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Nicolas Marie

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MULTIFUNCTION ESTIMATION IN A TIME-DISCRETIZED SKOROKHOD REFLECTION PROBLEM

NICOLAS MARIE[†]

[†]Université Paris Nanterre, CNRS, Modal'X, 92001 Nanterre, France.

ABSTRACT. This paper deals with a consistent estimator of the multifunction involved in a time-discretized Skorokhod reflection problem defined by a stochastic differential equation and a Moreau sweeping process.

1. INTRODUCTION

Let \mathbf{X}_t be the location recorded at time $t \in [0, T]$ of a target living in a convex body $\mathbf{C}(t)$ of \mathbb{R}^m , where $T > 0$ and $m \in \mathbb{N}^*$. Due to a lack of information on the target's behavior, $\mathbf{C}(t)$ may be unknown, and then needs to be estimated. For instance, if \mathbf{X}_t is the location of an animal at time t , then its territory $\mathbf{C}(t)$ is a priori unknown. Such an application in ethology has been investigated in Cholaquidis et al. [4] when the process \mathbf{X} is a reflected Brownian motion. More generally, \mathbf{X} may be modeled by the following continuous-time Skorokhod reflection problem defined by a stochastic differential equation and a Moreau sweeping process (see Moreau [8]):

$$(1) \quad \begin{aligned} \mathbf{X}_t &= \int_0^t b(\mathbf{X}_s) ds + \int_0^t \sigma(\mathbf{X}_s) dW_s + \mathbf{Y}_t \\ &\text{with } \begin{cases} -\dot{\mathbf{Y}}_t \in \mathcal{N}_{\mathbf{C}(t)}(\mathbf{X}_t) & |\mathrm{DY}| \text{-a.e.} \\ \mathbf{Y}_0(\cdot) \equiv \mathbf{x}_0 \end{cases} ; t \in [0, T], \end{aligned}$$

where

- b (resp. σ) is a Lipschitz continuous map from \mathbb{R}^m into itself (resp. $\mathrm{GL}_m(\mathbb{R})$; the space of the $m \times m$ invertible matrices with real entries), W is a m -dimensional Brownian motion and $\mathbf{C} : \mathbb{R}_+ \rightrightarrows \mathbb{R}^m$ is a convex-body-valued multifunction.

- For every convex body K of \mathbb{R}^m ,

$\mathcal{N}_K(\mathbf{x}) := \{\mathbf{s} \in \mathbb{R}^m : \forall \mathbf{y} \in K, \langle \mathbf{s}, \mathbf{y} - \mathbf{x} \rangle \leq 0\}$ is the normal cone of K at point $\mathbf{x} \in \mathbb{R}^m$.

- For every continuous function of bounded variation $\varphi : [0, T] \rightarrow \mathbb{R}^m$,

$$\dot{\varphi} := \frac{d\mathrm{D}\varphi}{d|\mathrm{D}\varphi|} \quad \text{with } \mathrm{D}\varphi \text{ the differential measure of } \varphi \text{ and } |\mathrm{D}\varphi| \text{ its variation measure.}$$

Key words and phrases. Skorokhod reflection problem ; Stochastic differential equation ; Set estimation.

Roughly speaking, a solution of Problem (1) behaves as a diffusion process when $\mathbf{X}_t \in \text{int}(\mathcal{C}(t))$, but when \mathbf{X}_t reaches $\partial\mathcal{C}(t)$, a minimal force \mathbf{Y}_t is applied to \mathbf{X}_t in order to keep it inside of $\mathcal{C}(t)$. About the existence and uniqueness of the solution of Problem (1), the reader may refer to Bernicot and Venel [1] and Castaing et al. [2]. Under appropriate regularity conditions on the multifunction \mathcal{C} (see Bernicot and Venel [1], Theorems 3.9 and 4.6), a uniformly convergent approximation of \mathbf{X} is defined by

$$(2) \quad \begin{cases} \bar{\mathbf{x}}_{j+1} = \Pi_{\mathcal{C}(t_{j+1})}(\bar{\mathbf{x}}_j + b(\bar{\mathbf{x}}_j)\Delta_n + \sigma(\bar{\mathbf{x}}_j)(\mathbf{W}_{t_{j+1}} - \mathbf{W}_{t_j})) \\ \bar{\mathbf{x}}_0 \equiv \mathbf{x}_0 \in \mathcal{C}(0) \end{cases} ; j \in J_n \setminus \{n\},$$

