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Robust nonsingular predefined-time terminal sliding mode control for perturbed chains of integrators

Yang Deng, Emmanuel Moulay, Vincent Léchappé, Zhang Chen, *Member, IEEE*, Bin Liang, *Senior Member, IEEE*, and Franck Plestan, *Member, IEEE*

Abstract—This paper provides a nonsingular predefined-time terminal sliding mode control scheme for perturbed chains of integrators. Based on the linear delayed feedback technique, a class of nonsingular smooth prescribed-time sliding surfaces with artificial time-delays is constructed, and the control scheme can achieve predefined-time control for perturbed chains of integrators with matched and mismatched disturbances. Especially, for double integrators, an initial-condition-based sufficient condition is provided to preserve the prescribed-time stability. Finally, several simulation results illustrate the performance of the proposed control scheme.

Index Terms—Chain of integrators, Terminal sliding mode control, Predefined-time stability, Delayed feedback.

I. INTRODUCTION

TERMINAL sliding mode control (TSMC) [1] is an important category of sliding mode control [2], which can not only establish the finite-time convergence towards a sliding surface but also stabilize the system dynamics in finite time. In recent years, fixed-time control [3] (whose settling time is uniformly bounded) has become a hot topic, which motivates the studies on fixed-time TSMC techniques. However, when designing a fixed-time TSMC, the singularity may arise in the controller design, which is undesirable in theory and application [4]. Therefore, developing singularity avoidance techniques is crucial for fixed-time TSMC algorithms.

A vast literature is available on the nonsingular fixed-time TSMC methods. In [4], a modified fixed-time sliding surface was firstly introduced, then a sinusoidal function was proposed to deal with the singularity. In [5], a nonsingular fixed-time TSMC based on the arctangent function and a switching sliding surface was put forward. In [6], a switching saturation function was introduced to avoid the singularity of a crossing term, and the controller was applied to a power system. In [7], a fixed-time tracking TSMC was designed for underwater vehicles, in which an error transformation was activated to map the states to a nonsingular space. In [8], [9], [10], the switching sliding surfaces were introduced, and the robust fixed-time

stability was established without singularity. In addition, the sinusoidal approach [4] and the switching technique [6] were also extended in [11], [12], [13], [14], [15], [16] for the fixed-time consensus control of second-order multi-agent systems. Concluding the nonsingular fixed-time TSMCs proposed in [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], one notices that they have the following drawbacks: (i) they must use switching sliding surfaces or switching control laws to eliminate the singularity; (ii) they only focused on double integrators or second-order systems, rather than high-order systems. To tackle the problems mentioned above, the time scaling/variable gain techniques were involved in [17], [18] to design predefined-time (fixed-time with maximal settling time assigned a priori by the user [19]) TSMC for chains of integrators with matched disturbances, and the singularity can be avoided without using additional switching mechanism.

In this paper, inspired by the linear periodic delayed feedback method [20], a class of nonsingular smooth prescribed-time (all trajectories with nonzero initial conditions converge to zero precisely at the prescribed instant [19]) sliding surfaces with artificial time-delays is designed, and the proposed TSMC can achieve predefined-time control for chains of integrators with matched and mismatched disturbances. Moreover, this paper also provides an initial-condition-based sufficient condition for the prescribed-time stabilization of perturbed double integrators. The contribution of this paper is threefold:

- The proposed TSMC can establish the robust predefined-time stability as the existing TSMCs [17], [18], and this paper also provides a sufficient condition to preserve the robust prescribed-time stability for double integrators.
- The proposed predefined-time TSMC is extended to deal with perturbed chains of integrators with matched and mismatched disturbances, whereas the existing TSMCs [17], [18] did not consider mismatched disturbances.
- The proposed method exerts the advantages of the “delayed feedback” concept in [20] while eliminating its drawback of robustness. According to [20, Remark 4], if one designs feedback controllers via [20, Theorem 1], then one cannot preserve robust predefined-time stability under perturbations/uncertainties. However, when this technique is used to design sliding surfaces, it can maintain robust stability since the sliding surfaces are perturbation-free and independent of the system model.

This paper is organized as follows. In Section II, the preliminaries are introduced. Next, Section III studies the predefined-time/prescribed-time TSMC of perturbed dou-

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ble integrators. The method in Section III is extended in Section IV to design predefined-time TSMC for chains of integrators. Thereafter, the simulation results are addressed in Section V, and a conclusion ends this paper in Section VI.

II. PRELIMINARIES

Notations. The n -order derivative of a function $f(t)$ is noted as $f^{(n)}(t)$. Let $\mathbf{C}^n([a, b])$ denote the set of functions with n -order continuous derivatives on $[a, b]$, and a \mathbf{C}^∞ function is an infinitely differentiable function. The combinatorial number formed by non-negative integers $n \geq m$ writes as C_n^m . The sign function $\text{sign}(\cdot)$ is defined by following the definitions in [21, pp. 4], and $|x|^p \triangleq |x|^p \text{sign}(x)$. The differential equations in this paper are understood in Filippov sense [22].

Definitions. Consider the following nonlinear system:

$$\dot{x} = f(t, x, u) \quad (1)$$

with $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and initial condition $x(0) = x_0$. If there exists a control law $u(x)$ and instants $T_c(x_0) \geq t_c(x_0) \geq 0$ such that:

$$t_c(x_0) = \inf\{T_c(x_0) : x(t) \equiv 0 \text{ for all } t \geq T_c(x_0)\}$$

then the solution $x(t, u)$ to (1) is finite-time stable with settling time less than $T_c(x_0)$ [23]. If $T_c(x_0)$ is uniformly bounded by $T_c(x_0) \leq T_c$ for all $x_0 \in \mathbb{R}^n$, then the solution $x(t, u)$ is fixed-time stable [3]. Next, if T_c is set a priori by the user as a parameter of the controller $u(t, x, T_c)$, then the solution $x(t, u)$ is predefined-time stable at $t = T_c$ [19]. Furthermore, if $t_c(x_0) = T_c(x_0) = T_c$ for all $x_0 \neq 0_{n \times 1}$, then the solution $x(t, u)$ is prescribed-time stable [19]. Finally, the stability is said to be robust if it is established despite perturbations [24]. The aforementioned definitions are summarized in Fig. 1.

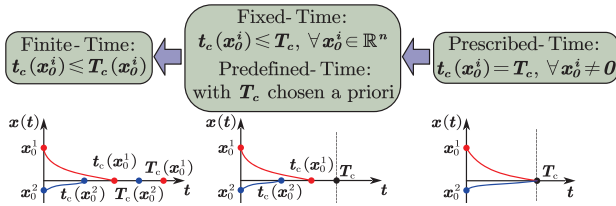


Fig. 1: Schematic of the four different types of stabilities.

