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SHARP-DIFFUSE INTERFACES MODEL WITH CROSS-DIFFUSION IN A FREE AQUIFER : NUMERICAL ANALYSIS OF A FINITE VOLUME SCHEME

AHMED AIT HAMMOU OULHAJ, ÉLOÏSE COMTE, MOUSSA MORY DIÉDHIYOU,
AND DAVID MALTESE

ABSTRACT. Sharp-diffuse interface is a new model for seawater intrusion problem which have the efficiency of the sharp interface approach and the physical realism of the diffuse interface. The introduction of the diffuse interface as a phase field in the new model makes it non-degenerate. Three-dimensional problem is reduced to a two-dimensional model involving a strongly coupled system of PDE of parabolic type describing the evolution of the depths of the interface between salt-and freshwater and the water table. We propose and analyze a finite volume scheme based on two-point flux approximation with upwind mobilities. The scheme preserves at the discrete level the main features of the continuous problem, namely the nonnegativity of the solutions, the decay of the energy and the control of the entropy and its dissipation. Based on these nonlinear stability results, we show that the scheme converges towards a weak solution to the problem. Numerical results are provided to illustrate the behavior of the model and of the scheme.

Keywords. Unsaturated porous media flow, seawater intrusion, nonlinear discretization, entropy stability, convergence analysis, cross-diffusion, unconfined aquifer

AMS subjects classification. 65M12, 65M08, 76S05

1. INTRODUCTION

Seawater intrusion in costal aquifers is a major problem for water supply. If they are densely populated in costal regions, this was because there existe in the groundwater, hydraulic exchanges between fresh groundwater and seawater, moreover the intensive extraction of freshwater leads for instance to local water table depression causing problems of saltwater intrusion in the aquifers. It's motivating the study of efficient and accurate models to simulate the displacement of seawater fronts for the optimal exploitation of freshwater in costal aquifer. Considerable work has been undertaken and for general information about seawater intrusion problems (see [8, 9, 10]). The static model based on Ghyben-Herzberg hypothesis do not enable to follow the evolution of the saltwater front and the existing dynamic models may be classified in four categories : The first approach is to consider that fresh and salt water are two miscible fluids. This modelling approach is physically correct and due to density contrast they tend to separate into two layers with a transition zone characterized by the variation of the salt concentration, but it has the disadvantage that it is impossible to follow explicitly the interface (see [15]). This approach is very heavy from theoretical and numerical points of view. The second consists in assuming that fresh and salt water are two immiscible fluids. That's a certain type of a sharp interface when the salty phase disappears. Nevertheless

the explicit tracking of the interface remains unworkable to implement without further assumptions (see also [15] in unsaturated media). The third approach is to assume that the fluids are immiscible and the domains occupied by each fluid are separated by an interface called sharp interface. Moreover, we consider that no mass transfer occurs between the fresh and the saltwater and the so-called Dupuit approximation. Physically, this approach is often reasonable but it enables to take into account and does not describe the behaviour of the transition zone. We refer to [8, 9, 10, 38, 37, 4] for more details about this approach. Recently, the fourth and new approach is a mixed approach (sharp-diffuse). This approach combining the simplicity of the sharp interface with the realism of diffuse interfaces. It have the advantage of respecting the physics of the problem and that of the computational efficiency. The two key assumptions of this approach are : there is no mass transfer between fresh and salt water and we suppose the existence of the diffuse interface which is modelled using a phase field approach, here an Allen-cahn type model in fluid-fluid context. The last assumption also highlighted a new term in the model which contains the thickness of the transition zone. We refer [16] for the derivation and mathematical analysis of sharp-diffuse interfaces in unconfined aquifer. In [17], there are the numerical simulations and some physical predictions from the model. The comparison between a mixed approach and a sharp interface approach has been done in [18]. In a confined aquifer, the comparison of a full 3D model for miscible and compressible displacements in slightly deformable porous media compared the mixed approach with numerical simulations results (see [14]).

In this present paper, we choose this new mixed (sharp-diffuse) interface approach for seawater intrusion in an unconfined aquifer. Thanks to this advantages, the sharp-diffuse model can be rewritten in the form of the formal asymptotic limit as the aspect ratio between the thickness and the horizontal length of the porous medium tends to zero. The new formulation described in [30] is a cross-diffusion system of degenerate parabolic equations for the evolution of the interface between saltwater and freshwater and the interface between the completely saturated part and the dry part in a partially saturated porous media. The mathematical analysis particularly the uniqueness of a cross-diffusion systems was done in [19]. We propose a finite volume scheme for a mixed approach based on a two-point flux approximation with upwind mobilities. We also present some numerical results.

The paper is organized as follows. In Section 2, we detail and present the continue problem, the classical entropy/dissipation property and the decrease energy functional along the time have made. The existence of nonnegative solution is shown in Section 3. Discrete counterparts of the entropy/entropy-dissipation and energy/energy-dissipation relations are established in Section 4. Section 5 is devoted to the convergence proof of the scheme. This proof is based first on the compactness of the sequence of approximate solutions and then on the identification of the limit. We finally present numerical experiments in Section 6, to illustrate the behaviour of the model and of the scheme.

2. PRESENTATION OF THE PROBLEM

2.1. Continuous problem. In our setting ξ is a nonnegative function expressing the height of the interface between the saltwater and the freshwater while $h \geq \xi$ is

the height of the interface separating the freshwater and the dry soil. We assume that the bottom of the porous medium, which is located at $b \neq 0$, is impermeable, cf. Figure 1. Moreover we assume quasi-horizontal displacements (Dupuit approximation), hence we get a 2D-vertically averaged model. This assumption is reasonable when the thickness of the aquifer is small compared to the horizontal length of the aquifer. This is a sharp interface approach without sources terms.

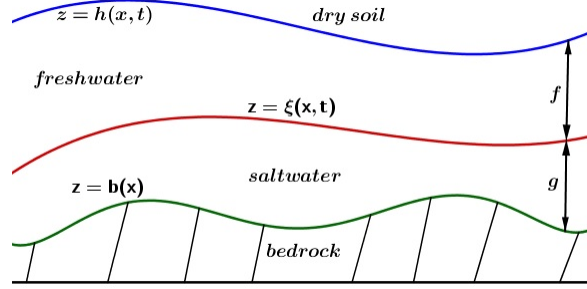


FIGURE 1. Description of an unconfined aquifer

The evolution of ξ and h is given by the following cross-diffusion system of degenerate parabolic equations

$$(1) \quad \begin{cases} \partial_t(h - \xi) - \nabla \cdot ((h - \xi)\nabla((1 - \varepsilon_0)h)) & = 0 \quad \text{in } \Omega \times (0, T) =: \Omega_T \\ \partial_t\xi - \nabla \cdot ((\xi - b)\nabla((1 - \varepsilon_0)h + \varepsilon_0\xi)) & = 0 \quad \text{in } \Omega \times (0, T), \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal open bounded subset and $T > 0$ is a finite time horizon. The parameter ε_0 is given by

$$\varepsilon_0 = \frac{\rho_s - \rho_f}{\rho_s},$$

where ρ_s (resp. ρ_f) is the mass density of the fluid saltwater (resp. freshwater) (assumed to be constant with $0 < \rho_f < \rho_s$). We set $f = h - \xi$, $g = \xi - b$ and

$$(2) \quad \nu = 1 - \varepsilon_0 = \frac{\rho_f}{\rho_s} \in (0, 1).$$

The system (1) then rewrites :

$$(3) \quad \begin{cases} \partial_t f - \nabla \cdot (\nu f \nabla(f + g + b)) & = 0 \quad \text{in } \Omega \times (0, T), \\ \partial_t g - \nabla \cdot (g \nabla(\nu f + g + b)) & = 0 \quad \text{in } \Omega \times (0, T). \end{cases}$$