where $\Pi_C(\cdot)$ is the orthogonal projection from \mathbb{R}^m onto a given convex body C of \mathbb{R}^m , $J_n := \{0, \dots, n\}$ with $n \in \mathbb{N}^*$, $\Delta_n := T/n$ with $T > 0$, and $t_j := j\Delta_n$ for every $j \in J_n$. Our paper, which is part of the copies-based statistical inference for diffusion processes (see Comte and Genon-Catalot [5], Marie and Rosier [7], Denis et al. [6], etc.), deals with the consistency of the estimator

$$\widehat{\mathcal{C}}_{N,j} := \text{conv}\{\bar{\mathbf{x}}_j^1, \dots, \bar{\mathbf{x}}_j^N\} ; j \in J_n \setminus \{0\}$$

of $\mathcal{C}(t_j)$, where $\text{conv}(\cdot)$ is the convex hull operator, and $\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^N$ are N independent copies of $\bar{\mathbf{x}}$. Precisely, Theorem 2.1 says that $\widehat{\mathcal{C}}_{N,j}$ converges to $\mathcal{C}(t_j)$ - with rate N - when $m = 1$, and Theorem 2.2 says that $\widehat{\mathcal{C}}_{N,j}$ converges pointwise to $\mathcal{C}(t_j)$ when $m > 1$ and the multifunction \mathcal{C} is decreasing (possibly constant). Finally, for another example of estimator computed from observations of the approximation scheme of a reflected diffusion model, the reader may refer to Cattiaux et al. [3].

Notations and basic definitions:

- The Euclidean norm (resp. distance) on \mathbb{R}^m is denoted by $\|\cdot\|$ (resp. d).
- For every $m \times m$ real matrix \mathbf{A} ,

$$\|\mathbf{A}\|_{\text{op}} := \sup_{\mathbf{x} \in \mathbb{R}^m} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}.$$

- For every $K \in \mathcal{P}(\mathbb{R}^m)$,

$$d(\cdot, K) := \inf\{d(\cdot, \mathbf{y}) ; \mathbf{y} \in K\}.$$

- Consider two normed vector spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$. For every Lipschitz continuous map $f : E \rightarrow F$,

$$\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{\|f(y) - f(x)\|_F}{\|y - x\|_E}.$$

2. MAIN RESULTS

First, the following theorem says that $\widehat{\mathcal{C}}_{N,j}$ converges to $\mathcal{C}(t_j)$ - with rate N - when $m = 1$.

Theorem 2.1. *Assume that there exists a constant $\mathfrak{m}_{\mathbf{C}} > 0$ such that, for every $t \in [0, T]$ and $x \in \mathbf{C}(t)$, $|x| \leq \mathfrak{m}_{\mathbf{C}}$. For any $j \in J_n \setminus \{0\}$,*

$$d_{\mathbf{H}}(\widehat{\mathbf{C}}_{N,j}, \mathbf{C}(t_j)) := \max\{\sup(\mathbf{C}(t_j)) - \sup(\widehat{\mathbf{C}}_{N,j}), \inf(\widehat{\mathbf{C}}_{N,j}) - \inf(\mathbf{C}(t_j))\} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0,$$

and if in addition σ is constant, then

$$Nd_{\mathbf{H}}(\widehat{\mathbf{C}}_{N,j}, \mathbf{C}(t_j)) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Now, consider two convex bodies K and L of \mathbb{R}^m such that $L \subset K$. If

$$d(\mathbf{x}, L) = 0 ; \forall \mathbf{x} \in \text{int}(K),$$

then $\text{int}(K) \subset L$, leading to $K = L$ because K and L are closed subsets of \mathbb{R}^m . For this reason, it makes sense to say that a sequence $(L_n)_{n \in \mathbb{N}}$ of random convex-compact subsets of K converges pointwise (in probability) to K if and only if,

$$d(\mathbf{x}, L_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 ; \forall \mathbf{x} \in \text{int}(K).$$

The following theorem says that $\widehat{\mathbf{C}}_{N,j}$ is pointwise consistent, for every $j \in J_n \setminus \{0\}$, when the multifunction \mathbf{C} is decreasing (possibly constant).

Theorem 2.2. *Assume that $\mathbf{C}(t) \subset \mathbf{C}(s)$ for every $s, t \in [0, T]$ satisfying $t \geq s$. Then, for every $j \in J_n \setminus \{0\}$, $\widehat{\mathbf{C}}_{N,j}$ converges pointwise to $\mathbf{C}(t_j)$:*

$$d(\mathbf{x}, \widehat{\mathbf{C}}_{N,j}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 ; \forall \mathbf{x} \in \text{int}(\mathbf{C}(t_j)).$$

APPENDIX A. PROOFS

A.1. Proof of Theorem 2.1. The proof of Theorem 2.1 relies on the following lemma.

