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# Why Navier-Stokes equations are wrong

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## Abstract

The completeness of the Navier-Stokes equations is addressed using two examples, a simple rotational motion and the turbulent Taylor-Green vortex; in both cases the solutions are not physical. The main artefacts and paradoxes of the Navier-Stokes equations are intrinsic to the notion of a continuous medium and to the representation of the velocity field in three-dimensional space in terms of components. The main error is inherent in the form of the inertia of these equations, more precisely in the Lamb vector, which induces the existence of fictitious forces, the Coriolis and centrifugal forces. In the presence of strong rotational effects, as in turbulence, the solutions of the Navier-Stokes equations are incorrect. On the other hand, the discrete law of motion, considered as an alternative to the Navier-Stokes equations [7], makes it possible to find the vast majority of solutions of the classical formalism, but excludes from the outset non-physical solutions. The inertia is represented as the sum of two contributions, a first curl-free component and a second component divergence-free, a Helmholtz-Hodge decomposition. The main difference between the two approaches is the abandonment of the concepts of classical mechanics, where the global frame of reference is replaced by Maxwell's local frame of reference. Rigid rotation thus becomes inertial motion in the same way as uniform translational motion in the Galilean frame of reference. Taylor-Green turbulent flow, the only interesting case where the two solutions diverge, is reinterpreted in the light of the conclusions drawn from the in-depth analysis of inertial effects.

## Keywords

Discrete Mechanics; Acceleration Conservation Principle; Hodge-Helmholtz Decomposition; Navier-Stokes equations; Galilean frame of reference

## 1. Introduction

The main attraction of a theory or formulation lies in its logical completeness. If a physically admissible solution challenges that theory, the whole edifice is weakened. To legitimately challenge the existing model, we need to explain the underlying reasons for the discrepancy between the model and the observations. The reconstruction of a more robust edifice must solve the problem without altering the properties of classical theory.

The Navier-Stokes equations are representative of the vast majority of physical observations and rightly constitute the standard model accepted by everyone. It is derived from the fundamental law of dynamics, Newton's second law. This law is based on the notion of a continuous medium in which all quantities are reduced to a single point in a three-dimensional Galilean frame of reference called  $\mathbb{R}^3$ . These concepts are themselves derived from the Weak Equivalence Principle (WEP) [44], where the mass associated with gravitational acceleration is equal to the mass associated with inertia. The theory of special and general relativity does not question these basic concepts, but introduces Lorentz invariance, which replaces Galilean invariance when the velocity is close to the celerity of light [14].

Discrete mechanics [7] presents itself as an alternative to the Navier-Stokes equations based on very different concepts, (i) mass is removed from the WEP by considering acceleration to be an absolute quantity, (ii) the Galilean frame of reference  $\mathbb{R}$  is replaced by a local frame of reference called Maxwell's frame of reference  $\mathbb{M}$ , (iii) the concepts of force, momentum and energy are also abandoned, and (iv) the derivation of the law of motion is based on the conservation of acceleration, energy per unit of mass and length. The analytical solutions and the results of

the simulations carried out with this new formulation are identical to those of the Navier-Stokes equations for a wide range of flows, compressible or incompressible, two-phase [5, 8], and so on.

The only issue addressed here is the potential discrepancy between the discrete formulation and the Navier-Stokes equations, which is strictly due to the physical model and not to the associated methodologies. At present, the only discrepancy observed concerns the turbulent Taylor-Green flow at a Reynolds number of  $Re = 1600$  [9], where the kinetic energy increases from the first instants. It is not easy to conclude which of the two models is correct, since the velocity fields evolve very closely with time and the overall energy decay obeys the classical law. In the many other simulations carried out, the solutions are identical, as are the many analytical solutions. The originality of this analysis lies (i) in the presentation of a unique and simple solution to the Navier-Stokes equations, (ii) in the physical and theoretical foundations of the discrepancies observed for the Taylor-Green vortex, and (iii) in the robust resolution of these problems using the discrete model.

The theoretical solution at the origin of the disagreement between the two models corresponds to the superposition of a radial source flow and a rigid rotating flow at constant angular velocity; the solution obtained is that of a steady flow of an inviscid fluid, where the spatial evolutions are essentially due to the inertial terms; the result is the same for a flow where the viscous term is described by a linear term. On the other hand, the inertial terms are non-linear and one of the main differences between the two models is their different form. In discrete mechanics, the dual contour of the inertial potential replaces the Lamb vector of classical mechanics; the former has zero divergence, while the nature of the latter is indeterminate. This major change is due not only to the Navier-Stokes equations, but to continuum mechanics itself. The essential consequence concerns the inertial or non-inertial nature of uniform rotational motion. In classical mechanics and in the theory of relativity, rotational motion at constant angular velocity is considered to be accelerated, i.e. non-inertial, whereas in discrete mechanics this motion is inertial.

The final discussion focuses on the observed inconsistency and analyses its causes in detail. In fact, the inconsistency of the simple solution reveals a complexity that goes far beyond the Navier-Stokes equations; it concerns the concept of inertia and its interpretation in a classical global frame of reference. The meaning of Newton's second law, which translates the conservation of momentum, can be called into question by abandoning the notion of mass, which is nevertheless present in relativity theory and in physics as a whole. Its equivalence to energy per unit of mass is sufficient to derive a kinematic law of motion.

The alternative to the Navier-Stokes equations [7] presents the main objections to the standard model of fluid mechanics. Most of them do not question its legitimacy in representing many fluid flow problems of all kinds. Some of these objections are covered by *artefact*. For example, the Navier-Stokes vector equation does not conserve mass by itself, but the addition of a special mass conservation equation allows this constraint to be maintained.

The aim of this article is to demonstrate the non-exhaustiveness of the Navier-Stokes model. This demonstration is based on a simple example which shows that a flow associated with a source and a uniform rotation leads to an inconsistency. The reasons for this inconsistency are examined in depth. The only known discrepancy between the Navier-Stokes equations and the discrete model is a turbulent flow, the Taylor-Green vortex at a Reynolds number of  $Re = 1600$ , which is used to explain the discrepancy in the light of the simple result described above.

## 2. Discrete mechanics framework

The discrete model has already been described in previous papers [5, 6, 7] but, given its non-standard nature, the main concepts of its derivation are briefly outlined.

### 2.1. Maxwell's local frame of reference

One of the cornerstones of discrete mechanics is the abandonment of the Galilean or inertial frame of reference of classical mechanics. The description of the cosmos from a global frame of reference is forgotten in favour of a quasi-local and quasi-instantaneous vision, where space is limited to what can be perceived over a limited period of time. Predicting a distant event is illusory, and any change of frame of reference is uncertain, so the notion of a change of frame of reference can be abandoned. It is reasonable to describe the evolution of a physical system over a period of time  $dt$  and a length  $dh$ , called the discrete horizon, these two quantities being linked by the celerity of the wave  $dh = c dt$ . The extension to a three-dimensional real domain must therefore be made from cause to effect.

Maxwell's brilliant idea [26] in 1865 is a means of reconstructing a local frame of reference which will make it possible to construct a new approach to mechanics. The Figure 1 shows an electrical conductor  $\Gamma$  of length  $dh = [a, b]$ , where  $a$  and  $b$  are the vertices of this rectilinear structure, called primal; this segment is oriented along the unit vector  $\mathbf{t}$ . The dual contour  $\Delta$  oriented along  $\mathbf{n}$  surrounds the segment  $\Gamma$  so that the two unit vectors are orthogonal,  $\mathbf{t} \cdot \mathbf{n} = 0$ . These two nested geometric structures form Maxwell's local frame of reference,  $\mathbb{M}$ .

