



**HAL**  
open science

# SHARP LARGE DEVIATIONS FOR THE NUMBER OF DESCENTS AND THE MAJOR INDEX IN A RANDOM PERMUTATION

Bernard Bercu, Michel Bonnefont, Luis Fredes, Adrien Richou

► **To cite this version:**

Bernard Bercu, Michel Bonnefont, Luis Fredes, Adrien Richou. SHARP LARGE DEVIATIONS FOR THE NUMBER OF DESCENTS AND THE MAJOR INDEX IN A RANDOM PERMUTATION. 2024. hal-04636515

**HAL Id: hal-04636515**

**<https://hal.science/hal-04636515v1>**

Preprint submitted on 5 Jul 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SHARP LARGE DEVIATIONS FOR THE NUMBER OF DESCENTS AND THE MAJOR INDEX IN A RANDOM PERMUTATION

BERNARD BERCU, MICHEL BONNEFONT, LUIS FREDES, AND ADRIEN RICHOU

ABSTRACT. The aim of this paper is to improve the large deviation principle for the number of descents in a random permutation by establishing a sharp large deviation principle of any order. We shall also prove a sharp large deviation principle of any order for the major index in a random permutation.

Sharp large deviations, random permutations, number of descents, major index  
60F10, 05A05

## 1. INTRODUCTION

Let  $\mathcal{S}_n$  be the symmetric group of permutations on the set of integers  $\{1, \dots, n\}$ . A permutation  $\pi_n \in \mathcal{S}_n$  is said to have a descent at position  $k \in \{1, \dots, n-1\}$  if  $\pi_n(k) > \pi_n(k+1)$ . Denote by  $D_n = D_n(\pi_n)$  and  $M_n = M_n(\pi_n)$  the random variables counting the number of descents and the major index of a permutation  $\pi_n$  chosen uniformly at random over the set  $\mathcal{S}_n$ . They are precisely defined, for all  $n \geq 2$ , by

$$(1.1) \quad D_n = \sum_{k=1}^{n-1} \mathbf{I}_{\{\pi_n(k) > \pi_n(k+1)\}} \quad \text{and} \quad M_n = \sum_{k=1}^{n-1} k \mathbf{I}_{\{\pi_n(k) > \pi_n(k+1)\}}.$$

On the one hand, it follows from a beautiful result from Tanny [7] that  $D_n$  shares the same distribution as the integer part of the Irwin-Hall distribution. More precisely, the distribution of  $D_n$  is nothing else than the one of the integer part of

$$(1.2) \quad S_n = \sum_{k=1}^n U_k$$

where  $(U_n)$  is a sequence of independent and identically distributed random variables with uniform distribution on the interval  $[0, 1]$ . On the other hand, it is well-known from an old result due to MacMahon [5] that  $M_n$  shares the same distribution as

---

UNIVERSITÉ DE BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UMR CNRS 5251, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE.

*E-mail addresses:* [bernard.bercu@math.u-bordeaux.fr](mailto:bernard.bercu@math.u-bordeaux.fr), [michel.bonnefont@math.u-bordeaux.fr](mailto:michel.bonnefont@math.u-bordeaux.fr), [luis.fredes@math.u-bordeaux.fr](mailto:luis.fredes@math.u-bordeaux.fr), [adrien.richou@math.u-bordeaux.fr](mailto:adrien.richou@math.u-bordeaux.fr).

*Date:* July 5, 2024.

*2020 Mathematics Subject Classification.* 60F10, 05A05.

*Key words and phrases.* Large deviations, concentration inequalities, random permutations.

The corresponding author is Bernard Bercu, email address: [bernard.bercu@math.u-bordeaux.fr](mailto:bernard.bercu@math.u-bordeaux.fr).

the number of inversions of the permutation  $\pi_n$ . Consequently, the distribution of  $M_n$  coincides with that of

$$(1.3) \quad \Sigma_n = \sum_{k=1}^n V_k$$

where  $(V_n)$  is a sequence of independent random variables such that for all  $n \geq 1$ ,  $V_n$  is uniformly distributed on the integers  $\{0, \dots, n-1\}$ . It is easy to see from (1.2) that for all  $n \geq 2$ ,

$$\mathbb{E}[D_n] = \frac{n-1}{2} \quad \text{and} \quad \text{Var}(D_n) = \frac{n+1}{12}.$$

Moreover, we also have from (1.3) that for all  $n \geq 2$ ,

$$\mathbb{E}[M_n] = \frac{n(n-1)}{4} \quad \text{and} \quad \text{Var}(M_n) = \frac{n(n-1)(2n+5)}{72}.$$

Using a totally different approach involving the number of leaves in a random recursive tree, it follows from Bryc, Minda and Sethuraman [4] that the sequence  $(D_n/n)$  satisfies a large deviation principle (LDP) with good rate function

$$(1.4) \quad I_D(x) = \sup_{t \in \mathbb{R}} \{xt - L_D(t)\}$$

where the cumulant generating function  $L_D$  is given by

$$(1.5) \quad L_D(t) = \log\left(\frac{\exp(t) - 1}{t}\right).$$

This result was improved in [2] by sharp large deviations and concentration inequalities for the sequence  $(D_n/n)$ . More recently, Méliot and Nikeghbali [6] have shown that the sequence  $(M_n/n^2)$  satisfies an LDP with good rate function

$$(1.6) \quad I_M(x) = \sup_{t \in \mathbb{R}} \{xt - L_M(t)\}$$

where, as expected,  $L_M$  is the integral of  $L_D$ ,

$$(1.7) \quad L_M(t) = \int_0^1 L_D(xt) dx.$$

The aim of this paper is to go deeper in the analysis of [2] and [6] by proving sharp large deviation principle (SLDP) of any order for the sequences  $(D_n)$  and  $(M_n)$  by use of the decompositions (1.2) and (1.3). One can observe that a SLDP of order zero for  $(M_n)$  was previously established in [6].

The paper is organized as follows. Section 2 is devoted our main results where we establish a SLDP of any order for the sequences  $(D_n)$  and  $(M_n)$ . The proof of the SLDP for the sequence  $(D_n)$  is given in Section 3 while the proof of the SLDP for the sequence  $(M_n)$  is carry out in Section 4.

## 2. MAIN RESULTS

Our first result improves the LDP for the sequence  $(D_n/n)$  previously established by Theorem 1.1 in [4] as well as Theorem 3.1 in [2]. For any positive real number  $x$ , denote  $\{x\} = \lceil x \rceil - x$ .

**Theorem 2.1.** *The number of descents  $(D_n)$  satisfies a SLDP of any order. More precisely, for any  $x$  in  $]1/2, 1[$  and for any integer  $p \geq 1$ , there exists a sequence  $d_{n,1}(x), \dots, d_{n,p}(x)$  such that for  $n$  large enough,*

$$(2.1) \quad \mathbb{P}\left(\frac{D_n}{n} \geq x\right) = \frac{\exp(-nI_D(x) - \{nx\}t_x)}{\sigma_x t_x \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^p \frac{d_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right) \right]$$

where the value  $t_x$  is the unique solution of  $L'_D(t_x) = x$ ,  $\sigma_x^2 = L''_D(t_x)$  and the coefficients  $d_{n,1}(x), \dots, d_{n,p}(x)$  are bounded in  $n$  and may be explicitly calculated as functions of the derivatives of  $L_D$  at point  $t_x$ . For example, the first coefficient  $d_{n,1}(x)$  is given by

$$d_{n,1}(x) = \frac{1}{\sigma_x^2} \left( -\frac{1}{t_x^2} - \frac{\{nx\}}{t_x} - \frac{\{nx\}^2}{2} - \frac{\{nx\}\ell_D(3)}{6\sigma_x^2} - \frac{\ell_D(3)}{2\sigma_x^2 t_x} + \frac{\ell_D(4)}{8\sigma_x^2} - \frac{5\ell_D^2(3)}{24\sigma_x^4} \right)$$

where  $\ell_D(3)$  and  $\ell_D(4)$  stand for the higher-order derivatives  $\ell_D(3) = L_D^{(3)}(t_x)$  and  $\ell_D(4) = L_D^{(4)}(t_x)$ .

