}(\mathcal{L}(D))$ , the Riemann-Roch Theorem states  $\ell(D) = \deg(D) + 1 - g + \ell(K - D)$ , where  $g$  is the genus of  $X$  and  $K$  is a canonical divisor of  $X$ . For this computation to be reflected in the chosen embedding  $X \subset \mathbb{P}^n$  of our curve, we assume that the curve is *projectively normal*, which means that the linear systems  $|dH|$  for any  $d \in \mathbb{N}$  are complete, where  $H$  is a hyperplane in  $\mathbb{P}^n$ . Concretely this means that, for any divisor  $D \in \text{Div}(X)$  that is linearly equivalent to the zero divisor  $\text{div} g$  of a form  $g \in \mathbb{C}[X]_d$ , there exists a form  $h \in \mathbb{C}[X]_d$  with  $D = \text{div} h$ .

We can now investigate, in the projectively normal case, the dimension of faces  $F \in \mathcal{F}$ .

**Lemma 3.3.1.** *Let  $X \subset \mathbb{P}^n$  be a totally real curve, and let  $\{0\} \neq F \subset P$  be a nonzero face. If  $X \subset \mathbb{P}^n$  is projectively normal and  $q \in \text{ri} F$ , then  $\dim F = \ell((\text{div} q)_{\mathbb{C}})$ .*

*Proof.* Since  $D = \text{div}(F) \in \text{Im} \Phi$  we have  $D = (\text{div} q)_{\mathbb{R}}$  for any  $q \in \text{ri} F$  by Theorem 3.2.5. As in Theorem 3.2.5 we write  $V_D = \{p \in R_2 : \text{div} p \geq 2D\}$ . For any  $q \in \text{ri} F$  consider the map:

$$V_D \longrightarrow \mathcal{L}((\text{div} q)_{\mathbb{C}}), \quad f \longmapsto f/q.$$

This is clearly well-defined and injective, and it is surjective since  $X \subset \mathbb{P}^n$  is projectively normal. Therefore  $\dim V_D = \ell((\text{div} q)_{\mathbb{C}})$ . By Theorem 3.2.5 we have  $\dim F = \ell((\text{div} q)_{\mathbb{C}})$ .  $\square$

Thanks to Lemma 3.3.1 and Theorem 3.2.5, when we consider a totally real projectively normal curve  $X \subset \mathbb{P}^n$ , we only need to consider nonnegative quadrics with a large number of real zeroes, in order to understand the dimension of all the small dimensional faces  $F \subset P$ . We make this statement precise in the following proposition for the extreme rays, i.e. for faces of dimension one.

**Proposition 3.3.2.** *Let  $X \subset \mathbb{P}^n$  be a totally real projectively normal curve of genus  $g$ . If  $\mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$ , then  $\deg((\text{div} q)_{\mathbb{R}}) \geq 2 \deg X - g$ . In particular, if all the real zeroes of  $q$  have multiplicity two, then  $q$  has at least  $\deg X - g/2$  real zeroes.*

*Proof.* If  $\mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$ , then combining Lemma 3.3.1 and Riemann's inequality we have:

$$1 = \ell((\text{div} q)_{\mathbb{C}}) \geq \deg((\text{div} q)_{\mathbb{C}}) - g + 1$$

This implies that:

$$2 \deg X - \deg((\text{div} q)_{\mathbb{R}}) = \deg((\text{div} q)_{\mathbb{C}}) \leq g$$

concluding the proof.  $\square$

The complete description of the extreme rays of  $P$  is difficult for curves of arbitrary genus, and we leave it for future investigation. However, in the genus one case, Proposition 3.3.2 implies that the extreme rays are all defined by nonnegative quadrics with the maximal number of real zeroes (counted with multiplicity). Genus one curves are already quite interesting, and we will study them in detail in Sections 4 to 6.

We conclude this section by showing an example of an extreme ray defined by a quadric with a nonmaximal number of real zeroes.

**Example 3.3.3.** Let  $X(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$  be the plane sextic defined by  $h = x_1^6 + x_2^6 - x_0^6 = 0$  and let  $q = x_1^2 + x_2^2 - x_0^2 \in \mathbb{R}[x_0, x_1, x_2]_2 \cong \mathbb{R}[X]_2$ . The curve  $X$  is a smooth plane curve of degree 6 and hence has genus  $g = 10$  from the genus-degree formula. So Proposition 3.3.2 implies that any nonnegative quadric defining an extreme ray of  $P$  has between  $2 \deg X - g = 2$  and 20 real zeroes, counted with multiplicity.

The curves  $h$  and  $q$  intersect with multiplicity two in the real points  $A_1 = (1 : 0 : 1)$ ,  $A_2 = (1 : 0 : -1)$ ,  $A_3 = (1 : 1 : 0)$  and  $A_4 = (1 : -1 : 0)$ , and with multiplicity two in the complex conjugate points  $A_5 = (0 : 1 : i)$  and  $A_6 = \sigma(A_5) = (0 : 1 : -i)$ . In other words,  $(\text{div} q)_{\mathbb{R}} = 2(A_1 + \dots + A_4)$  and  $(\text{div} q)_{\mathbb{C}} = 2(A_5 + \sigma(A_5))$ .

We know that  $X(\mathbb{R})$  is connected, and  $q \geq 0$  on  $X(\mathbb{R})$ , since the circle defined by  $q = 0$  is inside  $X(\mathbb{R})$ , which is a TV screen curve. By Proposition 3.3.2, to show that  $\mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$  it is sufficient to verify that  $\ell((\text{div} q)_{\mathbb{C}}) = \ell(2(A_5 + \sigma(A_5))) = 1$ . This can be done using the Cayley-Bacharach theorem, see e.g. [EGH96, Th. CB7] or using any computer algebra system. Alternatively, one can directly verify that the double vanishing at  $A_1, \dots, A_4$  imposes enough conditions to obtain a one-dimensional family of solutions.

Notice in particular that

$$2 \deg X - g = 2 < \deg(\text{div} q)_{\mathbb{R}} = 8 < 2 \deg X = 20$$

showing that  $\mathbb{R}_{\geq 0} \cdot q$  is an extreme ray defined by a quadric with a nonmaximal number of real zeroes, and not attaining the lower bound of the inequality in Proposition 3.3.2.

**3.4. Extreme rays with only real zeroes.** In this section, we study extreme rays defined by nonnegative quadrics with the maximal number of real zeroes for arbitrary genus curves. As discussed in Section 3.3, these are all the extreme rays in the genus one case. To do that, we will need to exploit properties of the Jacobian of a curve  $X$ , which we briefly recall. We consider in particular the Jacobian of an  $\mathbb{R}$ -curve  $X$ , for which we refer the reader to [GH81, Vin93, Kum19, Man20].

Let  $(X, \sigma)$  be a smooth irreducible  $\mathbb{R}$ -curve of genus  $g$ . For every  $f \in \mathbb{C}(X) \setminus \{0\}$ , we can consider the divisor  $\text{div} f$  of poles and zeroes. Such divisors form a normal subgroup of  $\text{Div}(X)$ , and the quotient  $\text{Cl}(X)$  is called the *divisor class group*. So two divisors  $D$  and  $D'$  represent the same class  $[D] = [D'] \in \text{Cl}(X)$  if and only if there exists  $f \in \mathbb{C}(X) \setminus \{0\}$  such that  $D = D' + \text{div} f$ . We denote  $\text{Cl}^d(X)$  the subgroup of divisor classes of degree  $d$ .  $\text{Cl}^0(X)$  is isomorphic to the Jacobian variety  $J = J(X) \cong \mathbb{C}^g/\Lambda$ , which is a (complex)  $g$ -torus. There is a natural action of  $\sigma$  on  $\text{Cl}^0(X) \cong J$ , which give  $J$  the structure of  $\mathbb{R}$ -variety. As usual, we denote  $J(\mathbb{R})$  the real part, i.e. the fixed points of  $\sigma$ . Concretely, a divisor class  $[D]$  is in  $\text{Cl}^0(X)(\mathbb{R}) \cong J(\mathbb{R})$  if the divisor  $\sigma(D)$  is linearly equivalent to  $D$ , which implies  $[\sigma(D)] = [D]$ .

We are especially interested in the 2-torsion points of  $J(\mathbb{R})$ .

**Definition 3.4.1.** We denote  $J_2 = \{\alpha \in J : 2\alpha = 0\}$  the subgroup of  $J$  consisting of 2-torsion points, and we denote the *real 2-torsion points* by  $J(\mathbb{R})_2 := J_2 \cap J(\mathbb{R})$ .

It is known that the number of real 2-torsion points depends on the genus  $g$  of  $X(\mathbb{R})$  and on the number of connected components  $r$  of  $X(\mathbb{R})$ .

**Lemma 3.4.2** ([GH81, § 5]). *If  $(X, \sigma)$  is a totally real  $\mathbb{R}$ -curve of genus  $g$  such that  $X(\mathbb{R})$  has  $r > 0$  connected components, then  $J(\mathbb{R})_2 \cong (\mathbb{Z}/2\mathbb{Z})^{g+r-1}$ .*

Among all the real 2-torsion points, we are interested in those that are defined by nonnegative rational functions.

**Lemma 3.4.3.** *Let  $\alpha \in J(\mathbb{R})_2$  and let  $D$  be a divisor representing  $\alpha$ . If there exists  $f \in \mathbb{R}(X)$  such that  $f \geq 0$  on  $X(\mathbb{R}) \cap \text{dom} f$  and  $\text{div} f = 2D$ , then for any  $D'$  representing  $\alpha$  there exists  $f' \in \mathbb{R}(X)$  such that  $f' \geq 0$  on  $X(\mathbb{R}) \cap \text{dom} f'$  and  $\text{div} f' = 2D'$ .*

*Proof.* Let  $D, D', f$  and  $\alpha$  be as in the statement. As  $[D] = \alpha = [D']$ , there exists  $g \in \mathbb{R}(X)$  such that  $D + \text{div} g = D'$ . Therefore we can take  $f' := fg^2$  as nonnegative rational function whose divisor is equal to  $2D'$ .  $\square$

**Definition 3.4.4.** We say that  $\alpha \in J(\mathbb{R})_2$  is *positive* if, for any divisor  $D$  representing  $\alpha$ , there exist  $f \in \mathbb{R}(X)$  such that  $f \geq 0$  on  $X(\mathbb{R}) \cap \text{dom} f$  and  $\text{div} f = 2D$  (see Lemma 3.4.3). We denote the subgroup of positive 2-torsion points by  $J(\mathbb{R})_2^+$ .

It is known that the number of positive 2-torsion points  $J(\mathbb{R})_2^+$  depends only on the genus of  $X(\mathbb{R})$ , and not on the topology of  $X(\mathbb{R})$ .

**Lemma 3.4.5** ([GM77, § 5]). *If  $(X, \sigma)$  is a totally real  $\mathbb{R}$ -curve of genus  $g$ , then  $J(\mathbb{R})_2^+ \cong (\mathbb{Z}/2\mathbb{Z})^g$ .*

*Proof.* Write  $X(\mathbb{R}) = Y_1 \sqcup \dots \sqcup Y_{r+1}$  as the disjoint union of its connected components. For all  $\alpha = [D] \in J(\mathbb{R})_2$ , we can choose  $f_\alpha \in \mathbb{R}(X)$  such that  $\text{div} f_\alpha = 2D$  and  $f_\alpha \geq 0$  on  $Y_1$ . The sign of any such  $f_\alpha$  on  $Y_2, \dots, Y_{r+1}$  only depends on  $\alpha$  because for any two rational functions  $f$  and  $f'$  such that  $\text{div}(f) = 2D$  and  $\text{div}(f') = 2D'$  and  $[D] = [D'] = \alpha \in J(X)$  differ by a square, see the proof of Lemma 3.4.3. We can therefore define the map  $J(\mathbb{R})_2 \rightarrow \{\pm 1\}^{r-1}$ , which associates to every  $\alpha$  the tuple of signs of  $f_\alpha$  on  $Y_2, \dots, Y_r$ . This is a well-defined group morphism, whose kernel is equal to  $J(\mathbb{R})_2^+$ . The morphism is surjective (see [GM77, § 5]), and thus Lemma 3.4.2 implies  $J(\mathbb{R})_2^+ \cong (\mathbb{Z}/2\mathbb{Z})^g$ .  $\square$

We are now ready to show how positive 2-torsion points can be used to characterize totally real divisors that arise as intersections of nonnegative quadrics on  $X \subset \mathbb{P}^n$  with the maximal number of real zeroes, which is equal to  $\text{deg} X$ .

