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Sparsity and L^1 -optimal control for linear control systems

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Abstract

For linear time-invariant systems, we take an interest in sparse controls, i.e., those whose support (or L^0 -norm) is of minimal Lebesgue measure. We propose to extend this research to Radon measures and, in particular, to controls presenting Dirac impulses. Then, we can find a minimizer which is impulsive (and so with a null support) and prove that there is no gap when we extend the problem from L^1 -controls to Radon measures. To refine our research, this reasoning leads us to study the minimization problems for L^1 and measure norms. We prove that impulsive controls are still pertinent for these problems: once again, we can find a minimizer and prove that there is no gap. In addition to the existence of a solution, we show that, by extending the research to Radon measures, we can find a solution of L^0 and L^1 -minimization problems in the form of a finite linear combination of Dirac impulses, and we give an expression of the maximal number of these impulses. We also propose an algorithm in order to compute an impulsive solution of the measure norm minimization problem.

Key words: Impulsive control, optimal control, sparsity.

1 Introduction

Context and problem statement. The investigation of the existence, uniqueness and determination of optimal control, i.e., a control which respects equations and conditions while minimizing a certain cost, goes back several decades (see e.g. [5,16,17,26]). In the same time, consideration of discontinuous controls, measure controls or impulsive controls, was made possible through the work of A. Bressan, F. Rampazzo or M. Motta (see e.g. [3,7,8,22,28]), among others. Thus, it seems natural that several impulsive optimal control problems were studied in the last few years (see e.g. [21,27,20,4,12]).

A natural cost to minimize is the duration of action of the control, which mathematically corresponds to the Lebesgue measure of its support or its L^0 -norm. A control minimizing this criterion is said to be *sparse* or *maximum hands-off* (see e.g. [25]). This problem is not convex and, in [24], M. Nagahara looks for conditions to have an equivalence, for linear time-invariant (LTI) systems, between the problem of L^0 -minimization and the one of L^1 -minimization which is convex and so easier to solve. By this method, he obtains, under some conditions (including a bounded condition on the control), optimal controls in a form called *bang-off-bang*, i.e., con-

trols switching between 0 and its bounds (see also the recent paper [1]). More generally, still for LTI systems and under a bounded condition, we can find the study of the L^p -minimization problem for $p \in [0, 1]$ in [14]. Controls considered in these papers are in the space L^1 and we would like to extend the research to the Radon measures set and, in particular, consider impulsive controls. More precisely, we take an interest in controls of purely impulsive form (i.e., which can be written as a linear combination of Dirac impulses). The L^0 -norm of a such control is equal to 0, and we could directly have a solution of the L^0 -minimization problem.

Main results. We consider the problem of state transfer for linear time-invariant LTI systems whose plant model is given by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x^0, \quad x(T) = x^1, \end{cases} \quad (\mathcal{S})$$

where for all $t \in [0, T]$, $x(t) \in \mathbb{R}^n$ describes the state and $u(t) \in L^1([0, T], \mathbb{R}^m) = L^1([0, T])^m$ the control (or input). The final time $T > 0$, the initial state $x^0 \in \mathbb{R}^n$ and the target $x^1 \in \mathbb{R}^n$ are given, and $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are (constant) matrices.

We say that a control is *feasible* if it steers the solution of the system from x^0 to x^1 in time T . In this paper, we prove that if there exist feasible L^1 -controls, then there exist feasible impulsive controls. More than that,

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we can guarantee that there is no gap when we extend the L^0 -minimization problem from the space L^1 to the set of Radon measures (Proposition 1). The construction of a sequence of feasible L^1 -controls approaching a given feasible impulsive control leads us to consider the L^1 -minimization problem to refine our research among all these feasible impulsive controls. This new problem is directly linked to the one of measure norm minimization, which always has an impulsive solution whose number of impulses can be bounded (Theorem 5). Once again, we can prove that there is no gap between these two problems (Proposition 7). Thanks to the Pontryagin's Maximum Principle (PMP), we can find sufficient conditions ensuring that any minimizer is necessarily of purely impulsive form (Theorem 17 and Corollary 18) and we propose, at the end of this paper, an algorithm based on coordinate descent method [19] and Bregman iterations [10] to compute an impulsive solution.

Organization of the paper. We give some notations used in this paper in Section 2. Section 3 refers to the extension of problem (P_0) from the L^1 -space to the Radon measures set. In order to refine our research, we try to adapt this reasoning to the L^1 -minimization problem in Section 4. In Section 5, thanks to a change of variable, we can use the (PMP) on the problem of measure norm minimization. Section 6 describes all the possible situations when $n = 2$ and $m = 1$. We build, in Section 7, an algorithm to approach an impulsive solution. Finally, Section 8 is dedicated to the conclusion.

2 Notations and preliminaries

To have controls to consider in our minimization problems, we assume that there is at least one feasible L^1 -control (Assumption 1).

Assumption 1 x^1 is reachable from x^0 in time T through a control $u \in L^1([0, T])^m$.

Among all feasible controls u (i.e., we can reach x^1 from x^0 in time T with u), we take an interest in those whose support (or L^0 -norm) is minimal. We call this problem (P_0) . If $u \in L^1([0, T])^m$ is feasible, then we must have

$$x^1 = e^{TA}x^0 + \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau.$$

We set $y = x^1 - e^{TA}x^0$ and we recall that for $u \in L^1([0, T])^m$, the input-to-state mapping is defined by

$$\Phi_T : u \in L^1([0, T])^m \mapsto \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau \in \mathbb{R}^n. \quad (1)$$

Thus, a control u is feasible if and only if $\Phi_T(u) = y$ and Assumption 1 means there exists $u \in L^1([0, T])^m$ such that $\Phi_T(u) = y$.

Before formulating problem (P_0) , we give some notations used in this paper. For $i \in \llbracket 1, m \rrbracket$, we call b_i the i -th column of the matrix B . C^\top denotes the transpose of a given matrix C . Moreover, we use different norms either for elements of \mathbb{R}^m or for functions of L^1 . Here we resume the different writings adopted. For $u = (u_1 \ u_2 \ \dots \ u_m)^\top \in \mathbb{R}^m$, we consider the norms $|u|_1 = \sum_{i=1}^m |u_i|$ and $|u|_\infty = \max_{i \in \llbracket 1, m \rrbracket} |u_i|$. For $u \in L^1([0, T])^m$, we consider the L^0 -norm (which is indeed not a norm because it does not satisfy the homogeneity condition) defined by

$$\|u\|_{L^0} = \mu_L \{t \in [0, T] : u(t) \neq 0\} = \mu_L \{\text{supp}(u)\},$$

where μ_L denotes the Lebesgue measure. We also consider the L^1 and the L^∞ norms given by $\|u\|_{L^1} = \int_0^T |u(t)|_1 dt$ and $\|u\|_{L^\infty} = \text{ess sup}_{t \in [0, T]} |u(t)|_\infty$.

Now, we can formulate the problem (P_0)

$$\inf \|u\|_{L^0} \quad (P_0) \\ \left| \begin{array}{l} u \in L^1([0, T])^m \text{ and } \Phi_T(u) = y. \end{array} \right.$$

In other words, we want to minimize the support of the control so it seems natural to consider impulsive controls whose support is null.

For that, we consider $\mathcal{M}([0, T])$ the set of finite Radon measures on $[0, T]$ endowed with the norm

$$\|u\|_{\mathcal{M}} = \sup_{\substack{\varphi \in C^0([0, T], \mathbb{R}) \\ \|\varphi\|_{L^\infty} \leq 1}} \int_{[0, T]} \varphi(t) du(t).$$

We can extend these notations to the multi-input case by considering $\mathcal{M}([0, T])^m$ the set of finite Radon measures on $[0, T]$ with values in \mathbb{R}^m (i.e., such that each component has to be in $\mathcal{M}([0, T])$). Then, the measure norm is obtained by summing the measure norm of each component.

Here, we take an interest in impulsive controls, i.e., composed of linear combinations of finite number of Dirac impulses. Let u be such a control. Then, there exist $N \in \mathbb{N}^*$, $(\alpha_i)_{i \in \llbracket 1, N \rrbracket} \in \mathbb{R}^m$ and $(t_i)_{i \in \llbracket 1, N \rrbracket} \in [0, T]$ such that $u = \sum_{i=1}^N \alpha_i \delta_{t_i} \in \mathcal{M}([0, T])^m$. Thus, we have

$$\|u\|_{\mathcal{M}} = \sum_{i=1}^N |\alpha_i|_1 = \sum_{i=1}^N \sum_{k=1}^m |\alpha_{i,k}|,$$

where $\alpha_{i,k}$ is the k -th component of $\alpha_i \in \mathbb{R}^m$.

By definition, a control $u \in L^1([0, T])^m$ has its components in $L^1([0, T])$ which can be canonically injected in $\mathcal{M}([0, T])$ via

$$L^1([0, T]) \rightarrow \mathcal{M}([0, T]) \\ u \mapsto \left(\varphi \in C^0([0, T], \mathbb{R}) \mapsto \int_0^T \varphi(t)u(t) dt \right).$$

Thus, we can canonically inject $L^1([0, T])^m$ in $\mathcal{M}([0, T])^m$ and extend the input-to-state mapping (1) to $\mathcal{M}([0, T])^m$ (as in [21])

$$\Phi_T : u \in \mathcal{M}([0, T])^m \mapsto \int_{[0, T]} e^{(T-\tau)A} B du(\tau) \in \mathbb{R}^n.$$

We still have that $u \in \mathcal{M}([0, T])^m$ is a feasible control if and only if $\Phi_T(u) = y$.

3 Existence of sparse control

The objective of this section is to prove Proposition 1 which ensures that, under Assumption 1, we have

$$0 = \inf \|u\|_{L^0} \quad \Big| \quad u \in L^1([0, T])^m \text{ and } \Phi_T(u) = y,$$

and, in addition to the existence of a feasible impulsive control, there is no gap when we extend the problem (P_0) from $L^1([0, T])^m$ to $\mathcal{M}([0, T])^m$.

Proposition 1 *If $x^0, x^1 \in \mathbb{R}^n$ and $T > 0$ satisfy Assumption 1, then for every nonempty set $\mathcal{T} \subset [0, T]$ and every $\varepsilon > 0$, there exists an impulsive control $u^* \in \mathcal{M}([0, T])$, which steers x^0 to x^1 in time T and such that $\text{dist}(\mathcal{T}, \text{supp}(u^*)) < \varepsilon$.*

Furthermore, for every feasible impulsive control $u^ \in \mathcal{M}([0, T])^m$, there exists a sequence of feasible controls $(u_k)_{k \in \mathbb{N}} \in L^1([0, T])^m$ such that*

$$\lim_{k \rightarrow \infty} \|u_k\|_{L^0} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u_k\|_{L^1} = \|u^*\|_{\mathcal{M}}.$$

We prove this proposition by constructing an appropriate sequence $(u_k)_{k \in \mathbb{N}}$ which vaguely converges to u^* . We recall the definition of the vague convergence below (see [6, Definition 30.1] or [11, Section 1.9] for more details).

Definition 2 (Vague (or weak-*) convergence) *If $m = 1$, saying that the sequence $(u_k)_{k \in \mathbb{N}} \in \mathcal{M}([0, T])$ converges vaguely to $u^* \in \mathcal{M}([0, T])$ means that for all f continuous from $[0, T]$ to \mathbb{R} ,*

$$\lim_{k \rightarrow \infty} \int_{[0, T]} f du_k = \int_{[0, T]} f du^*.$$

If $m \geq 1$, the sequence $(u_k)_{k \in \mathbb{N}} \in \mathcal{M}([0, T])^m$ converges vaguely to $u^ \in \mathcal{M}([0, T])^m$ if the sequence of each component of $(u_k)_{k \in \mathbb{N}}$ converges vaguely to the corresponding component of u^* . We note $u_k \xrightarrow{*} u^*$.*

Remark 3 *More generally, the vague convergence is a way of convergence defined on dual sets. Here,*

$\mathcal{M}([0, T])^m$ is the dual space of the set of continuous functions on $[0, T]$ endowed with the L^∞ -norm and each element of $L^1([0, T])^m$ is considered as an element of $\mathcal{M}([0, T])^m$. By using classical results on vaguely convergence and compactness (see e.g. [6, 11]), we have:

- *if $(u_k)_{k \in \mathbb{N}} \in \mathcal{M}([0, T])^m$ is such that $u_k \xrightarrow{*} u^* \in \mathcal{M}([0, T])^m$, then $\|u^*\|_{\mathcal{M}} \leq \liminf_{k \rightarrow \infty} \|u_k\|_{\mathcal{M}}$;*
- *if $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence of $\mathcal{M}([0, T])^m$ (i.e., $\sup_{k \in \mathbb{N}} \|u_k\|_{\mathcal{M}} < \infty$), then there exist a subsequence (u_{k_i}) and $u^* \in \mathcal{M}([0, T])^m$ such that $u_{k_i} \xrightarrow{*} u^*$.*

To prove Proposition 1, we need the following lemma, which will allow us to build the impulsive control u^* . We call \mathcal{R} the set of all possible final states with a L^1 -control by starting from $x^0 = 0$:

$$\mathcal{R} = \left\{ \int_0^T e^{(T-\tau)A} B u(\tau) d\tau : u \in L^1([0, T])^m \right\}.$$

We can notice that \mathcal{R} does not depend on T since we do not impose a bound on control u .