**Remark 2.1.** *The asymptotic expansion (2.1) is inspired by the seminal work of Bahadur and Rao [1] concerning the SLDP for the sample mean, see also [3] for the SLDP for quadratic forms of Gaussian stationary processes.*

Our second result goes deeper than the recent analysis of Méliot and Nikeghbali [6] on the large deviation properties of the major index of random standard tableaux. It is more complicated than our previous result as it involves the additional prefactor function

$$(2.2) \quad H(t) = \frac{1}{2}(L_D(t) - t).$$

**Theorem 2.2.** *The major index  $(M_n)$  satisfies a SLDP of any order. More precisely, for any  $x$  in  $]1/4, 1/2[$  and for any integer  $p \geq 1$ , there exists a sequence  $m_{n,1}(x), \dots, m_{n,p}(x)$  such that for  $n$  large enough,*

$$(2.3) \quad \mathbb{P}\left(\frac{M_n}{n^2} \geq x\right) = \frac{\exp(-nI_M(x) + H(t_x))}{\sigma_x t_x \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^p \frac{m_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right) \right]$$

where the value  $t_x$  is the unique solution of  $L'_M(t_x) = x$ ,  $\sigma_x^2 = L''_M(t_x)$  and the coefficients  $m_{n,1}(x), \dots, m_{n,p}(x)$  may be explicitly calculated as functions of the derivatives of  $L_D$ ,  $L_M$  and  $H$  at point  $t_x$ . For example, the first coefficient  $m_{n,1}(x)$  is given by

$$m_{n,1}(x) = \frac{t_x}{2} - \frac{t_x(1+t_x)}{24} - \{n^2x\}t_x + \frac{t_x\ell_D(1)}{12} \\ + \frac{1}{\sigma_x^2} \left( -\frac{1}{t_x^2} + \frac{h(1)}{t_x} - \frac{h(2)}{2} - \frac{h^2(1)}{2} - \frac{\ell_M(3)}{2\sigma_x^2 t_x} + \frac{\ell_M(4)}{8\sigma_x^2} + \frac{h(1)\ell_M(3)}{2\sigma_x^2} - \frac{5\ell_M^2(3)}{24\sigma_x^4} \right)$$

where for  $1 \leq k \leq 4$ ,  $\ell_D(k) = L_D^{(k)}(t_x)$ ,  $\ell_M(k) = L_M^{(k)}(t_x)$  and  $h(k) = H^{(k)}(t_x)$ .

**Remark 2.2.** We can even be more precise on the behavior of the coefficients  $d_{n,k}(x)$  and  $m_{n,k}(x)$  in Theorems 2.1 and 2.2. More precisely, it is possible to show that  $d_{n,k}(x)$  and  $m_{n,k}(x)$  are respectively polynomials on the bounded terms  $\{nx\}$  and  $\{n^2x\}$ , with coefficients only depending on  $t_x$  and on the derivatives of  $L_D$ ,  $L_M$  and  $H$  at point  $t_x$ .

### 3. PROOF OF THE SLDP FOR THE NUMBER OF DESCENTS

We saw in Section 1 that the distribution  $D_n$  coincides with that of the integer part of  $S_n$  given by (1.2). It is quite easy to compute the Laplace transform of  $S_n$ . As a matter of fact, as  $U_1, \dots, U_n$  are independent and identically distributed with uniform distribution on  $[0, 1]$ , we have for all  $t \in \mathbb{R}$ ,

$$(3.1) \quad \mathbb{E}[\exp(tS_n)] = \mathbb{E}\left[\prod_{k=1}^n \exp(tU_k)\right] = \prod_{k=1}^n \mathbb{E}[\exp(tU_k)] = \exp(nL_D(t))$$

where the cumulant generating function  $L_D$  is given by (1.5). Hence, as  $I_D(x) = xt_x - L_D(x)$ , we have for any  $x \in ]1/2, 1[$ ,

$$(3.2) \quad \begin{aligned} \mathbb{P}(D_n \geq nx) &= \mathbb{P}(S_n \geq \lceil nx \rceil), \\ &= \mathbb{E}_n \left[ \exp\left(-t_x S_n + nL_D(t_x)\right) \mathbb{1}_{\left\{\frac{S_n}{n} \geq x + \varepsilon_n\right\}} \right], \\ &= \exp(-nI_D(x)) \mathbb{E}_n \left[ \exp\left(-nt_x \left(\frac{S_n}{n} - x\right)\right) \mathbb{1}_{\left\{\frac{S_n}{n} \geq x + \varepsilon_n\right\}} \right], \end{aligned}$$

where  $\mathbb{E}_n$  stands the expectation under the new probability  $\mathbb{P}_n$  given by

$$(3.3) \quad \frac{d\mathbb{P}_n}{d\mathbb{P}} = \exp(t_x S_n - nL_D(t_x))$$

and  $\varepsilon_n = \{nx\}/n$ . Denote by  $V_n$  the standardized random variables under the new probability  $\mathbb{P}_n$ ,

$$V_n = \frac{\sqrt{n}}{\sigma_x} \left( \frac{S_n}{n} - x \right).$$

Moreover, let  $f_n$  and  $\Phi_n$  be the probability density function and the characteristic function of  $V_n$  under  $\mathbb{P}_n$ , respectively. One can easily see that for all  $v \in \mathbb{R}$ ,

$$f_n(v) = \sqrt{n}\sigma_x \exp(\sqrt{n}t_x\sigma_x v + nxt_x - nL_D(t_x)) g_n(\sqrt{n}\sigma_x v + nx)$$

where  $g_n$  is the probability density function of  $S_n$  under  $\mathbb{P}$ , that is to say the density of the Irwin-Hall distribution which is bounded with compact support. Thus,  $f_n$  is

clearly square integrable on  $\mathbb{R}$ . Accordingly, it follows from Parseval's identity that

$$\begin{aligned}
\mathbb{E}_n \left[ \exp \left( -nt_x \left( \frac{S_n}{n} - x \right) \right) \mathbb{I}_{\left\{ \frac{S_n}{n} \geq x + \varepsilon_n \right\}} \right] &= \mathbb{E}_n \left[ \exp \left( -t_x \sigma_x \sqrt{n} V_n \right) \mathbb{I}_{\left\{ V_n \geq \frac{\sqrt{n} \varepsilon_n}{\sigma_x} \right\}} \right], \\
&= \int_{\mathbb{R}} \exp \left( -\sigma_x t_x \sqrt{n} v \right) \mathbb{I}_{\left\{ v \geq \frac{\sqrt{n} \varepsilon_n}{\sigma_x} \right\}} f_n(v) dv, \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\exp \left( -(\sigma_x t_x \sqrt{n} + iv) \frac{\sqrt{n} \varepsilon_n}{\sigma_x} \right)}{\sigma_x t_x \sqrt{n} + iv} \Phi_n(v) dv, \\
(3.4) \quad &= \frac{\exp(-t_x \{nx\})}{2\pi \sigma_x t_x \sqrt{n}} \int_{\mathbb{R}} \left( \frac{\exp \left( -\frac{i\{nx\}v}{\sigma_x \sqrt{n}} \right)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} \right) \Phi_n(v) dv.
\end{aligned}$$

Therefore, we deduce from (3.2) and (3.4) that for any  $x \in ]1/2, 1[$ ,

$$(3.5) \quad \mathbb{P}(D_n \geq nx) = \frac{\exp(-nI_D(x) - t_x \{nx\})}{2\pi \sigma_x t_x \sqrt{n}} P_n(x)$$

where

$$(3.6) \quad P_n(x) = \int_{\mathbb{R}} \left( \frac{\exp \left( -\frac{i\{nx\}v}{\sigma_x \sqrt{n}} \right)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} \right) \Phi_n(v) dv.$$

