An alternative model of Euler equations based on conservation of acceleration

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Abstract

The conservation of acceleration, one energy per unit of mass and length, leads to a drastic reduction in the number of unknowns in the Euler model or the Navier-Stokes equations for compressible flows. The equations for the conservation of momentum, mass and energy, associated with constitutive laws, are replaced by a single law for the conservation of total energy, whose variable is acceleration, the only absolute quantity considered. Abandoning the notion of mass or density and the concepts of discrete mechanics leads to the derivation of a purely kinematic law. Intrinsic acceleration is expressed by two orthogonal terms, two accelerations, one divergence-free and the other curl-free, according to a Helmholtz-Hodge decomposition. The scalar and vector potentials of the acceleration embody the compression and rotation energies respectively, the two components of total energy per unit mass. The law of motion thus obtained possesses the remarkable properties of conservation and symmetries enshrined in Noether's theorem; it is essentially a naturally relativistic non-linear wave equation.

This physical model, whose formalism has already been validated for incompressible Navier-Stokes equations, is extended to compressible flows with or without shock. Some analytical solutions of compressible flows are selected to demonstrate the formal reduction in the number of quantities needed to describe a problem, whereas they are redundant in a classical description. This parsimonious view of the laws of fluid mechanics is presented as an alternative for simulating compressible flows.

Keywords

Euler's equations; Bernoulli theorem; Discrete Mechanics; Helmholtz-Hodge Decomposition; Conservation of Acceleration; Noether theorem

1 Introduction

The equations of fluid equilibrium established by L. Euler in 1757 [1] and by H. Navier in 1822 [2] have forever marked the history of the derivation of the equations of fluid mechanics. Today, the major works of this science [3, 4, 5, 6] translate the physical and mathematical concepts contained in these equations. The Navier-Stokes equations, in their various formulations, now embody the concepts needed to represent flows of all kinds. The most important advances in this field have been made in mathematics [7, 8, 9, 10] and numerical analysis [11, 12, 13, 14]. Various formulations of the Euler and Navier-Stokes equations exist, notably the stochastic approach, but the concept of a continuous medium remains largely predominant. Since Newton, the development of differential calculus and analysis has led to major advances in our understanding of these nonlinear laws.

Discrete mechanics is based on an original approach that abandons the notion of a continuous medium, built around the concept of a global $\mathbb{R}^3(x, y, z)$ frame of reference, in favor of a local frame of reference, called as Maxwell's frame of reference $\mathbb{M}$ [15]. This reference frame comprises two geometric structures: (i) a primal structure corresponding to a segment of length defined by its two extremities, and (ii) a dual structure, a closed contour surrounding the segment; this local reference frame mimics an electrical circuit surrounded by a loop. A direct current flows through the segment, inducing a secondary current in the loop, following Maxwell's idea of the unification of electrodynamics and magnetism [16]. The notion of a continuous medium in which all quantities are reduced to a point is abandoned, replaced by this local reference frame, which

can be homothetically reduced, but can never be reduced to a point. Vectors such as acceleration and velocity become scalars carried by the same oriented segment.

The Weak Equivalence Principle of is taken up and modified by excluding mass; indeed, Galileo's principle of the universal fall of bodies shows that inertial mass is equal to gravitational mass, very precisely. Discrete mechanics removes the notions of mass and density from the derivation of the equation of motion. The fundamental law of mechanics becomes an equality between accelerations, where the intrinsic acceleration is equal to the sum of the accelerations imposed on it. The second argument in favor of abandoning mass is the principle of equivalence of mass and energy laid down by special relativity, so there's no need to retain both concepts. The consequences are obviously substantial: the notions of momentum, force, pressure and all quantities involving mass or density are discarded.

These two physical principles have already been successfully applied to incompressible Navier-Stokes equations [17], but in the case of Euler equations for compressible flows, these restrictions are less obvious. Pressure is a thermodynamic quantity, density is one of the main variables in the Euler model, and total energy is expressed in terms of density. Moreover, the conservation of momentum is the main pillar of classical mechanics and even of relativity theory. The aim here is to show that Euler's equations contain redundancies that can be overcome by considering the conservation of acceleration.

2 Analysis of Euler equation redundancies

The Euler equations [1, 3, 4, 6] describe the evolution of a non-viscous, non-heat-conducting, irrotational flow of an ideal fluid. The compressible nature of the flow is classically attributed to the medium considered. But the propagation of a wave is linked to its celerity, which is a property of the wave and not of the medium; for example, the celerity c_0 of light in a vacuum is close to that of light in air or water, whereas the propagation of acoustic waves in these media is very different from c_0. The compressibility of a medium is a very real notion, and the isothermal χ_T or isentropic χ_S coefficient of compressibility is perfectly measurable. The link between wave velocity and fluid compressibility needs to be made with a degree of caution. Compression waves can of course be attenuated, which is a characteristic of the medium; in the case of isentropic evolution, attenuation is zero. The celerity and coefficient of compressibility are linked by the relationship:

$$c^2 = \left(\frac{dp}{d\rho}\right)_s = \gamma \left(\frac{dp}{d\rho}\right)_T = \frac{\gamma}{\rho\,\chi_T}, \tag{1}$$

where $\gamma = c_p/cv$ is the ratio of heats of mass at constant pressure and volume.

The Euler equations are composed of (i) a momentum conservation equation $\boldsymbol{q} = \rho\,\boldsymbol{v}$ (ii) a mass conservation law whose variable is the density ρ and (iii) an energy equation, e.g. the internal energy e. The system of equations must be closed by an additional thermodynamic relation, a state law of the form $f(p, \rho, e) = 0$ where e is the specific internal energy, or by other thermodynamic variables, temperature, enthalpy, etc. The system of equations in non-

conservative form reads as follows:

$$\begin{cases} \rho \dfrac{d\boldsymbol{v}}{dt} = -\nabla p, \\ \dfrac{d\rho}{dt} + \rho \, \nabla \cdot \boldsymbol{v} = 0, \\ \rho \dfrac{de}{dt} = -p \, \nabla \cdot \boldsymbol{v}, \\ e = f(p, \rho). \end{cases} \tag{2}$$

where $\boldsymbol{v}$ is, in this case, the velocity of $\mathbb{R}^3$ of the continuous medium concept, ρ the density, p the pressure. The conservative form is strictly equivalent to the nonconservative form.

Euler's system of equations (2) is specifically dedicated to isentropic flows. In the general case, the conservation of internal energy contains other terms,

$$\begin{cases} \rho \dfrac{de}{dt} = -p \, \nabla \cdot \boldsymbol{v} - \nabla \cdot \boldsymbol{\varphi} + \Phi + q, \\ \rho \dfrac{ds}{dt} = \dfrac{1}{T} \dfrac{de}{dt} + \dfrac{p}{\rho \, T} \nabla \cdot \boldsymbol{v}, \end{cases} \tag{3}$$

where $\boldsymbol{\varphi} = -k \nabla T$ is the heat flux described by Fourier's law, Φ is the dissipation function related to viscous effects and q is the heat density produced in the volume. Considering the evolution of entropy s over time, and adopting the assumptions $\boldsymbol{\varphi} = 0$, $\Phi = 0$ and $q = 0$, we find, by combining these two relations (3),

$$\frac{ds}{dt} = 0, \tag{4}$$

where entropy is constant along a trajectory.

The cases where $ds/dt > 0$ are multiple and linked to thermal effects imposed elsewhere or to viscous dissipation completely associated with the flow by the viscosity ν and the quantity $(\nabla \cdot \boldsymbol{v})^2$. All these effects can be accounted for by a single equation of motion based on the conservation of acceleration, energy per unit mass and length. Indeed, to the mechanical effects defined by the grouping p/ρ, we can add the thermal effects of diffusion and dissipation, thus creating new potentials while preserving the unity of the law of motion, whose variable is velocity. To focus on the main objective of reducing the number of unknowns and equations, the discussion is limited to isentropic flows.

In order to analyze the redundancies in Euler's equations in depth, we need to return to the two fundamental principles of mechanics, the Weak Equivalence Principle (WEP) and the Mass-Energy Equivalence Principle (MEEP) derived from special relativity. The WEP is derived from Galileo's law of the universal fall of bodies: the mass attributed to gravity is equal to the mass associated with inertia. This principle has been verified by numerous experiments, the most recent of which show that the error is one part in 10^{15} [18]. If we consider this principle to be verified, Newton's second law, the fundamental principle of dynamics, would be written as $\boldsymbol{\gamma} = \boldsymbol{g}$ where $\boldsymbol{g}$ is the acceleration of gravity and $\boldsymbol{\gamma}$ the intrinsic acceleration of the material medium or of a particle with or without mass. Transposed to any acceleration other than gravity, this law becomes $\boldsymbol{\gamma} = \boldsymbol{h}$ an equality between accelerations. In this case, the fundamental law of mechanics is a purely kinematic law, expressing that acceleration is equal to the sum of the accelerations imposed on it. Note that acceleration is expressed in $m\,s^{-2}$, i.e. an energy per unit of mass and length.

The MEEP is a major result of special relativity, which expresses the equivalence of mass m and energy E in the form $E = m\,c_0^2$, the famous formula of 20th-century physics. In this form, however, it's a simple proportionality, not a formal equivalence. The underlying expression is that $\phi = E/m = c_0^2$, energy per unit mass is equal to the celerity of light squared; ϕ is expressed in $m^2\ s^{-2}$ in the S.I. system. There is, however, a genuine equivalence, which translates into an equivalent form of the equations for the conservation of mass and energy per unit mass [19]. A logical result of this equivalence would be to choose between mass and energy to derive the equations of mechanics. The conservation of both quantities can only lead to redundancy and an excess of variables. The choice between mass and energy is easy to make, since mass represents only one form of energy, that linked to compression; thus, energy per unit mass becomes a quantity of prime importance in the foundation of the laws of physics.

The momentum $(\rho\,\boldsymbol{v})$ of the equation of motion of the system (2) is expressed in $kg\,m^{-2}\,s^{-1}$, a form of total energy, the product $m\,\boldsymbol{\gamma}$ is the energy per unit length. Dividing the first equation of the system (2) by ρ gives the ratio $\phi = p/\rho$, energy per unit mass. Although Euler's equations cannot be modified in this way, the representativeness of the equations' terms persists. Similarly, the internal energy equation can be rewritten as $de/dt = -p/\rho\,\nabla\cdot\boldsymbol{v} = -\phi\nabla\cdot\boldsymbol{v}$; in fact, the potential ϕ is none other than the specific internal energy. The state law itself can be expressed as $e = f(\phi)$. The law of conservation of mass is no longer useful if total energy is conserved according to MEEP. The potential $\phi = p/\rho$ also allows us to discard the energy conservation equation.

The solution fixed by the $(p, \rho, e, \boldsymbol{v})$ variables of the Euler equations can potentially be replaced by the $(\phi, \boldsymbol{v})$ variables. But note that the potential ϕ is an energy per unit mass equal to $\phi = e/\rho$. The laws of conservation of mass and energy are as follows:

$$\begin{cases} \dfrac{d\rho}{dt} = -\rho\,\nabla\cdot\boldsymbol{v}, \\[2ex] \dfrac{d\phi}{dt} = -\phi\,\nabla\cdot\boldsymbol{v}. \end{cases} \tag{5}$$

These laws represent the true equivalence between mass and energy, since mass itself can replace density in the first equation (5). Equivalence between quantities of different natures is ensured by the satisfaction of the same conservation equation. Conservation of mass is equivalent only to compression energy, and not to total energy, as some theories would have us believe. As the potential ϕ can be evaluated explicitly by the equation, the only quantity to be calculated is the velocity $\boldsymbol{v}$ by an equation of motion. The reduction in the quantities needed to derive the equations of mechanics is drastic. This analysis leads to a parsimonious view of Euler's equations.