Lemma A.1. *Consider $C = [I, S]$ with $I, S \in \mathbb{R}$ satisfying $I < S$, a \mathbb{R} -valued random variable H , and*

$$\widehat{C}_N := [\widehat{I}_N, \widehat{S}_N] \quad \text{with} \quad \widehat{I}_N = \min\{X_1, \dots, X_N\}$$

$$\text{and} \quad \widehat{S}_N = \max\{X_1, \dots, X_N\},$$

where X_1, \dots, X_N are N independent copies of $X = \Pi_C(H)$. Moreover, let Φ (resp. F) be the distribution function of H (resp. X).

(1) Assume that

$$(3) \quad \Phi((I, S)) \subset (0, 1).$$

Then,

$$d_{\mathbb{H}}(\widehat{\mathcal{C}}_N, C) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

(2) Assume that there exist $\varphi, \psi \in C^1(C; (0, 1))$ such that

$$(4) \quad \psi(\cdot) \leq \Phi(\cdot) \leq \varphi(\cdot) \quad \text{on } C.$$

Then,

$$Nd_{\mathbb{H}}(\widehat{\mathcal{C}}_N, C) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

The proof of Lemma A.1 is postponed to Subsubsection A.1.1.

Consider $j \in J_n \setminus \{n\}$, and let Φ_{j+1} be the distribution function of

$$\mathbb{H}_{j+1} := \bar{\mathbf{X}}_j + b(\bar{\mathbf{X}}_j)\Delta_n + \sigma(\bar{\mathbf{X}}_j)\mathbf{Z}_{j+1} \quad \text{with} \quad \mathbf{Z}_{j+1} = \mathbb{W}_{t_{j+1}} - \mathbb{W}_{t_j}.$$

First, since $\bar{\mathbf{X}}_j \in \mathcal{C}(t_j)$, $|\bar{\mathbf{X}}_j| \leq \mathfrak{m}_c$, and then

$$\underbrace{-\mathfrak{m}_c + \Delta_n \min_{|v| \leq \mathfrak{m}_c} b(v) + \sigma(\bar{\mathbf{X}}_j)\mathbf{Z}_{j+1}}_{=: m_n} \leq \mathbb{H}_{j+1} \leq \underbrace{\mathfrak{m}_c + \Delta_n \max_{|v| \leq \mathfrak{m}_c} b(v) + \sigma(\bar{\mathbf{X}}_j)\mathbf{Z}_{j+1}}_{=: M_n}.$$

So, by assuming that $\sigma(\cdot) > 0$ without loss of generality, for any $x \in \mathbb{R}$,

$$(5) \quad \begin{aligned} \mathbb{P}\left(\mathbf{Z}_{j+1} \leq -\frac{|x - M_n|}{m_\sigma}\right) &\leq \mathbb{P}(\sigma(\bar{\mathbf{X}}_j)\mathbf{Z}_{j+1} \leq x - M_n) \\ &\leq \Phi_{j+1}(x) \\ &\leq \mathbb{P}(\sigma(\bar{\mathbf{X}}_j)\mathbf{Z}_{j+1} \leq x - m_n) \leq \mathbb{P}\left(\mathbf{Z}_{j+1} \leq \frac{|x - m_n|}{m_\sigma}\right) \end{aligned}$$

with

$$m_\sigma = \min_{|x| \leq \mathfrak{m}_c} \sigma(x).$$

Since \mathbf{Z}_{j+1} is a Gaussian random variable, $\Phi_{j+1}(x) \in (0, 1)$, and then

$$d_{\mathbb{H}}(\widehat{\mathcal{C}}_{N,j}, \mathcal{C}(t_j)) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \quad \text{by Lemma A.1.(1).}$$

Now, assume that $\sigma(\cdot) \equiv \mathfrak{s}$ with $\mathfrak{s} > 0$. By Inequality (5),

$$\psi_{j+1}(\cdot) \leq \Phi_{j+1}(\cdot) \leq \varphi_{j+1}(\cdot) \quad \text{on } \mathbb{R},$$

where φ_{j+1} (resp. ψ_{j+1}) is the distribution function of $\xi_{j+1} := \mathfrak{s}\mathbf{Z}_{j+1} + M_n$ (resp. $\zeta_{j+1} := \mathfrak{s}\mathbf{Z}_{j+1} + m_n$).

