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Insights into the Stabilization of a Chain of Integrators via Delay-Difference Approximations [★]

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Abstract: This paper focuses on the output feedback stabilization of a chain of integrators using delay-difference operators to approximate derivative actions. The primary contribution of this study is the explicit computation of an upper bound for the time-delay, guaranteeing that the closed-loop system, employing the delay-difference approximation of the derivative action, maintains (exponential) stability. Illustrative numerical examples complete the presentation to demonstrate the efficiency of the proposed methodology. ©IFAC, all rights reserved.

Keywords: Delay-Difference, Chain of Integrators, Linear Systems, Delay Systems.

1. INTRODUCTION

The chain of integrators represents one of the simplest applications of delay-based controllers with the delay as a control parameter. Such a problem received a constructive solution 20 years ago, and necessary and sufficient conditions linking the number of delays and the chain length exist. More precisely, it was first shown (see, e.g., Niculescu and Michiels (2004)) that $n \in \mathbb{N}^*$ delay blocks can control a chain of n integrators, and later (see, for instance, Kharitonov et al. (2005)) that this condition is also necessary. For other approaches to this problem, we also refer to Mazenc (1997) for a Lyapunov functions approach, Mazenc et al. (2003) for an approach considering stabilizing bounded feedback, and Choi and Lim (2006) for a case considering the presence of unknown delays, as well as Chitour et al. (2020) for the perturbed case. For a deeper discussion on the idea of using delay as a control parameter as well as a guided tour of existing results in the literature, we refer to Michiels (2013).

This short note addresses one of the ideas proposed in Niculescu and Michiels (2004) using a different argument. On the one hand, we assume that the delays are *commensurate*, that is, of the form $\tau_2 = \ell\tau_1$, where $\ell \in \mathbb{N}^*$. This simple consideration reduces the number of control parameters from $2n - 1$ to $n + 1$, since there is only one delay to choose, making the process of tuning the controller simpler. On the other hand, the proposed construction leads to a closed-loop system that includes delay-dependent coefficients in the characteristic function.

The presence of delay-dependent coefficients poses a challenge in the analysis. While there is an extensive body of research on the stability analysis of time-delay systems (see, for example, Bellman and Cooke (1963), Michiels et al. (2002), Gu et al. (2003), and Diekmann et al. (2012)), studies incorporating delay-dependent coefficients are less prevalent (see, for instance, Beretta and Kuang (2002) and Jin et al. (2018) for some analytical results, as well as Wang (2012) and Lai et al. (2021) for some numerical results).

Additionally, through the arguments presented in Méndez-Barrios et al. (2024), and Méndez-Barrios et al. (2022), we conclude that the approximation of the derivative operator by delay-difference operator leads to a properly-posed closed-loop system for "small delays", and present a relatively simple methodology to explicitly compute the *delay margin* τ_r (see, for instance, Ma and Chen (2018)), which is an upper limit such that the closed-loop system remains stable for any $\tau \in (0, \tau_r)$, opening some perspectives in handling different system configurations.

Notations: Throughout this paper, the standard notations are used and explained (if necessary) the first time they are used.

2. PRELIMINARY RESULTS AND PROBLEM FORMULATION

We consider a differential equation of the form:

$$y^{(n)}(t) = u(t), \quad (1)$$

which is a chain of integrators of length n , therefore it can be stabilized by a control law of the form:

[★] This work is partially funded by CONACyT, under the grant CONACyT-929482.

$$u(t) = - \sum_{j=0}^{n-1} k_j y^{(j)}(t), \quad (2)$$

by choosing the parameters $\mathbf{k} = [k_0, k_1, \dots, k_{n-1}]$ such that the characteristic polynomial $F(\lambda)$, given by:

$$F(\lambda) = \lambda^n + \sum_{j=0}^{n-1} k_j \lambda^j, \quad (3)$$

is Hurwitz. Since we are interested in finding the delay margin, such an assumption is natural. Next, we consider the following approximation of the derivative in frequency-domain:

$$\lambda \approx \frac{1 - e^{-\lambda\tau}}{\tau}, \quad (4)$$

which corresponds to the following approximation:

$$y^{(n)}(t) \approx \frac{1}{\tau^n} \sum_{j=0}^n (-1)^j \binom{n}{j} y(t - j\tau).$$