**Proposition 3.4.6.** *Let  $X \subset \mathbb{P}^n$  be a totally real smooth irreducible projectively normal curve, with embedding given by the divisor  $E$ . Let  $D \in \text{Div}_{\geq 0}(X(\mathbb{R}))$  be an effective totally real divisor such that  $\text{deg} D = \text{deg} X$  and  $[2D] = [2E]$ . Then the following are equivalent.*

- (i) *There exists a quadric  $q \in R_2$  such that  $2D = (\text{div} q) = (\text{div} q)_{\mathbb{R}}$ .*
- (ii)  *$[D - E] \in J(\mathbb{R})$  is a 2-torsion point.*

*Moreover,  $q \in P_{X,2} = P$  if and only if  $[D - E] \in J(\mathbb{R})_2^+$ .*

*Proof.* For (i)  $\implies$  (ii), consider the rational function  $q/\ell^2$ , where  $\ell$  is the linear form with divisor  $\text{div } \ell = E$ . Then  $\text{div}(q/\ell^2) = 2D - 2E$ . If  $q \in P$ , then  $q/\ell^2$  is nonnegative on  $X(\mathbb{R})$  and thus  $[D - E] \in J(\mathbb{R})_2^+$ .

For (ii)  $\implies$  (i), as  $[D - E] \in J(\mathbb{R})_2$  there exists  $f \in \mathbb{R}(X)$  such that  $\text{div } f = 2D - 2E$ . Therefore  $2D = \text{div } f \ell^2$ , and by projective normality there exists  $q \in R_2$  such that  $2D = \text{div } q = (\text{div } q)_{\mathbb{R}}$ . Furthermore, if  $[D - E] \in J(\mathbb{R})_2^+$  then  $f$  is nonnegative on  $X(\mathbb{R})$  and thus  $q \in P$ , concluding the proof.  $\square$

**Theorem 3.4.7.** *Let  $X \subset \mathbb{P}^n$  be a totally real smooth irreducible projectively normal curve, with embedding given by the divisor  $E$ , and let  $D \in \text{Div}_{\geq 0}(X(\mathbb{R}))$  be a divisor of degree  $\deg D = \deg X$  with  $[2D] = [2E]$ . Denote  $P = P_{X,2}$  the cone of nonnegative quadrics on  $X(\mathbb{R})$ . Then the following are equivalent:*

- (i)  $F_D \subset P$  is an extreme ray of  $P$ ;
- (ii)  $[D - E]$  is a totally real positive 2-torsion point of the Jacobian.

*Proof.* Combine Proposition 3.4.6 and Theorem 3.2.5.  $\square$

Therefore, the existence of a nonnegative quadric with a maximal number of real zeroes can be read from the positive 2-torsion points in the Jacobian. However, to be able to apply Lemma 3.4.5 to count their number, we need to be sure that the torsion points are totally real, i.e. that their divisor classes can be represented using only points of  $X(\mathbb{R})$ .

**Definition 3.4.8.** Let  $X$  be an  $\mathbb{R}$ -curve. We say that a real divisor class  $\beta \in \text{Cl}^k(X)$  is *totally real* if there exists  $A_1, \dots, A_k \in X(\mathbb{R})$  such that  $\beta = [A_1 + \dots + A_k]$ . We denote by  $N(X)$  the smallest  $k$  such that all real divisor classes in  $\text{Cl}^k X$  are totally real.

The study of the invariant  $N(X)$  has been started by Scheiderer. [Sch00, Th. 2.7] in particular shows that  $N(X) < +\infty$  for every real curve  $X$ . While the study of the exact value invariant  $N(X)$  is still an open research question (except in the special cases of  $M$ -curves and  $M - 1$  curves [Mon03, Hui01]), the fact that  $N(X) < +\infty$  allows us to describe all extreme rays arising from forms with the maximal number of real zeroes, at least in sufficiently high degree  $2d$ .

**Lemma 3.4.9.** *Let  $X \subset \mathbb{P}^n$  be a totally real smooth irreducible projectively normal curve, with embedding given by the divisor  $E$ . For every  $d \in \mathbb{N}$ , and for every positive 2-torsion point  $\alpha \in J(\mathbb{R})_2^+$  such that  $\alpha + [dE]$  is totally real, there exist  $q_\alpha \in P_{X,2d}$  such that  $\text{div } q_\alpha = (\text{div } q_\alpha)_{\mathbb{R}}$  is totally real and  $\mathbb{R}_{\geq 0} \cdot q_\alpha$  is an extreme ray of  $P_{X,2d}$ .*

*Proof.* Let  $\ell \in R_1$  be such that  $\text{div } \ell = E$  and  $\text{div } \ell^d = dE$ . Since  $\alpha$  is a positive 2-torsion point, there exists  $f \in \mathbb{R}(X)$  nonnegative on  $X(\mathbb{R})$  such that  $\text{div } f = 2D$  and  $\alpha = [D]$ . As  $\alpha + [dE] = [D + dE]$  is totally real by hypothesis, there exists a totally real divisor  $D''$  such that  $[D + dE] = [D'']$ . Moreover, since the embedding is projectively normal and  $[2D''] = [2dE]$ , there exists  $q_\alpha \in R_2$  such that  $\text{div } q_\alpha = (\text{div } q_\alpha)_{\mathbb{R}} = 2D''$ . Finally, since  $\text{div } f = \text{div}(q_\alpha/\ell^{2d})$ , we have (up to a global sing change)  $q_\alpha \geq 0$  on  $X(\mathbb{R})$ , i.e.  $q_\alpha \in P_{X,2d}$ . From Theorem 3.4.7, we conclude that  $\mathbb{R}_{\geq 0} \cdot q_\alpha$  is an extreme ray of  $P_{X,2d} \cong P_{v_{n,d}(X),2}$ .  $\square$

**Corollary 3.4.10.** *Let  $X \subset \mathbb{P}^n$  be a totally real smooth irreducible curve of genus  $g$ . For all sufficiently large  $d$ , there are  $2^g$  distinct families of extreme rays of  $P_{X,2d}$  defined by forms with the maximal number of real zeroes, which are parametrized by the positive 2-torsion points  $J(\mathbb{R})_2^+ \cong (\mathbb{Z}/2\mathbb{Z})^g$ .*

*Proof.* Recall that  $P_{X,2d} \cong P_{v_{n,d}(X),2}$ , and if the divisor  $E$  gives the embedding  $X \subset \mathbb{P}^n$ , then  $dE$  gives the embedding of  $v_{n,d}(X)$ . Moreover, if  $d$  is big enough,  $v_{n,d}(X)$  is projectively normal and

$$d \cdot \deg X = \deg v_{n,d}(X) \geq N(X),$$

see Definition 3.4.8. This implies that for every  $\alpha \in J(\mathbb{R})_2^+$ , the divisor class  $\alpha + [dE]$  is totally real. Therefore, from Lemma 3.4.9 we deduce that for all  $\alpha \in J(\mathbb{R})_2^+$  there exist (a family of)  $q_\alpha \in P_{X,2d}$  such that  $\text{div}(q_\alpha)_{\mathbb{R}} = \text{div } q_\alpha$  and  $\mathbb{R}_{\geq 0} \cdot q_\alpha$  is an extreme ray. Finally, we can conclude from Lemma 3.4.5 that  $J(\mathbb{R})_2^+ \cong (\mathbb{Z}/2\mathbb{Z})^g$ .  $\square$

Corollary 3.4.10 also highlights the difference between sums of squares and nonnegative forms. Indeed, if the sums of squares and nonnegative cones are equal, then they have the same extreme rays. But sums of squares extreme rays correspond to only the zero of the Jacobian, which is a positive 2-torsion point. If the genus is at least one, then for all sufficiently large  $d$  there are positive 2-torsion points which correspond to non-sums of squares extreme rays, and thus we have  $\Sigma_{X,2d} \subsetneq P_{X,2d}$ . This is a particular case of [BSV16, Th. 1.1]. For a more detailed description in the genus one case, we refer to Proposition 4.1.1 and Theorem 4.2.3.

#### 4. NONNEGATIVE FORMS ON GENUS ONE CURVES

We now consider curves of genus 1. General results of Section 3 will specialize particularly nicely to this case. We begin with the example of nonnegative quadrics on cubic plane curves, where we illustrate our results explicitly.

**4.1. Plane cubics.** Plane cubics are the first example of genus one curves. For their basic properties, we refer e.g. to [Sha13, Ful89]. In this case, the analysis of  $P_{X,2}$  can be performed explicitly and completely. We include this analysis here to develop an intuition for Section 4.2, where it will be generalized to elliptic normal curves, and to provide a more accessible background for Section 6, where we focus on dual moment problems for plane cubics.

Plane cubics are curves in  $\mathbb{P}^2$  defined as zero locus of a form of degree three. In the following, we always assume that our plane cubics are smooth, irreducible, and totally real. Recall that the group law of a plane cubic  $X$  is induced by the isomorphism between the plane cubic and the Jacobian:

$$\begin{aligned} \varphi_{P_0} : X &\longrightarrow \text{Cl}^0 X \cong J_X \\ P &\longrightarrow [P - P_0] \end{aligned}$$

Hereafter, we consider  $P_0 = O$  to be a smooth real inflection point, so that our plane cubic has equation (Weierstrass form):

$$h = x_2^2 x_0 - (x_1 - a_1 x_0)(x_1 - a_2 x_0)(x_1 - a_3 x_0) = 0$$

with  $a_i \in \mathbb{C}$ .  $X$  is non-singular if and only if  $a_i \in \mathbb{C}$ 's are distinct.

Denote  $\oplus$  and  $\ominus$  the group operations in  $X$ . Then the point at infinity  $O$  is the identity, and

$$T_1 = (1 : a_1 : 0), T_2 = (1 : a_2 : 0), T_3 = (1 : a_3 : 0)$$

are the 3 nontrivial 2-torsion points for  $(X, \oplus)$  (see also Definition 3.4.1).

Since we are assuming that  $X$  is totally real, then at least one of the  $a_i$  is real, say  $a_1$ , while  $a_2$  and  $a_3$  can be either:

- (i) Complex conjugates  $a_2 = \overline{a_3}$ , and in this case  $X(\mathbb{R})$  is connected in the Euclidean topology; or
- (ii) both real (and distinct, if  $X$  is smooth), and in this case  $X(\mathbb{R})$  has two connected components. In such a case, we will always assume without loss of generality that  $a_1 < a_2 < a_3$ .

These are the two possible topologies for a genus one totally real curve. In the first case, there are only two real 2-torsion points,  $O$  and  $T_1$ , while in the second case there are four. This is consistent with Lemma 3.4.2.

We focus on quadratic forms on  $X$ , and we start by analyzing the extreme rays of  $P_{X,2}$ , which are given by forms with the maximal possible number of real zeroes. Given a quadric  $q \in \mathbb{R}[x_0, x_1, x_2]_2 \cong \mathbb{R}[x_0, x_1, x_2]_2 / (h)_2 = \mathbb{R}[X]_2$ , we denote  $\text{div} q = h \cdot q$  the intersection divisor of  $q$  and  $h$  (or of  $q$  and  $X = \mathcal{V}(h)$ ), see e.g. [Ful89]. If  $q$  is nonnegative on  $X(\mathbb{R})$ , then  $q$  has zeroes on  $X(\mathbb{R})$  with even multiplicity only. By Bézout's theorem, there are at most three such zeroes. We then deduce from [Ful89, Problem 5.41] that, given  $A_1, A_2, A_3 \in X(\mathbb{R})$ , there exists  $q \in \mathbb{R}[x]_2$  such that  $\text{div} q = 2(A_1 + A_2 + A_3)$  if and only if  $A_1 \oplus A_2 \oplus A_3$  is a 2-torsion point for  $(X, \oplus)$ .

If  $X(\mathbb{R})$  is connected, then  $\text{div} q = 2(A_1 + A_2 + A_3)$  implies that  $q$  is nonnegative. On the other hand, if  $X(\mathbb{R})$  is disconnected then  $\text{div} q = 2(A_1 + A_2 + A_3)$  does not imply that  $q$  is nonnegative. We now show that in the disconnected case, whether  $q$  is nonnegative or not depends on the 2-torsion point  $A_1 \oplus A_2 \oplus A_3$ .

If  $A_1 \oplus A_2 \oplus A_3 = O$ , then there exists a linear form  $\ell \in \mathbb{R}[x]_1$  such that  $\text{div} \ell = A_1 + A_2 + A_3$ , and thus  $q = a\ell^2$  for some  $a \in \mathbb{R}_{>0}$ . This shows that  $q$  is a sum of squares and thus nonnegative.