Since both ξ_{j+1} and ζ_{j+1} are Gaussian random variables, $\varphi_{j+1}, \psi_{j+1} \in C^1(\mathbb{R}; (0, 1))$, and then

$$Nd_{\mathbb{H}}(\widehat{\mathcal{C}}_{N,j}, \mathcal{C}(t_j)) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \quad \text{by Lemma A.1.(2).}$$

A.1.1. *Proof of Lemma A.1.* First of all, in order to prove Lemma A.1, let us establish the following suitable relationship between Φ and F :

$$(6) \quad F(x) = \Phi(x)\mathbf{1}_{I \leq x < S} + \mathbf{1}_{x \geq S}; \forall x \in \mathbb{R}.$$

For any $x \in \mathbb{R}$, since $\{I < H < S\} \subset \{X = H\}$,

$$\begin{aligned} F(x) &= \mathbb{P}(X \leq x, H \in (I, S)) + \mathbb{P}(X \leq x, H \notin (I, S)) \\ &= \mathbb{P}(H \leq x, I < H < S) + \mathbb{P}(X \leq x, H \leq I) + \mathbb{P}(X \leq x, H \geq S). \end{aligned}$$

Moreover,

$$\mathbb{P}(H \leq x, I < H < S) = \mathbb{P}(I < H \leq x)\mathbf{1}_{I < x < S} + \mathbb{P}(I < H < S)\mathbf{1}_{x \geq S}$$

and, since $\{H \leq I\} = \{X = I\}$ and $\{H \geq S\} = \{X = S\}$,

$$\mathbb{P}(X \leq x, H \leq I) + \mathbb{P}(X \leq x, H \geq S) = \mathbb{P}(H \leq I)\mathbf{1}_{x \geq I} + \mathbb{P}(H \geq S)\mathbf{1}_{x \geq S}.$$

Thus,

$$\begin{aligned} F(x) &= \mathbb{P}(I < H \leq x)\mathbf{1}_{I < x < S} + \mathbb{P}(H \leq I)\mathbf{1}_{x \geq I} \\ &\quad + (\mathbb{P}(I < H < S) + \mathbb{P}(H \geq S))\mathbf{1}_{x \geq S} \\ &= (\mathbb{P}(H \leq x) - \mathbb{P}(H \leq I))\mathbf{1}_{I < x < S} \\ &\quad + \mathbb{P}(H \leq I)(\mathbf{1}_{x=I} + \mathbf{1}_{x>I}) + \mathbb{P}(H > I)\mathbf{1}_{x \geq S} \\ &= \mathbb{P}(H \leq x)\mathbf{1}_{I \leq x < S} + \underbrace{(\mathbb{P}(H \leq I) + \mathbb{P}(H > I))}_{=1}\mathbf{1}_{x \geq S} = \Phi(x)\mathbf{1}_{I \leq x < S} + \mathbf{1}_{x \geq S}. \end{aligned}$$

Let us establish Lemma A.1.(1,2).

(1) First, let us show that H satisfies (3) if and only if

$$(7) \quad \Phi((-\infty, S)) \subset [0, 1) \quad \text{and} \quad \Phi((I, \infty)) \subset (0, 1].$$

On the one hand, if H satisfies (7), then

$$\begin{aligned} \Phi((I, S)) &= \Phi((-\infty, S) \cap (I, \infty)) \\ &\subset \Phi((-\infty, S)) \cap \Phi((I, \infty)) \subset [0, 1) \cap (0, 1] = (0, 1). \end{aligned}$$

On the other hand, (3) implies (7) because Φ is increasing. Now, let us prove that if

$$(8) \quad \Phi((-\infty, S)) \subset [0, 1),$$

then \widehat{S}_N is a converging estimator of S . By (6) and (8),

$$(9) \quad \sup\{x \in \mathbb{R} : F(x) < 1\} = S.$$

Let F_N be the distribution function of \widehat{S}_N . Since X_1, \dots, X_N are independent copies of X , for any $x \in \mathbb{R}$,

$$F_N(x) = \mathbb{P}\left(\bigcap_{i=1}^N \{X_i \leq x\}\right) = F(x)^N.$$

On the one hand, for $x < S$, $F(x) \in [0, 1)$ by (9), and then

$$\lim_{N \rightarrow \infty} F_N(x) = \lim_{N \rightarrow \infty} F(x)^N = 0.$$

On the other hand, for $x \geq S$, $F(x) = 1$ by (9), and then

$$\lim_{N \rightarrow \infty} F_N(x) = \lim_{N \rightarrow \infty} F(x)^N = 1.$$

So, F_N converges pointwise to $\mathbf{1}_{[S, \infty)}$ (the distribution function of $\omega \mapsto S$), leading to

