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An alternative to Maxwell's equation based on the conservation of acceleration

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Abstract

Equivalence between the discrete laws of electromagnetism and mechanics is revealed by the derivation of a non-linear equation intertwining electrical and magnetic effects. The alternative physical model to Maxwell's is based on the conservation of the acceleration of electric current, the only variable in the law of discrete electromagnetism. The modeling of this law is based solely on the essential elements of electrostatics and magnetostatics, where Ohm's and Ampère's laws are reformulated in a dynamic framework. The Maxwell-Thomson law is abandoned and the conservation of magnetic flux is ignored, this emancipation paves the way for the potential existence of magnetic monopoles. The Maxwell-Gauss equation expressing the conservation of electric flux is also set aside, as it is implicitly satisfied in the discrete formulation. The discrete point of view consists in observing that direct and induced currents flow on the same conductor without interacting in the steady state, and in an intertwined manner in the variable state. The sum of these two contributions forms the material derivative of the electric current density, the current acceleration. This quantity can be decomposed into a curl-free and a divergence-free components, a Helmholtz-Hodge decomposition to extract both direct and induced contributions. The law of discrete electromagnetism presents complete symmetry in the exchange of electric and magnetic fields. Inertial terms of the current acceleration contained in its material derivative are none other than the non-linear Lorentz forces. The new formulation is equivalent to the law of motion of discrete mechanics.

Keywords

Maxwell's equations; Weak Equivalence Principle; Conservation of Acceleration; Discrete Mechanics; Helmholtz-Hodge Decomposition; Special and General Relativity

1 Introduction

Maxwell's equations remain the reference model for electromagnetism, even though there are many tensor, covariant and potential formulations. The brilliant idea of J.C. Maxwell in 1865 [1] was to federate the existing laws of electrostatics and magnetism into a dynamic version. This reflects very well the coupling between direct and induced effects in interlaced circuits. A. Einstein's [2] demonstration of the invariance of these equations under a Lorentz transformation gave them a modernist status that persists to this day. The introduction of non-linear Lorentz forces [3] extended the scope of these equations to interactions with moving fluids. Excellent works, notably those by L. Landau and E. Lifchitz [4] are essential references for understanding the nature of these equations.

Attempts at an alternative to Maxwell's equations are few and far between. Jefimenko's equations [5] express the integral forms of the electric and magnetic fields of a distribution of electric charges by considering the notion of finite celerity of electromagnetic waves and that of a delay similar to Liénard and Wichert's time lapse [6]. The notion of time lapse is essential in physics, as it is the one that highlights the relationship between cause and effect. It is natural in electromagnetism, but is also present in all dynamic phenomena, such as mechanics. In a way, electric and magnetic fields are intertwined, even entangled. Some visions, such as that of E. Tonti [7, 8], envisage considering the structures of certain laws of physics, notably those of electromagnetism and fluid mechanics, by drawing out their similarities in the form of diagrams. Analogies between the Navier-Stokes and Maxwell equations have also been noted [9],

but for all that, the comparisons remain superficial. Analogies are not formal equivalences that unify physical laws. Other works of a similar nature examine mathematical properties in the context of classical [10] and relativistic [11] electrodynamics. An important problem examined in detail is that of the existence of monopoles [12]. For several decades, a discrete vision of electromagnetism has emerged, combining the properties of Maxwell's equations with those of differential geometry [13, 14, 15] but, for the most part, they correspond to the solution of these equations in the discrete framework of DEC (Discrete Exterior Calculus) or mimetic methods.

The aim here is to model electrical and magnetic phenomena in an original way, keeping in mind Maxwell's idea of federating the laws specific to these fields into a dynamic vision. The choice of Ampère's and Ohm's laws as the initial basis is a natural one, since they translate the two elementary phenomena of direct and induced currents. These laws are then transformed to make them compatible with the unsteady nature of coupled phenomena. The original choice of electric current density as the principal variable allows us to exclude the electric \mathbf{E} and magnetic \mathbf{B} fields as potential variables, which are merely consequences of the current density \mathbf{j} alone. Maxwell's equations are overdetermined, and it is essential to apply the principle of parsimony, i.e. to reduce to the strict minimum the quantities needed to describe the phenomena, without loss of information. The conservation of current density is the basic principle that allows us to derive a time-evolution equation as an acceleration. The fundamental law of discrete electromagnetism states that the acceleration of a current, of an elementary charge or of a material medium, is equal to the sum of the accelerations applied to it. Discrete mechanics [16] then becomes the appropriate framework for transposing the mechanical effects of the law of motion into electromagnetic effects in an equivalent law. This quantification paves the way for the proposal of a unique law in which fluid velocity and electric current are two formally equivalent quantities.

2 Comments on Maxwell's equations

2.1 Maxwell's model

Maxwell's mathematical model takes the form of a set of partial differential equations describing the behavior of electric and magnetic circuits, classical optics and, in general, classical electromagnetism. The microscopic Maxwell equations in vacuum are (i) the Maxwell-Thomson equation, (ii) the Maxwell-Gauss law, (iii) the Maxwell-Faraday law and (iv) the Maxwell-Ampère law,

$$\left\{ \begin{array}{l} \nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = \rho_e / \varepsilon_0, \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \end{array} \right. \quad \begin{array}{l} (1a) \\ (1b) \\ (1c) \end{array}$$

where ρ_e is the electric charge density, \mathbf{j} , the current density vector, \mathbf{E} the electric field vector and \mathbf{B} the magnetic field; the dielectric permittivity of vacuum is noted ε_0 and the magnetic permeability of vacuum μ_0 , these two quantities can be grouped together in the inverse of the celerity squared $c_0^2 = 1/\varepsilon_0 \mu_0$. The Lorentz force acting on the electric charge q whose velocity is equal to \mathbf{V} can be read as follows,

$$\mathbf{F} = q (\mathbf{E} + \mathbf{V} \times \mathbf{B}), \quad (2)$$

where the electric force associated with the electric field \mathbf{E} and where the product $\mathbf{V} \times \mathbf{B}$ of the velocity vector and the induced magnetic field represents the magnetic contribution to the total force \mathbf{F} . In addition to Maxwell's equations, there are the constitutive laws of Ohm's law and

the conservation of charge,

$$\begin{cases} \mathbf{j} = \sigma \mathbf{E}, & (3a) \\ \frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j} = 0, & (3b) \end{cases}$$

where \mathbf{j} is the current density vector and σ is the electrical conductivity.

Numerous books and manuals are devoted to these equations, including Landau & Lifchitz [4]. The Maxwell-Ampère law (1c) clearly shows two contributions

$$\mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (4)$$

where the second term is a current density called displacement current by Maxwell to extend Ampère's law to the variable regime; ε_0 is the permittivity of vacuum.

2.2 Maxwell's equations in terms of potentials

Maxwell's equations, where the fields $\mathbf{E} = -\nabla e$ and $\mathbf{B} = \nabla \times \mathbf{A}$ are coupled by terms *ad hoc*:

$$\begin{cases} \frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \nabla \left(\frac{\rho_e}{\varepsilon_0} \right) - \mu_0 \frac{\partial \mathbf{j}}{\partial t}, & (5a) \\ \frac{1}{c_0^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = \nabla \times (\mu_0 \mathbf{j}), & (5b) \end{cases}$$

or those that use the Lorentz gauge condition:

$$\varepsilon_0 \mu_0 \frac{\partial e}{\partial t} + \nabla \cdot \mathbf{A} = 0, \quad (6)$$

which can be written, with $1/c_0^2 = \varepsilon_0 \mu_0$, in the form of propagation equations on the two potentials e and \mathbf{A} :

$$\begin{cases} \frac{1}{c_0^2} \frac{\partial^2 e}{\partial t^2} - \nabla^2 e = \frac{\rho_e}{\varepsilon_0}, & (7a) \\ \frac{1}{c_0^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}. & (7b) \end{cases}$$

Thus, the two equations for the electric field \mathbf{E} and the magnetic field \mathbf{B} (5) cannot be reduced to a single equation. The fundamental reason for this is the strongly coupled nature of electric and magnetic effects, one inevitably driving the other when motion is no longer stationary or continuous. J.C. Maxwell understood this when he synthesized the existing laws of electrostatics and magnetostatics in a dynamic context. The electromagnetic equation must reflect the fact that the direct field and the induced field are two facets of the same phenomenon, defining a causal link between electric and magnetic fields.

2.3 Overdetermination of Maxwell's equations

The overdetermination of Maxwell's equations is obvious if we consider the number of unknowns - six in all, three for each of the electric \mathbf{E} and magnetic \mathbf{B} field components - compared to the number of equations, the components of the Maxwell-Faraday and Maxwell-Ampère equations and the two Gaussian equations. In fact, it's difficult to understand how the laws overlap when the combination of some of them leads to the satisfaction of the others. Certain vector identities based on the conservation of magnetic flux also make it possible to restrict the number of equations to close the system.

Some sequential equations in physics display the same characteristics of overdetermination of equations or variables. Constitutive laws can be used to close a system of equations, but this procedure also has its drawbacks. Indeed, in this case, there are inevitably overlaps of information that cause a time shift in the values of the variables. This is the case, for example, with the Euler equations for isentropic compressible flows, where the three components of the Euler equation - conservation of mass, energy equation and law of state - lead to a system that is well closed by the three components of momentum, density, pressure and temperature, but is still overdetermined; the energy equation can easily be shown to be unnecessary. In other words, an overdetermined system of equations may provide a solution, but this solution still contains inconsistencies, even if they go undetected.

2.4 Non-existence of monopoles

The search for magnetic monopoles has been the subject of sustained research in recent decades, both experimentally [17] and theoretically [18, 19], in nuclear physics. Other contributions [20, 12, 21, 22, 23, 24], provide a more general perspective on this issue. The potential existence of magnetic monopoles is examined in detail by K.A. Milton [25] in its theoretical and experimental aspects. The existence of monopoles is often postulated [25] as $\nabla \cdot \mathbf{B} = 4\pi\rho_m$, where ρ_m is the magnetic charge density. Let's consider one of the relations of Maxwell's equations, the Maxwell-Thomson law:

$$\nabla \cdot \mathbf{B} = 0. \quad (8)$$

To impose this condition on the gradient of a function, simply set $\mathbf{B} = \nabla \times \mathbf{A}$, where \mathbf{A} is the potential vector of \mathbf{B} . This rather too natural mathematical view of how to proceed is a roundabout way of eliminating another possible contribution of the magnetic field. The constraint (8) imposed by Maxwell's equations is abusive, as it excludes any possibility of predicting the existence of monopoles. It could be considered a limit if the celerity of light were infinite or if the stationary regime were reached; in fact, the celerity c_0 is very great but not infinite.

3 An alternative law of electromagnetic wave propagation

Modelling an alternative law to Maxwell's equations cannot be based on modifying them. Indeed, Maxwell's equations have a logic, and the fact that they have stood the test since their derivation proves their representativeness. The attempt presented here takes up modeling based on Ohm's and Ampère's laws, forgetting all the other laws of classical electromagnetism.

The microscopic version in terms of magnetic and electric fields, sometimes referred to as Maxwell's equations in vacuum, is complemented in classical electromagnetism by a macroscopic version more akin to a continuum approach. This difference does not exist in the approach presented here, which is essentially based on the conservation of acceleration. We must return to the profound meaning of the Weak Equivalence Principle (WEP) revisited by discrete mechanics, where the intrinsic acceleration of an isolated particle with or without mass, or of an element of a material medium, is equal to the sum of the accelerations imposed on it. The law proposed for electromagnetism is therefore a local equation that applies to a global material medium, and constitutes the very principle of many physics equations.