Hence, we obtain from (3.1) and (3.3) that

$$\begin{aligned}
\Phi_n(v) &= \mathbb{E}_n \left[ \exp \left( \frac{iv\sqrt{n}}{\sigma_x} \left( \frac{S_n}{n} - x \right) \right) \right], \\
&= \mathbb{E} \left[ \exp \left( \frac{ivS_n}{\sigma_x \sqrt{n}} - \frac{iv\sqrt{n}x}{\sigma_x} + t_x S_n - nL_D(t_x) \right) \right], \\
(3.7) \quad &= \exp \left( n \left( L_D \left( t_x + \frac{iv}{\sigma_x \sqrt{n}} \right) - L_D(t_x) \right) - \frac{iv\sqrt{n}x}{\sigma_x} \right).
\end{aligned}$$

Furthermore, the function  $L_D$  is infinitely differentiable in a complex neighbourhood of any point  $t_x > 0$ . Then, it follows from Taylor's theorem in the complex plane that for any integer  $p \geq 1$  and for all  $v \in \mathbb{R}$  such that  $|v| < \sigma_x \sqrt{n}$ ,

$$\log \Phi_n(v) = -\frac{iv\sqrt{n}x}{\sigma_x} + n \sum_{k=1}^{2p+3} \left( \frac{iv}{\sigma_x \sqrt{n}} \right)^k \frac{\ell_D(k)}{k!} + R_{n,p}(v)$$

where for all integer  $k \geq 1$ ,  $\ell_D(k) = L_D^{(k)}(t_x)$ , and  $R_{n,p}(v)$  is a complex remainder term. Since  $L_D'(t_x) = x$  and  $L_D''(t_x) = \sigma_x^2$ , the above Taylor expansion reduces to

$$(3.8) \quad \log \Phi_n(v) = -\frac{v^2}{2} + n \sum_{k=3}^{2p+3} \left( \frac{iv}{\sigma_x \sqrt{n}} \right)^k \frac{\ell_D(k)}{k!} + R_{n,p}(v).$$

Hereafter, one can easily see that for all  $t \neq 0$  and for all integer  $k \geq 1$ ,

$$(3.9) \quad L_D^{(k)}(t) = \sum_{\ell=1}^k \frac{a_{k,\ell}}{(e^t - 1)^\ell} + \frac{b_k}{t^k}$$

where the coefficients  $a_{k,\ell}$  are the positive integers given, for all  $1 \leq \ell \leq k$ , by

$$a_{k,\ell} = \frac{1}{\ell} \sum_{i=1}^{\ell} (-1)^{\ell-i} \binom{\ell}{i} i^k$$

and  $b_k = (-1)^k (k-1)!$ . In addition, for all  $t > 0$  and for all  $v \in \mathbb{R}$ ,  $|t + iv| \geq t$  and  $|\exp(t + iv) - 1| \geq \exp(t) - 1$ , which implies from (3.9) that for all  $k \geq 1$ ,

$$(3.10) \quad \left| L_D^{(k)}(t_x + iv) \right| \leq \sum_{\ell=1}^k \frac{a_{k,\ell}}{(e^{t_x} - 1)^\ell} + \frac{|b_k|}{t_x^k}.$$

It follows from (3.10) that for all  $v \in \mathbb{R}$  such that  $|v| < \sigma_x \sqrt{n}$ ,

$$(3.11) \quad |R_{n,p}(v)| \leq \frac{M_{x,p} v^{2p+4}}{n^{p+1}}$$

for some positive constant  $M_{x,p}$  that does not depend on  $v$ . Denote  $s_n = \varepsilon_n n^{1/6}$  where  $\varepsilon_n = n^{-\alpha}$  with  $0 < \alpha < 1/6$  and let

$$u_n = n \sum_{k=3}^{2p+3} \left( \frac{iv}{\sigma_x \sqrt{n}} \right)^k \frac{\ell_D(k)}{k!} + O\left( \frac{v^{2p+4}}{n^{p+1}} \right).$$

Since  $u_n$  is of order  $v^3/\sqrt{n}$ , we deduce from the standard Taylor expansion

$$\exp(u) = 1 + u + \cdots + \frac{u^{2p+1}}{(2p+1)!} + O(u^{2p+2})$$

that for all  $v \in \mathbb{R}$  such that  $|v| < s_n$ ,

$$(3.12) \quad \Phi_n(v) = \exp\left(-\frac{v^2}{2}\right) \left[ 1 + \sum_{k=1}^{2p+1} \frac{\psi_k(v)}{(\sqrt{n})^k} + O\left(\frac{\max(v^{2(p+2)}, v^{6(p+1)})}{n^{p+1}}\right) \right]$$

where the  $\psi_k$  are polynomials of degree  $3k$  and valuation  $k+2$ , with only odd powers for  $k$  odd and only even powers for  $k$  even. Hence, we obtain from (3.12) that for all  $v \in \mathbb{R}$  such that  $|v| < s_n$ ,

$$(3.13) \quad \frac{\Phi_n(v)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} = \exp\left(-\frac{v^2}{2}\right) \left[ 1 + \sum_{k=1}^{2p+1} \frac{\varphi_k(v)}{(\sqrt{n})^k} + O\left(\frac{\max(v^{2(p+1)}, v^{6(p+1)})}{n^{p+1}}\right) \right]$$

where the  $\varphi_k$  are also polynomials of degree  $3k$  and valuation  $k+2$ , with only odd powers for  $k$  odd and only even powers for  $k$  even. For example,

$$\begin{aligned} \varphi_1(v) &= -\frac{iv}{\sigma_x t_x} - \frac{iv^3 \ell_D(3)}{6\sigma_x^3}, \\ \varphi_2(v) &= -\frac{v^2}{\sigma_x^2 t_x^2} - \frac{v^4 \ell_D(3)}{6\sigma_x^4 t_x} + \frac{v^4 \ell_D(4)}{24\sigma_x^4} - \frac{v^6 \ell_D^2(3)}{72\sigma_x^6}. \end{aligned}$$

Note that in some special regimes of  $v$ , it may appear that some terms in the expansions (3.12) and (3.13) may be removed and added in one of the  $O$  terms. Note also that  $\varphi_k(v)$  is purely imaginary for  $k$  odd and real for  $k$  even. From now

on, the integral  $P_n(x)$ , given by (3.6), can be separated into three parts,  $P_n(x) = A_n(x) + B_n(x) + C_n(x)$  where

$$\begin{aligned} A_n(x) &= \int_{|v| < s_n} \left( \frac{\exp\left(-\frac{i\{nx\}v}{\sigma_x \sqrt{n}}\right)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} \right) \Phi_n(v) dv, \\ B_n(x) &= \int_{s_n \leq |v| \leq 2\sigma_x \sqrt{n}} \left( \frac{\exp\left(-\frac{i\{nx\}v}{\sigma_x \sqrt{n}}\right)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} \right) \Phi_n(v) dv, \\ C_n(x) &= \int_{|v| \geq 2\sigma_x \sqrt{n}} \left( \frac{\exp\left(-\frac{i\{nx\}v}{\sigma_x \sqrt{n}}\right)}{1 + \frac{iv}{\sigma_x t_x \sqrt{n}}} \right) \Phi_n(v) dv. \end{aligned}$$

We are going to show that the lower part  $A_n(x)$  of  $P_n(x)$  will complete the asymptotic expansion (2.1) via (3.5), while the middle part  $B_n(x)$  and the upper part  $C_n(x)$  of  $P_n(x)$  will play a negligible role. It follows from straightforward calculation previously made in section 4 of [2] that for all  $t > 0$  and for all  $v \in \mathbb{R}$ ,

$$(3.14) \quad \left| \frac{e^{t+iv} - 1}{t + iv} \right|^2 = \left( \frac{e^t - 1}{t} \right)^2 \left( \frac{t^2}{t^2 + v^2} \right) \left( 1 + \frac{2(1 - \cos(v))e^t}{(e^t - 1)^2} \right).$$

As  $2(1 - \cos(v)) \leq \min(v^2, 4)$ , (3.14) implies that

$$(3.15) \quad \left| \frac{e^{t+iv} - 1}{t + iv} \right|^2 \leq \left( \frac{e^t - 1}{t} \right)^2 \left( 1 - \left( \frac{(e^t - 1)^2 - t^2 e^t}{(e^t - 1)^2} \right) \frac{v^2}{t^2 + v^2} \right),$$

as well as

$$(3.16) \quad \left| \frac{e^{t+iv} - 1}{t + iv} \right|^2 \leq \left( \frac{e^t - 1}{t} \right)^2 \left( \frac{t^2}{t^2 + v^2} \right) \left( 1 + \frac{4e^t}{(e^t - 1)^2} \right).$$