$$\widehat{S}_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} S.$$

Finally, by assuming that H satisfies (3) (or equivalently (7)),

$$\widehat{I}_N \xrightarrow[N \rightarrow \infty]{\mathbb{P}} I,$$

and thus

$$\begin{aligned} d_{\mathbb{H}}(\widehat{C}_N, C) &= \sup_{x \in C} d(x, \widehat{C}_N) \quad \text{because } \widehat{C}_N \subset C \\ &= \max\{S - \widehat{S}_N, \widehat{I}_N - I\} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0. \end{aligned}$$

(2) First, since X is a $[I, S]$ -valued random variable, for every $x \in \mathbb{R}_+$,

$$\mathbb{P}(N(\widehat{S}_N - S) \leq x) = \mathbb{P}\left(X \leq S + \frac{x}{N}\right)^N = 1.$$

Now, by Equality (6), and since $\Phi(\cdot) \leq \varphi(\cdot)$ on $C = [I, S]$ with $\varphi \in C^1(C; (0, 1))$, for every $y \in [I, S)$,

$$\begin{aligned} 1 - \mathbb{P}(X > y) &= F(y) = \Phi(y) \\ &\leq \varphi(y) \underset{y \uparrow S}{=} \varphi(S) + \varphi'(S)(y - S) + o(y - S). \end{aligned}$$

Then, for any $x < 0$, and every $N \in \mathbb{N}^*$ such that $N \geq -x/(S - I)$,

$$\begin{aligned} \mathbb{P}(N(\widehat{S}_N - S) \leq x) &= \left(1 - \mathbb{P}\left(X > S + \frac{x}{N}\right)\right)^N \\ &\leq \varphi\left(S + \frac{x}{N}\right)^N = \varphi(S)^N \exp(N \log(1 + U_N(x))) \\ &\quad \text{with } U_N(x) = \frac{\varphi(S + x/N)}{\varphi(S)} - 1, \end{aligned}$$

and

$$N \log(1 + U_N(x)) \underset{N \rightarrow \infty}{=} \frac{\varphi'(S)}{\varphi(S)} x + o(1).$$

Since $\varphi(S) < 1$, the distribution function of $N(\widehat{S}_N - S)$ converges pointwise to $\mathbf{1}_{\mathbb{R}_+}$, and thus

$$(10) \quad N(S - \widehat{S}_N) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Since $\Phi(\cdot) \geq \psi(\cdot)$ on $C = [I, S]$ with $\psi \in C^1(C; (0, 1))$, as previously,

$$(11) \quad N(\widehat{I}_N - I) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

In conclusion, by (10) and (11),

$$Nd_{\mathbb{H}}(\widehat{C}_N, C) = \max\{N(S - \widehat{S}_N), N(\widehat{I}_N - I)\} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

A.2. Proof of Theorem 2.2. The proof of Theorem 2.2 relies on the following lemma.

Lemma A.2. *Consider a convex body C of \mathbb{R}^m , a \mathbb{R}^m -valued random variable H , and*

$$\widehat{C}_N := \text{conv}\{X_1, \dots, X_N\},$$

where X_1, \dots, X_N are N independent copies of $X = \Pi_C(H)$. Assume that, for every $\mathbf{x} \in \text{int}(C)$ and $\varepsilon > 0$,

$$(12) \quad \mathbb{P}_H(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(C)) > 0.$$

Then,

$$d(\mathbf{x}, \widehat{C}_N) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0; \quad \forall \mathbf{x} \in \text{int}(C).$$

The proof of Lemma A.2 is postponed to Subsubsection A.2.1.