Lemma 4 *By noting $r = \text{rk } \Phi_T$, where $\text{rk}(\cdot)$ denotes the rank, for almost every $t_1, \dots, t_r \in [0, T]$, we have*

$$\text{Span} \left\{ e^{(T-t_i)A} B v : i \in \llbracket 1, r \rrbracket, v \in \mathbb{R}^m \right\} = \mathcal{R}.$$

PROOF. By reasoning as in the proof of the Kalman condition in [29, Theorem 2.2.1], we have

$$\text{Span} \left\{ A^i B v : i \in \llbracket 0, n-1 \rrbracket, v \in \mathbb{R}^m \right\} = \mathcal{R}.$$

Thus, by using the Cayley-Hamilton theorem, we directly deduce that

$$\text{Span} \left\{ e^{(T-t_i)A} B v : i \in \llbracket 1, r \rrbracket, v \in \mathbb{R}^m \right\} \subset \mathcal{R}.$$

for all $t_1, \dots, t_r \in [0, T]$.

By reasoning as in [9, Proof of Theorem 7], there exists $\tau_1, \dots, \tau_r \in [0, T]$ and $v_1, \dots, v_r \in \mathbb{R}^m$ such that

$$\begin{aligned} \dim \text{Span} \left\{ e^{(T-t_i)A} B v : i \in \llbracket 1, r \rrbracket, v \in \mathbb{R}^m \right\} \\ \leq \dim \text{Span} \left\{ e^{(T-\tau_i)A} B v_i : i \in \llbracket 1, r \rrbracket \right\}. \end{aligned}$$

and we set $S = \text{Span} \left\{ e^{(T-\tau_i)A} B v_i : i \in \llbracket 1, r \rrbracket \right\}$. Then, for every $t \in [0, T]$ and every $v \in \mathbb{R}^m$, we have $e^{(T-t)A} B v \in S$. If not, there would exist $t_0 \in [0, T]$ and $v_0 \in \mathbb{R}^m$ such that $e^{(T-t_0)A} B v_0 \notin S$. Thus, vectors $e^{(T-\tau_i)A} B v_i$ (for $i \in \llbracket 0, r \rrbracket$) are $r+1$ elements of \mathcal{R}

whose dimension is r so they are linearly dependent, and we have a contradiction with the maximality of τ_1, \dots, τ_r and v_1, \dots, v_r . From this observation, we easily deduce that $\mathcal{R} = \text{Ran } \Phi_T \subset S$.

We have thus shown that there exist $\tau_1, \dots, \tau_r \in [0, T]$ and $v_1, \dots, v_r \in \mathbb{R}^m$ such that $\text{Span} \{e^{(T-\tau_i)A} B v_i : i \in \llbracket 1, r \rrbracket\} = \mathcal{R}$. In particular, there exist $P \in \mathbb{R}^{r \times n}$ such that

$$\det \left(P \left(e^{(T-\tau_1)A} B v_1, \dots, e^{(T-\tau_r)A} B v_r \right) \right) \neq 0.$$

Thus, the analytic application $(t_1, \dots, t_r) \mapsto \det \left(P \left(e^{(T-t_1)A} B v_1, \dots, e^{(T-t_r)A} B v_r \right) \right)$ is non-trivial. Hence, its zeros set is of null measure, this ensures that for almost every (t_1, \dots, t_r) , $\dim \text{Span} \{e^{(T-t_i)A} B v_i : i \in \llbracket 1, r \rrbracket\} = r$, which concludes this proof. \square

PROOF. (Proposition 1) Let $\mathcal{T} \subset [0, T]$ and $\varepsilon > 0$. By assumption, $y = x^1 - e^{TA} x^0 \in \mathcal{R}$ and, thanks to Lemma 4, there exist $t_1, \dots, t_r \in [0, T]$ and $\alpha_1, \dots, \alpha_r \in \mathbb{R}^m$ such that $\min_{i \in \llbracket 1, r \rrbracket} \text{dist}(t_i, \mathcal{T}) < \varepsilon$ and

$$y = \sum_{i=1}^r e^{(T-t_i)A} B \alpha_i = \int_{[0, T]} e^{(T-t)A} B du^*(t),$$

with $u^* = \sum_{i=1}^r \alpha_i \delta_{t_i} \in \mathcal{M}([0, T])^m$.

To prove the second part of the result, the idea to build the sequence $(u_k)_{k \in \mathbb{N}}$ is to consider a sequence $(v_k)_{k \in \mathbb{N}}$ based on u^* and, in order to make this sequence feasible, to add a sequence of corrective controls $(w_k)_{k \in \mathbb{N}}$.

For every $k \in \mathbb{N}^*$, we consider $I_k = [-\frac{1}{k}, \frac{1}{k}]$ and, for all $t \in [0, T]$,

$$v_k(t) = \sum_{i=1}^r \frac{\alpha_i}{\mu_L((I_k + t_i) \cap [0, T])} \chi_{I_k + t_i}(t),$$

where χ_I is the characteristic function of the set I . Thus, v_k describes, around each impulse $\alpha_i \delta_{t_i}$, a rectangle with an area equal to α_i . When $k \rightarrow \infty$, the height of these rectangles increases while the base decreases. Based on these considerations, it is easy to show that $v_k \xrightarrow{*} u^*$ when $k \rightarrow \infty$.

However, the controls v_k may not steer x^0 to x^1 in time T and, for all $k \in \mathbb{N}$, we define $y^k = \Phi_T(v_k)$. The vague convergence of v_k to u^* ensures that,

$$\lim_{k \rightarrow \infty} y^k = \lim_{k \rightarrow \infty} \Phi_T(v_k) = \Phi_T(u^*) = y.$$

Thus, we want to build a sequence of controls $w_k \in L^1([0, T])^m$ such that the sequence $(u_k)_{k \in \mathbb{N}}$ defined by $u_k = v_k + w_k$ respects the requested properties.

For all $\theta > 0$, we consider the sets

$$\mathcal{R}_\theta = \{ \Phi_\theta(f) : f \in L^1([0, \theta])^m \text{ s.t. } \|f\|_{L^\infty} \leq 1 \}.$$

These sets constitute an increasing sequence of compact sets containing 0 in their interior. Since $\lim_{k \rightarrow \infty} y^k = y$, for every large enough k , the time optimal control problem:

$$\theta_k = \min \theta \quad \left| \begin{array}{l} \Phi_\theta(f) = y - y^k, \\ f \in L^\infty([0, \theta], \mathbb{R}^m) \text{ and } \|f\|_{L^\infty} \leq 1, \end{array} \right.$$

admits a solution. In other words, θ_k is the index of the smallest set \mathcal{R}_θ containing $y - y^k$. The mapping $y - y^k \mapsto \theta_k$ is continuous in 0 and so $\lim_{k \rightarrow \infty} \theta_k = 0$. We note f_k a control associated to θ_k , i.e., such that $f_k \in L^1([0, \theta_k], \mathbb{R}^m)$ and $\Phi_{\theta_k}(f_k) = y - y^k$. We set,

$$w_k(t) = \begin{cases} 0 & \text{if } t \in [0, T - \theta_k), \\ f_k(t - (T - \theta_k)) & \text{if } t \in [T - \theta_k, T]. \end{cases}$$

We directly have that $w_k \xrightarrow{*} 0$ when $k \rightarrow \infty$. Then, for all $k \in \mathbb{N}$, we have $v_k + w_k \in L^1([0, T])^m$ and

$$\begin{aligned} \Phi_T(v_k + w_k) &= \int_0^T e^{(T-t)A} B (v_k(t) + w_k(t)) dt \\ &= \Phi_T(v_k) + \int_0^{\theta_k} e^{(\theta_k-s)A} B f_k(s) ds \\ &= \Phi_T(v_k) + \Phi_{\theta_k}(f_k) = y, \end{aligned}$$

which proves that $v_k + w_k$ is feasible. When $k \rightarrow \infty$, we have, $v_k \xrightarrow{*} u^*$ and $w_k \xrightarrow{*} 0$ so $v_k + w_k \xrightarrow{*} u^*$. \square

Thus, if x^1 is reachable from x^0 in time T with L^1 -controls, then we can always find impulsive feasible controls and so solutions of the L^0 -minimization problem. Proposition 1 seems to be showing a link between the L^1 -norm and the measure norm: let us use this new criterion to refine our research.

4 L^1 -optimal controls

4.1 Presentation of problems (P_1) and (P_M)

We decide to take an interest in the L^1 -minimization, and we consider the problem

$$\begin{aligned} \inf \|u\|_{L^1} &= \int_0^T |u(t)|_1 dt \\ \left| \right. & u \in L^1([0, T])^m \text{ and } \Phi_T(u) = y. \end{aligned} \quad (P_1)$$

To find solutions, this problem is generally studied with a condition of the type $\|u\|_{L^\infty} \leq M$ on the control (see e.g. [1]). In [14,24], authors proved the equivalence between (P_0) and (P_1) under some conditions:

- for the single-input case, it is shown in [24], that this equivalence holds if (A, B) is controllable and A is non-singular;
- for the multi-input case, we can find in [14] that if $A \begin{pmatrix} b_j & Ab_j & \dots & A^{n-1}b_j \end{pmatrix}$ is non-singular for all $j \in \llbracket 1, m \rrbracket$, then the equivalence holds.

Let assume that we are in one of these cases and consider a feasible control $v \in L^\infty([0, T])^m$ and u_M^* a solution of (P_0) and (P_1) , with the additional constraint $\|u_M^*\|_{L^\infty} \leq M$. It is in particular shown in [14, 24] that u_M^* is of bang-off-bang structure. We then have, for $M \geq \|v\|_{L^\infty}$,

$$M \|u_M^*\|_{L^0} = \|u_M^*\|_{L^1} \leq \|v\|_{L^1}.$$

Thus, $\|u_M^*\|_{L^0} \rightarrow 0$ when $M \rightarrow \infty$ (i.e., when we remove the constraint) which leads us to consider impulsive controls.

Let us prove that we can extend the problem (P_1) to $\mathcal{M}([0, T])^m$. We consider the problem,

$$\begin{aligned} \min \|u\|_{\mathcal{M}} \\ \left| \begin{array}{l} u \in \mathcal{M}([0, T])^m \text{ and } \Phi_T(u) = y. \end{array} \right. \end{aligned} \quad (P_{\mathcal{M}})$$

We can notice that this is a minimum and not an infimum because the limit is reached. Indeed, if we set

$$\ell = \inf \|u\|_{\mathcal{M}} \quad \left| \begin{array}{l} u \in \mathcal{M}([0, T])^m \text{ and } \Phi_T(u) = y, \end{array} \right.$$

and consider a sequence $(u_k)_{k \in \mathbb{N}}$ of feasible controls such that $\lim_{k \rightarrow \infty} \|u_k\|_{\mathcal{M}} = \ell$, then $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence of finite Radon measures and Remark 3 ensures that there exist a subsequence $(u_{k_i})_{i \in \mathbb{N}}$ and $u^* \in \mathcal{M}([0, T])^m$ such that $u_{k_i} \xrightarrow{*} u^*$. From that, we can deduce that:

- $\|u^*\|_{\mathcal{M}} \leq \liminf_{k_i \rightarrow \infty} \|u_{k_i}\| = \ell$ (see Remark 3);
- $\lim_{k_i \rightarrow \infty} \Phi_T(u_{k_i}) = \Phi_T(u^*) \in \mathcal{R}$ so u^* is feasible and $\|u^*\|_{\mathcal{M}} \geq \ell$.