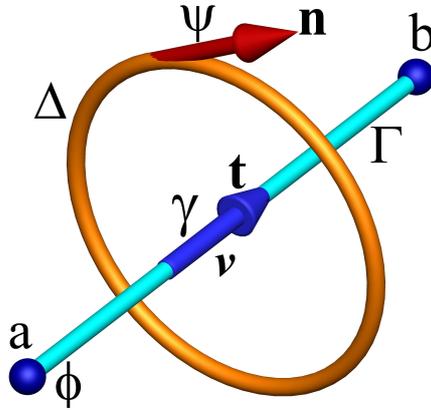


Figure 1: Maxwell local frame of reference: a straight segment  $\Gamma$  of length  $dh = [a, b]$  oriented along the unit vector  $\mathbf{t}$  forms the primal structure. The dual contour  $\Delta$  positively oriented by  $\mathbf{n}$  is such that  $\mathbf{t} \cdot \mathbf{n} = 0$ . Acceleration  $\boldsymbol{\gamma}$  and velocity  $\mathbf{v}$  are vectors carried by the  $\Gamma$  oriented segment; scalar potential  $\phi$  is assigned to its ends and vector potential  $\boldsymbol{\psi}$  is fixed on the  $\Delta$  contour.

A variable electric current flowing on  $\Gamma$  produces an induced current on  $\Delta$ . This is the result of Maxwell's unification of the laws of magnetism and electrodynamics established by his predecessors into a dynamic vision [26]. This model is used here to derive a mechanical equation as an alternative to Newton's fundamental law of mechanics. Maxwell's local frame of reference, shown in Figure 1, is a radical change from the point of view of the perception of space; the description of distant events is inaccessible by changing the frame of reference, interactions take place from cause to effect. If is the unit vector defined by the expression  $\mathbf{m} = \mathbf{t} \times \mathbf{n}$ , then Maxwell's local frame of reference  $(\mathbf{t}, \mathbf{n}, \mathbf{m})$  replaces the global frame of reference  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  of classical mechanics.

The acceleration  $\boldsymbol{\gamma}$  and the velocity  $\mathbf{v}$  are discrete quantities defined on the segment  $\Gamma$ ; they are both components of spatial vectors, the knowledge of which is not required, and scalars associated with the segment oriented by the unit vector  $\mathbf{t}$ . The scalars, such as the scalar potential  $\phi$ , are located at the vertices  $a$  and  $b$  of the original structure, and the pseudo-vectors, such as the vector potential  $\boldsymbol{\psi}$ , are carried by the dual contour  $\Delta$ , oriented positively by the unit vector  $\mathbf{n}$ . All local reference frames of Maxwell are assembled by their common vertices to tessellate the physical domain under consideration. By locating the variables in this reference frame, we can eliminate any interpolation in the derivation of the law of discrete motion. To

distinguish the discrete velocity  $\mathbf{v}$  from the velocity vector of the space  $\mathbb{R}^3(x, y, z)$  of classical continuum mechanics, the latter is denoted  $\mathbf{V}$ .

The information is transferred between the two geometric structures using four discrete operators, (i) the discrete divergence of the velocity  $\nabla \cdot \mathbf{v}$  associated with the vertices of the primal structure, obtained by summing the fluxes entering the dual volume, (ii) the primal curl of the velocity  $\nabla \times \mathbf{v}$ , calculated from the circulation on a contour formed by coplanar segments, (iii) the gradient of the scalar potential  $\nabla\phi = (\phi_b - \phi_a)/dh$ , which is the restriction to the segment  $\Gamma$  of the gradient vector of space, and (iv) the dual curl of the vector potential  $\nabla \otimes \boldsymbol{\psi}$ , also computed by Stokes' theorem on the dual contour  $\Delta$ . The operator  $\nabla \otimes$  denoting the dual curl is not the tensor product of classical mechanics; any confusion is avoided because the discrete formulation has no tensors of order greater than or equal to two and the vectors themselves are scalars associated with oriented segments. Note that the primal curl  $\nabla \times \mathbf{v}$  has a double representation, that of a scalar  $\omega = \nabla \times \mathbf{v} \cdot \mathbf{n} = \boldsymbol{\omega} \cdot \mathbf{n}$  located at the barycentre of the primal facet and that of a vector  $\boldsymbol{\omega}$  orthogonal to it. The geometric structures have several remarkable properties that mimic those of the continuous medium, including the identities  $\nabla \cdot \nabla \otimes \boldsymbol{\psi} = 0$  and  $\nabla \times \nabla\phi = 0$ .

## 2.2. Generic discrete equation of motion

The exegesis of the *Principia* [30] by specialists in the history of science highlights the status of external force attributed by I. Newton to the force of inertia; the latter must then have a cause, a real source in the context of absolute space. For Newton, the inertia of a mass and the force of inertia are identical. Without using the term acceleration, he introduced the term modification of the state of rest or uniform rectilinear velocity. Ernst Mach, for his part, formulated another version of this concept: the inertia of a mass would be all the other masses present in the universe. In fact, since Galileo, the universal law of falling bodies has closely linked inertial mass to gravitational mass - the weak equivalence principle (WEP). The theory of relativity does not change the meaning of inertia; mass, momentum and force are all representations that tend to federate the two abstractions. Nearly five centuries after Galileo, the notion of mass remains amalgamated with the notion of inertia.

Discrete mechanics breaks with this view and finally abandons the notion of mass to describe inertia in a new law of motion; from this perspective, inertia becomes a local concept, intrinsic to motion. In the absence of motion, the concept of inertia itself does not exist. Similarly, for uniform rectilinear motion and for rigid rotational motion, inertia is zero; in the other cases, motion is accelerated. It should be noted that, contrary to the definition given in classical mechanics or in the theory of relativity, uniform rotational motion is not accelerated [6]. In this context, inertia has nothing to do with mass; it is the local mean curvature of Bernoulli's inertial potential,  $\phi_i = |\mathbf{v}|^2/2$ . The example of the photon deflected by the gravitational attraction of the Sun or black holes shows that it has inertia even when its mass is zero. The spatial description of inertia is strictly contained in the Maxwell frame of reference.

The new interpretation of WEP leads to the fundamental law of kinematics  $\boldsymbol{\gamma} = \mathbf{h}$ , the intrinsic acceleration of a particle with or without mass or of a material medium is equal to the sum of the accelerations applied to it. The law of discrete motion derived earlier [7] is

$$\left\{ \begin{array}{l} \frac{d\mathbf{v}}{dt} = -\nabla(\phi^o - c_t^2 dt \nabla \cdot \mathbf{v}) + \nabla \otimes (\boldsymbol{\psi}^o - c_t^2 dt \nabla \times \mathbf{v}) + \mathbf{h}_s, \\ \alpha_l \phi^o - c_t^2 dt \nabla \cdot (\mathbf{v} - \mathbf{v}^o) \mapsto \phi^o, \\ \alpha_t \boldsymbol{\psi}^o - c_t^2 dt \nabla \times (\mathbf{v} - \mathbf{v}^o) \mapsto \boldsymbol{\psi}^o, \end{array} \right. \quad (1)$$

where  $\mathbf{h}_s = -\nabla\phi_s + \nabla \otimes \boldsymbol{\psi}_s$  is a generic source term, also formulated in a two-term Helmholtz-Hodge decomposition. The celerities of the longitudinal waves  $c_l$  and the transverse waves  $c_t$  are specific properties of these waves in the considered medium. The quantities  $\alpha_l$  and  $\alpha_t$  between 0 and 1 are the restitution factors of the longitudinal and transverse waves respectively and are related to the absorption coefficients of these waves. When these factors are equal to one, the dissipation is zero and the total energy is conserved.

The properties of this discrete law of motion, established earlier, are based in part on its form,  $\boldsymbol{\gamma} = -\nabla\phi + \nabla \otimes \boldsymbol{\psi}$ ; the intrinsic acceleration is written as a two-term Helmholtz-Hodge decomposition, the first curl-free and the second divergence-free. Despite its major differences from the Navier-Stokes equations, it can be used to find solutions to the latter, with the exception of the two examples presented here.