For every $j \in J_n \setminus \{n\}$, consider

$$\mathbf{H}_{j+1} := \bar{\mathbf{X}}_j + b(\bar{\mathbf{X}}_j)\Delta_n + \sigma(\bar{\mathbf{X}}_j)Z_{j+1} \quad \text{with} \quad Z_{j+1} = \mathbf{W}_{t_{j+1}} - \mathbf{W}_{t_j}.$$

The proof of Theorem 2.2 is dissected in three steps. Step 1 deals with a suitable control of $\|\mathbf{H}_{j+1} - \mathbf{x}\|$, depending on \mathbf{Z}_{j+1} and $\|\bar{\mathbf{x}}_j - \mathbf{x}\|$, for every $j \in J_n \setminus \{n\}$ and $\mathbf{x} \in \mathbf{C}(t_{j+1})$. In Step 2, the condition

$$\begin{aligned} \mathbb{P}_{\mathbf{H}_j}(\bar{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(\mathbf{C}(t_j))) &> 0 ; \\ \forall j \in J_n \setminus \{0\}, \forall \varepsilon > 0, \forall \mathbf{x} \in \text{int}(\mathbf{C}(t_j)) \end{aligned}$$

is checked in order to apply Lemma A.2 to $\widehat{\mathbf{C}}_{N,1}, \dots, \widehat{\mathbf{C}}_{N,n}$ in Step 3.

Step 1. For any $j \in J_n \setminus \{n\}$ and $\mathbf{x} \in \mathbf{C}(t_{j+1})$, since b and σ are Lipschitz continuous maps,

$$\begin{aligned} \|\mathbf{H}_{j+1} - \mathbf{x}\| &= \|\bar{\mathbf{x}}_j - \mathbf{x} + (b(\bar{\mathbf{x}}_j) - b(\mathbf{x}))\Delta_n \\ &\quad + (\sigma(\bar{\mathbf{x}}_j) - \sigma(\mathbf{x}))\mathbf{Z}_{j+1} + b(\mathbf{x})\Delta_n + \sigma(\mathbf{x})\mathbf{Z}_{j+1}\| \\ &\leq (1 + \|b\|_{\text{Lip}}\Delta_n + \|\sigma\|_{\text{Lip}}\|\mathbf{Z}_{j+1}\|)\|\bar{\mathbf{x}}_j - \mathbf{x}\| + \mathbf{R}_{j+1}(\mathbf{x}) \\ &\quad \text{with } \mathbf{R}_{j+1}(\mathbf{x}) = \|\sigma(\mathbf{x})\mathbf{Z}_{j+1} + b(\mathbf{x})\Delta_n\|. \end{aligned}$$

On the one hand, since $\sigma(\mathbf{x}) \in \text{GL}_m(\mathbb{R})$,

$$\begin{aligned} \|\mathbf{Z}_{j+1}\| &= \|\sigma(\mathbf{x})^{-1}(\sigma(\mathbf{x})\mathbf{Z}_{j+1} + b(\mathbf{x})\Delta_n - b(\mathbf{x})\Delta_n)\| \\ &\leq \|\sigma(\mathbf{x})^{-1}\|_{\text{op}}(\mathbf{R}_{j+1}(\mathbf{x}) + \|b(\mathbf{x})\Delta_n\|. \end{aligned}$$

On the other hand, since $\mathbf{C}(t_0), \dots, \mathbf{C}(t_n) \subset \mathbf{C}(0)$, and since $\mathbf{C}(0)$ is a compact subset of \mathbb{R}^m , there exists a constant $\mathbf{m}_\mathbf{C} > 0$ such that, for every $k \in J_n$ and $\mathbf{v} \in \mathbf{C}(t_k)$, $\|\mathbf{v}\| \leq \mathbf{m}_\mathbf{C}$. Thus,

$$(13) \quad \|\mathbf{H}_{j+1} - \mathbf{x}\| \leq \mathbf{c}_1 \|\bar{\mathbf{x}}_j - \mathbf{x}\| + \mathbf{c}_2 \mathbf{R}_{j+1}(\mathbf{x})$$

with

$$\begin{aligned} \mathbf{c}_1 &= 1 + \left(\|b\|_{\text{Lip}} + \|\sigma\|_{\text{Lip}} \sup_{\|\mathbf{v}\| \leq \mathbf{m}_\mathbf{C}} \{\|\sigma(\mathbf{v})^{-1}\|_{\text{op}}\|b(\mathbf{v})\|\} \right) \Delta_n \\ &\quad \text{and } \mathbf{c}_2 = 1 + 2\mathbf{m}_\mathbf{C}\|\sigma\|_{\text{Lip}} \sup_{\|\mathbf{v}\| \leq \mathbf{m}_\mathbf{C}} \|\sigma(\mathbf{v})^{-1}\|_{\text{op}}. \end{aligned}$$

Step 2. For every $j \in J_n \setminus \{0\}$ and $\mathbf{x} \in \text{int}(\mathbf{C}(t_j))$, there exists $\varepsilon_{j,\mathbf{x}} > 0$ such that

$$(14) \quad \bar{B}_d(\mathbf{x}, \varepsilon_{j,\mathbf{x}}) \subset \text{int}(\mathbf{C}(t_j)).$$