We now deal with the other cases. In the following, we denote  $\ell_{A_1, A_2}$  the line passing through  $A_1, A_2 \in X(\mathbb{R})$ , and  $\ell_A$  the line tangent to  $A \in X(\mathbb{R})$ . Recall from the definition of the group law that the third intersection point of  $\ell_{A_1, A_2}$  with  $X(\mathbb{R})$  is equal to  $\ominus A_1 \ominus A_2$ , i.e.  $\text{div} \ell_{A_1, A_2} = A_1 + A_2 + (\ominus A_1 \ominus A_2)$ . Assume that  $A_1 \oplus A_2 \oplus A_3 = T_i$  for some  $i \in \{1, 2, 3\}$  (recall that we have chosen indices in such a way  $a_1 < a_2 < a_3$ ). A direct computation shows that:

$$\text{div} q = \text{div} \frac{\ell_{A_1, A_2}^2 \ell_{A_1 \oplus A_2, A_3}^2}{\ell_{\ominus A_1 \oplus A_2, A_1 \oplus A_2}^2 \ell_{T_i}^2} \ell_O \ell_{T_i} \tag{1}$$

and since  $X$  is irreducible, the two expressions agree (up to a positive constant) on  $X(\mathbb{R})$ . Therefore,  $q$  is nonnegative if and only if  $\ell_O \ell_{T_i}$  is nonnegative. For  $i = 2, 3$  it is easy to show that  $\ell_O \ell_{T_i}$  changes sign on  $X(\mathbb{R})$ , and thus  $q$  is not nonnegative. On the other hand, for  $i = 1$  we have:

$$(\ell_{T_1}^2 + (a_2 - a_1)(a_3 - a_1)\ell_O^2)\ell_O \ell_{T_1} = x_2^2 \ell_O^2 + (a_2 + a_3 - 2a_1)\ell_O^2 \ell_{T_1}^2 \quad (2)$$

which shows that  $\ell_O \ell_{T_1}$ , and thus  $q$ , is nonnegative on  $X(\mathbb{R})$ .

In conclusion,  $A_1, A_2, A_3$  define a quadric  $q$  such that  $\text{div} q = 2(A_1 + A_2 + A_3)$  which is nonnegative if and only if:

- (i) either  $A_1 \oplus A_2 \oplus A_3 = O$ , and in this case  $q = \ell^2$  is a square;
- (ii) or  $A_1 \oplus A_2 \oplus A_3 = T$ , where  $T$  is the unique nontrivial real 2-torsion point such that  $\ell_O \ell_T$  is nonnegative on  $X(\mathbb{R})$ .

These quadrics span extreme rays of  $P = P_{X,2}$  since they have the maximal number of real zeroes: there are therefore two families of extreme rays defined by the two positive 2-torsion points  $O$  and  $T$ . This is consistent with Lemma 3.4.5.

We now study other faces. In particular, recall from Theorem 3.2.5 that all the faces  $F \subset P$  are of the form  $F = F_{A_1 + \dots + A_k}$ , with  $A_1, \dots, A_k \in X(\mathbb{R})$ .

If  $k \geq 4$ , then  $F_{A_1 + \dots + A_k} = \{0\}$  by Bézout's theorem. For  $k = 3$ , we deduce from the above discussion that either  $\dim F_{A_1 + A_2 + A_3} = 1$ , when the  $A_i$ 's are in special position, otherwise  $F_{A_1 + A_2 + A_3} = \{0\}$ .

For  $k = 2$ , we can consider the two additional points  $B_1 = O \ominus A_1 \ominus A_2$  and  $B_2 = T \ominus A_1 \ominus A_2$ . Then  $F_{A_1 + A_2 + B_1}$  and  $F_{A_1 + A_2 + B_2}$  are extreme rays, and  $F_{A_1 + A_2} = \text{cone}(F_{A_1 + A_2 + B_1}, F_{A_1 + A_2 + B_2})$  has dimension 2. Similarly, for  $k = 1$  we can show that  $\dim F_{A_1} = 4$ . In conclusion, we proved the following.

**Proposition 4.1.1.** *Let  $X \subset \mathbb{P}^2$  be a smooth totally real plane cubic, and denote  $P = P_{X,2}$  the convex cone of nonnegative quadratic forms. Let  $T$  be the unique nontrivial 2-torsion point of  $(X, \oplus)$  such that  $\ell_O \ell_T$  is nonnegative. Then all the proper faces of  $P$  are the following:*

- (i)  $F_A$  for  $A \in X(\mathbb{R})$ , and we have  $\dim F_A = 4$ ;
- (ii)  $F_{A_1 + A_2}$  for  $A_1, A_2 \in X(\mathbb{R})$ , and we have  $\dim F_{A_1 + A_2} = 2$ .
- (iii)  $F_{A_1 + A_2 + B_1}$  with  $A_1, A_2 \in X(\mathbb{R})$  and  $B_1 = O \ominus A_1 \ominus A_2$ ; in this case  $F_{A_1 + A_2 + B_1} = \mathbb{R}_{\geq 0} \cdot \ell^2$  is an extreme ray of the cone of sums of squares and nonnegative quadrics.
- (iv)  $F_{A_1 + A_2 + B_2}$  with  $A_1, A_2 \in X(\mathbb{R})$  and  $B_2 = T \ominus A_1 \ominus A_2$ ; in this case  $F_{A_1 + A_2 + B_2} = \mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of the nonnegative quadrics and  $q$  is not a sum of squares.

In particular, notice that all the extreme rays of  $P$  arise from quadrics with the maximal possible number of real zeroes, and all higher dimensional faces have the expected dimension, equal to  $6 = \dim \mathbb{R}[\mathbf{x}]_2$  minus two times the number of common zeroes of the quadrics in the face. Similar elementary considerations can be applied for higher degree nonnegative forms.

Let us also remark that Equations (1) and (2) can be combined to give explicit certificates of nonnegativity for all the extreme rays of  $P$ . Indeed, if  $\mathbb{R}_{\geq 0} \cdot q$  is an extreme ray of  $P$ , then Equations (1) and (2) imply that there exists  $\alpha \in \mathbb{R}_{>0}$  such that:

$$q = \alpha \frac{\ell_{A_1, A_2}^2 \ell_{A_1 \oplus A_2, A_3}^2 (x_2^2 \ell_O^2 + (a_2 + a_3 - 2a_1)\ell_O^2 \ell_{T_1}^2)}{\ell_{\ominus A_1 \ominus A_2, A_1 \oplus A_2}^2 \ell_{T_1}^2 (\ell_{T_1}^2 + (a_2 - a_1)(a_3 - a_1)\ell_O^2)} \text{ on } X(\mathbb{R})$$

where  $X$  is a totally real plane cubic given by the equation in Weirstrass form

$$x_2^2 x_0 - (x_1 - a_1 x_0)(x_1 - a_2 x_0)(x_1 - a_3 x_0) = 0$$

The above Artin-type certificates for extreme rays, can be combined in convex combinations to provide explicit certificates of nonnegativity for all nonnegative quadrics  $q \in P$ . We also remark that such certificates do not have the smallest possible degree: indeed, in [BSV19, Th. 1.1] is shown the existence of certificates with a denominator of degree two and numerator of degree four. On the other hand, the certificate given in Equation (2) is of minimal possible degree.



**4.2. Elliptic normal curves.** We now show how Proposition 4.1.1 can be generalized from plane cubics to elliptic normal curves.

**Definition 4.2.1.** An *elliptic normal curve* is a non-degenerate, smooth projective curve  $X \subset \mathbb{P}^n$  of genus one such that  $\deg X = n + 1$ .

Recall that every elliptic normal curve is automatically projectively normal, see e.g. [Har77, Ex. IV.4.2]. Using this fact it is possible to show that the Veronese embedding of an elliptic normal curve is again an elliptic normal curve: therefore we can restrict to the case of quadratic forms, as we did in Section 3.

As in Section 4.1, if the genus one curve  $X$  is totally real then  $X(\mathbb{R})$  has either one or two connected components. It follows from this that the invariant  $N(X)$ , introduced in Definition 3.4.8, is always equal to one.

**Lemma 4.2.2.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. Then  $N(X) = 1$ .*

*Proof.* If  $X(\mathbb{R})$  has one connected component, then  $X$  is an  $M - 1$  curve and  $N(X) = 1$  follows from [Mon03]. If  $X(\mathbb{R})$  has two connected components, then  $X$  is an  $M$ -curve and  $N(X) = 1$  follows from [Hui01].  $\square$

Lemma 4.2.2 implies, together with the fact that elliptic normal curves are projectively normal, that the divisor classes  $\alpha + [E]$  as in Lemma 3.4.9 are always totally real. This implies that every positive 2-torsion points (see Definition 3.4.4) determines a family of extreme rays, which are defined by quadrics with the maximal number of real zeroes. As we show in the following theorem, these are the only extreme rays of  $P_{X,2} = P$ , and we can furthermore describe all the faces of  $P_{X,2}$ .

**Theorem 4.2.3.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve, with embedding given by the divisor  $E$ . If we denote  $\{O, T\} = J(\mathbb{R})_2^+$  the positive 2-torsion points, then all the proper faces of  $P = P_{X,2}$  can be described as follows:*

- (i)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $1 \leq \deg D \leq n$ . In this case, we have  $\dim F_D = 2(n + 1 - \deg D)$ .
- (ii)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $n + 1$  such that  $[D - E] = O \in J(\mathbb{R})_2^+$ . In this case,  $F_D$  is an extreme ray of both  $\Sigma = \Sigma_{X,2}$  and  $P$ .
- (iii)  $F_D$ , where  $D$  is a totally real effective divisor of degree  $n + 1$  such that  $[D - E] = T \in J(\mathbb{R})_2^+$ . In this case,  $F_D$  is an extreme ray of  $P$ , but it is not in  $\Sigma = \Sigma_{X,2}$ .

*Proof.* From Theorem 3.2.5, all the faces of  $P$  are of the form  $F_D$  for some totally real effective divisor  $D$ . If  $\deg D > \deg X = n + 1$ , then  $F_D = \{0\}$ .

From Theorem 3.4.7, if  $\deg D = n + 1$  then  $F_D$  is an extreme ray if and only if  $[D - E]$  is a positive 2-torsion point of the Jacobian. Moreover, since  $X$  is projectively normal and  $N(X) = 1$  from Lemma 4.2.2, every positive 2-torsion point determines a family of extreme rays of  $P$  (see also Corollary 3.4.10).

If  $[D - E] = O \in J(\mathbb{R})_2^+$ , then there exist  $f \in \mathbb{R}(X)$  such that  $D = E + \text{div} f$ , and thus by projective normality there exists  $\ell \in R_1 = \mathbb{R}[X]_1$  such that  $\text{div} \ell = D$ . Therefore  $\text{div} \ell^2 = 2D$  and  $F_D = \mathbb{R}_{\geq 0} \cdot \ell^2$  is an extreme ray of both  $\Sigma$  and  $P$ . If  $[D - E] = T \in J(\mathbb{R})_2^+$ , then  $D$  is not the divisor of a linear form, and thus the extreme ray  $F_D$  is not generated by a square. This implies that  $F_D$  is not contained in  $\Sigma$ .

We now study higher-dimensional faces. We first show that, if  $D$  is a totally real effective divisor of degree  $1 \leq \deg D \leq n$ , then  $D \in \text{Im}(\Phi)$ , i.e. there exists  $q \in P$  such that  $(\text{div} q)_{\mathbb{R}} = D$  (see Lemma 3.2.7). Assume that  $\deg D = n = \deg X - 1$ . Using the isomorphism between  $X$  and its Jacobian, we can always find  $A_1 \neq A_2 \in X(\mathbb{R})$  such that  $D + A_1, D + A_2 \in J(\mathbb{R})_2^+$ . From Theorem 3.4.7, there exists  $q_1, q_2 \in P$  such that  $\text{div} q_i = (\text{div} q_i)_{\mathbb{R}} = D + A_i$ . Therefore  $q_1 + q_2$  is nonnegative and vanishes exactly on  $D$ , i.e.  $(\text{div}(q_1 + q_2))_{\mathbb{R}} = 2D$ , showing that  $D \in \text{Im} \Phi$ . A similar argument works for all totally real divisors of degree smaller than  $n$ .

The dimension of  $F_D$  can then be deduced using Lemma 3.3.1 and the Riemann-Roch theorem, as follows. Since  $D \in \text{Im} \Phi$ , then Lemmas 3.2.1 and 3.2.7 imply that for all  $q \in \text{ri} F_D$  we have  $(\text{div} q)_{\mathbb{R}} = 2D$ . It follows from Lemma 3.3.1 that  $\dim F_D = \ell((\text{div} q)_{\mathbb{C}})$  for all  $q \in \text{ri} F_D$ . Using the Riemann-Roch theorem and the fact that  $1 \leq \deg(\text{div} q)_{\mathbb{C}} = 2(n + 1 - \deg D)$ , we have:

$$\ell((\text{div} q)_{\mathbb{C}}) = \deg(\text{div} q)_{\mathbb{C}} + \ell(K - (\text{div} q)_{\mathbb{C}}) = 2(n + 1 - \deg D)$$

concluding the proof.  $\square$

Theorem 4.2.3 completely describes the facial structure of the cone of nonnegative quadrics on an elliptic normal curve. Using the fact that the Veronese embedding of an elliptic normal curve is an elliptic normal curve, Theorem 4.2.3 also describes all faces of  $P_{X,2d} \cong P_{v_n,d(X),2}$ . This is the first complete description of the facial structure of the nonnegative cone beyond Hilbert's cases [Sch21]. It is therefore the first non-rational example where such a description is shown.