On the one hand, one can observe that

$$\sigma_x^2 = L_D''(t_x) = \frac{(e^{t_x} - 1)^2 - t_x^2 e^{t_x}}{t_x^2 (e^{t_x} - 1)^2}.$$

Hence, we obtain from (3.7) and (3.15) that, as soon as  $|v| \leq 2\sigma_x \sqrt{n}$ ,

$$(3.17) \quad |\Phi_n(v)| \leq \left( 1 - \left( \frac{\sigma_x^2 t_x^2}{t_x^2 + 4\sigma_x^2} \right) \frac{v^2}{n} \right)^{n/2} \leq \exp\left(-B_x \frac{v^2}{2}\right)$$

where

$$B_x = \frac{\sigma_x^2 t_x^2}{t_x^2 + 4\sigma_x^2}.$$

Therefore, we get from (3.17) that

$$\begin{aligned} |B_n(x)| &\leq 2 \int_{s_n}^{2\sigma_x \sqrt{n}} \exp\left(-B_x \frac{v^2}{2}\right) dv, \\ &\leq \frac{2}{s_n} \int_{s_n}^{2\sigma_x \sqrt{n}} v \exp\left(-B_x \frac{v^2}{2}\right) dv, \\ (3.18) \quad &\leq \frac{2}{s_n B_x} \exp\left(-B_x \frac{s_n^2}{2}\right). \end{aligned}$$



The upper bound in (3.18) clearly implies that  $B_n(x)$  goes exponentially fast to 0. On the other hand, we find from (3.7) and (3.16) that

$$(3.19) \quad |\Phi_n(v)| \leq \left( \frac{t_x^2}{t_x^2 + \frac{v^2}{\sigma_x^2 n}} \right)^{n/2} \left( 1 + \frac{4e^{t_x}}{(e^{t_x} - 1)^2} \right)^{n/2}$$

which leads, for all  $n > 2$ , to

$$(3.20) \quad \begin{aligned} |C_n(x)| &\leq \left( t_x^2 + \frac{4t_x^2 e^{t_x}}{(e^{t_x} - 1)^2} \right)^{n/2} \int_{|v| \geq 2\sigma_x \sqrt{n}} \left( \frac{1}{t_x^2 + \frac{v^2}{\sigma_x^2 n}} \right)^{n/2} dv, \\ &\leq 2\sigma_x \sqrt{n} \left( t_x^2 + \frac{4t_x^2 e^{t_x}}{(e^{t_x} - 1)^2} \right)^{n/2} \int_{v \geq 2} \left( \frac{1}{t_x^2 + v^2} \right)^{n/2} dv, \\ &\leq \sigma_x \sqrt{n} \left( t_x^2 + \frac{4t_x^2 e^{t_x}}{(e^{t_x} - 1)^2} \right)^{n/2} \int_{v \geq 2} 2v \left( \frac{1}{t_x^2 + v^2} \right)^{n/2} dv, \\ &\leq \frac{\sigma_x \sqrt{n}}{(n-2)} (t_x^2 + 4) (C_x)^{n/2} \end{aligned}$$

where

$$C_x = \frac{t_x^2 (e^{t_x} + 1)^2}{(t_x^2 + 4)(e^{t_x} - 1)^2}.$$

One can easily see that for all  $t > 0$ ,  $t^2 e^t < (e^t - 1)^2$ , which ensures that for all  $x \in ]1/2, 1[$ ,  $0 < C_x < 1$ . Consequently, we obtain from (3.20) that  $C_n(x)$  also goes exponentially fast to 0. It only remains to evaluate the lower part  $A_n(x)$  of  $P_n(x)$ . We deduce from (3.6) and (3.13) that

$$(3.21) \quad A_n(x) = \int_{|v| < s_n} \exp \left( - \frac{i\{nx\}v}{\sigma_x \sqrt{n}} - \frac{v^2}{2} \right) \left[ 1 + \sum_{k=1}^{2p+1} \frac{\varphi_k(v)}{(\sqrt{n})^k} + O \left( \frac{1 + v^{6(p+1)}}{n^{p+1}} \right) \right] dv.$$

For all  $a \in \mathbb{R}$  and for any integer  $m \geq 0$ , denote

$$\begin{aligned} I_m(a) &= \int_{\mathbb{R}} v^{2m} \exp \left( - iav - \frac{v^2}{2} \right) dv, \\ J_m(a) &= \int_{\mathbb{R}} v^{2m+1} \exp \left( - iav - \frac{v^2}{2} \right) dv. \end{aligned}$$

It follows from standard Gaussian calculation that

$$\begin{aligned}
I_m(a) &= \exp\left(-\frac{a^2}{2}\right) \int_{\mathbb{R}} v^{2m} \exp\left(-\frac{1}{2}(v+ia)^2\right) dv, \\
&= \exp\left(-\frac{a^2}{2}\right) \int_{\mathbb{R}} (w-ia)^{2m} \exp\left(-\frac{w^2}{2}\right) dw, \\
&= \exp\left(-\frac{a^2}{2}\right) \sum_{k=0}^{2m} \binom{2m}{k} (-ia)^{2m-k} \int_{\mathbb{R}} w^k \exp\left(-\frac{w^2}{2}\right) dw, \\
&= \exp\left(-\frac{a^2}{2}\right) \sum_{k=0}^m \binom{2m}{2k} (-ia)^{2(m-k)} \int_{\mathbb{R}} w^{2k} \exp\left(-\frac{w^2}{2}\right) dw \\
&= \sqrt{2\pi} \exp\left(-\frac{a^2}{2}\right) \sum_{k=0}^m \binom{2m}{2k} (-1)^{n-k} a^{2(m-k)} \frac{(2k)!}{2^k k!} \\
(3.22) \quad &= \frac{\sqrt{2\pi}(2m)!}{2^m} \exp\left(-\frac{a^2}{2}\right) \sum_{k=0}^m \frac{(-1)^k (2a^2)^k}{(2k)!(m-k)!}.
\end{aligned}$$

By the same token,

$$(3.23) \quad J_m(a) = \frac{\sqrt{2\pi}(2m+1)!}{2^m} (-ia) \exp\left(-\frac{a^2}{2}\right) \sum_{k=0}^m \frac{(-1)^k (2a^2)^k}{(2k+1)!(m-k)!}.$$

It can be noticed that if we restrict the integration domains of  $I_m(a)$  and  $J_m(a)$  to the interval  $[-s_n, s_n]$ , and since  $s_n$  goes to infinity, we shall obtain the same expressions as in (3.22) and (3.23) with additional terms that are exponentially small. Hereafter, we obtain from (3.21) together with (3.22) and (3.23) with  $a = \{nx\}/\sigma_x\sqrt{n}$  that for any integer  $p \geq 1$ , there exists a sequence of reals numbers  $c_{n,1}(x), \dots, c_{n,p}(x)$  such that for  $n$  large enough,

$$(3.24) \quad A_n(x) = \sqrt{2\pi} \exp\left(-\frac{\{nx\}^2}{2\sigma_x^2 n}\right) \left[1 + \sum_{k=1}^p \frac{c_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right].$$