3 Discrete mechanics framework

3.1 Compressible kinematics

The abandonment of the notion of mass for the derivation of the laws of mechanics is suggested by the new interpretation of the Weak Equivalence Principle (WEP), where Newton's second law is translated into an equality between accelerations $\boldsymbol{\gamma} = \boldsymbol{h}$. From this point onwards, the concept of continuum dynamics disappears in its turn in favor of an extension of kinematics. As a result, the notion of compressibility, which was closely associated with the material medium through density variation, can now be perceived from a different angle. The primary quantity in this approach is acceleration, considered as an absolute quantity, intrinsically measurable at any place and at any time without an external reference frame. Velocity is a relative quantity

$\boldsymbol{v} = \boldsymbol{v}^o + \boldsymbol{\gamma}\, dt$ which can only be calculated if its value $\boldsymbol{v}^o$ at time t^o is itself known; this is the principle of relativity or the principle of inertia. Similarly, the displacement at time $t^o + dt$ must be defined from its value at time t^o, i.e. $\boldsymbol{u} = \boldsymbol{u}^o + \boldsymbol{v}\, dt$.

The material medium could be anything from solid to fluid to vacuum, and the laws of mechanics would apply to the propagation of waves of all kinds. Abandoning the notion of material medium in favor of the concepts of propagation and energy opens the way to particle mechanics, with or without mass. As with material media, a particle's motion is governed by the accelerations applied to it, rather than by forces, and its mass is therefore irrelevant to the description of its motion. The quantity that relates its own acceleration to the accelerations imposed on it is energy per unit mass $\boldsymbol{\phi}$,

$$\frac{d\phi}{dt} = -\phi\, \nabla \cdot \boldsymbol{v}, \tag{6}$$

which depends on the velocity divergence. In the case where $\nabla \cdot \boldsymbol{v} > 0$ motion undergoes compression or expansion when $\nabla \cdot \boldsymbol{v} < 0$. Contrary to the idea that this quantity is volumetric, this operator has a much broader physical meaning and applies to one dimension of space as well as to the motion of isolated particles. Consider the accelerated motion of two particles initially dh apart on a straight trajectory. If $\nabla \cdot \boldsymbol{v} > 0$, they diverge from each other, and when this divergence tends towards zero, the distance separating them becomes constant. The increment of energy acquired or given up by a particle during the time lapse dt can be evaluated from the relation (6), written $d\phi = -\phi\, dt\, \nabla \cdot \boldsymbol{v}$. Any particle or volume element in a material medium subjected to acceleration will see its energy increase or decrease according to the law (6). The massless photon obeys the same law, since it has a kinetic energy per unit mass $|\boldsymbol{v}|^2/2$. Compressible kinematics therefore dissociates the notion of material medium from that of propagation or motion.

The differentiation between the material medium and the wave suggests an unusual view of their interaction: the celerity of a material medium does not exist until wave propagation approaches it; the celerity of a medium at rest or at low velocity is not defined.

3.2 Geometric structures

The derivation of the discrete equation of motion requires a complete geometric framework; the notions of continuous medium, derivation at a point, analysis and continuum mechanics are abandoned and replaced by two structures called primal and dual, which can be inverted. The privileged directions of classical mechanics, like the notion of global frame of reference, are also abandoned. Figure 1 shows schematically the two elementary geometric structures; their orientation in space is not specified – one will simply define a local frame of reference by the orientations of the unit vectors $\mathbf{t}$ of the segment Γ and $\mathbf{n}$ orthogonal to the primal facet $\mathcal{S}$. The local frame of reference is called $\mathbb{M}$ in reference to the concepts of electromagnetism whose unification by Maxwell made it possible to understand the interactions of direct and induced currents in two interlaced circuits.

The oriented contour Γ^* of the primal facet is formed by the set of sides of the corresponding polygon; the primal $\mathcal{S}$ and dual Δ facets are considered to be planar, even if this is not always the case in practice. The contour of the dual facet is made up of all the oriented segments δ of the associated polygon. The vectors $\mathbf{t}$ and $\mathbf{n}$ are orthogonal by construction, $\mathbf{t} \cdot \mathbf{n} = 0$. The segment Γ of length $dh = [a, b]$, whose ends are a and b, is chosen to represent the component of velocity $\boldsymbol{v}$ and the component of acceleration $\boldsymbol{\gamma}$. The velocity vector and the acceleration vector will remain undefined; even if they can be reconstructed using their components, this will not be useful. Two functions will be used subsequently, ϕ a scalar defined at points a and b named scalar potential and $\boldsymbol{\psi}$ associated with the unit vector $\mathbf{n}$ named vector potential.

In addition, four discrete differential operators are introduced. The first is the operator $\nabla\phi$ applied to the scalar potential; it differs from the classical gradient vector because it only applies

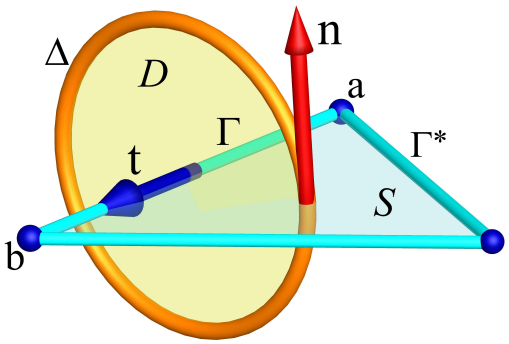


Figure 1. Maxwell local frame of reference: the primal structure is represented by the segment Γ of length $dh = [a, b]$ oriented by the unit vector $\mathbf{t}$; the planar primal surface $\mathcal{S}$ is constituted by the family Γ^* oriented by the vector $\mathbf{n}$ such that $\mathbf{t} \cdot \mathbf{n} = 0$. The closed contour Δ oriented by $\mathbf{n}$ forms the dual surface $\mathcal{D}$ whose normal is $\mathbf{t}$. The scalar potential is fixed at the a, b vertices of the primary structure and the vector potential is carried by the $\mathbf{n}$ normal.

to the component on the segment Γ, i.e. $\nabla\phi = (\phi_b - \phi_a)/dh$. Thus $\nabla\phi$ is both a scalar on the oriented segment and a vector. The second operator, named primal curl, is calculated from the circulation of the vector $\boldsymbol{v}$ on the contour Γ^* using Stokes' theorem, $\nabla \times \boldsymbol{v}$; it will then be assigned to a dual contour Δ and named $\boldsymbol{\psi}$, one of the curl components orthogonal to $\mathbf{n}$. Similarly, the dual curl operator $\nabla \otimes \boldsymbol{\psi}$ corresponds to a line integral on the dual contour, then assigned to the primal segment Γ; the symbol $\nabla\otimes$ chosen here to represent the dual curl is the most coherent, and poses no problem of compatibility with the classical tensor product, since the notion of tensor does not exist in discrete mechanics. Finally, the divergence operator corresponds to the integral on the dual volume of all the fluxes having the same vertex, $\nabla \cdot \boldsymbol{v}$; the result is then assigned to the vertex a. In a continuous context, $\boldsymbol{v}$ is a polar vector and $\boldsymbol{\psi}$ an axial or pseudo-vector. The two primal and dual curl operators are thus differentiated by the function to which they apply. This discrete framework is completed by two notions, (i) the time lapse $dt = t - t^o$ between the current time and the time t^o where the initial equilibrium is defined, and (ii) the discrete horizon $dh = c\,dt$, the length traveled by a wavefront of sound celerity c on a rectilinear trajectory.

The physical model has many properties, in particular the local and global orthogonality of the operators, $\nabla\phi \cdot \nabla\boldsymbol{\psi} = 0$, also verified by the discrete operators; the discrete model also mimics the properties of the continuum $\nabla \cdot \nabla \times \boldsymbol{\psi} = 0$ and $\nabla \times \nabla\phi = 0$, whatever the polygonal or polyhedral geometric topologies, structured or not structured [20].

3.3 Equation of motion

The modeling of the equation of motion is initiated by the fundamental law of dynamics expressed in terms of acceleration, $\boldsymbol{\gamma} = \boldsymbol{h}$ i.e., the intrinsic acceleration of the particle or material medium is equal to the sum of the accelerations applied to it; the notion of vector addition is restricted here to the Γ segment. Like the notion of mass, the concepts of force, pressure and power are abandoned. The integration of acceleration on the Γ segment of length dh is energy per unit mass. The sum of the imposed accelerations is modeled in two terms of a Helmholtz-Hodge decomposition [21], the first curl-free and the second divergence-free, $\boldsymbol{h} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$. The integral law of motion is as follows:

$$\int_\Gamma \boldsymbol{\gamma}_\phi \cdot \mathbf{t}\, dl = -\int_\Gamma \nabla\phi \cdot \mathbf{t}\, dl + \int_\Gamma \nabla \otimes \boldsymbol{\psi} \cdot \mathbf{t}\, dl. \tag{7}$$

The idea of deriving an equation of motion established for any inertial reference frame is

abandoned. The discrete equation of motion is introduced for the local reference frame $\mathbb{M}$ of length dh for a time period dt such that $\boldsymbol{x} \in [\boldsymbol{x}^o, \boldsymbol{x}^o + dh]$, $t \in [t^o, t^o + dt]$ where the quantities $\boldsymbol{v} \in [\boldsymbol{v}^o, \boldsymbol{v}^o + d\boldsymbol{v}]$, $\phi \in [\phi^o, \phi^o + d\phi]$ and $\boldsymbol{\psi} \in [\boldsymbol{\psi}^o, \boldsymbol{\psi}^o + d\boldsymbol{\psi}]$ are defined. Quantities with an exponent ^{o} refer to the same quantities at time t^o when mechanical equilibrium is assumed. The scalar ϕ^o and vector $\boldsymbol{\psi}^o$ potentials are called retarded potentials in reference to the retarded potentials of electromagnetism [22]; they are defined by:

$$\phi^o = -\int_0^{t^o} c_l^2 \, \nabla \cdot \boldsymbol{v} \, d\tau, \qquad \boldsymbol{\psi}^o = -\int_0^{t^o} c_t^2 \, \nabla \times \boldsymbol{v} \, d\tau, \tag{8}$$

where c_l and c_t are the longitudinal and transverse velocities respectively. They depend, of course, on the medium in which the waves propagate: the swell, a fluid or a solid, or a vacuum where, in this case, the celerity of light is equal to c_0. By explaining the increases $d\phi$ and $d\boldsymbol{\psi}$, the law of motion becomes:

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi^o + \frac{1}{2} \, |\boldsymbol{v}|^2 - c_l^2 \, dt \, \nabla \cdot \boldsymbol{v} \right) + \nabla \otimes \left(\boldsymbol{\psi}^o + \frac{1}{2} \, |\boldsymbol{v}|^2 \, \mathbf{n} - c_t^2 \, dt \, \nabla \times \boldsymbol{v} \right) + \boldsymbol{h}_s. \tag{9}$$

This equation (9) is formally a wave equation and can be reduced to the following. The term $\boldsymbol{h}_s$ represents the acceleration due to possible source terms, which will also be written as two terms of a Helmholtz-Hodge decomposition.