In this case, the controller $u_\tau(t)$ reads:

$$u_\tau(t) = - \sum_{j=0}^{n-1} \sum_{m=0}^j (-1)^m \frac{k_j}{\tau^j} \binom{j}{m} y(t - m\tau), \quad (5)$$

and includes past information over the interval $[t - n\tau, t)$, for all $t \geq 0$. The corresponding frequency-domain representation writes as:

$$C(\lambda) = \sum_{j=0}^{n-1} \sum_{m=0}^j (-1)^m \frac{k_j}{\tau^j} \binom{j}{m} e^{-m\tau\lambda}, \quad (6)$$

and the characteristic polynomial (3) rewrites as the following *quasipolynomial*:

$$F_n(\lambda) = \lambda^n + \sum_{j=0}^{n-1} \sum_{m=0}^j (-1)^m \frac{k_j}{\tau^j} \binom{j}{m} e^{-m\tau\lambda}. \quad (7)$$

Under the assumption that the chosen gains guarantee the closed-loop stability of the delay-free closed-loop system, our interest is to characterize the first (delay) interval $\mathcal{I} := (0, \tau_r)$ such that the quasipolynomial (7) preserves stability for any $\tau \in \mathcal{I}$. The delay τ_r represents the so-called *delay margin*.

With the remarks above, two issues must be addressed: (i) the properly-posed character of the closed-loop system for small delay values and (ii) an estimation of the delay margin.

2.1 Motivating example

Consider the case $n = 2$, then the characteristic quasipolynomial reads:

$$F_2(\lambda) = \lambda^2 + \frac{k_1}{\tau} + k_0 - k_1 \frac{e^{-\lambda\tau}}{\tau}. \quad (8)$$

Next, allow us to define the parameter p such that $p\tau = 1$, and rewrite (8) as following:

$$F_2(\lambda) = P_0(\lambda; p) + e^{-\lambda\tau} P_1(p).$$

It is clear that, for any $i\omega$ with $\omega \in \mathbb{R}$ such that $F_2(i\omega) = 0$, the following must hold:

$$P_0(i\omega; p)P_0(-i\omega; p) - P_1^2(p) = 0. \quad (9)$$

Eq. (9) states that the magnitude of P_0 and P_1 must be the same. From Eq. (9) the following polynomial is obtained:

$$\omega^4 - 2k_0\omega^2 + p(2k_0k_1 - 2k_1\omega^2) + k_0^2 = 0,$$

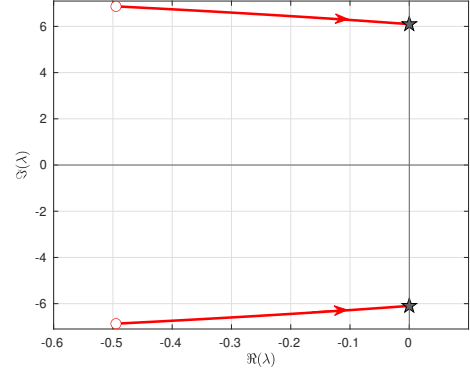


Fig. 1. Roots behavior as $\tau \in \mathcal{I} := (0, 0.5153)$ increases. The stars depict the crossing occurring at $\omega = 6.0966$.

which implies:

$$p(\omega) = \frac{\omega^2 - k_0}{2k_1}, \quad (10)$$

therefore $\tau(\omega) = p^{-1}(\omega)$. Finally, in order to identify the value of $\omega \in \mathbb{R}$ of interest, from Eq. (8) the following must hold:

$$\cos(\omega\tau(\omega)) = -\Re \left\{ \frac{P_0(i\omega; p(\omega))}{P_1(p(\omega))} \right\}, \quad (11)$$

$$\sin(\omega\tau(\omega)) = \Im \left\{ \frac{P_0(i\omega; p(\omega))}{P_1(p(\omega))} \right\}. \quad (12)$$