When we consider the mixed approach and the bedrock $b = 0$, we have the new terms which appear, thanks to the diffuse interface these terms enable to take into account the thickness (δ) of the two transition zones. We have the following cross-diffusion issuing from a mixed approach. The system is non-degenerate parabolic equations :

$$(4) \quad \begin{cases} \partial_t f - \nabla \cdot (\nu f \nabla(f + g)) - \nabla \cdot (\delta \nabla f) & = 0 \quad \text{in } \Omega \times (0, T), \\ \partial_t g - \nabla \cdot (g \nabla(\nu f + g)) - \nabla \cdot (\delta \nabla g) & = 0 \quad \text{in } \Omega \times (0, T). \end{cases}$$

It is supplemented with no-flux boundary conditions

$$(5) \quad \nabla f \cdot \mathbf{n} = \nabla g \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T),$$

where \mathbf{n} is the unit normal to the boundary $\partial\Omega$. Initial data are

$$(6) \quad f|_{t=0} = f_0, \quad g|_{t=0} = g_0,$$

with $f_0, g_0 \in L^\infty(\Omega)$ and

$$(7) \quad f_0, g_0 \geq 0, \quad \text{a.e } x \in \Omega.$$

Before describing more precisely our results, let us mention that the problems of kind (3), have been the object of several studies. The authors in [22, 23] studied the classical solutions of the system (3) (with $b = 0$). Moreover, weak solutions are established under different assumptions in [21, 31, 32].

We recall (see [21, 31, 32]) the definition of entropy functional :

$$\mathfrak{H}(f, g) = \int_{\Omega} \left[\Gamma(g) + \frac{1}{\nu} \Gamma(f) \right] dx, \quad \text{where } \Gamma(s) = s \log s - s + 1,$$

and of the energy functional :

$$\mathfrak{E}(f, g) = \int_{\Omega} \left[\frac{\nu}{2} (f + g + b)^2 + \frac{1 - \nu}{2} (g + b)^2 \right] dx.$$

Multiplying (formally) the first equation of (3) by $\frac{1}{\nu} \log f$ and the second equation by $\log g$, integrating over Ω and summing both equations, yields the classical entropy/dissipation property :

$$(8) \quad \frac{d}{dt} \mathfrak{H}(f, g) + \frac{1 - \nu}{2} \int_{\Omega} \left[(\nabla f)^2 + (\nabla g)^2 \right] dx \leq \frac{1}{2(\nu + 1)} \int_{\Omega} (\nabla b)^2 dx.$$

Moreover multiplying (formally) the first equation of (3) by $\nu(f + g + b)$ and the second equation by $\nu f + g + b$, integrating over Ω and summing both equations, yields that the energy functional decreases along time :

$$(9) \quad \frac{d}{dt} \mathfrak{E}(f, g) + \int_{\Omega} \left[\nu^2 f (\nabla(f + g + b))^2 + g (\nabla(\nu f + g + b))^2 \right] dx = 0.$$

Let us mention that the cross-diffusion systems are extensively presented in different domain as ecology, biology, chemistry and others. In [28] the author propose and analyze a finite volume scheme for the Patlak-Keller-Segel (PKS) chemotaxis model. In [11] the authors studie the PKS model with additional cross diffusion. We refer to [6] for the analysis of a finite volume method for a cross diffusion model in population dynamics. See [36, 34] for the numerical analysis for a seawater intrusion problem in a unconfined aquifer with finite element method approximation. In [1] the authors propose a finite element method and a finite volume method and compare the results given by these two methods. In [18] the authors address the question of global existence for the sharp interface approach. For an analysis of a finite volume scheme for two-phase immiscible flow in porous media, used in petroleum engineering, we can refer to these papers [35, 27].

In this work, we propose a finite volume scheme for the problem (3). This scheme is based on a two-point flux approximation with upwind mobilities. It is designed in order to preserve at the discrete level the main features of the continuous problem : the nonnegativity of the solutions, the decay of the energy (9), the control of the entropy and its dissipation (8).

2.2. The numerical scheme. In this section, we explicit the discretization of the problem (3)-(5) we will study in this paper (see [2]). The time discretization relies on backward Euler scheme, while the space discretization relies on a finite volume approach (see e.g [26]), with two-point flux approximation and upstream mobility.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, polygonal subset. An admissible mesh of Ω is given by a family \mathcal{T} of a control volumes (open and convex polygons), a family \mathcal{E} of edges and a family of points $(x_K)_{K \in \mathcal{T}}$ which satisfy Definition 9.1 in [26]. This definition implies that the straight line between two neighboring centers of cells (x_K, x_L) is orthogonal to the edge $\sigma = K|L$.

We distinguish the interior edges $\sigma \in \mathcal{E}_{\text{int}}$ and the boundary edges $\sigma \in \mathcal{E}_{\text{ext}}$. The set of edges \mathcal{E} equals the union $\mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$. For a control volume $K \in \mathcal{T}$, we denote by \mathcal{E}_K the set of its edges, by $\mathcal{E}_{K,\text{int}}$ the set of its interior edges and by $\mathcal{E}_{K,\text{ext}}$ the set of edges of K included in $\partial\Omega$.

Furthermore, we denote by d the distance in \mathbb{R}^2 and by m the Lebesgue measure in \mathbb{R}^2 or \mathbb{R} . We assume that the family of meshes satisfies the following regularity requirement : there exists $\zeta > 0$ such that for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}_{\text{int},K}$ with $\sigma = K|L$, it holds

$$(10) \quad d(x_K, \sigma) \geq \zeta d(x_K, x_L)$$

For all $\sigma \in \mathcal{E}_{\text{int},K}$ with $\sigma = K|L$, we define $d_\sigma = d(x_K, x_L)$ and the transmissibility coefficient

$$(11) \quad \tau_\sigma = \frac{m(\sigma)}{d_\sigma}, \quad \sigma \in \mathcal{E}.$$

The size of the mesh is defined by

$$\delta = \max_{K \in \mathcal{T}}(\text{diam}(K)).$$

In order to avoid heavier notations, we restrict our study to the case of a uniform time discretization of $(0, T)$. However, all the results presented in this paper can be extended to general time discretizations without any technical difficulty. In what follows, we assume that the spatial mesh is fixed and does not change with the time step. Let $T > 0$ be some final time and M_T the number of time steps. Then the time step size and the time points are given by, respectively,

$$\Delta t = \frac{T}{M_T}, \quad t^n = n\Delta t, \quad 0 \leq n \leq M_T.$$

We denote by \mathcal{D} an admissible space-time discretization of $\Omega_T = \Omega \times (0, T)$ composed of an admissible mesh \mathcal{T} of Ω and the values Δt and M_T . The size of this space-time discretization \mathcal{D} is defined by $\eta = \max(\delta, \Delta t)$.

The initial conditions are discretized by

$$(12) \quad f_{\mathcal{T}}^0 = \sum_{K \in \mathcal{T}} f_K^0 \mathbf{1}_K, \quad \text{where } f_K^0 = \frac{1}{m(K)} \int_K f_0(x) dx, \quad \forall K \in \mathcal{T},$$

$$(13) \quad g_{\mathcal{T}}^0 = \sum_{K \in \mathcal{T}} g_K^0 \mathbf{1}_K, \quad \text{where } g_K^0 = \frac{1}{m(K)} \int_K g_0(x) dx, \quad \forall K \in \mathcal{T},$$

and $\mathbf{1}_K$ is the characteristic function on K . Denoting by f_K^n and g_K^n approximations of the mean value of $f(\cdot, t^n)$ and $g(\cdot, t^n)$ on K , respectively. Taking for b_K the value of b in a fixed point of K (for instance, the center of gravity of K), where b is a

regular function and assume that $b_K \geq 0 \quad \forall K \in \mathcal{T}$. The discretization of problem (3) is given by the following set of nonlinear equations :

$$(14) \quad m(K) \frac{f_K^{n+1} - f_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma f_\sigma^{n+1} \nu \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) = 0,$$