Finally, $\|u^*\|_{\mathcal{M}} = \ell$ and, under Assumption 1, problem $(P_{\mathcal{M}})$ always has a minimizer.

The canonical injection of $L^1([0, T])^m$ into $\mathcal{M}([0, T])^m$ directly ensures that,

$$\begin{aligned} \inf \|u\|_{L^1} &\geq \min \|u\|_{\mathcal{M}} \\ \left| \begin{array}{l} u \in L^1([0, T])^m, \\ \Phi_T(u) = y, \end{array} \right. &\left| \begin{array}{l} u \in \mathcal{M}([0, T])^m, \\ \Phi_T(u) = y. \end{array} \right. \end{aligned} \quad (2)$$

As for (P_0) , we would like to check whether a gap occurs or not when we extend (P_1) to $(P_{\mathcal{M}})$. In other words, is relation (2) an equality? If $(P_{\mathcal{M}})$ has an impulsive solution, then we can reason as in the proof of Proposition 1 to answer this question.

4.2 Existence of impulsive solutions of $(P_{\mathcal{M}})$

The following theorem ensures that $(P_{\mathcal{M}})$ can be solved with an impulsive control and allows us to bound the number of Dirac impulses of this solution.

Theorem 5 *Assume that $x^0, x^1 \in \mathbb{R}^n$ and $T > 0$ satisfy Assumption 1. The minimization problem $(P_{\mathcal{M}})$ has a solution which is a linear combination of at most $\sum_{i=1}^m r_i$ Dirac impulses, where r_i is the rank of the Kalman matrix of the pair (A, b_i) .*

To prove this theorem, we first consider the single-input case ($m = 1$) then we break down the problem $(P_{\mathcal{M}})$ into m sub-problems.

4.2.1 Single-input case ($m = 1$)

When $m = 1$, the idea is to consider $y = x^1 - e^{TA}x^0$ as an element of the convex hull of the set \mathcal{C} defined by

$$\begin{aligned} \mathcal{C} = &\left\{ e^{(T-t)A}B : t \in [0, T] \right\} \\ &\cup \left\{ -e^{(T-t)A}B : t \in [0, T] \right\}, \end{aligned} \quad (3)$$

and to use the following lemma.

Lemma 6 *Let \mathcal{S} be a subset of a normed vector space of dimension d and $y \in \text{Conv}(\mathcal{S})$ (where $\text{Conv}(\cdot)$ is the convex hull). There exist $(\alpha_i)_{i \in \llbracket 1, d \rrbracket} \in \mathbb{R}^+$ and $(x_i)_{i \in \llbracket 1, d \rrbracket} \in \mathcal{S}$ such that $y = \sum_{i=1}^d \alpha_i x_i$ with $\sum_{i=1}^d \alpha_i \leq 1$.*

PROOF. The Caratheodory theorem [18, Theorem 4.3.24] ensures that there are $(\beta_i)_{i \in \llbracket 1, d+1 \rrbracket} \in \mathbb{R}^+$ and $(x_i)_{i \in \llbracket 1, d+1 \rrbracket} \in \mathcal{S}$ such that $y = \sum_{i=1}^{d+1} \beta_i x_i$ with $\sum_{i=1}^{d+1} \beta_i \leq 1$. We set $\beta_0 = 1 - \sum_{i=1}^{d+1} \beta_i \geq 0$ and we consider the following system (where $(\gamma_i)_{i \in \llbracket 0, d+1 \rrbracket}$ are the unknowns)

$$\sum_{i=1}^{d+1} \gamma_i x_i = 0 \quad \text{and} \quad \sum_{i=0}^{d+1} \gamma_i = 0.$$

There are $d+1$ equations for $d+2$ unknowns, so there is a non-trivial solution $(\gamma_i)_{i \in \llbracket 0, d+1 \rrbracket}$. In particular, there are (at least) one $\gamma_i > 0$ and one $\gamma_i < 0$ and, by taking the opposite if necessary, we can consider that $\gamma_0 \leq 0$. We consider the set

$$I = \left\{ \frac{\beta_i}{\gamma_i} : i \in \llbracket 1, d+1 \rrbracket \text{ and } \gamma_i > 0 \right\},$$

(which is not empty) and we note i_0 an index such that $\frac{\beta_{i_0}}{\gamma_{i_0}} = \min I$. For all $i \in \llbracket 1, d+1 \rrbracket$, we set $\alpha_i = \beta_i - \frac{\beta_{i_0}}{\gamma_{i_0}} \gamma_i$.

Thus, we have $\alpha_{i_0} = 0$,

$$\begin{aligned} \sum_{i=1}^{d+1} \alpha_i x_i &= \sum_{i=1}^{d+1} \left(\beta_i - \frac{\beta_{i_0}}{\gamma_{i_0}} \gamma_i \right) x_i \\ &= \sum_{i=1}^{d+1} \beta_i x_i - \frac{\beta_{i_0}}{\gamma_{i_0}} \sum_{i=1}^{d+1} \gamma_i x_i = y, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{d+1} \alpha_i &= \sum_{i=1}^{d+1} \left(\beta_i - \frac{\beta_{i_0}}{\gamma_{i_0}} \gamma_i \right) = \sum_{i=1}^{d+1} \beta_i - \frac{\beta_{i_0}}{\gamma_{i_0}} \sum_{i=1}^{d+1} \gamma_i \\ &= 1 - \beta_0 + \frac{\beta_{i_0}}{\gamma_{i_0}} \gamma_0 \leq 1. \end{aligned}$$

To prove the lemma, we have to check if the $(\alpha_i)_{i \in \llbracket 1, d+1 \rrbracket}$ are non-negative. Let $i \in \llbracket 1, d+1 \rrbracket$. There are three distinct cases:

- if $\gamma_i = 0$, then $\alpha_i = \beta_i \geq 0$;
- if $\gamma_i > 0$, $\frac{\beta_i}{\gamma_i} \geq \frac{\beta_{i_0}}{\gamma_{i_0}}$ by definition of i_0 and $\alpha_i \geq 0$;
- if $\gamma_i < 0$, then $\frac{\beta_i}{\gamma_i} \leq 0 \leq \frac{\beta_{i_0}}{\gamma_{i_0}}$ and $\alpha_i \geq 0$. \square

PROOF. (Theorem 5 ($m = 1$)) Let $u \in \mathcal{M}([0, T])$ such that $\Phi_T(u) = y$. Since we know the existence of a solution of $(P_{\mathcal{M}})$, in order to prove this proposition, we just have to build a control $v \in \mathcal{M}([0, T])$ which is a linear combination of at most $r = \dim \mathcal{R}$ Dirac impulses and such that $\Phi_T(v) = y$ and $\|v\|_{\mathcal{M}} \leq \|u\|_{\mathcal{M}}$.

We consider the set $\mathcal{C} \subset \mathcal{R}$ defined in (3) and we assume that $u \neq 0$ (otherwise, there is nothing to prove). Thus, we have

$$\begin{aligned} y &= \int_{[0, T]} e^{(T-t)A} B du(t) \\ &= \int_{[0, T]} e^{(T-t)A} B du^+(t) - \int_{[0, T]} e^{(T-t)A} B du^-(t), \end{aligned}$$

where, for every measurable set E , $u^\pm(E) = (|u(E)| \pm u(E))/2$. Hence, $y/\|u\|_{\mathcal{M}} \in \text{Conv}(\mathcal{C})$ and the Lemma 6 ensures that there are $(\alpha_i)_{i \in \llbracket 1, r \rrbracket} \in \mathbb{R}^+$, $(t_i)_{i \in \llbracket 1, r \rrbracket} \in [0, T]$ and $k \in \llbracket 1, r \rrbracket$ such that $\sum_{i=1}^r \alpha_i \leq 1$ and

$$\frac{1}{\|u\|_{\mathcal{M}}} y = \sum_{i=1}^k \alpha_i e^{(T-t_i)A} B - \sum_{i=k+1}^r \alpha_i e^{(T-t_i)A} B.$$

Setting $\varsigma_i = \alpha_i$ for $i \in \llbracket 1, k \rrbracket$ and $\varsigma_i = -\alpha_i$ for $i \in \llbracket k+1, r \rrbracket$, we have, $y = \|u\|_{\mathcal{M}} \sum_{i=1}^r \varsigma_i e^{(T-t_i)A} B$, with $\sum_{i=1}^r |\varsigma_i| \leq 1$. Finally, the control $v = \|u\|_{\mathcal{M}} \sum_{i=1}^r \varsigma_i \delta_{t_i}$ verifies $\Phi_T(v) = y$ and $\|v\|_{\mathcal{M}} = \|u\|_{\mathcal{M}} \sum_{i=1}^r |\varsigma_i| \leq \|u\|_{\mathcal{M}}$, which concludes this proof. \square

4.2.2 Multi-input case ($m \geq 1$)

PROOF. (Theorem 5 (general case)) Let $u \in \mathcal{M}([0, T])^m$ such that $\Phi_T(u) = y$. Just like the previous case, to prove this proposition, we have to build a control $v \in \mathcal{M}([0, T])^m$ which is a linear combination of at most $\sum_{i=1}^m r_i$ Dirac impulses and such that $\Phi_T(v) = y$ and $\|v\|_{\mathcal{M}} \leq \|u\|_{\mathcal{M}}$. For $i \in \llbracket 1, m \rrbracket$, we note u_i the i -th component of u . As before, we have

$$y = \int_{[0, T]} e^{(T-t)A} B du(t) = \sum_{i=1}^m \int_{[0, T]} e^{(T-t)A} b_i du_i(t),$$

and we set $y^i = \int_{[0, T]} e^{(T-t)A} b_i du_i(t)$. Then, we consider the problem

$$\begin{aligned} \min \|v_i\|_{\mathcal{M}} \\ \left| \begin{array}{l} v_i \in \mathcal{M}([0, T]) \text{ and } y^i = \int_{[0, T]} e^{(T-t)A} b_i dv_i(t). \end{array} \right. \end{aligned} \quad (P_{\mathcal{M}, i})$$

For every $i \in \llbracket 1, m \rrbracket$, y^i is reachable from 0 in time T (thanks to u_i) and Theorem 5 (case $m = 1$) ensures that there exist $(\alpha_j)_{j \in \llbracket 1, r_i \rrbracket} \in \mathbb{R}$ and $(t_j)_{j \in \llbracket 1, r_i \rrbracket} \in [0, T]$ such that the control $v_i = \sum_{j=1}^{r_i} \alpha_j \delta_{t_j}$ is a solution of $(P_{\mathcal{M}, i})$.

We go back to the problem $(P_{\mathcal{M}})$ and we consider the control v with values in \mathbb{R}^m whose components are the $(v_i)_{i \in \llbracket 1, m \rrbracket}$ issued of the m sub-problems $(P_{\mathcal{M}, i})$. By noting $R = \sum_{i=1}^m r_i$, thanks to the expression of the v_i , there are $(\gamma_j)_{j \in \llbracket 1, R \rrbracket} \in \mathbb{R}^m$ and times $(t_j)_{j \in \llbracket 1, R \rrbracket} \in [0, T]$ such that $v = \sum_{j=1}^R \gamma_j \delta_{t_j}$. Furthermore, we have

$$y = \sum_{i=1}^m y^i = \sum_{i=1}^m \int_{[0, T]} e^{(T-t)A} b_i dv_i(t) = \Phi_T(v).$$

Finally, for all $i \in \llbracket 1, m \rrbracket$, v_i is a solution of $(P_{\mathcal{M}, i})$ so $\|v\|_{\mathcal{M}} = \sum_{i=1}^m \|v_i\|_{\mathcal{M}} \leq \sum_{i=1}^m \|u_i\|_{\mathcal{M}} = \|u\|_{\mathcal{M}}$, which concludes this proof. \square

4.3 No gap between (P_1) and $(P_{\mathcal{M}})$

Now we know the existence of an impulsive solution u^* of $(P_{\mathcal{M}})$, we can build sequences (v_k) and (w_k) as in the proof of Proposition 1 to obtain:

- $(v_k)_{k \in \mathbb{N}^*} \in L^1([0, T])^m$;
- $(w_k)_{k \in \mathbb{N}^*} \in L^1([0, T])^m$;
- $v_k \xrightarrow{*} u^*$ and $w_k \xrightarrow{*} 0$;
- $\|v_k\|_{L^0} \rightarrow 0$ and $\|w_k\|_{L^0} \rightarrow 0$;
- $\|v_k\|_{L^1} \rightarrow \|u^*\|_{\mathcal{M}}$ and $\|w_k\|_{L^1} \rightarrow 0$.