Lemma 1 (Theorem 2 of [24]). *Consider the scalar system:*

$$\dot{x} = -k[x] \frac{\lambda x^2}{1+\mu x^2} + d, \quad (2)$$

with bounded perturbation $|d| \leq \delta$. If the parameters λ, μ satisfy $\chi = \frac{\lambda}{1+\mu} > 1$, and the gain k is sufficiently large with $k > \delta e^{\frac{\lambda}{2\epsilon}}$, then system (2) is fixed-time stable with a settling time less than $\frac{1}{(k-\delta)(\chi-1)} + \frac{1}{ke^{-\lambda/2\epsilon}-\delta}$.

Lemma 2 (Theorem 1 of [25]). *Consider the differentiation of a signal $f(t)$ satisfying $|f^{(n)}(t)| \leq \Delta$:*

$$\begin{aligned} \dot{z}_i &= -k_i \phi_i(z_0 - f) + z_{i+1}, \\ &\vdots \\ \dot{z}_n &= -k_n \phi_n(z_0 - f), \end{aligned} \quad (3)$$

in which $\phi_i(s) = \varphi_i \circ \dots \circ \varphi_2 \circ \varphi_1(s)$ is a composition of the following functions: $\varphi_i(s) = \kappa_i[s] \frac{r_{0,i+1}}{r_{0,i}} + \rho_i[s] \frac{r_{\infty,i+1}}{r_{\infty,i}}$, with $r_{0,i} = 1 - (n-i)d_0$, $r_{\infty,i} = 1 - (n-i)d_\infty$, and $-1 \leq d_0 \leq d_\infty < 1/(n-1)$. If $d_0 = -1$, and k_i, κ_i, ρ_i are appropriately chosen, then there exist positive constants $p_0, p_\infty, \eta_0, \eta_\infty$ such that the exact differentiation $z_i(t) \equiv f^{(i-1)}(t)$ is established after a uniformly bounded settling time T_O satisfying:

$$T_O \leq \frac{p_0}{d_0 \eta_\infty} \left(\frac{p_\infty d_0}{p_0 d_\infty} - 1 \right) \left(\frac{\eta_0}{\eta_\infty} \right)^{\frac{1}{\left(\frac{p_\infty d_0}{p_0 d_\infty} - 1 \right)}}. \quad (4)$$

Moreover, under the following parameter scaling method:

$$\kappa_i \mapsto \left(\frac{L^n}{\alpha} \right)^{\frac{d_0}{r_{0,i}}} \kappa_i, \quad \rho_i \mapsto \left(\frac{L^n}{\alpha} \right)^{\frac{d_\infty}{r_{\infty,i}}} \rho_i, \quad k_i \mapsto L^i k_i, \quad (5)$$

the settling time T_O and the Lipschitz constant Δ can be scaled to T_O/L and $\alpha\Delta$, respectively.

Remark 1. Lemmas 1–2 can be extended to predefined-time versions. For Lemma 1, since $0 < \frac{1}{k-\delta} \leq \frac{1}{ke^{-\lambda/2\epsilon}-\delta}$, one can set:

$$k \geq \left(\frac{\chi}{h(\chi-1)} + \delta \right) e^{\frac{\lambda}{2\epsilon}} \quad (6)$$

to ensure $\frac{1}{(k-\delta)(\chi-1)} + \frac{1}{ke^{-\lambda/2\epsilon}-\delta} \leq h$, which leads to the robust predefined-time stability at $t = h$. For Lemma 2, one can apply the parameter tuning method [25, Section III-C] and (5) to assign an arbitrary pair of (T_O, Δ) that establishes predefined-time exact differentiation.

Inspired by the “delayed feedback” concept in [20], this paper designs prescribed-time stable sliding surfaces with artificial delays in the next lemma, and all h appearing in the remainder of this paper are tuning parameters of sliding surfaces, rather than the physical delays studied in [26].

Lemma 3. *Consider the time-delay scalar system:*

$$\dot{\zeta}(t) = a\zeta(t) - K(t)\zeta(t-h) \quad (7)$$

with $a \in \mathbb{R}$ and $\zeta(\theta) \in \mathbf{C}^1([-h, 0])$. If $K(t)$ is defined as:

$$K(t) = R_h(t) e^{ah} W e^{-2at}, \quad (8)$$

in which the globally \mathbf{C}^∞ function $R_h(t)$ satisfies:

$$R_h(t) = \begin{cases} 0, & t \in (-\infty, h] \cup [2h, \infty) \\ M^{-\frac{1}{(t-h)(2h-t)}}, & t \in (h, 2h) \end{cases} \quad (9)$$

with $M > 1$, and

$$W = \left[\int_h^{2h} R_h(\theta) e^{-2a\theta} d\theta \right]^{-1}, \quad (10)$$

then system (7) is prescribed-time stable at $t = 2h$ such that:

$$\zeta(t) \equiv 0, \quad \forall t \geq 2h. \quad (11)$$

Proof. From (8)–(9), one observes that $K(t) = 0$ for all $t \in [0, h]$, so (7) is reduced to a linear system on $t \in [0, h]$ with the solution:

$$\zeta(t) = e^{at} \zeta(0), \quad t \in [0, h] \quad (12)$$

which implies $\zeta(\theta - h) = e^{a(\theta-h)}\zeta(0)$ for all $\theta \in [h, 2h]$. Next, combine (7)–(8)–(9) provides the following solution:

$$\begin{aligned}\zeta(t) &= e^{a(t-h)}\zeta(h) - \int_h^t e^{a(t-\theta)}K(\theta)\zeta(\theta-h)d\theta \\ &= e^{at} \left[1 - \int_h^t R_h(\theta)e^{-2a\theta}d\theta \cdot W \right] \zeta(0), \quad t \in [h, 2h].\end{aligned}\quad (13)$$