The potentials must be updated to reach equilibrium in the current state; the corresponding updates are written:

$$\text{updates} \begin{cases} \alpha_l \, \phi^o - c_l^2 \, dt \, \nabla \cdot (\boldsymbol{v} - \boldsymbol{v}^o) \longmapsto \phi^o, \\ \\ \alpha_t \, \boldsymbol{\psi}^o - c_t^2 \, dt \, \nabla \times (\boldsymbol{v} - \boldsymbol{v}^o) \longmapsto \boldsymbol{\psi}^o, \\ \\ \boldsymbol{v} = \boldsymbol{v}^o + \boldsymbol{\gamma} \, dt, \\ \\ \boldsymbol{u} = \boldsymbol{u}^o + \boldsymbol{v} \, dt, \end{cases} \tag{10}$$

where α_l and α_t are respectively the attenuation factors of longitudinal and transverse waves, due to their dissipation by the medium. The symbol $\longmapsto$ simply translates the actualization of potentials at time $t^o + dt$. This explicit incremental process naturally reveals only the shear energy $\nabla \times (\boldsymbol{v} - \boldsymbol{v}^o)$ and not the complete rotation $\nabla \times \boldsymbol{v}$; in fact, a uniform rotational motion characterized by the velocity $\boldsymbol{v} = \boldsymbol{\Omega} \times \mathbf{r}$ such that $\nabla \times \boldsymbol{v} = cte$ generates no dissipation. The same reasoning applies to a uniformly expanding motion such that $\nabla \cdot \boldsymbol{v} = cte$ [23].

Inertia is a fundamental element in the equations of fluid mechanics. In Eulerian variables, the material derivative is composed of the partial derivative in time and the inertia terms themselves noted $\boldsymbol{\kappa}$. These non-linear terms oppose the variation of translational and rotational motion in time and space, and conflict with the other terms of the equation of motion that ensure its equilibrium.

In classical mechanics, inertia terms are expressed from the velocity $\boldsymbol{v}$ in a three-dimensional space $\mathbb{R}^3(x, y, z)$. They take different forms, (i) $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$, (ii) $\nabla \cdot (\boldsymbol{v} \, \boldsymbol{v}) - \boldsymbol{v} \, \nabla \cdot \boldsymbol{v}$ or (iii) $\nabla(|\boldsymbol{v}|^2/2) - \boldsymbol{v} \times \nabla \times \boldsymbol{v}$; the latter term is none other than the Lamb vector $\boldsymbol{\mathcal{L}} = -\boldsymbol{v} \times \boldsymbol{\omega}$ with $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$.

In discrete mechanics, the velocity $\boldsymbol{v}$ considered is that carried by the segment Γ of the Figure 1, it is a scalar on a segment oriented by $\mathbf{t}$. Inertia is expressed differently [24], it's the sum of two orthogonal contributions of a Helmholtz-Hodge decomposition; the material derivative reads:

$$\boldsymbol{\gamma} \equiv \frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \left(\frac{|\boldsymbol{v}|^2}{2} \right) - \nabla \otimes \left(\frac{|\boldsymbol{v}|^2}{2} \, \mathbf{n} \right) = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\kappa}, \tag{11}$$

where $\boldsymbol{\kappa}$ is also carried by the segment Γ. This term can be assimilated to a curvature, that of the inertial potential, $\phi_i = |\boldsymbol{v}|^2/2$. As with the curvature of a geometric surface, the two curvature terms can add up, as in the case of the surface of a sphere, cancel each other out, as in the case of a minimal surface (a catenoid, for example), or take on any value. The case of zero inertia $\boldsymbol{\kappa} = 0$ is linked (i) either to zero velocity $\boldsymbol{v}$ or (ii) to a sum where the two terms annihilate; the motion is then qualified as non-inertial even if each term is non-zero. Some stationary motions, such as Couette or Poiseuille, have this characteristic.

The very special case of uniform rotation mentioned above is of great importance. It is considered inertial in classical mechanics and generates a fictitious term, the centrifugal acceleration. The expression (11) applied to this rotational motion leads to $\boldsymbol{\kappa} = 0$, the motion is non-inertial in discrete mechanics. This distinction and the generation of fictitious accelerations (centrifugal, Coriolis) are due to the very foundations of mechanics, to the concept of a continuous medium where all quantities must be represented at a single point. Since we're dealing with vectors, it becomes necessary to consider their components in the three-dimensional space $\mathbb{R}^3(x, y, z)$. A null vector must therefore have all three of its components equal to zero, which is an overabundant constraint.

The two inertia components of $\boldsymbol{\kappa}$, $\nabla(|\boldsymbol{v}|^2/2)$ and $\nabla \otimes (|\boldsymbol{v}|^2/2\,\mathbf{n})$ can be non-zero but equal, and their difference zero. This is the case, for example, with the Poiseuille flow. Note that these two components are carried by the same Γ segment, and that the operation is a simple sum. This result reveals a troubling aspect of classical mechanics and relativity, which use spatial reference frames. The age-old concept of the continuous medium, from which differential calculus, integration and mathematical analysis in general derive, implicitly introduces constraints on the representation of reality. Maxwell's reference frame allows us to reduce the direct and induced actions of any phenomenon to a single representation on the Γ segment. The case of uniform rotational motion is emblematic of the conceptual difference between the two approaches.

Euler's equations (2) in non-conservative form allow us to recover the usual meaning of the variables in the new approach. First, dividing the internal energy equation by the density, we find $de/dt = -(p/\rho)\,\nabla \cdot \boldsymbol{v}$ is the law (6) noting that $\phi = p/\rho$, the very definition of scalar potential acceleration. The system conservation equation (2) translates the equivalence between ϕ and the density ρ. Finally, dividing the vector equation by the density gives $d\boldsymbol{v}/dt = -(1/\rho)\,\nabla p$; it is therefore not possible to transform this relationship directly into the form $d\boldsymbol{v}/dt = -\nabla\phi$. The underlying reason lies in the very nature of the concept of momentum $\boldsymbol{q} = (\rho\,\boldsymbol{v})$, which inconsistently combines two quantities, one of which is a scalar and the other a vector. In discrete mechanics, $\boldsymbol{q}$ has no meaning or representation in $\mathbb{M}$ the local frame of reference; only acceleration and velocity are carried by the γ segment. Some other thermodynamic quantities can be deduced from potential and velocity, notably entalpy, $h = p/\rho + |\boldsymbol{v}|^2/2$, which is identified by the Bernoulli potential $\phi^o_B = \phi^o + |\boldsymbol{v}|^2/2$.

The law of motion (9) is a wave equation which can be easily transformed by substituting the displacement variable $\boldsymbol{u} = \boldsymbol{u}^o + \boldsymbol{v}\,dt$ into velocity and using the classic vector calculus formula $\nabla^2\boldsymbol{u} = \nabla\,(\nabla \cdot \boldsymbol{u}) - \nabla \otimes (\nabla \times \boldsymbol{u})$:

$$\frac{1}{c_l^2}\,\frac{d^2\boldsymbol{u}}{dt^2} - \nabla^2\,(\boldsymbol{u} - \boldsymbol{u}^o) = 0. \tag{12}$$

In a Lagrangian description, the second derivative of matter in time becomes a partial derivative and we find an Alembert's law $\Box\,(\boldsymbol{u} - \boldsymbol{u}^o) = 0$ since, in fact, the solution $\boldsymbol{u}^o$ does not depend on time in a Lagrangian description since it is defined only at time t^o. This law applies to both displacement and velocity $\Box\,(\boldsymbol{v} - \boldsymbol{v}^o) = 0$:

$$\frac{1}{c_l^2}\,\frac{d^2\boldsymbol{v}}{dt^2} - \nabla^2\,(\boldsymbol{v} - \boldsymbol{v}^o) = 0. \tag{13}$$

The $\boldsymbol{v}^o$ term corresponds to a retarded field similar to that of electromagnetism [22]. The equation (12) is therefore structurally a relativistic law [23] like the wave equation or Maxwell's equations. But the difference lies in the presence of two non-linearities, $\nabla(|\boldsymbol{v}|^2/2)$ and $\nabla \otimes (|\boldsymbol{v}|^2/2\,\mathbf{n})$ included in the material derivative. Thus, the discrete equation (13) is relativistic, and the classical demonstration is based on the invariance of the equations by a Lorentz transformation [23].

At this point, it is worth comparing the concepts of discrete mechanics and continuum mechanics to clarify how the discrete formalism takes a different path from that of special relativity. The latter is based on the notion of a continuous medium and a change of reference frame in three-dimensional space. Lorentz transformations, like the quadrivector formalism of relativity, are defined for a space $\mathbb{R}^3(x, y, z)$ and time t, grouped together within the concept of space-time. However, the treatment of each direction of space is not homogeneous: the direction of wave propagation is privileged, while the other two orthogonal directions are unaffected. What's more, for the Lorentz transformation, the coordinates transverse to the relative motion of the $\mathbb{R}$ and $\mathbb{R}'$ frames are assumed to be invariant, $y' = y$ and $z' = z$. It is therefore no longer necessary to consider these orthogonal directions, and the whole analysis of relativity theory can be limited to the direction of wave propagation and, of course, time. In discrete mechanics, the formulas for transforming laws from a $\mathbb{R}$ frame of reference to a $\mathbb{R}'$ frame of reference no longer make sense, and interactions from one local $\mathbb{M}$ frame of reference to another take place through cause and effect. A further restriction applies: these interactions are necessarily linked to a single direction in space, that of the Γ path oriented by the $\mathbf{t}$ unit vector for irrotational propagation. This leaves only two fundamental quantities: a direction in space and time. These arguments confirm the value of replacing the global reference frame $\mathbb{R}$ of classical mechanics with Maxwell's local reference frame $\mathbb{M}$.

3.4 Noether theorem

The equation of motion (9) has two orthogonal components, (i) a curl-free term corresponding to irrotational flow and (ii) a divergence-free term characteristic of solenoidal flows. These are two Lagrangians, each giving rise to two actions: potential energy represented by a potential (scalar or vector) and kinetic energy characterized by the divergence or curl of the velocity. The law (9) thus represents a dynamic system with two oscillators, each of which exchanges not only potential energy for kinetic energy, but also compression energy for rotational energy. This latter transfer mode is only possible if the system includes non-linear inertial terms. The two orthogonal contributions are then intertwined.

The complete form of the law of motion makes it possible to apply Noether's theorem to equations formed by Lagrangians or Hamiltonians. This theorem confers on these equations properties of symmetry and conservation that are invaluable for balancing certain quantities in physics. Each invariance of the system is linked to the conservation of a particular quantity. Time invariance ensures the conservation of total energy, invariance by directional translation designates the conservation of compression energy, and invariance by rotation affirms the conservation of angular momentum. Discrete mechanics is based on the conservation of accelerations, i.e. the intrinsic acceleration of a particle or material medium is equal to the sum of compressive and rotational accelerations; Noether's theorem is therefore formulated in terms of accelerations, but the nature of invariances and symmetries remains unchanged.