Finally, the ω^* of interest corresponds to one such that $p^{-1}(\omega^*) > 0$ and Eq. (11) and Eq. (12) hold. To illustrate this result, consider $\mathbf{k} = [10, 7]$ which places the roots of the *delay-free* closed-loop characteristic polynomial at $s_{1,2} = \{-2, -5\}$, therefore stabilizing the system. Replacing the derivatives with the approximation and computing, we obtain the following system of equations:

$$\begin{cases} \tau(\omega) = \frac{14}{\omega^2 - 10}, \\ \cos(\omega\tau(\omega)) = -1, \\ \sin(\omega\tau(\omega)) = 0, \end{cases}$$

from where we obtain $\mathcal{I} = (0, 0.5153)$, with the crossing occurring at $\omega = 6.0966$. This result can be further validated numerically, as depicted in Fig. 1. The presented approach follows closely the method proposed by Walton and Marshall (1987). However, in Walton and Marshall's work, the case where the coefficients of the quasipolynomial explicitly depend on the delay parameter τ is not considered. To the best of the authors' knowledge, although some case studies exist, the first method leading to an explicit algorithm for studying generic delay-dependent coefficients under the assumption of analytic characteristic functions including a single delay can be found in Beretta and Kuang (2002). It should be mentioned that this result cannot be extended straightforwardly for multiple delay cases, even if the delays are commensurate.

2.2 Problem formulation

With the above discussion in mind, the problem we address on this note is the following:

Problem 1. For a chain of integrators of length $n \in \mathbb{N}^*$ and an stabilizing controller $u(t)$ of the form (2), determine:

- The properly-posedness of the closed-loop system after approximating the derivatives on $u(t)$ via a delay-difference approximation.
- The first interval $\mathcal{I} := (0, \tau_r)$ such that for any $\tau \in \mathcal{I}$ the closed-loop system preserves its stability properties after replacing the derivatives on the controller by their corresponding delay-difference approximations.

3. MAIN RESULTS

In the following, we aim to analytically solve Problem 1.

3.1 Properly-posed closed-loop systems

Recent studies have focused on the improperly-posed nature of the approximation of derivative action (see, for example, Méndez-Barrios et al. (2024) and Michiels (2022)). For a more general discussion on improperly-posed operators, we refer to Georgiou and Smith (1989)). We have the following definition, retrieved from Méndez-Barrios et al. (2024):

Definition 3.1. (Improperly-posed system). Consider the LTI SISO system (1). Suppose that $u(t)$ of the form (2) is a stabilizing controller and is replaced by $u_\tau(t)$ given in (5). If there exists a sequence of real numbers $(\tau_n)_{n \in \mathbb{N}^*}$, $\tau_n \rightarrow 0_+$ when $n \rightarrow \infty$ such that for all $\epsilon > 0$, there exists some positive integer n_ϵ , with $\tau_{n_\epsilon} < \epsilon$ and $u_\tau(t)$ non-stabilizing controller, the controller is called an *improperly-posed* controller for “small” delays. In this case, the closed-loop system is *improperly-posed* for “small” delays.

In other words, “improperly-posed” systems correspond to the situations when the closed-loop system is sensitive to delay variations $\tau = \epsilon$, with $\epsilon > 0$, but sufficiently “small”.

Remark 3.1. As one may expect from the definition above, the “properly-posed” for “small” delays case refers to the situation when the closed-loop system is robust against small delay variations.

Bearing this definition and these remarks in mind, by analogy to the discussion proposed in Méndez-Barrios et al. (2024) and Méndez-Barrios et al. (2022), a *sufficient condition* for guaranteeing that the closed-loop system with a delay-difference approximation of the derivative action is “properly-posed” is the relative degree of the system being greater than one.

In this spirit, we have the following result:

Proposition 3.1. Consider the characteristic function $F(\lambda)$, given by (3). Then, if we replace the derivative in the controller by its corresponding delay-difference approximation, the obtained characteristic function $F_n(\lambda)$, given by (7), preserves stability for sufficiently “small” delays.

Proof. First, consider the transfer function of the open-loop system (see Fig. 2). It simply writes as:

$$G(\lambda) = \left(\frac{1}{\lambda^n} \right) \left(\sum_{j=0}^{n-1} k_j \lambda^j \right).$$

Next, we replace only one of the λ terms by:

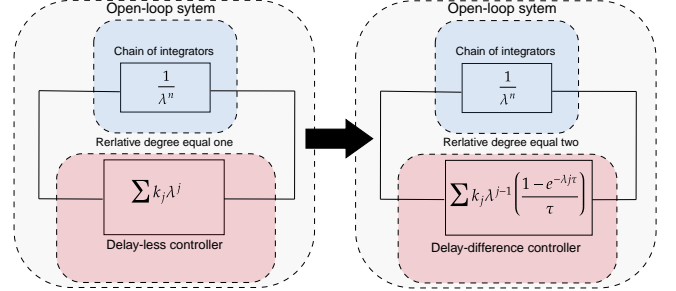


Fig. 2. Open-loop system relative degree reduction.