Thus, we deduce this following proposition, which ensures that there is no gap between (P_1) and $(P_{\mathcal{M}})$.

Proposition 7 *We have*

$$\inf \|u\|_{L^1} = \min \|u\|_{\mathcal{M}} \quad (4)$$

$$\left| \begin{array}{l} u \in L^1([0, T])^m, \\ \Phi_T(u) = y, \end{array} \right| \quad \left| \begin{array}{l} u \in \mathcal{M}([0, T])^m, \\ \Phi_T(u) = y. \end{array} \right|$$

Furthermore, if u^* is an impulsive solution of $(P_{\mathcal{M}})$, then there is a sequence of feasible controls $(u_k) \in L^1([0, M], \mathbb{R}^m)$ such that

$$u_k \xrightarrow{*} u^* \quad \text{and} \quad \lim_{k \rightarrow \infty} \|u_k\|_{L^1} = \|u^*\|_{\mathcal{M}}.$$

Remark 8 *There is no guarantee that this impulsive control will be unique or give a better result than continuous controls. For example, we consider $\dot{x}(t) = u(t)$ with $n = m = 1$. We can prove that the set of L^1 -minimizers for this system is composed of all elements u of $\mathcal{M}([0, T])$ such that u is of constant sign and $u([0, T]) = \int_{[0, T]} du = x^1 - x^0$. Thus, among these minimizers, we can find impulsive controls (with a potentially infinite number of impulses), C^∞ controls, Cantor functions or even a mix of these varieties.*

5 Use of Pontryagin's maximum principle

In this section, we search for sufficient conditions to ensure that any minimizer of $(P_{\mathcal{M}})$ is necessarily of purely impulsive form by using the Pontryagin's maximum principle (PMP) (see e.g. [2,15,26]) on the problem $(P_{\mathcal{M}})$.

Remark 9 *Here, we do not necessarily have the controllability of system (S) but we can rewrite it according to a basis of \mathbb{R}^n adapted to the accessible set \mathcal{R} (see [30, Theorem 3.11]). By restraining to \mathcal{R} , we obtain a controllable system with a state in dimension r . The $n - r$ remaining components are not affected by the control and, thanks to Assumption 1, they will follow the trajectory imposed by this last system. So, even if that means working in dimension r , in the following, we consider that the pair (A, B) is controllable.*

5.1 Change of variable

By its "almost all" aspect, the PMP does not allow us to obtain optimal impulsive controls. Thus, we use the same change of variable as in [8] or [21, p. 31] to "reparametrize" the time t and use the PMP.

We remind that for all $u \in \mathbb{R}^m$, $|u|_1 = \sum_{i=1}^m |u_i|$ where u_i are the components of u . The idea is to set:

- $s(t) = t + \int_0^t |u(\tau)|_1 d\tau$ with $S = s(T)$;
- $y(s(t)) = x(t)$ for all $t \in [0, T]$;
- $w_0(s(t)) = \frac{1}{1+|u(t)|_1} = \frac{1}{s'(t)}$;

- $w(s(t)) = \frac{1}{1+|u(t)|_1} u(t) = \frac{1}{s'(t)} u(t)$;
- $\dot{y}_0 = w_0$ with $y_0(0) = 0$.

Thus, for all $s \in [0, S]$, we have $w_0(s) + |w(s)|_1 = 1$ with $w_0(s) \in [0, 1]$ and all components of $w(s)$ in $[-1, 1]$. Moreover, we have,

$$y_0(s) = \int_0^s w_0(\sigma) d\sigma = \int_0^s 1 d\tau = t,$$

which particularly ensures that $w_0 \not\equiv 0$ on $[0, S]$. Finally, we can rewrite the system (S) in the form

$$\begin{cases} \dot{y}(s) = w_0(s)Ay(s) + Bw(s), \\ \dot{y}_0(s) = w_0(s). \end{cases} \quad (5)$$

We want to minimize the quantity $\int_0^t |u(t)|_1 dt = S - T$ with T given so finally, we are interested in

$$\min S \quad \left| \begin{array}{l} y(0) = x^0 \text{ and } y(S) = x^1, \\ y_0(0) = 0 \text{ and } y_0(S) = T, \\ (w_0, w) \in L^\infty([0, S]) \times L^\infty([0, S])^m, \\ \forall s \in [0, S], w_0(s) \geq 0 \text{ and } w_0(s) + |w(s)|_1 = 1, \\ (y_0, y) \text{ solution of (5)}, \end{array} \right. \quad (6)$$

which is a problem on which we can apply the PMP.

Remark 10 *Let us represent this change of variable in the case of an impulsive control $u = \alpha \delta_\tau$ for $\alpha \in \mathbb{R}^m$ and $\tau \in [0, T]$. From the foregoing, we have*

$$s(t) = \begin{cases} t & \text{if } 0 \leq t \leq \tau, \\ t + |\alpha|_1 & \text{if } \tau < t \leq T. \end{cases}$$

We could have a problem of continuity to use this change of variable, but we can show that the trajectory $x(\cdot)$ of (S) corresponds to the solution $y(\cdot)$ of the new problem (see [21, p. 30-31] for the case $m = 1$).

Below, for $m = 1$, we have drawn the controls $t \mapsto u(t)$, $s \mapsto w_0(s)$ and $s \mapsto w(s)$. We can notice that this change of variable "freezes" the impulse's moment (during a length equal to the intensity of the impulse), see Fig. 1.

5.2 First deductions of the PMP

The objective of this paragraph is to prove the following proposition, which we will interpret after.

Proposition 11 *Assume that (w_0, w, y_0, y) is a minimizer of (6). There exists $q : [0, S] \rightarrow \mathbb{R}^n$ absolutely continuous and such that for almost all $s \in [0, S]$:*

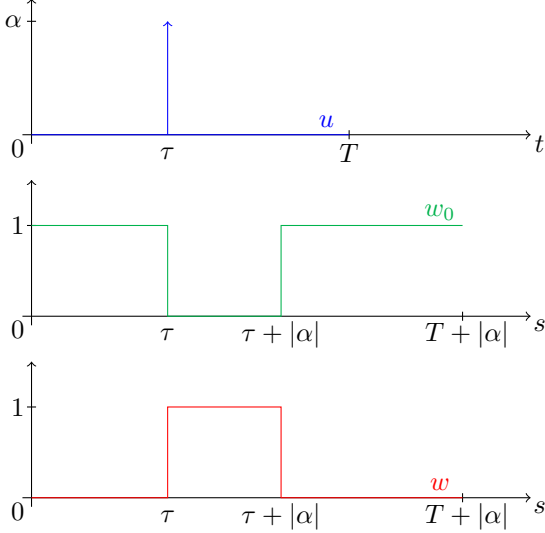


Fig. 1. Illustration of the change of variable given in Section 5.1, see Remark 10.

- (i) $\dot{q}(s) = -w_0(s)A^\top q(s)$;
- (ii) $|B^\top q(s)|_\infty \leq 1$;
- (iii) if $w_0(s) = 0$, then $|B^\top q(s)|_\infty = 1$;
- (iv) if $|B^\top q(s)|_\infty < 1$, then $w_0(s) = 1$.

PROOF. (Proposition 11, item (i)) The PMP ensures that if the control (w_0, w) associated to the trajectory $y(\cdot)$ is optimal on $[0, S]$, then there exist two absolutely continuous functions $q : [0, S] \rightarrow \mathbb{R}^n$ and $q_0 : [0, S] \rightarrow \mathbb{R}$ and $c \leq 0$ such that (q, q_0, c) is non-trivial and for almost all $s \in [0, S]$

$$\begin{aligned} \dot{q}(s) &= -\frac{\partial H}{\partial y}(y(s), y_0(s), q(s), q_0(s), c, w_0(s), w(s)), \\ \dot{q}_0(s) &= -\frac{\partial H}{\partial y_0}(y(s), y_0(s), q(s), q_0(s), c, w_0(s), w(s)), \end{aligned}$$

where the Hamiltonian H is defined by

$$H(y, y_0, q, q_0, c, w_0, w) = \langle q, w_0 A y + B w \rangle + q_0 w_0 + c.$$

It follows that q_0 is constant and q satisfies item (i). \square

In order to prove items (ii), (iii) and (iv) of Proposition 11, we need the following two lemmas. In these lemmas, we consider the assumptions of Proposition 11, and use the first deductions made in the proof of item (i) of Proposition 11.

Lemma 12 *For almost all $s \in [0, S]$ we have*

$$w_0(s) = \begin{cases} 0 & \text{if } \langle q(s), A y(s) \rangle + q_0 < |B^\top q(s)|_\infty, \\ 1 & \text{if } \langle q(s), A y(s) \rangle + q_0 > |B^\top q(s)|_\infty. \end{cases}$$

PROOF. When $(y, y_0, q, q_0, c, w_0, w)$ is optimal, the maximization condition of the PMP ensures¹ that for almost all $s \in [0, S]$,

$$\begin{aligned} H(y, y_0, q, q_0, c, w_0, w) &= \max_{\substack{0 \leq v_0 \leq 1 \\ |v|_1 = 1 - v_0}} H(y, y_0, q, q_0, c, v_0, v) \\ &= \max_{0 \leq v_0 \leq 1} \max_{|v|_1 = 1 - v_0} H(y, y_0, q, q_0, c, v_0, v). \end{aligned}$$

So we must have $\langle B^\top q, w \rangle = (1 - w_0) |B^\top q|_\infty$ and the Hamiltonian becomes,

$$\begin{aligned} H(y, y_0, q, q_0, c, w_0, w) &= \max_{0 \leq v_0 \leq 1} (v_0 (\langle q, A y \rangle + q_0) + (1 - v_0) |B^\top q|_\infty + c) \end{aligned} \quad (7)$$

which concludes the proof. \square

Lemma 13 *We have $c \neq 0$.*

Thus, up to a rescaling, we can assume that $c = -1$, and, according to (7), $q(\cdot)$ and q_0 satisfy for almost all $s \in [0, S]$,

$$\begin{aligned} H(y, y_0, q, q_0, -1, w_0, w) &= w_0 (\langle q, A y \rangle + q_0) + (1 - w_0) |B^\top q|_\infty - 1 = 0. \end{aligned}$$

PROOF. The final time S is not fixed, and the system is autonomous, so for almost all $s \in [0, S]$, $H(y(s), y_0(s), q(s), c, w_0(s)) = 0$. Let us prove Lemma 13 by contraposition: we assume that $c = 0$. Thanks to Lemma 12, we can distinguish three cases:

- if $\langle q, A y \rangle + q_0 = |B^\top q|_\infty$, we have $H = |B^\top q|_\infty = 0$, that is to say $B^\top q = 0$;
- if $\langle q, A y \rangle + q_0 < |B^\top q|_\infty$, we have $w_0 = 0$ and so $H = |B^\top q| = 0$, that is to say $B^\top q = 0$;
- if $\langle q, A y \rangle + q_0 > |B^\top q|_\infty$, we have $w_0 = 1$ so $H = \langle q, A y \rangle + q_0 = 0$ and, according to our assumption, we must have $|B^\top q| < 0$ which is a contradiction.

Finally, we must have $B^\top q(s) = 0$ for almost all $s \in [0, S]$ and we use the same reasoning as in [21, p. 34] by considering the set E defined by

$$E = \{s \in [0, S] : B^\top q(s) = 0\} \cap \{s \in [0, S] : w_0(s) > 0\}.$$

As $w_0 \not\equiv 0$, $\mu_L(E) > 0$ and, by noting E^d the set of density points of E , we have $\mu_L(E^d) = \mu_L(E)$. Then, we consider $s \in E^d$ and a sequence $(s_k)_{k \in \mathbb{N}} \in E$ which converges to s such that $|[s_k, s] \cap E| > 0$ for all $k \in \mathbb{N}$.