We also remark that we do not explicitly see the distinction between the two possible topologies of  $X(\mathbb{R})$  in Theorem 4.2.3, as the theorem applies to both the connected and disconnected case. In particular, we are not using the real 2-torsion points of the Jacobian which are not positive. Such points exist when  $X(\mathbb{R})$  is disconnected, see Lemmas 3.4.2 and 3.4.5, and they will play a key role in Section 5 for the study of Carathéodory number of the of the dual cone  $P^\vee$ .

We conclude the section presenting two useful lemmas about the signature signs of quadrics on connected components of elliptic normal curves, that will be needed in Section 5.

**Lemma 4.2.4.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve whose real locus has two connected components. If  $Y \subset X(\mathbb{R})$  is a connected component, then there exists  $q \in P_{Y,2} \setminus P_{X,2}$  (i.e.  $q$  is nonnegative on  $Y$  but changes sign on  $X(\mathbb{R})$ ) such that  $\text{div} q = (\text{div} q)_{\mathbb{R}} = 2D$ , with  $\text{supp}(D)$  consist of  $\deg X = n + 1$  distinct points of  $Y$ .*

*Proof.* Fix  $A_0 \in X(\mathbb{R})$ . Let  $Y \subset X(\mathbb{R})$  be a connected component, and let  $A \in Y$  be the (unique) point such that  $[A - A_0] = \alpha \in J(\mathbb{R})_2 \setminus J(\mathbb{R})_2^+$  (such a point exists since genus one curves are isomorphic to their Jacobian). Therefore, it follows from [Ful89, Problem 5.41] that there exists  $q_A \in R_2$  such that  $\text{div} q_A = 2(n + 1)A$ . As  $[A - A_0]$  is not a positive 2-torsion point,  $q_A$  changes sign on  $X(\mathbb{R})$ , and we may assume that  $q_A \in P_{Y,2} \setminus P_{X,2}$ . Moving one pair of points at a time along  $Y$ , it is possible to continuously perturb the  $(n + 1)$ -tuple  $(A, \dots, A)$  to  $(A_1, \dots, A_{n+1}) \in Y^{n+1}$ , in such a way the  $A_i$ 's are pairwise distinct and  $A_1 \oplus \dots \oplus A_{n+1} = A$ . Therefore there exists  $q \in P_{Y,2} \setminus P_{X,2}$ , perturbation of  $q_A$ , such that  $\text{div} q = 2(A_1 + \dots + A_{n+1})$ , concluding the proof.  $\square$

**Lemma 4.2.5.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve whose real locus has two connected components. If  $X(\mathbb{R}) = Y_1 \sqcup Y_2$  and  $B_1, \dots, B_n \in Y_1$ , then there exists  $q \in R_2$  such that:*

- (i)  $(\text{div} q)_{\mathbb{R}} = 2(B_1 + \dots + B_n)$ ;
- (ii)  $q$  is nonnegative on  $Y_1$ ;
- (iii)  $q$  is strictly negative on  $Y_2$ .

*Proof.* Consider the non-positive 2-torsion points  $\{T_2, T_3\} = J(\mathbb{R})_2 \setminus J(\mathbb{R})_2^+$ , and define  $A_i = T_i \ominus B_1 \ominus \dots \ominus B_n$  for  $i = 2, 3$ . Therefore there exist  $q_2, q_3 \in R_2$  such that  $\text{div} q_i = (\text{div} q_i)_{\mathbb{R}} = 2(B_1 + \dots + B_n + A_i)$  for  $i = 2, 3$ . As  $T_2, T_3$  are non-positive 2-torsion points, then  $q_2, q_3$  change sign on  $X(\mathbb{R})$ . In particular, we can assume that  $q_2, q_3$  are nonnegative on  $Y_1$  and nonpositive on  $Y_2$ . We then define  $q = q_1 + q_2$ . Notice that:

- (i)  $q$  vanishes with multiplicity two at  $B_1, \dots, B_n$ , since both  $q_2$  and  $q_3$  vanish there with multiplicity two;
- (ii)  $q$  is nonnegative on  $Y_1$ , since both  $q_2$  and  $q_3$  are nonnegative on  $Y_1$ ;
- (iii) As  $A_2 \neq A_3$ ,  $q$  does not vanish at any point of  $X(\mathbb{R})$  except  $B_1, \dots, B_n$ , and thus  $q$  is strictly negative on  $Y_2$ .

This concludes the proof.  $\square$

## 5. CARATHÉODORY NUMBERS FOR GENUS ONE CURVES

We now investigate the Carathéodory number  $\mathcal{C} = \mathcal{C}_{X,2}$  of  $P^\vee = P_{X,2}^\vee$ , where  $X \subset \mathbb{P}^n$  is a totally real elliptic normal curve. We show that the Carathéodory number depends on the topology of the real locus  $X(\mathbb{R})$ , using a technique inspired by Hilbert's proof on ternary quartics (see [BPSV19] for a modern exposition).

Recall from Section 2.3 that  $P^\vee$  is the moment cone, i.e. the convex cone of linear functionals on  $R_2$  that can be expressed as integration with respect to a measure supported on  $X(\mathbb{R})$ , or equivalently the convex hull of point evaluations on  $X(\mathbb{R})$ . The Carathéodory number of  $P^\vee$  is the minimal natural number  $\mathcal{C}$  of point evaluations needed to represent all the linear functionals  $L \in P^\vee$ .

Using standard conic duality, we can use Theorem 4.2.3 to describe linear functionals  $L \in \partial P^\vee$  on the boundary of the moment cone. Indeed, if  $L \in \partial P^\vee$  then by conic duality there exists  $q \in P$  such that  $L(q) = 0$ . More precisely, Theorem 4.2.3 implies the following corollary.

**Corollary 5.0.1.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. Let  $\{O, T\} = J(\mathbb{R})_2^+$  be the positive 2-torsion points. We have*

$$\begin{aligned} \partial P^\vee = & \left\{ \sum_{i=1}^k \lambda_i \text{ev}_{A_i} \mid 1 \leq k \leq n, A_1, \dots, A_k \in X(\mathbb{R}), \lambda_1, \dots, \lambda_k \in \mathbb{R}_{>0} \right\} \\ & \sqcup \left\{ \sum_{i=1}^{n+1} \lambda_i \text{ev}_{A_i} \mid A_1, \dots, A_{n+1} \in X(\mathbb{R}) \text{ distinct}, \lambda_1, \dots, \lambda_{n+1} \in \mathbb{R}_{>0}, A_1 \oplus \dots \oplus A_{n+1} = O \right\} \\ & \sqcup \left\{ \sum_{i=1}^{n+1} \lambda_i \text{ev}_{A_i} \mid A_1, \dots, A_{n+1} \in X(\mathbb{R}) \text{ distinct}, \lambda_1, \dots, \lambda_{n+1} \in \mathbb{R}_{>0}, A_1 \oplus \dots \oplus A_{n+1} = T \right\} \end{aligned}$$

**5.1. Critical values and boundary points.** Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. For  $Y \subset X(\mathbb{R})$  a connected component of  $X(\mathbb{R})$ , let  $P_{Y,2d} \subset R_{2d}$  be the cone of nonnegative forms on  $Y$ . As  $Y$  is not (Euclidean) dense in  $X(\mathbb{R})$ , we have  $P_{Y,2d} \supsetneq P_{X,2d}$  and  $P_{Y,2d}^\vee \subsetneq P_{X,2d}^\vee$ .

We will use the following notation:

- $Z := v_{n,2}(X(\mathbb{R}))$ ;
- If  $S$  is any set,

$$\text{cone}_k(S) := \left\{ \sum_{i=1}^k \lambda_i s_i : s_i \in S, \lambda_i \in \mathbb{R}_{\geq 0} \right\}$$

denotes the set of conic combinations of points of  $S$  which use at most  $k$  points from  $S$ .

As we have  $v_{n,2}(A) \cong \text{ev}_A \in R_2^*$ , then we can characterize the Carathéodory number as:

$$\mathcal{C} = \mathcal{C}_{X,2} = \min \left\{ k \in \mathbb{N} \mid \text{cone}_k(Z) = P^\vee \right\}$$

**Lemma 5.1.1.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. Then  $n+1 = \deg X \leq \mathcal{C}_{X,2} \leq \deg X + 1 = n+2$ .*

*Proof.* The proof is a consequence of the following standard argument, see e.g. [dK21, Th. 4.8].

Let  $L = \sum_{i=1}^{n+1} \text{ev}_{A_i}$  be such that the  $A_i$  are distinct and  $A_1 \oplus \dots \oplus A_{n+1} \in J(\mathbb{R})_2^+$ . Then  $L \in \partial(P^\vee)$  from Corollary 5.0.1. We deduce e.g. from [Sch17, Prop. 18.12] that  $\mathcal{C}(L) = n+1$ , and thus  $\mathcal{C} \geq n+1$ . Corollary 5.0.1 shows also that  $\mathcal{C}(L) \leq n+1$  for all  $L \in \partial P^\vee$ . Now, let  $L \in \text{int}(P^\vee)$ . Since  $P^\vee$  is pointed and closed, for all  $A \in X(\mathbb{R})$  there exists  $\lambda_A \in \mathbb{R}_{\geq 0}$  such that  $L - \lambda_A \text{ev}_A \in \partial P^\vee$ . By the above,  $\mathcal{C}(L - \lambda \text{ev}_A) \leq n+1$  holds for all  $A \in X(\mathbb{R})$ , and thus  $\mathcal{C}(L) \leq n+2$ .  $\square$

An equivalent result to Lemma 5.1.1 was proven in [dK21, Th. 4.8] for general affine curves with compact real locus, and considering forms of sufficiently high degree  $2d$ . Lemma 5.1.1 shows that there are only two possibilities for the Carathéodory number: In the following, we show that the exact value depends on the topology of  $X(\mathbb{R})$ . For the proof, we will use the following simple observation.

**Lemma 5.1.2.** *Let  $A \subset B \subset \mathbb{R}^N$  be such that  $A$  is closed,  $\text{int} A \neq \emptyset$  and  $B$  is convex. Then  $A = B$  if and only if  $\partial A \subset \partial B$ .*

*Proof.* Clearly, if  $A = B$  then  $\partial A = \partial B$ .

For the converse implication, we show that  $A \neq B$  implies  $\partial A \subsetneq \partial B$ . Let  $x \in \text{int} A \subset \text{int} B$  and  $y \in B \setminus A$ . Consider now the segment between  $x$  and  $y$ . Since  $B$  is convex, this segment is contained in  $B$ , and all the points in the relative interior of the segment belong to the interior of  $B$ , see e.g. [Roc70, Th. 6.1]. As  $A$  is closed,  $x \in \text{int} A$  and  $y \in B \setminus A$ , we can find  $z$  in the relative interior of the segment such that  $z \in \partial A$ . Therefore,  $z \in \partial A \cap \text{int} B$ , proving that  $\partial A \subsetneq \partial B$ .  $\square$

We are now ready to determine the Carathéodory number in the case of connected curves, or more generally for the moment cone on connected components of the real locus.

**Proposition 5.1.3.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve, and let  $Y$  be a connected component of  $X(\mathbb{R})$ . Then  $P_{Y,2}^\vee = \text{cone}_{n+1}(v_{n,2}(Y))$ , or, in other words,  $\mathcal{C}_{Y,2} = \deg X = n+1$ .*

*Proof.* Let  $Z = v_{n,2}(Y)$ . Our goal is to apply Lemma 5.1.2 to  $\text{cone}_{n+1}(Z) \subset \mathbb{P}_{Y,2}^\vee$ , to show that they are equal. First, notice that  $\mathbb{P}_{Y,2}^\vee$  is convex and that  $\text{cone}_{n+1}(Z)$  is closed. Second,  $\text{cone}_{n+1}(Z)$  has nonempty interior. Indeed, the proof of [BT15, Th. 2], which shows that the smallest typical rank (with arbitrary real coefficients) is equal to the complex generic rank, can be generalized to the case of nonnegative coefficients (see also [QCL16, Lem. 37]). As  $n+1$  is the complex generic rank (see [Lan84] and also Section 5.2), the above implies that  $\text{cone}_{n+1}(Z)$  has nonempty interior.