Consider the facts that $R_h(h) = R_h(2h) = 0$, $R_h(\theta) > 0$ for all $\theta \in (h, 2h)$, and $e^{-2a\theta} > 0$ for all $\theta \in [h, 2h]$, the integral $\int_h^t R_h(\theta)e^{-2a\theta}d\theta$ is non-negative and strictly increasing on $t \in [h, 2h]$ for all $a \in \mathbb{R}$, implying that $W > 0$ according to (10). Combine the aforementioned results and (10) leads to:

$$1 - \int_h^t R_h(\theta)e^{-2a\theta}d\theta \cdot W \geq 0, \quad t \in [h, 2h] \quad (14)$$

where equality holds if and only if $t = 2h$ due to (10) and the strict monotonicity mentioned above, so the right-hand side of (13) equals zero if and only if $t = 2h$. Thus, (12)–(13) reveal that $\zeta(t)$ reaches zero precisely at the prescribed instant $t = 2h$ for all $\zeta(0) \neq 0$. Finally, definitions (8)–(9) indicate that $K(t) = 0$ for all $t \geq 2h$, so (7) is reduced to:

$$\dot{\zeta}(t) = a\zeta(t), \quad \forall t \geq 2h \quad (15)$$

with $\zeta(2h) = 0$, meaning $\zeta(t) = 0$ is maintained as (11). \square

III. DOUBLE INTEGRATOR

Consider the following perturbed double integrator:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d\end{aligned}\quad (16)$$

in which the perturbation satisfies $|d| \leq \delta$. Lemma 3 is applied to build the following prescribed-time sliding surface:

$$s(t) = x_2(t) - [ax_1(t) - K_0(t)x_1(t-h)] \quad (17)$$

where a, h are defined in Lemma 3, and

$$K_0(t) = K(t-h) \quad (18)$$

with $K(t)$ defined by (8)–(9)–(10). From (18) and (8)–(9)–(10), one confirms that $K_0(t)$ is a globally C^∞ function [27, Chapter 6.3]. Thus, the following TSMC can be designed:

$$\begin{aligned}u(t) &= u_{eq}(t) + u_{FxT}(t), \\ u_{eq}(t) &= ax_2(t) - \dot{K}_0(t)x_1(t-h) - K_0(t)x_2(t-h), \\ u_{FxT}(t) &= -k[s(t)]^{\frac{\lambda s(t)^2}{1+\mu s(t)^2}}.\end{aligned}\quad (19)$$

In the sequel, the following results are declared to ensure the nonsingularity and robust predefined-time stability.

Lemma 4. *For any positive integer n , the gain $K_0(t)$ and its n -order derivative $K_0^{(n)}(t)$ are bounded for all $t \geq 0$.*

Proof. Due to definitions (8)–(9) and (18), it is sufficient to prove that the gain $K(t)$ defined by (8)–(9)–(10) satisfies this property on interval $t \in [h, 2h]$.

Firstly, the monotonicity of $R_h(t)$ defined in (9) ensures that $R_h(t) \geq 0$ for all $t \in [h, 2h]$ and $R_h(t) \geq R_h(5h/4) =$

$R_h(7h/4) = M^{-\frac{16}{3h^2}} > 0$ for all $t \in [5h/4, 7h/4]$, then it leads to:

$$\int_h^{2h} R_h(\theta)e^{-2a\theta}d\theta \geq \left(\int_{\frac{5h}{4}}^{\frac{7h}{4}} R_h(\theta)d\theta \right) \epsilon \geq M^{-\frac{16}{3h^2}} \frac{h\epsilon}{2} > 0$$

with $\epsilon \triangleq \min\{e^{-2ah}, e^{-4ah}\} > 0$. Thus, the gain W defined in (10) is bounded by $0 < W < M^{\frac{16}{3h^2}} \frac{2}{h\epsilon}$, then $K(t)$ is bounded on $t \in [h, 2h]$ since $R_h(t)$ and e^{-2at} are also bounded on this interval. Secondly, consider the fact that $K(t)$ is globally C^∞ from (8)–(9) and [27, Chapter 6.3], so $K^{(n)}(t)$ must be continuous on $t \in [h, 2h]$. Thus, the Boundedness and Extreme value theorems [28, Theorems 3.11–3.12] ensure that $K^{(n)}(t)$ is bounded on the closed interval $t \in [h, 2h]$, which guarantees the boundedness of $K_0(t)$ and $K_0^{(n)}(t)$ for all $t \geq 0$. \square

Theorem 1. *Consider the double integrator (16) controlled by the nonsingular TSMC (8)–(9)–(10)–(17)–(19). If $\chi = \frac{\lambda}{1+\mu} > 1$ and k is sufficiently large to satisfy (6), then the closed-loop system (16)–(19) is robust predefined-time stable at $t = 3h$.*

Proof. Combining (16)–(17) and (19) yields:

$$\begin{aligned}\dot{s}(t) &= u(t) + d(t) - ax_2(t) + \dot{K}_0(t)x_1(t-h) + K_0(t)\dot{x}_1(t-h) \\ &= -k[s(t)]^{\frac{\lambda s(t)^2}{1+\mu s(t)^2}} + K_0(t)[\dot{x}_1(t-h) - x_2(t-h)] + d(t).\end{aligned}\quad (20)$$

On $t \in [0, h)$, definitions (8)–(9)–(10) and (18) imply that $K_0(t) = 0$, and the delayed terms in (17)–(20) are not activated; for $t \geq h$, one deduces that $\dot{x}_1(t-h) - x_2(t-h) = 0$ from (16). Thus, (20) is reduced to $\dot{s} = -k[s]^{\frac{\lambda s^2}{1+\mu s^2}} + d$ for all $t \geq 0$, and $s(t)$ is robust predefined-time stabilized at $t = h$ under parameter tuning (6) based on Lemma 1 and Remark 1. For all $t \in [0, h]$, the sliding surface (17) is reduced to:

$$s(t) = x_2(t) - ax_1(t) = \dot{x}_1(t) - ax_1(t), \quad \forall t \in [0, h]. \quad (21)$$

Consider different arrival points at the sliding surface (21):

- Case 1: (x_1, x_2) reaches the sliding surface at the origin;
- Case 2: (x_1, x_2) reaches the sliding surface elsewhere,

different system behaviors are obtained: with Case 1, (x_1, x_2) must stay at the origin and satisfy $x_1(h) = 0$; with Case 2, (x_1, x_2) will slide on the sliding surface generated by (21) and $s(t) = 0$, which leads to $x_1(h) \neq 0$ due to the linearity.

For all $t \geq h$, the delayed term in (17) is activated, consider (17) and $\dot{x}_1 = x_2$, the sliding motion $s(t) = 0$ leads to:

$$\dot{x}_1(t) = ax_1(t) - K_0(t)x_1(t-h), \quad \forall t \geq h. \quad (22)$$

With Case 1, the dynamics of (22) are initialized at $x_1(h) = 0$, so one has $x_1(t) = 0$ for all $t \geq h$ from the proof of Lemma 3. With Case 2, the definition (18) and Lemma 3 ensure that (22) can make $x_1(t)$ prescribed-time stabilized at $t = 3h$. By concluding the two cases, the stabilities of $s(t)$, $x_1(t)$, and the expression $s(t) = x_2(t) - ax_1(t)$ for all $t \geq 3h$ imply that $x_2(t) = 0$ for all $t \geq 3h$, deriving the robust predefined-time stability of the closed-loop system (16)–(19) at $t = 3h$.