$$\lambda \approx \frac{1 - e^{-\lambda\tau}}{\tau},$$

therefore, as shown in Fig. 2, we obtain:

$$G_1(\lambda) = \left(\frac{1}{\lambda^n} \right) \left(\sum_{j=0}^{n-1} k_j \lambda^{j-1} \left(\frac{1 - e^{-\lambda\tau}}{\tau} \right) \right).$$

Note that the transformed system now has only one delay, and it can be interpreted as the interconnection between a transfer function with the relative degree equal to two and the corresponding transfer of the delay-difference approximation. Since the relative degree is two, the property that the closed-loop system is properly-posed follows from Theorem 1 in Méndez-Barrios et al. (2022).

By repeating the above procedure, replacing the terms λ one by one, we get:

$$G_{n-2}(\lambda) = \left(\frac{1}{\lambda^n} \right) \left(\sum_{j=0}^{n-1} \sum_{m=0}^j (-1)^m \frac{k_j}{\tau^j} \binom{j}{m} e^{-m\tau\lambda} \right),$$

which has a relative degree equal to n with an interconnection of the transfer function $1/\lambda^n$ with the corresponding delay blocks, and, by using the same arguments, it follows that the closed-loop scheme is also properly-posed. ■

3.2 Stabilizing delays computation

As already mentioned, the main objective of this note is to find the first interval of stability \mathcal{I} for given stabilizing gains in the delay-free case. To this aim, we propose a recursive procedure based on the one presented in Walton and Marshall (1987). Such methodology allows to rewrite the quasipolynomial (7) as a polynomial on p and ω , where $p \in \mathbb{R}_+$ is a parameter such that $p\tau = 1$ and $\omega \in \mathbb{R}_+$ is such that $F_n(i\omega) = 0$. Since the closed-loop characteristic function is properly-posed, there is no restriction in imposing the delay τ strictly positive, but not too “small”. The methodology requires the $n - 1$ delays to be *commensurate*, that is, $\tau_\ell = \ell\tau$ with $\ell \in \mathbb{N}^*$. Therefore, the control design at the end requires only $n + 1$ parameters, which are the n stabilizing gains, and only one delay value τ .

First, we rewrite the characteristic quasipolynomial (7) as a *two-parameter* problem:

$$F_n(\lambda) = P_0(\lambda; p) + \sum_{j=1}^{n-1} P_j(p) e^{-j\tau\lambda}, \quad (13)$$

where, as mentioned above, the parameters p and τ should verify the equality $p\tau = 1$. Note that, due to the structure of the approximation, regardless of the value of n , the

polynomials P_j for $j = \{1, 2, \dots, n-1\}$ will always only depend on the parameter p . Furthermore, each of the polynomials (including P_0) will always be of degree $n-1$ in p . Now, since p is real, if $\lambda = i\omega$ is a characteristic root, $\lambda = -i\omega$ is also a root, and we can write:

$$P_0(i\omega; p)F_n(-i\omega) - e^{i\omega\tau(n-1)}P_{n-1}(p)F_n(i\omega) = 0.$$

Note that after this manipulation, there are no longer terms in $e^{-(n-1)\lambda\tau}$ and we obtain:

$$\widehat{F}_{n-1}(\omega; p) = \widehat{P}_{0,1}(\omega; p) + \sum_{j=1}^{n-2} \widehat{P}_{j,1}(\omega; p)e^{ij\omega\tau},$$

which will also include a term $P_{n-1}^2(p)$. Note that the process can be repeated for the obtained $\widehat{F}_{n-1}(\omega; p)$ until we obtain a polynomial in (ω, p) . The term $\widehat{P}_{j,k}$ denotes the polynomial P_j obtained after repeating the reduction process above k -times. Moreover, since the product in the last iteration will correspond to a product between complex conjugates, the obtained polynomial will only have real coefficients and will be of degree $2(n-1)$ on p .