Therefore, in order to apply Lemma 5.1.2 we only need to show that  $\partial \text{cone}_{n+1}(Z) \subset \partial(\mathbb{P}_{Y,2}^\vee)$ . For this, we proceed similarly to Hilbert's proof that every ternary quartic is a sum of at most three squares, see e.g. [BPSV19]. To make the analogy clearer, we identify the space  $R_2^*$  with  $\mathcal{I}(X)^\perp \subset \mathbb{R}[\mathbf{x}]_2$  using the apolar or Bombieri-Weil inner product  $\langle \cdot, \cdot \rangle_{BW}$ , see Section 5.2 for the definition and more details. If  $B = (B_0, \dots, B_n), v = (v_0, \dots, v_n) \in \mathbb{R}^{n+1}$  and  $\ell_B = B_0x_0 + \dots + B_nx_n$ , then  $\langle \ell_B^2, q \rangle_{BW} = \text{ev}_B(q) = q(B)$  and  $\langle 2\ell_v \ell_B, q \rangle_{BW} = \partial_v q(B)$  for all  $q \in \mathbb{R}[\mathbf{x}]_2$ .

We can therefore consider the maps

$$\begin{aligned} \phi: (\mathbb{R}^{n+1})^{n+1} &\longrightarrow \mathbb{R}[\mathbf{x}]_2 & \psi: \widehat{Y}^{n+1} &\longrightarrow \mathcal{I}(X)_2^\perp \cong R_2^* \\ \mathbf{A} = (A_1, \dots, A_{n+1}) &\longmapsto \sum_{i=1}^{n+1} \ell_{A_i}^2 & \mathbf{A} = (A_1, \dots, A_{n+1}) &\longmapsto \sum_{i=0}^n \ell_{A_i}^2 \end{aligned}$$

By definition,  $\psi$  is the restriction of  $\phi$  to the affine cone over  $Y$ . Our goal is to show that  $\partial \text{cone}_3(Z) \cong \partial \text{Im } \psi$  is included in  $\partial(\mathbb{P}_{Y,2}^\vee)$ . The differential of  $\phi$  is well-known (see e.g. [BPSV19, Lem. 2.2]), and therefore, by restriction we have:

$$\begin{aligned} d\psi_{\mathbf{A}}: T_{\mathbf{A}}(\widehat{Y}^{n+1}) &\cong T_{A_1} \widehat{Y} \times \dots \times T_{A_{n+1}} \widehat{Y} \longrightarrow T_{\psi(\mathbf{A})}(\mathcal{I}(X)_2^\perp) \cong \mathcal{I}(X)_2^\perp \\ \mathbf{v} = (v_1, \dots, v_{n+1}) &\longmapsto 2 \sum_{i=1}^{n+1} \ell_{v_i} \ell_{A_i} \end{aligned}$$

Notice that  $\dim T_{\mathbf{A}}(\widehat{Y}^{n+1}) = \dim \mathcal{I}(X)_2^\perp = 2n+2$ , and therefore  $d\psi_{\mathbf{A}}$  is a square linear map. In particular, if  $\psi(\mathbf{A}) \in \partial \text{Im } \psi$  then  $d\psi_{\mathbf{A}}$  does not have full rank. Using again the Bombieri-Weil inner product, this is equivalent to the fact that there exists  $q \in \mathbb{R}[\mathbf{x}]_2 \setminus \mathcal{I}(X)_2^\perp$  such that  $\langle \ell_{v_i} \ell_{A_i}, q \rangle_{BW} = 0$  for all  $\mathbf{v}$ . Setting  $v_i = A_i$  (this is possible since  $\widehat{Y}$  is a cone), we have  $\langle \ell_{A_i}^2, q \rangle_{BW} = q(A_i) = 0$  for all  $i = 1, \dots, n+1$ . Letting  $v_i$  vary in the tangent space, we see that  $q$  double vanishes at  $A_1, \dots, A_{n+1}$ .

There are then two cases.

- (i) If the  $A_i$ 's are distinct,  $q$  (now seen as a nonzero element of  $R_2 = \mathbb{R}[\mathbf{x}]_2 / \mathcal{I}(X)_2$ ) is uniquely determined by the double vanishing at  $A_1, \dots, A_{n+1}$ , i.e.  $\text{div } q = (\text{div } q)_{\mathbb{R}} = 2(A_1 + \dots + A_{n+1})$ . Since  $Y$  is connected,  $q$  does not change sign on  $Y$ , and therefore we can take  $q$  nonnegative on  $Y$ . By conic duality, this means that  $\sum_{i=1}^{n+1} \text{ev}_{A_i} \cong \sum_{i=1}^{n+1} \ell_{A_i}^2 \in \partial(\mathbb{P}_{Y,2}^\vee)$ .
- (ii) If the  $A_i$ 's are not distinct, we can always find a linear form  $\ell \in R_1$  vanishing on all of them, and we can take  $q = \ell^2$ . As in the previous point, since  $q = \ell^2$  is nonnegative on  $Y$  we have  $\sum_{i=1}^{n+1} \text{ev}_{A_i} \cong \sum_{i=1}^{n+1} \ell_{A_i}^2 \in \partial(\mathbb{P}_{Y,2}^\vee)$ .

Since the above two cases cover all the possible  $\mathbf{A}$  such that  $\psi(\mathbf{A}) \in \partial \text{Im } \psi \cong \partial \text{cone}_{n+1}(Z)$ , we have shown that  $\partial \text{cone}_{n+1}(Z) \subset \partial(\mathbb{P}_{Y,2}^\vee)$ . We can therefore use Lemma 5.1.2 to conclude that  $\text{cone}_{n+1}(Z) = (\mathbb{P}_{Y,2}^\vee)$ . In other words, we have shown that  $C_{Y,2} = n+1$ , finishing the proof.  $\square$

We are now ready to prove the main result of this section.

**Theorem 5.1.4.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. Then:*

- (i) *if  $X(\mathbb{R})$  is connected, then  $C_{X,2} = \deg X = n+1$ ;*
- (ii) *if  $X(\mathbb{R})$  is disconnected, then  $C_{X,2} = \deg X + 1 = n+2$ .*

*Moreover the set of  $L \in \mathbb{P}^\vee$  such that  $\mathcal{C}(L) = C_{X,2}$  has nonempty interior.*

*Proof.* Assume that  $X(\mathbb{R})$  is connected. Then the result follows directly from Proposition 5.1.3.

Now assume that  $X(\mathbb{R})$  is disconnected, and set  $Z = \nu_{n,2}(X(\mathbb{R}))$ . By Lemma 5.1.1 we only need to show that  $\mathcal{C} = \mathcal{C}_{X,2} > n + 1$ , or in other words that

$$\text{cone}_{n+1}(\nu_{n,2}(X(\mathbb{R}))) = \text{cone}_{n+1}(Z) \subsetneq P_{X,2} = P.$$

While in the proof of Proposition 5.1.3 we have shown that  $\partial \text{cone}_{n+1}(Z) \subset \partial(P_{Y_1,2}^\vee)$  if  $X(\mathbb{R})$  is connected, hereafter we want to show that  $\partial \text{cone}_{n+1}(Z) \subsetneq \partial(P_{X,2}^\vee)$ , i.e. that there exists  $L \in \partial \text{cone}_{n+1}(Z) \cap \text{int}(P_{X,2}^\vee)$ . This implies that  $\text{cone}_{n+1}(Z) \subsetneq P_{X,2}$ , which is our claim. In order to prove the existence of  $L \in \partial \text{cone}_{n+1}(Z) \cap \text{int}(P_{X,2}^\vee)$ , we are going to use the fact that if  $X(\mathbb{R})$  is disconnected, then there exists non-positive 2-torsion points, see Lemmas 3.4.2 and 3.4.3, through Lemma 4.2.4.

Let  $Y_1$  be a connected component of  $X(\mathbb{R})$ , and let  $Z_1 = \nu_{n,2}(Y_1)$ . By Lemma 4.2.4, there exists  $q \in P_{Y_1,2} \setminus P_{X,2}$  such that  $\text{div} q = (\text{div} q)_{\mathbb{R}} = 2(A_1 + \dots + A_{n+1})$ , with  $A_i \in Y_1$  distinct. Let  $L = \text{ev}_{A_1} + \dots + \text{ev}_{A_{n+1}} \in \text{cone}_{n+1}(Z_1)$ . We notice that  $L \in \partial(P_{Y_1,2})^\vee$ , since  $L(q) = 0$  and  $q \in P_{Y_1,2}$ . Furthermore,  $L \in \text{int}(P_{X,2}^\vee)$ : indeed, if  $L \in \partial(P_{X,2}^\vee)$ , there would exist  $\tilde{q} \in P_{X,2}$  such that  $L(\tilde{q}) = 0$ . However such a  $\tilde{q}$  does not exist, since  $A_1, \dots, A_{n+1}$  do not add to a positive 2-torsion point. In conclusion, we have shown that

$$L \in \text{cone}_{n+1}(Z_1) \cap \partial(P_{Y_1,2})^\vee \cap \text{int}(P_{X,2}^\vee)$$

We now show that such an  $L$  cannot be represented in  $\text{cone}_{n+1}(Z)$  using points in  $Y_2 = X(\mathbb{R}) \setminus Y_1$ .

Write  $L = \sum_{i=1}^{n+1} \text{ev}_{B_i}$ , and assume that  $B_1, \dots, B_n \in Y_1$ ,  $B_{n+1} \in Y_2 = X(\mathbb{R}) \setminus Y_1$ . By Lemma 4.2.5, there exists  $\tilde{q} \in R_2$  such that:

- $\tilde{q}$  vanishes at  $B_1, \dots, B_n$  (with multiplicity two);
- $\tilde{q} \geq 0$  on  $Y_1$ ;
- $\tilde{q} < 0$  on  $Y_2$ .

Therefore we have:

$$L(\tilde{q}) = \left( \sum_{i=1}^{n+1} \text{ev}_{A_i} \right) (\tilde{q}) = \tilde{q}(A_1) + \dots + \tilde{q}(A_{n+1}) \geq 0$$

$$L(\tilde{q}) = \left( \sum_{i=1}^{n+1} \text{ev}_{B_i} \right) (\tilde{q}) = \tilde{q}(B_1) + \dots + \tilde{q}(B_{n+1}) = \tilde{q}(B_{n+1}) < 0$$

a contradiction. A similar argument works assuming  $B_1, \dots, B_k \in Y_1$  and  $B_{k+1}, \dots, B_{n+1} \in Y_2$  for all  $k = 0, \dots, n$ .

Thus any representation of  $L$  as the sum of  $n + 1$  point evaluations only uses points of  $Y_1$ . This implies that, in a sufficiently small neighborhood of  $L$ ,  $\text{cone}_{n+1}(Z)$  coincides with  $\text{cone}_{n+1}(Z_1)$ . In other words, in a sufficiently small neighborhood of  $L$ , all convex combinations of  $n + 1$  evaluations at points of  $X(\mathbb{R})$  are convex combinations of  $n + 1$  evaluations at points of  $Y_1 \subsetneq X(\mathbb{R})$ .

Furthermore, notice that  $\text{cone}_{n+1}(Z_1) = P_{Y_1,2}^\vee$  by Proposition 5.1.3. As  $L$  defines a supporting hyperplane for  $q \in \partial P_{Y_1,2}$ , every point in a representation of  $L$  belongs to the zero locus of  $q$  on  $X(\mathbb{R})$ . This implies that the representation  $L = \text{ev}_{A_1} + \dots + \text{ev}_{A_{n+1}}$  is unique in  $\text{cone}_{n+1}(Z_1) = P_{Y_1,2}^\vee$ , and by the above it is also unique in  $\text{cone}_{n+1}(Z)$ <sup>1</sup>.

Therefore we see that in a sufficiently small neighborhood  $U$  of  $L$ , we have

$$\text{cone}_{n+1}(Z) \cap U = \text{cone}_{n+1}(Z_1) \cap U = P_{Y_1,2}^\vee \cap U$$

But  $(P_{Y_1,2})^\vee$  is convex, and thus  $\text{cone}_{n+1}(Z) \cap U$  lies completely inside one half-space defined by the tangent space  $T_L \partial \text{cone}_{n+1}(Z) \cap U$  to  $L \in \partial(P_{Y_1,2})^\vee \cap U = \partial \text{cone}_{n+1}(Z) \cap U$ . Recall also that  $L \in \text{int}(P_{X,2}^\vee)$ : therefore, going in the normal direction  $\mathcal{N}_L \partial \text{cone}_{n+1}(Z)$  to  $T_L \partial \text{cone}_{n+1}(Z)$  we can find an open set included in  $P_{X,2}^\vee \setminus \text{cone}_{n+1}(Z)$ , concluding the proof.  $\square$

As  $P_{X,2d} \cong P_{\nu_{n,d}(X),2}$ , we can extend Theorem 4.2.3 to higher degree forms.

**Corollary 5.1.5.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve. Then:*

<sup>1</sup>Another way to show that the representation of  $L$  is unique is to use the fact that the general point of the  $(n + 1)$ -secant variety of an elliptic normal curve is contained in two distinct secant spaces, see Section 5.2 and [CC06], or equivalently that a generic  $L$  has two distinct decompositions. See also Section 5.2 for more details.