The last part proves the nonsingularity of control law (19). Firstly, $u_{FxT}(t)$ is nonsingular due to the proof of [24, Theorem 2]. Secondly, Lemma 4 ensures that $K_0(t)$, $\dot{K}_0(t)$ are bounded, then $u_{eq}(t)$ is also nonsingular. Thus, the whole control law (19) is nonsingular. \square

Remark 2. Control law (19) contains delayed terms $x_1(t-h)$, $x_2(t-h)$, so one must expand the initial conditions of x_1, x_2 on $[-h, 0)$ to ensure that (19) is well-defined. As stated in the proof, the expanded initial conditions cannot affect the closed-loop dynamics (20) nor the predefined-time stability since the terms depending on $x_1(\theta), x_2(\theta), \theta \in [-h, 0)$ are switched off on $t \in [0, h]$ by $K_0(t) = 0$ for all $t \in [0, h]$. Thus, any sufficiently smooth initial condition that ensures the continuity of $x_1(t), x_2(t)$ at $t = 0$, e.g., $x_1(\theta), x_2(\theta) \in C^1([-h, 0])$, can be chosen for the controller design.

Due to the existence of Case 1, Theorem 1 can only lead to robust predefined-time stability even if the sliding surface (17) is prescribed-time stable. Next, Proposition 1 provides a sufficient condition to preserve robust prescribed-time stability.

Proposition 1. Consider the double integrator (16) controlled by the nonsingular TSMC (8)–(9)–(10)–(17)–(19). If $\chi = \frac{\lambda}{1+\mu} > 1$, k is sufficiently large to satisfy (6), and a is determined by the initial conditions such that:

$$a = \begin{cases} x_2(0)/x_1(0), & \text{if } x_1(0) \neq 0 \\ 0, & \text{if } x_1(0) = 0 \end{cases} \quad (23)$$

then the closed-loop system (16)–(19) is robust prescribed-time stable at $t = 3h$.

Proof. The proof is made by considering the following cases: 1) $x_1(0) \neq 0$ and $a = x_2(0)/x_1(0)$. Under this circumstance, (21) implies $s(0) = 0$, and it indicates that the trajectories of the closed-loop system evolve on the sliding surface $s(t) \equiv 0$ for all $t \geq 0$ under the robust TSMC (19). Thus, the linearity of (21) implies that $x_1(h) = e^{ah}x_1(0) \neq 0$. 2) $x_1(0) = 0$ and $a = 0$. In this case, (21) is reduced to $s(t) = x_2(t)$ for all $t \in [0, h]$, and the closed-loop system becomes:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k[x_2]^{\frac{\lambda x_2^2}{1+\mu x_2^2}} + d, \quad t \in [0, h]. \end{cases} \quad (24)$$

If $x_2(0) \neq 0$ and k is tuned by (6), then (24), Lemma 1, and Remark 1 ensure that $x_2(t)$ converges monotonically to 0 before $t = h$, so $x_1(h) = x_1(0) + \int_0^h x_2(\theta)d\theta = \int_0^h x_2(\theta)d\theta$ is nonzero and has the same sign as $x_2(0)$.

In conclusion, the sufficient condition (23) ensures that $x_1(h) \neq 0$ for all initial conditions satisfying $(x_1(0), x_2(0)) \in \mathbb{R}^2 \setminus (0, 0)$, and the robust prescribed-time stability at $t = 3h$ is thus guaranteed by Lemma 3 and the proof of Theorem 1. \square

Remark 3. The parameter a computed by (23) may be positive and make the steady state sliding surface $s(t) = x_2(t) - ax_1(t) = 0$ unstable for all $t \geq 3h$. In theory, $a > 0$ can still make $x_1(t), x_2(t)$ stay at the origin since they have already reached it at $t = 3h$, but it could render the closed-loop system sensitive to fluctuations in practice. For this case, one can set $a < 0$ for all $t > 3h$ in practice. Since the parameter switching is conducted at the equilibrium point $x_1(3h) = x_2(3h) = 0$, it can preserve the stability for all $t \geq 3h$. Similar switching techniques were also used in [17], [18].

The main results of Theorem 1 and Proposition 1 have the following properties:

- Theorem 1 implies the predefined-time stability at $t = 3h$, and the settling time can be reduced by tuning h small. Next, Proposition 1 provides a sufficient condition to ensure prescribed-time stability. Moreover, the singularity is eliminated thanks to [24, Theorem 2] and Lemma 4.
- The controller achieves advantage complementation for methods [20], [24]: the controller in [24, Proposition 2] is robust but leads to practical fixed-time stability due to the use of a discontinuous and non-differentiable sliding surface [29]; the controller in [20] is smooth but cannot preserve robust predefined-time stability under perturbation or model uncertainties [20, Remark 4], e.g., if the second line of (16) writes as $\dot{x}_2 = f(x) + g(x)u + d$ with uncertain functions $f(x), g(x)$. The proposed method uses Lemma 3 (inspired by the “delayed feedback” concept of [20]) to construct a prescribed-time perturbation-free smooth sliding surface with artificial time-delays, and the robustness can be improved by increasing k according to Lemma 1 and the invariance condition [30].

IV. CHAINS OF INTEGRATORS

In this section, the core idea of Theorem 1 is firstly extended to the robust nonsingular predefined-time TSMC for chains of integrators thanks to Lemma 4 and the fact that $K(t)$ defined by (8)–(9)–(10) is globally C^∞ ; then the control paradigm is extended to deal with Lipschitz mismatched disturbances.

A. Predefined-time TSMC with matched disturbances

In this subsection, one considers the nonsingular predefined-time TSMC for the following perturbed n -order integrator:

$$\begin{aligned} \dot{x}_i &= x_{i+1} \\ &\vdots \\ \dot{x}_n &= u + d \end{aligned} \quad (25)$$

with bounded perturbation satisfying $|d| \leq \delta$.

