



HAL
open science

Extended angle conditions for the study of the asymptotic behaviour of some continuous and discrete forced dynamical systems

Haythem Cheikh

► **To cite this version:**

Haythem Cheikh. Extended angle conditions for the study of the asymptotic behaviour of some continuous and discrete forced dynamical systems. 2024. hal-04637289

HAL Id: hal-04637289

<https://hal.science/hal-04637289>

Preprint submitted on 5 Jul 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Extended angle conditions for the study of the asymptotic behaviour of some continuous and discrete forced dynamical systems

HAYTHEM CHEIKH

Laboratoire Équations aux Dérivées Partielles, LR03ES04,
Faculté des sciences de Tunis, Université Tunis El Manar,
2092 El Manar, Tunisia,

and

Laboratoire M2N
CNAM, 292 rue Saint-Martin
75003 Paris, France.
haythemcheikh8@gmail.com

Abstract

In this paper we study the asymptotic behaviour of the bounded solution of the following forced system and its discretization

$$\varepsilon u'' + |u'|^\alpha u' + \nabla F(u) = g(t)$$

with conditions on F and g . We present similar forced angle conditions for both continuous and discrete cases, and then we apply it to examples from the literature.

Mathematics Subject Classification 2010 (MSC2010): 34E10, 37N30, 40A05, 35L05, 35L70, 65L07.

Key words: Angle condition, Lojasiewicz's inequality, Discretization, single limit-point convergence, stability, convergence rates.

1 Introduction

In this paper we study the asymptotic behaviour of the following differential equation and its discretization

$$\varepsilon u'' + |u'|^\alpha u' + \nabla F(u) = g(t), \tag{1}$$

where $F : \mathbb{R}^d \rightarrow \mathbb{R}$, $g : \mathbb{R}_+ \rightarrow \mathbb{R}^d$, $\varepsilon \geq 0$ and $\alpha \geq 0$. The main question that we want to deal with is whether a bounded solution of (1) or its implicit discretization converges to an equilibrium point. It is now well known that convergence or nonconvergence may occur.

There are some previous works in the literature on this subject in continuous situation:

- when $\varepsilon = 0$, $\alpha = 0$, F is analytic and $g = 0$ (see [20, 21]);
- when $\varepsilon = 1$, $\alpha = 0$, F is analytic and $g = 0$ (see [14, 13]);
- when $\varepsilon = 1$, $\alpha > 0$ is small enough, F is analytic and $g = 0$ (see [7]);
- when $\varepsilon = 1$, $\alpha > 0$ small enough, F is analytic and g in some sense in L^1 (see [5]).

There are also previous works in the discrete situation:

- when $\varepsilon = \alpha = 0$, $g_n = 0$ (g_n is a discretization of g) and F is analytic (see [1] for explicit discretization and [3, 22] for implicit discretization);
- when $\varepsilon = 1$, $\alpha = 0$, $g_n = 0$ and F is analytic (see [2]);
- when $\varepsilon = 1$, $\alpha = 0$, g_n is in l^2 and F is analytic (see [11]);
- when $\varepsilon = 1$, $\alpha > 0$ is small enough, $g_n = 0$ and F is analytic (see [18]);
- when $\varepsilon = 1$, $\alpha > 0$ small enough, g_n in some sense in l^1 and F is analytic (see [6]).

Note that convergence of bounded solutions does not always occur, as illustrated in [23, 19, 13, 17] in the continuous case and in [1] in the discrete case.

An important tool used to obtain the convergence is the so-called angle condition. Let us quickly review this approach. Equation (1) (with $g = 0$) can be written as a first-order system

$$\dot{u}(t) + \mathcal{F}(u(t)) = 0, \quad t \geq 0, \quad (2)$$

where $\mathcal{F} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuous function. Let $\mathcal{E} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a function of class C^1 .

If there exists $\sigma > 0$ such that

$$\forall u \in \mathbb{R}^N : \quad \langle \mathcal{E}'(u), \mathcal{F}(u) \rangle \geq \sigma \|\mathcal{E}'(u)\| \|\mathcal{F}(u)\|,$$

then we say that \mathcal{E} and \mathcal{F} satisfy an angle condition.

This condition first appeared in [1]. It proved that every solution of (1) with $\varepsilon = \alpha = 0$ et $g = 0$ satisfies this angle condition. Chill et al. [8] generalized this result for second order system by proving that every solution of (1) with $\varepsilon = 1$ and $\alpha = 0$, $g = 0$ satisfies this angle condition. Haraux and Jendoubi [13] propose the more general inequality

$$\exists \beta \geq 0 \mid \forall u \in \mathbb{R}^N : \quad \langle \mathcal{E}'(u), \mathcal{F}(u) \rangle \geq \sigma \|\mathcal{E}'(u)\|^{1+\beta} \|\mathcal{F}(u)\|.$$

to cover the case of equation (1) with $\alpha > 0$.

In this paper, we present similar forced angle conditions for both continuous and discrete cases (forced in the sense that $g \neq 0$), and then we apply it to examples from the literature. With these angle conditions (see (3) and (29)) and under the assumption that some energy satisfies the Lojasiewicz inequality, we give new proofs of our preceding results given in [6]. In order to do so, we prove two main abstract theorems (Theorem 2.2 and Theorem 3.2). Our proofs are inspired by the ideas in [9, 5, 11] for the continuous case and in [22, 11] for

the discrete case.

Note that the natural condition for a forced system is

$$\forall t \in \mathbb{R}_+ : \quad -\frac{d}{dt}[\phi(u(t))] + Cf(t) \geq \sigma \|\nabla\phi(u(t))\|^{\beta+1} \|u'(t)\|,$$

with $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$.

Let us also mention that, to our best knowledge, angle conditions in the forced case have never been used before.

This article is organized as follows:

Section 2 deals with the continuous case. In Subsection 2.1, we specify an abstract result for the continuous case. In Subsection 2.2, we apply the result of Subsection 2.1 to (1) with various cases depending on the values of ε and α .

Section 3 deals with the discrete case.

In Subsection 3.1, we specify an abstract result for the discrete case. In Subsection 3.2, we apply the result of Subsection 3.1 in an implicit discretization of (1) with various cases depending on the values of ε and α .

This method allows us to get the same results as in [11]. However, we obtain different results from those in [5]. It is not surprising, since our conditions on g are different from the ones in [5].

2 Continuous case

2.1 An abstract result

In this section we give the definition of an angle condition and we prove that functions satisfying this condition converge.

Definition 2.1. Let $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$.

We say that a function $x \in C^1(\mathbb{R}_+, \mathbb{R}^N)$ satisfies a forced continuous angle condition relative to ϕ and f if there exist constants $C > 0$, $\beta > 0$ and $\lambda > 0$ such that

$$\forall t \in \mathbb{R}_+ : \quad -\frac{d}{dt}[\phi(x(t))] + Cf(t) \geq \lambda \left(\|\nabla\phi(x(t))\|^{\beta+2} + \|x'(t)\|^{\beta+2} \right). \quad (3)$$

We now state and prove the main result of this part:

Theorem 2.2. *Let x be a function that satisfies a forced continuous angle condition relative to functions ϕ , f and constants C , β , λ . We assume that ϕ satisfies a Lojasiewicz-type inequality (see [20, 21]), that is, there exists $\theta \in (0, \frac{1}{2}]$ such that*

$$\begin{aligned} \forall a \in \mathbb{R}^N, \exists \gamma_a > 0, \exists \sigma_a > 0 \mid \forall u \in \mathbb{R}^N : \|u - a\| < \sigma_a \\ \implies \|\nabla\phi(u)\| \geq \gamma_a |\phi(u) - \phi(a)|^{1-\theta}, \end{aligned} \quad (4)$$

and that there is a constant $\delta > 0$ such that f satisfies

$$\int_0^{+\infty} f(s)ds < +\infty \text{ and } \sup_{t \in \mathbb{R}_+} \left(t^{1+\beta+\delta} \int_t^{+\infty} f(s)ds \right) < +\infty. \quad (5)$$

Assume also that $(\beta + 1)(1 - \theta) < 1$.