- (i) if  $X(\mathbb{R})$  is connected, then  $C_{X,2d} = \deg X = d(n+1)$ ;
- (ii) if  $X(\mathbb{R})$  is disconnected, then  $C_{X,2d} = \deg X + 1 = d(n+1) + 1$ .

*Proof.* Apply Theorem 5.1.4 to the elliptic normal curve  $v_{n,d}(X)$ . □

**5.2. Discussion: Carathéodory numbers and Waring decompositions.** We hereafter summarize the equivalence between the computation of the Carathéodory numbers and the minimal rank of a certain Waring-type decomposition.

We briefly introduce the *apolar* or *Bombieri-Weil* inner product [IK99]. Let  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ . The inner product  $\langle \cdot, \cdot \rangle_{BW}$  on  $\mathbb{K}[\mathbf{x}]_k = \mathbb{K}[x_0, \dots, x_n]_k$  is defined on monomials  $\mathbf{x}^\alpha, \mathbf{x}^\beta$  (and extended by linearity) as:

$$\langle \mathbf{x}^\alpha, \mathbf{x}^\beta \rangle_{BW} := \frac{1}{k!} \partial_\alpha \mathbf{x}^\beta(0)$$

One can show that, if  $A = (A_0, \dots, A_k) \in \mathbb{K}^{n+1}$  and  $f \in \mathbb{K}[\mathbf{x}]_k$ , then  $\langle \ell_A^k, f \rangle_{BW} = f(A) = \text{ev}_A(f)$ , where  $\text{ev}_A: \mathbb{K}[\mathbf{x}]_k \rightarrow \mathbb{R}$  denotes the usual point evaluation and  $\ell_A = A_0 x_0 + \dots + A_n x_n$ . Now let  $Z \subset \mathbb{P}^n$  be a smooth, non-degenerate algebraic variety. If  $L \in (\mathbb{K}[\mathbf{x}]/I(Z))_k^* \cong I(Z)_k^\perp \subset \mathbb{K}[\mathbf{x}]_k$ ,  $A_i \in Z(\mathbb{K})$  and  $a_i \in \mathbb{K}$ , then:

$$L = \sum_{i=1}^r a_i \text{ev}_{A_i} \iff f_L = \sum_{i=1}^r a_i \ell_{A_i}^k$$

where  $f_L \in I(Z)^\perp \subset \mathbb{K}[Z]$  is the unique polynomial representing  $L = \langle f_L, \cdot \rangle_{BW}$  using the Riesz representation theorem. If  $Z = \mathbb{P}^n$ , such a decomposition of a form as a sum of powers of linear forms is called a *Waring decomposition* of rank  $r$  over  $\mathbb{K}$  (assuming that the  $A_i$ 's are pairwise distinct), and the minimal such  $r$  is called the *Waring rank* over  $\mathbb{K}$  of  $f$ . More generally, when  $Z \subset \mathbb{P}^n$  is a non-degenerate algebraic variety, such a minimal  $r$  is called the *Z-rank* of  $f$  over  $\mathbb{K}$ . If we write  $Z = v_{n,2d}(X)$  for the second Veronese embedding, the Carathéodory number  $C_{X,2d}(L)$  correspond the *nonnegative Z-rank* of  $f_L$ , i.e. to the *Z-rank* over  $\mathbb{R}_{\geq 0}$ , see e.g. [QCL16, BS16, Ang17, ABC18]. We say that a nonnegative *Z-rank*  $r$  is *typical* if the set of forms of nonnegative *Z-rank* equal to  $r$  has nonempty interior. Using this terminology, Corollary 5.1.5 can be rephrased as follows.

**Corollary 5.2.1.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve and denote  $Z = v_{n,2d}(X)$ . Then, for linear functionals in the moment cone  $P_{X,2d}^\vee$ :*

- (i) if  $X(\mathbb{R})$  is connected, then the maximal nonnegative *Z-rank* is  $d(n+1)$ ;
- (ii) if  $X(\mathbb{R})$  is disconnected, then the maximal nonnegative *Z-rank* is  $d(n+1) + 1$ .

Furthermore, the maximal nonnegative *Z-rank* is always typical.

Let us also remark on another connection between our study of Carathéodory numbers and secant varieties. The proof of Theorem 5.1.4 relies on the study of the boundary points of the set of linear functionals obtained using at most  $n+1$  point evaluations. In particular, we study the tangent space at the boundary, which is of a smaller dimension than the expected one. In the language of tensor decomposition, those points define what is called *Terraccini locus*. For more information and precise definitions on the Terraccini locus, we refer the reader to the recent work [ACO23] and references therein.

We finally state some lemmas that will be useful in Section 6, about the geometry of the representations of linear functional as sum of point evaluations. The first one is a particular case of [CC06, Prop. 5.2].

**Lemma 5.2.2.** *Let  $X \subset \mathbb{P}^n$  be an elliptic normal curve,  $Z = v_{n,2d}(X)$ , and*

$$L = \text{ev}_{A_1} + \dots + \text{ev}_{A_{d(n+1)}} \in (\mathbb{C}[\mathbf{x}]_{2d}/I(X)_{2d})^*$$

*If  $A_1, \dots, A_{d(n+1)}$  are distinct and  $A_1 \oplus \dots \oplus A_{d(n+1)}$  is not a 2-torsion point of  $X$ , then  $L$  has two distinct representations of *Z-rank* (over  $\mathbb{C}$ ) equal to  $d(n+1)$ .*

*Proof.* The statement is equivalent to the fact that the general point of the  $d(n+1)$ -secant variety of  $Z$  is contained in two distinct  $d(n+1)$ -secant spaces. We can then conclude as in the proof of [CC06, Prop. 5.2], applied to the elliptic normal curve  $Z$ . □

We now adapt Lemma 5.2.2 to the *nonnegative Z-rank*: in Lemma 5.2.3 we treat the generic case, while in Lemma 5.2.4 we consider special cases.

**Lemma 5.2.3.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve and  $Z = v_{n,2d}(X(\mathbb{R}))$ . All linear functions  $L \in \text{cone}_{d(n+1)}(Z) \setminus \partial \text{cone}_{d(n+1)}(Z)$  (which is a Euclidean dense subset of  $\text{cone}_{d(n+1)}(Z)$ ) admit exactly two different representations:*

$$L = \sum_{i=1}^{d(n+1)} a_i \text{ev}_{A_i} = \sum_{i=1}^{d(n+1)} b_i \text{ev}_{B_i}$$

with  $a_i, b_i \in \mathbb{R}_{\geq 0}$  and  $A_i, B_i \in X(\mathbb{R})$ .

*Proof.* Let  $L = \sum_{i=1}^{d(n+1)} a_i \text{ev}_{A_i} \in \text{cone}_{d(n+1)}(Z) \setminus \partial \text{cone}_{d(n+1)}(Z)$ . Then by Lemma 5.2.2  $L$  admits another, unique, distinct representation  $L = \sum_{i=1}^{d(n+1)} b_i \text{ev}_{B_i}$ , with  $b_i \in \mathbb{C}$  and  $B_i \in X$ . Our goal is to show that  $b_i \in \mathbb{R}_{\geq 0}$  and  $B_i \in X(\mathbb{R})$ .

In the following, we denote  $\sigma(\cdot)$  the action of complex conjugation. Assume that  $\{B_1, \dots, B_{d(n+1)}\} \neq \{\sigma(B_1), \dots, \sigma(B_{d(n+1)})\}$ . Then:

$$L = \sum_{i=1}^{d(n+1)} a_i \text{ev}_{A_i} = \sum_{i=1}^{d(n+1)} b_i \text{ev}_{B_i} = \sum_{i=1}^{d(n+1)} \sigma(b_i) \text{ev}_{\sigma(B_i)}$$

are three distinct decompositions, contradicting Lemma 5.2.2.

Therefore we have two cases: either  $B_i \in X(\mathbb{R})$ , or  $\sigma(B_i) = B_j$  for some  $j = j(i)$ , and in such a case we also have  $b_i = \sigma(b_j)$ . We now show that the second possibility leads to a contradiction. Pick a real linear form  $\ell$  vanishing at all the points  $\{B_1, \dots, B_{d(n+1)}\}$  except  $B_i$  and  $\sigma(B_i)$ . This imposes  $d(n+1) - 2$  conditions in the  $d(n+1)$ -dimensional space of linear forms on  $v_{n,d}(X(\mathbb{R}))$ , and thus we can furthermore assume that  $\ell(B_i)^2 = -(b_i + \sigma(b_i)) = \sigma(\ell(\sigma(B_i)))^2$ . Therefore:

$$\begin{aligned} L(\ell^2) &= \sum_j a_j \ell(A_j)^2 \geq 0 \\ L(\ell^2) &= b_i \ell(B_i)^2 + \sigma(b_i) \ell(\sigma(B_i))^2 = -(b_i + \sigma(b_i))^2 < 0 \end{aligned}$$

which is a contradiction.

Therefore  $B_i \in X(\mathbb{R})$  for all  $i$ . We now want to show that  $b_i \in \mathbb{R}_{\geq 0}$ . For this, pick a real linear form  $\ell$  vanishing at all the points  $\{B_1, \dots, B_{d(n+1)}\}$  except  $B_i$ . Therefore:

$$0 \leq \sum_{j=1}^{d(n+1)} a_j \ell(A_j)^2 = L(\ell^2) = \sum_{j=1}^{d(n+1)} b_j \ell(B_j)^2 = b_i \ell(B_i)^2$$

showing that  $b_i \in \mathbb{R}_{\geq 0}$  and concluding the proof.  $\square$

**Lemma 5.2.4.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve and  $Z = v_{n,2d}(X(\mathbb{R}))$ . All linear functionals  $L \in \partial \text{cone}_{d(n+1)}(Z) \setminus \text{cone}_{d(n+1)-1}(Z)$  admit at most two representations as in Lemma 5.2.3.*

*Proof.* Notice that if  $L \in \partial \text{cone}_{d(n+1)}(Z)$  then  $L$  is a critical value of the map  $\psi$  (extended to the full real locus  $X(\mathbb{R})$ ) in the proof of Proposition 5.1.3. More precisely, since  $L$  is in the boundary, for every  $\mathbf{A} \in \widehat{X}(\mathbb{R})^{d(n+1)}$  in the preimage of  $L$ , the differential  $d\psi_{\mathbf{A}}$  is not surjective. Furthermore, since  $L \notin \text{cone}_{d(n+1)-1}(Z)$ , then every representation of  $L$  uses  $d(n+1)$  atoms which add to a 2-torsion point (see again the proof of Proposition 5.1.3).

Consider then, for every  $\alpha \in J(\mathbb{R})_2$ , the set of  $d(n+1)$ -uples which add to  $\alpha$ :

$$P_\alpha := \{\mathbf{A} = (A_1, \dots, A_{d(n+1)}) \in \widehat{X}(\mathbb{R})^{d(n+1)} \mid A_1 \oplus \dots \oplus A_{d(n+1)} = \alpha\}$$

and the restriction  $\psi_\alpha$  of  $\psi$  to this subset:

$$\begin{aligned} \psi_\alpha : P_\alpha &\longrightarrow \mathcal{I}(X)_{2d}^\perp \cong R_{2d}^* \\ \mathbf{A} = (A_1, \dots, A_{d(n+1)}) &\longmapsto \sum_{i=1}^{d(n+1)} \ell_{A_i}^2 \end{aligned}$$

As in the proof of Proposition 5.1.3, analyzing the differential of  $\psi_\alpha$  we can show that, if the  $A_i$ 's are distinct, then the codimension of the image of  $d\psi_\alpha$  is one. A dimension count shows then that, if  $L \in \partial \text{cone}_{d(n+1)}(Z) \setminus \text{cone}_{d(n+1)-1}(Z)$ , the fiber  $\psi_\alpha^{-1}(L)$  is finite (or empty). Since every representation of  $L \in \partial \text{cone}_{d(n+1)}(Z) \setminus \text{cone}_{d(n+1)-1}(Z)$  uses distinct atoms which add to a 2-torsion point, we have shown that  $L$  admits finitely many representations.

Finally, by semicontinuity (see e.g. [Har77, Theorem III.12.8]) we can conclude that  $L$  admits at most two decompositions.  $\square$

In the previous lemmas, we focused on representations using  $d(n+1)$  points, showing that they are generically finitely many. On the other hand, when we use  $d(n+1) + 1$  points a generic linear functional  $L$  admits infinitely many representations. More precisely, we can show that every linear functional in the relative interior of  $P_{X,2d}^\vee$  has a 2-dimensional family of representations using  $d(n+1) + 1$  points.