It's important to remember that the conservation of mass is subordinate to that of compression energy. Indeed, the relativity theory principle of equivalence of mass and energy must be restricted to compressional energy alone, and can be expressed as $E/m = c_l^2$ where c_l is the celerity of compressional waves in the general case, or $c_l = c_0$ is the celerity of light in vacuum in this particular case. In discrete mechanics, this principle simply becomes $\phi^o = c_l^2$.

The effects of rotation or transverse propagation are not directly included in the theory of special relativity, although the possible existence of gravitational waves was predicted as early as 1916 by A. Einstein and demonstrated in 2015. In discrete mechanics, the existence of polarizable transverse waves is a reality translated by the dual curl term of the vector potential in the equation (9). Understanding this phenomenon does not require recourse to the notion of mass, unlike general relativity, which also implies the curvature of space-time. Here, the only fundamental quantities are length and time, and they are disjoint; time is never dilated, but flows smoothly along a positively oriented axis (time invariance), and lengths are never contracted. The thought experiments of the last century linked to the theory of relativity have given way to physical and mathematical arguments based on the conservation of energy and Noether's theorem.

3.5 Proposal of an alternative to the Euler equations

The discrete formulation (9) can easily replace the incompressible Navier-Stokes equations, since density is no longer a variable and pressure itself is a Lagrangian. The Euler equations closely associate momentum, internal energy and density, as well as an equation of state to close the system of equations, which makes the justification of the equations' equivalence less obvious. However, the form of the equation (9) divides the total energy into two contributions, one representing compression energy and the second embodying shear, or more generally rotational, energy. This law of motion reveals two contributions to the evolution of velocity along the Γ segment, a curvature-free component of the form $\nabla\phi$ and a divergence-free one with $\nabla\otimes\boldsymbol{\psi}$; they can be shown to be orthogonal. It is then possible to suppress either of these contributions if the flow is solenoidal ($\nabla\cdot\boldsymbol{v}=0$) or irrotational ($\nabla\times\boldsymbol{v}=0$). This condition does not exclude the dissipation of compressional waves, even if the flow is considered to be non-viscous ($\nu=0$) and non-heat-conducting ($k=0$).

Let's consider here only the irrotational contribution such that:

$$\left\{\begin{array}{l} \dfrac{\partial\boldsymbol{v}}{\partial t}=-\nabla\left(\phi^o+\dfrac{1}{2}\,|\boldsymbol{v}|^2-dt\,c_l^2\,\nabla\cdot\boldsymbol{v}+\phi_s\right), \\ \\ \alpha_l\,\phi^o-c_l^2\,dt\,\nabla\cdot\boldsymbol{v}\longmapsto\phi^o, \end{array}\right. \tag{14}$$

where the scalar potential of the source term is integrated into the gradient operator, $\boldsymbol{h}_s=\nabla\phi_s$. The fluid under consideration is defined by only two physical characteristics, the celerity c_l of longitudinal waves and α_l the attenuation factor of these waves. If the wave is attenuated, the compression energy per unit mass ϕ^o is updated from the expression $\alpha_l\,\phi^o-c_l^2\,dt\,\nabla\cdot\boldsymbol{v}$. If the flow is isentropic, then $\alpha_l=1$. The medium's celerity c_l can be a function of another variable, and simply needs to be known in space and time.

The irrotational nature of the flow implies that it is defined for a single dimension of space; indeed, both the velocity vector $\boldsymbol{v}$ and the gradient operator have the rectilinear segment Γ as their oriented support. Interactions from one segment to the next take place only by cause and effect through the common vertices of consecutive segments. The flow itself may be two- or three-dimensional, but the propagation model is one-dimensional.

The law (14) is a non-linear wave equation, and the corresponding inertia term can be integrated with the Bernoulli potential $\phi_B^o=\phi^o+|\boldsymbol{v}|^2/2$; in this case, it is the latter that is updated, and the scalar potential is then deduced by difference, $\phi^o=\phi_B^o-|\boldsymbol{v}|^2/2$. This procedure, often used for incompressible flows, has been extended to compressible flows.

No constitutive law is added to this equation of motion, whereas a law of state is essential for the classical Euler equations. The velocity itself can be replaced by the potential, $c_l^2=\phi^o$. For flows involving travelling shock waves, it is necessary to advect the scalar potential using an appropriate transport method, e.g. a TVD scheme and a limiter [25].

If the motion is stationary, the time derivative disappears and Bernoulli's law is recovered,

$$\phi^o + \frac{1}{2}\,|\boldsymbol{v}|^2 - dt\, c_l^2\, \nabla\cdot\boldsymbol{v} + \phi_s = cte, \tag{15}$$

on a trajectory. In addition to Bernoulli's law for incompressible flows, a term can be used to represent all irrotational flows in general. In addition, the system (14) is, like Bernoulli's, a law expressing the evolution of flow along a trajectory. The extension to several dimensions of space is made causally by the connections of the Γ segments through their common vertices.

The equation (14) replaces Euler's system of equations (2), which has several advantages: (i) the mass or density ρ no longer appears in this equation of motion, which is one of the structural assumptions of discrete mechanics where the mass conservation equation is no longer necessary (ii) pressure is replaced by the scalar potential $\phi = p/\rho$, (iii) the inertia term is split into two components at the divergence-free and cur-free and only the latter is conserved in this irrotational flow hypothesis.

Flows with a progressive shock wave require an advection step for the scalar potential ϕ^o. In this case, the classical schemes used to solve approximate Riemann problems apply to this scalar. The conservative advection of this potential is dissociated from the conservation of the compression energy fixed by equation (14) and can be achieved using a Lagrangian method.

An energy equation associated with the (14) law is no longer necessary, as it already expresses the conservation of the mechanical energy of compression. The fact that both a momentum conservation equation and an energy conservation equation are conserved in the Euler system is superfluous. As with the equation of discrete motion (9), the constitutive laws are excluded. Only the longitudinal celerity c_l defines whether the transformation is isothermal, isentropic or other. This quantity may depend on the temperature of the material. This quantity may depend on the temperature of the material. This quantity may depend on the temperature or other variables of the problem at hand. In the case of isentropic evolution, it can be directly defined by energy, i.e. $c_l = \sqrt{p/\rho} = \sqrt{\phi^o}$.

In general, compressible flows treated by Euler's equations can be solved using the law (14). This form well characterizes irrotational flows linked to one-dimensional evolutions of the variables in the problem posed. Simulation in two or three dimensions of space takes place from cause to effect, from one segment of the primitive structure to another. This equation of motion is hyperbolic and can be used both for continuous compressible flows and in the presence of shock waves. The jump condition $[[\phi^o]]$ on energy per unit mass is implicitly ensured by the equation of motion on velocity $\boldsymbol{v}$ and by energy advection ϕ^o for a progressive shock wave.

3.6 Shock-wave propagation

Let's consider a one-dimensional elastic medium whose longitudinal celerity is equal to c_l, the celerity of waves, the celerity of sound in a solid, or c_0 the celerity of light in a vacuum. At the initial instant t^o, the medium is compressed by subjecting its boundary located at $\boldsymbol{x} = 0$ to an imposed normal velocity $\boldsymbol{v}$. What is the energy required to maintain this velocity $\boldsymbol{v}$ for a time dt? This value is easy to determine if the phenomenon under consideration is hyperbolic; indeed, no disturbance can be felt at a distance dh greater than that defined by the celerity, i.e. $dh = c_l\, dt$, where dh is the discrete horizon. Calculation of the energy per unit mass ϕ^o corresponding to propagation can be approximated by modeling the compression of a one-dimensional medium. In discrete mechanics, we find an expression for the energy per unit mass, the retarded potential ϕ^o of the form:

$$\phi^o = \int_{t^o}^{t^o+dt} c_l^2\, \nabla\cdot\boldsymbol{v}\, d\tau, \tag{16}$$

between the initial time t^o and its current value $t^o + dt$.

It is possible to assign orders of magnitude to each term of the expression (16), (i) the velocity is equivalent to $\boldsymbol{v}$, the value imposed in $x = 0$, (ii) the lapse time dt is of the order of magnitude of $\tau = dh/c_l$ and (iii) the characteristic length equal to dh, the discrete horizon. A simple analysis leads to the expression of the energy per unit mass required to propagate a wave of celerity c_l at a velocity level equal to $\boldsymbol{v}$, we find $\phi^o = c_l\, \boldsymbol{v}$.

In one dimension of space, the velocity of a particle or material medium cannot exceed the celerity of the medium; for example, the velocity of a massless photon cannot exceed the celerity of light c_0 on a straight trajectory. The equivalent phenomenon is observed for a gas of acoustic celerity c_l, which cannot exceed this value in a tube of constant cross-section. So, if $\boldsymbol{v} = c_l$, the energy value is equal to $\phi = c_l^2$. In the case of light propagating in a vacuum, this expression becomes $\phi = E/m = c_0^2$ where E is total energy and m is mass. This expression is none other than the famous special relativity formula $E = m\, c_0^2$. Note, however, that this is a simple proportionality of mass and energy, contrary to popular belief. There is indeed a Principle of Equivalence that expresses a homology between mass and energy, but its demonstration relies on the conservation laws of both. What's more, we're not talking about total energy, but only expansion energy. Discrete mechanics states that the intrinsic acceleration of a particle or material medium is equal to the sum of the expansion and rotation accelerations. The former is linked to the divergence of velocity $\nabla \cdot \boldsymbol{v}$ and the latter is linked to its rotation $\nabla \times \boldsymbol{v}$. In addition, the acceleration extended to a Γ segment of length dh is the energy,

$$\phi = \int_\Gamma \boldsymbol{\gamma} \cdot \mathbf{t}\, dl, \tag{17}$$

where $\mathbf{t}$ is the unit vector associated with the rudder segment. The discrete law of motion is written $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$ where the symbol $\nabla\otimes$ corresponds to the dual curl; it replaces the fundamental law of mechanics expressed in terms of momentum. Thus, the law of acceleration expresses total conservation, the sum of the effects of compression and rotation, and the conservation of angular momentum in classical mechanics.

To clarify the notion of energy required for the propagation of a compressional wave, we adopt the alternative law of Euler's equations (14) for a single dimension of space,

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi_B^o - c_l^2\, dt\, \nabla \cdot \boldsymbol{v} \right), \\ \\ \phi_B^o - c_l^2\, dt\, \nabla \cdot \boldsymbol{v} \longmapsto \phi_B^o, \end{cases} \tag{18}$$

where the attenuation factor α_l is zero; the symbol $\longmapsto$ reflects the updating of the retarded Bernoulli potential ϕ_B^o. This potential corresponds to the sum of the potential energy per unit mass and $|\boldsymbol{v}|^2/2$ the kinetic energy per unit mass; the last term reflects the exchanges between the compression energy and the kinetic and potential energies. These terms form an oscillator that allows energy to change nature without dissipation. The vector equation (18) is, in the absence of rotational terms, a law of conservation of total energy.