Firstly, define $s_0 = x_1$ with expanded initial condition $s_0(\theta) \in C^n([-h, 0])$ according to Remark 2, it leads to $s_0^{(i)} = x_{i+1}$ for all $i \in \{1, \dots, n-1\}$. Next, define the sliding surfaces:

$$s_i(t) = \dot{s}_{i-1}(t) - [as_{i-1}(t) - K_i(t)s_{i-1}(t-h)], \quad (26)$$

with initial conditions expanded by $s_i(\theta) \in C^{n-i}([-h, 0])$ under Remark 2, and the C^∞ gains $K_i(t)$ satisfy:

$$K_i(t) = K(t - (2n - 2i - 1)h) \quad (27)$$

with $K(t)$ defined by (8)–(9)–(10). Thus, one can compute $s_i^{(j)}(t)$ by taking the j -order derivative of (26) as follows:

$$s_i^{(j)}(t) = s_{i-1}^{(j+1)}(t) - \left[as_{i-1}^{(j)}(t) - \sum_{r=0}^j C_j^r K_i^{(r)}(t) s_{i-1}^{(j-r)}(t-h) \right], \quad (28)$$

where $\{(i, j) \in \mathbb{N}^2 | 1 \leq i \leq n-1, 0 \leq j \leq n-i-1\}$.

Remark 4. Equations (26) and (28) should be gradually updated under the following recursive procedure: one firstly sets $i = 1$, then computes s_1 by (26) and $\dot{s}_1, \dots, s_1^{(n-2)}$ through (28); next one sets $i = 2$ and determines s_2 then $s_2^{(j)}$ for all $j \in \{1, \dots, n-3\}$, etc.

Based on (26) and (28), the control law is designed by:

$$u(t) = -k[s_{n-1}(t)]^{\frac{\lambda s_{n-1}(t)^2}{1+\mu s_{n-1}(t)^2}} + \sum_{i=1}^{n-1} \left[as_{i-1}^{(n-i)}(t) - \sum_{r=0}^{n-i} C_{n-i}^r K_i^{(r)}(t) s_{i-1}^{(n-i-r)}(t-h) \right]. \quad (29)$$

Theorem 2. Consider the perturbed n -order integrator (25) controlled by the nonsingular TSMC (8)–(9)–(10)–(26)–(27)–(29). If $\chi = \frac{\lambda}{1+\mu} > 1$ and k is sufficiently large to satisfy (6), then the sliding variables (26)–(28) satisfy:

$$s_i(t) \equiv 0, \quad \forall t \geq (2n - 2i - 1)h \quad (30)$$

for all $i \in \{1, \dots, n-1\}$, and the closed-loop system (25)–(29) is robust predefined-time stable at $t = (2n - 1)h$.

Proof. Firstly, by applying a similar analysis with Remark 2, it is possible to obtain that the sliding surfaces and control law in (26)–(29) are well-defined. Then, combining (26) (with $i = n - 1$) and (28) (by taking i from $n - 2$ to 1 and setting $j = n - i - 1$) provides:

$$s_{n-1}(t) = s_0^{(n-1)}(t) - \sum_{i=1}^{n-1} \left[as_{i-1}^{(n-i-1)}(t) - \sum_{r=0}^{n-i-1} C_{n-i-1}^r K_i^{(r)}(t) s_{i-1}^{(n-i-r-1)}(t-h) \right]. \quad (31)$$

Differentiating (31) along the trajectories of (25) yields:

$$\dot{s}_{n-1}(t) = u(t) + d(t) - \sum_{i=1}^{n-1} \left[as_{i-1}^{(n-i)}(t) - \sum_{r=0}^{n-i} C_{n-i}^r K_i^{(r)}(t) s_{i-1}^{(n-i-r)}(t-h) \right], \quad (32)$$

in which the relation

$$\begin{aligned} \frac{d}{dt} \left[\sum_{r=0}^{n-i-1} C_{n-i-1}^r K_i^{(r)}(t) s_{i-1}^{(n-i-r-1)}(t-h) \right] \\ = \sum_{r=0}^{n-i} C_{n-i}^r K_i^{(r)}(t) s_{i-1}^{(n-i-r)}(t-h) \end{aligned}$$

is involved. Plugging (29) into (32) and applying the similar arguments as in the proof of Theorem 1 and Remark 2 leads to:

$$\dot{s}_{n-1}(t) = -k[s_{n-1}(t)]^{\frac{\lambda s_{n-1}(t)^2}{1+\mu s_{n-1}(t)^2}} + d(t), \quad \forall t \geq 0 \quad (33)$$

which further implies that $s_{n-1}(t) \equiv 0$ for all $t \geq h$ under parameter tuning (6), Lemma 1, and Remark 1.

As long as $s_{n-1}(t)$ is stabilized, the definition (26) yields:

$$\dot{s}_{n-2}(t) = as_{n-2}(t) - K_{n-1}(t)s_{n-2}(t-h), \quad \forall t \geq h$$

which makes $s_{n-2}(t)$ predefined-time stabilized at $t = 3h$ according to Lemma 3 and the definition $K_{n-1}(t) = K(t-h)$ from (27). Sequentially, consider (26) for all $i \in \{n-3, \dots, 0\}$, the definition (27), and Lemma 3, it leads to:

$$\dot{s}_i(t) = as_i(t) - K_{i+1}(t)s_i(t-h), \quad \forall t \geq (2n - 2i - 3)h$$

then (30) is derived by applying Lemma 3. Thus, the system trajectories reach and stay at the origin (that is the unique equilibrium point of (25) with $s_0 = x_1 = 0$) for all $t \geq (2n - 1)h$, which derives the robust predefined-time stability.

Finally, the singularity avoidance can be achieved by considering Lemma 4 and repeating the proof of Theorem 1. \square

B. Predefined-time TSMC with mismatched disturbances

In this subsection, the main results of Theorem 2 are extended to the following perturbed n -order integrators with mismatched disturbances:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + d_i \\ &\vdots \\ \dot{x}_n &= u + d_n. \end{aligned} \quad (34)$$

Assumption 1. The mismatched disturbances d_i ($1 \leq i \leq n - 1$) have up to $(n - i + 1)$ -order bounded derivatives, and the matched disturbance d_n is bounded with $|d_n| \leq \delta$.

As proposed in [31], [32], to achieve the fixed-time TSMC of (34), a series of fixed-time disturbance observers is designed, one firstly considers the auxiliary variables:

$$\Phi_i = x_i(t) - x_i(0) - \int_0^t x_{i+1}(\theta) d\theta = \int_0^t d_i(\theta) d\theta, \quad (35)$$

then designs the fixed-time observers based on Lemma 2:

$$\begin{aligned} \dot{z}_{ij_i} &= -k_{ij_i} \phi_{ij_i}(z_{i1} - \Phi_i) + z_{i(j_i+1)}, \\ &\vdots \\ \dot{z}_{in_i} &= -k_{in_i} \phi_{in_i}(z_{i1} - \Phi_i), \end{aligned} \quad (36)$$

with $\phi_{ij_i}(\cdot)$ defined in Lemma 2, $i \in \{1, \dots, n - 1\}$, $n_i = n - i + 2$, and $j_i \in \{1, \dots, n_i\}$. From Lemma 2 and Remark 1, for any fixed settling time $T_O > 0$ chosen a priori, it is possible to scale the gains $k_{ij_i}, \kappa_{ij_i}, \rho_{ij_i}$ by using [25, Section III-C] and (5) to establish the following online observations:

$$z_{ij_i}(t) \equiv d_i^{(j_i-2)}(t), \quad \forall t \geq T_O \quad (37)$$

with $j_i \geq 2$.