### 2.3. A local law without limits

The discrete law of motion (2) is a strong local formulation without any interaction *a priori* with the boundary conditions. The law of motion (1) is not necessarily related to the boundary conditions as long as the longitudinal and transverse waves do not reach the limits of the global primal structure constituted by the set of interconnected Maxwell reference frames. If this is the case, or if we are looking for a stationary solution, it is then necessary to set certain constraints. In discrete mechanics, these constraints differ significantly from those applied to define the boundary conditions associated with the Navier-Stokes equations. As the physical model is built entirely on the local Maxwell frame of reference, the constraints must be applied locally. They are of two types only, (i) a flow injected on the vertices of the primal structure and (ii) a shear stress imposed on the segment  $\Gamma$ . The value of the flow  $q$  is a quantity expressed in  $m^2 s^{-2}$ , an energy per unit mass that can be related to a flow rate; the shear stress  $\mathbf{r}$  is, in the same way, an energy per unit mass. The flow rate can be associated with a velocity  $\mathbf{v}_0$  to evaluate the parameter  $q = dt c_l^2 \nabla \cdot \mathbf{v}_0$  in a form equivalent to the compressive energy term. Similarly, the shear stress is expressed from the kinematic viscosity,  $\mathbf{r} = \nu \nabla \cdot \mathbf{v}_0$ . The corresponding accelerations  $-\nabla q$  and  $\nabla \otimes \mathbf{r}$  are projected onto the segment  $\Gamma$  as intrinsic acceleration  $\boldsymbol{\gamma}$  and velocity  $\mathbf{v}$ . A Dirichlet condition on the velocity  $\mathbf{v} = \mathbf{v}_0$  can be imposed directly on a segment  $\Gamma$  even if this procedure is not in accordance with the spirit of the formulation ; indeed, the velocity is a relative quantity which can only be deduced from its value at the preceding instant,  $\mathbf{v} = \mathbf{v}^o + \boldsymbol{\gamma} dt$ , it is therefore the acceleration which must be imposed. The law of motion incorporating these conditions is:

$$\frac{d\mathbf{v}}{dt} = -\nabla (\phi^o - c_l^2 dt \nabla \cdot \mathbf{v} + q) + \nabla \otimes (\boldsymbol{\psi}^o - c_t^2 dt \nabla \times \mathbf{v} + \mathbf{r}). \quad (2)$$

The parameters  $q$  and  $\mathbf{r}$  are applied respectively to each vertex and to each segment within the physical domain or to each element constituting the boundaries of the latter. These two quantities are sufficient to reproduce any type of boundary conditions usually applied when solving the Navier-Stokes equations. The partial derivatives in space describing the boundary conditions are replaced by discrete operators consistent with the formulation of the law of motion.

Certain notions of mathematical analysis to establish the properties of the Navier-Stokes equations are no longer necessary, for example that of considering the decay of a solution to infinity. The intrinsic hyperbolic nature of the law of motion (1) rules out any possibility of reaching infinity in a finite time. Classical variational formulations aim to replace the Navier-Stokes equation by an equivalent formulation, obtained by integrating the equation multiplied by any test function. It is then necessary to integrate by parts, which gives rise to boundary conditions. These boundary terms are then eliminated by the assumed regularity of the desired solution or by *ad hoc* boundary conditions.

### 2.4. On the properties of Lamb's vector

The inertia term of the Navier-Stokes equations,  $\mathcal{K} = \mathbf{V} \cdot \nabla \mathbf{V}$ , is strictly equal to the rotational form  $\mathcal{K} = \nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}$  where the latter term is none other than the Lamb vector,  $\mathcal{L} = -\mathbf{V} \times \boldsymbol{\omega}$  with  $\boldsymbol{\omega} = \nabla \times \mathbf{V}$ . Applying the divergence and primal curl operators to the inertia vector  $\mathcal{K}$  reveals its specific properties,

$$\begin{cases} \nabla \cdot \mathcal{K} = \mathbf{V} \cdot \nabla (\nabla \cdot \mathbf{V}) + (\nabla \cdot \mathbf{V})^2 - 2 I_2, \\ \nabla \times \mathcal{K} = \mathbf{V} \cdot \nabla (\nabla \times \mathbf{V}) - \nabla \times \mathbf{V} \cdot \nabla \mathbf{V}, \end{cases} \quad (3)$$

where  $\mathbf{V} \cdot \nabla \delta$  and  $\mathbf{V} \cdot \nabla \boldsymbol{\omega}$  are transport terms for the divergence  $\delta = \nabla \cdot \mathbf{V}$  and the vortex. The term  $I_2$  is the second invariant of the tensor  $\nabla \mathbf{V}$  *a priori* non-zero, including for incompressible flow. The additional terms  $(\nabla \cdot \mathbf{V})^2$ ,  $-2 I_2$  and  $\boldsymbol{\omega} \cdot \nabla \mathbf{V}$ , mathematically exact whatever the classical formulation of inertia, have an uncertain physical significance. The physical analysis of the divergence of the Lamb vector is already a topic addressed in the literature [20, 7] but its curl is just as instructive.

Let's take the expression of Stokes' theorem in the framework of classical mechanics,

$$\int_{\partial \mathcal{S}} \mathbf{V} \cdot \mathbf{t} \, dl = \int_{\mathcal{S}} \nabla \times \mathbf{V} \cdot \mathbf{n} \, ds, \quad (4)$$

where  $\mathbf{t}$  is positively oriented according to the choice of orientation of the normal  $\mathbf{n}$  to a facet of the surface  $\mathcal{S}$  and where  $\partial \mathcal{S}$  is its edge (Figure 2).

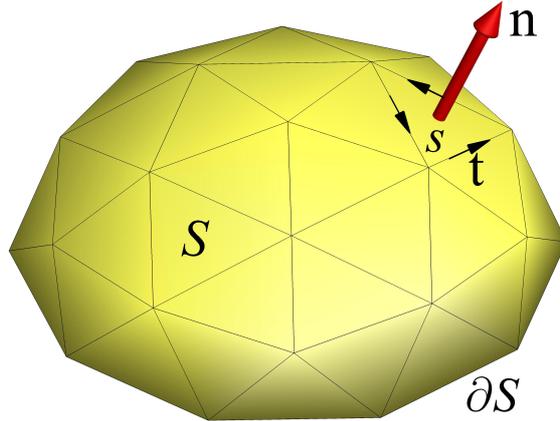


Figure 2: Stokes' theorem applied to a surface  $\mathcal{S}$  bounded by the edge  $\partial \mathcal{S}$ ; the surface is composed of planar facets  $\mathbf{s}$  oriented by the normal  $\mathbf{n}$  while each segment of the surface and the edge are oriented along the unit vector  $\mathbf{t}$ .

If the approach adopted is that of discrete mechanics, the physical meaning given to Stokes' Theorem (4) persists as long as the edge length and the surface  $\mathcal{S}$  are themselves significant. These quantities can be as small as necessary, but in no case can they be reduced to a point. Establishing theorems on surfaces or volumes, as in classical mechanics, to extrapolate local laws is a difficult process to master, since the notion of orientation is first ignored and then reintroduced by the need to use a three-dimensional orthonormal frame of reference. In the present case, the reduction of the length of the edges and of the surface of each facet of  $\mathbf{s}$  leads, to the nearest second order, to a velocity defined in a plane of this facet; by extension, considering all facets, the velocity of  $\mathbf{V}$  is defined only on the surface of  $\mathcal{S}$  and not in the space of  $\mathbb{R}^3$ ; this velocity is the restriction of the space velocity on  $\mathcal{S}$ . Thus the operator  $\nabla \times \mathbf{V}$  makes no physical sense when defined per point in the context of the continuous medium concept; whatever the approach adopted and whatever the dimension of space, this operator must be defined for a fixed

surface  $\mathcal{S}$ . The application of Stokes' theorem then depends on defining the surface velocity  $\mathbf{V}$  such that  $\mathbf{V} \cdot \mathbf{n}$  on each facet  $\mathbf{s}$  of the surface  $\mathcal{S}$ .