Then either $\lim_{t \rightarrow +\infty} \|x(t)\| = +\infty$, or there exists $x^* \in \mathbb{R}^N$ such that $\lim_{t \rightarrow +\infty} x(t) = x^*$.

Precisely, in the second case there exists a constant $A > 0$ such that

$$\forall t > 0: \quad \|x(t) - x^*\| \leq At^{-\mu} \text{ where } \mu = \min \left(\frac{1 - (1 - \theta)(\beta + 1)}{-1 + (1 - \theta)(\beta + 2)}, \frac{\delta}{\beta + 2} \right). \quad (6)$$

Remark 2.3. If ϕ satisfies (4) for some exponent $\theta \in (0, \frac{1}{2}]$, then, by changing σ and γ if necessary, it is easy to see that ϕ also satisfies (4) for every exponent $\theta' \in (0, \theta]$.

Proof. By modifying the constants θ, σ and γ in the inequality (4) (see Remark 2.3) we can assume that $\theta \in (\frac{\beta}{\beta+1}, \frac{1}{2})$ is small enough so that:

$$\frac{\delta}{\beta + 2} \geq \frac{1 - (1 - \theta)(\beta + 1)}{-1 + (1 - \theta)(\beta + 2)}.$$

This clearly comes from the fact that $\frac{1}{2} \geq \theta > \frac{\beta}{\beta+1}$ and the study of the function $s \rightarrow \frac{1 - (1-s)(\beta+1)}{-1 + (1-s)(\beta+2)}$.

Using this last inequality, we observe that

$$\theta \leq \frac{\beta + \frac{\delta(\beta+1)}{\beta+2}}{\beta + \delta + 1}. \quad (7)$$

Assume that $\|x(t)\| \not\rightarrow +\infty$ as $t \rightarrow +\infty$. Then $x(t)$ has an accumulation point x^* in \mathbb{R}^N .

Let us define

$$\forall t \geq 0: \quad \psi(t) = \phi(x(t)) - \phi(x^*) + C \int_t^{+\infty} f(s)ds.$$

Using (3), we get

$$\forall t \geq 0: \quad -\psi'(t) \geq \lambda \left(\|\nabla \phi(x(t))\|^{\beta+2} + \|x'(t)\|^{\beta+2} \right).$$

Then there exists a constant $\lambda' > 0$ such that

$$\forall t \geq 0: \quad -\psi'(t) \geq \lambda' (\|\nabla \phi(x(t))\| + \|x'(t)\|)^{\beta+2}. \quad (8)$$

The last inequality implies that ψ is nonincreasing. Moreover, since x^* is an accumulation point of $x(t)$, it follows by continuity of ψ that $\psi(t)$ decreases towards 0. If there is t_0 such

that $\psi(t_0) = 0$, then ψ must be constant because $\psi(t)$ decreases towards 0. By using (8), we can therefore deduce that x is constant which proves the result in this case. Otherwise, without loss of generality we can assume that

$$\forall t \in \mathbb{R}_+ : \quad 0 < \psi(t) \leq 1 \text{ and } 0 \leq C \int_t^{+\infty} f(s) ds \leq 1. \quad (9)$$

For all $t \geq 0$ such that $\|x(t) - x^*\| < \sigma_{x^*}$ (σ_{x^*} as in (4)), we have

$$(\psi(t))^{1-\theta} \leq |\phi(x(t)) - \phi(x^*)|^{1-\theta} + \left(C \int_t^{+\infty} f(s) ds \right)^{1-\theta}.$$

We can apply the inequality (4) and we obtain

$$(\psi(t))^{1-\theta} \leq C_1 \|\nabla\phi(x(t))\| + \left(C \int_t^{+\infty} f(s) ds \right)^{1-\theta} \quad (10)$$

where $C_1 = \frac{1}{\gamma_{x^*}}$.

Let $\zeta = (\beta + 1)\theta - \beta$. We have $\zeta = 1 - (1 - \theta)(\beta + 1)$. From $\beta \in [0, \frac{\theta}{1-\theta})$, it follows that $\zeta > 0$.

Now, let $\underline{t} \geq 0$ be such that $\|x(\underline{t}) - x^*\| < \sigma_{x^*}$. We distinguish two cases. First case, we assume that

$$\|x'(\underline{t})\| + \|\nabla\phi(x(\underline{t}))\| \leq \left(C \int_{\underline{t}}^{+\infty} f(s) ds \right)^{1-\theta}. \quad (11)$$

Using (7) and (9), we obtain

$$\|x'(\underline{t})\| + \|\nabla\phi(x(\underline{t}))\| \leq \left(C \int_{\underline{t}}^{+\infty} f(s) ds \right)^{\frac{1+\frac{\delta}{\beta+2}}{\beta+\delta+1}}.$$

Thanks to (5), there exists a constant $C_2 > 0$ such that

$$\|x'(\underline{t})\| \leq C_2 \underline{t}^{-(1+\frac{\delta}{\beta+2})}. \quad (12)$$

Second case, we assume that

$$\|x'(\underline{t})\| + \|\nabla\phi(x(\underline{t}))\| > \left(C \int_{\underline{t}}^{+\infty} f(s) ds \right)^{1-\theta}. \quad (13)$$

Then combining (10) and (13), there exists a constant $C_3 > 0$ such that

$$(\psi(\underline{t}))^{1-\theta} \leq C_3(\|x'(\underline{t})\| + \|\nabla\phi(x(\underline{t}))\|). \quad (14)$$

We have

$$\begin{aligned} -\frac{d}{dt}\psi_{/t=\underline{t}}^\zeta &= -\zeta\psi'(\underline{t})(\psi(\underline{t}))^{\zeta-1} \\ &= -\zeta\psi'(\underline{t})(\psi(\underline{t}))^{-(1-\theta)(\beta+1)}. \end{aligned}$$

By using (8) and (14) we obtain

$$-C_4\frac{d}{dt}\psi_{/t=\underline{t}}^\zeta \geq \|x'(\underline{t})\| \quad (15)$$

with $C_4 = \frac{C_3^{\beta+1}}{\zeta^\lambda}$.

So, in both cases for all t such that $\|x(t) - x^*\| < \sigma_{x^*}$, we have

$$\|x'(t)\| \leq -C_4\frac{d}{dt}\psi^\zeta(t) + C_2t^{-(1+\frac{\delta}{\beta+2})}. \quad (16)$$

Note that the function on the right-hand side is integrable on $(1, +\infty)$.

Now, let \bar{t} be sufficiently large such that

$$\|x(\bar{t}) - x^*\| < \frac{\sigma_{x^*}}{2} \text{ and } C_4\psi^\zeta(\bar{t}) + C_2\int_{\bar{t}}^{+\infty} s^{-(1+\frac{\delta}{\beta+2})} ds < \frac{\sigma_{x^*}}{2}. \quad (17)$$

Define

$$t^+ = \sup \{t \geq \bar{t} : \|x(s) - x^*\| < \sigma_{x^*} \forall s \in [\bar{t}, t)\}.$$

Let us assume by contradiction that $t^+ < +\infty$, so that $\|x(t^+) - x^*\| = \sigma_{x^*}$. For all $t \in [\bar{t}, t^+)$, estimate (16) is satisfied, and due to the choice of \bar{t} , we obtain

$$\|x(t) - x(\bar{t})\| \leq \int_{\bar{t}}^t \|x'(s)\| ds \leq \frac{\sigma_{x^*}}{2}. \quad (18)$$

Applying the triangle inequality,

$$\|x(t^+) - x^*\| \leq \|x(t^+) - x(\bar{t})\| + \|x(\bar{t}) - x^*\| < \sigma_{x^*}.$$

This leads to a contradiction. Therefore, $t^+ = +\infty$, estimate (18) is valid for all $t \geq \bar{t}$, then

$\int_{\bar{t}}^{+\infty} \|x'(s)\| ds < +\infty$ which implies that $x(t)$ has a limit as t tends to infinity.