¹ The time s is omitted in the expression of H to make computations clearer.

Let $k \in \mathbb{N}$. By definition, we have $B^\top q(s) = B^\top q(s_k) = 0$ and, since $B^\top q \in W^{1,\infty}([0, S])$, we obtain that

$$\begin{aligned} 0 &= B^\top q(s) - B^\top q(s_k) = \int_{s_k}^s B^\top \dot{q}(\sigma) d\sigma \\ &= \int_{s_k}^s -w_0(\sigma) B^\top A^\top q(\sigma) d\sigma, \end{aligned}$$

according to the (already proved) item (i) of the Proposition 11. Let $i \in \llbracket 1, m \rrbracket$. We note $a_i(\sigma)$ the i -th component of $B^\top A^\top q(\sigma) \in \mathbb{R}^m$. Thus, we have

$$\int_{s_k}^s w_0(\sigma) a_i(\sigma) d\sigma = 0.$$

But $w_0 \not\equiv 0$ with $w_0 \geq 0$, so either $a_i(\sigma)$ is identically equal to 0 or its sign changes on $[s_k, s]$. In all cases, there exists $\sigma_k \in [s_k, s]$ such that $a_i(\sigma_k) = 0$. When $k \rightarrow \infty$, we have $\sigma_k \rightarrow s$ and, by continuity, $a_i(s) = 0$. We can generalize for all i and obtain that $B^\top A^\top q(s) = 0$. By repeating this operation, we show that $B^\top (A^\top)^k q(s) = 0$ for all $s \in E^d$ and all $k \in \llbracket 0, n-1 \rrbracket$. Thus, the Kalman condition due to the controllability of the system (see Remark 9) ensures that we have $q(s) = 0$ for all $s \in E^d$. Then, item (i) of Proposition 11 ensures that

$$q(s) = e^{\left(\int_s^\sigma w_0(\tau) d\tau\right) A^\top} q(\sigma),$$

and, by choosing $\sigma \in E^d$, we finally obtain that $q \equiv 0$. Then, the Hamiltonian is $H = w_0 q_0 = 0$ and, since $w_0 \not\equiv 0$, we have $q_0 = 0$. But (q, q_0, c) must be non-trivial, and we obtain the desired contradiction. \square

PROOF. (Proposition 11, items (ii) to (iv))

Lemma 13 ensures that for almost all $s \in [0, S]$, we have

$$w_0 (\langle q, Ay \rangle + q_0) + (1 - w_0) |B^\top q|_\infty = 1.$$

Using Lemma 12, we can distinguish three cases:

- if $\langle q(s), Ay(s) \rangle + q_0 < |B^\top q(s)|_\infty$, then $w_0(s) = 0$ and $H = |B^\top q|_\infty - 1 = 0$, so $|B^\top q(s)|_\infty = 1$;
- if $\langle q(s), Ay(s) \rangle + q_0 = |B^\top q(s)|_\infty$, then $H = |B^\top q|_\infty - 1 = 0$, so $|B^\top q(s)|_\infty = 1$;
- if $\langle q(s), Ay(s) \rangle + q_0 > |B^\top q(s)|_\infty$, then $w_0(s) = 1$ and $H = \langle q(s), Ay(s) \rangle + q_0 - 1 = 0$, so $|B^\top q(s)|_\infty < \langle q(s), Ay(s) \rangle + q_0 = 1$.

We deduce from it that for almost all $s \in [0, S]$, we have $|B^\top q(s)|_\infty \leq 1$ (item (ii)). Furthermore, we obtain the two following equivalences:

$$\begin{aligned} \langle q(s), Ay(s) \rangle + q_0 \leq |B^\top q(s)|_\infty &\iff |B^\top q(s)|_\infty = 1, \\ \langle q(s), Ay(s) \rangle + q_0 > |B^\top q(s)|_\infty &\iff |B^\top q(s)|_\infty \neq 1. \end{aligned}$$

Now, we have,

$$\begin{aligned} E_0 &= \{s \in [0, S] : w_0(s) = 0\} \\ &\subset \{s \in [0, S] : |B^\top q(s)|_\infty = 1\}, \quad (\text{item (iii)}) \\ E_1 &= \{s \in [0, S] : w_0(s) = 1\} \\ &\supset \{s \in [0, S] : |B^\top q(s)|_\infty \neq 1\}, \quad (\text{item (iv)}) \end{aligned}$$

which concludes the proof. \square

5.3 Sufficient conditions

We recall that our objective is to find sufficient conditions in order to obtain an impulsive control by using the PMP as previously. According to Remark 10, it seems that we have an impulsive control when $w_0(s) \in \{0, 1\}$ for almost all $s \in [0, T]$ and so, by using items (iii) and (iv) of Proposition 11, we would like to avoid the case where $w_0(s) > 0$ and $|B^\top q(s)|_\infty = 1$ on an interval of $[0, S]$. Indeed, we have the following lemma.

Lemma 14 *By using the notations and assumptions of Proposition 11, if, for almost all $s \in [0, S]$, $|B^\top q(s)|_\infty = 1$ involves $w_0(s) = 0$, then the L^1 -optimal control found by the PMP (in the previous part) is a finite linear combination of Dirac impulses.*

PROOF. We assume that, for almost all $s \in [0, S]$, $|B^\top q(s)|_\infty = 1$ involves $w_0(s) = 0$. Thus, items (ii), (iii) and (iv) of Proposition 11 ensure that for almost all $s \in [0, S]$,

$$w_0(s) = \begin{cases} 0 & \text{if } |B^\top q(s)|_\infty = 1, \\ 1 & \text{if } |B^\top q(s)|_\infty \neq 1. \end{cases}$$

Item (i) of Proposition 11 and the expression of y_0 (obtained during the change of variable) allow us to write that, for almost all $t \in [0, T]$,

$$\begin{aligned} q(s(t)) &= \exp\left(\int_{s(t)}^S w_0(\sigma) d\sigma A^\top\right) q(S) \\ &= \exp((y_0(S) - y_0(s(t))) A^\top) q(S) \\ &= \exp((T - t) A^\top) q(S) =: p(t). \end{aligned}$$

Then, for all $i \in \llbracket 1, m \rrbracket$, we consider the following functions on $[0, T]$

$$f_i(t) = b_i^\top p(t) - 1 \quad \text{and} \quad g_i(t) = b_i^\top p(t) + 1,$$

and the set

$$\begin{aligned}\mathcal{T} &= \{t \in [0, T] : w_0(s(t)) = 0\} \\ &= \{t \in [0, T] : |B^\top q(s(t))|_\infty = 1\} \\ &= \{t \in [0, T] : \exists i \in \llbracket 1, m \rrbracket \text{ s.t. } |b_i^\top q(s(t))| = 1\} \\ &= \{t \in [0, T] : \exists i \in \llbracket 1, m \rrbracket \text{ s.t. } f_i(t) = 0 \text{ or } g_i(t) = 0\}.\end{aligned}$$

None of the functions f_i and g_i can be identically null, since otherwise, we would have $\mathcal{T} = [0, T]$ and $w_0 \equiv 0$, which is not. Moreover, these functions are clearly analytic so they vanish a finite number of times and two cases can occur:

- if, for all $i \in \llbracket 1, m \rrbracket$ and all $t \in [0, T]$, $f_i(t) \neq 0$ and $g_i(t) \neq 0$, then for all $s \in [0, S]$, we have $|B^\top q(s)|_\infty \neq 1$, that is to say $w_0(s) = 1$ and so $u \equiv 0$;
- otherwise, there are $N \in \mathbb{N}^*$ and $\tau_1, \dots, \tau_N \in [0, T]$ such that $\mathcal{T} = \{\tau_1, \dots, \tau_N\}$. We directly deduce that

$$\{s \in [0, S] : w_0(s) = 0\} = y_0^{-1}(\{\tau_1, \dots, \tau_N\}).$$

Since y_0 is continuous and non-decreasing, we obtain that this set is a finite union of closed intervals, which concludes the proof. \square

Remark 15 *If all eigenvalues of A are real, as in the proof of [21, Proposition 5.2.5], by using [17, Exercice 16], we can show that, for $i \in \llbracket 1, m \rrbracket$, each f_i and each g_i has at most n zeros (counted with multiplicity) and we have $N \leq 2nm$.*

To find sufficient conditions to avoid the case where $w_0(s) > 0$ and $|B^\top q(s)|_\infty = 1$ on an interval of $[0, S]$, we reason as in [5, Proof of Theorem 6-13] by assuming that

$$\mu_L(\{s \in [0, S] : w_0(s) > 0 \text{ and } |B^\top q(s)|_\infty = 1\}) > 0.$$

For all $i \in \llbracket 1, m \rrbracket$, we consider the sets

$$E_i = \{s \in [0, S] : w_0(s) > 0 \text{ and } |b_i^\top q(s)|_\infty = 1\}.$$

Thus, we have $E = \bigcup_{i=1}^m E_i$ and $\sum_{i=1}^m \mu_L(E_i) \geq \mu_L(E) > 0$. Thus, there exists $i \in \llbracket 1, m \rrbracket$ such that $\mu_L(E_i) > 0$ and we can write $E_i = E_i^+ \cup E_i^-$ where

$$\begin{aligned}E_i^+ &= \{s \in [0, S] : w_0(s) > 0 \text{ and } b_i^\top q(s) = 1\}, \\ E_i^- &= \{s \in [0, S] : w_0(s) > 0 \text{ and } b_i^\top q(s) = -1\}.\end{aligned}$$

Hence, we have $\mu_L(E_i^+) > 0$ or $\mu_L(E_i^-) > 0$ and we assume that $\mu_L(E_i^+) > 0$ (the other case is similar). We can differentiate the function $s \mapsto b_i^\top q(s)$ on the set E_i^d of density points of E_i^+ and, by reasoning as in the proof

of Lemma 13, we deduce that, for all $s \in E_i^d$ and all $k \in \mathbb{N}^*$, $b_i^\top (A^k)^\top q(s) = 0$. This implies

$$C_i^\top A^\top q(s) = 0 \quad \forall s \in E_i^d, \quad (8)$$

where, as in [5, Theorem 6-13], we have considered the matrix $C_i = \begin{pmatrix} b_i & Ab_i & \dots & A^{n-1}b_i \end{pmatrix}$.

But, by definition of E_i , $q(s) \neq 0$ and, for (8) to hold, it is necessary that $C_i^\top A^\top$ be a singular matrix. Thus, we must have the relation

$$\det(C_i^\top A^\top) = \det(C_i^\top) \det(A^\top) = 0. \quad (9)$$

Before formulating a theorem about the desired sufficient conditions, we recall the definition of a normal linear system given by [5, Definition 6-15].

Definition 16 (Normality)² *The system (\mathcal{S}) (or the pair (A, B)) is normal if, for all $i \in \llbracket 1, m \rrbracket$, the system $\dot{x}(t) = Ax(t) + b_i u(t)$ is controllable.*

Thus, if (A, B) is normal and A is non-singular, then (9) cannot hold, and we obtain the following theorem.

Theorem 17 *If (A, B) is normal and A is non-singular, then any minimizer of $(P_{\mathcal{M}})$, for the system (\mathcal{S}) , is a finite linear combination of Dirac impulses.*

Corollary 18 *In the single-input case ($m = 1$), if (A, B) is controllable and A is non-singular³, then any minimizer of $(P_{\mathcal{M}})$, for the system (\mathcal{S}) , is a finite linear combination of Dirac impulses.*

6 Characterization for $n = 2$ and $m = 1$

In Section 3, we have proved Theorem 5 by using convex combinations. Here, we assume that $n = 2$ and $m = 1$ and we want to study this convex set in order to characterize the moments of the impulses. For all $t_1, t_2 \in [0, T]$, we set $\mathcal{C}([t_1, t_2]) = \mathcal{C}_+([t_1, t_2]) \cup \mathcal{C}_-([t_1, t_2])$, where

$$\mathcal{C}_+([t_1, t_2]) = \left\{ e^{(T-t)A} B : t \in [t_1, t_2] \right\} \subset \mathbb{R}^2,$$

and $\mathcal{C}_-([t_1, t_2])$, its symmetrical to the origin. We simply note $\mathcal{C}_+ = \mathcal{C}_+([0, T])$ (idem for \mathcal{C}_- and \mathcal{C}). Thus, the convex hull of \mathcal{C} is the set of all $y \in \mathbb{R}^2$ which can be reached from the origin in time T with an impulsive control u such that $\|u\|_{\mathcal{M}} \leq 1$. Obviously, we can adapt this maximum thanks to a rescaling.