Let's apply the curl operator to the Lamb vector,  $-\nabla \times (\mathbf{V} \times \boldsymbol{\omega})$ :

$$\nabla \times (\mathbf{V} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \nabla \mathbf{V} - \boldsymbol{\omega} \nabla \cdot \mathbf{V} - \mathbf{V} \cdot \nabla \boldsymbol{\omega} + \mathbf{V} \nabla \cdot \boldsymbol{\omega}. \quad (5)$$

This expression (5) can be simplified by considering (i) that the flow is incompressible  $\nabla \cdot \mathbf{V} = 0$ , (ii) that the vortex divergence is zero  $\nabla \cdot \boldsymbol{\omega} = 0$ , that is,

$$\nabla \times \mathcal{L} = -\nabla \times (\mathbf{V} \times \boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \nabla \mathbf{V}. \quad (6)$$

The Navier-Stokes equation for incompressible flow with constant viscosity becomes:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{V} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{V} + \nu \nabla^2 \boldsymbol{\omega}, \quad (7)$$

but in two-dimensional space the velocity gradient is a vector defined in the plane and  $\boldsymbol{\omega} \cdot \nabla \mathbf{V} = 0$ . Applying the curl operator to the components of the Navier-Stokes equations reveals a significant difference between two- and three-dimensional motion, but if we accept that the curl operator is associated with a single surface, the term  $\boldsymbol{\omega} \cdot \nabla \mathbf{V}$  is always zero. This paradox is resolved by considering that the velocity vector  $\mathbf{V}$  and its gradient  $\nabla \mathbf{V}$  are defined by each plane surface and that its curl operator is orthogonal to it. Under these conditions, Lamb's vector is a plane vector because the vector product  $\mathbf{V} \times \boldsymbol{\omega}$  is orthogonal to the surface  $\mathbf{s}$ . The rotational form of the equation reduces to

$$\frac{d\boldsymbol{\omega}}{dt} = \nu \nabla^2 \boldsymbol{\omega}, \quad (8)$$

an advection-diffusion equation, whatever the spatial dimension of the problem posed.

The notion of a local rotation associated with a point has no physical meaning; it's an artefact of the concept of a continuous medium. From a practical point of view, calculating the three components of the curl of a vector from its partial derivatives leads to its evaluation in terms of averages that depend on the position of the variables on the geometric structures. In the general case, Lamb's vector,  $\mathcal{L}$ , is neither divergence-free nor curl-free, even for a divergence-free velocity field, and is neither a gradient field nor a solenoidal field. A class of zero-inertia problems  $\mathbf{V} \cdot \nabla \mathbf{V} = 0$ , e.g. Laminar channel flow, Poiseuille plane flow, leads us to define the Lamb vector as a gradient field  $\nabla(|\mathbf{V}|^2/2) = \mathbf{V} \times \nabla \times \mathbf{V}$ ; this vector is orthogonal to both the planar and axial velocity and is therefore equal and opposite to the velocity gradient. This vector equality is ensured for each component of the  $\mathbb{R}^3$  coordinate system. The way in which the inertia term becomes non-zero in laminar flow is undoubtedly related to the balance between these two terms and is one of the keys to the loss of flow stability. The indeterminacy of the nature of the Lamb vector makes the physical significance of this term for its interpretation in general, and for turbulence in particular, more uncertain.

Another way of describing inertia is proposed by discrete mechanics [4], where the concept is translated into a Helmholtz-Hodge decomposition of the inertial potential. In this formalism there is no distinction between two-dimensional and three-dimensional motion: the law of motion is spatially invariant.

## 2.5. Inertia as a mean curvature of Bernoulli potential

In discrete mechanics, the notion of curvature must be redefined by considering the planar facet  $\mathbf{s}$  bounded by the partial contour  $\mathbf{s}$  of the Figure (2); vectors are scalars on segments  $\Gamma$

oriented by  $\mathbf{t}$  and the facet is oriented by its normal  $\mathbf{n}$ . Hence Stokes' theorem,

$$\int_{\Gamma} \mathbf{v} dl = \int_{\mathbf{s}} \nabla \times \mathbf{v} ds, \quad (9)$$

becomes an identity vector. From the discrete point of view, the quantity  $\mathbf{v}$  is constant on the segment and  $\nabla \times \mathbf{v}$  is also constant on the whole facet; this vector is carried by the unit vector  $\mathbf{n}$ . The dual curl operator applied to the potential vector  $\boldsymbol{\psi}$  projects the result onto the segment  $\Gamma$ ,

$$\int_{\Delta} \nabla \otimes \boldsymbol{\psi} ds = \int_{\Gamma} \mathbf{w} dl, \quad (10)$$

where  $\mathbf{v}$  and  $\mathbf{w}$  can thus benefit from the simple addition operation. This specific property eliminates the artefacts of classical mechanics based on approximations of the one-point derivation.

Let's consider the separation of the kinetic energies into two components, the translational kinetic energy, denoted  $\Phi_k$ , and the rotational kinetic energy, denoted  $\Psi_k$ . These are the integrals on the primal segment  $\Gamma$  defined by the local energies per unit mass  $\phi_k$  and  $\boldsymbol{\psi}_k$ ,

$$\begin{cases} \Phi_k = \int_{\Gamma} \nabla \phi_k dl = \int_{\Gamma} \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) dl, \\ \Psi_k = \int_{\Gamma} \nabla \otimes \boldsymbol{\psi}_k dl = \int_{\Delta} \nabla \otimes \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) dl, \end{cases} \quad (11)$$

The global kinetic energy per unit mass is then the sum  $\Phi_k + \Psi_k$  of the Maxwell reference frame  $\mathbb{M}$  shown in Figure 1. An essential feature of this model of conservation of kinetic energy is the unique projection of the two energies of translation and rotation onto the  $\Gamma$ -segment, allowing a simple addition. In fact, the conservation of kinetic energy can be expressed in terms of two Bernoulli theorems, referred to here as the primal theorem and the dual theorem:

$$\begin{cases} \phi_k + \frac{|\mathbf{v}|^2}{2} = cte \text{ along the } \Gamma \text{ path,} \\ \boldsymbol{\psi}_k + \frac{|\mathbf{v}|^2}{2} \mathbf{n} = cte \text{ along the } \Delta \text{ path.} \end{cases} \quad (12)$$

The mutual transformation of the kinetic energies  $\phi_k$  and  $\boldsymbol{\psi}_k$  is a complex process because the operators associated with them, the gradient and the dual curl, are orthogonal; like the direct and induced currents that Maxwell unified to establish the laws of electromagnetism, the two kinetic energies can only be exchanged in the case of unsteady motion. Let's consider the case of rotational motion at constant angular velocity  $\boldsymbol{\Omega} = \boldsymbol{\omega} \mathbf{t}$  so that the velocity is equal to  $\boldsymbol{\omega} r \mathbf{n}$  for a domain of surface  $\mathcal{D}$  of circular contour  $\Delta$  and radius  $R$ . The kinetic energy of rotation is carried by the segment  $\Gamma$ ,

$$\int_{\Gamma} \frac{|\mathbf{v}|^2}{2} \mathbf{t} dl = - \int_{\mathcal{D}} \nabla \otimes \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) ds = \pi \boldsymbol{\omega}^2 R^3, \quad (13)$$

where it is the dual curl  $\nabla \otimes$  that projects the result onto the segment.

Another important aspect of the discrete formulation is the absence of mass. Whereas in classical mechanics kinetic energy is expressed by a volume integration of  $\rho |\mathbf{V}|^2/2$ , in discrete mechanics the local kinetic energy is expressed by  $e_k = |\mathbf{V}|^2/2$ , an energy per unit mass. The rotational energy, on the other hand, depends on the moment of inertia  $J_{\Gamma}$  and not on the mass,

and is written as  $E_k = J_\Gamma \boldsymbol{\omega}^2/2$ . Moment of inertia is to rotational motion what mass is to translational motion. By abandoning the concepts of mass and moment of inertia, it is possible to combine translational and rotational energies and, equivalently, translational and angular accelerations, the sum of which equals the intrinsic acceleration  $\boldsymbol{\gamma}$ .