Next, the terminal sliding surfaces are designed. Based on the outputs of (36), one firstly defines the sliding variable $s_0 = x_1$ and expands its initial condition with $s_0(\theta) \in \mathbf{C}^n([-h, 0])$ based on Remark 2, then estimates $s_0^{(i)} = x_{i+1} + \sum_{j=1}^i d_j^{(i-j)}$ by using the following approximation:

$$\widehat{s_0^{(i)}} = x_{i+1} + \sum_{j=1}^i z_{j(i+2-j)}, \quad (38)$$

for all $i \in \{1, \dots, n - 1\}$. In addition, the initial conditions of (38) are expanded to satisfy $\widehat{s_0^{(i)}}(\theta) \in \mathbf{C}^{n-i}([-h, 0])$ according to Remark 2. Similar to (26), it is possible to define the terminal sliding surfaces by using:

$$s_i(t) = \widehat{s}_{i-1}(t) - [as_{i-1}(t) - K_i(t)s_{i-1}(t-h)], \quad (39)$$

and expand their initial conditions $s_i(\theta) \in \mathbf{C}^{n-i}([-h, 0])$ by Remark 2, and the \mathbf{C}^∞ time-varying gains $K_i(t)$ are set to:

$$K_i(t) = K(t - 2(n - i)h) \quad (40)$$

with $K(t)$ defined by (8)–(9)–(10). Next, the derivatives of (39) are estimated as:

$$\widehat{s}_i^{(j)}(t) = \begin{cases} \widehat{s}_{i-1}^{(j+1)}(t) - \left[a\widehat{s}_{i-1}^{(j)}(t) - \sum_{r=0}^j C_j^r K_i^{(r)}(t) \widehat{s}_{i-1}^{(j-r)}(t-h) \right], & j > 0 \\ \widehat{s}_i(t), & j = 0 \end{cases} \quad (41)$$

with expanded initial condition $\widehat{s}_i^{(j)}(\theta) \in \mathbf{C}^{n-i-j}([-h, 0])$ selected by Remark 2, and $\{(i, j) \in \mathbb{N}^2 | 1 \leq i \leq n-1, 0 \leq j \leq n-i-1\}$.

Finally, the following control law is designed to accomplish the robust nonsingular predefined-time TSMC for system (34):

$$\begin{aligned} u(t) &= u_{eq}(t) + u_{FXT}(t), \\ u_{eq}(t) &= - \sum_{i=1}^{n-1} z_{i(n+2-i)}(t) + \sum_{i=1}^{n-1} \left[a\widehat{s}_{i-1}^{(n-i)}(t) - \sum_{r=0}^{n-i} C_{n-i}^r K_i^{(r)}(t) \widehat{s}_{i-1}^{(n-i-r)}(t-h) \right], \\ u_{FXT}(t) &= -k \left[s_{n-1}(t) \right]^{\frac{\lambda s_{n-1}(t)^2}{1+\mu s_{n-1}(t)^2}}. \end{aligned} \quad (42)$$

Thereafter, the convergence results of the state estimation (38)–(41) and the closed-loop stability analysis are stated in Lemma 5 and Theorem 3, respectively.

Lemma 5. *If the settling time T_O in (37) satisfies $T_O \leq h$, then the following relation:*

$$\widehat{s}_i^{(j)}(t) = s_i^{(j)}(t), \quad \forall t \geq h \quad (43)$$

holds for all $\{(i, j) \in \mathbb{N}^2 | 0 \leq i \leq n-1, 0 \leq j \leq n-i-1\}$.

Proof. The proof is based on induction. Firstly, let $i = 0$, if $T_O \leq h$ is satisfied, then the base case of (43):

$$\widehat{s}_0^{(j)}(t) = s_0^{(j)}(t), \quad \forall t \geq h \quad (44)$$

is ensured by (37)–(38) for all $0 \leq j \leq n-1$. Next, suppose that the inductive hypothesis $\widehat{s}_i^{(j)}(t) = s_i^{(j)}(t)$ holds for all admissible j on interval $t \geq h$, then (39) leads to:

$$s_{i+1}(t) = \dot{s}_i(t) - [a s_i(t) - K_{i+1}(t) s_i(t-h)] \quad (45)$$

for all $t \geq h$, which further implies:

$$s_{i+1}^{(j)}(t) = s_i^{(j+1)}(t) - \left[a s_i^{(j)}(t) - \sum_{r=0}^j C_j^r K_{i+1}^{(r)}(t) s_i^{(j-r)}(t-h) \right] \quad (46)$$

for all $t \geq h$. Recall the definition in (41), it leads to:

$$\widehat{s}_{i+1}^{(j)}(t) = \widehat{s}_i^{(j+1)}(t) - \left[a\widehat{s}_i^{(j)}(t) - \sum_{r=0}^j C_j^r K_{i+1}^{(r)}(t) \widehat{s}_i^{(j-r)}(t-h) \right]. \quad (47)$$

Comparing the right-hand sides of (46)–(47) for all $t \geq h$, one notices that their first and second terms are equivalent under the inductive hypothesis. Next, consider the delayed terms of (46)–(47), they are eliminated by the gain $K_i(t) = 0$ for all $i \in \{1, \dots, n-1\}$ on $t \in [0, 2h)$ according to (40) and (8)–(9)–(10), and they are equivalent on $t \geq 2h$ due to $\widehat{s}_i^{(j-r)}(t-h) =$

$s_i^{(j-r)}(t-h)$ for all $t \geq 2h$ from the inductive hypothesis. Hence, it leads to $\widehat{s}_{i+1}^{(j)}(t) = s_{i+1}^{(j)}(t)$ for all $t \geq h$, so assertion (43) holds for all admissible pairs (i, j) by induction. \square

Theorem 3. *Consider the perturbed n -order integrator (34) controlled by the nonsingular TSMC (8)–(9)–(10)–(36)–(40)–(42). If Assumption 1 is fulfilled, the inequality $\chi = \frac{\lambda}{1+\mu} > 1$ is satisfied, k is sufficiently large to satisfy (6), and the parameters $k_{ij_i}, \kappa_{ij_i}, \rho_{ij_i}$ of (36) are appropriately tuned and scaled by (5) to ensure $T_O \leq h$ for each observer, then the sliding variables (39)–(38) satisfy:*

$$s_i(t) \equiv 0, \quad \forall t \geq 2(n-i)h \quad (48)$$

for all $i \in \{1, \dots, n-1\}$. Then, the closed-loop system converges towards the perturbed equilibrium trajectory $(x_1, x_2, \dots, x_n) = (0, -d_1, \dots, -\sum_{i=1}^{n-1} d_i^{(n-i-1)})$ before the predefined instant $t = 2nh$.

