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Efficient methods for the solution of boundary integral equations on fractal antennas

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**Abstract**

This work focuses on construction of efficient numerical methods for wave scattering by fractal antennas, see [3]. It builds on the theoretical basis proposed in the recent work [1], which establishes boundary integral (BIE) formulations for solving sound-soft Helmholtz scattering problems on fractal screens. An important feature of such formulations is the use of the Hausdorff measure on fractals instead of the standard Lebesgue’s measure. This adds an extra dimension to the two classical difficulties encountered with numerical BEM simulations, namely the evaluation of boundary integrals and the fact that the underlying matrices are dense. Our idea is to exploit the Hausdorff measure’s self-similar structure in order to deal with these difficulties.

**Keywords:** Integral Equation, Fractal, Quadrature,  $\mathcal{H}$ -matrix.

**1 Geometry of fractal scatterers**

Let  $\mathcal{S} = \{S_\ell : x \mapsto \rho_\ell T_\ell x + b_\ell : \ell \in \mathbb{I}_L\}$  with  $\mathbb{I}_L := \{0, \dots, L - 1\}$ , a set of contractive similarities in  $\mathbb{R}^n, n = 2, 3$ , where  $0 \leq \rho_\ell < 1, T_\ell$  is an orthogonal matrix, and  $b_\ell \in \mathbb{R}^n$ . Then the scatterer  $\Gamma$  is the attractor of these similarities, namely it is the unique non-empty compact set, such that  $\Gamma = \bigcup_\ell S_\ell(\Gamma)$ , see [1, Sec. 2.3].

This type of fractals can be seen as the limit of an iterative process of decreasing pre-fractals  $F_p \supset F_{p+1}$ , where  $F_{p+1} := \bigcup_\ell S_\ell(F_p)$ , see Fig. 1, We will additionally assume that the IFS satisfy

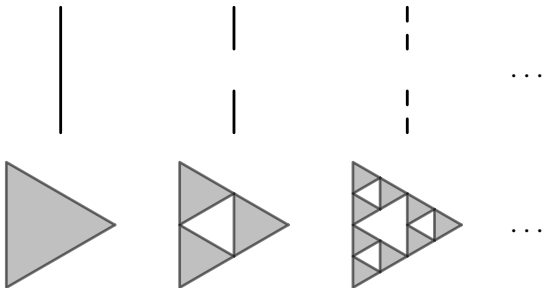


Figure 1: Converging sequences of pre-fractals. Top: Cantor set, Bottom: Sierpiński triangle

the *open set condition* (OSC) [1, Sec. 2.3]. In this case the Hausdorff dimension  $d$  of the fractal is the solution of  $\sum_\ell \rho_\ell^d = 1$ . Moreover, the  $d$ -Hausdorff measure  $\mu$  of the fractal  $\Gamma$  is a non-zero finite number (which we normalize to 1).

**2 Setting of the problem**

Given a wavenumber  $\omega > 0$  and an incident field  $u^{\text{in}}$ , we are interested in the scattered field:  $u^{\text{sc}} \in H_{\text{loc}}^2(\mathbb{R}^n \setminus \Gamma)$  satisfying

$$\begin{cases} -\Delta u^{\text{sc}} - \omega^2 u^{\text{sc}} = 0, & \text{in } \mathbb{R}^n \setminus \Gamma, \\ u^{\text{sc}} = -u^{\text{in}}, & \text{on } \Gamma, \\ u^{\text{sc}} \text{ outgoing,} & \text{at } \infty. \end{cases} \quad (1)$$

As for standard scattering problem, one can reformulate (1) using *Boundary Integral Equations*, see [1, Thm. 4.2]: Find  $\varphi : \Gamma \mapsto \mathbb{C}$  such that

$$\int_\Gamma G(x, y)\varphi(y) \, d\mu_y = u^{\text{in}}(x), \quad \forall x \in \Gamma, \quad (2)$$

where  $G$  is the Green function of the Helmholtz equation. From  $\varphi$ , one then recovers  $u^{\text{sc}}$  via

$$u^{\text{sc}}(x) = - \int_\Gamma G(x, y)\varphi(y) \, d\mu_y, \quad x \in \mathbb{R}^n \setminus \Gamma.$$

**3 Space discretization**

Eq. (2) is discretized, in its variational form, using the Galerkin method called *Hausdorff-measure Boundary Element Method*, see [1].

Thanks to OSC, we built a partition of  $\Gamma$  into sub-fractals  $\Gamma_{\mathbf{m}} := S_{m_1} \circ \dots \circ S_{m_p}(\Gamma)$ , where  $\mathbf{m} = (m_1, \dots, m_p) \in \mathbb{I}_L^p$ . The Galerkin approximation space is then built with piece-wise constant functions:

$$\mathbb{V}_p := \text{span}\{\mathbf{1}_{\Gamma_{\mathbf{m}}} \mid \mathbf{m} \in \mathbb{I}_L^p\}$$

This leads to the matrix system  $\mathbf{A}\Phi = U^{\text{in}}$  where the coefficients of the matrix  $\mathbf{A}$  are

$$\mathbf{A}_{\alpha, \beta} := \iint_{\Gamma_\alpha \times \Gamma_\beta} G(x, y) \, d\mu_y \, d\mu_x. \quad (3)$$

Therefore, the difficulties to be addressed are:

- compute efficiently  $\mathbf{A}_{\alpha, \beta}$  (Section 4);
- invert efficiently the matrix  $\mathbf{A}$  (Section 5).