and

$$(15) \quad m(K) \frac{g_K^{n+1} - g_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma g_\sigma^{n+1} \left(\nu(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) = 0,$$

for $K \in \mathcal{T}$ and $0 \leq n \leq M_T - 1$, where

$$(16) \quad f_\sigma^{n+1} = \begin{cases} f_K^{n+1} & \text{if } (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \geq 0, \\ f_L^{n+1} & \text{if } (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) < 0. \end{cases}$$

and

$$(17) \quad g_\sigma^{n+1} = \begin{cases} g_K^{n+1} & \text{if } \nu(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \geq 0, \\ g_L^{n+1} & \text{if } \nu(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) < 0, \end{cases}$$

where $x^+ = \max(0, x)$. We next define the numerical approximation $(f_{\mathcal{D}}, g_{\mathcal{D}})$ of (f, g) by

$$f_{\mathcal{D}}(x, t) = \sum_{\substack{K \in \mathcal{T} \\ 0 \leq n \leq M_T - 1}} f_K^{n+1} \mathbf{1}_{K \times (t^n, t^{n+1}]}(x, t), \quad \text{and} \\ g_{\mathcal{D}}(x, t) = \sum_{\substack{K \in \mathcal{T} \\ 0 \leq n \leq M_T - 1}} g_K^{n+1} \mathbf{1}_{K \times (t^n, t^{n+1}]}(x, t).$$

We also define approximations of the gradients $\nabla^{\mathcal{D}} f_{\mathcal{D}}$ and $\nabla^{\mathcal{D}} g_{\mathcal{D}}$ of f and g , respectively. To this end, we introduce that : for $K \in \mathcal{T}$

- If $\sigma = K|L \in \mathcal{E}_{\text{int},K}$, $\mathfrak{D}_{K,L}$ is the cell ("diamond") whose vertices are given by x_K, x_L and the end points of the edge $\sigma = K|L$.
- $\mathfrak{D}_{K,\sigma} = \mathfrak{D}_{K,L} \cap K$ is the cell ("triangle") whose vertices are given by x_K and the end points of the edge $\sigma = K|L$.

The approximate gradient $\nabla^{\mathcal{D}} S_{\mathcal{D}}$ (with $S = f$, or $S = g$) is a piecewise constant function, defined in Ω_T by

$$(18) \quad \nabla^{\mathcal{D}} S_{\mathcal{D}}(x, t) = -\frac{m(\sigma)}{m(\mathfrak{D}_{K,L})} (S_K^{n+1} - S_L^{n+1}) \nu_{K,L}, \quad x \in \mathfrak{D}_{K,L}, \quad t \in (t^n, t^{n+1}],$$

where $\nu_{K,L}$ is the unit vector normal to σ and outward to K .

2.3. Main results and outline of the paper. The scheme (14)-(17) amounts to a nonlinear system to be solved at each time step. The existence of a solution to this system is therefore non trivial. The first result we highlight is thus the existence of a nonnegative solution to the scheme (14)-(17), the stability in terms of the discrete entropy and the decay of the discrete energy.

Theorem 2.1. *There exists (at least) one solution $(f_K^{n+1}, g_K^{n+1})_{K \in \mathcal{T}, n \in \{0, \dots, M_T-1\}}$ to the scheme (14)-(17). Moreover, $f_K^n \geq 0$, $g_K^n \geq 0$ for all $K \in \mathcal{T}$ and for all $n \in \{0, \dots, M_T\}$ and there exists C depending only on Ω, f_0, g_0, ν and b such that*

$$\sup_{n \in \{0, \dots, M_T-1\}} \mathfrak{H}^{n+1} + \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \tau_\sigma \left[(f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right] \leq C(1+T),$$

and

$$\begin{aligned} & \sup_{n \in \{0, \dots, M_T-1\}} \mathfrak{E}^{n+1} \\ & + \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} \nu^2 \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \\ & + \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \leq C. \end{aligned}$$

Our second result concerns the convergence of the scheme to a weak solution of (3)-(5). Let $(\mathcal{D}_m)_{m>0}$ be a family of admissible space-time discretization of Ω_T . We denote by $(\mathcal{T}_m)_{m>0}$ the corresponding meshes of Ω , with $\text{size}(\mathcal{T}_m) = \delta_m \rightarrow 0$, as $m \rightarrow 0$. We define $(f_m, g_m) := (f_{\mathcal{D}_m}, g_{\mathcal{D}_m})$ the sequence of approximate solutions constructed on the discretization \mathcal{D}_m . We set $\nabla^m := \nabla^{\mathcal{D}_m}$.

Theorem 2.2. *Let $(\mathcal{D}_m)_{m>0}$ be a sequence of admissible discretizations satisfying (10) uniformly in m and $\lim_{m \rightarrow +\infty} \eta_m = 0$. Let (f_m, g_m) be a sequence of finite volume solutions to (14)-(17). Then there exists (f, g) such that, up a subsequence,*

$$\begin{aligned} f_m & \longrightarrow f \quad \text{in } L^r(\Omega_T), \forall r < 4, \quad \text{and } \nabla^m f_m \longrightarrow \nabla f \quad \text{weakly in } L^2(\Omega_T)^2, \\ g_m & \longrightarrow g \quad \text{in } L^r(\Omega_T), \forall r < 4, \quad \text{and } \nabla^m g_m \longrightarrow \nabla g \quad \text{weakly in } L^2(\Omega_T)^2, \end{aligned}$$

and $(f, g) \in L^2(0, T; H^1(\Omega))^2$ is a weak solution to (3)-(5) in the following sense

$$(19) \quad \int_0^T \int_\Omega (f \partial_t \psi - \nu f \nabla(f+g+b) \cdot \nabla \psi) \, dx \, dt + \int_\Omega f_0 \psi(\cdot, 0) \, dx = 0,$$

$$(20) \quad \int_0^T \int_\Omega (g \partial_t \psi - g \nabla(\nu f + g + b) \cdot \nabla \psi) \, dx \, dt + \int_\Omega g_0 \psi(\cdot, 0) \, dx = 0,$$

for all test functions $\psi \in C_0^\infty(\Omega \times [0, T])$.

3. EXISTENCE OF A NONNEGATIVE DISCRETE SOLUTIONS

First all, we prove the positivity of the discrete solutions. This estimate allows to prove the existence of a solution to the nonlinear system (14)-(17).

Proposition 3.1. *For all $K \in \mathcal{T}, n \geq 0$,*

$$(21) \quad f_K^n \geq 0, \quad g_K^n \geq 0,$$

hence

$$(22) \quad \sum_{K \in \mathcal{T}} m(K) f_K^n = \sum_{K \in \mathcal{T}} m(K) f_K^0 = \|f_0\|_{L^1(\Omega)},$$

and

$$(23) \quad \sum_{K \in \mathcal{T}} m(K) g_K^n = \sum_{K \in \mathcal{T}} m(K) g_K^0 = \|g_0\|_{L^1(\Omega)}.$$

We will prove the existence of a solution, we follow the methodology proposed in [25], using a topological degree argument [20, 33].