Proof. Recall the relation:

$$\begin{aligned} \sum_{r=0}^{n-i} C_{n-i}^r K_i^{(r)}(t) \widehat{s}_{i-1}^{(n-i-r)}(t-h) &= \\ \sum_{r=0}^{n-i-1} C_{n-i-1}^r K_i^{(r)}(t) \widehat{s}_{i-1}^{(n-i-r)}(t-h) &+ \\ + \sum_{r=0}^{n-i-1} C_{n-i-1}^r K_i^{(r+1)}(t) \widehat{s}_{i-1}^{(n-i-r-1)}(t-h), & \end{aligned} \quad (49)$$

then consider the state definitions (38)–(41), the control law (42), and apply the same technique as in (31)–(32) leads to the following system dynamics:

$$\begin{aligned} \dot{s}_{n-1}(t) &= \sum_{i=1}^{n-1} [\dot{z}_{i(n+1-i)}(t) - z_{i(n+2-i)}(t)] + a \sum_{i=1}^{n-1} \left[\widehat{s}_{i-1}^{(n-i)}(t) - s_{i-1}^{(n-i-1)}(t) \right] + \sum_{i=1}^{n-1} \sum_{r=0}^{n-i-1} C_{n-i-1}^r K_i^{(r)}(t) \\ &\times \left[\widehat{s}_{i-1}^{(n-i-r-1)}(t-h) - s_{i-1}^{(n-i-r)}(t-h) \right] \\ &- k \left[s_{n-1}(t) \right]^{\frac{\lambda s_{n-1}(t)^2}{1+\mu s_{n-1}(t)^2}} + d_n(t). \end{aligned} \quad (50)$$

On interval $t \in [0, h)$, the arguments in [33], [25], [24] ensure that the closed-loop dynamics (50) cannot escape in finite time. Then, Lemma 2 and Lemma 5 provide that the first and second error terms of (50) vanish for all $t \geq h$. Next, the third term of (50) is equal to 0 since $K_i^{(r)}(t) = 0$ for all $t \in [0, 2h)$ by definition (40), and the delayed error term becomes zero for all $t \geq 2h$ thanks to Lemma 5. Hence, (50) is reduced to:

$$\dot{s}_{n-1}(t) = -k \left[s_{n-1}(t) \right]^{\frac{\lambda s_{n-1}(t)^2}{1+\mu s_{n-1}(t)^2}} + d_n(t), \quad \forall t \geq h \quad (51)$$

and $s_{n-1}(t)$ is stabilized before $t = 2h$ by virtue of Lemma 1 and the sufficiently large k tuned by (6).

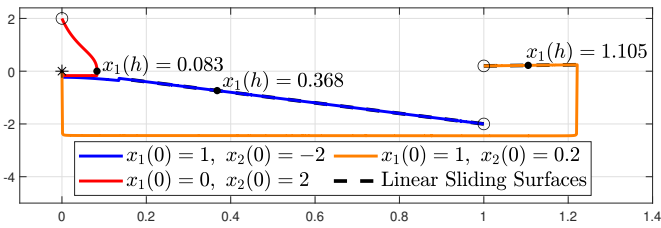
The proof can be completed by combining the definition (39), Lemma 5, and the proof of Theorem 2. \square

V. SIMULATION RESULTS

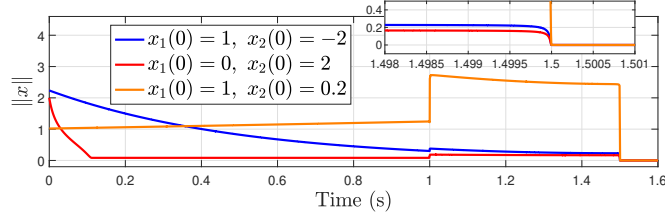
In this section, Proposition 1 and Remark 3 are firstly verified by subsection V-A. Next, subsection V-B studies the TSMC for perturbed triple integrators by using Theorem 3.

A. Simulation results of double integrator

Consider the prescribed-time TSMC of the perturbed double integrator (16) under matched disturbance $d = 1.5 + \sin(2t)$ (with $\delta = 2.5$) at $T_c = 1.5s$, for which the control law (19) and the parameter tuning (23) are designed. Three initial conditions are considered in the simulation: the first one is $(x_1(0), x_2(0)) = (1, -2)$, the second one is $(x_1(0), x_2(0)) = (0, 2)$, and the last one is $(x_1(0), x_2(0)) = (1, 0.2)$. The parameters of the sliding surface (17) are set to $h = T_c/3 = 0.5s$ and $M = 1.00001$. Then one determines $a = -2, W = 0.0848$ from (23)–(10) for the first case, $a = 0, W = 2.0017$ for the second, and $a = 0.2, W = 2.6975$ for the last. The parameters of $u_{FXT}(t)$ in (19) are set to $k = 18, \lambda = 3, \mu = 1.2$ by $\chi = \frac{\lambda}{1+\mu} > 1$ and (6) to satisfy $\frac{1}{(k-\delta)(\chi-1)} + \frac{1}{ke^{-\lambda/2e-\delta}} \leq h$. The simulation results are presented in Fig. 2.



(a) Phase portraits under different initial conditions.



(b) State norms versus time under different initial conditions.

Fig. 2: Prescribed-time TSMC of double integrator (16).

In the first case $(x_1(0), x_2(0)) = (1, -2)$, one determines $a = x_2(0)/x_1(0) = -2$ by (23), then the trajectories start evolution on the linear sliding surface $s = x_2 - ax_1 = 0$ and thus are generated by the linear dynamics $\dot{x}_1 = ax_1$ on interval $t \in [0, h]$. Hence, one observes from Fig. 2a that $x_1(h) = e^{ah}x_1(0) = 0.368 \neq 0$, which ensures the prescribed-time stability at $t = 3h = 1.5s$ thanks to Proposition 1.