² Normality is not only a notion concerning linear systems. One can find its definition in [25, Definition 7].

³ In [23], for $B \in \mathbb{R}^n$, the pair (A, B) is said *non-singular* when (A, B) is controllable and A is non-singular.

We recall that every point of a convex set can be expressed as a convex combination of points on its boundary. Thus, we want to study $\Delta = \mathcal{C} \cap \partial \text{Conv}(\mathcal{C})$.

Before giving a result, let us interpret Δ and these notations through an example: we consider $T = \frac{\pi}{2}$,

$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. On Fig. 2, we have drawn

the different sets introduced in this section (on the left, arrows follow $t \in [0, T]$). It would seem that there exists $\tau \in (0, T)$ such that $\Delta = \mathcal{C}([0, \tau])$. Thus, any point $y \in \text{Conv}(\mathcal{C})$ can be reached thanks to an impulsive control u consisting of (at most) two impulses in $[0, \tau]$ and such that $\|u\|_{\mathcal{M}} \leq 1$.

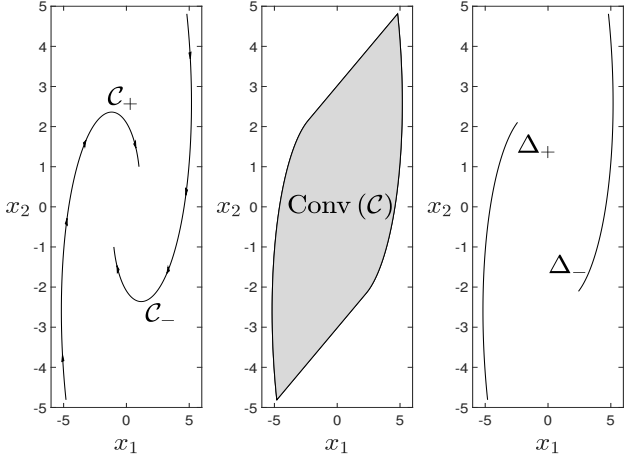


Fig. 2. The sets $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ (on the left), $\text{Conv}(\mathcal{C})$ (in the middle) and $\Delta = \Delta_+ \cup \Delta_-$ (on the right).

Thanks to convex combinations, for y given, finding a such control becomes a geometry game. The appearance of \mathcal{C} (and so of Δ) directly depends on the eigenvalues of A , and we obtain the same conclusions by change of basis. Thus, we distinguish cases according to the Jordan matrix similar to A . We do not give an extensive analytic demonstration, but just some ideas to see the link between the eigenvalues of A and a characterization of the moments of the impulses.

If A is singular, we can show that \mathcal{C}_+ and \mathcal{C}_- are segments, symmetrical to the origin. Thus, $\text{Conv}(\mathcal{C})$ is a parallelogram (possibly flat) and $\Delta = \mathcal{C}$ (see Fig. 3).

In the following, we assume that $0 \notin \sigma(A)$ (where $\sigma(A)$ denotes the spectrum of A). Moreover, the substitution $s = T - t$ allows us to consider fewer cases by assuming that A has at least one eigenvalue with nonnegative real part: it will be sufficient to “reverse” the conclusions.

Two distinct real eigenvalues. Thus, we consider

$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1 > 0$ and $\lambda_2 \in \mathbb{R}^*$. The controllability of (A, B) ensures that the two components of B are not null. We can show that \mathcal{C}_+ is the representation of a function whose study leads to two cases:

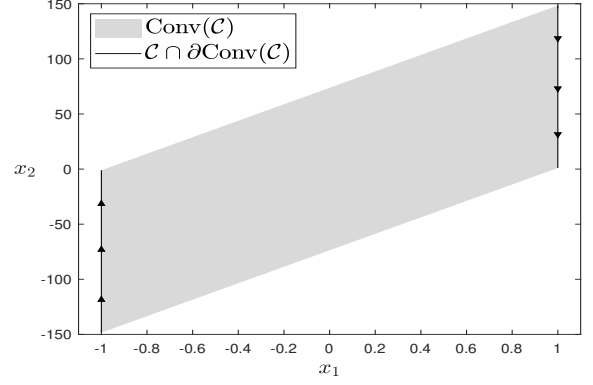


Fig. 3. A singular.

- if $\lambda_2 > 0$, then there exists $\tau \in [0, T]$ such that $\Delta = \mathcal{C}([0, \tau])$ (see Fig. 4);
- if $\lambda_2 < 0$, then Δ is reduced to four points corresponding to the times $t = 0$ and $t = T$ (see Fig. 5).

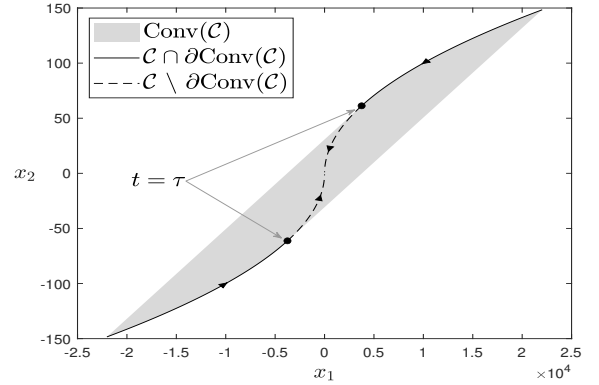


Fig. 4. Two distinct positive eigenvalues.

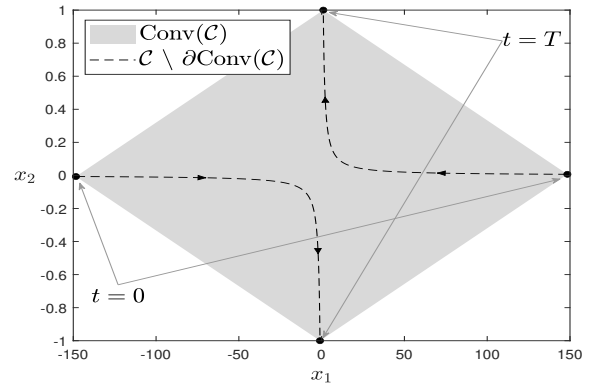


Fig. 5. Two real eigenvalues with opposite signs.

One double real eigenvalue. Thus, we consider $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda > 0$. The controllability of (A, B) ensures that the second component of B is not null. By studying

the function represented by \mathcal{C}_+ in the plan (x_2, x_1) , we obtain that there exists $\tau \in [0, T]$ such that $\Delta = \mathcal{C}([0, \tau])$ (see Fig. 6);

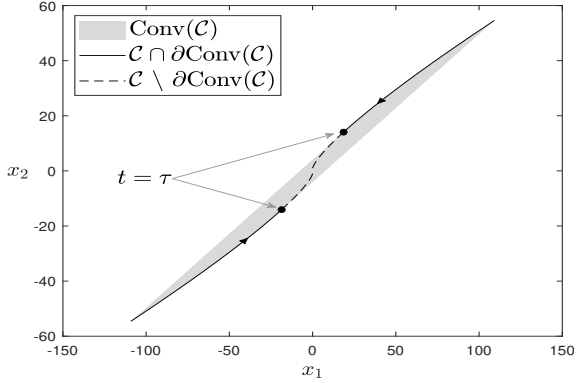


Fig. 6. One double positive eigenvalue

Two complex conjugate eigenvalues. Thus, we consider $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ with $\alpha \geq 0$ and $\beta \in \mathbb{R}^{+*}$. The controllability of (A, B) ensures that $B = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^\top \neq 0$. The equations of \mathcal{C}_+ are given by

$$\begin{cases} x_1(t) = e^{\alpha(T-t)} (b_1 \cos(\beta(T-t)) + b_2 \sin(\beta(T-t))), \\ x_2(t) = e^{\alpha(T-t)} (b_2 \cos(\beta(T-t)) - b_1 \sin(\beta(T-t))). \end{cases}$$

Thus, for all $t \in [0, T]$, the point $(x_1(t), x_2(t)) \in \mathcal{C}_+$ is on the circle of center $(0, 0)$ and radius $e^{\alpha(T-t)} \sqrt{b_1^2 + b_2^2}$.

- if $\alpha = 0$, then \mathcal{C} is included in a circle and $\Delta = \mathcal{C}$ (see Fig. 7);
- if $\alpha > 0$, then \mathcal{C}_+ is a spiral which get closer to the origin and there exists $\tau \in [0, T]$ such that $\Delta = \mathcal{C}([0, \tau])$ (see Fig. 8);

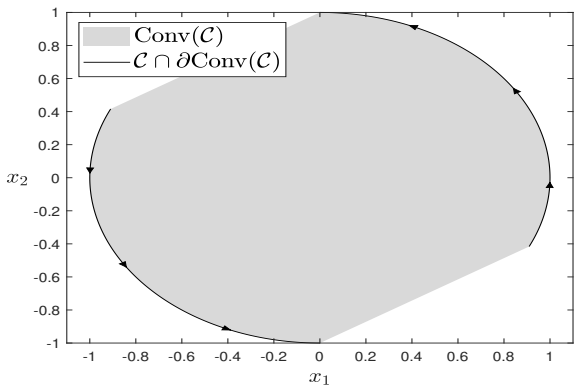


Fig. 7. Complex conjugate eigenvalues with null real part.

Conclusions. Table 1 summarizes the different results.

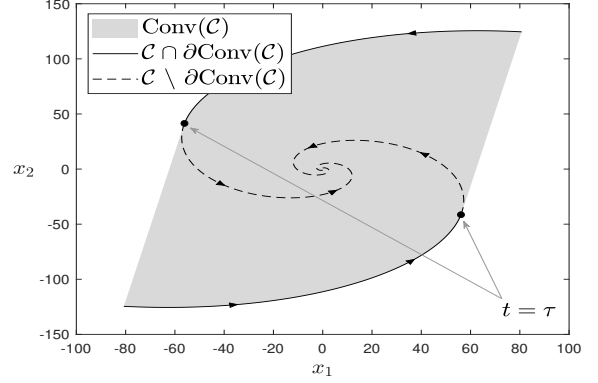


Fig. 8. Complex conjugate eigenvalues with positive real part.

Table 1

Summary of the results obtained in Section 6.

Eigenvalues of A		Description of Δ
A singular ($0 \in \sigma(A)$)		$\Delta = \mathcal{C}([0, T])$
Two distinct real eigenvalues	positive	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([0, \tau])$
	negative	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([\tau, T])$
	of different signs	$\Delta = \mathcal{C}(\{0\}) \cup \mathcal{C}(\{T\})$
One double real eigenvalue	positive	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([0, \tau])$
	negative	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([\tau, T])$
Two complex conjugate eigenvalues	positive real part	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([0, \tau])$
	negative real part	$\exists \tau \in [0, T]$ s.t. $\Delta = \mathcal{C}([\tau, T])$
	zero real part	$\Delta = \mathcal{C}([0, T])$

7 Numerical computations

In this section, we propose an algorithm converging to an impulsive solution of $(P_{\mathcal{M}})$. For more facilities, we assume that (A, B) is controllable.

7.1 Strategy

We reason as in [19]. Here there are the main ideas:

- (1) To approach a solution of $(P_{\mathcal{M}})$, we “cut” $[0, T]$ in N intervals, and we impose that the impulses occur at times $t_i = \frac{iT}{N}$ for $i \in \llbracket 0, N \rrbracket$. Then, we search for elements of \mathbb{R}^m (corresponding to the weights of the impulses) which allow us to approach u_N , the “discretized” solution of $(P_{\mathcal{M}})$ with imposed impulses at times t_i . Intuitively, we can imagine that when $N \rightarrow \infty$, we do not impose the moment of the impulses anymore, and so we approach the desired solution (see paragraph 7.2).
- (2) Instead of solving directly $(P_{\mathcal{M}})$, we take an interest in a minimization problem in the form

$$\min_{\mu \in (\mathbb{R}^m)^{N+1}} |\mu| + \lambda |y - L\mu|_2^2,$$

where $|\cdot|_2$ denotes the Euclidean norm. We can show that we obtain a solution of $(P_{\mathcal{M}})$ when $\lambda \rightarrow \infty$ (see paragraph 7.3).