The local law of electromagnetism is based on the coupling model between electrical and magnetic phenomena, which only makes sense in a time-varying regime. Direct and induced currents are interwoven in a complex geometric structure. The core of this structure, known as Maxwell's local reference frame, considers a Γ conductor and a current coil arranged around the conductor.

3.1 Maxwell's local frame of reference

The Maxwell local reference frame shown in Figure 1 is a geometric structure made up of a primal structure, a rectilinear segment Γ bounded by two ends a and b where its length $dh = [a, b]$ is called the discrete horizon because the time needed to cross this distance is equal to $dt = dh/c$ where c is the celerity of the electromagnetic wave. The dual structure is the Δ circuit surrounding the Γ conductor. Each structure is oriented respectively by the unit vectors \mathbf{t} for the primal segment and \mathbf{n} for the dual contour, so that $\mathbf{t} \cdot \mathbf{n} = 0$ by construction. The scalar potential ϕ is located on the a or b vertices of the segment and the vector potential ψ is assigned to the dual contour. Other Γ segments are connected by their vertices to form a Γ^* family delimiting a flat polygonal surface (not shown) called primal facet \mathcal{S} whose oriented normal is \mathbf{n} . The physical domain of Ω is thus tessellated by a set of Maxwell reference frames interconnected by their common vertices. The Γ and Δ circuits are thus interwoven to form an unstructured mesh of polyhedral cells with any number of faces. The only unknown in an electromagnetism problem lies on the Γ segment, namely the intrinsic acceleration $\gamma = d\mathbf{v}/dt$, which here represents the current density derivative $d\mathbf{j}/dt$. This geometric structure has the property of being entirely symmetrical when we exchange the roles of Γ and Δ , i.e. those of the electric and magnetic fields themselves.

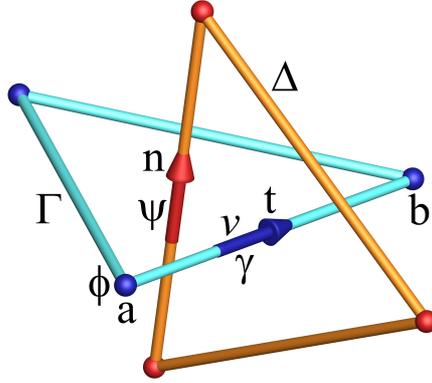


Figure 1. Maxwell's local reversible frame of reference: a line segment Γ of length $dh = [a, b]$ oriented along the unit vector \mathbf{t} forms the primal structure. The dual contour Δ positively oriented by \mathbf{n} is such that $\mathbf{t} \cdot \mathbf{n} = 0$. The acceleration γ and velocity \mathbf{v} are vectors carried by the oriented segment Γ ; the scalar potential ϕ is assigned to its extremities and the vector potential ψ is fixed on the contour Δ .

Four discrete differential operators correspond to these structures: (i) the gradient operator $\nabla\phi = (\phi_b - \phi_a)/dh$, which is the restriction on Γ of the classical gradient, (ii) the velocity divergence $\nabla \cdot \mathbf{v}$, which represents the flux of the vector \mathbf{v} across the dual surface \mathcal{D} orthogonal to the segment, (iii) the primal curl $\nabla \times \mathbf{v}$ calculated as the circulation of this vector along the Γ^* contour, whose result is a vector oriented along \mathbf{n} and (iv) the dual curl of the vector potential, $\nabla \otimes \psi$, whose result is projected onto the Γ segment. This operator $\nabla \otimes$ is not the tensor product of classical analysis. Moreover, the notion of second-order or higher-order tensor does not exist in discrete mechanics, and vectors themselves are scalars on oriented segments. Remarkable vector identities due to the construction of this set, $\nabla \times \nabla\phi = 0$ and $\nabla \cdot \nabla \otimes \psi = 0$ give the modeling orthogonality properties essential to a realistic description of physical phenomena. These discrete operators are the only ones necessary and sufficient to describe all interactions between electric and magnetic currents.

In this structural context, the modeling of physical phenomena is carried out while ignoring

the concepts of classical mechanics, mass m or density ρ , force, pressure, global energy and the classical notions of one-point derivation, integration, differential calculus and mathematical analysis in general. Thus, the modeling of phenomena relies solely on concepts of differential geometry to derive the law of discrete mechanics [16].

3.2 Modeling of currents

The following modeling of the discrete electromagnetic equation is not a variant of Maxwell's equations, but is based solely on Ohm's and Ampère's laws, extended to the dynamic regime. The other laws used by Maxwell and federated in his model are not taken into account. In particular, the Maxwell-Thomson law, $\nabla \cdot \mathbf{B} = 0$, which expresses that the magnetic flux is always conservative, is not an acceptable constraint *a priori*, it's a law *ad hoc*. The proposed approach aims to reduce the number of unknowns and equations in electromagnetism, as the various forms of Maxwell's equations are overdetermined. The principle of parsimony is applied to establish a model reduced to the bare minimum, but complete from a physical point of view. Similarly, the Lorentz electromagnetic force should emerge naturally from the model without being added *a posteriori*. The material derivative of the electric current density $d\mathbf{j}/dt$ is the starting point of the proposed approach, symbolizing the acceleration of electric charges in a conductor in the same way as the acceleration of a fluid flowing through a channel; both phenomena are therefore governed by the same fundamental law of discrete mechanics.

The total current \mathbf{j} cannot be the sum of the direct current \mathbf{j}_{dir} and the induced current \mathbf{j}_{ind} in steady state, as this makes no sense. On the other hand, when the current is variable, the two direct and induced currents are nested, and the total current is the sum of the other two, $\mathbf{j} = \mathbf{j}_{dir} + \mathbf{j}_{ind}$ both flowing on the same Γ segment. The electric field can be defined by its potential noted here e in the form $\mathbf{E} = -\nabla e$ and the induction field \mathbf{B} is associated with the magnetic permeability of the medium μ . These quantities are linked by Ohm's law and Ampère's law,

$$\begin{cases} \mathbf{j}_{dir} = -\sigma \nabla e, & (9a) \\ \mathbf{j}_{ind} = \frac{1}{\mu} \nabla \otimes \mathbf{B}, & (9b) \end{cases}$$

where σ is the local electrical conductivity and μ the local magnetic permeability. These two laws are only valid in steady state. To obtain a law for the evolution of the total current \mathbf{j} , we must first model the evolution of the electric and magnetic potentials. Note that the vector potential is the induction field \mathbf{B} itself, whose flux is not conservative *a priori*.

$$\begin{cases} c_e \frac{de}{dt} = -\nabla \cdot \mathbf{j}_{dir}, & (10a) \\ c_m \frac{d\mathbf{B}}{dt} = \nabla \times \mathbf{j}_{ind}. & (10b) \end{cases}$$

where $c_e = C_e/\mathcal{V}$ is the electrical capacitance per unit volume expressed in $s^4 A^2 / kgm^5$, or farad per unit volume; c_m is its magnetic equivalent expressed in $s^3 A^2 / kgm^3$. By integrating the expressions of the potentials (9), these relationships become diffusion equations,

$$\begin{cases} \frac{de}{dt} = a_e \nabla^2 e, & (11a) \\ \frac{d\mathbf{B}}{dt} = a_m \nabla^2 \mathbf{B}, & (11b) \end{cases}$$

where $a_e = \sigma/c_e$ is a diffusion coefficient expressed in $m^2 s^{-1}$ and $a_m = 1/\mu c_m$ is a magnetic diffusion coefficient. The last relation implies that $\nabla \cdot \mathbf{B} = 0$.

These diffusion equations are only a certain representation of reality. Although they take time scales into account, they in no way reflect the propagation of elastic and magnetic waves, whose celerity is equal to c_0 in a vacuum. On the other hand, they are representative of phenomena with large time constants, such as the diffusion of electric current in a metallic conductor, where the velocity of electrons is very low, of the order of magnitude of a few centimetres per second. For very small time constants, the diffusion equations become erroneous, like those for heat transfer based on Fourier's law or Stokes' law based on Newton's viscous law. In fact, phenomenological laws such as Fourier's law for heat transfer, Newton's law describing the behavior of viscous fluids or Ohm's law are integrated into partial differential equations to become evolutionary laws, such as the heat equation with temperature or enthalpy as the variable, Stokes' equation for momentum, and so on. These are generally diffusion or advection-diffusion equations. Unfortunately, the corresponding solutions contain artefacts, as wave velocity is not taken into account on very small time scales, since wave celerity is implicitly infinite. Discrete mechanics introduces the notion of time at small scales and transforms these equations into propagation laws; for example, the heat equation is transformed into a propagation law at small time constants, but recovers the classical behavior of diffusion at large time constants [26]. Under these conditions, the laws must be transformed into propagation laws. Assuming $a = dt c^2$ where dt is the observation time of the phenomenon, we find:

$$\left\{ \begin{array}{l} \frac{1}{c^2} \frac{d^2 e}{dt^2} - \nabla^2 e = 0, \\ \frac{1}{c^2} \frac{d^2 \mathbf{B}}{dt^2} - \nabla^2 \mathbf{B} = 0, \end{array} \right. \quad (12a)$$

$$\left\{ \begin{array}{l} \frac{1}{c^2} \frac{d^2 e}{dt^2} - \nabla^2 e = 0, \\ \frac{1}{c^2} \frac{d^2 \mathbf{B}}{dt^2} - \nabla^2 \mathbf{B} = 0, \end{array} \right. \quad (12b)$$

a wave equations.

At this stage, however, direct and induced currents are not nested, so it is necessary for the total current $\mathbf{j} = \mathbf{j}_{dir} + \mathbf{j}_{ind}$ to be the only variable of a discrete electromagnetic law in the form of an evolution law, considering its material derivative $d\mathbf{j}/dt$ whose direct and induced components can be read,

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla \left(\frac{d(\sigma e)}{dt} \right), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes \left(\frac{d(\mathbf{B}/\mu)}{dt} \right). \end{array} \right. \quad (13a)$$

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla \left(\frac{d(\sigma e)}{dt} \right), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes \left(\frac{d(\mathbf{B}/\mu)}{dt} \right). \end{array} \right. \quad (13b)$$

To obtain the scalar and vector potentials of the current acceleration $d\mathbf{j}/dt$, it is necessary to consider the generalized form of Ohm's law, $e = R C de/dt - L C d^2/dt^2$ where R is the resistance in ohm, C the electrical capacitance in farad and L the electrical inductance in henry; this generalized Ohm's law has been extended to periodic currents. In this case, the hyperbolic character of this law is not reduced to an additional term, but results from the physical modeling of the undulatory nature of electric and magnetic waves. Thus, the law of evolution of electric potential can be reduced to $de/dt = e/R C$ where the quantity $\tau_e = R C$ is a time constant expressing the characteristic time of the velocity of electric current in a conductor. By analogy, there is a time $\tau_m = R_m C_m$ with which the magnetic wave information propagates, where R_m and C_m are quantities that don't exist in classical electromagnetism because the induction field \mathbf{B} is constrained to satisfy the zero divergence condition, $\nabla \cdot \mathbf{B} = 0$. The relations (13) become,

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla \left(\frac{\sigma}{R C} e \right), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes \left(\frac{1}{\mu R_m C_m} \mathbf{B} \right), \end{array} \right. \quad (14a)$$

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla \left(\frac{\sigma}{R C} e \right), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes \left(\frac{1}{\mu R_m C_m} \mathbf{B} \right), \end{array} \right. \quad (14b)$$

where direct and induced currents $\mathbf{j}_{dir}^o = -\nabla(\sigma e)$ and $\mathbf{j}_{ind}^o = -\nabla \otimes (B/\mu)$ when the phenomenon becomes stationary, at large time constants; indeed, a current \mathbf{j} is the sum of its value at time

t^o and the product of acceleration and time, $\mathbf{j} = \mathbf{j}^o + \boldsymbol{\gamma} dt$. While the velocity of the direct current \mathbf{j}_{dir} may be low in metallic conductors, it becomes equal to the celerity of the electric wave c_0 in a vacuum. In the case of magnetic wave propagation, this is not the case, as there is no accumulation of charges in a monopole, which is why it is wrong to assert the conservation of magnetic flux.