Now, we will prove the convergence rate (6).

We follow the ideas in [12]. We define

$$J_1 = \{t > 0 : (11) \text{ holds}\} \text{ and } J_2 = \{t > 0 : (13) \text{ holds}\}.$$

By (8) and (14), there exists a constant $C_5 > 0$ such that, for every $t \in J_2$

$$\begin{aligned} -\frac{d}{dt}\psi^{1-(1-\theta)(\beta+2)}(t) &= -[1 - (1 - \theta)(\beta + 2)]\psi'(t) (\psi(t))^{-(1-\theta)(\beta+2)} \\ &\geq C_5. \end{aligned} \tag{19}$$

To get (6), let us assume first that either $[t_0, +\infty) \subset J_1$ or $[t_0, +\infty) \subset J_2$ for some $t_0 > 0$. In the first case we have, for any $t \geq t_0$,

$$\begin{aligned} \|x(t) - x^*\| &\leq \int_t^{+\infty} \|x'(s)\| ds \\ &\stackrel{\text{by (12)}}{\leq} \int_t^{+\infty} C_2 s^{-(1+\frac{\delta}{\beta+2})} ds \\ &\leq C_6 t^{-\frac{\delta}{\beta+2}}. \end{aligned}$$

If $[t_0, +\infty) \subset J_2$, we get by integrating (19) from t_0 to t .

$$\begin{aligned} \psi(t) &\leq \left(C_5 (t - t_0) + (\psi(t_0))^{1-(1-\theta)(\beta+2)} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}} \\ &\leq C_7 t^{\frac{-1}{-1+(1-\theta)(\beta+2)}}. \end{aligned}$$

Using (15) we get

$$\begin{aligned} \|x(t) - x^*\| &\leq \int_t^{+\infty} \|x'(s)\| ds \\ &\leq C_4 (\psi(t))^\zeta \\ &\leq C_4 C_7^\zeta t^{\frac{-\zeta}{-1+(1-\theta)(\beta+2)}}. \end{aligned}$$

The case where neither J_1 nor J_2 contain a half-line must specifically be handled in order to finish the proof. Because J_2 is an open set by definition, there exists a countable family of disjoint open intervals (a_n, b_n) such that $J_2 = \bigcup_{n=0}^{\infty} (a_n, b_n)$.

Let us consider $t \in J_2$. Let n^* be the integer n such that $t \in (a_n, b_n)$. Note that t can be chosen such that $a_{n^*} \geq 1$. We get by integrating (19) from a_{n^*} to t

$$\begin{aligned} \psi(t) &\leq \left(C_5 (t - a_{n^*}) + (\psi(a_{n^*}))^{1-(1-\theta)(\beta+2)} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}} \\ &\leq \left(C_5 (t - a_{n^*}) + (\psi(a_{n^*}))^{1-(1-\theta)(\beta+2)} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}}. \end{aligned}$$

We obtain, by using (5), (10) and (11)

$$(\psi(a_{n^*}))^{1-\theta} \leq \left(C_8 a_{n^*}^{-(1+\beta+\delta)} \right)^{(1-\theta)},$$

and, thus,

$$\psi(t) \leq \left(C_5 (t - a_{n^*}) + \left(C_8^{1-(1-\theta)(\beta+2)} a_{n^*}^{-(1+\beta+\delta)(1-(1-\theta)(\beta+2))} \right) \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}}.$$

From (7), it follows that

$$(1 + \beta + \delta)[-1 + (1 - \theta)(\beta + 2)] \geq 1.$$

Replacing C_5 by $C'_5 = \min \left(C_5, C_8^{1-(1-\theta)(\alpha+2)} \right)$, we obtain

$$\psi(t) \leq (C'_5 t)^{\frac{-1}{-1+(1-\theta)(\beta+2)}}.$$

Note that this last inequality holds for every $t \in (a_{n^*}, \infty) \cap J_2$.

We have, according to (12) and (15), for every $t > a_{n^*}$

$$\begin{aligned} \|x(t) - x^*\| &\leq \int_{(t,+\infty) \cap J_1} \|x'(s)\| ds + \int_{(t,+\infty) \cap J_2} \|x'(s)\| ds \\ &\leq \int_{(t,+\infty) \cap J_1} C_2 s^{-(1+\frac{\delta}{\beta+2})} ds + \int_{(t,+\infty) \cap J_2} -C_4 \frac{d}{ds} \psi^\zeta(s) ds \\ &\leq C_6 t^{-\frac{\delta}{\beta+2}} + C_4 (C'_5 t)^{\frac{-\zeta}{-1+(1-\theta)(\beta+2)}}. \end{aligned}$$

We, therefore, obtain the convergence rate (6) again. \square

2.2 Applications

In this section, we implement our approach on the following examples of continuous functions.

Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a C^2 function. We define

$$S = \{u^* \in \mathbb{R}^d / \nabla F(u^*) = 0\}.$$

We assume that:

$$\begin{aligned} \exists \theta \in (0, \frac{1}{2}] \forall u^* \in S \exists r_a > 0 \exists c_a > 0 \mid \forall u \in \mathbb{R}^d : \|u - u^*\| < r_a \\ \implies \|\nabla F(u)\| \geq c_a |F(u) - F(u^*)|^{1-\theta}. \end{aligned} \quad (20)$$

Proposition 2.4 ([18]). *The Assumption (20) holds if one of the following two cases is satisfied:*

- F is a polynomial function, or
- F is analytic and S is a compact set.

2.2.1 A first order gradient system

We consider the differential system:

$$U'(t) + \nabla F(U(t)) = g(t) \quad (21)$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a continuous function.

Theorem 2.5. *Let $U \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^d)$ be a function which satisfies (21), and assume that there exists a constant $\delta > 0$ such that g satisfies*

$$\int_0^{+\infty} \|g(s)\|^2 ds < +\infty \text{ and } \sup_{t \in \mathbb{R}_+} \left(t^{1+\delta} \int_t^{+\infty} \|g(s)\|^2 ds \right) < +\infty.$$

Then there exists $U^ \in \mathbb{R}^d$ such that $\lim_{t \rightarrow +\infty} U(t) = U^*$.*

Moreover, there exists a constant $A > 0$ such that for all $t > 0$, we have

$$\|U(t) - U^*\| \leq At^{-\mu} \text{ where } \mu = \min\left(\frac{\theta}{1-2\theta}, \frac{\delta}{2}\right).$$

Proof. We define $\phi := F$, $x := U$, $f := \|g\|^2$ and $\beta = 0$.

We have for all $t \geq 0$:

$$\begin{aligned} -\frac{d}{dt}\phi(x(t)) + \frac{1}{2}f(t) &= -\langle \nabla F(U(t)), U'(t) \rangle + \frac{1}{2}\langle g(t), g(t) \rangle \\ &= -\langle \nabla F(U(t)), U'(t) \rangle + \frac{1}{2}\langle U'(t) + \nabla F(U(t)), U'(t) + \nabla F(U(t)) \rangle \\ &= \frac{1}{2}\left(\|\nabla F(U(t))\|^2 + \|U'(t)\|^2\right) \\ &= \frac{1}{2}\left(\|\nabla\phi(x(t))\|^{\beta+2} + \|x'(t)\|^{\beta+2}\right). \end{aligned}$$

Then the function x satisfies (3) with the function ϕ and the statement follows from Theorem 2.2. \square

2.2.2 A second order gradient system

We consider the differential system:

$$U''(t) + \|U'(t)\|^\alpha U'(t) + \nabla F(U(t)) = g(t) \quad (22)$$

where $1 > \alpha > 0$ and $g : \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is a continuous function.

Theorem 2.6. Let $U \in W^{2,\infty}(\mathbb{R}_+, \mathbb{R}^d)$ be a function which satisfies (22), and assume that
(i) $\alpha \in (0, \frac{\theta}{1-\theta})$,
(ii) there exists a constant $\delta > 0$ such that g satisfies

$$\int_0^{+\infty} \|g(s)\| ds < +\infty \text{ and } \sup_{t \in \mathbb{R}_+} \left(t^{1+\alpha+\delta} \int_t^{+\infty} \|g(s)\| ds \right) < +\infty.$$

Then there exists $U^* \in \mathbb{R}^d$ such that $\lim_{t \rightarrow +\infty} U(t) = U^*$.