- (3) Bregman iterations help us to obtain a solution of $(P_{\mathcal{M}})$ from a solution of this new problem (see paragraph 7.3).
- (4) Practically, to solve this new problem, we use an algorithm based on the coordinate descent method (see paragraph 7.4).

7.2 Time discretization and convergence

Let us show that the discretized solution converges to a solution of $(P_{\mathcal{M}})$ when $N \rightarrow \infty$.

For $N \in \mathbb{N}^*$, we consider a discretization \mathcal{T}_N of $[0, T]$ such that

- (i) $\mathcal{T}_N \subset \mathcal{T}_{N+1}$;
- (ii) for all $t \in [0, T]$, $\lim_{N \rightarrow \infty} \text{dist}(t, \mathcal{T}_N) = 0$;
- (iii) $\tilde{\mathcal{T}} = \bigcap_{N \in \mathbb{N}^*} \mathcal{T}_N = \{\tilde{t}_1, \dots, \tilde{t}_k\}$ with:
 $\text{rk} \left(e^{(T-\tilde{t}_1)A} B, \dots, e^{(T-\tilde{t}_k)A} B \right) = n$.

This last assumption is not so strong since the controllability of (A, B) ensures that for almost all $\tilde{t}_1, \dots, \tilde{t}_k \in [0, T]$, we have $\text{rk} \left(e^{(T-\tilde{t}_1)A} B, \dots, e^{(T-\tilde{t}_k)A} B \right) = n$ (see Lemma 4). We set $u^* = \sum_{i=1}^{N^*} \mu_i^* \delta_{\tau_i^*}$ a solution of $(P_{\mathcal{M}})$ and we consider:

- $u_N^* = \sum_{i=1}^{N^*} \mu_i^* \delta_{\tau_i}$ where $\tau_i \in \text{argmin}_{\tau \in \mathcal{T}_N} |\tau_i^* - \tau|$;
- $u_N = \sum_{t_i \in \mathcal{T}_N} \mu_i \delta_{t_i}$, a solution of $(P_{\mathcal{M}})$ with imposed impulses according to the discretization \mathcal{T}_N .

When $N \rightarrow \infty$, for all $i \in \llbracket 1, N^* \rrbracket$, assumption (ii) ensures that $\tau_i \rightarrow \tau_i^*$ and so $u_N^* \xrightarrow{*} u^*$. But, according to its construction, u_N^* may not be a feasible control. To “correct” this, we consider a measure v_N whose support is included in $\tilde{\mathcal{T}}$ and such that $\Phi_T(u_N^* + v_N) = y$, that is to say $\Phi_T(v_N) = y - \Phi_T(u_N^*)$.

Since u^* is a solution of $(P_{\mathcal{M}})$, $\Phi_T(u^*) = y$. From $u_N^* \xrightarrow{*} u^*$, we have that $\lim_{N \rightarrow \infty} \Phi_T(u_N^*) = \Phi_T(u^*) = y$ and we deduce that

$$\lim_{N \rightarrow \infty} \Phi_T(v_N) = \lim_{N \rightarrow \infty} \sum_{t \in \tilde{\mathcal{T}}} e^{(T-t)A} B v_N(t) = 0.$$

Thus, since $\text{rk} \left(e^{(T-\tilde{t}_1)A} B, \dots, e^{(T-\tilde{t}_k)A} B \right) = n$, we deduce that v_N can be chosen such that $\lim_{N \rightarrow \infty} \|v_N\|_{\mathcal{M}} = 0$ and so $u_N^* + v_N \xrightarrow{*} u^*$.

By construction, u_N is solution of $(P_{\mathcal{M}})$ among the measures presenting impulses at times $t \in \mathcal{T}_N$. The control $u_N^* + v_N$ only has impulses at times $t \in \mathcal{T}_N$ so

$$\|u_N\|_{\mathcal{M}} \leq \|u_N^* + v_N\|_{\mathcal{M}} \leq \|u_N^*\|_{\mathcal{M}} + \|v_N\|_{\mathcal{M}}.$$

But $\|u_N^*\|_{\mathcal{M}} = \|u^*\|_{\mathcal{M}}$ and $\|v_N\|_{\mathcal{M}} \rightarrow 0$ so there is $C \in \mathbb{R}$ independent of N such that $\|u_N\|_{\mathcal{M}} \leq C$. We deduce that there is $\tilde{u} \in \mathcal{M}([0, T])$ such that $u_N \xrightarrow{*} \tilde{u}$. Since u^* is optimal, we have $\|\tilde{u}\|_{\mathcal{M}} \geq \|u^*\|_{\mathcal{M}}$. But we

also have

$$\begin{aligned} \|\tilde{u}\|_{\mathcal{M}} &\leq \liminf_{N \rightarrow \infty} (\|u_N^*\|_{\mathcal{M}} + \|v_N\|_{\mathcal{M}}) \\ &= \liminf_{N \rightarrow \infty} \|u_N^*\|_{\mathcal{M}} = \|u^*\|_{\mathcal{M}}. \end{aligned}$$

Thus, $\|\tilde{u}\|_{\mathcal{M}} = \|u^*\|_{\mathcal{M}}$ which proves that, when $N \rightarrow \infty$, the minimizer of the discretized version of $(P_{\mathcal{M}})$ converges (up to the extraction of a subsequence) to a minimizer of $(P_{\mathcal{M}})$.

As announced in the strategy, we decide, for $N \in \mathbb{N}^*$, to cut $[0, T]$ in N intervals (of same length) $[t_i, t_{i+1}]$ where $t_i = \frac{iT}{N}$ with $i \in \llbracket 0, N \rrbracket$. We note \mathcal{U}_N the set of controls in the form of linear combination of Dirac impulses at times t_i on $[0, T]$. In other words, we have

$$\begin{aligned} \mathcal{U}_N &= \left\{ \sum_{i=0}^N \alpha_i \delta_{t_i} : \forall i \in \llbracket 0, N \rrbracket, t_i = \frac{iT}{N} \right. \\ &\quad \left. \text{and } \alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,m})^\top \in \mathbb{R}^m \right\}. \end{aligned}$$

Modulo extraction, this discretization of $[0, T]$ respects the assumptions of this proof. Thus, for N given, we want to write an algorithm allowing us to approach u_N , solution of $(P_{\mathcal{M}})$ constrained in \mathcal{U}_N . Large values of N make u_N closest to a general solution of $(P_{\mathcal{M}})$ (i.e., without a restriction to \mathcal{U}_N).

7.3 Reformulation of $(P_{\mathcal{M}})$

For $N \in \mathbb{N}^*$ given, we are interested in the “discretized” minimization problem

$$\begin{aligned} \min \|u\|_{\mathcal{M}} \\ \left| \begin{array}{l} u \in \mathcal{U}_N \text{ and } \Phi_T(u) = y. \end{array} \right. \end{aligned} \quad (P_{\mathcal{M}}^N)$$

To a control $u = \sum_{i=0}^N \alpha_i \delta_{t_i} \in \mathcal{U}_N$, we can associate the vector $\mu \in \mathbb{R}^{m(N+1)}$ presenting the m components of each $N+1$ elements α_i associated to the impulses of u . Thus, this “discretized” problem can be rewritten in the form

$$\begin{aligned} \min |\mu|_1 = \sum_{j=1}^{m(N+1)} |\mu_j| \\ \left| \begin{array}{l} \mu \in \mathbb{R}^{m(N+1)} \text{ and } L\mu = y, \end{array} \right. \end{aligned}$$

where μ_j are the components of $\mu \in \mathbb{R}^{m(N+1)}$ and

$$L = \left(e^{(T-t_0)A} B, \dots, e^{(T-t_N)A} B \right) \in \mathbb{R}^{n \times m(N+1)}.$$

We note μ^* a solution of this problem, and we consider the problem

$$\min_{\mu \in \mathbb{R}^{m(N+1)}} J_\lambda(\mu) = |\mu|_1 + \lambda |y - L\mu|_2^2, \quad (P_\lambda)$$

where $\lambda > 0$ is a scalar parameter. We note μ_λ a solution of this problem, and we have that

$$|\mu_\lambda|_1 + \lambda |y - L\mu_\lambda|_2^2 \leq |\mu^*|_1 + \lambda |y - L\mu^*|_2^2 = |\mu^*|_1.$$

Since $\lambda |y - L\mu_\lambda|_2^2 \geq 0$, we deduce that $|\mu_\lambda|_1 \leq |\mu^*|_1$ for all $\lambda > 0$.

By denoting u_λ (respectively u^*) the control of \mathcal{U}_N associated to μ_λ (respectively to μ^*), we obtain that $\|u_\lambda\|_{\mathcal{M}} \leq \|u^*\|_{\mathcal{M}}$ for all $\lambda > 0$. Thus, there is $u_\infty \in \mathcal{U}_N$ such that (up to the extraction of a subsequence) $u_\lambda \xrightarrow{*} u_\infty$ when $\lambda \rightarrow \infty$. By considering μ_∞ the element of $\mathbb{R}^{m(N+1)}$ associated to u_∞ , we directly have that $|\mu_\infty|_1 \leq |\mu^*|_1$ and, since $|y - L\mu_\infty|_2^2 = 0$, the optimality of μ^* ensures that $|\mu_\infty|_1 = |\mu^*|_1$.

To obtain a solution of $(P_{\mathcal{M}}^N)$ from one of (P_λ) , we use, as in [19] (and initiated in [10]), the Algorithm 1 based on Bregman iterations. This method allows us to approach the solution with a few numbers of iterations and for reasonable values of λ (see [19]).

Algorithm 1 Bregman iterations method

Require: $L \in \mathbb{R}^{n \times M}$, $y \in \mathbb{R}^n$, $\lambda > 0$, tolerance

Ensure: $\mu \in \operatorname{argmin} \{|\mu|_1 : \mu \in \mathbb{R}^M \text{ s.t. } L\mu = y\}$

- 1: $y^0 \leftarrow 0$
 - 2: $\mu^0 \leftarrow 0$
 - 3: **while** $\frac{|L\mu^k - y|_2}{|y|_2} > \text{tolerance}$ **do**
 - 4: $y^{k+1} \leftarrow y + y^k - L\mu^k$
 - 5: $\mu^{k+1} \leftarrow \operatorname{argmin}_{\mu \in \mathbb{R}^M} \left\{ |\mu|_1 + \lambda |L\mu - y^{k+1}|_2^2 \right\}$
 - 6: **end while**
-

7.4 Coordinate descent method

To complete this algorithm, we have to describe a method to solve (P_λ) (line 5 of Algorithm 1). We choose to use the coordinate descent method: at each iteration, we stabilize all components of μ except one, which we update to approach the solution. So this method asks two questions:

- How to choose the component of μ that we want to update? We call this way of selection the sweep pattern.
- How to update the chosen component of μ ?

We do not give details of computation, but the algorithm can be found in Appendix A.

Sweep pattern. The function J_λ is convex and lower semicontinuous. So $\mu^* = \operatorname{argmin}_\mu J_\lambda(\mu)$ if and only if $0 \in (\partial J_\lambda(\mu^*))_i$ for all $i \in \llbracket 1, m(N+1) \rrbracket$, where $\partial J_\lambda(\mu^*)_i$ is the subdifferential of J_λ in μ^* and $(\partial J_\lambda(\mu^*))_i$ its i -th component.

We decide to update the i -th component of μ such that $(\partial J_\lambda(\mu))_i$ is the farthest (from 0) component of $\partial J_\lambda(\mu)$.

Expression for the update. In dimension 1, the solution of

$$\min_{x \in \mathbb{R}} (|x| + \lambda(x - f)^2)$$

is given by $x = \operatorname{shrink}(f, \frac{1}{2\lambda})$, where

$$\operatorname{shrink}(f, \mu) = \begin{cases} f - \mu & \text{if } f > \mu, \\ 0 & \text{if } -\mu \leq f \leq \mu, \\ f + \mu & \text{if } f < -\mu. \end{cases}$$

For $\mu \in \mathbb{R}^{m(N+1)}$ and $i \in \llbracket 1, m(N+1) \rrbracket$, we express what depends on the i -th component of μ in the expression of $J_\lambda(\mu)$ to find a formula to update the chosen component.