At this stage, the material derivatives of the second member are considered as electric and magnetic potentials, which take the form $\phi_e = \sigma e/\tau_e$ and $\boldsymbol{\psi}_m = \mathbf{B}/(\mu \tau_m)$. These are quantities that accumulate electrical and magnetic energies over time, and are defined to within a constant. The time lapse dt is not a simple differential element, but has a real physical meaning associated with the incremental vision of the formulation. When this time lapse is significant, we find the laws relating to continuous or slowly varying currents of Ohm's and Ampère's laws (9). The sequential description of the evolution of current potentials ϕ_e and $\boldsymbol{\psi}_m$ at time $t^o + dt$, is discretized as $\phi_e = \phi_e^o + d\phi_e$ and $\boldsymbol{\psi}_m = \boldsymbol{\psi}_m^o + d\boldsymbol{\psi}_m$ where ϕ_e^o and $\boldsymbol{\psi}_m^o$ are the retarded potentials [6] and where $d\phi_e$ and $d\boldsymbol{\psi}_m$ are the corresponding increments.

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla (\phi_e^o - dt c^2 \nabla \cdot \mathbf{j}_{dir}), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes (\boldsymbol{\psi}_m^o - dt c^2 \nabla \cdot \mathbf{j}_{ind}). \end{array} \right. \quad (15a)$$

$$\left\{ \begin{array}{l} \frac{d\mathbf{j}_{dir}}{dt} = -\nabla (\phi_e^o - dt c^2 \nabla \cdot \mathbf{j}_{dir}), \\ \frac{d\mathbf{j}_{ind}}{dt} = \nabla \otimes (\boldsymbol{\psi}_m^o - dt c^2 \nabla \cdot \mathbf{j}_{ind}). \end{array} \right. \quad (15b)$$

Currents are only defined by additive constants; indeed, the superposition of a uniform current does not modify the interactions modeled by an equation possessing the property of relativity. Similarly, a uniform velocity \mathbf{v}^o does not alter the mechanical equilibrium defined by Newton's law, the principle of inertia or Newton's first law. It is therefore the velocity that is the discrete integral of the acceleration $\mathbf{v} = \mathbf{v}^o + dt \boldsymbol{\gamma}$ and not acceleration which is the material derivative of velocity; in the latter case, the contribution of \mathbf{v}^o disappears, resulting in a loss of information about the current state of the system. The quantity \mathbf{v}^o will be linked to a state of equilibrium at time t^o . Motion is called inertial or Galilean motion, as is the electric current \mathbf{j} , which must be written as $\mathbf{j} = \mathbf{j}^o + dt \boldsymbol{\gamma}$ where $\boldsymbol{\gamma}$ is the acceleration of the electric current. The direct current thus becomes $\mathbf{j}_{dir} = \mathbf{j}_{dir}^o + dt \boldsymbol{\gamma}_{dir}$ and similarly for the induced current, $\mathbf{j}_{ind} = \mathbf{j}_{ind}^o + dt \boldsymbol{\gamma}_{ind}$. Note also that the curl of a gradient is zero, so $\nabla \cdot \mathbf{j}_{dir} = 0$, and that the divergence of a dual curl is also zero, so $\nabla \cdot \mathbf{j}_{ind} = 0$. These vector identities, such as $\nabla \cdot \mathbf{j}_{dir} = \nabla \cdot \mathbf{j}$ and $\nabla \times \mathbf{j}_{ind} = \nabla \times \mathbf{j}$, allow us to establish a law of motion for the total current. The discrete and alternative law of Maxwell's equations can then be written as a single equation for a single variable, the total current,

$$\frac{d\mathbf{j}}{dt} = -\nabla (\phi_e^o - dt c^2 \nabla \cdot \mathbf{j}) + \nabla \otimes (\boldsymbol{\psi}_m^o - dt c^2 \nabla \times \mathbf{j}). \quad (16)$$

As scalar and vector potentials are not variables, these two quantities are updated incrementally to calculate current potentials based on retarded potentials.

$$\left\{ \begin{array}{l} \phi_e = \phi_e^o - dt c^2 \nabla \cdot \mathbf{j}, \\ \boldsymbol{\psi}_m = \boldsymbol{\psi}_m^o - dt c^2 \nabla \times \mathbf{j}, \end{array} \right. \quad (17a)$$

$$\left\{ \begin{array}{l} \phi_e = \phi_e^o - dt c^2 \nabla \cdot \mathbf{j}, \\ \boldsymbol{\psi}_m = \boldsymbol{\psi}_m^o - dt c^2 \nabla \times \mathbf{j}, \end{array} \right. \quad (17b)$$

where the current density \mathbf{j} present in these updates is the solution of the equation (16).

The equation of discrete electromagnetism (16) associated to potential updates (17) constitutes a law of conservation of current acceleration; indeed the ratio $\mathbf{v} = \mathbf{j}/\rho_e$ is a velocity and its material derivative $d(\mathbf{j}/\rho_e)/dt$ is an acceleration.

It expresses the conservation of total energy, the sum of direct and induced energies. The conservation of the electric charge q or of the electric charge density ρ_e is written from the divergence of the current density $\nabla \cdot \mathbf{j}$ in the form $\nabla \rho_e/dt + \nabla \cdot \mathbf{j} = 0$. However, electric charge is subject to displacement or advection in space, so the partial derivative must be replaced by

the material derivative. The balance of electrical energy over an elementary volume ϕ_e that we follow during its movement leads to a similar expression,

$$\left\{ \begin{array}{l} \frac{d\rho_e}{dt} + \rho_e \nabla \cdot (\mathbf{j}/\rho_e) = 0, \\ \frac{d\phi_e}{dt} + \phi_e \nabla \cdot (\mathbf{j}/\rho_e) = 0, \end{array} \right. \quad (18a)$$

$$\left\{ \begin{array}{l} \frac{d\rho_e}{dt} + \rho_e \nabla \cdot (\mathbf{j}/\rho_e) = 0, \\ \frac{d\phi_e}{dt} + \phi_e \nabla \cdot (\mathbf{j}/\rho_e) = 0, \end{array} \right. \quad (18b)$$

where ϕ_e is an energy per unit mass and is none other than the celerity squared c^2 . These two relations relate to different variables, but they are structurally identical; their variables are therefore equivalent.

Just as in special and general relativity the equivalence between mass and energy $E = m c^2$, the previous laws express the equivalence between charge and electrical energy, $E = \tilde{q} c^2$ with $\tilde{q} = q \rho / \rho_e$ where q is the electric charge, ρ the density and ρ_e the charge density. Thus \tilde{q} , the charge by unit mass, plays the same role as the mass m in mechanics. However, it should be remembered that these equivalences are in fact only proportionalities and that it is preferable to write $E/m = c^2$ and $E/\tilde{q} = c^2$, these are energies which are written $\phi = c^2$ in a general way, an energy per unit of mass.

Under these conditions, the role of mass and electric charge is no longer necessary for the derivation of the laws of physics, which already express the conservation of energy through these equivalences. The real cause of these equivalences lies in the profound meaning of the Principle of Weak Equivalence (WEP) revisited by discrete mechanics, where the proper acceleration of a particle with or without mass, or of an element of material medium, is equal to the sum of the accelerations imposed on it by the exterior, $\boldsymbol{\gamma} = \mathbf{h}$. Newton's second law becomes an equality of accelerations. It should be noted, however, that the formula $E/m = c^2$ or $E/\tilde{q} = c^2$ represents only one component of total acceleration, that corresponding to directional or direct effects, but rotational effects must also be included in the expression of total acceleration, $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$.

The terms $d\phi_e = -dt c^2 \nabla \cdot \mathbf{j}$ and $d\boldsymbol{\psi}_m = -dt c^2 \nabla \times \mathbf{j}$ are respectively accumulated in scalar and vector potentials representing electric and magnetic charges. Electric charges are located on the vertices of the Figure 1 structure, and magnetic charges are associated with each of the primal facets. Conservation of electric charge $d\rho_e + \rho_e \nabla \cdot \mathbf{j} = 0$ and of the magnetic charge $d\rho_m + \rho_m \nabla \cdot \mathbf{j} = 0$ are already implicitly included in the equation (16), so there's no need to add them to this law. As with direct effects, where there is a phase shift between the electric current and the corresponding field, induced effects present the same type of phase shift between the magnetic field \mathbf{B} and the induced current. The existence of magnetic monopoles is not predicted by classical electromagnetism or the theory of relativity, and no elementary particle with a magnetic monopole has ever been observed. The discrete law of electromagnetism (16) actually shows that the classical conservation constraint of Maxwell's equations $\nabla \cdot \mathbf{B} = 0$ is not strictly imposed; this would be the case if the celerity c were infinite, which is not the case since $c = c_0 \approx 10^8 \text{ m.s}^{-1}$. As the discrete law shows, the phase shift is of the order of magnitude of $c^2 \approx 10^{15} \text{ s}$. Remember that the weak equivalence principle is only valid to within 10^{-15} . In 1931, Paul Dirac [27] demonstrated its theoretical existence within the framework of quantum physics, but for the time being, classical or relativity theories and quantum physics are irreconcilable on this point, preventing any unification of fundamental laws. G. Lochak [28] gives another interpretation, but this problem remains open for the time being.

In addition to the potential existence of monopoles, the law (16) differs widely from Maxwell's equations in all their versions: classical, tensorial, relativistic in Minkowski space, in potentials. The complex nature of direct and induced currents leads to a drastic reduction of equations and variables to a single variable, the current \mathbf{j} and a single equation (16). Scalar and vector potentials are accumulators, not new variables. The law of motion is non-linear, and the non-linearities are those of the material derivative $\boldsymbol{\gamma} = d\mathbf{j}/dt$; the addition of the Lorentz force is not

necessary, as it is already integrated into the material derivative on the current. The acceleration on the current density reveals a formal Helmholtz-Hodge decomposition, $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$, the sum of a curl-free term and a second divergence-free term. It is therefore possible to establish an alternative formulation of Maxwell's equations, where the true variable is a generalized velocity.