**Lemma 5.2.5.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve and let  $L \in \text{int} P_{X,2d}^\vee$ . Then  $L$  has a 2-dimensional family of representations using  $d(n+1) + 1$  point evaluations.*

*Proof.* Considering the Veronese reembedding  $v_{n,d}(X)$ , we can restrict to the case of quadrics, i.e.  $d = 1$ . As is in the proof of Proposition 5.1.3, consider the map

$$\begin{aligned} \psi: \widehat{X}(\mathbb{R})^{n+2} &\longrightarrow \mathcal{I}(X)_2^\perp \cong R_2^* \\ \mathbf{A} = (A_1, \dots, A_{n+2}) &\longmapsto \sum_{i=1}^{n+2} \ell_{A_i}^2 \end{aligned}$$

and its differential

$$\begin{aligned} d\psi_{\mathbf{A}}: T_{\mathbf{A}}(\widehat{X}(\mathbb{R})^{n+2}) &\cong T_{A_1} \widehat{X}(\mathbb{R}) \times \dots \times T_{A_{n+2}} \widehat{X}(\mathbb{R}) \longrightarrow T_{\psi(\mathbf{A})}(\mathcal{I}(X)_2^\perp) \cong \mathcal{I}(X)_2^\perp \\ \mathbf{v} = (v_1, \dots, v_{n+2}) &\longmapsto 2 \sum_{i=1}^{n+2} \ell_{v_i} \ell_{A_i} \end{aligned}$$

From Corollary 5.1.5, we deduce that  $\text{Im } \psi = P_{X,2}^\vee$ , which is a  $2(n+1)$ -dimensional convex cone. We want to show that, for  $L \in \text{int} P_{X,2}^\vee$ ,  $\dim \psi^{-1}(L)$  has dimension two.

As in the proof of Proposition 5.1.3, we notice that  $d\psi_{\mathbf{A}}$  is not surjective if and only if there exists a double vanishing quadratic form  $q \in R_2$  at  $\mathbf{A} = (A_1, \dots, A_{n+2})$ . As  $\deg \text{div } q = \deg q \cdot \deg X = 2(n+1)$ , this can happen if and only if the  $A_i$ 's are not distinct and they admit a double vanishing quadratic form through them. Therefore, if the  $A_i$ 's are distinct then  $\mathbf{A}$  is a regular point of  $\psi$ , and

$$\dim \psi^{-1}(\psi(\mathbf{A})) = \dim \widehat{X}(\mathbb{R})^{n+2} - \dim R_2^* = 2$$

It is then sufficient to show that every linear functional in the relative interior of  $P_{X,2}^\vee$  is the image of a regular point of  $\psi$ . We proceed as in the proof of Lemma 5.1.1: let  $L \in \text{int} P_{X,2}^\vee$ . For all  $A \in \widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$ , there exists  $\lambda = \lambda(A) \in \mathbb{R}_{>0}$  such that

$$L - \lambda^2 \text{ev}_A = L - \lambda^2 \ell_A^2 = q - \ell_{\lambda A}^2 \in \partial P_{X,2}^\vee$$

and  $L - \ell_{\lambda A}^2$  can be represented in a unique way as a sum of  $n+1$  point evaluations (see the proof of Lemma 5.1.1). Therefore, if the  $n+1$  points are distinct and different from  $A$ , then  $L$  is the image of a regular point of  $\psi$ , as desired. If the  $n+1$  points and  $A$  are not distinct for all  $A \in X(\mathbb{R})$ , then  $L$  would admit a representation as a sum of  $n+1$  point evaluations using every  $A \in X(\mathbb{R})$ , contradicting Lemmas 5.2.3 and 5.2.4. Thus  $L$  has a representation using  $n+2$  distinct point evaluations, and it is the image of a regular point of  $\psi$ , as desired.  $\square$

We note that the proof of Lemma 5.2.5 can be simplified if  $X(\mathbb{R})$  is connected. Indeed, in this case all the linear functionals in the relative interior of  $P_{X,2}^\vee$  are regular values of  $\psi$ , and we can immediately conclude that all the fibers have dimension two. On the other hand, if  $X(\mathbb{R})$  is disconnected then  $\psi$  has critical values in the interior of  $P_{X,2}^\vee$ , and the proof cannot be easily simplified.

We now use Lemma 5.2.5 to show that there exist representations of linear functionals in  $P_{X,2d}^\vee$  whose atoms avoid finitely many points in  $X(\mathbb{R})$ .



**Lemma 5.2.6.** *Let  $X \subset \mathbb{P}^n$  be a totally real elliptic normal curve and let  $L$  be in the interior of  $\mathbb{P}_{X,2d}^\vee$ . If*

$$\mathcal{B} = \{B_1, \dots, B_m\} \subset \widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$$

*then there exists a representation of  $L$  using  $d(n+1) + 1$  atoms that do not belong to  $\mathcal{B}$ .*

*Proof.* As in the proof of Lemma 5.2.5, we can restrict to the case  $d = 1$ . Let  $\psi$  be the map in the proof of Lemma 5.2.5. Assume by contradiction that every representation of  $L \in \text{int } \mathbb{P}_{X,2}^\vee \subset R_2^*$  uses  $\text{ev}_{B_j}$  for some  $B_j \in \mathcal{B}$ . Then, if we let  $\mathcal{A}_{B_j}(L) \subset (\widehat{X}(\mathbb{R}) \cap \mathbb{S}^n)^{n+1}$  be the  $n+1$ -tuples of representing atoms of  $L$  which include  $B_j$ , we have:

$$\psi^{-1}(L) = \mathcal{A}_{B_1}(L) \cup \dots \cup \mathcal{A}_{B_m}(L)$$

We deduce from Lemma 5.2.5 that there exists  $j$  such that  $\dim \mathcal{A}_{B_j}(L) = 2$ . In particular, we have a two-dimensional family of representations of the form:

$$L = \lambda \text{ev}_{B_j} + \sum_{i=1}^{n+1} \gamma_i(\lambda) \text{ev}_{A_i(\lambda)}$$

where  $\gamma_i(\lambda) \in \mathbb{R}_{\geq 0}$ ,  $A_i(\lambda) \in \widehat{X}(\mathbb{R}) \cap \mathbb{S}^n$  and  $\lambda \in \mathbb{R}_{>0}$ . This implies that for some  $\lambda \in \mathbb{R}$  there exists a one-dimensional family of representations for  $L - \lambda \text{ev}_{B_j}$  using  $n+1$  point evaluations, in contradiction to Lemma 5.2.3. This concludes the proof.  $\square$

**Remark 5.2.7.** We make some remarks on the previous results.

- (i) In the proof of Lemma 5.2.3, we are implicitly using the fact that, if  $L \in \mathbb{P}_{X,2d}^\vee$ , then the *moment or catalecticant matrix*  $M_d(L)$  is positive definite, see Section 6.1.
- (ii) Assuming that  $X(\mathbb{R})$  is connected, we can deduce from Corollary 5.2.1 that the rank of the moment or catalecticant matrix  $M_d(L)$  is equal to the nonnegative  $Z$ -rank of  $L$ : in general, the rank of  $M_d(L)$  is only a lower bound for the rank.
- (iii) Given a general  $L$ , in Lemma 5.2.3 (and in Lemma 5.2.2) the atoms of the two different possible representations form the zero locus of a form  $q = q_L \in R_{2d}$  on the elliptic normal curve, i.e.

$$\text{div } q = A_1 + \dots + A_{d(n+1)} + B_1 + \dots + B_{d(n+1)}.$$

This shows that the set of possible atoms forms a self-associated set for the Gale transform, see [EP00]. Therefore, given  $L$  and a representing atoms  $A_1, \dots, A_{d(n+1)}$ , the problem of determining the second set of representing atoms  $B_1, \dots, B_{d(n+1)}$  could be investigated using the geometry of the Gale transform.

## 6. THE MOMENT PROBLEM FOR PLANE CUBICS

In this section, we consider *affine* genus one curves, and see how the results in Section 5 can be exploited to solve the *moment problem* on such curves. In particular, we show how the topology of the real locus and the number of points at infinity can be used to characterize the *flat extension* degree. In the following, affine curves and their real loci will be denoted by  $X$  and  $X(\mathbb{R})$  respectively, while their projective closures will be denoted by  $\overline{X}$  and  $\overline{X}(\mathbb{R})$ .

In recent years, the case of *plane* curves has been particularly investigated, see [Fia11, Zal21, Zal22, Zal23, Bha20]. For ease of presentation and comparison, we restrict ourselves to the same case. i.e. we consider a smooth totally real affine plane cubic  $X(\mathbb{R}) \subset \mathbb{R}^2$ .

**6.1. Flat extensions and almost flat extensions.** We now briefly introduce the relevant definitions and notations for this section.

Let  $X(\mathbb{R}) \subset \mathbb{R}^2$  be a totally real affine plane cubic. Given

$$L \in R_{\leq 2d}^* = \mathbb{R}[X]_{\leq 2d}^* = \left( \mathbb{R}[x, y]_{\leq d} / I(X)_{\leq d} \right)^*$$

we can consider the bilinear form

$$\begin{aligned} M_d(L): R_{\leq d} \times R_{\leq d} &\longrightarrow \mathbb{R} \\ (q_1, q_2) &\longmapsto L(q_1 q_2) \end{aligned}$$

Equivalently,  $M_d(L)$  is the linear map:

$$\begin{aligned} M_d(L): R_{\leq d} &\longrightarrow R_{\leq d}^* \\ q &\longmapsto L \circ m_q \end{aligned}$$

where  $m_q: R_{\leq d} \rightarrow R_{\leq 2d}$  denotes the multiplication by  $q \in R_{\leq d}$ . We call the matrix representing  $M_d(L)$  (with respect to any basis of  $R_{\leq d}$ ) the *moment matrix* of  $L$ . We refer to [Sch17] for more details. These matrices are also known as *Catalecticant matrices* [IK99] (in the case  $X = \mathbb{R}^n$ ) or *Hankel matrices* [Mou18]. We also notice that  $\text{rank } M_d(L) \leq \dim R_{\leq d} = 3d$ .

Similarly to the homogeneous case, we denote

$$P_{X, \leq 2d} = \{q \in R_{\leq 2d} : q(A) \geq 0 \text{ for all } A \in X(\mathbb{R})\}.$$

Notice that, if  $L \in P_{X, \leq 2d}^\vee$ , then the moment matrix  $M_d(L)$  is positive semidefinite.

As  $R_{2d} \cong R_{\leq 2d}$  by homogenization, we also have  $P_{X, \leq 2d} \cong P_{\bar{X}, 2d}$  and  $P_{X, \leq 2d}^\vee \cong P_{\bar{X}, 2d}^\vee$ . To emphasize the fact that we are working in an affine setting, hereafter we use the notations  $R_{\leq 2d}$  and  $P_{X, \leq 2d}$ .

We denote  $\mathcal{M}_{2d}(X(\mathbb{R}))$  the *moment cone*, i.e. the convex cone of linear functionals acting on polynomials of degree  $\leq 2d$  which are induced by a Borel measure supported on  $X(\mathbb{R})$ . Using the Richter-Tchakaloff theorem [Sch17, Th. 1.24], we can describe  $\mathcal{M}_{2d}(X(\mathbb{R}))$  using only conic sums of point evaluations at  $X(\mathbb{R})$ :

$$\mathcal{M}_{2d}(X(\mathbb{R})) = \text{cone}(\text{ev}_A : A \in X(\mathbb{R}))$$

If  $X(\mathbb{R})$  is compact then  $\mathcal{M}_{2d}(X(\mathbb{R})) = P_{X, 2d}^\vee$ . On the other hand, if  $X(\mathbb{R})$  is not compact, then  $\mathcal{M}_{2d}(X(\mathbb{R}))$  is not closed and the inclusion  $\mathcal{M}_{2d}(X(\mathbb{R})) \subsetneq P_{X, \leq 2d}^\vee$  is proper: this is always the case for plane cubics  $X(\mathbb{R}) \subset \mathbb{R}^2$ .

We are now ready to recall the notion of flat extension.

**Definition 6.1.1.** A *positive flat extension* of  $L \in R_{\leq 2d}^*$  is any  $\tilde{L} \in R_{\leq 2d+2}^*$  such that:

- (i)  $\tilde{L}|_{R_{\leq 2d}} = L$ ;
- (ii)  $\text{rank } M_{d+1}(\tilde{L}) = \text{rank } M_d(L)$ ;
- (iii)  $M_{d+1}(\tilde{L})$  is positive semidefinite.

Testing the existence of a flat extension is the standard algorithmic tool to verify if  $L$  is a moment linear functional or not, see e.g. [CF98, Sch17, Mou18]. Our definition of flat extension is slightly different from others present in the literature for the case of curves (see e.g. [Fia11]), in terms of how we enforce the atoms of any representation of  $L$  to lie on  $X(\mathbb{R})$ . We refer to Remark 6.1.4 for a comparison.