Figure 2 shows the evolution of the Bernoulli scalar potential $\phi_B^o = \phi^o + |\boldsymbol{v}|^2/2$ as a function of the abscissa $\boldsymbol{x}$ obtained by solving equation (18) where the celerity is equal to $c_l = \sqrt{\phi_B^o}$; the spatial approximation is $n = 64$ meshes and the time step chosen is such that $CFL = 0.5$. In this context, it is necessary to advect the ϕ_B^o potential here, using a Lax-Wendroff scheme and a Superbee limiter. The equation of motion provides the value of the potential ϕ_B^o as a function of velocity divergence, and its advection is achieved by a transport equation:

$$\frac{\partial \phi_B^o}{\partial t} = -\boldsymbol{v} \cdot \nabla \phi_B^o. \tag{19}$$

The wavefront is well positioned at the expected value, close to $\boldsymbol{x}_0 = c_l\, dt$. The characteristic of this model is associated with the variation in the celerity of the medium, which is not equal to

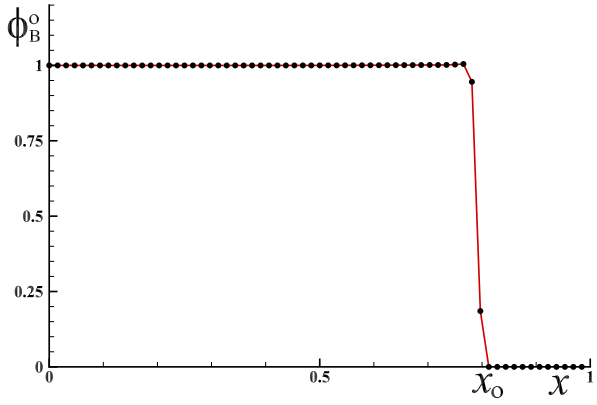


Figure 2. Evolution of energy as a function of x where $\phi_B^o = c_l\, \boldsymbol{v} = 1$, $c_l = \sqrt{\phi_B^o}$, and $\boldsymbol{x}_0 = dh = c_l\, dt = 0.8$ obtained by numerical simulation from equation (18) with spatial approximation $n = 64$.

c_l over the entire domain but varies as a function of $\boldsymbol{x}$; it is zero until the wave reaches the value $\boldsymbol{x}_0$. The wave therefore propagates in a hypercompressible medium ($c_l = 0$) and only acquires its value in the presence of the wave. The same advection scheme used without solving the equation (18) shows a numerical downstream diffusion characteristic of transport by conventional schemes, TVD or not. From a physical point of view, this suggests that in the absence of a wave, the medium does not have a well-defined celerity.

The attenuation factor of longitudinal waves α_l is correlated to their dissipation in the form of heat; this process has nothing to do with the notion of viscosity inherent in fluid shear and the viscous dissipation associated with the term $\nu\, \nabla \times \boldsymbol{v}$, energy per unit mass. Longitudinal wave dissipation is described by the quantity $\phi_d = (1 - \alpha_l)\, dt\, c_l^2\, \nabla \cdot \boldsymbol{v}$, also an energy per unit mass corresponding to the time lapse dt between two mechanical equilibria. Unlike incompressible flows, the local divergence of velocity is a real quantity that characterizes compression energy. In the case of periodic waves, attenuation is low in gases and very low in liquids, and depends on frequency. Dissipation in these media can be accurately measured; indeed, velocity divergence is very low for these periodic waves, as it is for continuous compressible flows. In the case of shock-type discontinuities, the situation is quite different, since $\nabla \cdot \boldsymbol{v}$ undergoes a sudden change at the moment of shock, whereas this quantity is almost zero before and after the shock. This phenomenon is described in the literature as entropy production [26, 9, 27]. The conservative nature of the equation (18) makes it possible to evaluate the energy dissipated during the time interval dt, especially if the shock wave is stationary; for a travelling wave, this exercise proves more difficult. In most numerical simulations of shock wave flows, this dissipation phenomenon is neglected when isentropic flow is considered. This will be the case here, as the problem of dissipation at a shock wave has not yet been solved in this approach.

The equation (9) has remarkable properties due to its structure derived from its initial formulation. It is a nonlinear wave equation [23], relativistic by nature. In particular, it verifies the limitation of the velocity $\boldsymbol{v}$ to the celerity of the medium c_l in one dimension of space, whatever the medium considered (solid, fluid, vacuum) and the nature of the wave (swell, acoustic waves, light). This essential characteristic shows that the phenomenon corresponding to the limitation of the velocity of a gas in a duct of constant cross-section fixed by Hugoniot's theorem, is of the same nature as that linked to the velocity of a photon limited to the celerity of light c_0 [28, 29, 30]. These limitations do not depend on the level of acceleration imposed.

3.7 Rankine-Hugoniot conditions

In classical mechanics, shock wave discontinuities are represented by Rankine-Hugoniot conditions when the equations are formulated in a conservative formulation. They express the conservation of mass, momentum and energy through the shock; considering a motion attached to the shock, they are written in a form,

$$\begin{cases} \rho_l \, \boldsymbol{v}_l = \rho_r \, \boldsymbol{v}_r, \\ \rho_l \, |\boldsymbol{v}_l|^2 + p_l = \rho_r \, |\boldsymbol{v}_r|^2 + p_r, \\ h_l + \dfrac{|\boldsymbol{v}_l|^2}{2} = h_r + \dfrac{|\boldsymbol{v}_r|^2}{2}, \end{cases} \tag{20}$$

the subscripts l and r correspond to the quantities upstream and downstream of the wave, where enthalpy $h = e + p/\rho$ is an energy per unit mass and e is internal energy per unit mass. Simplifying, we obtain

$$e_l + \frac{|\boldsymbol{v}_l|^2}{2} + \frac{p}{\rho}\bigg|_l = e_r + \frac{|\boldsymbol{v}_r|^2}{2} + \frac{p}{\rho}\bigg|_r , \tag{21}$$

a relationship which, for isentropic flow, translates the conservation of total enthalpy through the shock.

In discrete mechanics, discontinuities such as contact surfaces and shock waves are treated differently. The absence of mass in the equations is compensated for by defining the scalar potential $\phi^o = p/\rho$. By varying the longitudinal celerity as a function of this potential, we can find exact solutions to problems without shock waves. For a shock wave, the jump condition on density $[[\rho]]$ no longer applies, having been abandoned by the discrete formulation, and the same applies to momentum $[[\rho \boldsymbol{v}]]$, leaving only the condition on energy. For a straight shock and isentropic flow, the jump relations are already synthesized in the law of motion by the gradient operator,

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi^o + \frac{|\boldsymbol{v}|^2}{2} + c_l^2 \, dt \, \nabla \cdot \boldsymbol{v} \right), \tag{22}$$

which, in classical mechanics, is none other than the conservation of total enthalpy, $h = p/\rho + e + |\boldsymbol{v}|^2/2 = cte$ along a trajectory where the scalar potential is equal to $\phi^o = p/\rho$. The internal energy is indeed that derived from Euler's equations $de/dt = -p/\rho \, \nabla \cdot \boldsymbol{v}$ when velocity is defined by $c_l^2 = p/\rho$. In the case where the flow is no longer isentropic, the internal energy e contains other contributions relating to heat transfer and dissipation. But here, only isentropic flow of an inviscid, non-heat-conducting fluid has been considered.

The law of motion (22) expresses the conservation of total enthalpy through a stationary shock, when $\partial \boldsymbol{v}/\partial t = 0$. All terms in this equation are accelerations, energies per unit mass and length, and constitute a law of conservation of acceleration. The discrete nature of this equation means that it can be interpreted as a jump condition on the Γ segment of length $dh = [a, b]$,

$$\frac{\partial \boldsymbol{v}}{\partial t} = \nabla H \equiv \frac{[[H]]}{dh}, \tag{23}$$

where H is the generating or total enthalpy. In fact, the gradient operator is the restriction of the classical operator to the Γ segment only. This law of motion therefore intrinsically incorporates jump conditions for all types of discontinuity, and of course remains valid for continuous flows. The case of oblique shock waves does not arise in the same way. The shock is necessarily located

between two vertices, and its treatment is limited to a single dimension of space. Interactions between segments are causal, via their common vertices.

3.8 Practical implementation

The alternative system of Euler equations (14) consists of a single equation to solve and an explicit update of the retarded scalar potential ϕ^o. The solution of the problem is fixed by the quantities $(\boldsymbol{v}, \phi)$. The number of unknowns is equal to n_e, the number of segments on which velocities are calculated, while the scalar potential is updated on the n_v vertices of the primary structure from the divergence of velocities. This system is autonomous, requiring no conservation of mass, no energy equation and no constitutive laws. Velocity is directly accessible, as it is linked to the scalar potential by the relation $c_l^2 = \phi^o$. Boundary conditions are set on the velocity $\boldsymbol{v}$ at the edge of the physical domain, or on the flow rate imposed by the potential ϕ^o at the edge. The initial condition on the potential is given if known, i.e. the energy per unit initial mass; it can be set to zero, as it is defined to within one constant. In the case of an progressive shock wave, it is necessary to advect the ϕ^o field using an Eulerian or Lagrangian method.

If the variables (ρ, p, e) are absolutely necessary, then the initial value of the scalar potential will be $\phi^o = p^o/\rho^o$. Once the system has been solved (14), the current value of $\boldsymbol{v}$ is used to update the new value of the potential, $\phi = \phi^o - dt\, c_l^2\, \nabla \cdot \boldsymbol{v}$ but also the new value of the density, $\rho = \rho^o - dt\, \rho^o\, \nabla \cdot \boldsymbol{v}$; the current pressure is then deduced, $p = \rho\, \phi$. The internal energy per unit mass is equal to the potential itself, so the enthalpy per unit mass can be evaluated by its definition $h = p/\rho + |\boldsymbol{v}|^2/2 = \phi + |\boldsymbol{v}|^2/2$ and the absolute temperature determined by the relation $h = c_p\, T$ where c_p is the specific heat at constant pressure.

Reducing the quantities $(\rho, p, e, \boldsymbol{v})$ of classical mechanics to just two equivalent quantities $(\phi^o, \boldsymbol{v})$ intrinsically ensures the conservation of total energy and, implicitly, that of mass. The conservation of acceleration rather than momentum represents a radical change in the derivation of the equations of mechanics.

4 Test cases

The aim of this section is to demonstrate that the number of variables in a compressible flow problem, with or without shock, can be found by drastically limiting their number. The abandonment of the notion of mass in the revisited Weak Equivalence Principle leads to the elimination of density from the quantities used in mechanics equations, including Euler's equations. Despite the significant variations in density in compressible flows at high Mach numbers, the restriction already applied to incompressible flows is extended to the Euler equations. The equivalence between mass and energy in the theory of special relativity is a reality that applies to the equations of fluid mechanics. The law of discrete motion (9) intrinsically conserves total energy and does not require an adjoint law of mass conservation. The examples detailed here are not numerical test cases, even if quantitative results do exist, but they do demonstrate the interest of the discrete formulation from the point of view of the physical model alone. These cases have theoretical solutions that enable us to directly demonstrate the reduction in the number of variables.

4.1 Equilibrium of a column of gas

The very simple case of the static and dynamic equilibrium of a perfect gas is chosen to show that the number of variables $(\rho, p, T, \boldsymbol{v})$ of classical mechanics can be reduced to just the quantities $(\phi, \boldsymbol{v})$ for the discrete formulation.