Moreover, there exists a constant $A > 0$ such that for all $t > 0$, we have

$$\|U(t) - U^*\| \leq At^{-\mu} \text{ where } \mu = \min \left(\frac{1 - (1-\theta)(\alpha+1)}{-1 + (1-\theta)(\alpha+2)}, \frac{\delta}{\alpha+2} \right).$$

Proof. We define

$$\phi_\varepsilon(u, v) = \frac{1}{2} \|v\|^2 + F(u) + \varepsilon \|\nabla F(u)\|^\alpha \langle \nabla F(u), v \rangle, \quad (23)$$

$x = (U, U')$, $f = \|g\|$ and $\beta = \alpha$. Using [5] and the fact that (U, U') is bounded, we can find $\varepsilon = \bar{\varepsilon} > 0$ small enough such that there exists constant $\gamma > 0$ satisfying

$$-\frac{d}{dt} \phi_{\bar{\varepsilon}}(x(t)) + C \|g(t)\| \geq \gamma [\|U'(t)\| + \|\nabla F(U(t))\|]^\alpha. \quad (24)$$

Let us show that x satisfies (3) with the function $\phi_{\bar{\varepsilon}}$. We have

$$\begin{aligned} \nabla \phi_{\bar{\varepsilon}}(u, v) = \\ \left(\begin{array}{c} \nabla F(u) + \bar{\varepsilon} \|\nabla F(u)\|^\alpha \nabla^2 F(u) \cdot v + \bar{\varepsilon} \alpha \|\nabla F(u)\|^{\alpha-2} \langle \nabla F(u), v \rangle \nabla^2 F(u) \cdot \nabla F(u) \\ v + \bar{\varepsilon} \|\nabla F(u)\|^\alpha \nabla F(u) \end{array} \right). \end{aligned}$$

Since (U, U') is bounded, there exists a constant $C_9 > 0$ such that

$$\forall t \in \mathbb{R}_+ : \quad \|\nabla \phi_{\bar{\varepsilon}}(x(t))\| \leq C_9 [\|U'(t)\| + \|\nabla F(U(t))\|]. \quad (25)$$

On the other hand

$$\begin{aligned} \|x'(t)\| &= \|(U'(t), U''(t))\| \\ &= \|(U'(t), -\|U'\|^\alpha - \nabla F(U(t)) + g(t))\| \\ &\leq C_{10} [\|U'(t)\| + \|\nabla F(U(t))\|] \end{aligned}$$

where we used the fact that U' and g are bounded.

By combining this last inequality with (24) and (25), we get that x satisfies (3).

Now, let us show that $\phi_{\bar{\varepsilon}}$ satisfies (4).

Let $B \subset \mathbb{R}^d \times \mathbb{R}^d$ be a ball containing $\{(U(t), U'(t)), t \in \mathbb{R}_+\}$. We have $\forall (u, v) \in B$

$$\begin{aligned}
& \|\nabla\phi_{\bar{\varepsilon}}(u, v)\| \\
= & \|\nabla F(u) + \bar{\varepsilon}\|\nabla F(u)\|^\alpha \nabla^2 F(u) \cdot v + \bar{\varepsilon}\alpha\|\nabla F(u)\|^{\alpha-2} \langle \nabla F(u), v \rangle \nabla^2 F(u) \cdot \nabla F(u)\| \\
& + \|v + \bar{\varepsilon}\|\nabla F(u)\|^\alpha \nabla F(u)\| \\
\geq & \|\nabla F(u)\| - \bar{\varepsilon}\|\|\nabla F(u)\|^\alpha \nabla^2 F(u) \cdot v + \bar{\varepsilon}\alpha\|\nabla F(u)\|^{\alpha-2} \langle \nabla F(u), v \rangle \nabla^2 F(u) \cdot \nabla F(u)\| \\
& + \|v\| - \bar{\varepsilon}\|\|\nabla F(u)\|^\alpha \nabla F(u)\| \\
\geq & (1 - \bar{\varepsilon}C_{11})[\|v\| + \|\nabla F(u)\|].
\end{aligned}$$

By choosing a smaller value for $\bar{\varepsilon} > 0$, there exists $\rho > 0$ such that

$$\forall (u, v) \in B \quad \|\nabla\phi_{\bar{\varepsilon}}(u, v)\| \geq \rho[\|v\| + \|\nabla F(u)\|]. \quad (26)$$

If (a,b) is not a critical point of $\phi_{\bar{\varepsilon}}$, then, using the continuity of $\phi_{\bar{\varepsilon}}$, it is evident that $\phi_{\bar{\varepsilon}}$ satisfies (4).

Consider $(a, b) \in B$ to be a critical point of $\phi_{\bar{\varepsilon}}$. Then $\nabla F(a) = 0$ and $b = 0$. By (20)

$$\exists r_a > 0 \exists c_a > 0 \forall u \in \mathbb{R}^d : \|u - a\| < r_a \implies \|\nabla F(u)\| \geq c_a |F(u) - F(a)|^{1-\theta}. \quad (27)$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
[\phi_{\bar{\varepsilon}}(u, v) - \phi_{\bar{\varepsilon}}(a, 0)]^{1-\theta} &= \left[\frac{1}{2}\|v\|^2 + F(u) - F(a) + \bar{\varepsilon}\|\nabla F(u)\|^\alpha \langle \nabla F(u), v \rangle\right]^{1-\theta} \\
&\leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\|^{(\alpha+1)(1-\theta)} \|v\|^{1-\theta}.
\end{aligned} \quad (28)$$

By using Young's inequality, we get

$$\|\nabla F(u)\|^{(\alpha+1)(1-\theta)} \|v\|^{1-\theta} \leq \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta-\alpha(1-\theta)}}.$$

Hence

$$[\phi_{\bar{\varepsilon}}(u, v) - \phi_{\bar{\varepsilon}}(a, 0)]^{1-\theta} \leq \|v\|^{2(1-\theta)} + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\|^{\frac{1-\theta}{\theta-\alpha(1-\theta)}}.$$

Using (27) and the fact that $2(1-\theta)$ and $\frac{1-\theta}{\theta-\alpha(1-\theta)}$ are larger than 1, we obtain for all $(u, v) \in B$ with $\|v\| \leq 1$ and $\|u - a\| < r_a$

$$\begin{aligned}
[\phi_{\bar{\varepsilon}}(u, v) - \phi_{\bar{\varepsilon}}(a, 0)]^{1-\theta} &\leq \|v\| + |F(u) - F(a)|^{1-\theta} + \|\nabla F(u)\| + \|v\| \\
&\leq \left(2 + \frac{1}{c_a}\right) [\|\nabla F(u)\| + \|v\|] \\
&\stackrel{\text{by (26)}}{\leq} \frac{1}{\rho} \left(2 + \frac{1}{c_a}\right) \|\nabla\phi_{\bar{\varepsilon}}(u, v)\|.
\end{aligned}$$

Then the function x satisfies (3) with the function $\phi_{\bar{\varepsilon}}$ and the statement follows from Theorem 2.2. \square

Remark 2.7. In [11], the equation (22) was studied with $\alpha = 0$, where it was assumed that F satisfies (20) and g satisfies $\int_0^{+\infty} \|g(s)\|^2 ds < +\infty$ and

$\sup_{t \in \mathbb{R}_+} \left(t^{1+\delta} \int_t^{+\infty} \|g(s)\|^2 ds \right) < +\infty$. By considering ϕ_ε defined by (23) with $\alpha = 0$ and using [11] and following the same proof as in the previous section, we get the same convergence result as in the previous theorem with $\alpha = 0$

3 Discrete case

3.1 An abstract result

In this section we prove an abstract theorem for the discrete case. For this we define the angle condition and we prove that sequences satisfying this condition converge.