The rank of the flat extension determines the number of atoms needed to represent  $L$  as conic combination of point evaluations on  $X(\mathbb{R})$ , see e.g. [Sch17, Th. 17.36]. Moreover, if a flat extension is found, then the flat extension can be further extended from degree  $d+1$  to an arbitrary degree. Notice also that, since  $\tilde{L}|_{R_{\leq 2d}} = L$ , the matrix  $M_d(L)$  can be identified with the matrix  $M_d(\tilde{L})$ , which is a submatrix of  $M_{d+1}(\tilde{L})$ .

In other words, the rank of a flat extension gives the (affine) Carathéodory number of a linear functional. We define the Carathéodory number for the affine plane curve  $X$  as in the projective case, as follows:

$$\mathcal{C}_{X, \leq 2d} := \min \{k \in \mathbb{N} \mid \text{cone}_k(v_{2,d}(X(\mathbb{R}))) = \mathcal{M}_{2d}(X(\mathbb{R}))\}$$

Note that we are not allowed to use atoms at infinity, i.e. point evaluations from  $\bar{X}(\mathbb{R}) \setminus X(\mathbb{R})$ .

In the following, we will also need a slight generalization of Definition 6.1.1.

**Definition 6.1.2.** An *almost flat extension* of  $L \in R_{\leq 2d}$  is any  $\tilde{L} \in R_{\leq 2d+4}^*$  such that:

- (i)  $\tilde{L}|_{R_{\leq 2d}} = L$ ;
- (ii)  $\text{rank } M_{d+2}(\tilde{L}) \leq \text{rank } M_d(L) + 1$ ;
- (iii)  $M_{d+2}(\tilde{L})$  is positive semidefinite.

**Remark 6.1.3.** If  $\tilde{L}$  is a flat extension of  $L$ , then  $\tilde{L}$  can be further extended to an almost flat extension of  $L$ . More precisely, if  $L$  admits a flat extension  $\tilde{L}$ , then  $\tilde{L} \in R_{\leq 2d+2}^*$  admits a flat extension  $\tilde{\tilde{L}} \in R_{\leq 2d+4}^*$  such that:

$$\text{rank } M_{d+2}(\tilde{\tilde{L}}) = \text{rank } M_{d+1}(\tilde{L}) = \text{rank } M_d(L).$$

On the contrary, if  $L$  does not admit a flat extension, but it admits an almost flat extension  $\tilde{L}$ , we necessarily have:

$$\text{rank } M_{d+2}(\tilde{L}) = \text{rank } M_{d+1}(\tilde{L}) = \text{rank } M_d(L) + 1.$$

In Section 6.2 we show that we can characterize solutions to the moment problem on plane cubics using almost flat extensions.

**Remark 6.1.4.** We compare our definitions with the ones existing in the moment literature, e.g. in [Fia11, Th. 1.1]. The main difference is that, instead of working with all the monomials up to a certain degree and imposing extra conditions for the representing measure to be supported on the plane curve, we directly work in the quotient  $R_{\leq 2d} = \mathbb{R}[x, y]_{\leq 2d} / I(X)_{\leq 2d}$ . We also speak of the flat extension of a linear functional, instead of a flat extension of the corresponding moment matrix. Concretely, a  $p$ -pure moment matrix in [Fia11, Th. 1.1] corresponds to a positive definite moment matrix in this manuscript, and a *positive, recursively generated flat moment matrix extension* in [Fia11, Th. 1.1] simply corresponds to our flat extension.

**6.2. The moment problem for plane cubics.** We are now ready to show how to solve the moment problem, i.e. how to characterize membership  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$ , using almost flat extensions, see Definition 6.1.2.

**Proposition 6.2.1.** *Let  $X(\mathbb{R}) \subset \mathbb{R}^2$  be the affine real locus of a totally real plane cubic, whose projectivization  $\overline{X}$  is smooth. Given  $L \in R_{\leq 2d}^*$ , the following are equivalent:*

- (i)  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$ ;
- (ii) there exists an almost flat extension  $\tilde{L} \in R_{\leq 2d+4}$ .

*Proof.* (i)  $\implies$  (ii) Recall that, by homogenization,  $P_{X, \leq 2d} \cong P_{\overline{X}, 2d}$ . Therefore, given  $L \in \mathcal{M}_{2d}(X(\mathbb{R})) \subset P_{X, \leq 2d}^\vee$ , it follows from Corollary 5.1.5 that we can always write

$$L = \sum_{i=1}^{3d+1} a_i \text{ev}_{A_i}$$

where, for all  $i$ ,  $a_i \geq 0$  and  $A_i$  belongs to  $\overline{X}(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ , the real locus of the projective closure  $\overline{X}$  of  $X$ . From Lemma 5.2.6, we deduce that there exists a representation for which  $A_i \in X(\mathbb{R})$  (we are excluding the finitely many points at infinity). Therefore  $L$  admits the almost flat extension  $\tilde{L} = \sum_{i=1}^{3d+1} a_i \text{ev}_{A_i} \in R_{\leq 2d+4}^*$ .

(i)  $\implies$  (ii) If  $L$  admits an almost flat extension  $\tilde{L}$ , then either  $\text{rank } M_{d+1}(\tilde{L}) = \text{rank } M_d(L)$  or  $\text{rank } M_{d+2}(\tilde{L}) = \text{rank } M_{d+1}(\tilde{L})$ . From the flat extension theorem, we deduce that, in both cases,  $L$  can be represented as an atomic measure using point evaluations in  $X(\mathbb{R})$ , which implies  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$ .  $\square$

We showed in Proposition 6.2.1 that, extending the degree by 4 and considering an almost flat extension, it is possible to characterize whether  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$  or not. A natural question then arises: is it necessary to consider almost flat extensions, or can we replace them with ordinary flat extensions? In the following, we are going to show how the answer to this question depends on the topology of  $X(\mathbb{R})$  and the number of points at infinity of the projective closure  $\overline{X}(\mathbb{R})$ .

We start showing that flat extension is sufficient when the curve is connected and has only one point at infinity.

**Theorem 6.2.2.** *Let  $X$  be an affine, smooth, totally real plane cubic. Assume that  $X(\mathbb{R})$  is connected and that the projective closure  $\overline{X}(\mathbb{R})$  contains only one real point at infinity. Then, given  $L \in R_{\leq 2d}$ , the following are equivalent:*

- (i)  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$ ;
- (ii)  $L$  has a positive flat extension  $\tilde{L} \in R_{\leq 2d+2}^*$ .

Moreover, the affine and projective Carathéodory numbers are equal:  $\mathcal{C}_{X, \leq 2d} = \mathcal{C}_{\overline{X}, 2d} = 3d$ .

*Proof.* For (ii)  $\implies$  (i), we can proceed as in Proposition 6.2.1.

For (i)  $\implies$  (ii), it follows from Theorem 5.1.4 that  $L$  can be written using at most  $3d$  points from  $\overline{X}(\mathbb{R})$ . We now show that at least one such representation has all points in the affine part  $X(\mathbb{R})$ .

If  $L \in \partial \mathcal{M}_{2d}(X(\mathbb{R}))$ , it follows e.g. from [Sch17, Prop. 18.12] that  $L$  has a unique representation as a sum of point evaluations in  $X(\mathbb{R})$ . This unique representation uses at most  $3d$  point evaluations, and thus we can construct a flat extension of  $L$  using such a representation.

If  $L \in \text{int } \mathcal{M}_{2d}(X(\mathbb{R}))$ , then Lemma 5.2.3 implies that  $L$  admits two different representations as sum of  $3d$  point evaluations. Such distinct representations cannot share any point, and therefore the fact that  $\overline{X}(\mathbb{R})$  has a unique point at infinity implies that one of the two uses only points in  $X(\mathbb{R})$ . We can therefore construct a flat extension for  $L$  from this representation with only points in  $X(\mathbb{R})$ .

The fact that  $\mathcal{C}_{X, \leq 2d} = \mathcal{C}_{\overline{X}, 2d}$  follows from the previous points, as the existence of representations using points from  $\overline{X}(\mathbb{R})$  implies that existence of representations using points from  $X(\mathbb{R})$  with the same number of atoms.  $\square$

We now construct an example which illustrates that  $\overline{X}(\mathbb{R})$  having a unique point at infinity is a necessary assumption in Theorem 6.2.2.

**Example 6.2.3.** Let  $\widetilde{X}(\mathbb{R}) \subset \mathbb{R}^2$  be the real locus of the affine totally real plane cubic, and assume that  $\widetilde{X}(\mathbb{R})$  is connected. Let  $L \in \text{int } \mathcal{M}_{2d}(\widetilde{X}(\mathbb{R})) \subset \mathbb{P}_{\widetilde{X}, \leq 2d}^\vee$ .

We deduce from Lemma 5.2.3 that  $L$  admits exactly two distinct representations

$$L = \sum_{i=1}^{3d} a_i \text{ev}_{\widetilde{A}_i} = \sum_{i=1}^{3d} b_i \text{ev}_{\widetilde{B}_i}$$

using  $3d$  points form the projective closure of  $\widetilde{X}(\mathbb{R})$ . Now consider the (real) line between  $A_1$  and  $B_1$ , and apply a projective linear change of coordinates in such a way the line between  $\widetilde{A}_1$  and  $\widetilde{B}_1$  is sent to the line at infinity. Let  $X(\mathbb{R}) \subset \mathbb{R}^2$  be the affine real locus of  $\widetilde{X}(\mathbb{R})$  under this change of coordinates.

We have  $L \in \text{int } \mathcal{M}_{2d}(X(\mathbb{R})) = \text{int } \mathcal{M}_{2d}(\widetilde{X}(\mathbb{R}))$ , and as before  $L$  admits only two representations

$$L = \sum_{i=1}^{3d} a_i \text{ev}_{A_i} = \sum_{i=1}^{3d} b_i \text{ev}_{B_i} \in \mathbb{P}_{\widetilde{X}, \leq 2d}^\vee = \mathbb{P}_{X, \leq 2d}^\vee$$

where  $A_i, B_i$  denote the images of  $\widetilde{A}_i, \widetilde{B}_i$  after the change of coordinates above. Notice that both representations use a point at infinity, i.e.  $A_1$  and  $B_1$  respectively, by construction.

This shows that  $L \in \text{int } \mathcal{M}_{2d}(X(\mathbb{R}))$  cannot be represented using  $3d$  evaluations at points of the affine curve  $X(\mathbb{R})$ , and thus  $L$  does not admit a flat extension. Therefore, we cannot remove the condition that  $\overline{X}(\mathbb{R})$  has a unique point at infinity from Theorem 6.2.2, to conclude that the flat extension condition is sufficient. Notice also that the above discussion implies that  $\mathcal{C}_{X, \leq 2d}(L) = 3d + 1$ , but  $\mathcal{C}_{\overline{X}, 2d} = 3d$ , and thus  $\mathcal{C}_{\overline{X}, 2d} < \mathcal{C}_{X, \leq 2d}$ . It is therefore not possible to compute the affine Carathéodory numbers using solely the projective information.

We conjecture that such pathological behavior happens for all affine plane cubics with at least two real points at infinity.

We now turn our attention to the disconnected case.

**Theorem 6.2.4.** Let  $X(\mathbb{R}) \subset \mathbb{R}^2$  be the affine real locus of a totally real plane cubic, whose projectivization  $\overline{X}$  is smooth. Assume that  $X(\mathbb{R})$  has two connected components. Then, given  $L \in \mathcal{R}_{\leq 2d}$ , the following are equivalent:

- (i)  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$ ;
- (ii)  $L$  has a positive almost flat extension  $\widetilde{L} \in \mathcal{R}_{\leq 2d+4}^*$ .

The almost flat extension condition cannot be replaced by flat extension in (ii). Moreover, the affine and projective Carathéodory numbers are equal:  $\mathcal{C}_{X, \leq 2d} = \mathcal{C}_{\overline{X}, 2d} = 3d + 1$ .

*Proof.* By Proposition 6.2.1, we only need to show that there exists  $L \in \mathcal{M}_{2d}(X(\mathbb{R}))$  which does not admit a flat extension. For this, it is sufficient to show that  $\mathcal{C}_{\overline{X}, 2d}(L) > 3d$ . By Theorem 5.1.4, such an  $L$  exists when  $X(\mathbb{R})$  has two connected components. This also implies that  $\mathcal{C}_{X, \leq 2d} = \mathcal{C}_{\overline{X}, 2d} = 3d + 1$ .  $\square$

We notice that the solution of the moment problem proposed in Theorem 6.2.4 with the almost flat extension condition is much more complicated than the usual flat extension one, exploited in Theorem 6.2.2 and [Fia11, Zal21, Zal22, Zal23, Bha20]. However, this increased complexity cannot be avoided, as it is a direct consequence of the higher Carathéodory number.

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