Proposition 3.2. *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$. There exists (at least) one solution to the scheme (14)-(17).*

Proof. Let $\mu \in [0, 1]$ and define $(f_{K,\mu}^{n+1}, g_{K,\mu}^{n+1})_K$ as the solution of the scheme : $\forall K \in \mathcal{T}$

$$m(K) \frac{f_{K,\mu}^{n+1} - f_K^n}{\Delta t} + \mu \sum_{\sigma \in \mathcal{E}_{\text{int}}, K} \tau_\sigma f_{\sigma,\mu}^{n+1} \nu \left((f_{K,\mu}^{n+1} - f_{L,\mu}^{n+1}) + (g_{K,\mu}^{n+1} - g_{L,\mu}^{n+1}) + (b_K - b_L) \right) = 0,$$

$$m(K) \frac{g_{K,\mu}^{n+1} - g_K^n}{\Delta t} + \mu \sum_{\sigma \in \mathcal{E}_{\text{int}}, K} \tau_\sigma g_{\sigma,\mu}^{n+1} \left(\nu (f_{K,\mu}^{n+1} - f_{L,\mu}^{n+1}) + (g_{K,\mu}^{n+1} - g_{L,\mu}^{n+1}) + (b_K - b_L) \right) = 0.$$

Reproducing the proof of Proposition 3.1, one can show that

$$f_{K,\mu}^{n+1} \geq 0, \quad \text{and} \quad g_{K,\mu}^{n+1} \geq 0 \quad \forall \mu \in [0, 1],$$

hence

$$\sum_{K \in \mathcal{T}} m(K) f_{K,\mu}^{n+1} = \sum_{K \in \mathcal{T}} m(K) f_K^0 = \|f_0\|_{L^1(\Omega)},$$

and

$$\sum_{K \in \mathcal{T}} m(K) g_{K,\mu}^{n+1} = \sum_{K \in \mathcal{T}} m(K) g_K^0 = \|g_0\|_{L^1(\Omega)}.$$

Therefore, for all $K \in \mathcal{T}$, one has

$$(24) \quad 0 \leq f_{K,\mu}^{n+1} \leq \frac{\|f_0\|_{L^1(\Omega)}}{m(K)} \leq \frac{\|f_0\|_{L^1(\Omega)}}{\min_{K \in \mathcal{T}} m(K)} := m_f,$$

and

$$(25) \quad 0 \leq g_{K,\mu}^{n+1} \leq \frac{\|g_0\|_{L^1(\Omega)}}{m(K)} \leq \frac{\|g_0\|_{L^1(\Omega)}}{\min_{K \in \mathcal{T}} m(K)} := m_g.$$

Define the compact subset $\mathcal{K} = [-1, m_f + 1]^{\#\mathcal{T}} \times [-1, m_g + 1]^{\#\mathcal{T}}$ of $\mathbb{R}^{\#\mathcal{T}} \times \mathbb{R}^{\#\mathcal{T}}$ and define the function $\mathcal{H}((f_K, g_K)_K, \mu) : \mathbb{R}^{\#\mathcal{T}} \times \mathbb{R}^{\#\mathcal{T}} \times [0, 1] \rightarrow \mathbb{R}^{\#\mathcal{T}} \times \mathbb{R}^{\#\mathcal{T}}$ by :

$\forall K \in \mathcal{T}$,

$$\begin{aligned} \mathcal{H}((f_K, g_K)_K, \mu) = & \left(m(K) \frac{f_{K,\mu}^{n+1} - f_K^n}{\Delta t} \right. \\ & + \mu \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma f_{\sigma,\mu}^{n+1} \nu \left((f_{K,\mu}^{n+1} - f_{L,\mu}^{n+1}) + (g_{K,\mu}^{n+1} - g_{L,\mu}^{n+1}) + (b_K - b_L) \right), \\ & \left. m(K) \frac{g_{K,\mu}^{n+1} - g_K^n}{\Delta t} + \mu \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma g_{\sigma,\mu}^{n+1} \left(\nu (f_{K,\mu}^{n+1} - f_{L,\mu}^{n+1}) + (g_{K,\mu}^{n+1} - g_{L,\mu}^{n+1}) + (b_K - b_L) \right) \right). \end{aligned}$$

The function \mathcal{H} is uniformly continuous on $\mathcal{K} \times [0, 1]$ and it follows from (24) that for all $\mu \in [0, 1]$, the nonlinear system

$$(26) \quad \mathcal{H}((f_K, g_K)_K, \mu) = (0, 0),$$

cannot admit any solution on $\partial\mathcal{K}$. Therefore, the corresponding topological degree $\delta(\mathcal{H}, \mathcal{K})(\mu)$ is constant w.r.t μ . For $\mu = 0$ the linear system $\mathcal{H}((f_K, g_K)_K, 0)$ admits a unique solution and the topological degree is equal to 1. Hence, the nonlinear system (26) admits at least one solution for $\mu = 1$, ensuring the existence of a solution to the scheme (14)-(17). \square

4. ENTROPY AND ENERGY ESTIMATES

The goal of this section is to establish discrete counterparts to the entropy/entropy-dissipation estimate (8) and energy/energy-dissipation estimate (9). In what follows, (f_K^n, g_K^n) denotes a solution to the scheme (14)-(17). The proof of Theorem 2.1 is based on suitable estimates, which are shown below. This section also contains some results that will be useful in the convergence proof of Section 5.

4.1. Discrete $L^2(0, T; H^1(\Omega))$ semi-norm. We first have to define the space $\mathcal{X}(\mathcal{D})$ the solution belongs to and the discrete $L^2(0, T; H^1(\Omega))$ semi-norm.

Definition 4.1. *We denote by $\mathcal{X}(\mathcal{D})$ the functional space :*

$$\mathcal{X}(\mathcal{D}) = \left\{ u \in L^\infty(\Omega_T) / u \text{ is constant on } K \times (t^n, t^{n+1}] \right\}.$$

Definition 4.2. (*Discrete $L^2(0, T; H^1(\Omega))$ semi-norm*) *We define the discrete $L^2(0, T; H^1(\Omega))$ semi-norm on $\mathcal{X}(\mathcal{D})$ by :*

$$|u|_{1,\mathcal{D}} = \left(\sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma (u_K^{n+1} - u_L^{n+1})^2 \right)^{1/2}.$$

Remark 4.3. Note that

$$(27) \quad \|\nabla^{\mathcal{D}} u\|_{L^2(\Omega_T)} = \sqrt{2} |u|_{1,\mathcal{D}}.$$

4.2. Entropy estimate. We introduce a discrete version of entropy functional :

$$\mathfrak{H}^n := \mathfrak{H}(f_K^n, g_K^n) = \sum_{K \in \mathcal{T}} m(K) \left(\frac{1}{\nu} \Gamma(f_K^n) + \Gamma(g_K^n) \right).$$

Proposition 4.4. (*Entropy stability*) For all $n \in \{0, \dots, M_T - 1\}$, one has

$$(28) \quad \begin{aligned} \mathfrak{H}^{n+1} - \mathfrak{H}^n + \frac{1-\nu}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \left[(f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right] \\ \leq \frac{\Delta t}{2(\nu+1)} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma (b_K - b_L)^2. \end{aligned}$$

Proof. We multiply (14) by $\Delta t \frac{\log f_K^{n+1}}{\nu}$ and summing over $K \in \mathcal{T}$ and (15) by $\Delta t \log g_K^{n+1}$ and summing over $K \in \mathcal{T}$, provides that :

$$A + B + C = 0,$$

where

$$\begin{aligned} A &= \sum_{K \in \mathcal{T}} m(K) \left[\frac{1}{\nu} (f_K^{n+1} - f_K^n) \log f_K^{n+1} + (g_K^{n+1} - g_K^n) \log g_K^{n+1} \right], \\ B &= \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}, K}} \tau_\sigma f_\sigma^{n+1} \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) \log f_K^{n+1}, \\ C &= \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}, K}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) \log g_K^{n+1}. \end{aligned}$$

By the convexity of Γ , we find that

$$\mathfrak{H}^{n+1} - \mathfrak{H}^n = \sum_{K \in \mathcal{T}} m(K) \left[\frac{1}{\nu} (\Gamma(f_K^{n+1}) - \Gamma(f_K^n)) + \Gamma(g_K^{n+1}) - \Gamma(g_K^n) \right] \leq A.$$

We can rewrite B and C as :

$$B = \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) \times \\ (\log f_K^{n+1} - \log f_L^{n+1}),$$

$$C = \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right) \times \\ (\log g_K^{n+1} - \log g_L^{n+1}).$$