Proposition 3.2. Consider the characteristic function $F_n(\omega)$ given by (7) with $n \in \mathbb{N}^*$. Then, the pairs $(\tau, \omega) \in \mathbb{R}_+^2$ for which there exists a root of the form $\lambda = i\omega$ of $F_n(\lambda)$, with $n \in \mathbb{N}^*$, correspond to pairs for which the following holds:

$$\widehat{P}_{0,n-1}(\omega, p) + \widehat{P}_{1,n-1}(\omega, p) = 0, \quad (14)$$

$$\cos(\omega p(\omega)) = -\Re \left\{ \frac{\widehat{P}_{0,n-2}(\omega; p(\omega))}{\widehat{P}_{1,n-2}(\omega; p(\omega))} \right\}, \quad (15)$$

$$\sin(\omega p(\omega)) = \Im \left\{ \frac{\widehat{P}_{0,n-2}(\omega; p(\omega))}{\widehat{P}_{1,n-2}(\omega; p(\omega))} \right\}, \quad (16)$$

where $p(\omega)$ is the solution of (14) in p , and $\tau(\omega) = p^{-1}(\omega)$.

Sketch of proof. First, the need for Eq. (14) comes directly from the reduction process employed. Next, Eq. (15) and (16) can be obtained by noting that in the second to last iterations, one always obtains an equation of the form $P_0 + e^{-i\omega p(\omega)}P_1 = 0$ and separating it on its real and imaginary part. ■

Corollary 3.0.1. Consider the characteristic function $F_n(\omega)$ given by (7) with $n \in \mathbb{N}^*$. Assume that $\tau \in (0, \tau_r) =: \mathcal{I}$, where τ_r corresponds to the smallest positive value of τ such that there exist $\omega \in \mathbb{R}_+$ for which the following holds:

$$\widehat{P}_{0,n-1}(\omega, p) + \widehat{P}_{1,n-1}(\omega, p) = 0,$$

$$\cos(\omega p(\omega)) = -\Re \left\{ \frac{\widehat{P}_{0,n-2}(\omega; p(\omega))}{\widehat{P}_{1,n-2}(\omega; p(\omega))} \right\},$$

$$\sin(\omega p(\omega)) = \Im \left\{ \frac{\widehat{P}_{0,n-2}(\omega; p(\omega))}{\widehat{P}_{1,n-2}(\omega; p(\omega))} \right\},$$

where $\tau = p^{-1}(\omega)$. Then, all the characteristic roots of $F_n(\omega)$ will be located at \mathbb{C}_- .

Remark 3.2. Note that Corollary 3.0.1 only tells us that as long as $\tau \in \mathcal{I}$ the system preserves its stability. However, it does not tell anything about the uniqueness of such a stable interval. That is, more intervals $\mathcal{I}_j = (\tau_-, \tau_+)$ with

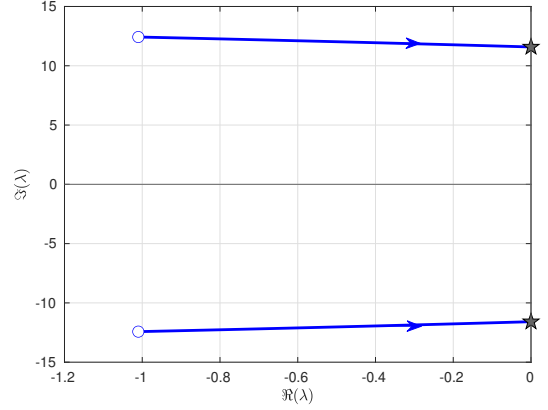


Fig. 3. Root behavior of the chain of three integrators in closed-loop with the delay-difference based controller as $\tau \in \mathcal{I} := (0, 0.1154)$ increases.

$0 < \tau_- < \tau_+$ such that the system preserve stability may exist.

4. ILLUSTRATIVE EXAMPLES

Example 1. (Triple integrator). Consider the case $n = 3$, for which after replacing the derivatives in the controller, we obtain:

$$P_0(\lambda; p) = \lambda^3 + k_2 p^2 + k_1 p + k_0,$$

$$P_1(p) = -2k_2 p^2 - k_1 p,$$

$$P_2(p) = k_2 p^2.$$

The proposed reduction process leads to the following:

$$\begin{aligned} \widehat{F}_1(\omega; p) &= p^4 (4k_0 k_1^2 k_2 - 4k_2^2 \omega^6) + p^3 (8k_0^2 k_1 k_2 + 2k_0 k_1^3) \\ &\quad + p^2 (4k_0^3 k_2 + 5k_0^2 k_1^2 + 4k_0 k_2 \omega^6 + k_1^2 \omega^6) \\ &\quad + p (4k_0^3 k_1 + 4k_0 k_1 \omega^6) + 2k_0^2 \omega^6 + \omega^{12} + k_0^4 = 0. \end{aligned}$$