Definition 3.1. Let $\phi \in C^1(\mathbb{R}^N, \mathbb{R})$ and $(f_n)_{n \in \mathbb{N}}$ be a non-negative sequence. We say that a sequence (x_n) of elements of \mathbb{R}^N satisfies a forced discretized angle condition relative to ϕ and $(f_n)_{n \in \mathbb{N}}$ if there exist constants $C > 0$, $\beta \geq 0$ and $\lambda > 0$ such that

$$\forall n \in \mathbb{N} : \quad \phi(x_n) - \phi(x_{n+1}) + C f_{n+1} \geq \lambda [\|\nabla \phi(x_{n+1})\|^{\beta+2} + \|x_{n+1} - x_n\|^{\beta+2}]. \quad (29)$$

We now state and prove the main result of this part:

Theorem 3.2. *Let (x_n) be a sequence that satisfies a forced discretized angle condition relative to function ϕ , sequence (f_n) and constants C, β, λ . We assume that ϕ satisfies a Lojasiewicz-type inequality, that is, there exists $\theta \in (0, \frac{1}{2}]$ such that*

$$\forall a \in \mathbb{R}^N, \exists \gamma_a > 0, \exists \sigma_a > 0 \mid \forall u \in \mathbb{R}^N : \|u - a\| < \sigma_a \implies \|\nabla \phi(u)\| \geq \gamma_a |\phi(u) - \phi(a)|^{1-\theta} \quad (30)$$

and that there is a constant $\delta > 0$ such that (f_n) satisfies

$$\sum_{k=0}^{+\infty} f_k < \infty \text{ and } \sup_{n \in \mathbb{N}} \left(n^{1+\delta+\beta} \sum_{k=n}^{+\infty} f_k \right) < \infty. \quad (31)$$

Additionally, we assume that $(\beta + 1)(1 - \theta) < 1$.

Then either $\lim_{n \rightarrow +\infty} \|x_n\| = +\infty$, or there exists $x^* \in \mathbb{R}^N$ such that $\lim_{n \rightarrow +\infty} x_n = x^*$.

Precisely, in the second there exists a constant $A > 0$ such that

$$\forall n \in \mathbb{N}^* : \quad \|x_n - x^*\| \leq A n^{-\mu} \text{ where } \mu = \min \left(\frac{1 - (1 - \theta)(\beta + 1)}{-1 + (1 - \theta)(\beta + 2)}, \frac{\delta}{\beta + 2} \right). \quad (32)$$

Proof. We recall that it can be assumed that $\theta \in (\frac{\beta}{\beta+1}, \frac{1}{2})$ satisfies

$$\frac{\delta}{\beta+2} \geq \frac{1 - (1-\theta)(\beta+1)}{-1 + (1-\theta)(\beta+2)}.$$

So that

$$\theta \leq \frac{\beta + \frac{\delta(\beta+1)}{\beta+2}}{\beta + \delta + 1}. \quad (33)$$

Now we assume that $\lim_{n \rightarrow +\infty} \|x_n\| \neq +\infty$. The sequence (x_n) then admits an accumulation point x^* .

Let for all $n \in \mathbb{N}$:

$$\psi_n = \phi(x_n) - \phi(x^*) + C \sum_{k=n+1}^{+\infty} f_{k+1}.$$

Using (29), we get

$$\psi_n - \psi_{n+1} \geq \lambda(\|\nabla\phi(x_{n+1})\|^{\beta+2} + \|x_{n+1} - x_n\|^{\beta+2}). \quad (34)$$

Then there exists a constant $\lambda' > 0$ such that

$$\psi_n - \psi_{n+1} \geq \lambda'(\|\nabla\phi(x_{n+1})\| + \|x_{n+1} - x_n\|)^{\beta+2}. \quad (35)$$

So that $(\psi_n)_n$ is nonincreasing. Let $n_k \rightarrow +\infty$ such that $x_{n_k} \rightarrow x^*$. We clearly have $\psi_{n_k} \rightarrow 0$ as $k \rightarrow +\infty$. So $\psi_n \rightarrow 0$ as $n \rightarrow +\infty$ and $\psi_n \geq 0$ for all $n \geq 0$.

Without loss of generality we assume that

$$\forall n \in \mathbb{N}: \quad 0 \leq \psi_n \leq 1 \text{ and } 0 \leq C \sum_{k=n+1}^{+\infty} f_{k+1} \leq 1. \quad (36)$$

We have, since $0 < \theta < 1$,

$$(\psi_{n+1})^{1-\theta} \leq |\phi(x_{n+1}) - \phi(x^*)|^{1-\theta} + \left(C \sum_{k=n+1}^{+\infty} f_{k+1} \right)^{(1-\theta)}. \quad (37)$$

If $\|x_{n+1} - x^*\| < \sigma_{x^*}$ (σ_{x^*} as in (30)), we can apply the inequality (30) and we obtain from (37)

$$(\psi_{n+1})^{1-\theta} \leq C_1 \|\nabla\phi(x_{n+1})\| + \left(C \sum_{k=n+1}^{+\infty} f_{k+1} \right)^{(1-\theta)} \quad (38)$$

where $C_1 = \frac{1}{\gamma_{x^*}}$. We recall that

$$\zeta = (\beta + 1)\theta - \beta.$$

We have

$$\begin{aligned}
\zeta - \frac{1}{\beta+2} &= \frac{(\beta+1)(\beta+2)\theta - \beta(\beta+2) - 1}{\beta+2} \\
&\leq \frac{\frac{1}{2}(\beta+1)(\beta+2) - \beta(\beta+2) - 1}{\beta+2} \\
&= \frac{-\frac{1}{2}(\beta^2 + \beta)}{\beta+2} \\
&\leq 0,
\end{aligned}$$

so that

$$0 < \zeta \leq \frac{1}{\beta+2}. \quad (39)$$

Now, consider $n \in \mathbb{N}^*$ such that $\|x_{n+1} - x^*\| < \sigma_{x^*}$.
First case, we assume that

$$\|x_{n+1} - x_n\| + \|\nabla\phi(x_{n+1})\| \leq \left(C \sum_{k=n+1}^{+\infty} f_{k+1} \right)^{1-\theta}. \quad (40)$$

Using (33) and (36), we obtain

$$\|x_{n+1} - x_n\| + \|\nabla\phi(x_{n+1})\| \leq \left(C \sum_{k=n+1}^{+\infty} f_{k+1} \right)^{\frac{1+\frac{\delta}{\beta+2}}{\beta+\delta+1}}.$$

Thanks to (31), there is a constant $C_2 > 0$ such that

$$\|x_{n+1} - x_n\| \leq C_2 n^{-(1+\frac{\delta}{\beta+2})}. \quad (41)$$

Second case, we assume that

$$\|x_{n+1} - x_n\| + \|\nabla\phi(x_{n+1})\| > \left(C \sum_{k=n+1}^{+\infty} f_{k+1} \right)^{1-\theta}. \quad (42)$$

As in [22] and [11], we distinguish two cases:

- If $\psi_{n+1} \leq \psi_n/2$, according to (34), there holds

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \left(\frac{1}{\lambda} \right)^{\frac{1}{\beta+2}} [\psi_n - \psi_{n+1}]^{\frac{1}{\beta+2}} \\
&\stackrel{\text{by (39)}}{\leq} \left(\frac{1}{\lambda} \right)^{\frac{1}{\beta+2}} [\psi_n]^\zeta \\
&\leq C_3 \left[(\psi_n)^\zeta - (\psi_{n+1})^\zeta \right],
\end{aligned} \quad (43)$$

where

$$C_3 = \left(\frac{1}{\lambda}\right)^{\frac{1}{\beta+2}} (1 - 2^{-\zeta})^{-1}.$$

- If $\psi_{n+1} > \psi_n/2$, we have

$$\begin{aligned} (\psi_n)^\zeta - (\psi_{n+1})^\zeta &= \int_{\psi_{n+1}}^{\psi_n} \zeta s^{\zeta-1} ds \\ &\geq \zeta (\psi_n)^{\zeta-1} [\psi_n - \psi_{n+1}] \\ &\geq \zeta 2^{\zeta-1} (\psi_{n+1})^{\zeta-1} [\psi_n - \psi_{n+1}]. \end{aligned} \quad (44)$$