The coefficients  $c_{n,1}(x), \dots, c_{n,p}(x)$  are bounded in  $n$  and may be explicitly calculated using the exact expression of the polynomials  $\varphi_k$  in (3.21) in conjunction with (3.22) and (3.23). Finally, it follows from (3.5) and (3.24) that for any  $x \in ]1/2, 1[$ ,

$$\begin{aligned}
\mathbb{P}(D_n \geq nx) &= \frac{\exp\left(-nI_D(x) - t_x\{nx\} - \frac{\{nx\}^2}{2\sigma_x^2 n}\right)}{\sqrt{2\pi}\sigma_x t_x \sqrt{n}} \left[1 + \sum_{k=1}^p \frac{c_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right], \\
&= \frac{\exp(-nI_D(x) - t_x\{nx\})}{\sqrt{2\pi}\sigma_x t_x \sqrt{n}} \left[1 + \sum_{k=1}^p \frac{d_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right],
\end{aligned}$$

which completes the proof of Theorem 2.1.  $\square$

## 4. PROOF OF THE SLDP FOR THE MAJOR INDEX

We saw in Section 1 that the distribution  $I_n$  coincides with that  $\Sigma_n$  given by (1.3). Let  $V_1, \dots, V_n$  be independent random variables such that for all  $1 \leq k \leq n$ ,  $V_k$  is uniformly distributed on the integers  $\{0, \dots, k-1\}$ . We have for all  $t \in \mathbb{R}$ ,

$$\begin{aligned}
\mathbb{E}[\exp(t\Sigma_n)] &= \mathbb{E}\left[\prod_{k=1}^n \exp(tV_k)\right] = \prod_{k=2}^n \left(\frac{1}{k} \sum_{\ell=0}^{k-1} e^{t\ell}\right), \\
&= \exp\left(\sum_{k=2}^n \log\left(\frac{1}{k} \sum_{\ell=0}^{k-1} e^{t\ell}\right)\right), \\
&= \exp\left(\sum_{k=2}^n \log\left(\frac{1}{k} \left(\frac{e^{tk} - 1}{e^t - 1}\right)\right)\right), \\
&= \exp\left(\sum_{k=1}^n \left(\log\left(\frac{e^{tk} - 1}{tk}\right) - \log\left(\frac{e^t - 1}{t}\right)\right)\right) \\
(4.1) \quad &= \exp\left(\sum_{k=1}^n L_D(tk) - nL_D(t)\right).
\end{aligned}$$

Consequently, we obtain from (4.1) that for all  $t \in \mathbb{R}$ ,

$$(4.2) \quad \mathbb{E}\left[\exp\left(\frac{t\Sigma_n}{n}\right)\right] = \exp(nL_n(t))$$

where

$$(4.3) \quad L_n(t) = \frac{1}{n} \sum_{k=1}^n L_D\left(\frac{tk}{n}\right) - L_D\left(\frac{t}{n}\right).$$

We have for all  $t \in \mathbb{R}$ ,

$$(4.4) \quad L_D(t) = \sum_{k=1}^{\infty} \frac{B_k t^k}{kk!} = \frac{t}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} t^{2k}}{2k(2k)!}.$$

The coefficients  $B_k$  in (4.4) are the well-known Bernoulli numbers. We have taken only even values since the odd Bernoulli numbers are zero except  $B_1 = 1/2$ . It follows from the Euler-Maclaurin formula that

$$\begin{aligned}
\sum_{k=1}^n L_D\left(\frac{tk}{n}\right) &= \int_0^n L_D\left(\frac{tx}{n}\right) dx + \frac{1}{2}(L_D(t) - L_D(0)) + \Delta_{n,p}(t) + R_{n,p}(t), \\
(4.5) \quad &= n \int_0^1 L_D(tx) dx + \frac{1}{2}L_D(t) + \Delta_{n,p}(t) + R_{n,p}(t)
\end{aligned}$$

where

$$\begin{aligned}
\Delta_{n,p}(t) &= \sum_{k=1}^p \frac{B_{2k}}{(2k)!} \left(\frac{t}{n}\right)^{2k-1} \left(L_D^{(2k-1)}(t) - L_D^{(2k-1)}(0)\right), \\
(4.6) \quad &= \frac{t}{12n} \left(L'_D(t) - \frac{1}{2}\right) + \sum_{k=2}^p \frac{B_{2k}}{(2k)!} \left(\frac{t}{n}\right)^{2k-1} L_D^{(2k-1)}(t)
\end{aligned}$$

and the remainder term

$$(4.7) \quad R_{n,p}(t) = \frac{1}{(2p+1)!} \left(\frac{t}{n}\right)^{2p+1} \int_0^n L_D^{(2p+1)}\left(\frac{tx}{n}\right) \mathcal{B}_{2p+1}(x - [x]) dx$$

where  $\mathcal{B}_p$  stands for the Bernoulli polynomial of order  $p$ . We have for any  $k \geq 0$ ,

$$(4.8) \quad |R_{n,p}^{(k)}(t)| = O\left(\frac{1}{n^{2p}}\right).$$

Hence, we deduce from (4.3) and (4.5) that for all  $t \in \mathbb{R}$ ,

$$(4.9) \quad L_n(t) = L_M(t) + \frac{1}{n}H(t) + \frac{t}{2n} - L_D\left(\frac{t}{n}\right) + \frac{1}{n}\Delta_{n,p}(t) + \frac{1}{n}R_{n,p}(t)$$

where the function  $L_M$  is given in (1.7) and the function  $H$  is defined in (2.2). Hereafter, the strategy is quite different and more difficult to handle than the one for the descents. First of all, we have for any  $x \in ]1/4, 1/2[$ ,

$$(4.10) \quad \mathbb{P}(M_n \geq n^2x) = \mathbb{P}(\Sigma_n \geq n^2x) = \sum_{k=\lceil n^2x \rceil}^{s_n} \mathbb{P}(\Sigma_n = k)$$

where  $s_n = n(n-1)/2$ . Moreover, we have from (4.2) that for all  $t, v \in \mathbb{R}$ ,

$$\exp(nL_n(n(t+iv))) = \mathbb{E}\left[\exp((t+iv)\Sigma_n)\right] = \sum_{k=0}^{s_n} \exp((t+iv)k) \mathbb{P}(\Sigma_n = k),$$

which implies that for all  $t, v \in \mathbb{R}$  and for all  $k \geq 0$ ,

$$(4.11) \quad \mathbb{P}(\Sigma_n = k) = \exp(-tk) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(nL_n(n(t+iv))) \exp(-ikv) dv.$$

One can observe that the identity (4.11) is also true for all  $k \geq s_n$ . Hence, we deduce from Fubini's theorem together with (4.10) and (4.11) that for all  $t > 0$ ,

$$(4.12) \quad \mathbb{P}(M_n \geq n^2x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(nL_n(n(t+iv))) \sum_{k=\lceil n^2x \rceil}^{+\infty} \exp(-k(t+iv)) dv.$$

Accordingly, if we choose  $t = t_x/n$ , which is positive since  $x > 1/4$ , we find from (4.12) that the probability  $\mathbb{P}(M_n \geq n^2x)$  can be separated into two terms,

$$(4.13) \quad \mathbb{P}(M_n \geq n^2x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(nL_n(t_x + inv)) \Lambda_n(t_x + inv) dv = A_n(x) B_n(x),$$

where

$$(4.14) \quad A_n(x) = \exp(n(L_n(t_x) - xt_x))$$

and

$$(4.15) \quad B_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(n(L_n(t_x + inv) - L_n(t_x) + xt_x)) \Lambda_n(t_x + inv) dv$$

with

$$(4.16) \quad \Lambda_n(t_x + inv) = \sum_{k=\lceil n^2 x \rceil}^{+\infty} \exp(-k(t_x + inv)/n) = \frac{\exp(-\frac{\lceil n^2 x \rceil}{n}(t_x + inv))}{1 - \exp(-\frac{1}{n}(t_x + inv))}.$$

Concerning the first term  $A_n(x)$ , the point  $t_x$  is chosen such that  $L'_M(t_x) = x$  which ensures that  $I_M(x) = xt_x - L_M(t_x)$ , leading via (4.14) to

$$(4.17) \quad A_n(x) = \exp(-nI_M(x)) \exp\left(n(L_n(t_x) - L_M(t_x))\right).$$

Moreover, we have from (4.9) that

$$(4.18) \quad n(L_n(t_x) - L_M(t_x)) = H(t_x) + \frac{t_x}{2} - nL_D\left(\frac{t_x}{n}\right) + \Delta_{n,p}(t_x) + R_{n,p}(t_x).$$

We already saw from (4.4) that

$$(4.19) \quad nL_D\left(\frac{t_x}{n}\right) = \frac{t_x}{2} + n \sum_{k=1}^p \frac{B_{2k}}{2k(2k)!} \left(\frac{t_x}{n}\right)^{2k} + O\left(\frac{1}{n^{2p}}\right).$$