Consider a column of height $h = 1m$ initially containing an ideal gas, assumed to be viscosity-free, of constant initial density $\rho_0 = 1\ kg\, m^{-3}$ at pressure $p_0 = 10^5\ Pa$; at time $t = 0$, a gravity

field is applied along the vertical, $\boldsymbol{g} = -g\,\mathbf{e}_y$. Under the action of gravity, the fluid falls with an acceleration $d\boldsymbol{v}/dt = -\boldsymbol{g}$ but this movement is counteracted by the compression of the lower part of the fluid.

Fluid motion tends towards equilibrium, corresponding to zero velocity and pressure and density fields satisfying the law of fluid statics. In dynamics and in the absence of viscosity, the energy initially released persists indefinitely in the form of waves. When the motion is highly compressible, the density is no longer a constant and must be calculated using the state law for the chosen evolution, isothermal or isentropic for example.

In discrete mechanics, the unsteady solution can be obtained from the discrete equation (9) and the stationary solution is inherited from the temporal convergence of the former. In one dimension of space, the dual rotation terms are zero and the system of equations becomes:

$$\begin{cases} \boldsymbol{\gamma} = -\nabla\left(\phi^o - dt\, c_l^2\, \nabla\cdot\boldsymbol{v}\right) + \boldsymbol{g}, \\ \\ \phi^o - c_l^2\, dt\, \nabla\cdot\boldsymbol{v} \longmapsto \phi^o. \end{cases} \tag{24}$$

The scalar potential ϕ is always defined by the relation $\phi = p/\rho$, but the celerity c_l must be expressed as a function of the chosen evolution.

4.1.1 Static equilibrium

In discrete mechanics, the law of fluid statics $-\nabla\phi + \boldsymbol{g} = 0$ leads to a solution $\phi(y)$ to within one constant, whatever the state law and thermodynamic evolution; if gravity is constant, the solution of the problem is fixed by the law $\phi(y) = g\,y$.

In classical mechanics, the solution of static equilibrium can be obtained by combining the equation of fluid statics, $-\nabla p + \rho\,\boldsymbol{g} = 0$, the perfect gas law of state $p = \rho\,r\,T = \rho\,(p_0/\rho_0)$ and conservation of mass; for an isothermal evolution, this reads:

$$\begin{cases} p(y) = a\,\dfrac{p_0}{\rho_0}\,exp(-\rho_0/p_0\;g\;y), \\ \\ \rho(y) = a\,exp(-\rho_0/p_0\;g\;y), \end{cases} \tag{25}$$

with $a = \rho_0^2\;g\;h/p_0/(1 - exp(\rho_0/p_0\;h))$.

The static equilibrium solution is obtained numerically from the system (24) where the celerity is equal to c_l and the potential equal to $\phi = p/\rho$. Figure 3 shows the stationary solution obtained for a gravity $g = 10^5\;m\;s^{-2}$ with a spatial approximation of $n = 16$ points uniformly distributed over the domain $[0, 1]$ and a time step of $dt = 10^{-3}\;s$ for which the waves attenuate rapidly. The solution $\phi(y)$ is compared with the classical theoretical solution (25) (p, ρ) for an isothermal evolution imposed by heat exchange with the exterior. The case of an isentropic evolution would be obtained in the same way by considering the isentropic compressibility coefficient $\chi_s = \gamma\,\chi_T$. Conservation of mass is intrinsically ensured by the system (24) without having to solve the continuity equation. Moreover, the state law, like the other constitutive laws, is not necessary; it is implicitly integrated into the expression of the celerity $c_l(\phi) = \sqrt{\phi}$.

In this case, the evolution of pressure and density follow strictly the same law, leading to a linear law for the scalar potential $\phi^o(y)$. For isentropic evolution, the ratio p/ρ^γ is a constant. Knowing ϕ makes it easy to deduce the density, as the evolution laws (5) are equivalent, depending only on the divergence of the velocity; the pressure can then be calculated by the expression $p = \rho\,\phi$. In short, the discrete formulation leads to an analytical solution, whereas the classical approach requires the use of multiple equations.

4.1.2 Dynamic evolution

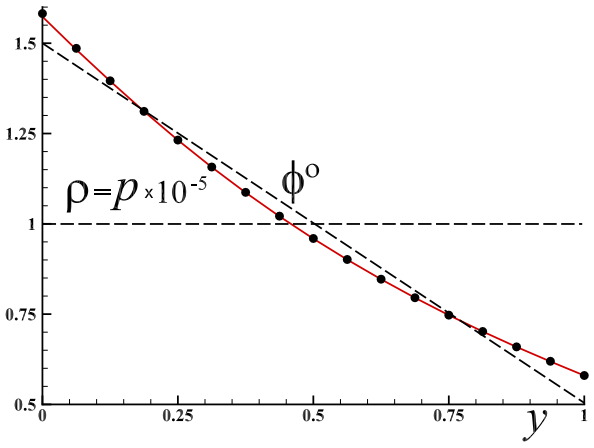


Figure 3. Steady solution for compressible equilibrium in a fluid column; density, pressure in function of y with $g = 10^5\ m\ s^{-2}$, celerity $c = \sqrt{p/\rho}$ and spatial approximation $n = 16$. The dotted black line represents the scalar potential $\phi^o = p/\rho$, the static solution of the discrete formulation (24).

The two independent parameters dt and c_l must be chosen according to the physics to be represented. Let's consider here the case of the dynamic equilibrium of a column of gas of height $h = 1\,m$ initially at uniform pressure subjected, at time t^o, to gravity $\boldsymbol{g} = -g\,\mathbf{e}_y$ when dt is small enough to observe periodic changes in velocity over time. For $g = 100\,m\,s^{-2}$ and $c_l = 10^3\,m\,s^{-1}$, the system (9) is integrated over a total time of $10^{-2}\ s$ with a time step of $dt = 10^{-7}\ s$ and a spatial approximation of $n = 1000$.

Figure 4 illustrates the phenomenon of gas oscillation in the cavity as a function of the value of the base potential $\phi(0,t)$. Compression-relaxation waves propagate with celerity c_l and a round trip requires a time of $dt = 2\,h/c_l = 2\,10^{-3}\ s$; reflections off the horizontal wall generate no attenuation. In theory, oscillations should persist indefinitely without attenuation ($\alpha_l = 0$), but in practice there is a slight attenuation due to numerical resolution.

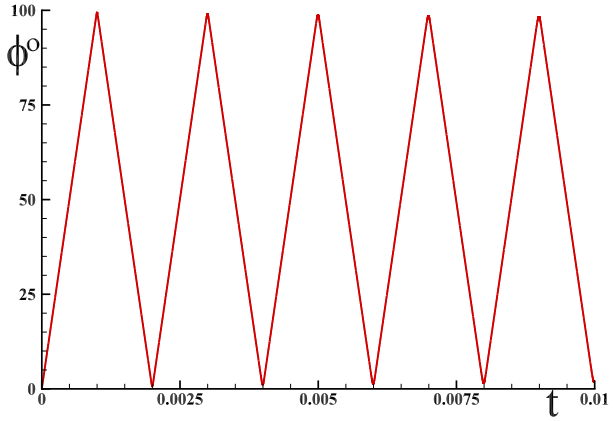


Figure 4. Dynamic solution for compressible equilibrium in a fluid column; the scalar potential ϕ at $y = 0$ is plotted as a function of time for $dt = 10^{-7}\ s$, gravity $g = 100\,m\,s^{-2}$ and velocity $c_l = 10^3\,m\,s^{-1}$

Thus, the truly physical compressible behavior of mechanical equations can only be obtained

if the time steps used are compatible with wave propagation in the medium under consideration, where the celerity c_l is imposed *a priori*. If dh is the discrete horizon, then the time step must be much smaller than the quantity dh/c_l. Conversely, if purely incompressible behavior is required, then $c_l >> dh/dt$ must be imposed, since it is the time step dt that is imposed for physical reasons. Both parameters of the equation (9) are physical quantities, and in particular dt is not a numerical time step but expresses the temporal observation scale of the phenomenon.

4.2 Injection of an ideal gas into a cavity

Let us consider the case of a square cavity of dimension unity, which, at a time t^o contains a supposedly ideal gas whose pressure p^o, density ρ^o, temperature T^o and mass m^o are uniform. According to the considered transformation, isothermal or isentropic, the evolutions of these quantities according to time are perfectly known from the state law $p = \rho\, r\, T$. The injection of the gas in the cavity will be very slow in order to eliminate the important propagation effects at small time scales. Viscosity effects will be neglected throughout this section.

The objective is to show that, since the state of the system is known at time t^o by its energy per unit mass ϕ^o, it is possible to determine its evolution at time t, i.e. $\phi(t)$, from the celerity c_l alone. For this purpose, the equation of motion (9) is restricted to compressibility effects and the computationally discrete solution is compared to the solution of the continuous problem.

4.2.1 Isothermal injection

Consider a very slow injection of the fluid existing in the density domain ρ with a velocity $V_0 = -0.01\ \mathbf{e}_y$ through a surface S for a time dt. The solution of this problem, obtained by considering the evolution as isothermal and the perfect gas, is simple and gives the evolution of the problem variables as a function of time:

$$\begin{cases} p(t) = p^o\, e^{-\nabla\cdot\boldsymbol{v}\ (t-t^o)} \\ \\ \rho(t) = \rho^o\, e^{-\nabla\cdot\boldsymbol{v}\ (t-t^o)} \end{cases} \tag{26}$$

In order to find these results, the discrete equation is taken in the form:

$$\begin{cases} \boldsymbol{\gamma} = -\nabla\left(\phi^o - dt\, c_l^2\, \nabla\cdot\boldsymbol{v}\right) \\ \\ \phi^o - dt\, c_l^2\, \nabla\cdot\boldsymbol{v} \longmapsto \phi^o \end{cases} \tag{27}$$

For an isothermal transformation, the celerity is equal to $c_l = \sqrt{r\, T_0} = 293.428$ with $T_0 = 300$. The resolution of this system from $\phi^o(0) = 0$ leads to a uniform and constant velocity divergence in time equal to $\nabla\cdot\boldsymbol{v} = -V_0\, S/[\Omega]$. The actualization of the potential ϕ^o noted by the symbol $\longmapsto$ can be realized with a Gear type time scheme of any order but, if an exponential scheme is used, the obtained results restore the theoretical solution to the machine accuracy.

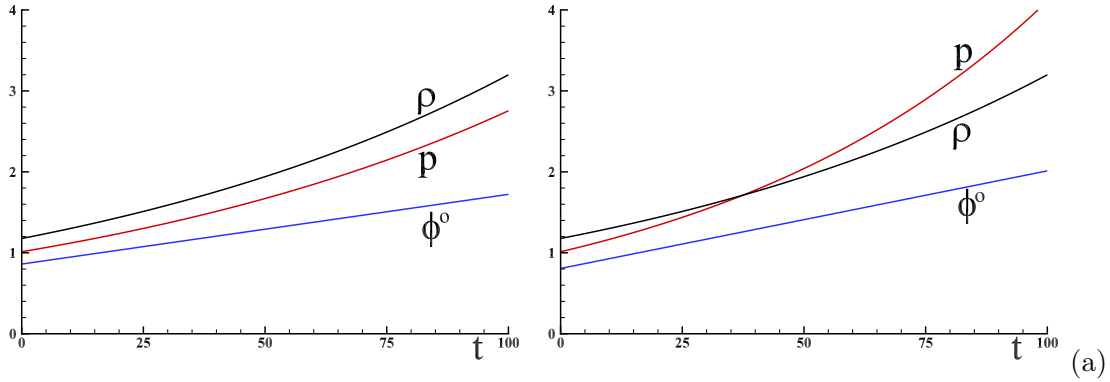
Figure 5a shows the linear time evolution of the potential $\phi(t)$ as well as those of the pressure $p(t)$ and the density $\rho(t)$. These last two quantities are not necessary to solve the problem and are simply obtained from the definition $\phi = p/\rho$ by fixing one of these quantities.