In discrete mechanics, inertia plays a very special role: it is completely integrated into the other terms of the law of motion in the same form. For a velocity field  $\boldsymbol{v}$  the material derivative is written  $d\boldsymbol{v}/dt = \partial\boldsymbol{v}/\partial t + \boldsymbol{\kappa}$  with

$$\boldsymbol{\kappa} = \nabla \left( \frac{1}{2} |\boldsymbol{v}|^2 \right) - \nabla \otimes \left( \frac{1}{2} |\boldsymbol{v}|^2 \mathbf{n} \right), \quad (14)$$

where  $\boldsymbol{\kappa}$  is defined by the inertial potential  $\phi_i = |\boldsymbol{v}|^2/2$ . Its deeper meaning is related to the spatial curvature of this potential, where the two orthogonal terms  $-\nabla\phi_i + \nabla \otimes (\phi_i \mathbf{n})$  are equivalent to those of the curvature of a surface in space [8]. Their values may add up, as in the case of a sphere, or cancel out, as in the case of a minimal surface such as a catenoid. In astrophysics, as in capillary effects, the energy minimisation that leads to a spherical drop in equilibrium leads to quasi-spherical planets under the influence of gravity, although these phenomena are completely dissociated. The specific form of inertia (14) corresponding to the curvature of the Bernoulli potential does not depend on the reference frame chosen; whatever the orientation of the segment  $\Gamma$ , the result on  $\boldsymbol{\kappa}$  is the same. If the vector  $\mathbf{m}$  is defined by  $\mathbf{m} = \mathbf{t} \times \mathbf{n}$ , then the rotation of the reference frame  $(\mathbf{t}, \mathbf{m})$  centred on the segment  $\Gamma$  around  $\mathbf{n}$  does not change the mean curvature  $\boldsymbol{\kappa}$ , a result of differential geometry stated by Euler's theorem [16, 10]. The special properties associated with the mean curvature are as follows

$$\begin{cases} \nabla \cdot \boldsymbol{\kappa} = \nabla^2 \left( \frac{|\boldsymbol{v}|^2}{2} \right), \\ \nabla \times \boldsymbol{\kappa} = \nabla^2 \left( \frac{|\boldsymbol{v}|^2}{2} \mathbf{n} \right), \end{cases} \quad (15)$$

given the orthogonality relations of the discrete operators, which mimic those of the continuous medium. Both terms of the expression (14) can be simultaneously zero, but some flows have a mean curvature by compensation of two terms; this is the case of Couette flow or Poiseuille flow.

The inertia term in classical mechanics  $\nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}$  consists of a first term, that of Bernoulli's law, and a second term called Lamb's vector  $\boldsymbol{\mathcal{L}} = -\mathbf{V} \times \nabla \times \mathbf{V}$ , which has a non-zero divergence and generates fictitious non-physical forces, but which are necessary to ensure mechanical equilibrium in a Galilean frame of reference. It is the local frame of reference  $\mathbb{M}$  that leads to the possibility of replacing the Lamb vector by a dual curl such that  $\nabla \cdot (\nabla \otimes \boldsymbol{\psi}_i) = 0$ .

Noting  $\delta = \nabla \cdot \boldsymbol{v}$  the divergence of velocity and  $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$  its primal curl,  $\phi_i = |\boldsymbol{v}|^2/2$  and  $\boldsymbol{\psi}_i = |\boldsymbol{v}|^2/2 \mathbf{n}$  the scalar and vector potentials of inertia and  $\phi^o$ , the scalar potential of acceleration and  $\boldsymbol{\psi}^o = -\nu \nabla \times \boldsymbol{v}$  its vector potential, the applications of the divergence and curl operators to the law of motion take the following forms:

$$\begin{cases} \nabla \cdot \boldsymbol{\gamma} \equiv \frac{\partial \delta}{\partial t} + \nabla^2 \phi_i = -\nabla^2 \phi^o, \\ \nabla \times \boldsymbol{\gamma} \equiv \frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla^2 \boldsymbol{\psi}_i = -\nabla^2 \boldsymbol{\psi}^o. \end{cases} \quad (16)$$

The quantity  $\delta$  is located on the vertices of the primal geometric structure and  $\boldsymbol{\omega}$  is both the vortex vector and the vector carried by the normal to a facet  $\boldsymbol{\omega} = \omega \mathbf{n}$  with  $\omega$ , a scalar located at the barycentre of the facet. In this sense, the relations (16) are advection-diffusion transport

equations for the divergence  $\delta$  and the vortex  $\omega$ .

These forms (16) must be used with great care, as they are not laws of motion. Application of the operators transforms the discrete fundamental law of motion, eliminating contributions that may be important in some cases. It should be noted that the application of the divergence and primal curl operators to the law of motion gives them an equivalent role and a symmetry that does not exist in classical mechanics. Indeed, the application of the divergence and primal curl operators to the law of motion itself leads to different results from those obtained with the Navier-Stokes equations.

Inertia (14) is the cornerstone of discrete mechanics, underpinning the global formalism of the law of motion where each contribution participates in the orthogonal Helmholtz-Hodge decomposition. The equation of motion then becomes

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \phi^o + \frac{|\mathbf{v}|^2}{2} - c_t^2 dt \nabla \cdot \mathbf{v} \right) + \nabla \otimes \left( \psi^o + \frac{|\mathbf{v}|^2}{2} \mathbf{n} - c_t^2 dt \nabla \times \mathbf{v} \right), \quad (17)$$

where scalar and vector potentials become Bernoulli potentials,  $\phi_B^o = \phi^o + \phi_i$  and  $\psi_B^o = \psi^o + \phi_i \mathbf{n}$ . Thus, the Lagrangian formulation (1) and the Eulerian form (17) are represented as a two-term orthogonal Helmholtz-Hodge decomposition. The strict separation of solenoidal and irrotational terms does not exist in classical mechanics; for example, the Navier-Stokes equations are composed of terms that are neither divergence-free nor curl-free; moreover, this equation is always accompanied by a mass conservation equation, whereas the equation (17) implicitly ensures this conservation. In the case of a Newtonian viscous fluid, the vector potential  $\psi^o$  is zero, as the accumulation of viscous stresses is no longer possible, and the grouping  $dt c_t^2$  is replaced by the kinematic viscosity  $\nu$ .

## 2.6. Inertia for superposition of two motions

Equation (17) is generic and applies to all physical phenomena covered by classical field theory, including the motion of fluids and solids, and wave propagation of all kinds. In the general case of couplings between different physical phenomena, the relative celerities of each of these phenomena lead to additional non-linear terms for the inertia term. The analysis can be performed for two velocities  $\mathbf{u}$  and  $\mathbf{w}$ , both associated with the same segment  $\Gamma$ ; the nonlinear term then takes the form,

$$(\mathbf{u} \pm \mathbf{w})^2 = |\mathbf{u}|^2 \pm 2 \mathbf{u} \cdot \mathbf{w} + |\mathbf{w}|^2, \quad (18)$$

where the scalar product  $\mathbf{u} \cdot \mathbf{w}$  is here a simple product carried by the same segment  $\Gamma$ . Noticing that  $\partial(\mathbf{u} + \mathbf{w})/\partial t = \partial\mathbf{u}/\partial t + \partial\mathbf{w}/\partial t$ , the material derivative becomes  $d(\mathbf{u} + \mathbf{w})/dt = \partial\mathbf{u}/\partial t + \kappa$  with

$$\kappa = \nabla \left( \frac{1}{2} (\mathbf{u} + \mathbf{w})^2 \right) - \nabla \otimes \left( \frac{1}{2} (\mathbf{u} + \mathbf{w})^2 \mathbf{n} \right) = \begin{cases} \nabla \left( \frac{1}{2} |\mathbf{u}|^2 \right) - \nabla \otimes \left( \frac{1}{2} |\mathbf{u}|^2 \mathbf{n} \right) \\ + \nabla \left( \frac{1}{2} |\mathbf{w}|^2 \right) - \nabla \otimes \left( \frac{1}{2} |\mathbf{w}|^2 \mathbf{n} \right) \\ + \nabla (\mathbf{u} \cdot \mathbf{w}) - \nabla \otimes (\mathbf{u} \cdot \mathbf{w} \mathbf{n}). \end{cases} \quad (19)$$

The inertia  $\kappa$  comprises three Helmholtz-Hodge decompositions, (i) that of the inertia associated with the vector  $\mathbf{u}$ , (ii) the same terms for the velocity  $\mathbf{w}$  and (iii) the inertial terms of the scalar product  $\mathbf{u} \cdot \mathbf{w}$ . The first two groups correspond to the inertial terms specific to each component. Since the discrete velocities  $\mathbf{u}$  and  $\mathbf{w}$  are carried by the same segment  $\Gamma$ , the scalar product is a simple product; however, if the velocities in space are orthogonal, the product of

the components on the same segment is also zero.