3.3 The discrete law of motion

Newton's second law of dynamics has been reformulated by discrete mechanics in the form $\boldsymbol{\gamma} = \mathbf{h}$ where the intrinsic acceleration $\boldsymbol{\gamma}$ of a particle with or without mass is equal to the sum of the accelerations applied to it; acceleration is a conservative quantity. With a view to eventual unification, it becomes necessary to establish a law in which the main variable and the other unified quantities could take on different meanings depending on the field of physics concerned. The term velocity \mathbf{v} will be used for the main variable, which may refer to fluid velocity, electric current density, heat flux, etc. Table 1 shows the correspondence between the quantities used in discrete mechanics and those classically used in electromagnetism. The quantities in the table are respectively charge density ρ_e , permittivity ε , magnetic permeability μ , electrical conductivity σ , electric potential e and electric current density \mathbf{j} . The longitudinal celerity c_l differs from the transverse celerity c_t , even though in the case of electromagnetism in a vacuum $c_l = c_t = c_0$ is equal to the celerity of light.

\mathbf{v}	ϕ	$\boldsymbol{\psi}$	c_l^2	c_t^2
\mathbf{j}/ρ_e	$(\sigma/\tau_e) e$	$(1/(\mu \tau_m)) \mathbf{B}$	$1/(\varepsilon \mu)$	$1/(\varepsilon \mu)$

Table 1. Correspondence between the quantities of discrete mechanics and those of electromagnetism, where \mathbf{j} is the current density, e the electric potential, \mathbf{B} the magnetic induction, ρ_e the electric charge density, ε the electric permittivity, μ the magnetic permeability and σ the electric conductivity. The quantities $\tau_e = R C$ and $\tau_m = R_m C_m$ are the respective time constants for the evolution of electric and magnetic currents.

Thus, the discrete law of motion, established in fluid mechanics as an alternative to the Navier-Stokes or Euler equations [16, 29], in solid mechanics to replace the Navier-Lamé equation [30], or for heat transfer [26], can be extended to electromagnetism as an alternative to Maxwell's equations. As the derivation of the general law has already been performed and presented in some of these articles, its final form is given directly,

$$\frac{d\mathbf{v}}{dt} = -\nabla(\phi^o - c_l^2 dt \nabla \cdot \mathbf{v}) + \nabla \otimes (\boldsymbol{\psi}^o - c_t^2 dt \nabla \times \mathbf{v}) + \mathbf{h}_s, \quad (19)$$

where ϕ^o and $\boldsymbol{\psi}^o$ are the retarded scalar and vector potentials. The source term \mathbf{h}_s represents a possible acceleration applied on the same segment Γ of the primal structure; it will itself be written in two terms of a Helmholtz-Hodge decomposition. The two potentials are updated respectively from the divergence of the velocity and its primal curl from the expressions,

$$\begin{cases} \alpha_l \phi^o - c_l^2 dt \nabla \cdot (\mathbf{v} - \mathbf{v}^o) \mapsto \phi^o, & (20a) \\ \alpha_t \boldsymbol{\psi}^o - c_t^2 dt \nabla \times (\mathbf{v} - \mathbf{v}^o) \mapsto \boldsymbol{\psi}^o, & (20b) \\ \mathbf{v} - \boldsymbol{\gamma} dt \mapsto \mathbf{v}^o, & (20c) \\ \mathbf{u} - \mathbf{v} dt \mapsto \mathbf{u}^o, & (20d) \end{cases}$$

where the symbol \mapsto denotes the update of the quantity concerned and will take the place of the retarded quantity when the vector equation is next solved. The quantities α_l and α_t are the

restitution factors for longitudinal and transverse waves; for example, $\alpha = 0$ corresponds to a totally dissipative medium. The quantity \mathbf{v}^o is the retarded velocity at time t^o .

The transformation of the law of electromagnetism (16) into a law (19) is not a simple analogy, but a quantification of certain laws of physics derived from the fundamental principle of mechanics. The Table 1 is used to formulate the mathematical problem from the properties of the medium considered and the boundary conditions, and to solve it before returning to the specific variables using the same table. However, it is not necessary to use classical physical quantities, as the solution of the problem (\mathbf{v}, ϕ, ψ) physically represents the result, each unified quantity already having a precise meaning.

The law (19) has certain properties associated with the symmetry observed when the roles of electric and magnetic fields are reversed. According to Noether's theorem [31], these symmetries define properties of invariance in time and space associated with the conservation of certain quantities. Invariance in time defines the conservation of total energy; invariance by rotation in space, the conservation of angular momentum; and invariance by translation, the conservation of momentum. The difference with Noether's theorem is the formulation of the equation in terms of accelerations and its natural Helmholtz-Hodge decomposition. The first two terms on the right-hand side of this equation are locally orthogonal, giving it special structural properties.

3.4 Lorentz acceleration and inertia

The Lorentz force can be derived in different ways, depending on the formalism chosen, the initial Maxwell-Lorentz formalism, the Lagrangian formulation or the space-time formulation. This section shows that Lorentz acceleration results from a non-linear superposition. Inertia is a complex concept in mechanics, involving the non-linear interaction of one field with another or with itself. Maxwell's equations of electromagnetism, originally linear, were supplemented by a force generated by the interaction of a velocity field and a magnetic field; when a charged particle moves at velocity \mathbf{w} in an electromagnetic field, it is subject to the Lorentz force,

$$\frac{d(m\mathbf{v})}{dt} = -q (\mathbf{E} + \mathbf{w} \times \mathbf{B}) \quad (21)$$

where q is the particle's elementary charge in coulomb, \mathbf{w} its velocity, \mathbf{E} the electric field in Volt per meter, \mathbf{B} the magnetic field in Tesla, $\mathbf{q} = m\mathbf{w}$ the momentum where $m = \gamma m_0$ is the relativistic mass, m_0 the mass of the particle at rest and $\gamma = 1/\sqrt{1 - v^2/c_0^2}$ the Lorentz factor. The first term on the right-hand side corresponds to acceleration by the direct field, and the second term to a non-linear response called the magnetic force. The latter deflects the particle in the opposite direction to the electric field gradient, and is used to collimate electron beams in accelerators. If the particle is uncharged ($q = 0$), the actions of an electric and magnetic field on it are zero. If it carries an electric charge, it accelerates and interacts with these fields. In this context, it is not possible to directly transpose the Lorentz force in the discrete approach, as the \mathbf{B} field is attached to each facet and directed along its \mathbf{n} oriented normal, whereas classical electromagnetism defines all quantities per point in the continuous medium approach. Moreover, the \mathbf{B} field is not conservative *a priori*. In classical mechanics, the velocity of the particle \mathbf{w} is defined per point in a three-dimensional space, as is the induction field \mathbf{B} , and the vector product $\mathbf{w} \times \mathbf{B}$ must then be projected into a system of orthogonal axes.

In fact, the law of discrete motion (19) is representative of both fluid motion [16] and electromagnetic currents when expressed from the unified variables of the Table 1. The velocity of a particle is denoted \mathbf{w} and the velocity of the unified electric current $\mathbf{v} = \mathbf{j}/\rho_e$. In discrete mechanics, inertia takes on another representation [32] and the material derivative is written:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{v}|^2}{2} \mathbf{n} \right), \quad (22)$$

when only one phenomenon is considered, fluid mechanics or electromagnetism. When coupled, however, the two currents flow along the same segment at velocities \mathbf{w} and \mathbf{v} . While the linear terms are the same, the inertia corresponding to the interaction between the two phenomena includes additional terms that need to be specified. Vector calculus shows that the sum of two vectors squared can be written,

$$(\mathbf{w} + \mathbf{v})^2 = |\mathbf{w}|^2 + 2 \mathbf{w} \cdot \mathbf{v} + |\mathbf{v}|^2, \quad (23)$$

where appears the scalar product $2 \mathbf{w} \cdot \mathbf{v}$; in this case, the velocities are carried by the same segment and this scalar product becomes a simple product; moreover, the vectors themselves are scalars associated with oriented segments. If we omit the inertia terms specific to each of the two phenomena, the inertia corresponding to the coupling terms can be written as a Helmholtz-Hodge decomposition,

$$\kappa_L = \nabla (\mathbf{w} \cdot \mathbf{v}) - \nabla \otimes (\mathbf{w} \cdot \mathbf{v} \mathbf{n}), \quad (24)$$

where the Lorentz scalar potential $\phi_L = \mathbf{w} \cdot \mathbf{v}$ linked to the vertices of the primal structure and the vector potential $\psi_L = \mathbf{w} \cdot \mathbf{v} \mathbf{n}$ is located on a facet and oriented along \mathbf{n} . This is the discrete form of Lorentz acceleration. The first term can also be written as $\mathbf{v} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{v}$ or $(\mathbf{v}/\mathbf{w}) \cdot \nabla (|\mathbf{w}|^2/2) + (\mathbf{w}/\mathbf{v}) \cdot \nabla (|\mathbf{v}|^2/2)$ expresses the advection of a current by the fluid velocity and its opposite equivalent in the longitudinal direction oriented by the vector \mathbf{t} . The second term is also an advection term in the \mathbf{n} direction. Proper inertia terms already represent the advection of a flow by itself.

3.5 Conservation of charges

The conservation of mass in mechanics, the conservation of electric charges and possibly magnetic charges are not necessary laws in discrete mechanics. These laws are already implicit in the unified law of motion.

Let's consider first the conservation of mass in mechanics established for a volume Ω that we follow during its motion; in the absence of internal sources, the density ρ satisfies the equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (25)$$

which completes the Navier-Stokes and Euler equations. These equations, based on the conservation of momentum $\mathbf{q} = \rho \mathbf{v}$ do not intrinsically conserve mass; the law of conservation of mass or density is absolutely necessary under these conditions. In discrete mechanics, the conservation of acceleration $\boldsymbol{\gamma} = d\mathbf{v}/dt$ expresses both the conservation of mass and the conservation of angular momentum. Acceleration itself is energy per unit mass and length; the equivalence of mass and compressive energy derived from the theory of relativity means that the concept of mass can be completely abandoned in the formulation of the law of discrete motion [16]. Newton's second fundamental law is revisited and becomes an equality between accelerations, $\boldsymbol{\gamma} = \mathbf{h}$, the intrinsic acceleration of a particle with or without mass or of a material medium is equal to the sum of the accelerations imposed on it.

In electromagnetism, the conservation of electric charge is written,

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (26)$$

where ρ_e is the electric charge density and \mathbf{j} the current density.

However, the electric charge or its surface density ρ_e is a scalar which must be transported in an Eulerian view and the partial derivative is then replaced by the total derivative,

$$\frac{d\rho_e}{dt} + \rho_e \nabla \cdot \left(\frac{\mathbf{j}}{\rho_e} \right) = \frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{j} + \left(\frac{\mathbf{j}}{\rho_e} \right) \cdot \nabla \left(\frac{1}{\rho_e} \right) = 0, \quad (27)$$

where the last term is similar to the displacement current of the electric charge introduced by Maxwell to ensure consistency with Ampère's law in the variable regime. The conservation of electric charge equations in classical electromagnetism (26) and in discrete electromagnetism (27) differ on this point. In fact, this difference has little impact on the formulation presented because, as in mechanics, this conservation law is not necessary, it is already implicitly contained in the law of conservation of current (16). Thus ρ_e is no longer a variable in the problem, which makes it possible to restrict the total number of unknowns in electromagnetism. In the unified version, the quantity $\mathbf{v} = \mathbf{j}/\rho_e$ is identified as a velocity.

The conservation of magnetic charge ρ_m could also be evaluated from the magnetic field, since the constraint on the induced field is no longer guaranteed, $\nabla \cdot \mathbf{B} \neq 0$. The law of conservation of magnetic charge ρ_m will take the same form as the expression (27). Thus, in all cases, these conservation laws are excluded from the discrete formulation but what they express is already integrated into the unified general law of motion (19).