Let us estimate $(\psi_{n+1})^{\zeta-1}$ from the below. Combining (38) and (42), there exists a constant $C_4 > 0$ such that

$$(\psi_{n+1})^{1-\theta} \leq C_4 (\|x_{n+1} - x_n\| + \|\nabla\phi(x_{n+1})\|) \quad (45)$$

then

$$(\psi_{n+1})^{1-\zeta} = (\psi_{n+1})^{(1-\theta)(\beta+1)} \leq C_4^{\beta+1} (\|x_{n+1} - x_n\| + \|\nabla\phi(x_{n+1})\|)^{\beta+1}. \quad (46)$$

Using (35) together with (44) and (46), we obtain

$$(\psi_n)^\zeta - (\psi_{n+1})^\zeta \geq \frac{\zeta 2^{\zeta-1}}{C_4^{\beta+1}} \lambda' \|x_{n+1} - x_n\|. \quad (47)$$

By combining (41) and (43) and (47), we can deduce that for all $n \geq 0$ such that $\|x_{n+1} - x^*\| < \sigma_{x^*}$

$$\|x_{n+1} - x_n\| \leq C_2 n^{-(1+\frac{\delta}{\beta+2})} + C_5 [(\psi_n)^\zeta - (\psi_{n+1})^\zeta] \quad (48)$$

where

$$C_5 = \max\left(\frac{C_4^{\beta+1}}{\zeta 2^{\zeta-1} \lambda'}, C_3\right).$$

Let $n_0 \in \mathbb{N}^*$ be sufficiently large such that

$$\|x_{n_0} - x^*\| < \frac{\sigma_{x^*}}{3}$$

and

$$C_2 \sum_{n=n_0}^{+\infty} n^{-(1+\frac{\delta}{\beta+2})} + C_5 (\psi_{n_0})^\zeta < \frac{\sigma_{x^*}}{3}. \quad (49)$$

We define $N_0 \geq n_0$ the largest integer such that $\|x_n - x^*\| < \frac{2\sigma_{x^*}}{3}$ for all $n_0 \leq n \leq N_0$ (we take $N_0 = +\infty$ if $\|x_n - x^*\| < \frac{2\sigma_{x^*}}{3}$ for all $n \geq n_0$).

Suppose, by contradiction, that $N_0 < +\infty$. According to (34), we obtain

$$\begin{aligned}
\|x_{N_0+1} - x^*\| &\leq \|x_{N_0+1} - x_{N_0}\| + \|x_{N_0} - x^*\| \\
&\leq C_3 (\psi_{N_0})^{\frac{1}{\beta+2}} + \|x_{N_0} - x^*\| \\
&\leq C_3 (\psi_{n_0})^{\frac{1}{\beta+2}} + \|x_{N_0} - x^*\| \\
&\leq C_5 (\psi_{n_0})^\zeta + \|x_{N_0} - x^*\| \\
&\stackrel{\text{by (36) and (39)}}{<} \sigma_{x^*},
\end{aligned}$$

where we used (49) and the fact that $(\psi_n)_n$ is nonincreasing. Thus, we can apply (48) for all $n \in \{n_0, \dots, N_0\}$ and summing from $n = n_0$ to $n = N_0$, we get

$$\begin{aligned}
\sum_{n=n_0}^{N_0} \|x_{n+1} - x_n\| &\leq C_2 \sum_{n=n_0}^{+\infty} n^{-(1+\frac{\delta}{\beta+2})} + C_5 (\psi_{n_0})^\zeta \\
&< \frac{\sigma_{x^*}}{3}.
\end{aligned} \tag{50}$$

So, $\|x_{N_0+1} - x^*\| \leq \frac{\sigma_{x^*}}{3} + \|x_{n_0} - x^*\| < \frac{2\sigma_{x^*}}{3}$, which contradicts the definition of N_0 .

As a result, $N_0 = +\infty$, estimate (50) remains true, and the sequence $(x_n)_n$ converges to x^* .

Now, we will prove the convergence rate (32).

If $\psi_n = 0$ for some $n_0 \in \mathbb{N}$, then $\psi_n = 0$ for $n \geq n_0$ and by using (35) we obtain that $x_{n+1} = x_n$ for $n \geq n_0$, we deduce $x_n = x^*$ for n large enough and (32) is clearly true.

Now, we assume that $\psi_n > 0$ for all $n \geq 0$ and let $n_0 \in \mathbb{N}^*$ large enough so that $\|x_{n+1} - x^*\| < \sigma$ for all $n \geq n_0$.

Let us define

$$J_1 = \{n \geq n_0 : (40) \text{ holds}\} \text{ and } J_2 = \{n \geq n_0 : (42) \text{ holds}\}.$$

We remark that J_1 and J_2 are disjoint and $J_1 \cup J_2 = \{n \geq n_0\}$. Let $n \in J_2$.

If $\psi_{n+1} \leq \frac{\psi_n}{2}$ then

$$\begin{aligned}
(\psi_{n+1})^{1-(1-\theta)(\beta+2)} - (\psi_n)^{1-(1-\theta)(\beta+2)} &\geq (\psi_n)^{1-(1-\theta)(\beta+2)} \times (2^{-1+(1-\theta)(\beta+2)} - 1) \\
&\stackrel{\text{by (36)}}{\geq} 2^{-1+(1-\theta)(\beta+2)} - 1.
\end{aligned}$$

If $\psi_{n+1} > \frac{\psi_n}{2}$ we have

$$\begin{aligned}
&(\psi_{n+1})^{1-(1-\theta)(\beta+2)} - (\psi_n)^{1-(1-\theta)(\beta+2)} \\
&= [-1 + (1-\theta)(\beta+2)] \int_{\psi_{n+1}}^{\psi_n} s^{-(1-\theta)(\beta+2)} ds \\
&\geq [-1 + (1-\theta)(\beta+2)] (\psi_n)^{-(1-\theta)(\beta+2)} [\psi_n - \psi_{n+1}] \\
&\geq [-1 + (1-\theta)(\beta+2)] 2^{-(1-\theta)(\beta+2)} (\psi_{n+1})^{-(1-\theta)(\beta+2)} [\psi_n - \psi_{n+1}].
\end{aligned}$$

By using (35) and (45), we have

$$(\psi_{n+1})^{1-(1-\theta)(\beta+2)} - (\psi_n)^{1-(1-\theta)(\beta+2)} \geq [-1 + (1-\theta)(\beta+2)]2^{-(1-\theta)(\beta+2)} \times \frac{\lambda'}{C_4^{\beta+2}}.$$

Therefore, in the two cases, for all $n \in J_2$, we obtain that

$$(\psi_{n+1})^{1-(1-\theta)(\beta+2)} - (\psi_n)^{1-(1-\theta)(\beta+2)} \geq C_6. \quad (51)$$

for some constant $C_6 > 0$. According to (43) and (47) we also get for all $n \in J_2$,

$$\|x_{n+1} - x_n\| \leq C_5 ((\psi_n)^\zeta - (\psi_{n+1})^\zeta). \quad (52)$$

Furthermore, estimate (41) is available for all $n \in J_1$.

We distinguish now between three cases.

First case: if J_1 contains $\{n \geq n_1\}$ for some $n_1 \geq n_0$, then we obtain that

$$\|x_n - x^*\| \leq \sum_{k=n}^{+\infty} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{+\infty} C_2 k^{-(1+\frac{\delta}{\beta+2})} \leq C_7 n^{-\frac{\delta}{\beta+2}},$$

for all $n \geq n_1$.

Second case: we assume that J_2 contains $\{n \geq n_1\}$ for some $n_1 \geq n_0$.