First, for every $\varepsilon > 0$ and $\mathbf{x} \in \text{int}(\mathcal{C}(t_1))$, by (14), since $\sigma(\mathbf{x}_0) \in \text{GL}_m(\mathbb{R})$, and since $[\mathbf{W}_{t_1}]_1, \dots, [\mathbf{W}_{t_1}]_m$ are i.i.d. Gaussian random variables,

$$\begin{aligned} & \mathbb{P}_{\mathbf{H}_1}(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(\mathcal{C}(t_1))) \\ & \geq \mathbb{P}(\|\mathbf{x}_0 - \mathbf{x} + b(\mathbf{x}_0)\Delta_n + \sigma(\mathbf{x}_0)\mathbf{W}_{t_1}\| \leq \varepsilon \wedge \varepsilon_{1,\mathbf{x}}) \\ & \geq \mathbb{P}(\|\mathbf{W}_{t_1} + \sigma(\mathbf{x}_0)^{-1}(\mathbf{x}_0 - \mathbf{x} + b(\mathbf{x}_0)\Delta_n)\| \leq \|\sigma(\mathbf{x}_0)\|_{\text{op}}^{-1}(\varepsilon \wedge \varepsilon_{1,\mathbf{x}})) \geq \mu_{1,\mathbf{x}}(\varepsilon) \end{aligned}$$

with

$$\begin{aligned} \mu_{1,\mathbf{x}}(\varepsilon) &= \prod_{k=1}^m \mathbb{P}(|[\mathbf{W}_{t_1}]_k + [\sigma(\mathbf{x}_0)^{-1}(\mathbf{x}_0 - \mathbf{x} + b(\mathbf{x}_0)\Delta_n)]_k| \\ & \leq (\sqrt{m}\|\sigma(\mathbf{x}_0)\|_{\text{op}})^{-1}(\varepsilon \wedge \varepsilon_{1,\mathbf{x}}) > 0. \end{aligned}$$

Now, consider $j \in J_n \setminus \{0, n\}$, and assume that for every $\varepsilon > 0$ and $\mathbf{x} \in \text{int}(\mathcal{C}(t_j))$,

$$(15) \quad \mathbb{P}_{\mathbf{H}_j}(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(\mathcal{C}(t_j))) > 0.$$

For any $\varepsilon > 0$ and $\mathbf{x} \in \text{int}(\mathcal{C}(t_{j+1}))$, by Inequality (13), by (14), since \mathbf{Z}_{j+1} (and then \mathbf{R}_{j+1}) is independent of $\mathcal{F}_j = \sigma(\mathbf{W}_{t_0}, \dots, \mathbf{W}_{t_j})$, and since $\overline{\mathbf{X}}_j$ is \mathcal{F}_j -measurable,

$$\begin{aligned} & \mathbb{P}_{\mathbf{H}_{j+1}}(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(\mathcal{C}(t_{j+1}))) \\ & \geq \mathbb{P}(\|\mathbf{H}_{j+1} - \mathbf{x}\| \leq \varepsilon \wedge \varepsilon_{j+1,\mathbf{x}}) \\ & \geq \mathbb{P}(\mathbf{c}_1\|\overline{\mathbf{X}}_j - \mathbf{x}\| \leq 2^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}}), \mathbf{c}_2\mathbf{R}_{j+1}(\mathbf{x}) \leq 2^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}})) \\ & \geq \mathbb{P}(\|\overline{\mathbf{X}}_j - \mathbf{x}\| \leq (2\mathbf{c}_1)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}}), \mathbf{H}_j \in \text{int}(\mathcal{C}(t_j))) \\ & \quad \times \mathbb{P}(\mathbf{R}_{j+1}(\mathbf{x}) \leq (2\mathbf{c}_2)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}})). \end{aligned}$$

On the one hand, since $\overline{\mathbf{X}}_j = \mathbf{H}_j$ on $\{\mathbf{H}_j \in \text{int}(\mathcal{C}(t_j))\}$, and since $\mathbf{x} \in \mathcal{C}(t_{j+1}) \subset \mathcal{C}(t_j)$, by (15),

$$\begin{aligned} & \mathbb{P}(\|\overline{\mathbf{X}}_j - \mathbf{x}\| \leq (2\mathbf{c}_1)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}}), \mathbf{H}_j \in \text{int}(\mathcal{C}(t_j))) \\ & = \mathbb{P}_{\mathbf{H}_j}(\overline{B}_d(\mathbf{x}, (2\mathbf{c}_1)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}})) \cap \text{int}(\mathcal{C}(t_j))) > 0. \end{aligned}$$