3.6 Equations in terms of potentials

Like Maxwell's equations, discrete mechanics can lead to a different form by applying the divergence operator to the equation (19). The rotation term disappears and:

$$\nabla \cdot \boldsymbol{\gamma} = \nabla \cdot \left(\frac{d\mathbf{v}}{dt} \right) = \frac{d}{dt} (\nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{v})^2, \quad (28)$$

assuming c_l constant; the result is a scalar potential equation,

$$\nabla^2 \phi = -\frac{d}{dt} (\nabla \cdot \mathbf{v}) - (\nabla \cdot \mathbf{v})^2, \quad (29)$$

but updating ϕ from the same equation system allows us to derive $c_l^2 \nabla \cdot \mathbf{v} = -d\phi/dt$ and, replacing:

$$\nabla^2 \phi - \frac{1}{c_l^2} \frac{d^2 \phi}{dt^2} = \left(\frac{1}{c_l^2} \frac{d\phi}{dt} \right)^2. \quad (30)$$

Now let's apply the primal curl operator to the acceleration $\boldsymbol{\gamma}$:

$$\nabla \times \boldsymbol{\gamma} = \nabla \times \left(\frac{d\mathbf{v}}{dt} \right) = \frac{d}{dt} (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{v})^2 \mathbf{n}. \quad (31)$$

Considering the identity $c_l^2 \nabla \times \mathbf{v} = -d\boldsymbol{\psi}/dt$ and taking into account the equality $\nabla \otimes \boldsymbol{\psi} = \nabla(\nabla \cdot \boldsymbol{\psi}) - \nabla^2 \boldsymbol{\psi}$, as the pseudo-vector $\boldsymbol{\psi}$ is solenoidal $\nabla \cdot \boldsymbol{\psi} = 0$, we can simplify the equation obtained and recapitulate the formulation in potentials:

$$\left\{ \begin{array}{l} \nabla^2 \phi - \frac{1}{c_l^2} \frac{d^2 \phi}{dt^2} = \left(\frac{1}{c_l^2} \frac{d\phi}{dt} \right)^2, \end{array} \right. \quad (32a)$$

$$\left\{ \begin{array}{l} \nabla^2 \boldsymbol{\psi} - \frac{1}{c_l^2} \frac{d^2 \boldsymbol{\psi}}{dt^2} = \left(\frac{1}{c_l^2} \frac{d\boldsymbol{\psi}}{dt} \right)^2 \mathbf{n}. \end{array} \right. \quad (32b)$$

If we compare Maxwell's equations (7) in terms of potentials e and \mathbf{A} with those of the system (32) on ϕ and $\boldsymbol{\psi}$, we find a similarity, but the second derivatives in time become material second derivatives. The difference corresponds to the advection of quantities at velocity \mathbf{v} ; these inertial effects, not taken into account in Maxwell's equations, may be second-order, but it cannot be denied that every medium or elementary particle is subject to acceleration due to variations in its velocity. Indeed, all media possess a finite compressibility characterized by the celerity c_l or

c_t ; these potential equations translate the exchange of compression energy from one point of a conductor to another. The physical significance of this phenomenon is best illustrated in the equation of motion (19), where the equilibrium potentials ϕ^o and ψ^o represent instantaneous exchange accumulators with their respective deviators $dt c_t^2 \nabla \cdot \mathbf{v}$ and $dt c_t^2 \nabla \times \mathbf{v}$. At first order, in a vacuum, if we neglect the effects of inertia, we find expressions for potentials derived directly from Maxwell's equations. Another difference is that in discrete mechanics ϕ and ψ are expressed in the same fundamental units, length and time, and above all are potentials of the same quantity, acceleration.

Indeed, the application of operators to an equation, whatever they may be, inevitably leads to a loss of information; this degradation is due to the elimination of certain terms from the initial equation. The intrinsic properties of discrete vision verify the two essential equalities $\nabla \times \nabla \phi = 0$ and $\nabla \cdot (\nabla \otimes \psi) = 0$, but for all that it's important to keep both contributions in the same equation of motion. By solving the equation of motion, we obtain the acceleration and velocity potentials in a single step. The retarded potentials ϕ^o and ψ^o represent long-term persistent quantities, such as permanent magnetization or hysteresis effects.

3.7 A relativistic equation

The equation of discrete motion (19) applied to fluid flow, stress in solids, and heat transfer on small time and space scales [26] is also valid for light propagation. The velocities c_l and c_t are then equal to the celerity of light c_0 . If we set $\mathbf{v} = \mathbf{v}^o + \boldsymbol{\gamma} dt$ and $\mathbf{u} = \mathbf{u}^o + \mathbf{v} dt$ where \mathbf{u} is the displacement field, we find a propagation equation:

$$\frac{d^2 \mathbf{u}}{dt^2} = -\nabla (\phi^o - c_0^2 \nabla \cdot \mathbf{u}) + \nabla \times (\psi^o - c_0^2 \nabla \times \mathbf{u}) + \mathbf{h}_s, \quad (33)$$

where $\boldsymbol{\gamma} = d^2 \mathbf{u} / dt^2$ is a material second derivative containing non-linear terms in an Eulerian view. The source term itself can be expressed as a Helmholtz-Hodge decomposition, $\mathbf{h}_s = \nabla \phi_s + \nabla \otimes \psi_s$ where ϕ_s and ψ_s are the potentials of \mathbf{h}_s . The discrete equation of motion thus appears as an alternative to the Maxwell-Lorentz equations including directly applied accelerations, and in particular gravitational acceleration, whether or not the particle's mass is zero.

The symmetrical form of the vector equations (33) shows a retarded time ([6]) at the origin of the Jefimenko equations [33, 5], which justifies the existence of a causal link between electric and magnetic fields. This equation concerns only the displacement \mathbf{u} , a scalar quantity on the oriented segment Γ ; updates are used to explicitly calculate the retarded potentials ϕ^o and ψ^o , the energies per unit mass of the direct and induced effects. The only unknown in the single equation (33) is velocity, or in this case, displacement. Considering the celerity of light c_0 as an invariant, we can transform the operators by the classical formula of vector analysis:

$$\nabla^2 \mathbf{u} = \nabla \nabla \cdot \mathbf{u} - \nabla \times \nabla \times \mathbf{u}, \quad (34)$$

to obtain an equation where the second member is solely a function of the retarded potentials without the source terms:

$$\frac{d^2 \mathbf{u}}{dt^2} - c_0^2 \nabla^2 \mathbf{u} = -\nabla \phi^o + \nabla \otimes \psi^o. \quad (35)$$

This equation is an alternative to Maxwell's equations; it contains the Lorentz force by considering the non-linear terms of the material derivative; in a Lagrangian description of motion, the material derivative becomes a partial derivative. By abandoning the concept of a memory model and deleting the other terms on the right-hand side of the equation (35), the equation becomes a vacuum propagation equation, an Alembertian equation, $\square \mathbf{u} = 0$. In the general case, the equation is non-linear and the second member of (35) is none other than the retarded

displacement \mathbf{u}^o , hence:

$$\frac{1}{c_0^2} \frac{d^2 \mathbf{u}}{dt^2} - \nabla^2 (\mathbf{u} - \mathbf{u}^o) = 0. \quad (36)$$

The Lagrangian form $\square \mathbf{u} = 0$ then allows us to attribute relativistic character to the law of discrete motion; this d'Alembert equation and, by extension, the equation (19) is indeed invariant under a Lorentz transformation. The corresponding proof has already been given in [34].

3.8 The potential existence of monopoles

The question of whether all vectors, with or without physical significance, can be decomposed into a part with zero divergence and another part with curl-free? The answer is yes, but in the general case this decomposition has no particular physical significance, e.g. the scalar and vector potentials of velocity \mathbf{v} are not those of acceleration $\boldsymbol{\gamma} = -\nabla\phi + \nabla\psi$. While $\boldsymbol{\gamma}$ is an absolute quantity, velocity has only a relative significance; their respective potentials do not have the same importance in physics. Requiring a field to be curl-free or divergence-free has important consequences for the meaning of the laws of physics. The transposition of the induction field \mathbf{B} into a unified quantity is $\boldsymbol{\psi}^o$, the potential vector of acceleration, and its definition is written:

$$\boldsymbol{\psi}^o = - \int_0^{t^o} c_0^2 \nabla \times \mathbf{v} d\tau. \quad (37)$$

There is no reason to assume that $\boldsymbol{\psi}^o$ is a curl; the accumulation over time (37) of this potential implies that it is retarded like those of Liénard-Wiechert [6]. The limiting case considered for $c \rightarrow \infty$ induces that magnetic charge accumulation is not possible, the potential vector $\boldsymbol{\psi}^o = -c_t^2 dt \nabla \times \mathbf{v}$ or, in classical notation, $\mathbf{B} = \nabla \times \mathbf{A}$. Since $\boldsymbol{\psi}^o$ is not a curl, this retarded potential opens the way to the possible existence of monopoles. The intrinsic acceleration $\boldsymbol{\gamma}$ of the particle or material medium is not temporally equal to the accelerations imposed by the left-hand side of the law of motion; there is a time lag due to the non-infinite celerity of light. Discrete mechanics corresponds to a local model in which information flows from cause to effect between the intrinsic acceleration and the two orthogonal terms on the right-hand side of the law of motion $\boldsymbol{\gamma} = -\nabla\phi + \nabla\psi$. Another important difference concerns the relationship between scalar and vector potentials. In discrete mechanics, they are expressed from the same velocity as $\phi^o \propto \nabla \cdot \mathbf{v}$ and $\boldsymbol{\psi}^o \propto \nabla \times \mathbf{v}$, which is not the case for e and \mathbf{A} , which are coupled using the Maxwell-Faraday equation in the form $\mathbf{E} = -\nabla e - \partial \mathbf{A} / \partial t$.

So, assigning a stationary function to the magnetic field, for example $\mathbf{B} = q/r^2 \mathbf{r}$, is a possibility but is not obvious *a priori*. Even if the mathematical operations are legal, the physical implications are difficult to interpret. Discrete mechanics requires (i) that we abandon mass m and electric charge density ρ_e and magnetic charge density ρ_m , (ii) that we interpret electric and magnetic fields as one and the same current field, and (iii) that only a vector equation on this field can ensure the coupling between these two contributions.

The physical interpretation of these discrete mechanical concepts is illustrated by the diagram in Figure 1. The segment Γ is traversed in time dt over a distance equal to the discrete horizon $dh = c_0 dt$ by the direct current under the effect of a potential gradient $(\phi_b - \phi_a)/dh$. Similarly, the induced current represented by $\nabla \otimes \boldsymbol{\psi}$ does not instantaneously cross the contour of Δ ; during this time, at least the $\boldsymbol{\psi}$ field is not a curl-free field. The analogy with fluid mechanics can be used, noting that the rotation of a viscous fluid in a rotating cylindrical cavity only involves all the fluid it contains in a rigid rotation for a time that depends on the conditions of the experiment [32]. The velocity field is constant in rotation, $\nabla \times \mathbf{v} = Cte$ only after a long delay. The laws of physics seem to show a fundamental coherence between apparently disparate phenomena. The extension of the laws of mechanics to electromagnetism seems to confirm this

similarity. The link between discrete mechanics and quantum physics has yet to be made, but the possible existence of magnetic monopoles by $\nabla \cdot \mathbf{B} \neq 0$ opens up this potential way.