### 2.7. Case of a uniform rotational motion

Let's consider the case of a single motion defined by the components  $\mathbf{w} = \boldsymbol{\Omega} \times \mathbf{r} = \boldsymbol{\omega} r e_\theta$  where  $\boldsymbol{\Omega}$  is the constant rotation vector of the  $Oz$  axis,  $\theta$ , the polar angle and  $\mathbf{r}$  the distance vector from the axis. The introduction of the velocity  $\mathbf{w}$  into the law of discrete motion (1) is as follows:

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \left( \frac{|\mathbf{w}|^2}{2} \right) - \nabla \otimes \left( \frac{|\mathbf{w}|^2}{2} \mathbf{n} \right) = -\nabla (\phi^o - c_l^2 dt \nabla \cdot \mathbf{w}) + \nabla \otimes (\boldsymbol{\psi}^o - \nu \nabla \times \mathbf{w}). \quad (20)$$

Calculating each term in this equation (20) leads to simplifications,

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{w}}{\partial t} = 0; \quad \nabla \cdot \mathbf{w} = 0, \\ \nabla \left( \frac{|\mathbf{w}|^2}{2} \right) - \nabla \otimes \left( \frac{|\mathbf{w}|^2}{2} \mathbf{n} \right) = 0, \\ -\nabla \otimes (\nu \nabla \times \mathbf{w}) = 0, \end{array} \right. \quad (21)$$

where, in particular, inertia is zero by compensation. Thus, the sum of all terms is zero, including the potential,  $\phi^o = 0$  since the divergence of  $\mathbf{w}$  is zero, and the scalar potential  $\boldsymbol{\psi}^o = 0$  since motion is inviscid.

This result is the basis for the extension of Galilean or inertial invariance to discrete mechanics based on Maxwell's local frame of reference. Rigid rotational motion is simply unobservable by the law of motion. However, due to non-linearities, the interactions between uniform rotation  $\mathbf{w}$  and any other motion  $\mathbf{u}$  are non-zero.

Subtracting equation (21) from equation (1) where  $\mathbf{v}$  is replaced by  $\mathbf{u} + \mathbf{w}$ , the equation remains:

$$\frac{d\mathbf{u}}{dt} + \nabla (\mathbf{u} \cdot \mathbf{w}) - \nabla \otimes (\mathbf{u} \cdot \mathbf{w} \mathbf{n}) = -\nabla (\phi^o - c_l^2 dt \nabla \cdot \mathbf{u}) + \nabla \otimes (\boldsymbol{\psi}^o - \nu \nabla \times \mathbf{u}). \quad (22)$$

The equation (22) is composed of the classical terms of the law of generic motion (1) and two non-linear terms based on the scalar product  $(\mathbf{u} \cdot \mathbf{w})$ . This term is also a simple product, since  $\mathbf{u}$  and  $\mathbf{w}$  are carried by the same segment  $\Gamma$ . The quantity  $\boldsymbol{\kappa} = \nabla (\mathbf{u} \cdot \mathbf{w}) - \nabla \otimes (\mathbf{u} \cdot \mathbf{w} \mathbf{n})$  is the mean curvature of the scalar potential  $(\mathbf{u} \cdot \mathbf{w})$ . The curvature  $\boldsymbol{\kappa}$  can be zero for two reasons, (i) the scalar product  $\mathbf{u} \cdot \mathbf{w} = 0$  is zero or, (ii) the two terms of the curvature are opposite. While the first case is rarely encountered, the second can be observed in many flows of real practical interest, such as Poiseuille flow. The laminar flow corresponding to the Poiseuille solution persists as long as the Reynolds number is below a critical value  $Re_c \approx 2300$ , after which it becomes unstable and then turbulent; the appearance of instabilities is strongly linked to the growth of the non-linear terms and therefore to the curvature  $\boldsymbol{\kappa}$ . Indeed, the other terms of the equation of motion do not change fundamentally and, moreover, the Poiseuille solution is independent of the viscosity  $\nu$  and therefore of the Reynolds number; under certain conditions, it is possible to maintain laminar flow well beyond  $Re_c$ . The subcritical bifurcation of the solutions is probably linked to the  $bm\kappa$  curvature, but this concomitance has not yet been established.

### 3. On a simple motion

The Navier-Stokes equations must be verified identically when a physically admissible solution is injected into them. Similarly, if a physical solution is sought from the Navier-Stokes equations

by assigning physical assumptions and boundary conditions, it must respect these constraints and be physically acceptable. Numerous textbooks [2, 41] describe the classical solutions of the Navier-Stokes equations; they are all consistent and representative of reality. However, completeness can only be achieved when no solution can fail the mathematical equation and the underlying physical model. If an acceptable and experimentally feasible solution fails to satisfy the equation, the whole edifice is weakened.

This section presents a motion composed of two superimposed solutions whose non-linear interaction is not zero, a motion corresponding to a point source and a uniform rotation at constant velocity. This motion can be realised experimentally in the form of a flow around a cylinder of radius  $R_1$  rotating around its axis  $Oz$ . The volume of the fluid is limited by a cylinder of radius  $R_2$  and the same axis. In steady mode, all the fluid is driven in a rigid rotation of angular velocity  $\boldsymbol{\Omega}$ . At the same time, the same fluid is injected by the inner cylinder with a flow rate  $D$ , while the outlet to the outer cylinder corresponds to free conditions. This motion is treated as inviscid, but introducing the viscosity of the fluid would change absolutely nothing. Only inertia governs the behaviour of the solution in terms of steady-state velocity and pressure. It is of course possible to find the solution to this problem directly from the Navier-Stokes equations and the boundary conditions, but this physical solution corresponds to the superposition of two motions, those of a uniform rotational motion  $\mathbf{W}$  and a radial motion  $\mathbf{U}$ .

Let us therefore consider two incompressible stationary motions in a three-dimensional space  $\mathbf{U}$  and  $\mathbf{W}$  associated with an orthonormal frame of reference  $\mathbb{R}^3(r, \theta, z)$ . The sum of these two motions is called  $\mathbf{V} = \mathbf{U} + \mathbf{W}$ . The field  $\mathbf{U} = a/r \mathbf{e}_r$  corresponds to a radial flow with origin  $Oz$  where  $a$  is a constant and  $\mathbf{W} = \boldsymbol{\Omega} \times \mathbf{r} = \omega r \mathbf{e}_\theta$  is a rigid rotating flow around  $Oz$  of velocity of rotation  $\boldsymbol{\Omega}$ . The first flow is accelerated and non-inertial and the second is also considered to be non-inertial in the context of classical mechanics. The velocity field  $\mathbf{V}$  thus defined is physically admissible and corresponds to an inviscid flow. The two fields of which it is composed are orthogonal,  $\mathbf{U} \cdot \mathbf{W} = 0$ ; the field  $\mathbf{U}$  has zero divergence and zero curl and  $\mathbf{W}$  has zero divergence and constant curl. The velocity field  $\mathbf{U}$  can be studied in the context of potential holomorphic functions representing irrotational perfect fluid flows from the function  $f(z) = D/2\pi \text{Ln}z$  where  $D = 2\pi a$  is the flow rate of a source centred at the origin and  $z = x + iy$ , the complex variable. The uniform motion  $\mathbf{W}$  is not irrotational and cannot be studied using the same formalism. The velocity field,

$$\mathbf{V}(r) = \frac{a}{r} \mathbf{e}_r + \omega r \mathbf{e}_\theta, \quad (23)$$

can be used to calculate streamlines, which are plane spirals with decreasing pitch in a fixed frame of reference. This motion is strictly axisymmetric and the velocity components do not depend on the polar angle.