As we can see the evolution of $\phi(t)$ is linear while $p(t)$ and $\rho(t)$ correspond to the two exponential laws (26). Moreover $\phi(t)$ is defined only with respect to a constant which represents the initial energy contained in the system. The expression of the mass contained at time t can

be obtained simply and compared to that of the energy per unit mass:

$$\begin{cases} \dfrac{d\phi}{\phi} = -dt\,\nabla\cdot\boldsymbol{v} \Longrightarrow \phi(t) = \phi^o\, e^{-\nabla\cdot\boldsymbol{v}\ (t-t^o)} \\ \\ \dfrac{dm}{m} = -dt\,\nabla\cdot\boldsymbol{v} \Longrightarrow m(t) = m^o\, e^{-\nabla\cdot\boldsymbol{v}\ (t-t^o)} \end{cases} \tag{28}$$

We find the equivalence between mass and energy of the special relativity theory but here it is the energy per unit of mass. The mass does not depend on the absolute velocity but on the divergence of the velocity which allows to apply the principle of relativity without having to know the mass at rest; it is enough to know the mass at time t^o. The operator $\nabla\cdot\boldsymbol{v}$ filters a possible uniform velocity but this concept of relativity requires to know the velocity at time t^o to deduce its value at time t.



(a)

(b)

Figure 5. Isothermal (a) and isentropic (b) injection in a cavity, time evolution of the pressure $10^{-5}\times p(t)$, of the density $\rho(t)$ and of the potential $10^{-5}\times\phi(t)$; the computationally solutions are exact.

4.2.2 Adiabatic injection

Consider a very slow injection of the fluid existing in the density domain ρ with a velocity $V_0 = -0.01\,\mathbf{e}_y$ through a surface S for a time t. The solution of this problem, obtained by considering the evolution of the perfect gas as adiabatic, is simple and gives the evolution of the problem variables as a function of time:

$$\begin{cases} p_1 = p_0\, e^{-\gamma\,\nabla\cdot\boldsymbol{v}\,t} \\ \\ \rho_1 = \rho_0\, e^{-\nabla\cdot\boldsymbol{v}\,t} \\ \\ T_1 = T_0\, e^{-(\gamma-1)\,\nabla\cdot\boldsymbol{v}\,t} \end{cases} \tag{29}$$

The solved discrete equation is written:

$$\begin{cases} \boldsymbol{\gamma} = -\nabla\left(\phi^o - dt\, c_l^2\,\nabla\cdot\boldsymbol{v}\right) \\ \\ \phi^o - dt\, c_l^2\,\nabla\cdot\boldsymbol{v} \longmapsto \phi^o \end{cases} \tag{30}$$

For an isentropic transformation, the celerity is equal to $c_l = \sqrt{\gamma\, r\, T_0} = 347.189$ with $T_0 = 300$. The divergence of the uniform and constant velocity in time is equal to $\nabla \cdot \boldsymbol{v} = -V_0\, S/[\Omega]$. The quantity ρ is not useful for the integration of the equation of motion; it can be calculated *a posteriori* from the conservation of mass equation:

$$\frac{1}{\rho}\,\frac{d\rho}{dt} = -\nabla \cdot \boldsymbol{v} \tag{31}$$

The time integration of this equation leads to an accuracy that depends on the scheme used, for example the order one or two of a Gear scheme. However, the form of this equation suggests an exponential type of time integration scheme:

$$\begin{cases} \rho^o = \rho^o\, e^{-dt\, \nabla \cdot \boldsymbol{v}} \\ \\ p^o = p^o\, e^{-\gamma\, dt\, \nabla \cdot \boldsymbol{v}} \end{cases} \tag{32}$$

The solution is entirely defined by the velocity equal to $\boldsymbol{v} \cdot \mathbf{e}_y = V_0\, y$ and the scalar potential ϕ^o which is defined as a constant; the pressure and density can be determined *a posteriori* by integration (32). Figure 5b shows the solution obtained by calculation from the system (30). The temporal evolution of the potential is linear because the divergence of the velocity in the whole cavity is uniform and constant. The isentropic compression leads to non-linear variations of the pressure and density. We find exactly, with the precision of a machine, the theoretical solution of the problem. For this problem where the divergence of velocity is constant and uniform in the cavity, the celerity c_l and the potential ϕ^o are the only quantities needed to predict the state of the physical system. This observation is generalized to two-phase flows where only the celerity is the characteristic used. Pressure and density are related quantities that are replaced by the potential ϕ^o.

4.3 Theoretical solution of flow in Sod shock tube

The Sod shock tube problem [25, 31] is a classical test case for computationally testing Riemann solvers. The time evolutions of the main quantities, pressure, density, velocity, energy, can be described by solving the Euler equations in one dimension of space. The propagation velocity is different according to the region of space, a rarefaction region, a contact discontinuity and a shock-type discontinuity. Figure 6 illustrates the evolution of pressure, density, velocity and energy along the $\boldsymbol{x}$ abscissa for a time $t = 0.2$.

The objective is not to find these results by computation but to consider only the theoretical solution by transforming the usual variables $(p, \rho, \boldsymbol{v}, e)$ by the variables of the discrete law $(\phi^o, \boldsymbol{v})$. Indeed, the ratio p/ρ is none other than the energy $\phi^o = p/\rho$. Thus the resolution of the equation allows us to obtain the velocity $\boldsymbol{v}$ then its divergence $\nabla \cdot \boldsymbol{v}$ and to explicitly update the potential ϕ^o. The latter is then advected to the velocity $\boldsymbol{v}$ at each time step. Figure 7 shows the evolution of the velocity as a function of $\boldsymbol{x}$ and of the potential ϕ^o for the same time $t = 0.2$.

The flow considered is isentropic and the attenuation factor α_l is equal to unity. In this case the longitudinal celerity of the equation is expressed directly from the energy, $c^2 = \phi^o$. We find again, without surprise, the law of equivalence between mass and energy $E/m = c^2$ of special relativity. The velocity profile is identical to the one obtained from Euler's equations and the potential ϕ^o is deduced from the variations of p and ρ. Note that in the rarefaction zone, the profile of ϕ^o is linear while those of ρ and p are not.

Thus, the equation of discrete mechanics appears as an alternative to Euler's equations. The resolution of several equations in a sequence, momentum, mass conservation, conservation of energy associated with state law leads to an overabundant number of variables and to a recognized risk that certain quantities are not conserved. Discrete mechanics law already expresses the

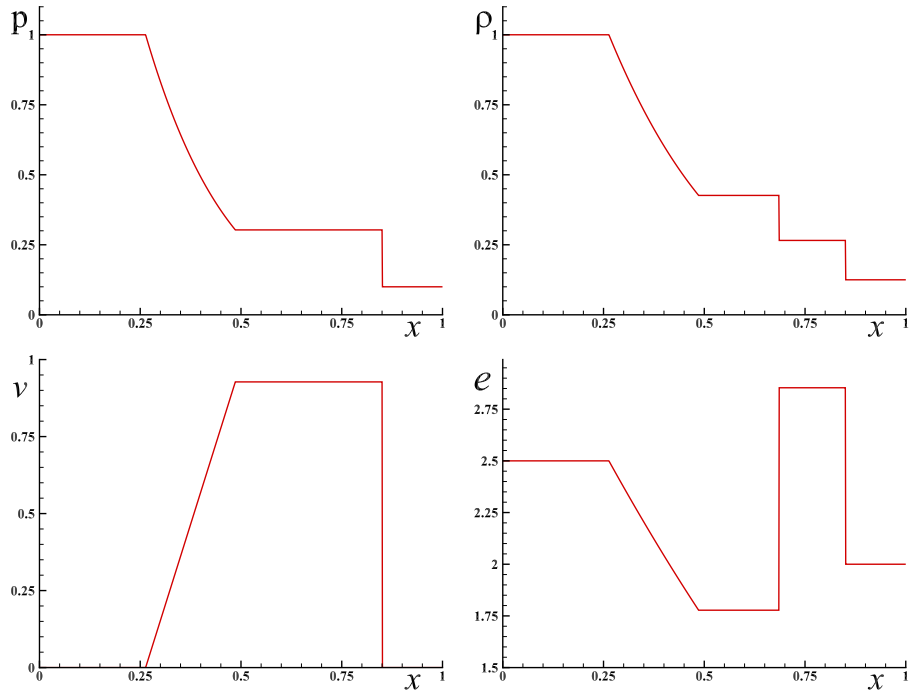


Figure 6. Sod tube, theoretical solution in Euler formulation $(p, \rho, \boldsymbol{v}, e)$ for $t = 0.2$.

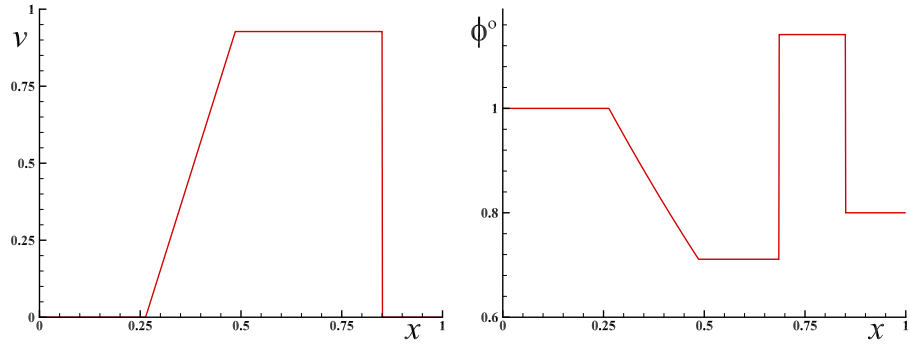


Figure 7. Sod tube, theoretical solution in discrete formulation $(\boldsymbol{v}, \phi^o)$ for $t = 0.2$.

conservation of acceleration, an energy by unit of mass and length, so it is useless to associate an additional energy law to it. Similarly, the state law is replaced by the attribution of a celerity c which is expressed by the potential ϕ^o itself. It is possible (but not necessary) to return to the usual variables $(p, \rho, \boldsymbol{v}, e)$ by using $\nabla \cdot \boldsymbol{v}$ to calculate ρ then $p = \rho\, \phi^o$.

Reducing the number of variables and equations corresponding to an inviscid compressible flow problem from Euler's equations can be interpreted as the search for the minimum number of quantities to describe the same problem. This is the principle of parsimony. The computationally correct solution was also obtained by directly solving the discrete equation of motion [32], page 155.