3.9 A drastic reduction in the number of variables

The principle of parsimony applied to physics consists in minimizing the number of unknowns and equations without losing the information or representativeness of the phenomena. The development phase of physics, and in particular electrostatics and magnetism, led to the creation of specific laws, followed by a phase of assembling these laws by coupling. This is what Maxwell achieved by introducing a dynamic vision. However, the number of physical quantities was not reduced, nor was the number of equations. In a three-dimensional space, the number of unknowns is six, the three components of the electric and magnetic fields, \mathbf{E} and \mathbf{B} but eight equations, the three components of Ampère's and Faraday's equations and Gauss's two equations. Under the right conditions, certain redundancies can be eliminated. Some vector identities, $\nabla \cdot \nabla \times \mathbf{B} = 0$ and $\nabla \cdot \nabla \times \mathbf{E} = 0$ also allow us to reduce the system to six equations with six unknowns. However, if we rewrite them in terms of vectors and scalar potentials, the equations are underdetermined due to the gauge fixation. Generally speaking, there are still difficulties in interpreting Maxwell's initial set, and problems of consistency in the boundary conditions to be adopted on each of the unknowns. Of course, we must add the impossibility of explaining certain phenomena in quantum physics, notably those involving photons.

The discrete electromagnetic equation is unique, and the only unknown is the unified velocity \mathbf{v} . The scalar and vector potentials of the current acceleration, ϕ and $\boldsymbol{\psi}$ respectively, are then updated from the divergence of the total velocity $\nabla \cdot \mathbf{v}$ and its primal curl $\nabla \times \mathbf{v}$. The retarded acceleration potentials ϕ^o and $\boldsymbol{\psi}^o$ are the energies per unit mass accumulated over time as a result of transfers between the direct and induced fields. Since the frame of reference of discrete mechanics is local, references to the components of a vector, tensor or quadri-vector no longer have the same meaning as in traditional field theory.

An essential advantage of discrete mechanics is that it substantially reduces the number of fundamental units needed to describe unknowns and potentials. Only two fundamental units are used in the unified version, those of length and time, namely the meter and the second in the S.I. international system. The very many definitions attached to electrical and magnetic phenomena are no longer useful in this context, and only the velocities c_l and c_t assimilated to the celerity of light c_0 in vacuum, the unified velocity expressed in ms^{-1} and the potentials, energies per unit of mass expressed in m^2s^{-2} , are needed to model a problem of electromagnetism.

3.10 Differences and convergences

Maxwell's equations are partial differential equations in time and space; it is possible to reformulate them in the language of exterior algebra if we know how to determine the analogues of derivation operations on differential forms. They can also be found in the form of tensors, e and \mathbf{A} potentials, or in a form used in special relativity, quadri-vectors. The discrete equation of motion (19) is very different from these classical forms of electromagnetism. The key point is that this equation does not call into question previous results, but we can expect significant differences in form, as the discrete formulation is based directly on Ohm's and Ampère's laws.

A fundamental difference lies in the treatment of instationarity in the equations. In Maxwell's equations, they are associated with temporal variations in the electric potential $\partial\mathbf{E}/\partial t$ in the Maxwell-Ampere equation and in the induced magnetic field $\partial\mathbf{B}/\partial t$ in the Maxwell-Faraday equation; they appear as quantities *ad-hoc* that enable us to represent the evolutions of these quantities but also of all those to which they are associated. In discrete mechanics, it is the acceleration of the current $d\mathbf{j}/dt$ that generates all the temporal evolutions of the variables; from a physical point of view, it is the variations in acceleration that generate the strong coupling

between \mathbf{E} and \mathbf{B} . These fields therefore become solenoidal for \mathbf{B} , $\nabla \cdot \mathbf{B} = 0$, and irrotational for \mathbf{E} , $\nabla \times \mathbf{E} = 0$ or $\mathbf{E} = -\nabla e$ and $\mathbf{B} = \nabla \times \mathbf{A}$, the induced magnetic field \mathbf{B} does not accumulate magnetic charges. The electric field \mathbf{E} is a polar vector defined by the gradient of the electric potential $\mathbf{E} = \nabla e$, while the induced magnetic field \mathbf{B} is a pseudo-vector associated with the normal \mathbf{n} of the primal topology; these two fields are not expressed in the same units. In discrete mechanics or discrete electromagnetism, the vectors $\nabla\phi$ and $\nabla \otimes \psi$ are two true vectors carried by Γ whose sum is equal to the acceleration γ , the three vectors being expressed in the same units.

Whether or not electric and magnetic fields are time-dependent, the gauge conditions associated with Maxwell's equations are quite complex to define and apply. In discrete mechanics, the two fields $\gamma_\phi = -\nabla\phi$ and $\gamma_\psi = \nabla \otimes \psi$ are orthogonal and therefore exchange nothing directly, these two accelerations γ_ϕ and γ_ψ are independent. The exchange mechanism is in fact complex: when one of the two fields is no longer in equilibrium with the other, it's the γ acceleration that varies, redistributing electric currents into magnetic fields, or vice versa. If $\gamma = 0$ two orthogonal fields can only be locally equal to a constant.

Another remarkable difference is the absence of inertia terms in Maxwell's equations; in a discrete medium, they are written as $\nabla(|\mathbf{v}|^2/2) - \nabla \otimes (|\mathbf{v}|^2/2\mathbf{n})$. It should be noted that the dual curl operator does not exist in a continuous medium; it is similar to the gradient operator but in directions orthogonal to it in two dimensions of space; in a discrete medium, it corresponds to the circulation of an axial vector along the dual contour Δ of Figure 1. Inertia exists whether or not the medium has mass, as in the case of photons for light. Velocity variations due to inertia - those of electrons, photons and matter - can be of very different values in different cases. It should be noted that the Navier-Lamé equations do not include inertial terms, which are practically negligible in the usual cases, but this does not mean that they do not exist; the displacement of the medium must always be accompanied by inertial effects if there are spatio-temporal variations in velocity.

By combining the Maxwell-Gauss and Maxwell-Ampère equations, we obtain the law of conservation of electric charge density $\partial\rho_e/\partial t + \nabla \cdot \mathbf{j} = 0$; this equation is identical in fluids and solids. Although the treatment of instationarity is different, the discrete mechanics equation has the general characteristics of Maxwell's original equations. The need to unify the equations of physics requires that an equation be formulated on the basis of a single variable - the velocity of the fluid, solid or electric current. In the absence of variations in electric and magnetic fields, Maxwell's equations decouple and the magnetostatic equations give rise to different treatments, with the Coulombian approach favoring the scalar potential e and the Amperian approach favoring the vector potential \mathbf{A} . In discrete mechanics, since the two terms are orthogonal, they must also be separately equal to zero. Similarly, in electrostatics, an electric potential can be defined in the same way as in discrete mechanics.

4 Conclusion

The proposed formulation is already an alternative formulation of the Navier-Stokes, Euler and Navier-Lamé equations; in these cases, the celerity is that of acoustic waves in fluids or solids. The structure of this alternative equation is very close to the concepts developed from Maxwell's reference frame illustrated in Figure 1; in the case of electromagnetism, where direct and induced currents are naturally intertwined, the formulation coupling longitudinal waves to transverse waves is entirely relevant.

The originality of this proposal lies in the derivation of an electromagnetic equation based on Ampère's and Ohm's primary laws, postulating the conservation of current density acceleration. The transposition of the law of discrete mechanics, where it is acceleration, the material derivative of velocity, that is conserved, led to the recovery of the structure of a Helmholtz-Hodge

decomposition, a sum of a component curl-free and another divergence-free. Another striking result concerns the non-conservation of magnetic flux, the induction field is not *a priori* at zero divergence, so this assumption turns out to be unnecessary. It is not true at very low time constants and, like many laws of physics interpreted on bases where phenomena are slow, becomes erroneous under very different conditions. This observation paves the way for the possibility of one day proving the existence of magnetic monopoles. Abandoning the constraint of magnetic flux conservation makes the law of motion totally symmetrical, with the roles of electric and magnetic fields interchangeable.

The principle of parsimony applied to the derivation of a discrete law of electromagnetism has enabled a drastic reduction in the quantities needed to model these phenomena. If this law is validated and accepted, its main interest will probably lie in reducing the number of variables and also the number of fundamental units needed to describe them; only two fundamental units are required, those of length and time. Contrary to the postulate of the theory of special relativity, these two concepts are strictly disjoint; indeed, apart from a few special cases such as the celerity of light in a vacuum, the velocity of a particle or a material medium is not equal to the celerity of the medium.

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Appendix - Test cases

The cases dealt with in this section do not constitute a validation of the discrete electromagnetism model, but simply serve to illustrate the founding aim of this approach by showing a few examples of applications. It is only by multiplying the number of test cases that we can be sure of the representativeness of the physical model and, ultimately, of its completeness.

A - Magnetic field created by a wire of infinite length

This very simple case corresponds to a stationary phenomenon derived from magnetostatics; a current I flows in an electrical conductor of infinite length and very small radius; it has an electrical conductivity σ and the permeability of the external medium is equal to that of vacuum μ_0 . The degeneracy of the equation of motion (19) gives the magnetostatic equation in terms of potentials:

$$-\nabla\phi + \nabla \otimes \boldsymbol{\psi} = 0. \quad (38)$$

The two quantities $\phi(x)$ and $\boldsymbol{\psi}(r)$ are functions of different variables, and the two fields of equation (38) are orthogonal. Stokes' theorem and the fundamental theorem of the integral mean value allow us to write:

$$\int_0^{2\pi} \boldsymbol{\psi} \cdot \mathbf{t} \, dl = \int_a^b \nabla\phi \, dx. \quad (39)$$

With a constant, zero in this case since the magnetic field lines are closed, the solution to this problem is as follows:

$$\boldsymbol{\psi} \cdot \mathbf{n} = \frac{(\phi_b - \phi_a)}{2 \pi r}. \quad (40)$$

By replacing the potentials by the usual electromagnetic variables and noting that $(e_b - e_a) = I/\sigma$ we find the result classically obtained by Biot and Savart's law in the form of the \mathbf{n} component of the magnetic field:

$$B(r) = \frac{\mu_0 I}{2 \pi r}. \quad (41)$$

What's interesting about the equation (38) and its solution (40) involving the two potentials ϕ and $\boldsymbol{\psi}$, is that it's expressed using just two fundamental units, time and space. In electromagnetism, the equations and their solutions involve the other fundamental units, mass M and intensity A . From the point of view of the unification of the laws of physics, this would remain a difficulty; indeed, it is not an analogy that is sought, but a single equation.

In the general context of electromagnetism, it is the discrete equation (19) that must be integrated directly in space and time; it allows us to find the solution of the velocity \mathbf{v} by knowing its value at the previous time t^o . Any perturbation of the magnetic or electric field, or any variation in a source term or boundary condition, generates an acceleration $\boldsymbol{\gamma}$ which extends to the two components of its Hodge-Helmholtz decomposition. Understanding the behavior of this discrete equation is highly complex, and it would be illusory to attempt to explain it with trivial reasoning. It's enough to find the results obtained classically with the previously established laws of physics, which doesn't mean that the equations have to be identical. For example, the discrete mechanics equation is significantly different from the Navier-Stokes equation, but the solutions are the same. Although different from one of Maxwell's forms, the equation of discrete motion must allow us to find the same results as in electromagnetism. The formalism presented to unify certain equations of physics cannot, of course, escape the concepts of general relativity introduced by Einstein to integrate those of Newton and many others before him, notably Galileo.