It follows from the convexity of \exp that

$$a(\log a - \log b) \geq a - b \geq b(\log a - \log b) \quad \forall a, b \in [0, +\infty[,$$

where we have used the convention $\log(0) = -\infty$ and $0 \log(0) = 0$. Hence, in view of the definition (16) and (17) of the upwind mobilities, one has

$$\begin{aligned} B \geq \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \left((f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})(f_K^{n+1} - f_L^{n+1}) \right. \\ \left. + (b_K - b_L)(f_K^{n+1} - f_L^{n+1}) \right), \end{aligned}$$

$$C \geq \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left((g_K^{n+1} - g_L^{n+1})^2 + \nu (g_K^{n+1} - g_L^{n+1})(f_K^{n+1} - f_L^{n+1}) \right. \\ \left. + (b_K - b_L)(g_K^{n+1} - g_L^{n+1}) \right).$$

Combining these inequalities, one deduces that

$$\mathfrak{H}^{n+1} - \mathfrak{H}^n + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})^2 + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (g_K^{n+1} - g_L^{n+1})^2 \\ + (\nu + 1) \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})(g_K^{n+1} - g_L^{n+1}) \leq D,$$

where

$$D = -\Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L) \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) \right].$$

Using the Young inequality, one has

$$D \leq \frac{1}{2\epsilon} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L)^2 + \frac{\epsilon}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) \right]^2,$$

for all $\epsilon > 0$. We choose $\epsilon = 1 + \nu$, we have

$$D \leq \frac{1}{2(\nu + 1)} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L)^2 \\ + \frac{\nu + 1}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left[(f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right] \\ + (\nu + 1) \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left[(f_K^{n+1} - f_L^{n+1})(g_K^{n+1} - g_L^{n+1}) \right].$$

Finally, one has

$$\mathfrak{H}^{n+1} - \mathfrak{H}^n + \frac{1 - \nu}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})^2 \\ + \frac{1 - \nu}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (g_K^{n+1} - g_L^{n+1})^2 \leq \frac{1}{2(\nu + 1)} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L)^2,$$

where $1 - \nu > 0$ thanks to (2). \square

Corollary 4.5. *There exists C_1 depending only on T, Ω, f_0, g_0, ν and b such that*

$$(29) \quad \mathfrak{H}^{M_T} + \frac{1 - \nu}{2} \left(|f_{\mathcal{D}}|_{1, \mathcal{D}}^2 + |g_{\mathcal{D}}|_{1, \mathcal{D}}^2 \right) \leq C_1.$$

Proof. Summing (28) over $n = 0, \dots, M_T - 1$ provides

$$\begin{aligned} \mathfrak{H}^{M_T} + \frac{1-\nu}{2} \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \left[(f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right] \\ \leq \mathfrak{H}^0 + \frac{1}{2(\nu+1)} \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma (b_K - b_L)^2. \end{aligned}$$

As a consequence of Jensen's inequality, one has

$$\mathfrak{H}^0 = \int_{\Omega} \left[\frac{1}{\nu} \Gamma(f_K^0) + \Gamma(g_K^0) \right] dx \leq \int_{\Omega} \left[\frac{1}{\nu} \Gamma(f_0) + \Gamma(g_0) \right] dx < +\infty,$$

and

$$\sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma (b_K - b_L)^2 \leq T |\Omega| \|\nabla b\|_{\infty}^2,$$

concluding the proof of Corollary 4.5. \square

We obtain immediately thanks to (27), the following discrete $L^2(\Omega_T)$ estimate on the discrete gradients :

$$(30) \quad \|\nabla^{\mathcal{D}} f_{\mathcal{D}}\|_{L^2(\Omega_T)}^2 + \|\nabla^{\mathcal{D}} g_{\mathcal{D}}\|_{L^2(\Omega_T)}^2 \leq 2C_1.$$

4.3. Energy estimate. The current subsection is devoted to the proof of the discrete energy estimate. We introduce a discrete version of energy functional :

$$\mathfrak{E}^n := \mathfrak{E}(f_K^n, g_K^n) = \sum_{K \in \mathcal{T}} m(K) \left(\frac{\nu}{2} (f_K^n + g_K^n + b_K)^2 + \frac{1-\nu}{2} (g_K^n + b_K)^2 \right).$$

Proposition 4.6. *For all $n \in \{0, \dots, M_T - 1\}$, one has*

$$(31) \quad \begin{aligned} \mathfrak{E}^{n+1} + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} \nu^2 \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \\ + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \leq \mathfrak{E}^n. \end{aligned}$$

Proof. We multiply (14) (resp. (15)) by $\Delta t \nu (f_K^{n+1} + g_K^{n+1} + b_K)$ (resp. $\Delta t (\nu f_K^{n+1} + g_K^{n+1} + b_K)$) and sum over $K \in \mathcal{T}$. Summing both equalities and reorganizing the sums, we get $A + B = 0$, where

$$\begin{aligned} A = \sum_{K \in \mathcal{T}} m(K) \left[\nu (f_K^{n+1} - f_K^n) (f_K^{n+1} + g_K^{n+1} + b_K) \right] \\ + \sum_{K \in \mathcal{T}} m(K) \left[(g_K^{n+1} - g_K^n) (\nu f_K^{n+1} + g_K^{n+1} + b_K) \right], \end{aligned}$$

$$\begin{aligned}
B &= \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} \nu^2 \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \\
&\quad + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2.
\end{aligned}$$

One has

$$\begin{aligned}
A &= \sum_{K \in \mathcal{T}} m(K) \left[\nu \left((f_K^{n+1} + g_K^{n+1} + b_K) - (f_K^n + g_K^n + b_K) \right) \left(f_K^{n+1} + g_K^{n+1} + b_K \right) \right] \\
&\quad + \sum_{K \in \mathcal{T}} m(K) \left[(1 - \nu) \left((g_K^{n+1} + b_K) - (g_K^n + b_K) \right) \left(g_K^{n+1} + b_K \right) \right].
\end{aligned}$$

We use the following inequality : $(a - b)a \geq \frac{1}{2}(a^2 - b^2)$, $\forall a, b \in \mathbb{R}$, to get

$$\begin{aligned}
A &\geq \sum_{K \in \mathcal{T}} m(K) \left[\frac{\nu}{2} \left((f_K^{n+1} + g_K^{n+1} + b_K)^2 - (f_K^n + g_K^n + b_K)^2 \right) \right] \\
&\quad + \sum_{K \in \mathcal{T}} m(K) \left[\frac{1 - \nu}{2} \left((g_K^{n+1} + b_K)^2 - (g_K^n + b_K)^2 \right) \right] = \mathfrak{E}^{n+1} - \mathfrak{E}^n.
\end{aligned}$$

□

Remark 4.7. In the discrete counterpart (31) of (9), the equality is replaced by an inequality. But as well as in the continuous setting, the function \mathfrak{E} decreases along time.