After numerical computation, the obtained interval is given by $\mathcal{I} = (0, 0.1154)$, with the crossing occurring at $\omega = 11.58$, as depicted in Fig. 3. Next, let the control gains $\mathbf{k} = [40, 38, 11]$, such that the closed-loop characteristic roots of the delay-free closed-loop system are real and located at $s_{1,2,3} = \{-2, -4, -5\}$. The corresponding frequency-domain control law reads:

$$C(\lambda) = \widehat{k}_2 e^{-2\lambda\tau} + \widehat{k}_1 e^{-\lambda\tau} + \widehat{k}_0,$$

where $\widehat{k}_0 = k_0 + k_2/\tau^2 + k_1/\tau$, $\widehat{k}_1 = -2k_2\tau^2 - k_1/\tau$ and $\widehat{k}_2 = k_2/\tau^2$, and by choosing $\tau = 0.1$ the spectral abscissa¹ of the system is located at $\lambda = -1.9 \pm 0.4i$.

It is worth mentioning that in Niculescu and Michiels (2004), contrary to this approach, the control gains tend to be "small". However, their proposed pole placement, while successfully stabilizing the system, requires the placement of a multiple root at a "small" value $\epsilon \rightarrow 0$, that is, $\lambda = -\epsilon$ is a root of multiplicity n , which has the drawback of rendering slow the closed-loop dynamics.

Example 2. (High-order chain & closed-loop time response). Consider the case $n = 4$, for which the closed-loop system can be represented as in Fig. 4. Consider the gains $\mathbf{k} = [24, 50, 35, 10]$, which will place the delay-free closed-loop characteristic roots at $s_{1,2,3,4} = \{-1, -2, -3, -4\}$, and let $\tau = 0.01$.

¹ the right-most root

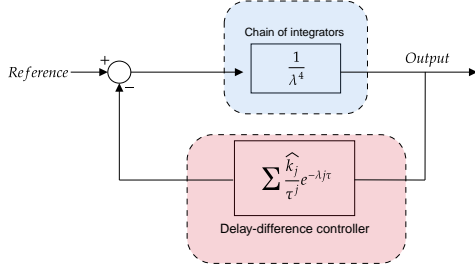
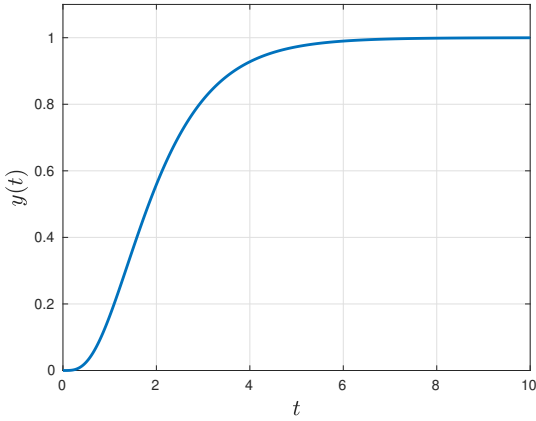
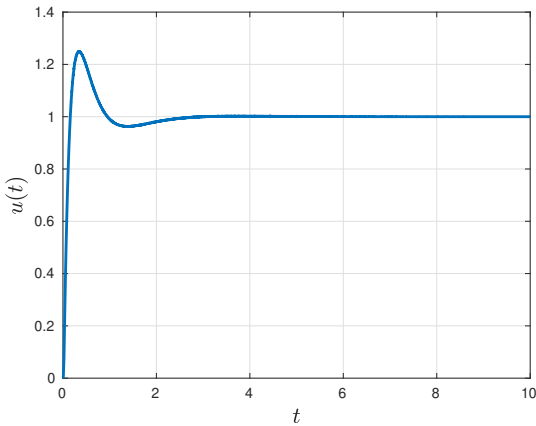


Fig. 4. Chain of four integrators in closed-loop with the delay-difference approximation based controller.

The closed-loop system's time response is depicted in Fig. 5a, where we can observe that the proposed controller stabilized the given chain. Moreover, one thing that is worth noting is that, as depicted in Fig. 5b, despite the "high control gains" produced in the approximations process, the "overshoot" of the control output stays "small".