On the other hand, since $\sigma(\mathbf{x}) \in \text{GL}_m(\mathbb{R})$, and since $[\mathbf{Z}_{j+1}]_1, \dots, [\mathbf{Z}_{j+1}]_m$ are i.i.d. Gaussian random variables,

$$\begin{aligned} & \mathbb{P}(\mathbf{R}_{j+1}(\mathbf{x}) \leq (2\mathbf{c}_2)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}})) \\ & \geq \mathbb{P}(\|\sigma(\mathbf{x})\|_{\text{op}}\|\mathbf{Z}_{j+1} + \sigma(\mathbf{x})^{-1}b(\mathbf{x})\Delta_n\| \leq (2\mathbf{c}_2)^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}})) \geq \mu_{j+1,\mathbf{x}}(\varepsilon) \end{aligned}$$

with

$$\begin{aligned} \mu_{j+1,\mathbf{x}}(\varepsilon) &= \prod_{k=1}^m \mathbb{P}(|[Z_{j+1}]_k + [\sigma(\mathbf{x})^{-1}b(\mathbf{x})]_k \Delta_n| \\ &\leq (2c_2\sqrt{m}\|\sigma(\mathbf{x})\|_{\text{op}})^{-1}(\varepsilon \wedge \varepsilon_{j+1,\mathbf{x}}) > 0. \end{aligned}$$

Step 3 (conclusion). By Step 2, for any $j \in J_n \setminus \{0\}$,

$$\mathbb{P}_{H_j}(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(\mathbf{C}(t_j))) > 0; \forall \varepsilon > 0, \forall \mathbf{x} \in \text{int}(\mathbf{C}(t_j)).$$

Thus, by Lemma A.2, $\widehat{\mathbf{C}}_{N,j}$ converges pointwise to $\mathbf{C}(t_j)$:

$$d(\mathbf{x}, \widehat{\mathbf{C}}_{N,j}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0; \forall \mathbf{x} \in \text{int}(\mathbf{C}(t_j)).$$

A.2.1. *Proof of Lemma A.2.* Consider $\mathbf{x} \in C$ and

$$\delta_N(\mathbf{x}) := \min_{i=1}^N d(\mathbf{x}, X_i).$$

Since $\{X_1, \dots, X_N\} \subset \widehat{\mathbf{C}}_N$ by the definition of $\widehat{\mathbf{C}}_N$,

$$\begin{aligned} d(\mathbf{x}, \widehat{\mathbf{C}}_N) &= \inf_{\mathbf{y} \in \widehat{\mathbf{C}}_N} d(\mathbf{x}, \mathbf{y}) \\ &\leq \min_{\mathbf{y} \in \{X_1, \dots, X_N\}} d(\mathbf{x}, \mathbf{y}) = \delta_N(\mathbf{x}). \end{aligned}$$

Moreover, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\delta_N(\mathbf{x}) > \varepsilon) &= \mathbb{P}\left(\bigcap_{i=1}^N \{d(\mathbf{x}, X_i) > \varepsilon\}\right) \\ &= \mathbb{P}(d(\mathbf{x}, X) > \varepsilon)^N = (1 - F_{\mathbf{x}}(\varepsilon))^N, \end{aligned}$$

where $F_{\mathbf{x}}$ is the distribution function of $d(\mathbf{x}, X)$, and

$$\begin{aligned} F_{\mathbf{x}}(\varepsilon) &= \mathbb{P}(d(\mathbf{x}, X) \leq \varepsilon, H \in \text{int}(C)) + \mathbb{P}(d(\mathbf{x}, X) \leq \varepsilon, H \notin \text{int}(C)) \\ &\geq \mathbb{P}(d(\mathbf{x}, H) \leq \varepsilon, H \in \text{int}(C)) \\ &\quad \text{because } \{H \in \text{int}(C)\} \subset \{X = H\}. \end{aligned}$$

Thus,

$$\mathbb{P}(d(\mathbf{x}, \widehat{\mathbf{C}}_N) > \varepsilon) \leq (1 - \mathbb{P}_H(\overline{B}_d(\mathbf{x}, \varepsilon) \cap \text{int}(C)))^N.$$

In conclusion, by Inequality (12),

$$d(\mathbf{x}, \widehat{\mathbf{C}}_N) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

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Email address: nmarie@parisnanterre.fr