Corollary 4.8. *There exists C_2 depending only on f_0, g_0, b, ν and Ω such that*

$$\|f_{\mathcal{D}}\|_{L^\infty(0,T;L^2(\Omega))} + \|g_{\mathcal{D}}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_2.$$

Proof. Summing (31) over $n = 0, \dots, M_T - 1$, we obtain immediately thanks to the positivity of f_σ^{n+1} and g_σ^{n+1} that

$$\mathfrak{E}^n \leq \mathfrak{E}^0.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\mathfrak{E}^0 \leq \sum_{K \in \mathcal{T}} m(K) \left(\frac{\nu + \nu^2}{2} (f_K^0)^2 + 2(g_K^0)^2 + 2b_K^2 \right).$$

Moreover, for $s = f$, or g one has

$$\sum_{K \in \mathcal{T}} m(K) (s_K^0)^2 \leq \|s_0\|_{L^2(\Omega)}^2, \quad \text{and} \quad \sum_{K \in \mathcal{T}} m(K) b_K^2 \leq |\Omega| \|b\|_\infty^2.$$

Hence

$$\mathfrak{E}^0 \leq \|f_0\|_{L^2(\Omega)}^2 + 2\|g_0\|_{L^2(\Omega)}^2 + 2|\Omega| \|b\|_\infty^2 < +\infty.$$

On the other hand since b_K, f_K and g_K are nonnegative for all $K \in \mathcal{T}$, then we have

$$\mathfrak{E}^n \geq \sum_{K \in \mathcal{T}} m(K) \frac{\nu}{2} \left[(f_K^n)^2 + (g_K^n)^2 \right].$$

We deduce that

$$\sum_{K \in \mathcal{T}} m(K) \left[(f_K^n)^2 + (g_K^n)^2 \right] \leq \frac{2}{\nu} \mathfrak{E}^n \leq \frac{2}{\nu} \mathfrak{E}^0,$$

concluding the proof of Corollary 4.8. \square

Then we deduce from Proposition 4.6 that

Corollary 4.9. *There exists C_3 depending only on f_0, g_0, b, ν and Ω such that*

$$\begin{aligned} & \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} \nu^2 \left((f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 + \\ & \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{int} \\ \sigma = K|L}} \tau_\sigma g_\sigma^{n+1} \left(\nu (f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right)^2 \leq C_3. \end{aligned}$$

5. CONVERGENCE ANALYSIS

This section is devoted to the compactness of the approximate solution. Our goal is to show the strong compactness of the sequences $(f_m)_{m>0}$ and $(g_m)_{m>0}$ in $L^2(\Omega_T)$ and the weak compactness in $L^2(\Omega_T)^2$ of the approximate gradient of f_m and g_m defined by (18). As a first step, we show in §5.1 the appropriate compactness properties on the reconstructed discrete solutions. Then we identify in §5.2 the limit value (whose existence is ensured thanks to the compactness properties) as the weak solution to the problem (3).

5.1. Compactness properties of discrete solutions. As it is classical for unsteady problems, we need to prove some time-compactness for the approximate solutions. We make use of the time-compactness result for degenerate parabolic equations proposed in [7], as an alternative to the classical technique that consists in estimating the time-translates (see [5] in the continuous setting and [26] in the discrete setting).

Lemma 5.1. *There exists C_4 depending only on ζ, T, f_0, g_0, ν and b such that*

$$(32) \quad \sum_{n=0}^{M_T-1} \sum_{K \in \mathcal{T}} m(K) (f_K^{n+1} - f_K^n) \varphi(x_K, t_{n+1}) \leq C_4 \|\nabla \varphi\|_{L^\infty(\Omega_T)}, \quad \varphi \in \mathcal{C}_c^\infty(\Omega_T).$$

$$(33) \quad \sum_{n=0}^{M_T-1} \sum_{K \in \mathcal{T}} m(K) (g_K^{n+1} - g_K^n) \varphi(x_K, t_{n+1}) \leq C_4 \|\nabla \varphi\|_{L^\infty(\Omega_T)}, \quad \varphi \in \mathcal{C}_c^\infty(\Omega_T).$$

Proof. For the sake of readability, we denote by $\varphi_K^{n+1} = \varphi(x_K, t_{n+1})$ for all $K \in \mathcal{T}$ and all $n \in \{0, \dots, M_T - 1\}$. We multiply the scheme (14) by $\Delta t \varphi_K^{n+1}$ and sum for $K \in \mathcal{T}$, for $n \in \{0, \dots, M_T - 1\}$. This yields :

$$A = B,$$

where

$$A = \sum_{n=0}^{M_T-1} \sum_{K \in \mathcal{T}} m(K) (f_K^{n+1} - f_K^n) \varphi_K^{n+1},$$

and

$$B = -\nu \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma f_\sigma^{n+1} \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right] (\varphi_K^{n+1} - \varphi_L^{n+1}).$$

Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |B|^2 &\leq \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma f_\sigma^{n+1} (\varphi_K^{n+1} - \varphi_L^{n+1})^2 \\ &\quad \times \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma f_\sigma^{n+1} \nu^2 \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right]^2. \end{aligned}$$

Moreover $f_\sigma^{n+1} \in [\min(f_K^{n+1}, f_L^{n+1}), \max(f_K^{n+1}, f_L^{n+1})]$, hence

$$(34) \quad 0 \leq f_\sigma^{n+1} \leq f_K^{n+1} + f_L^{n+1}, \quad \forall \sigma \in \mathcal{E}, \forall n \in \{0, \dots, M_T - 1\}.$$

Using Corollary 4.9, we get

$$|B|^2 \leq C_3 \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma (f_K^{n+1} + f_L^{n+1}) d(x_K, x_L)^2 \|\nabla \varphi\|_{L^\infty(\Omega_T)}^2.$$

Observe that in two space dimensions,

$$(35) \quad \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}, K}} d_\sigma m(\sigma) \leq 2 \sum_{K \in \mathcal{T}} m(K).$$

By (11), (10) one has

$$\begin{aligned} &\sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma (f_K^{n+1} + f_L^{n+1}) d(x_K, x_L)^2 \\ &\leq \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} m(\sigma) d(x_K, x_L) (f_K^{n+1} + f_L^{n+1}) \\ &\leq \sum_{n=0}^{M_T-1} \Delta t \sum_{K \in \mathcal{T}} f_K^{n+1} \sum_{\sigma \in \mathcal{E}_{\text{int}, K}} m(\sigma) d(x_K, x_L) \\ &\leq \frac{2T}{\zeta} \sum_{K \in \mathcal{T}} m(K) f_K^{n+1} \leq \frac{2T}{\zeta} \|f_0\|_{L^1(\Omega)}, \end{aligned}$$

thanks to the mass conservation (22). Hence

$$(36) \quad \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} m(\sigma) d(x_K, x_L) f_\sigma^{n+1} \leq \frac{2T}{\zeta} \|f_0\|_{L^1(\Omega)}.$$

This concludes the proof of (32). The proof of (33) is similar. \square

We can apply the [7, Theorem 3.9], we conclude that

Proposition 5.2. *Let $(\mathcal{D}_m)_{m \geq 1}$ be a family of admissible space-time discretization of Ω_T such that (10) holds. Let $(f_m)_m$ and $(g_m)_m$ be the corresponding sequence of discrete solution to the scheme (14)-(17), then up to an unlabeled subsequence, there exist $f \in L^2(0, T; H^1(\Omega))$ and $g \in L^2(0, T; H^1(\Omega))$ such that*

$$f_m \xrightarrow{m \rightarrow +\infty} f, \quad \text{a.e in } \Omega_T, \quad \text{and} \quad g_m \xrightarrow{m \rightarrow +\infty} g, \quad \text{a.e in } \Omega_T.$$

$(f_m)_{m > 0}$ and $(g_m)_{m > 0}$ are uniformly bounded in $L^\infty(0; T, L^2(\Omega))$ thanks to Corollary 4.8. By Corollary 4.5 and Sobolev embedding, $(f_m)_{m > 0}$ and $(g_m)_{m > 0}$ are bounded in $L^2(0, T; L^p(\Omega))$, with $p < +\infty$. Then, thanks to Riez-Thorin theorem, $(f_m)_{m > 0}$ and $(g_m)_{m > 0}$ are bounded in $L^r(\Omega_T)$ with $2 < r < 4$. Hence $(f_m)_{m > 0}$ and $(g_m)_{m > 0}$ are equi-integrable in $L^r(\Omega_T)$. Applying the Vitali's convergence theorem we deduce that

Lemma 5.3. *Keeping the assumption and notations of Proposition 5.2, one has*

$$f_m \xrightarrow{m \rightarrow +\infty} f, \quad \text{strongly in } L^r(\Omega_T), \quad \text{for all } r < 4,$$

and

$$g_m \xrightarrow{m \rightarrow +\infty} g, \quad \text{strongly in } L^r(\Omega_T), \quad \text{for all } r < 4.$$

We show now the weak compactness in $L^2(\Omega_T)^2$ of the approximate gradient of f_m and g_m defined in (18) :