(a) Time response of the closed-loop system represented in Fig. 4



(b) Control output of the system depicted in Fig. 4

Fig. 5. Time evolution of the system's output (a) and control's output (b).

The corresponding polynomials P_j are given by:

$$\begin{aligned} P_0(\lambda; p) &= \lambda^4 + k_3 p^3 + k_2 p^2 + k_1 p + k_0, \\ P_1(p) &= -3k_3 p^3 - 2k_2 p^2 - k_1 p, \\ P_2(p) &= 3k_3 p^3 + k_2 p^2, \\ P_3(p) &= -k_3 p^3, \end{aligned}$$

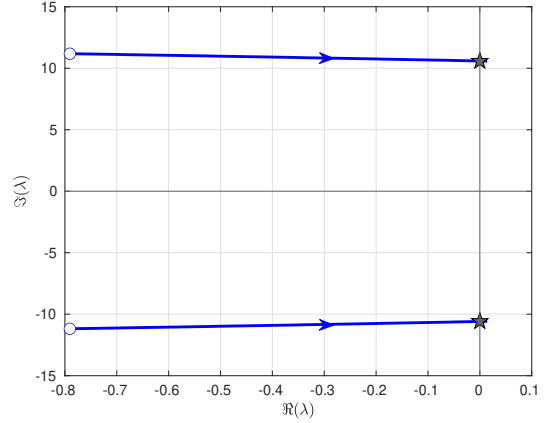


Fig. 6. Characteristic roots behavior for the closed-loop system with $n = 4$ for $\tau \rightarrow 0.08$

leading to the delay interval $\mathcal{I} := (0, 0.08)$ with the crossing at $\omega = 10.58$, as illustrated in Fig. 6. The choice of $\tau = 0.01 \in \mathcal{I}$ and its temporal response are consistent with these computations.

Example 3. (More than one stabilizing interval). While the main objective of the note is to find a *delay margin* for the approximation, the interval $\mathcal{I} = (0, \tau_r)$ is not necessarily the only stable interval that exists.

Consider the motivating example from Section 2, where for $n = 2$ and $\mathbf{k} = [10, 7]$, we have $\mathcal{I} = (0, 0.5153)$. However, further investigation shows that the system preserves stability for $\mathcal{I}_1 = (2, 2.36)$. Fig. 7 depicts the evolution of the system roots, highlighting the existence of three crossings. More precisely, the first one corresponds to the first stability loss, namely $\tau_r = 0.5153$, and the second one defines the second stable interval, that is $\tau = 2$, and finally the last one corresponds to the upper bound of the second interval, that is $\tau = 2.3636$.

One advantage of this phenomenon is the possibility of stabilizing the system using *smaller* gains. Nonetheless, it should be mentioned that, on the one hand, the characteristic roots will not be close to those of the delay-free version of the system, and, on the other hand, to the best of the author's knowledge, there is no analytical result that allows certifying whether or not a second stabilizing delay interval exists in the general case.

5. CONCLUSIONS

In this paper, we proposed an effective methodology that allows computing the delay margin of the delay-difference approximation of the derivative action in the context of a chain of integrators. The obtained results were further verified via numerical examples. Future research efforts will focus on finding whether or not conditions exist, such as at least two different delay intervals guaranteeing stability since, in general, larger delays also reduce the size of the required gains.

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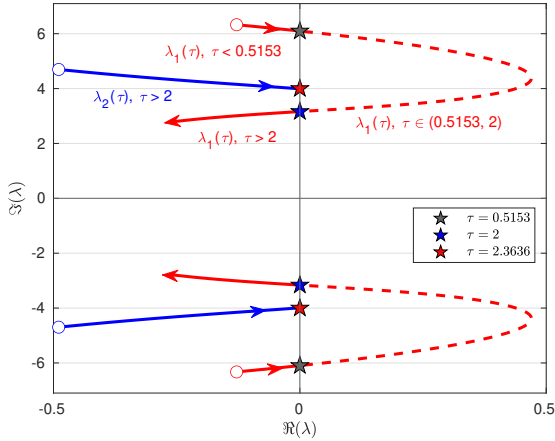


Fig. 7. Characteristic roots of the closed-loop system for $n = 2$, $\mathbf{k} = [10, 7]$ and $\tau \in (0.5, 2.36)$. The dashed line, corresponding to $\tau \in (0.5153, 2)$ indicates the values of τ for which the system is unstable.

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