Consequently, by summing (51) from n_1 to $n-1$, we find

$$\begin{aligned} \psi_n &\leq \left(C_6(n - n_1) + (\psi_{n_1})^{1-(1-\theta)(\beta+2)} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}} \\ &\leq C_8 n^{\frac{-1}{-1+(1-\theta)(\beta+2)}}, \end{aligned}$$

for all $n \geq n_1$. By summing (52) from $n \geq n_1$ to ∞ , we obtain that

$$\|x_n - x^*\| \leq \sum_{k=n}^{+\infty} \|x_{k+1} - x_k\| \leq C_5 (\psi_n)^\zeta \leq C_5 C_8^\zeta n^{\frac{-\zeta}{-1+(1-\theta)(\beta+2)}},$$

for all $n \geq n_1$. So, we again get (32).

If neither of the two cases above applies, we have two sequences of positive integers, $(n_p)_{p \in \mathbb{N}^*}$ and $(m_p)_{p \in \mathbb{N}^*}$, such that $J_2 \cap \{n \geq n_1\} = \cup_{p \in \mathbb{N}} \{n_p, \dots, m_p\}$, with $n_1 > n_0, n_p \leq m_p$ and $n_p - 1 \notin J_2, \forall p \geq 1$. Let $n \in \{n_p, \dots, m_p\}$. By summing (51) from n_p to $n-1$, we find

$$\psi_n \leq \left(C_6(n - n_p) + (\psi_{n_p})^{1-(1-\theta)(\beta+2)} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}}. \quad (53)$$

We have, according to (31), (38) and (40)

$$(\psi_{n_p})^{1-\theta} \leq (C_9 n_p^{-(1+\beta+\delta)})^{(1-\theta)}.$$

Using this in (53), we get

$$\psi_n \leq \left(C_6(n - n_p) + C_9^{1-(1-\theta)(\beta+2)} n_p^{(1+\beta+\delta)[-1+(1-\theta)(\beta+2)]} \right)^{\frac{-1}{-1+(1-\theta)(\beta+2)}}.$$

From (33), it can be deduced that

$$(1 + \beta + \delta)[-1 + (1 - \theta)(\beta + 2)] \geq 1.$$

Replacing C_6 by $C'_6 = \min(C_6, C_9^{1-(1-\theta)(\beta+2)})$, we find

$$\psi_n \leq (C'_6 n)^{\frac{-1}{-1+(1-\theta)(\beta+2)}} \quad \forall n \in \{n_p, \dots, m_p\}.$$

By using (41) and (52), we have, for all $n \geq n_1$

$$\begin{aligned} \|x_n - x^*\| &\leq \sum_{k=n}^{+\infty} \|x_{k+1} - x_k\| \\ &\leq \sum_{k \geq n, k \in J_1} C_2 k^{-(1+\frac{\delta}{\beta+2})} + \sum_{k \geq n, k \in J_2} C_5 ((\psi_k)^\zeta - (\psi_{k+1})^\zeta) \\ &\leq C_7 n^{-\frac{\delta}{\beta+2}} + C_5 (C'_6 n)^{\frac{-\zeta}{-1+(1-\theta)(\beta+2)}}. \end{aligned}$$

We once more find the convergence rate (32). □

3.2 Applications

In this section, we implement our approach on the following examples. We find the same result in [11] and [6].

Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function of class C^2 . We define

$$S = \{u^* \in \mathbb{R}^d \mid \nabla F(u^*) = 0\}.$$

We assume that:

$$\exists L_F > 0 \quad \forall u, v \in \mathbb{R}^d \quad \|\nabla F(u) - \nabla F(v)\| \leq L_F \|u - v\|, \quad (54)$$

$$\begin{aligned} \exists \theta \in (0, \frac{1}{2}] \quad \forall u^* \in S \quad \exists r_a > 0 \quad \exists c_a > 0 \quad \mid \quad \forall u \in \mathbb{R}^d : \|u - u^*\| < r_a \\ \implies \|\nabla F(u)\| \geq c_a |F(u) - F(u^*)|^{1-\theta}. \end{aligned} \quad (55)$$

3.2.1 A first order gradient system

We consider a sequence $u_n \in \mathbb{R}^d$ which satisfies:

$$\begin{cases} \frac{u_{n+1} - u_n}{\Delta t} = -\nabla F(u_{n+1}) + g_{n+1} \\ u_0 \in \mathbb{R}^d \end{cases} \quad (56)$$

where $(g_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^d .

Theorem 3.3. *Let $(u_n)_{n \in \mathbb{N}}$ be a sequence satisfying (56), and assume that*

(i) $(u_n)_{n \in \mathbb{N}}$ is bounded,

(ii) $1 > \frac{L_F}{2} \Delta t$,

(iii) there exists a constant $\delta > 0$ such that $(g_n)_{n \in \mathbb{N}}$ satisfies

$$\sum_{k=0}^{+\infty} \|g_k\|^2 < \infty \text{ and } \sup_{n \in \mathbb{N}} \left(n^{1+\delta} \sum_{k=n}^{+\infty} \|g_{k+1}\|^2 \right) < \infty.$$

Then there exists $u^* \in S$ such that $\lim_{n \rightarrow +\infty} u_n = u^*$. Additionally, there exists a constant $A > 0$ such that for all $n > 0$, we have

$$\|u_n - u^*\| \leq A n^{-\mu} \text{ where } \mu = \min \left(\frac{\theta}{1 - 2\theta}, \frac{\delta}{2} \right).$$

Proof. We define $\phi := F$, $x_n := u_n$, $f_n := \|g_n\|^2$ and $\beta = 0$. Using Section 4 from [11] we can find constant $\gamma > 0$ satisfying

$$\phi(x_n) - \phi(x_{n+1}) + C \|g_{n+1}\| \geq \gamma [\|x_{n+1} - x_n\|^2 + \|\nabla F(x_{n+1})\|^2].$$

Then the sequence (x_n) satisfies (29) with the function ϕ and the statement follows from Theorem 3.2. □

3.2.2 A second order gradient system

We consider a sequence $(u_n, v_n) \in \mathbb{R}^d \times \mathbb{R}^d$ which satisfies:

$$\begin{cases} \frac{u_{n+1} - u_n}{\Delta t} = v_{n+1} \\ \frac{v_{n+1} - v_n}{\Delta t} = -\|v_{n+1}\|^\alpha v_{n+1} - \nabla F(u_{n+1}) + g_{n+1} \\ u_0, v_0 \in \mathbb{R}^d \end{cases} \quad (57)$$

where $1 > \alpha > 0$ and $(g_n)_{n \in \mathbb{N}}$ is a sequence in \mathbb{R}^d .

Theorem 3.4. Let $(u_n, v_n)_{n \in \mathbb{N}}$ be a sequence satisfying (57), and assume that

(i) $(u_n)_{n \in \mathbb{N}}$ is bounded,

(ii) there exists a constant $c_F > 0$ such that F satisfies

$$\forall u, v \in \mathbb{R}^d \quad \langle \nabla F(u) - \nabla F(v), u - v \rangle \geq -c_F \|u - v\|^{\alpha+2},$$

(iii) $1 > \frac{c_F}{2} (\Delta t)^{\alpha+1}$ and $\alpha \in (0, \frac{\theta}{1-\theta})$,

(iv) there exists a constant a constant $\delta > 0$ such that $(g_n)_{n \in \mathbb{N}}$ satisfies

$$\sum_{k=0}^{+\infty} \|g_k\| < \infty \text{ and } \sup_{n \in \mathbb{N}} \left(n^{1+\delta+\alpha} \sum_{k=n}^{+\infty} \|g_{k+1}\| \right) < \infty.$$

Then there exists $u^* \in S$ such that $\lim_{n \rightarrow +\infty} u_n = u^*$. Additionally, there exists a constant $A > 0$ such that for all $n > 0$, we have

$$\|u_n - u^*\| \leq A n^{-\mu} \text{ where } \mu = \min \left(\frac{1 - (1 - \theta)(\alpha + 1)}{-1 + (1 - \theta)(\alpha + 2)}, \frac{\delta}{\alpha + 2} \right).$$