Proposition 5.4. *Keeping the assumption and notations of Proposition 5.2, one has*

$$\nabla^m f_m \xrightarrow{m \rightarrow +\infty} \nabla f \quad \text{weakly in } L^2(\Omega_T)^2, \quad \text{and} \quad \nabla^m g_m \xrightarrow{m \rightarrow +\infty} \nabla g \quad \text{weakly in } L^2(\Omega_T)^2.$$

Proof. Thanks to (30) $\nabla^D f_D$ and $\nabla^D g_D$ are bounded in $L^2(\Omega_T)^2$, then there exists a subsequence of $\nabla^D f_D$ and of $\nabla^D g_D$ (still labeled $\nabla^D f_D$ and $\nabla^D g_D$) and two function $\Theta, \Xi \in L^2(\Omega_T)^2$ such that

$$\nabla^m f_m \xrightarrow{m \rightarrow +\infty} \Theta, \quad \text{weakly in } L^2(\Omega_T)^2,$$

and

$$\nabla^m g_m \xrightarrow{m \rightarrow +\infty} \Xi, \quad \text{weakly in } L^2(\Omega_T)^2.$$

We refer to [13, 24] to prove that $\Theta = \nabla f$ and $\Xi = \nabla g$. □

5.2. Identification as a weak solution.

Proposition 5.5. *Let (f, g) be as in Proposition 5.2, then f and g are the weak solution to (3)-(5) in the sense of (19) and (20).*

Proof. Let $\psi \in C_0^\infty(\bar{\Omega} \times [0, T])$ be a test function and $\psi_K^{n+1} = \psi(x_K, t_{n+1})$ for all $K \in \mathcal{T}$ and $n \in \{0, \dots, M_T - 1\}$. We first establish (19) from (14) and to obtain (20) from (15) is similar. In order to prove that f is a weak solution, we multiply (14) by $\Delta t_m \psi_K^n$ and sum over $n \in \{0, \dots, M_T - 1\}$ and $K \in \mathcal{T}$, we obtain

$$A_m + B_m = 0,$$

where

$$A_m = \sum_{n=0}^{M_T-1} \sum_{K \in \mathcal{T}} m(K) (f_K^{n+1} - f_L^{n+1}) \psi_K^n,$$

$$B_m = \nu \sum_{n=0}^{M_T-1} \Delta t_m \sum_{K \in \mathcal{T}} \psi_K^n \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma f_\sigma^{n+1} \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) + (b_K - b_L) \right].$$

Note that $\psi_K^{M_T} = 0$ for all $K \in \mathcal{T}$, then a discrete integration part yields

$$A_m = - \sum_{n=0}^{M_T-1} \Delta t_m \sum_{K \in \mathcal{T}} m(K) \frac{\psi_K^{n+1} - \psi_K^n}{\Delta t_m} f_K^{n+1} - \sum_{K \in \mathcal{T}} m(K) f_K^0 \psi_K^0$$

$$= - \int_0^T \int_\Omega f_m(\delta\psi)_m \, dx \, dt - \int_\Omega f_m^0 \psi_m(\cdot, 0) \, dx,$$

where the function $\delta\psi_m(x_K, t) = \frac{\psi_K^{n+1} - \psi_K^n}{\Delta t_m}$, if $(x_K, t) \in K \times (t^n, t^{n+1})$. Thanks to the regularity of ψ , the function $\delta\psi_m$ converges uniformly towards $\partial_t \psi$ on Ω_T . Moreover, we have

$$f_m \longrightarrow f \quad \text{in } L^2(\Omega_T) \text{ as } m \rightarrow \infty.$$

Therefore

$$(37) \quad \int_0^T \int_\Omega f_m(\delta\psi)_m \, dx \, dt \longrightarrow \int_0^T \int_\Omega f(x) \partial_t \psi \, dx \, dt \quad \text{as } m \rightarrow \infty.$$

Moreover, f_m^0 converges strongly in $L^1(\Omega)$ towards the initial data f_0 and $\psi_m(\cdot, 0)$ converges uniformly towards $\psi(\cdot, 0)$. Therefore, we get that

$$(38) \quad \int_\Omega f_m^0 \psi_m(\cdot, 0) \, dx \longrightarrow \int_\Omega f_0(x) \psi(\cdot, 0) \, dx \quad m \rightarrow \infty.$$

We deduce from (37) and (38) that

$$A_m \longrightarrow - \int_0^T \int_\Omega f(x) \partial_t \psi \, dx \, dt - \int_\Omega f_0(x) \psi(\cdot, 0) \, dx \quad m \rightarrow \infty.$$

We introduce the term

$$E_m = \int_{\Omega_T} \bar{f}_{\mathcal{D}_m} \nabla^m u_m \cdot \nabla \psi \, dx \, dt,$$

where

$$\bar{f}_{\mathcal{D}}(x, t) = f_\sigma^{n+1} \quad \forall (t, x) \in (t^n, t^{n+1}] \times \mathcal{D}_{K,L}, \quad \text{and } u = f + g + b.$$

We have $f_m \xrightarrow{m \rightarrow +\infty} f$, strongly in $L^2(\Omega_T)$. Let us to prove that

$$\bar{f}_{\mathcal{D}_m} := \bar{f}_m \xrightarrow{m \rightarrow +\infty} f, \quad \text{strongly in } L^2(\Omega_T).$$

Since $\|\bar{f}_m - f\|_{L^2(\Omega_T)} \leq \|\bar{f}_m - f_m\|_{L^2(\Omega_T)} + \|f_m - f\|_{L^2(\Omega_T)}$, it is sufficient to prove that $\|\bar{f}_m - f_m\|_{L^2(\Omega_T)} \rightarrow 0$, as $\delta_m \xrightarrow{m \rightarrow +\infty} 0$. One has

$$\begin{aligned} \|\bar{f}_m - f_m\|_{L^2(\Omega_T)}^2 &= \sum_{n=0}^{M_T-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int},K}} m(\mathfrak{D}_{K,\sigma})(f_K^{n+1} - f_\sigma^{n+1})^2 \\ &\leq \sum_{n=0}^{M_T-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int},K}} m(\mathfrak{D}_{K,\sigma})(f_K^{n+1} - f_L^{n+1})^2 \\ &\leq \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\mathfrak{D}_{K,L})(f_K^{n+1} - f_L^{n+1})^2 \\ &\leq \frac{1}{2} \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma (f_K^{n+1} - f_L^{n+1})^2 d_\sigma^2 \leq \frac{C_1}{2} \delta_m^2. \end{aligned}$$

Since $\nabla^m u_m$ converges weakly in $L^2(\Omega_T)$ to ∇u , since \bar{f}_m converges strongly in $L^2(\Omega_T)$ to f , we have

$$E_m \longrightarrow \int_{\Omega_T} f \nabla u \cdot \nabla \psi \, dx \, dt \quad \text{as } m \rightarrow \infty.$$

Let us to prove $E_m - \bar{E}_m$ tends to 0 as $m \rightarrow \infty$, where

$$\bar{E}_m = \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1})(\psi_K^n - \psi_L^n).$$

Using the definition of discrete gradient (18), we have

$$E_m = \sum_{n=0}^{M_T-1} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} f_\sigma^{n+1} \int_{t^n}^{t^{n+1}} \int_{\mathfrak{D}_{K,L}} \frac{m(\sigma)}{m(\mathfrak{D}_{K,L})} (u_K^{n+1} - u_L^{n+1}) \nabla \psi \cdot \nu_{L,K} \, dx \, dt.$$

Therefore by the definition of τ_σ ,

$$\begin{aligned} E_m - \bar{E}_m &= \sum_{n=0}^{M_T-1} \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1}) \int_{t^n}^{t^{n+1}} \left(\frac{\psi_K - \psi_L}{d(x_K, x_L)} \right. \\ &\quad \left. - \frac{1}{m(\mathfrak{D}_{K,L})} \int_{\mathfrak{D}_{K,L}} \nabla \psi \cdot \nu_{L,K} \, dx \right) dt. \end{aligned}$$