As in Section 3, for all integer  $k \geq 1$ , denote  $\ell_D(k) = L_D^{(k)}(t_x)$  and  $\ell_M(k) = L_M^{(k)}(t_x)$ . One needs to be very careful here. We already saw that  $t_x$  is chosen such that  $L'_M(t_x) = x$  and we set  $\sigma_x^2 = L''_M(t_x)$ . This will not provide tractable information on  $\ell_D(k)$  as we recall from (1.7) that  $L_M$  is the integral of  $L_D$ ,

$$(4.20) \quad L_M(t) = \int_0^1 L_D(xt) dx = \frac{t}{4} + \sum_{k=1}^{\infty} \frac{B_{2k} t^{2k}}{2k(2k+1)!}.$$

However, (4.6) reduces to

$$(4.21) \quad \Delta_{n,p}(t_x) = \sum_{k=1}^p \frac{B_{2k} \ell_D(2k-1)}{(2k)!} \left(\frac{t_x}{n}\right)^{2k-1} - \frac{t_x}{24n}.$$

Hence, it follows from (4.17) together with the three contributions (4.18), (4.19) and (4.21) that

$$(4.22) \quad A_n(x) = \exp(-nI_M(x) + H(t_x)) \left[ 1 + \sum_{k=1}^p \frac{a_k(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right) \right]$$

where the prefactor was previously defined in (2.2) and  $a_1(x), \dots, a_p(x)$  can be explicitly calculated as functions of the derivatives of  $L_D$  at point  $t_x$ . For example,  $a_1(x)$  and  $a_2(x)$  are given by

$$a_1(x) = -\frac{t_x(1+t_x)}{24} + \frac{t_x \ell_D(1)}{12},$$

$$a_2(x) = \frac{a_1(x)^2}{2}.$$

We shall now focus our attention on the second term  $B_n(x)$ . By the change of variable  $w = n\sqrt{n}v$ , we have from (4.15) that

$$B_n(x) = \frac{1}{2\pi n\sqrt{n}} \int_{-\pi n\sqrt{n}}^{\pi n\sqrt{n}} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) + xt_x\right)\right) \Lambda_n\left(t_x + \frac{iw}{\sqrt{n}}\right) dw.$$

Denote  $s_n = \varepsilon_n n^{1/6}$  where  $\varepsilon_n = n^{-\alpha}$  with  $0 < \alpha < 1/6$ . As previously done,  $B_n(x)$  can be separated into three terms,

$$B_n(x) = \frac{1}{\sqrt{2\pi n}} (C_n(x) + D_n(x) + E_n(x))$$

where

$$\begin{aligned} C_n(x) &= \frac{1}{n\sqrt{2\pi}} \int_{|w| < s_n} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) + xt_x\right)\right) \Lambda_n\left(t_x + \frac{iw}{\sqrt{n}}\right) dw, \\ D_n(x) &= \frac{1}{n\sqrt{2\pi}} \int_{s_n \leq |w| \leq a\sqrt{n}} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) + xt_x\right)\right) \Lambda_n\left(t_x + \frac{iw}{\sqrt{n}}\right) dw, \\ E_n(x) &= \frac{1}{n\sqrt{2\pi}} \int_{a\sqrt{n} \leq |w| \leq \pi n\sqrt{n}} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) + xt_x\right)\right) \Lambda_n\left(t_x + \frac{iw}{\sqrt{n}}\right) dw \end{aligned}$$

and  $a$  is a positive parameter that will be chosen later. It is well-known that for all  $z \in \mathbb{C}$  with  $z \neq 0$ ,

$$\frac{1}{1 - \exp(-z)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{B_k z^k}{k!}.$$

Consequently, we obtain from (4.16) that

$$\begin{aligned} C_n(x) &= \frac{\exp(-\tau_n(x))}{n\sqrt{2\pi}} \int_{|w| < s_n} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) - \frac{iwx}{\sqrt{n}}\right)\right) \\ &\quad \exp\left(-\frac{i\{n^2x\}w}{n\sqrt{n}}\right) \left(1 - \exp\left(-\frac{t_x}{n} - \frac{iw}{n\sqrt{n}}\right)\right)^{-1} dw, \\ &= \frac{\exp(-\tau_n(x))}{t_x\sqrt{2\pi}} \int_{|w| < s_n} \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) - \frac{iwx}{\sqrt{n}}\right)\right) \\ (4.23) \quad &\quad \exp\left(-\frac{i\{n^2x\}w}{n\sqrt{n}}\right) \left(1 + \frac{iw}{t_x\sqrt{n}}\right)^{-1} \sum_{k=0}^{\infty} \frac{B_k t_x^k}{k! n^k} \left(1 + \frac{iw}{t_x\sqrt{n}}\right)^k dw, \end{aligned}$$

where

$$\tau_n(x) = \frac{1}{n} \{n^2x\} t_x.$$

It follows Taylor's theorem that for all  $|w| < s_n$ ,

$$(4.24) \quad L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) - \frac{iw}{\sqrt{n}} L'_n(t_x) = \sum_{j=2}^{2p+3} \frac{1}{j!} \left(\frac{iw}{\sqrt{n}}\right)^j L_n^{(j)}(t_x) + \zeta_{n,p}(w)$$

where the remainder  $\zeta_{n,p}(w)$  satisfies for all  $w \in \mathbb{R}$  such that  $|w| < s_n$ ,

$$(4.25) \quad |\zeta_{n,p}(w)| \leq \frac{M_{x,p} w^{2p+4}}{n^{p+2}}$$

for some constant positive  $M_{x,p}$  that does not depend on  $w$ . We obtain from (4.4), (4.6), (4.8) and (4.9) that

$$\begin{aligned} L'_n(t_x) &= L'_M(t_x) + \frac{H'(t_x)}{n} + \frac{1}{2n} - \frac{1}{n}L'_D\left(\frac{t_x}{n}\right) + \frac{1}{n}\Delta'_{n,p}(t_x) + \frac{1}{n}R'_{n,p}(t_x), \\ &= x + \frac{H'(t_x)}{n} - \frac{1}{24n^2} + \frac{1}{n} \sum_{k=1}^p \frac{B_{2k}}{(2k)!} \left(\frac{t_x}{n}\right)^{2k-1} (\ell_D(2k) - 1) \\ &\quad + \frac{1}{n^2} \sum_{k=1}^p \frac{(2k-1)B_{2k}}{(2k)!} \ell_D(2k-1) \left(\frac{t_x}{n}\right)^{2k-2} + O\left(\frac{1}{n^{2p+1}}\right), \end{aligned}$$

which reduces to

$$(4.26) \quad L'_n(t_x) = x + \frac{H'(t_x)}{n} - \frac{1}{24n^2} + \sum_{k=1}^p \frac{\psi_{k,1}(x)}{n^{2k}} + O\left(\frac{1}{n^{2p+1}}\right),$$

where

$$\psi_{k,1}(x) = \frac{B_{2k}(t_x)^{2k-2}}{2k(2k-2)!} \left( \frac{t_x(\ell_D(2k) - 1)}{2k-1} + \ell_D(2k-1) \right).$$

By the same token, for any  $j \geq 2$ ,

$$(4.27) \quad \begin{aligned} L_n^{(j)}(t_x) &= L_M^{(j)}(t_x) + \frac{H^{(j)}(t_x)}{n} - \frac{1}{n^j}L_D^{(j)}\left(\frac{t_x}{n}\right) + \frac{1}{n}\Delta_{n,p}^{(j)}(t_x) + \frac{1}{n}R_{n,p}^{(j)}(t_x), \\ &= \ell_M(j) + \frac{H^{(j)}(t_x)}{n} + \sum_{k=1}^p \frac{\psi_{k,j}(x)}{n^{2k}} + O\left(\frac{1}{n^{2p+1}}\right), \end{aligned}$$

where the coefficients  $\psi_{k,j}(x)$  may be explicitly calculated as functions of the derivatives of  $L_D$  at point  $t_x$ . For example, in the special case  $j = 2$ , as  $\ell_M(2) = \sigma_x^2$ ,

$$(4.28) \quad L_n''(t_x) = \sigma_x^2 + \frac{H''(t_x)}{n} + \sum_{k=1}^p \frac{\psi_{k,2}(x)}{n^{2k}} + O\left(\frac{1}{n^{2p+1}}\right),$$

where  $\psi_{1,2}(x) = (t_x \ell_D(3) + 2\ell_D(2) - 1)/12$  and for  $k \geq 2$ ,

$$\psi_{k,2}(x) = \frac{B_{2k}(t_x)^{2k-3}}{2k(2k-3)!} \left( \frac{t_x^2 \ell_D(2k+1)}{(2k-1)(2k-2)} + \frac{t_x(2\ell_D(2k) - 1)}{(2k-2)} + \ell_D(2k-1) \right).$$