B - Magnetic field around a permanent magnet

The first example of magnetostatics involves making a permanent magnet and calculating the residual magnetic field. Equation (19) is a continuous memory model that includes the possibility of representing the permanent magnetization of a ferromagnetic metal. Let's consider the case of a magnetic field \mathbf{B} created by a permanent magnet \mathbf{M} . The induced magnetic field $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ where \mathbf{H} is the excitation magnetic field, \mathbf{M} is the magnetization field and μ_0 is the vacuum permeability; in the case where $\mathbf{H} = 0$, the induction field \mathbf{B} is the residual flux density.

Magnetization is achieved by applying a magnetic field to a piece of ferromagnetic material. The system of equations (19) is solved by maintaining a constant field \mathbf{B} and updating the equilibrium vector potential $\boldsymbol{\psi}^o$ with the transverse restitution factor $\alpha_t = 1$. At the end of this phase, the magnet is permanent and $\boldsymbol{\psi}^o$ maintains a constant field \mathbf{M} inside the magnet; the value of the field $\boldsymbol{\psi}^o$ can be obtained with conventional electromagnetic variables. The second phase of the experiment consists in studying the magnetic field induced \mathbf{B} outside the magnet in a vacuum or in air, always maintaining (19) a field $\boldsymbol{\psi}^o = y \mathbf{e}_z$ generating a velocity equal to $\mathbf{v} = \nabla \otimes \boldsymbol{\psi}^o = 1 \mathbf{e}_x$ inside the solid domain. The scalar potential ϕ^o updated over time by the velocity divergence is not zero, but it has no influence on the magnetic field; this velocity field becomes at zero divergence (Coulomb gauge) both inside and outside the magnet. Figure 2 shows the magnetic field generated by a permanent magnet in a vacuum. This $\boldsymbol{\psi}$ field has zero divergence and corresponds to the field that gave rise to it if the material is perfectly remanent.

We can see that the magnetic flux current lines $\boldsymbol{\psi}^o$ and the equipotentials of ϕ^o are orthogonal.

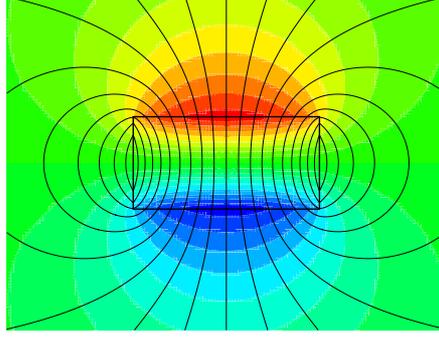


Figure 2. A magnet made of ferromagnetic material, represented by the rectangle, produces a magnetic field on the outside. The field imposed in the magnet is equal to $\psi^o = y \mathbf{e}_z$. The magnetic field in the entire domain is represented by a color gradient, and equipotentials by lines.

C -Induced currents in a cylindrical conductor

When medium and high-frequency alternating currents flow in a conductor, they exhibit a skin effect. The skin effect is due to the opposing eddy currents induced by the changing magnetic field resulting from these alternating currents. The highest current density is found at the surface of the conductor. In good conductors such as metals, the skin depth is given by:

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}, \quad (42)$$

where μ is the magnetic permeability of the material, σ is the conductivity of the conductor and $\omega = 2 \pi f$ the angular frequency of the current.

Let's consider the case of a cylindrical conductor of indefinite length and radius R subjected to an alternating current of frequency f . The solution to this problem is obtained from Maxwell's equations, where the time-dependent terms are preserved. The current density has three components in a cylindrical orthogonal coordinate system, but only the component along z is non-zero, $\mathbf{I} = (0, 0, I_z)$. Assuming $\nu = 1/\mu \sigma$, the equation reduced to one component of the Laplacian vector is written:

$$\frac{\partial I_z}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left(\nu r \frac{\partial I_z}{\partial r} \right). \quad (43)$$

For a temporal solution in the general case, equation (43) can be solved by separating the time and space variables to give a solution of the form:

$$I_z(z, t) = \sum_{n=1}^{\infty} [A_n J_0(k_n r) + B_n K_0(k_n r)] \exp\{-k_n^2 t\}, \quad (44)$$

where the families of constants (A_n, B_n, k_n) are determined by the boundary conditions and the initial condition. In the established periodic regime, it is possible to pose $I_z(r, t) = \Re\{\underline{I}_z(r) \exp(j\omega t)\}$ and the solution is then:

$$\underline{I}_z(r) = \underline{A} J_0(kr) + \underline{B} K_0(kr), \quad (45)$$

where \underline{A} and \underline{B} are complex constants to be determined from the average current to be imposed to obtain a surface current at $r = R$ equal to I_M and the boundary condition at $r = 0$. In the case of the cylindrical conductor, the solution is as follows:

$$I_z(r) = I_M \frac{J_0(kr)}{J_0(kR)}, \quad (46)$$

The electromagnetic quantities $1/(\varepsilon \mu)$ are replaced here by $\nu = dt c_t^2$. Non-linear inertial terms are not included in this presentation, but they exist even though they are not present in Maxwell's equations. This equation, deduced from the vector equation of the system (19), can be solved in time or frequency space. The numerical solution of this problem is sought directly from the discrete equation of motion (19) without any gauge condition, as it allows us to obtain the solution of the problem, the current density \mathbf{v} and the magnetic field $\boldsymbol{\psi}$, while maintaining the current field \mathbf{v} at zero divergence $\nabla \cdot \mathbf{v} = 0$.

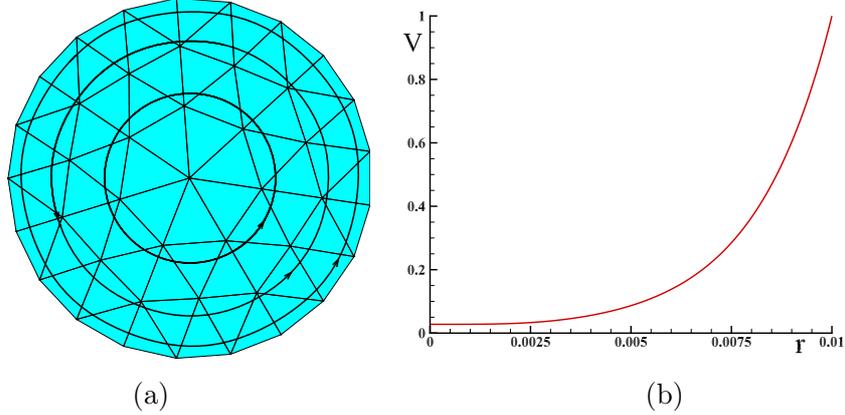


Figure 3. (a) Magnetic field current lines for an unstructured 63 coarse mesh based on regular triangles. (b) Reduced rms current density induced by an alternating electric current of frequency f in a cylindrical conductor.

The simulation was carried out in the time regime, and the effective current density was calculated over a large number of periods. The simulation parameters are $\nu = 1/(\mu \mu) = 10^{-2}$, $f = 1000$, $R = 10^{-2}$ and $dt = 10^{-6}$. For these values, the penetration depth is $\delta = 1.78 \cdot 10^{-3} m$. No quantitative comparison has been made, but convergence in time and space should be of order two, as for all other simulations performed with this formulation. Simulations can be carried out in 1D, in (r, z) , 2D in (r, θ) , 3D in structured or unstructured meshes (polygonal, polyhedral) whose cells have any number of flat faces. The vector potential field $\boldsymbol{\psi}^o(r)$ in Figure 3a represents the induced magnetic field \mathbf{B} at zero divergence in the (r, θ) plane of a cylindrical conductor. The unstructured mesh of coarse regular triangles is used to represent the magnetic flux lines. The rms current (velocity) in the conductor is associated with the mean magnetic field $\boldsymbol{\psi}^o$; the equation corresponding to the steady state is written:

$$\nabla \times (\boldsymbol{\psi}^o - \nu \nabla \times \mathbf{v}) = 0. \quad (47)$$

Figure 3b shows the skin effect, the non-uniform distribution of current density for a moderate-frequency alternating current.

D - Solenoid coil

The N turns of a cylindrical coil of axis z , radius R and length L are traversed by a current of intensity I creating a magnetic field \mathbf{H} whose component on θ depends on the variables (r, z) . The aim is to find the magnetic field and the electric field from the system of equations (19) in the framework of unified variables transposing the magnetic field \mathbf{H} into a vector potential $\boldsymbol{\psi}$ and the electric field defined by the electric potential e transposed into a scalar potential ϕ .

The current in the coil is represented by a constant source term S_b on a rectangle in the plane (r, z) of length $L = 0.1$ and area $\mathcal{A} = 10^{-3}$. The imposed source term is equal to $S_b = 10^3 \mathbf{e}_z$ which enables us to calculate a parameter called δ from the current intensity and the number of turns of the coil, i.e. $\delta = N I/2 L = 125$. This value is associated with the value of the magnetic field linearly dependent on r created in the coil of mean radius $R = 0.025$. Figure 4 shows the axisymmetric solution of the component on θ of the magnetic field $\psi(r, z)$ and the electric potential $\phi(r, z)$ around the coil. The boundaries of the domain have been chosen far enough from the coil to obtain a coil-independent solution.

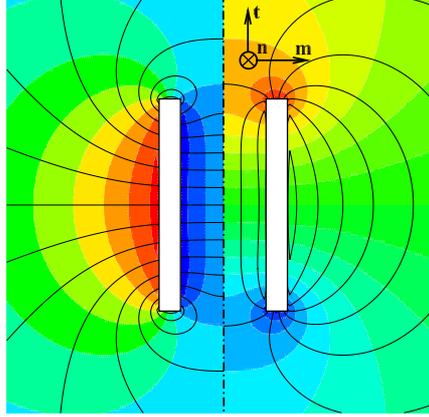


Figure 4. Cylindrical electromagnet coil represented in the (r, z) plane orthogonal to the unit vector \mathbf{n} such that $\mathbf{n} = \mathbf{t} \times \mathbf{m}$. The electric field \mathbf{E} is mainly oriented along \mathbf{m} , the magnetic field \mathbf{B} (ψ) along \mathbf{n} and the current density \mathbf{j} (\mathbf{v}) along \mathbf{t} .

The solution converges in a few iterations to obtain a current field \mathbf{v} with zero divergence at machine accuracy. At all points, the orthogonality of the electric and magnetic flux lines is verified. As expected, in the coil, the magnetic field is practically uniform along z and the electric potential varies linearly; to a first approximation, their evolutions can be written as:

$$\begin{cases} \phi^o(z) = -\delta z, & (48a) \\ \psi(r) = \nu \nabla \times \mathbf{v} = \frac{\delta}{2} r \mathbf{e}_\theta. & (48b) \end{cases}$$

Throughout the domain, the stationary solution (ϕ, ψ) depends on (r, z) . It satisfies the following equation:

$$-\nabla \phi^o - \nabla \otimes (\nu \nabla \times \mathbf{v}) = 0, \quad (49)$$

and the solution obtained by the formalism presented is, once transposed, identical to that of the classical equations of electromagnetism. However, several fundamental conceptual differences appear as soon as the equations are derived. The case presented or similar cases are generally associated with the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ where \mathbf{A} is the potential vector of the magnetic field. Here, the constraint $\nabla \cdot \mathbf{v} = 0$ is not strictly guaranteed, the order of magnitude of $\nabla \cdot \mathbf{v}$ depends on the celerity of the wave c_l ; even if $c_l = c_0$ the celerity of light in vacuum, the constraint is not imposed *a priori* but obtained in the course of the resolution. Similarly, the generalised magnetic field $\psi^o = \nu \nabla \times \mathbf{v}$ is in fact of zero divergence since ν is a constant on each facet of the primitive geometric topology but here it is the vector potential of the acceleration. All the non-linear terms in the equation of motion have been retained in the solution, even though they are negligible in the proposed test case. Finally, the solution of an electromagnetism problem in

the discrete context is always represented by the gradient of a potential ϕ^o and the dual curl of a vector potential ψ^o , two orthogonal fields on the whole geometric topology.