In the second case $(x_1(0), x_2(0)) = (0, 2)$, one sets $a = 0$ by (23), so the reaching dynamics (24) force the trajectories of (x_1, x_2) to rotate clockwise on the phase plane and reach the x_1 -axis at an equilibrium point $(0.083, 0)$ satisfying $x_1(h) \neq 0$ as shown in Fig. 2a. Thereafter, on $t \in [h, 3h]$, the imposed dynamics (22) is activated and make (x_1, x_2) move to the desired equilibrium point $(0, 0)$ at $t = 3h = 1.5s$.

In the last case $(x_1(0), x_2(0)) = (1, 0.2)$, one firstly sets $a = x_2(0)/x_1(0) = 0.2 > 0$ by (23). Under the same arguments as the first case, one has $x_1(h) = e^{ah}x_1(0) = 1.105 \neq 0$, then (x_1, x_2) reach the origin $(0, 0)$ at $t = 3h = 1.5s$ due to Proposition 1. Next, for all $t > 3h = 1.5s$, the practical parameter switching $a = -1$ is conducted according to Remark 3, in order to preserve the stability in numerical implementation, and one observes from Fig. 2a that the trajectories of (x_1, x_2) stay at the origin $(0, 0)$ after the reaching phase.

Finally, the aforementioned explanations are verified by Fig. 2b, in which the norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ converges to zero precisely at the prescribed instant $t = T_c = 3h = 1.5s$ and stays at zero thereafter, illustrating the effectiveness of Proposition 1 and Remark 3.

B. Simulation results of triple integrator

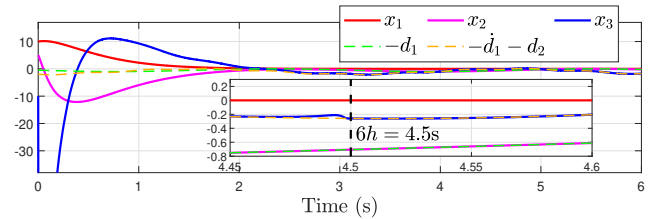
Consider the following perturbed triple integrator:

$$\begin{aligned}\dot{x}_1 &= x_2 + d_1 \\ \dot{x}_2 &= x_3 + d_2 \\ \dot{x}_3 &= u + d_3\end{aligned}\quad (52)$$

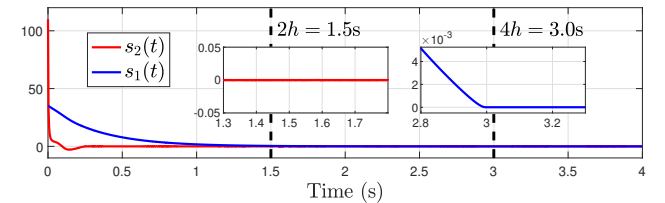
with $d_1 = 0.5 + 0.5 \sin(2t)$, $d_2 = 1 + 0.2 \sin(10t)$, $d_3 = 1 + \cos(2t)$, and $\delta = 2$. The control law is designed by (42) with $n = 3$. The parameters of the sliding surfaces are chosen such that $h = 0.75s$, $a = -3$, and $M = 1.01$, then one determines $W = 9.6795 \times 10^{-4}$ offline by (10). The parameters of $u_{FXT}(t)$ in (42) are set to $k = 22, \lambda = 5, \mu = 3$ by $\chi = \frac{\lambda}{1+\mu} > 1$ and (6) to satisfy $\frac{1}{(k-\delta)(\chi-1)} + \frac{1}{ke^{-\lambda/2e-\delta}} \leq h$, and the parameters of the disturbance observers (36) are appropriately tuned by [25, Section III-C] to satisfy $T_O = 0.65s \leq h$.

Two simulations are conducted to verify the main results of Theorem 3. The first one (with initial condition $(10, 5, -10)$) shows the effectiveness of the control scheme (42), whereas the second one is carried out to illustrate the robust predefined-time tracking under different initial conditions.

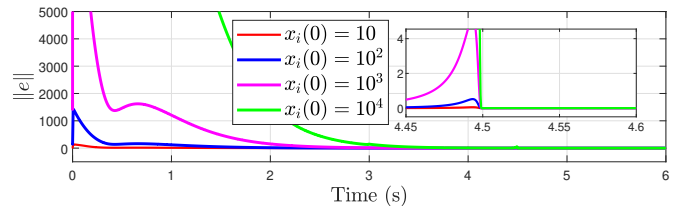
In the first simulation, one observes from Figs. 3a–3b that



(a) States x_1, x_2, x_3 and disturbances $-d_1, -d_2 - \dot{d}_1$ versus time.



(b) Terminal sliding variables s_1, s_2 versus time.



(c) Error norm $\|e\|$ versus time with initial conditions $x_i(0) = 10^1, 10^2, 10^3, 10^4$ for all $i \in \{1, 2, 3\}$.

Fig. 3: Predefined-time TSMC of triple integrator (52).

the sliding variable s_2 is stabilized before $t = 2h = 1.5s$

under the fixed-time reaching control law $u_{FXT}(t)$ in (42), then s_1, s_0 are predefined-time stabilized by the sliding surfaces (39) before $t = 4h = 3.0\text{s}$ and $t = 6h = 4.5\text{s}$, respectively. Finally, Fig. 3a shows that the trajectories of the closed-loop system converge towards the perturbed equilibrium trajectory $(0, -d_1, -d_2 - \dot{d}_1)$ before the predefined instant $t = 6h = 4.5\text{s}$. Thus, the simulation results are in accordance with the theoretical results of Theorem 3.

In the second simulation, one sets the initial conditions of $x_i(0)$ to $10^1, 10^2, 10^3, 10^4$ for all $i \in \{1, 2, 3\}$ in order to verify the predefined-time convergence. Please see Fig. 3c and the zoomed figure, the error norm $\|e\| = \sqrt{x_1^2 + (x_2 + d_1)^2 + (x_3 + d_2 + \dot{d}_1)^2}$ converges to zero before the predefined instant $t = 6h = 4.5\text{s}$ under different initial conditions, illustrating that the robust predefined-time tracking is realized by the control scheme.

VI. CONCLUSIONS

This paper provides a class of robust nonsingular predefined-time/prescribed-time TSMC methods. Firstly, a class of nonsingular smooth prescribed-time sliding surfaces is designed, and a robust predefined-time TSMC is developed for perturbed double integrators, then a sufficient condition is provided to preserve the robust prescribed-time stability. Secondly, the control scheme is extended to perturbed chains of integrators, and the compensation for Lipschitz mismatched disturbances is considered. In the future, the control method will be developed to deal with nonlinear triangular systems, and the experimental validation of the proposed method will be implemented on electro-mechanical robotic systems [34].

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