On the one hand, since the straight line (x_K, x_L) is orthogonal to σ , we have $x_K - x_L = d(x_K, x_L) \nu_{L,K}$. It follows from the regularity of ψ that

$$\begin{aligned} \frac{\psi_K^n - \psi_L^n}{d(x_K, x_L)} &= \nabla \psi(t^n, x_L) \cdot \nu_{L,K} + O(\delta) \\ &= \nabla \psi(t, x) \cdot \nu_{L,K} + O(\eta), \quad \forall (t, x) \in (t^n, t^{n+1}) \times \mathfrak{D}_{K,L}. \end{aligned}$$

By taking the mean value over $\mathfrak{D}_{K,L}$, there exists a constant $C_5 > 0$, depending only on ψ , such that

$$\left| \int_{t^n}^{t^{n+1}} \left(\frac{\psi_K^n - \psi_L^n}{d(x_K, x_L)} - \frac{1}{m(\mathfrak{D}_{K,L})} \int_{\mathfrak{D}_{K,L}} \nabla \psi \cdot \nu_{L,K} dx \right) dt \right| \leq C_5 \Delta t \eta.$$

On the other hand, one has $m(\sigma) = \sqrt{\tau_\sigma} \sqrt{m(\sigma) d(x_K, x_L)}$. Hence by Cauchy-schwarz inequality, we have

$$\begin{aligned} & \left| \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1}) \right|^2 \\ & \leq \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1})^2 \times \sum_{n=0}^{M_T-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} m(\sigma) d(x_K, x_L) f_\sigma^{n+1}. \end{aligned}$$

Using Corollary 4.9 and (36) we conclude that

$$|E_m - \bar{E}_m|^2 \leq \frac{2T}{\zeta} \|f_0\|_{L^1(\Omega)} C_2 C_5^2 \eta^2 \longrightarrow 0, \quad \text{as } \eta \rightarrow 0.$$

This ensures that B_m converges towards $\nu \int_{\Omega_T} f \nabla u \cdot \nabla \psi dx dt$, as $m \rightarrow +\infty$. \square

6. NUMERICAL RESULTS

Let us provide some illustrations of the behavior of the numerical scheme (14)-(17). The scheme leads to a nonlinear system that we solve thanks to the Newton-Raphson method. In our test case, the domain is the unit square, i.e., $\Omega = (0, 1)^2$. We consider an admissible triangular mesh made of 14336 triangles. An illustration of a mesh type used here is given in Figure 2. The numerical analysis of the scheme was carried out for a uniform time discretization of $(0, T)$ only in order to avoid heavy notations. In order to increase the robustness of the algorithm and to ensure the convergence of the Newton-Raphson iterative procedure, we used an adaptive time step procedure in the practical implementation. More precisely, we associate a maximal time step $\Delta t_{\text{max}} = 0.00004$ for the mesh. If the Newton-Raphson method fails to converge after 30 iterations —we choose that the ℓ^∞ norm of the residual has to be smaller than 10^{-10} as stopping criterion—, the time step is divided by two. If the Newton-Raphson method converges, the time step is multiplied by two and projected on $[0, \Delta t_{\text{max}}]$. The first time step Δt is equal to Δt_{max} in the test case presented below. We perform the numerical experiments with the following data

$$b(x, y) = \max \left(0, \frac{1}{2} \left(1 - 16(x - 1/2)^2 \right) \left((\cos(\pi y) + 2) \right) \right), \quad \nu = 0.9.$$

As an initial condition we take

$$f_0(x, y) = \begin{cases} \frac{1}{2} & \text{if } x \leq \frac{1}{4}, \\ 0 & \text{elsewhere,} \end{cases} \quad g_0(x, y) = \begin{cases} b\left(\frac{1}{2}, 0\right) - b(x, y) - \left(x - \frac{1}{2}\right) & \text{if } x \leq \frac{1}{2}, \\ 0 & \text{elsewhere.} \end{cases}$$

Figure 3 shows the evolution of $\xi(x, t) = b(x) + g(x, t)$ (red) and $h(x, t) = b(x) + g(x, t) + f(x, t)$ (blue) at time $t = 0, t = 0.2, t = 0.72$ and $t = 12$. There is

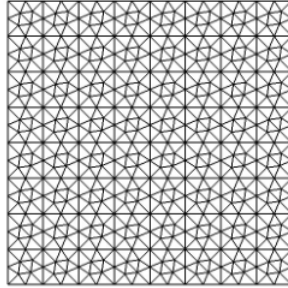


FIGURE 2. Mesh type used in the numerical test

convergence towards an equilibrium state, with horizontal interfaces as expected (see [21]).

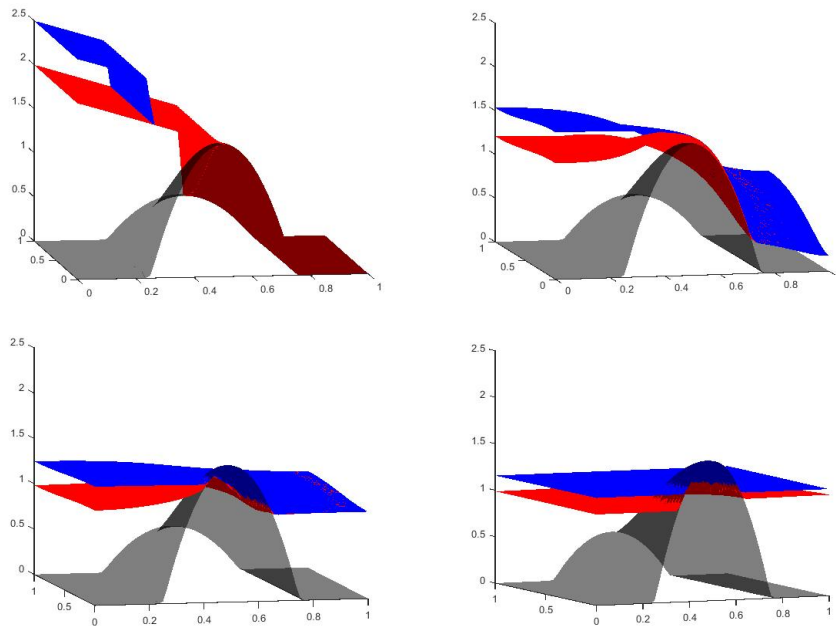
FIGURE 3. Behaviour of the model at $t = 0$, $t = 0.2$, $t = 0.72$, $t = 12$

Figure 4 shows the evolution of the energy along time

7. CONCLUSION

We proposed and analyzed a finite volume scheme for solving the seawater intrusion model. It preserves at the discrete level the main features of the continuous problem : the nonnegativity of the solutions, the decay of the energy, the control of the entropy and its dissipation. Moreover, we were able to carry out a full convergence analysis based on compactness arguments.

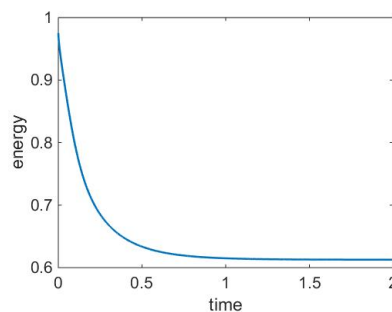


FIGURE 4. Evolution of the energy along time

Let us mention that to derive the problem (3), the authors in [30] assume that, for simplification, the porous medium is isotropic. Moreover in our work the mesh satisfies the so-called orthogonality condition (see, e.g., [26, Definition 9.1] so that the two-point flux approximation is consistent. We know that the finite volume method with two-point flux approximation, used here, does not allow to handle the anisotropic case. Nevertheless in order to treat this case, we can use for instance a Control Volume Finite Element scheme (CVFE) [29, 12, 3].

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