Proof. We define

$$\phi_\varepsilon(u, v) := \frac{1}{2} \|v\|^2 + F(u) + \varepsilon \|\nabla F(u)\|^\alpha \langle \nabla F(u), v \rangle \quad (58)$$

as the continuous case, $x_n := (u_n, v_n)$, $f_n := \|g_n\|$ and $\beta = \alpha$. Given that (u_n) is bounded, using (57), (v_n) is also bounded. Using Lemma 3.8 from [6] and the fact that the sequence (x_n) is bounded, we can find a small enough $\varepsilon = \bar{\varepsilon} > 0$ such that there exists a constant $\gamma > 0$ satisfying

$$\phi_{\bar{\varepsilon}}(x_n) - \phi_{\bar{\varepsilon}}(x_{n+1}) + C \|g_{n+1}\| \geq \gamma [\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]^{\alpha+2}. \quad (59)$$

Let us show that when (u_n) is a bounded sequence, (x_n) satisfies (29) with the function $\phi_{\bar{\varepsilon}}$. Indeed, a simple computation gives

$$\begin{aligned} \nabla \phi_{\bar{\varepsilon}}(u, v) = \\ \left(\begin{array}{c} \nabla F(u) + \bar{\varepsilon} \|\nabla F(u)\|^\alpha \nabla^2 F(u) \cdot v + \bar{\varepsilon} \alpha \|\nabla F(u)\|^{\alpha-2} \langle \nabla F(u), v \rangle \nabla^2 F(u) \cdot \nabla F(u) \\ v + \bar{\varepsilon} \|\nabla F(u)\|^\alpha \nabla F(u) \end{array} \right). \end{aligned}$$

Since we assume that (u_n) is bounded, there exists a constant $C_{10} > 0$ such that

$$\forall n \in \mathbb{N} \quad \|\nabla \phi_{\bar{\varepsilon}}(x_{n+1})\| \leq C_{10} [\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]. \quad (60)$$

Besides

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(u_{n+1} - u_n, v_{n+1} - v_n)\| \\ &= \|(\Delta t v_{n+1}, -\Delta t \|v_{n+1}\|^\alpha v_{n+1} - \Delta t \nabla F(u_{n+1}) + \Delta t g_{n+1})\| \\ &\leq C_{11} [\|v_{n+1}\| + \|\nabla F(u_{n+1})\|]. \end{aligned}$$

where we used the fact that (v_n) and (g_n) are bounded. By combining this last inequality with (59) and (60), we deduce that (x_n) satisfies (29).

By using the same function ϕ_{ε} as in the continuous case, we obtain that ϕ_{ε} satisfies the Lojasiewicz inequality (30) and the statement follows from Theorem 3.2. \square

Remark 3.5. In [11], the equation (57) was studied with $\alpha = 0$, where it was assumed that F satisfies (54) and (55), with $\alpha = 0$, and (g_n) satisfies $\sum_{k=0}^{+\infty} \|g_k\|^2 < \infty$ and

$\sup_{n \in \mathbb{N}} \left(n^{1+\delta} \sum_{k=n}^{+\infty} \|g_{k+1}\|^2 \right) < \infty$. By considering ϕ_{ε} defined by (58) with $\alpha = 0$ and using Lemma 3.3 of [11] and following the same proof as in the previous section, we get the same convergence result as in the previous theorem with $\alpha = 0$.

Acknowledgements

This work forms part of the author's PhD thesis, supervised by Professors Thierry Horsin and Mohamed Ali Jendoubi, to whom the author expresses sincere gratitude. The author would also like to thank Professor Ralph Chill for his remarks and comments which improve this work.

References

- [1] P.-A. Absil, R. Mahony and B. Andrews, Convergence of the iterates of descent methods for analytic cost functions. *SIAM J. Optim.* **16** (2005), no. 2, 531–547.
- [2] N. Alaa and M. Pierre, Convergence to equilibrium for discretized gradient-like systems with analytic features. *IMA J. Numer. Anal.* **33** (2013), 1291–1321.
- [3] H. Attouch and J. Bolte, On the convergence of the proximal algorithm for nonsmooth functions involving analytic features, *Math. Program., Ser. B*, **116** (2009), 5-16.
- [4] I. Ben Hassen, Decay estimates to equilibrium for some asymptotically autonomous semilinear evolution equations. *Asymptotic Anal.* **69** (2010), no. 1-2, 31-44 .
- [5] I. Ben Hassen and L. Chergui, Convergence of global and bounded solutions of some nonautonomous second order evolution equations with nonlinear dissipation. *J. Dyn. Differ. Equations* **23** (2011), no. 2, 315-332 .
- [6] H. Cheikh, Convergence to equilibrium for solutions of some forced discretized second-order gradient-like systems. To appear in *Mathematical Control and Related Fields*.
- [7] L. Chergui, Convergence of global and bounded solutions of a second order gradient like system with nonlinear dissipation and analytic nonlinearity. *J. Dynam. Differential Equations* **20** (2008), 643–652.

- [8] R. Chill, A. Haraux and M.A. Jendoubi, Applications of the Łojasiewicz-Simon, gradient inequality to gradient-like evolution equations. *Anal. Appl.*, Singap. **7** (2009), 351–372.
- [9] R. Chill and M.A. Jendoubi, Convergence to steady states in asymptotically autonomous semilinear evolution equations. *Nonlinear Anal.* **53** (2003), no. 7-8, 1017–1039.
- [10] D. D’Acunto and K. Kurdyka, Explicit bounds for the Łojasiewicz exponent in the gradient inequality for polynomials. *Ann. Pol. Math.* **87**, (2005), 51–61.
- [11] M. Grasselli and M. Pierre, Convergence to equilibrium of solutions of the backward Euler scheme for asymptotically autonomous second-order gradient-like systems. *Commun. Pure Appl. Anal.* **11** (2012), 2393–2416.
- [12] M. Grasselli, H. Petzeltova and G. Schimperna, Convergence to stationary solutions for a parabolic-hyperbolic phase-field system, *Commun. Pure Appl. Anal.*, **5** (2006), 827-838.
- [13] A. Haraux and M.A. Jendoubi, The convergence problem for dissipative autonomous systems - classical methods and recent advances. *SpringerBriefs in Mathematics, Springer, Bcam, 2015*.
- [14] A. Haraux and M.A. Jendoubi, Convergence of solutions of second-order gradient-like systems with analytic nonlinearities. *J. Differential Equations* **144** (1998), 313–320.
- [15] A. Haraux and M.A. Jendoubi, Decay estimates to equilibrium for some evolution equations with an analytic nonlinearity. *Asymptotic Anal.* **26** (2001), 21–36.
- [16] A. Haraux and M.A. Jendoubi, Asymptotics for a second order differential equation with a linear, slowly time-decaying damping term. *Evolution Equations and Control Theory*, **2** (2013), 461–470.
- [17] T. Horsin and M.A. Jendoubi, Non-genericity of initial data with punctual ω -limit set. *Arch. Math.* **114** (2020), 185–193 .
- [18] T. Horsin and M.A. Jendoubi, Asymptotics for some discretizations of dynamical systems, application to second order systems with non-local nonlinearities. *Communications on Pure & Applied Analysis*, **21** (2022), no. 3, 999–1025.
- [19] M.A. Jendoubi and P. Polacik, Non-stabilizing solutions of semilinear hyperbolic and elliptic equations with damping. *Proc. R. Soc. Edinb., Sect. A, Math.* **133** (2003), 1137–1153 .
- [20] S. Łojasiewicz, Une propriété topologique des sous ensembles analytiques réels. *Colloques internationaux du C.N.R.S #117. Les équations aux dérivées partielles* (1963).
- [21] S. Łojasiewicz, *Ensembles semi-analytiques, I.H.E.S. notes* (1965).

- [22] B. Merlet and M. Pierre, Convergence to equilibrium for the backward Euler scheme and applications. *Commun. Pure Appl. Anal.* **9** (2010), no. 3, 685–702 .
- [23] J. Palis and W. de Melo, *Geometric theory of dynamical systems: An introduction.* Springer-Verlag, New-York (1982).