In discrete mechanics, the velocity is defined on each segment  $\Gamma$  of the primal structure; the values assigned are equal to  $\mathbf{u} = a/r \mathbf{e}_r$  and  $\mathbf{w} = \omega r \mathbf{e}_\theta$ .

### 3.1. With Navier-Stokes equations

This motion is perfectly admissible and can be carried out experimentally; it must be an exact solution of the Navier-Stokes equations in a steady incompressible formulation. To check this, we inject this field  $\mathbf{V}$  into the Navier-Stokes equations; the only unknown factor remains the pressure  $p(r, \theta)$  in polar coordinates. The inertia contained in the derivative of the material can be written in different ways:

$$\mathbf{V} \cdot \nabla \mathbf{V} = \nabla \left( \frac{|\mathbf{V}|^2}{2} \right) - \mathbf{V} \times \nabla \times \mathbf{V}, \quad (24)$$

where the last term is the Lamb vector, which is neither a gradient nor a curl vector. It plays

an important role in rotating flows, including turbulence, in the energy cascade of vortices. It is the second form of inertia (24) that is used, but the conclusions are exactly the same as with the first form. In particular, if the inertia is zero, the two terms of the second form are equal.

Since the motion is stationary, the acceleration  $\boldsymbol{\gamma}$  in a fixed frame of reference is the material derivative of the velocity  $\mathbf{V} = a/r \mathbf{e}_r + \boldsymbol{\omega} r \mathbf{e}_\theta$  is reduced to inertia,

$$\boldsymbol{\gamma} = \boldsymbol{\kappa} = \nabla \left( \frac{|\mathbf{V}|^2}{2} \right) - \mathbf{V} \times \nabla \times \mathbf{V} = \left( -\frac{a^2}{r^3} - \boldsymbol{\omega}^2 r \right) \mathbf{e}_r + \frac{2 a \boldsymbol{\omega}}{r} \mathbf{e}_\theta, \quad (25)$$

calculated from the two components of the Navier-Stokes equations, i.e. taking into account the assumptions made,

$$\begin{cases} V_r \frac{\partial V_r}{\partial r} - \frac{|V_\theta|^2}{r} = \left( -\frac{a^2}{r^3} - \boldsymbol{\omega}^2 r \right) \mathbf{e}_r, \\ V_r \frac{\partial V_\theta}{\partial r} + \frac{V_r V_\theta}{r} = \left( \frac{a \boldsymbol{\omega}}{r} + \frac{a \boldsymbol{\omega}}{r} \right) \mathbf{e}_\theta. \end{cases} \quad (26)$$

Given the incompressibility of motion, all that remains are the pressure gradients in the second members of the Navier-Stokes equations:

$$\begin{cases} \rho \left( -\frac{a^2}{r^3} - \boldsymbol{\omega}^2 r \right) = -\frac{\partial p}{\partial r} \Rightarrow \frac{p}{\rho} = -\left( \frac{a^2}{2 r^2} - \frac{\boldsymbol{\omega}^2 r^2}{2} \right) + f(\theta), \\ \rho \left( \frac{2 a \boldsymbol{\omega}}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} \Rightarrow \frac{p}{\rho} = -2 a \boldsymbol{\omega} \theta + g(r), \end{cases} \quad (27)$$

where successive integrations over  $r$  and  $\theta$  can be used to calculate the functions  $f(\theta)$  and  $g(r)$  in order to obtain the pressure  $p(r, \theta)$ :

$$\frac{p}{\rho} = -\frac{a^2}{2 r^2} + \frac{\boldsymbol{\omega}^2 r^2}{2} - 2 a \boldsymbol{\omega} \theta + cte, \quad (28)$$

where  $\rho$  is a constant.

The same calculation can be done in a rotating frame of reference  $\mathbb{R}'$  where  $\mathbf{V}' = \mathbf{W}$ ; noting  $\boldsymbol{\gamma}'$ , the acceleration in the rotating frame of reference, we read the acceleration  $\boldsymbol{\gamma}$  in the fixed frame of reference:

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}' + 2 \boldsymbol{\Omega} \times \mathbf{V}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \left( -\frac{a^2}{r^3} - \boldsymbol{\omega}^2 r \right) \mathbf{e}_r + \frac{2 a \boldsymbol{\omega}}{r} \mathbf{e}_\theta, \quad (29)$$

and the result for pressure (28) is of course identical. Note that the sum of the fictitious Coriolis and centrifugal accelerations  $2 \boldsymbol{\Omega} \times \mathbf{V}' + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = 2 a \boldsymbol{\omega}/r \mathbf{e}_\theta - \boldsymbol{\omega}^2 r \mathbf{e}_r$  is non-zero even if the two fields  $\mathbf{U}$  and  $\mathbf{W}$  are orthogonal. It is possible to calculate the Lamb vector separately,

$$-\mathbf{V} \times \nabla \times \mathbf{V} = 2 \boldsymbol{\Omega} \times \mathbf{V}' + 2 \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = 2 \boldsymbol{\Omega} \times (\mathbf{V}' + (\boldsymbol{\Omega} \times \mathbf{r})) = 2 \boldsymbol{\Omega} \times \mathbf{V}, \quad (30)$$

to find  $-\mathbf{V} \times \nabla \times \mathbf{V} = -2 \boldsymbol{\omega}^2 r \mathbf{e}_r + 2 a \boldsymbol{\omega}/r \mathbf{e}_\theta$ , a non-zero term.

The properties of this vector were stated by H. Lamb in 1895 [23] and it is used in many problems in fluid mechanics. The application of the divergence and curl operators is of particular interest, especially in turbulence [20]. By applying the divergence operator to the Lamb vector,  $\nabla \cdot (-\mathbf{V} \times \nabla \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \nabla \times \mathbf{V} - |\nabla \times \mathbf{V}|^2$ , These two terms, called bending and enstrophy respectively, are thought to play an important role in turbulent rotational flows. In the case

presented the divergence of the Navier-Stokes equations can be read,

$$\nabla \cdot (\nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}) = 2a^2/r^4 - 2\omega^2 = -\nabla^2 \left( \frac{p}{\rho} \right), \quad (31)$$

where the Laplacian of the second member gives the quantity  $p/\rho$ ,

$$\frac{p}{\rho} = -\frac{a^2}{2r^2} + \frac{\omega^2 r^2}{2} + f(\theta), \quad (32)$$

such that  $f(\theta)$  is a polynomial of order one; we naturally find the solution (28).

The application of the curl operator to the Navier-Stokes equations for an inviscid flow leads to  $\nabla \times (\nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}) + \nabla \nabla(p/\rho) = 0$  or  $\nabla \times (-\mathbf{V} \times \nabla \times \mathbf{V}) = 0$ . The Lamb vector  $\mathcal{L} = -\mathbf{V} \times \nabla \times \mathbf{V}$  is not zero but its curl is zero and  $\mathcal{L}$  can only be, for this example, the gradient of a scalar potential  $\mathcal{L} = \nabla \Phi$  with  $\Phi = a^2/r^2 + 2a\omega\theta$  to the nearest constant.