4.4 Isentropic nozzle flow

4.4.1 Classical mechanics solution

The flow in a de Laval nozzle is continuous if the downstream conditions are met, the regime is subsonic up to the throat and then becomes supersonic. The solution to this problem can easily be found from Euler's equations and for a quasi one-dimensional flow. Consider a convergent-divergent de Laval nozzle whose cross-section S(x) is equal to S_c at the throat; the reduced cross-section is denoted by the ratio $\mathcal{S}(x) = S(x)/S_c$. The fluid is a perfect gas of density ρ where $\gamma = c_p/c_v$ is the ratio of the specific heat at constant pressure to the specific heat at constant density and χ_S is the adiabatic compressibility coefficient; the gas velocity is given by the expression $c = \sqrt{1/\rho\,\chi_S} = \gamma\, rT$. The one-dimensional flow is assumed to be isentropic, and the ratio of specific heats is the sum of two contributions, one corresponding to the isothermal evolution of density for a change in pressure, and the second attributed to the variation in temperature under the effect of an increase in pressure:

$$\gamma = \frac{c_p}{c_v} = \left(1 + \frac{\beta^2\, T}{\rho\, c_v \chi_T}\right), \tag{33}$$

where β is the coefficient of expansion, χ_T the coefficient of isothermal compressibility and T the absolute temperature. The evolution of the fluid is isentropic, which induces $p\,\rho^{-\gamma} = p_i\,\rho_i^{-\gamma}$ where p_i and ρ_i are respectively the pressure and density under generating conditions.

For a stationary flow, the mass flow $q = \rho V S$ is a constant in the nozzle; the reduced velocity is defined by $\mathcal{V} = V/V_c = q/(\rho_c\, S_c)$. Similarly, the generating conditions p_i, the density ρ_i and the celerity c_i are used to reduce the real variables to define the reduced quantities p/p_i, ρ/ρ_i and c/c_i. The notations of the reduced quantities adopted now are the same in order to reduce the number of symbols. The analytical solution of this problem (p, ρ, T, V) can be easily calculated as a function of the Mach number $M(x) = V(x)/c(x)$ where $c(x)$ is the local celerity:

$$\begin{cases} \dfrac{p}{p_i} = \left(1 + \dfrac{\gamma-1}{2}\, M^2\right)^{-\gamma/(\gamma-1)}, \\[2ex] \dfrac{\rho}{\rho_i} = \left(1 + \dfrac{\gamma-1}{2}\, M^2\right)^{-1/(\gamma-1)}, \\[2ex] \dfrac{T}{T_i} = \dfrac{c^2}{c_i^2} = \left(1 + \dfrac{\gamma-1}{2}\, M^2\right)^{-1}. \end{cases} \tag{34}$$

Variables reduced by conditions at the nozzle throat where Mach number $M = 1$ and velocity $c_c = c$ can be calculated from the relations (34) or found in many textbooks. For a Mach number equal to $M = 1$ at throat level, we find the dimensionless quantities, mass flow $\mathbf{q} = \rho_c/\rho_i = 0.6339381453$, $c_c^2 = \phi_c^o = p_c/\rho_c = 0.8333333333$ and $p_c/p_i = 0.5282817877$. The problem data are therefore the flow rate $\mathbf{q}$, the nozzle profile $\mathcal{S}(x)$ and the potential at the origin ϕ_i^o corresponding to the generating enthalpy h_i.

Figure 8 shows the solution obtained from the classical analytical solution, with the nozzle profile calculated in such a way as to impose a linear evolution of the Mach number $M\ in[0, 3]$.

This classical solution of one-dimensional theory of irrotational perfect fluid flows lends itself very well to the reduction of the number of variables by discrete mechanics. Of course, the number of equations required is also limited to the law of motion alone, considering intrinsic acceleration as energy per unit mass and length.

4.4.2 Discrete mechanics solution

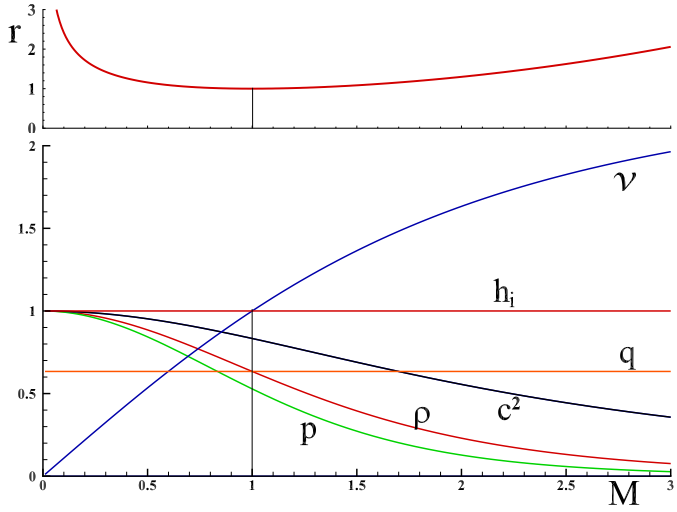


Figure 8. de Laval nozzle; top: profile of the nozzle where $r = \sqrt{\mathcal{S}(x)}$ with $x = M$; bottom: variations as a function of local Mach number in pressure p/p_i, density ρ/ρ_i, temperature $T/T_i = c^2/c_i^2$, velocity $\boldsymbol{\mathcal{V}} = \mathrm{V}/\mathrm{V_c}$, mass flow $\mathbf{q} = \rho_c/\rho_i = cte$.

In the discrete formulation, the one-dimensional equation of motion (9) without viscous term is written:

$$\begin{cases} \dfrac{\partial \boldsymbol{v}}{\partial t} = -\nabla \left(\phi^o + \dfrac{|\boldsymbol{v}|^2}{2} - dt\, c^2\, \nabla \cdot (\mathcal{S}\, \boldsymbol{v}) \right), \\ \\ \phi^o - \phi^o\, dt\, \nabla \cdot (\mathcal{S}\, \boldsymbol{v}) \longmapsto \phi^o, \end{cases} \tag{35}$$

where $\boldsymbol{v}(x)$ is the velocity on a Γ segment, $c(x)$ the longitudinal velocity of the gas and $\mathcal{S}(x)$ a dimensionless parameter characterizing the reduced cross-section of the nozzle. In the case of stationary flow, equation (35) provides a relationship equivalent to Bernoulli's law:

$$\phi^o + \frac{|\boldsymbol{v}|^2}{2} - dt\, c^2\, \nabla \cdot (\mathcal{S}\, \boldsymbol{v}) = cte, \tag{36}$$

where $\phi^o = p/\rho$ is the scalar potential, c is the longitudinal velocity and $\mathcal{S}$ is the reduced cross-section of the nozzle. The quantity $\phi^o_B = \phi^o + |\boldsymbol{v}|^2/2$, called Bernoulli potential, is the invariant generating potential on a streamline, comparable to the generating enthalpy $h_i = h + |\boldsymbol{v}|^2/2$.

When the section is constant, or the converging part of the nozzle is followed by a channel of constant section, $\mathcal{S} = 1$, the upstream subsonic flow becomes sonic at the throat and then remains equal to a Mach number value $M = 1$, the phenomenon of sonic blockage. The transposition of relativity theory to particle motion effectively limits the velocity of photons to the celerity c_0 of light in a vacuum. The homology is more complete if we consider the energy of relativity theory and the famous relation $E = m\, c_0^2$; in discrete mechanics, this relation becomes $\phi^o = E/m = c_0^2$. Mass is in fact a proportionality coefficient; the equivalence between mass and energy is a reality, but not illustrated by this relationship. If the cross-section has curvilinear aspects, as in the convergent-divergent nozzle, the motion becomes supersonic in fluid mechanics, but this concept can be transposed to other laws of physics.

Figure 9 shows the solution of equation (35); the velocity evolution is that which corresponds to the theoretical solution of Euler's equations for a perfect gas in a quasi one-dimensional approach. The scalar potential replaces the ratio of pressure and density $\phi^o(x) = p(x)/\rho(x)$,

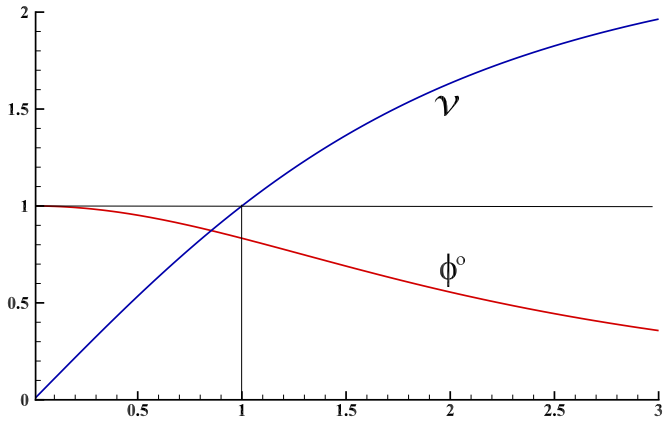


Figure 9. Isentropic expansion of a perfect gas in a de Laval nozzle; variations in potential $\phi^o = c^2/c_i^2$ and velocity $\boldsymbol{\mathcal{V}}$ as a function of local Mach number.

which in this case is also the celerity squared, $\phi^o = c^2$.

Thus, the solution of the classical problem (p, ρ, T, V) based on Euler's equations, which comprises four equations, the conservation of momentum, the energy equation, and the conservation of mass equation completed by a state law, is replaced here by a single energy conservation law whose solution is materialized by the velocity $\boldsymbol{v}$. From here, the scalar potential ϕ^o is explicitly updated, and thus the local velocity $c^2/c_i^2 = \phi^o$.

5 Conclusion

Like the Navier-Stokes equations, the Euler model includes conservation equations for momentum, energy and mass, as well as a state law and as many variables: velocity, density, internal energy and pressure. These equations are solved in sequence, which can lead to inconsistent results. Indeed, these three equations all reflect some form of energy conservation, and Euler's model introduces redundancies in the equations and, consequently, in the variables.

The alternative model developed on the basis of discrete mechanics clearly shows that the number of equations to be solved reduces to a single one, and that the variable is the velocity localized on a segment Γ of $\mathbb{M}$, the local frame of reference. Energy per unit mass is simply the ratio of pressure to density $\phi^o = p/\rho$, i.e. the specific internal energy e. The quantity ϕ is not an additional variable, as it is calculated from the divergence of velocity. The celerity itself is deduced from the scalar potential ϕ^o. The analytical examples of compressible flows used to demonstrate the reduction of the various quantities cover both continuous compressible cases and cases with discontinuities. Although not necessary, the quantities $(\boldsymbol{v}, \phi^o)$ can be used to return to classical variables $(\boldsymbol{v}, p, \rho, e)$.

The reduction of Euler's model was only possible by returning to two fundamental principles of mechanics. The first is an old one: Galileo's universal law of free fall, now known as the of Weak Equivalence Principle, in which the mass attributed to inertia is defined as equal to the mass linked to gravity. This principle, precisely verified, is a postulate of discrete mechanics, which extends it to all types of acceleration. Newton's second law becomes a purely kinematic law of equality between proper and imposed accelerations. The second fundamental principle is that of special relativity, which establishes the equivalence between mass and energy. Although the law used to demonstrate this is a simple proportionality, the equivalence does exist and is reflected in conservation laws of the same form. Note that acceleration is energy per unit mass and length, so Euler's model includes excess quantities.

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