Then, we deduce from (4.26), (4.27) and (4.28) that

$$\begin{aligned} n \left( L_n \left( t_x + \frac{iw}{\sqrt{n}} \right) - L_n(t_x) - \frac{iwx}{\sqrt{n}} \right) &= \frac{iwH'(t_x)}{\sqrt{n}} - \frac{iw}{24n\sqrt{n}} - \frac{w^2\sigma_x^2}{2} - \frac{w^2H''(t_x)}{2n} \\ &+ \sum_{j=3}^{2p+3} \frac{1}{j!} \left( \frac{iw}{\sqrt{n}} \right)^j (n\ell_M(j) + H^{(j)}(t_x)) + \sum_{j=1}^{2p+3} \frac{n}{j!} \left( \frac{iw}{\sqrt{n}} \right)^j \left( \sum_{k=1}^p \frac{\psi_{k,j}(x)}{n^{2k}} + O\left(\frac{1}{n^{2p+1}}\right) \right) \\ &+ n\zeta_{n,p}(w). \end{aligned}$$

Consequently, similarly to (3.21), we obtain from (4.23) that

$$(4.29) \quad C_n(x) = \frac{\exp(-\tau_n(x))}{t_x \sqrt{2\pi}} \int_{|w| < s_n} \exp\left(-\frac{w^2 \sigma_x^2}{2} - \frac{i\{n^2 x\}w}{n\sqrt{n}}\right) \left[1 + \sum_{k=1}^{2p+1} \frac{\varphi_k(w)}{(\sqrt{n})^k} + O\left(\frac{1+w^{6(p+1)}}{n^{p+1}}\right)\right] dw$$

where the  $\varphi_k(w)$  are polynomials in odd powers of  $w$  for  $k$  odd, and polynomials in even powers of  $w$  for  $k$  even. For example,

$$\begin{aligned} \varphi_1(w) &= iwH'(t_x) - \frac{iw}{t_x} - \frac{iw^3 \ell_M(3)}{6}, \\ \varphi_2(w) &= \frac{t_x}{2} - \frac{w^2}{t_x^2} - \frac{w^2 H''(t_x)}{2} - \frac{w^2 (H'(t_x))^2}{2} + \frac{w^2 H'(t_x)}{t_x} - \frac{w^4 \ell_M(3)}{6t_x} \\ &\quad + \frac{w^4 \ell_M(4)}{24} + \frac{w^4 H'(t_x) \ell_M(3)}{6} - \frac{w^6 \ell_M^2(3)}{72}. \end{aligned}$$

Note also that  $\varphi_k(v)$  is purely imaginary for  $k$  odd and real for  $k$  even. Therefore, we deduce from (4.29), (3.22) and (3.23) that

$$\begin{aligned} C_n(x) &= \frac{\exp\left(-\tau_n(x) - \frac{\{nx\}^2}{2\sigma_x^2 n^3}\right)}{\sigma_x t_x} \left[1 + \sum_{k=1}^p \frac{c_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right], \\ &= \frac{1}{\sigma_x t_x} \left[1 + \sum_{k=1}^p \frac{d_{n,k}(x)}{n^k} + O\left(\frac{1}{n^{p+1}}\right)\right], \end{aligned}$$

where the coefficients  $c_{n,1}(x), \dots, c_{n,p}(x)$  may be explicitly calculated using the exact expression of the polynomials  $\varphi_k$  in (4.29). For example,  $c_{n,1}(x)$  is given by

$$\begin{aligned} c_{n,1}(x) &= \frac{t_x}{2} - \frac{1}{\sigma_x^2 t_x^2} - \frac{H''(t_x)}{2\sigma_x^2} - \frac{(H'(t_x))^2}{2\sigma_x^2} + \frac{H'(t_x)}{\sigma_x^2 t_x} - \frac{\ell_M(3)}{2\sigma_x^4 t_x} \\ &\quad + \frac{\ell_M(4)}{8\sigma_x^4} + \frac{H'(t_x) \ell_M(3)}{2\sigma_x^4} - \frac{5\ell_M^2(3)}{24\sigma_x^6}. \end{aligned}$$

In order to complete the proof of Theorem 2.2, it only remains to show that the two last terms  $D_n(x)$  and  $E_n(x)$  will play a negligible role. We start by noticing that

$$(4.30) \quad |D_n(x)| \leq \frac{\chi(x, n)}{n\sqrt{2\pi}} \int_{s_n \leq |w| \leq a\sqrt{n}} \left| \exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x)\right)\right) \right| dw,$$

where we use the inequality  $\exp(nxt_x) |\Lambda_n(t_x + iw/\sqrt{n})| \leq \chi(x, n)$  with

$$(4.31) \quad \chi(x, n) = \exp(-\tau_n(x)) \left| 1 - \exp\left(\frac{-t_x}{n}\right) \right|^{-1} \leq \frac{ne^{t_x}}{t_x}.$$



From now on, we focus on the integral appearing in (4.30). We have from (4.3) that

$$\begin{aligned} L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x) &= \frac{1}{n} \sum_{k=1}^n \left( L_D\left(\frac{k}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{kt_x}{n}\right) \right) \\ &\quad - \left( L_D\left(\frac{1}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{t_x}{n}\right) \right), \end{aligned}$$

which leads to

(4.32)

$$\exp\left(n\left(L_n\left(t_x + \frac{iw}{\sqrt{n}}\right) - L_n(t_x)\right)\right) = \frac{\prod_{k=1}^n \exp\left(L_D\left(\frac{k}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{kt_x}{n}\right)\right)}{\exp\left(n\left(L_D\left(\frac{1}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{t_x}{n}\right)\right)\right)}.$$

Hereafter, our goal is to find a suitable upper bound for the numerator in equation (4.32). It follows from inequality (3.15) that as soon as  $|w| \leq a\sqrt{n}$ ,

$$(4.33) \quad \left| \prod_{k=1}^n \exp\left(L_D\left(\frac{k}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{kt_x}{n}\right)\right) \right| \leq \exp\left(-\sum_{k=1}^n C\left(\frac{kt_x}{n}\right) \frac{w^2}{2n}\right)$$

where for all  $t > 0$ ,

$$C(t) = \frac{t^2}{t^2 + a^2} L_D''(t).$$

One can observe that  $L_D''(t)$  is a strictly decreasing function. Consequently, we deduce from (4.33) together with a simple Maclaurin-Cauchy test that

$$(4.34) \quad \left| \prod_{k=1}^n \exp\left(L_D\left(\frac{k}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{kt_x}{n}\right)\right) \right| \leq \exp\left(-D_{a,x} L_D''(t_x) \frac{w^2}{2}\right)$$

where  $D_{a,x}$  is the positive constant given by

$$D_{a,x} = \int_0^1 \frac{(t_x y)^2}{(t_x y)^2 + a^2} dy.$$

Hence, it will be enough to show that the denominator of equation (4.32) is bounded away from zero to deduce that  $D_n(x)$  is exponentially negligible. We have from Taylor's theorem in the complex plane that

$$nL_D\left(\frac{z}{n}\right) = \frac{z}{2} + O\left(\frac{z^2}{n}\right).$$

Therefore, for all  $w \in \mathbb{R}$  such that  $|w| \leq a\sqrt{n}$ ,

$$n\left(L_D\left(\frac{1}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{t_x}{n}\right)\right) = \frac{iw}{2\sqrt{n}} + \frac{1}{n}O(1)$$

where the  $O(1)$  term is uniform in  $w$ . It implies that there exists a positive constant  $E_{a,x}$  such that, for all  $|w| \in [s_n, a\sqrt{n}]$ ,

$$(4.35) \quad \exp\left(n\left(L_D\left(\frac{1}{n}\left(t_x + \frac{iw}{\sqrt{n}}\right)\right) - L_D\left(\frac{t_x}{n}\right)\right)\right) \geq E_{a,x}.$$