E - A case of electromagnetic levitation

The selected test problem was the subject of a code validation benchmark in 2008, based on experiments by Kurz et al. dating from 1996 [35, 36]. An aluminum cylinder defined by the domain Ω_c with conductivity $\sigma = 3.410^7 Sm^{-1}$ and mass $m = 0.107kg$ is levitated by two coaxial cylindrical coils; the electrodynamic levitation device is shown in Figure 5a. The inner coil has $N = 960$ turns and the outer coil $N = 576$ turns; a sinusoidal current flows through them in the opposite direction $I(t) = I_0 \sin(2\pi f_c t)$ with $I_0 = 20 A$ and $f_c = 50Hz$. The source term in the turns is given by $S_b = \pm N/\mathcal{A} I(t)$ where \mathcal{A} is the cross-sectional area of each coil in the (r, z) plane. The plate is positioned at $z = 3.8 \cdot 10^{-3}m$ at the initial instant. The simulation is carried out on a Ω domain large enough to avoid perturbations due to boundary conditions; these are of the homogeneous Neuman type on velocity. The mesh is composed of $8 \cdot 10^4$ quadrangles to represent the (r, z) plane in axisymmetric cylindrical coordinates. The time step is that of the discrete model $dt = 10^{-4}s$ and the simulation represents a total time of $1.8 s$.

Several strategies are possible for modeling the phenomena, from direct monolithic simulation coupling all the electromagnetic and mechanical equations, to more or less partial decoupling of these equations. First of all, it should be noted that the frequency of the current $f_c = 50 Hz$ is much higher than that of the mechanical oscillations generated by the competition between gravitational acceleration and acceleration due to electrodynamic effects. It is then possible to decouple electromagnetism from mechanical actions by averaging velocity and scalar and vector potentials over a sufficient number of periods. The commonly accepted assumption is that the induced magnetic field is unaffected by mechanical oscillations. The filtered potentials $\overline{\phi^o}$ and $\overline{\psi^o}$ will then be used to calculate the upward acceleration generated by the alternating current in the coils. The unsteady equations (19), including the non-linear inertial terms, are integrated in time. Figure 5b shows the mean scalar potential field $\overline{\phi^o}(r, z)$ for a coil-plate distance equal to $z = 1.1 \cdot 10^{-3}m$. The decreasing potential values with z in the plate region clearly show that the greater the height z , the weaker the force exerted.

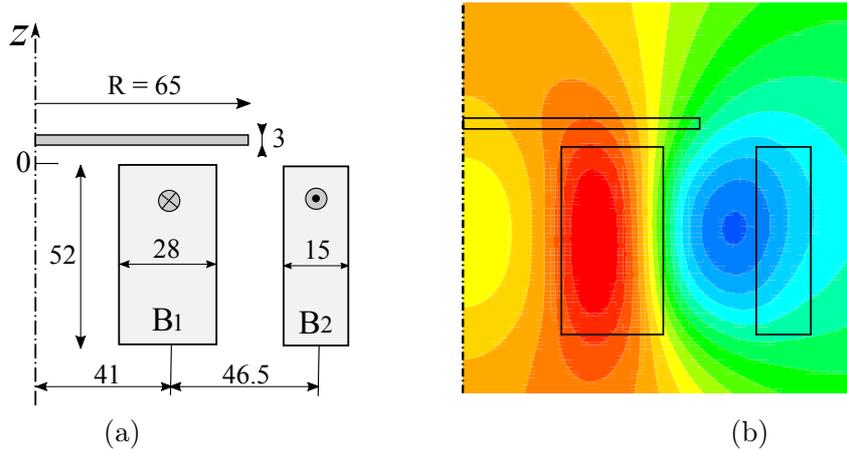


Figure 5. Levitation of a cylindrical plate; (a) electrodynamic levitation device with dimensions in mm ; (b) mean scalar potential $\overline{\phi^o}(r, z)$.

The mechanical force exerted by the magnetic field in the plate is classically expressed as

the integral over Ω_c of the vector product $\mathbf{j} \times \mathbf{B}$. In discrete mechanics, this vector product corresponds to one of the non-linear terms in the equation of motion $\nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})$, which is also an acceleration. These non-linear effects are reflected in the vector potential $\overline{\boldsymbol{\psi}}^\circ$ but also in the scalar potential ϕ° . The electromagnetic acceleration on the plate in (r, θ, z) coordinates can therefore be calculated from the $-\nabla \overline{\phi}^\circ$ acceleration:

$$\mathbf{h}_m(z) = -\frac{1}{[\Omega_c]} \int_{\Omega_c} \nabla \overline{\phi}^\circ dv. \quad (50)$$

Given the mean field $\overline{\phi}^\circ(r, z)$ calculated above, we see that $\mathbf{h}_m(z)$ decreases as the distance between the coils and the plate increases; to first order, its variation can be estimated by the law $\mathbf{h}(z) = -1.7 \cdot 10^3 z + 29.2$. The acceleration of gravity is equal to $\mathbf{g} = -9.81 \mathbf{e}_z$. If \mathbf{v} is the vertical velocity of the levitating cylindrical plate, we can calculate its evolution over time using the fundamental law of mechanics in terms of accelerations $d\mathbf{v}/dt = \mathbf{h}_m(z) - \mathbf{g}$ with $dz/dt = \mathbf{v}$. In this case, the vertical plate position solution obtained with a second-order time scheme oscillates indefinitely around its equilibrium position equal to $z = 11.4 \cdot 10^{-3} m$ with a frequency of $f = 6.5 Hz$. Induced second-order electromagnetic or mechanical effects may be responsible for the reduced amplitude of experimentally observed oscillations [35, 36, 37]. The geometric configuration of the electrodynamic levitation device shows the proximity of the first coil to the plate; the average acceleration during the first cycle is of the order of magnitude of $\mathbf{v} \approx 0.14 m s^{-1}$ but an acceleration of the order of magnitude of $(2\pi f)^2$, i.e. nearly 150 times that of gravity. The damping effects are probably due to positive and negative pressure variations between the plate and the coils.

To clarify the influence of air flows on the levitating plate's motion over time, we need to solve the equation of motion, assuming that the coils are solid obstacles and that the plate moves in a reciprocating motion. The solution in air is described by the equation (19) where \mathbf{V} is the fluid velocity, $\phi^\circ = p/\rho$ is the scalar potential where p is the pressure and ρ the density, $\nu = \mu/\rho$ is the kinematic viscosity; the celerity c_l here is that of sound in air. The equation becomes:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla \left(\frac{|\mathbf{V}|^2}{2} \right) - \nabla \otimes \left(\frac{|\mathbf{V}|^2}{2} \mathbf{n} \right) = -\nabla (\phi^\circ - dt c_l^2 \nabla \cdot \mathbf{V}) - \nabla \otimes (\nu \nabla \times \mathbf{V}). \quad (51)$$

The potential field ϕ° (the pressure) in a limited area around the plate is given in Figure 6 for a time $t_0 = 0.04 s$ for which the plate is located at a height of $z = 11 \cdot 10^{-3} m$. From the instantaneous solution, it is possible to estimate the pressure difference generated by the rise of the plate at an imposed velocity of $\mathbf{V} = 0.05 m s^{-1}$ for air considered as a compressible perfect gas of celerity $c_l \approx 340 m s^{-1}$.

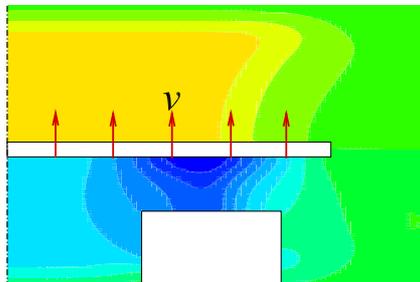


Figure 6. Levitation of an aluminium plate; instantaneous pressure field at time $t = t_0 + 8 \cdot 10^{-5} s$ in the vicinity of the plate as it moves upwards. The mean difference on either side of the plate is equal to $\Delta\phi^\circ = [-35, 15]$ or $\Delta p = [-41, 18]$, about $50 Pa$.

The time $\tau = 8 \cdot 10^{-5} \text{ s}$ allows the wave to propagate beyond the distance separating the plate from the first coil. Coherent phenomena are observed where the pressure is higher on the upper part of the plate but above all there is a significant depression between the two elements. From the mass $m = 0.107 \text{ kg}$ of the plate and its surface, it is possible to calculate the instantaneous negative acceleration due to the pressure difference Δp , i.e. $\gamma_p \approx 18 \text{ m s}^{-2}$. Compared with the electromagnetic and gravitational accelerations, the acceleration due to the pressure difference is not negligible. The coupling between the plate and the internal coil therefore acts as a damper: when the plate rises, the coil exerts a negative acceleration and when it falls, an overpressure induces a positive acceleration. Although they represent only a small part of this contribution, viscous movements produce dissipation transformed into heat. As the velocities considered are low, these effects depend, to a first approximation, linearly on the velocity of the plate \mathbf{v} . In order to model these phenomena, the assumption of linearity is adopted: damping is described by a constant η . The oscillator is therefore composed of the accelerations due to the electromagnetic fields, gravity and damping in the form:

$$\begin{cases} \frac{d\mathbf{v}}{dt} = \mathbf{h}_m - \mathbf{g} - \eta \mathbf{v}, & (52a) \\ \frac{dz}{dt} = \mathbf{v}. & (52b) \end{cases}$$

In the absence of a complete simulation of the mechanical and electromagnetic effects, the constant η was chosen to best represent the decay of the oscillations; only the frequency $f = 6.5 \text{ Hz}$ was calculated from the electromagnetic simulation. Figure 7 compares the model results with those obtained experimentally [35, 36].

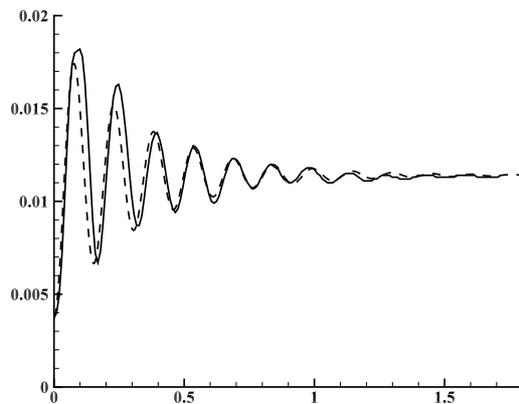


Figure 7. Levitation of an aluminum plate; measured height (solid line) [35, 36] compared with calculated height (dotted line)

The results of the simulations carried out using the proposed physical model are consistent with those in the literature, for example [37]. This analysis allows us to understand the role of unsteady compressible air flows around the levitating plate, but it is only qualitative. Fluid-structure interactions coupled to electromagnetism can be treated monolithically by considering that the electromagnetic and mechanical equations are the same. However, the differences in orders of magnitude between the different frequencies of the problem make this approach more difficult to implement and take us out of the context of validating the physical model presented.