#### 4 High order quadrature

The most versatile method to compute integral on  $\Gamma$  is the one-point barycenter rule [1]. However, this method is of low order. Our goal is to construct higher order quadratures by exploiting the property that, thanks to OSC,

$$\int_{\Gamma} f(x) d\mu_x = \sum_{\ell} \rho_{\ell}^d \int_{\Gamma} f(S_{\ell}(x)) d\mu. \quad (4)$$

We concentrate on the case where  $f$  is a continuous function in an open set  $\mathcal{G} \supset \Gamma$ . Take a space  $\mathcal{P}$  of polynomials, with  $\dim \mathcal{P} = J$ , and quadrature points  $x_1, \dots, x_J \in \mathcal{G}$  that are  $\mathcal{P}$ -unisolvent, which allows defining Lagrange polynomials  $\mathcal{L}_i$  such that  $\mathcal{L}_i(x_j) = \delta_{ij}$ . We look for the quadrature formula

$$\int_{\Gamma} f(x) d\mu_x \approx Q_J(f) := \sum_j w_j f(x_j).$$

Our idea is to look for weights  $w_j$  that integrate exactly the constant function

$$\sum_j w_j = 1 \quad (5)$$

and respect the property (4) in the sense that  $Q_J(f) = \sum_{\ell} \rho_{\ell}^d Q_J(f \circ S_{\ell})$  which is shown to be equivalent to

$$\mathbf{S} \mathbf{w} = \mathbf{w}, \quad \mathbf{w} = (w_j) \in \mathbb{R}^J,$$

where  $\mathbf{S}_{ij} = \sum_{\ell} \rho_{\ell}^d \mathcal{L}_i(S_{\ell}(x_j))$ . Therefore, we compute the weights by solving

( $\mathcal{P}$ ) Find  $\mathbf{w} \in \text{Ker}(\mathbf{S} - \mathbf{I})$  that satisfies (5).

**Theorem 1** (i) The matrix  $\mathbf{S}$  always admits the eigenvalue 1. (ii) If the space  $\mathcal{P}$  is  $\mathcal{S}$ -invariant, in the sense that  $\mathcal{P} \circ S_{\ell} \subset \mathcal{P}$ , for any  $\ell$ , the eigenvalue 1 is simple and all other eigenvalues of  $\mathbf{S}$  are strictly less than 1 in modulus. In addition, the quadrature formula is exact in  $\mathcal{P}$ .

Note that the  $\mathcal{S}$ -invariance property is satisfied with  $\mathcal{P} = \mathbb{P}_k$  and also for  $\mathcal{P} = \mathbb{Q}_k$  when  $T_{\ell} = \text{diag}(\pm 1, \dots, \pm 1)$  for any  $\ell$ . In these cases, Theorem 1 shows that ( $\mathcal{P}$ ) characterizes  $\mathbf{w}$  and that the iterated power method can be applied to  $\mathbf{S}$  for computing  $\mathbf{w}$ . Moreover, the quadrature formula being exact in  $\mathcal{P}$ , the quadrature error for the integral of  $f$  is directly related to its interpolation error in  $\mathcal{P}$ .

**Remark 2** For  $\mathcal{S}$ -invariant  $\mathcal{P}$  the quadrature weights are not necessarily positive, but they satisfy the following bound  $\sum_j |w_j| \leq \Lambda_J$ , where  $\Lambda_J$  is the Lebesgue constant of the points  $(x_j)$ .

When  $\mathcal{P} = \mathbb{Q}_k$ , in the general case, numerical evidence suggests that 1 is still a simple eigenvalue and the largest in modulus. As the quadrature is no longer exact in  $\mathcal{P}$ , an additional error, that can be analyzed, is committed.

Various numerical experiments will be presented at the conference.

#### 5 A self-similar $\mathcal{H}$ -matrix approach

To deal with the density of the BEM matrices, we use  $\mathcal{H}$ -matrices, which are based on block low-rank approximations. For our applications, the low-rank block partitioning of the matrix is guided by the self-similarity of the obstacle.

Moreover, we propose a method to assemble low-rank blocks efficiently. To fix the ideas, let us consider the case of  $\omega = 0$  and  $n = 3$  (3d Laplace case). The corresponding integral kernel  $G$  satisfies the homogeneity relation

$$G(S_{\ell}(x), S_{\ell}(y)) = \rho_{\ell}^{-1} G(x, y), \quad \forall \ell \in \mathbb{I}_L.$$

This relation allows re-using low-rank approximation of a small number of large blocks to construct effectively the low-rank approximation of the remaining matrix blocks. Our current efforts are dedicated to implementation and extending these ideas to the low-frequency Helmholtz equation.

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