Then, by using (4.31), (4.34) and (4.35) in (4.30), we obtain that

$$\begin{aligned} |D_n(x)| &\leq \frac{2}{t_x \sqrt{2\pi} E_{a,x}} \int_{s_n}^{a\sqrt{n}} \frac{w}{s_n} \exp\left(-D_{a,x} L_D''(t_x) \frac{w^2}{2}\right) dw, \\ &\leq \frac{2}{s_n t_x \sqrt{2\pi} E_{a,x} D_{a,x} L_D''(t_x)} \exp\left(-D_{a,x} L_D''(t_x) \frac{s_n^2}{2}\right), \end{aligned}$$

which clearly means that  $D_n(x)$  goes exponentially fast to zero. It remains to carry out a similar analysis for the last term  $E_n(x)$ . By the same lines as for inequality (4.30), we have, up to a change of variable,

$$(4.36) \quad |E_n(x)| \leq \frac{\chi(x, n)}{\sqrt{2\pi n}} \int_{a \leq |w| \leq \pi n} \left| \exp\left(n\left(L_n\left(t_x + iw\right) - L_n\left(t_x\right)\right)\right) \right| dw.$$

It follows from (4.32) together with (3.14) that

$$\begin{aligned} &\frac{1}{n} \log \left| \exp\left(n\left(L_n\left(t_x + iw\right) - L_n\left(t_x\right)\right)\right) \right|^2 \\ (4.37) \quad &= \frac{1}{n} \sum_{k=1}^n \log \left( 1 + \frac{\exp\left(\frac{kt_x}{n}\right)}{\left(\exp\left(\frac{kt_x}{n}\right) - 1\right)^2} 2\left(1 - \cos\left(\frac{kw}{n}\right)\right) \right) \\ &\quad - \log \left( 1 + \frac{\exp\left(\frac{t_x}{n}\right)}{\left(\exp\left(\frac{t_x}{n}\right) - 1\right)^2} 2\left(1 - \cos\left(\frac{w}{n}\right)\right) \right). \end{aligned}$$

We already saw in Section 3 that for all  $v \in \mathbb{R}$ ,  $2(1 - \cos(v)) \leq \min(v^2, 4)$ . Moreover, it is well-known that for all  $v \geq 0$ ,  $(e^v - 1)^2 \geq v^2$ . Consequently, we deduce from Jensen's inequality that

$$\begin{aligned} &\frac{1}{n} \sum_{k=1}^n \log \left( 1 + \frac{\exp\left(\frac{kt_x}{n}\right)}{\left(\exp\left(\frac{kt_x}{n}\right) - 1\right)^2} 2\left(1 - \cos\left(\frac{kw}{n}\right)\right) \right) \\ &\leq \log \left( 1 + \frac{1}{n} \sum_{k=1}^n \frac{\exp\left(\frac{kt_x}{n}\right)}{\left(\exp\left(\frac{kt_x}{n}\right) - 1\right)^2} 2\left(1 - \cos\left(\frac{kw}{n}\right)\right) \right), \\ &\leq \log \left( 1 + \frac{e^{t_x}}{t_x^2} \frac{1}{n} \sum_{k=1}^n \min\left(w^2, \frac{4n^2}{k^2}\right) \right). \end{aligned}$$

Therefore, we obtain from the previous inequality and a simple Maclaurin-Cauchy test that as soon as  $|w| \geq a > 2$ ,

$$\begin{aligned} & \frac{1}{n} \sum_{k=1}^n \log \left( 1 + \frac{\exp\left(\frac{kt_x}{n}\right)}{\left(\exp\left(\frac{kt_x}{n}\right) - 1\right)^2} 2 \left(1 - \cos\left(\frac{kx}{n}\right)\right) \right) \\ & \leq \log \left( 1 + \frac{e^{t_x}}{t_x^2} \int_0^1 \min\left(w^2, \frac{4}{y^2}\right) dy \right) \leq \log \left( 1 + \frac{4e^{t_x}}{t_x^2} |w| \right). \end{aligned}$$

Furthermore, we have for all  $v \in \mathbb{R}$  such that  $|v| \leq \pi$ ,  $2(1 - \cos(v)) \geq (2v/\pi)^2$ . It implies that for all  $w \in \mathbb{R}$  such that  $|w| \leq \pi n$ ,

$$\begin{aligned} & \log \left( 1 + \frac{\exp\left(\frac{t_x}{n}\right)}{\left(\exp\left(\frac{t_x}{n}\right) - 1\right)^2} 2 \left(1 - \cos\left(\frac{w}{n}\right)\right) \right) \\ & \geq \log \left( 1 + \frac{\exp\left(\frac{t_x}{n}\right)}{\left(\exp\left(\frac{t_x}{n}\right) - 1\right)^2} \frac{4w^2}{n^2\pi^2} \right) \geq \log \left( 1 + \frac{4}{\pi^2 t_x^2 e^{t_x}} w^2 \right) \end{aligned}$$

where the last inequality comes from the fact that for all  $t > 0$ ,  $e^t - 1 \leq te^t$ . Hence, putting these two inequalities into (4.37) and recalling that  $|w| \geq a$ , we obtain that

$$(4.38) \quad \left| \exp \left( n \left( L_n(t_x + iw) - L_n(t_x) \right) \right) \right|^2 \leq \left( \frac{1 + \frac{4e^{t_x}}{at_x^2} w^2}{1 + \frac{4}{\pi^2 t_x^2 e^{t_x}} w^2} \right)^{n/2}.$$

Hereafter, we can choose  $a = 2\pi^2 e^{2t_x} > 2$ , leading for all  $|w| \geq a$ , to

$$(4.39) \quad \left( \frac{1 + \frac{4e^{t_x}}{at_x^2} w^2}{1 + \frac{4}{\pi^2 t_x^2 e^{t_x}} w^2} \right) \leq \left( \frac{1 + \frac{2}{\pi^2 t_x^2 e^{t_x}} a^2}{1 + \frac{4}{\pi^2 t_x^2 e^{t_x}} a^2} \right).$$

Denote by  $q_{a,x}$  the upper bound in (4.39). We clearly have  $0 < q_{a,x} < 1$ . Finally, we deduce from (4.36) together with (4.31), (4.38) and (4.39) that

$$|E_n(x)| \leq \frac{n\sqrt{2\pi n}e^{t_x}}{t_x} (q_{a,x})^{n/2},$$

which means that  $E_n(x)$  goes exponentially fast to zero.  $\square$

## REFERENCES

- [1] R. R. Bahadur and R. Ranga Rao. On deviations of the sample mean. *Ann. Math. Statist.*, 31:1015–1027, 1960.
- [2] B. Bercu, M. Bonnefont, and A. Richou. Sharp large deviations and concentration inequalities for the number of descents in a random permutation. *J. Appl. Probab.*, 61(3):1–27, 2024.
- [3] B. Bercu, F. Gamboa, and M. Lavielle. Sharp large deviations for Gaussian quadratic forms with applications. *ESAIM Probab. Statist.*, 4:1–24, 2000.
- [4] W. Bryc, D. Minda, and S. Sethuraman. Large deviations for the leaves in some random trees. *Adv. in Appl. Probab.*, 41(3):845–873, 2009.
- [5] P. A. MacMahon. Two applications of general theorems in combinatory analysis. *Proc. Lond. Math. Soc.*, 15:314–321, 1917.
- [6] P. L. Meliot and A. Nikeghbali. Asymptotics of the major index of a random standard tableau. *arXiv:2210.13022*, 2024.
- [7] S. Tanny. A probabilistic interpretation of Eulerian numbers. *Duke Math. J.*, 40:717–722, 1973.