7.5 Results on discretized 1D heat equation

We consider the 1D heat equation, for $t > 0$ and $\xi \in (0, 1)$, defined by

$$\begin{cases} \partial_t \psi(t, \xi) = \partial_\xi^2 \psi(t, \xi), \\ \partial_\xi \psi(t, 0) = 0, \\ \psi(t, 1) = u(t), \\ \psi(0, \xi) = \psi^0(\xi), \end{cases} \quad (10)$$

where $\psi^0 \in L^2([0, 1])$ is the initial state. As in [21], we consider the finite-difference spatial discretization of this problem. Thus, we can rewrite (10) in the form of the system (\mathcal{S}) with

$$A = (n-1)^2 \begin{pmatrix} -2 & 2 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

and $B = (n-1)^2 \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\top \in \mathbb{R}^n$ where $n > 2$ is the number of discretization points and $x_i(t)$ (the i -th component of $x(t)$) stands for $\psi\left(t, \frac{i-1}{n-1}\right)$. Moreover, the pair (A, B) is controllable.

On Fig.9, we have drawn the impulsive control (composed of 10 impulses) obtained thanks to Algorithm 1 for the control time $T = 1$, initial condition $\psi^0(\xi) = \sin(\pi\xi) + \frac{1}{4} \sin(10\pi\xi)$ and target $\psi^1(\xi) = 0$, and we have set the relaxation parameter λ to 50. For the discretization, we have chosen $N = 1000$ and $n = 101$. On Fig.10, we have represented the evolution of the temperature with this control for $\xi \in [0, 1]$:

- before the first impulse on Fig.10(a);

- between the first impulse and just before the second one on Fig.10(b);
- between the second impulse and the fifth one on Fig.10(c);
- between the sixth impulse and the last one on Fig.10(d).

8 Conclusion

In this paper, we proved the interest of impulsive controls in L^0 or L^1 -minimization problems for linear systems. On one hand, when we authorize Radon measure controls, there exists a minimizer which is impulsive and, on the other hand, when we are restricted to $L^1([0, T])^m$, we can still approach this impulsive minimizer. In other words, there is no gap when we extend these minimization problems from $L^1([0, T])^m$ to $\mathcal{M}([0, T])^m$. An impulsive optimal control is usually not unique, but we can find one with a number of impulses bounded by:

- the rank of the Kalman matrix of the pair (A, B) for the L^0 -minimization problem;
- $\sum_{i=1}^m r_i$ (where r_i is the rank of the Kalman matrix of the pair (A, b_i)) for the L^1 -minimization problem.

Furthermore, the obtained results are confirmed by some numerical experiments.

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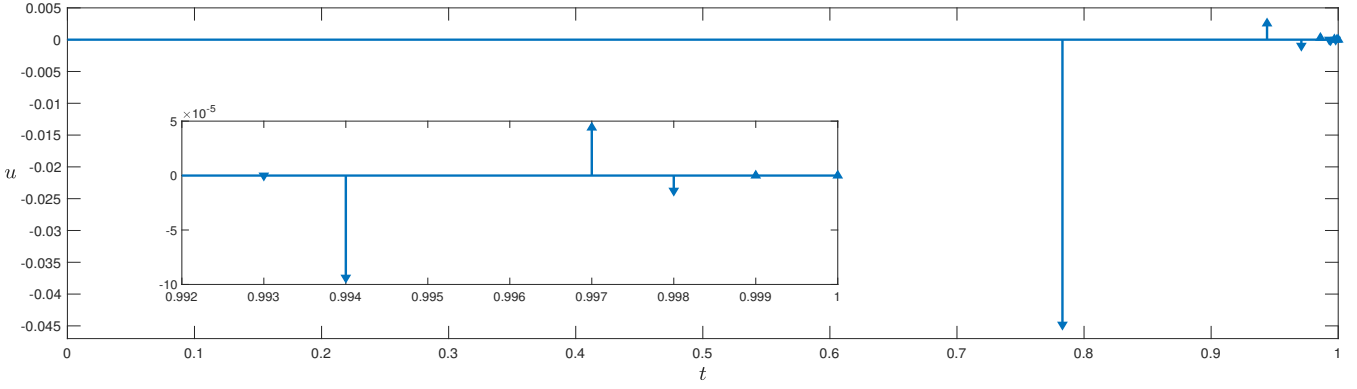


Fig. 9. An optimal impulsive control for the discretized 1D heat equation.

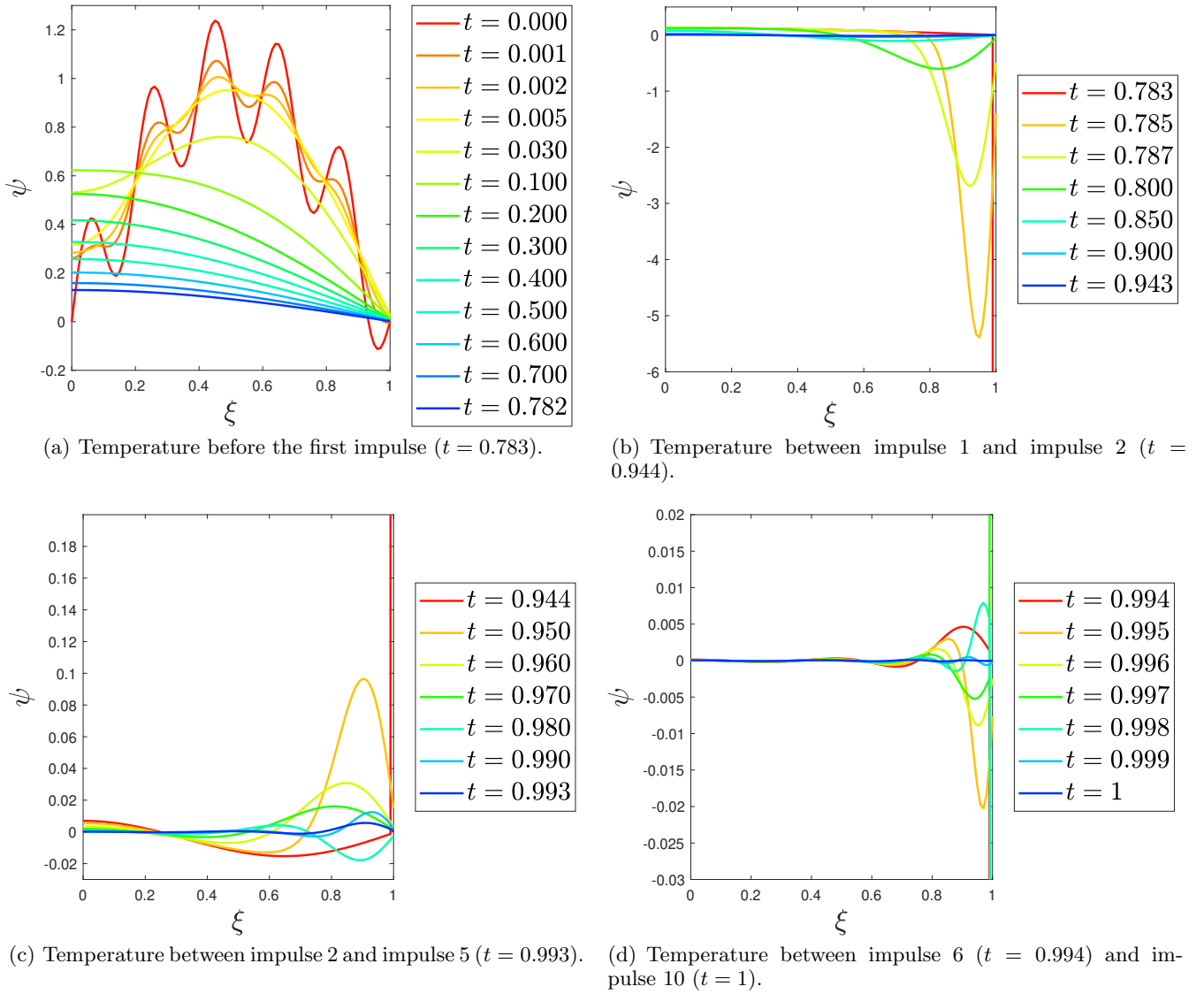


Fig. 10. Evolution of the temperature with the obtained control.

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A Algorithm for coordinate descent method

Algorithm 2 allows us to update the vector μ in Bregman iterations (line 5 of Algorithm 1). We give here some explications on it.

Algorithm 2 Coordinate descent method

Require: $L \in \mathbb{R}^{n \times M}$, $y \in \mathbb{R}^n$, $\lambda > 0$, tolerance

Ensure: $\mu \in \operatorname{argmin}_{\mu \in \mathbb{R}^M} \left\{ |\mu|_1 + \lambda \|L\mu - y\|_2^2 \right\}$

```

1:  $\mu \leftarrow 0 \in \mathbb{R}^M$ 
2:  $\Lambda \leftarrow -L^\top y \in \mathbb{R}^M$ 
3:  $G \leftarrow 0 \in \mathbb{R}^M$ 
4: for  $i \in \llbracket 1, M \rrbracket$  do
5:   if  $|2\lambda\Lambda(i) + 1| \geq 1$  then
6:      $G(i) \leftarrow |2\lambda\Lambda(i) + 1| - 1$ 
7:   end if
8: end for
9:  $i^* \leftarrow \operatorname{argmax}_{i \in \llbracket 1, M \rrbracket} G(i)$ 
10:  $d \leftarrow (L^\top L)(i^*, i^*)$ 
11:  $\eta \leftarrow \mu(i^*)$ 
12:  $\mu(i^*) \leftarrow \frac{1}{d} \operatorname{shrink}(\mu(i^*)d - \Lambda(i^*); \frac{1}{2\lambda})$ 
13:  $k \leftarrow 0$ 
14: while  $|G|_\infty > \text{tolerance}$  do
15:    $\Lambda(i^*) \leftarrow \Lambda(i^*) + (\mu(i^*) - \eta) L^\top L(:, i^*)$ 
16:   for  $i \in \llbracket 1, M \rrbracket$  do
17:     if  $\mu(i) = 0$  then
18:       if  $|2\lambda\Lambda(i) + 1| \geq 1$  then
19:          $G(i) \leftarrow |2\lambda\Lambda(i) + 1| - 1$ 
20:       else
21:          $G(i) \leftarrow 0$ 
22:       end if
23:     else
24:        $G(i) \leftarrow |2\lambda\Lambda(i) + \operatorname{sign}(\mu(i))|$ 
25:     end if
26:    $i^* \leftarrow \operatorname{argmax}_{i \in \llbracket 1, M \rrbracket} G(i)$ 
27:    $d \leftarrow (L^\top L)(i^*, i^*)$ 
28:    $\eta \leftarrow \mu(i^*)$ 
29:    $\mu(i^*) \leftarrow \frac{1}{d} \operatorname{shrink}(\mu(i^*)d - \Lambda(i^*); \frac{1}{2\lambda})$ 
30:   end for
31:    $k \leftarrow k + 1$ 
32: end while

```

Variable Λ (lines 2 and 15) is an help for computations. By reasoning as in [19], we can find an induction relation on it (line 15) and save computations. For this reason, we have to initiate variables (lines 1 to 12) before the main loop (lines 13 to 32).

Each component of $G \in \mathbb{R}^M$ is the smallest element (in absolute value) of the corresponding component of the subdifferential of J_λ . The biggest components of G indicate the index of the component of μ to be updated (lines 9 and 26).

Variable d (lines 10 and 27) is the coefficient on the i^* -th row and the i^* -th column of the matrix $L^\top L$ and takes

part in the update of μ (lines 12 and 29). Variable η is a save of the i^* -th component of μ before the update: we need it in the induction relation of Λ .

Remark 19 *When this algorithm updates a new component of μ , it might lose time between two close impulses and produce a “dirty” representation of μ . To avoid that, we can solve (thanks to CVX⁴ for example) the restrained minimization problem before the update of a new component.*



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⁴ see [13] and <https://cvxr.com/cvx>.