The expression of the pressure field (28) reveals a singularity: the pressure depends linearly on the angle  $\theta$ ; at each revolution, the pressure is incremented by the constant value  $-2\rho a\omega$ . This is physically impossible because the field must be axisymmetric and periodic and satisfy the condition  $p(r, \theta + 2\pi) = p(r, \theta)$ . Furthermore, the pressure increases as  $r^2$  when the radius increases, which is physically impossible. These defects represent a real flaw in the representativeness of the Navier-Stokes equations, which do not satisfy the completeness requirement. The reasons for this situation, discussed below, are deeply rooted in the continuum formalism.

### 3.2. In discrete mechanics

In discrete mechanics, the velocities are limited to the components of the segments of the primal structure and are written as  $\mathbf{u} = a/r \mathbf{e}_r$  and  $\mathbf{w} = \omega r \mathbf{e}_\theta$  where the unit vector  $\mathbf{t}$  is replaced by the unit vectors of the global frame of reference for greater clarity and above all to facilitate comparison. In the case of the superposition of the two motions, the sum of the coupled non-linear terms of the discrete formulation is zero,  $\nabla(\mathbf{u} \cdot \mathbf{w}) - \nabla \otimes (\mathbf{u} \cdot \mathbf{w} \mathbf{n}) = 0$ . In fact, the curvature of the inertial potential  $\phi_i = |\mathbf{v}|^2/2$  of the sum of the two contributions  $\mathbf{v}$  is written:

$$\boldsymbol{\kappa} = \nabla \left( \frac{|\mathbf{v}|^2}{2} \right) - \nabla \otimes \left( \frac{|\mathbf{v}|^2}{2} \mathbf{n} \right) = \left( \frac{a^2}{r^3} - \omega^2 r \right) \mathbf{t} - \left( \frac{a^2}{r^3} - \omega^2 r \right) \mathbf{m} = 0, \quad (33)$$

each of the two terms is non-zero but the sum is zero. Whatever the orientations of the unit vectors  $\mathbf{t}$  and  $\mathbf{m}$ , this result is the same because these orthogonal vectors form a basis of the tangent plane of the potential  $\phi_i$ . Under these conditions, the material derivatives of the local fixed and moving reference frames are equal to:

$$\frac{d\mathbf{v}}{dt} \equiv \frac{d\mathbf{u}}{dt} = -\frac{a^2}{r^3} \mathbf{t}, \quad (34)$$

then  $\mathbf{t} = \mathbf{e}_r$ . The scalar potential of acceleration  $\phi^o = p/\rho$  is equal to:

$$\phi^o = -\frac{a^2}{2r^2} + cte, \quad (35)$$

and the Bernoulli potential  $\phi_B^o = \phi^o + |\mathbf{u}|^2/2 = 0$ . The fact that the Bernoulli scalar potential, an energy per unit mass, is equal to  $\phi_B^o = 0$  is perfectly legitimate because the motion defined by the field  $\mathbf{v}$  is already established and no energy is needed to maintain this motion over time, it is inviscid.

Applying the divergence operator to the discrete equation of motion (22) immediately eliminates the dual curl terms and the contribution to the divergence of the scalar product  $\mathbf{u} \cdot \mathbf{w}$

with  $\mathbf{w} = \omega r \mathbf{e}_\theta$  of inertia is zero and all that remains is the contribution of the radial field  $\mathbf{u}$ ,  $\nabla \cdot (\nabla(|\mathbf{u}|^2/2) - \nabla \otimes (|\mathbf{u}|^2/2 \mathbf{n})) = 2 a^2/r^4$ , i.e.

$$\nabla^2 (\phi^o + |\mathbf{u}|^2/2) = \nabla^2 \phi_B^0 = 0. \quad (36)$$

Similarly, the application of the primal curl operator to the equation (22) leads, given the disappearance of the gradient of the scalar potential, to  $\nabla \times (\nabla(|\mathbf{v}|^2/2) - \nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})) = \nabla \times \boldsymbol{\kappa} = 0$ .

A change of frame of reference, in the context of discrete mechanics, would not change the result because the notion of a Galilean frame of reference for translational motions with constant divergence has been extended to rotational motions with constant curl [6].

### 3.3. Comparison of solutions

Figure 3a shows, for the Navier-Stokes equations, the pressure field  $p(r)/\rho$  and the spiral streamlines in a fixed frame of reference in the case where the two components  $\mathbf{U}$  and  $\mathbf{W}$  are associated with the same axis.

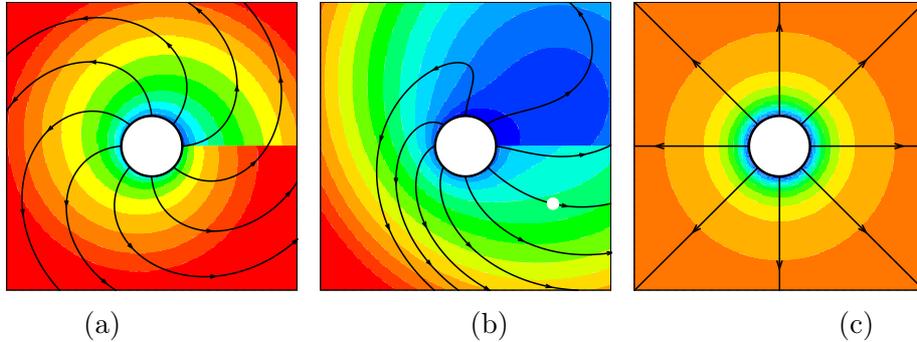


Figure 3: Navier-Stokes equations: current lines corresponding to the velocity field  $\mathbf{V}(r) = a/r \mathbf{e}_r + \omega r \mathbf{e}_\theta$  and pressure  $\mathbf{e}_\theta \mathbf{e}_\theta$  and pressure field  $p(r)$  projected in a fixed Cartesian frame of reference  $\mathbb{R}$  of classical mechanics, (a) for aligned axes and (b) for a uniformly right-shifted axis of rotation  $\boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_0)$  (its centre is symbolised by a white point). Discrete mechanics: (c) potential field  $\phi^o$  and current lines of the field  $\mathbf{u}(r)$  in the local frame of reference of Maxwell.

Current lines describe spirals generated by the superposition of a uniform rotational motion and a flow due to the injection of a fluid from a source located at the origin or by a fluid flow on a circular contour with a fixed radius. The terms of inertia are responsible for the interaction between the two motions. The use of a moving frame of reference linked to rotation would not change much, apart from the direction of the helices observed with respect to the fixed frame of reference. The pressure given by the expression (28) shows a discontinuity for  $\theta = 0$  or  $\theta = 2\pi$  when the problem is necessarily axisymmetric, so this solution to the Navier-Stokes equations is incorrect. Furthermore, the pressure increases in  $p \propto r^2$  as the radius goes to infinity, which makes no physical sense.

If the centre of rotation of the uniform rotational motion is shifted to a position  $\mathbf{r}_0$ , the velocity field becomes  $\boldsymbol{\Omega} \times (\mathbf{r} - \mathbf{r}_0)$  and the current lines shown in Figure 3b are of course modified compared to the previous case; the motion loses its axial symmetry and so does the pressure. Physically, the change in the position of the centre of rotation of the motion is defined by  $\mathbf{W} = \omega r \mathbf{e}_\theta$  should not change anything because the primal curl  $\nabla \times \mathbf{W} = 2 \mathbf{e}_\theta \omega \mathbf{n}$  is constant throughout space, including the origin and infinity. The fictitious Coriolis and centrifugal forces allow a different description of motion because of the change in point of view, but are not fundamentally different.

In discrete mechanics, the potential field  $\phi(r)$  and the current lines represented in Figure 3c remain identical in a fixed or moving reference frame. In this case, the scalar product  $\mathbf{u} \cdot \mathbf{w}$  is not zero but the sum  $\boldsymbol{\kappa} = \nabla(\mathbf{u} \cdot \mathbf{w}) - \nabla \otimes (\mathbf{u} \cdot \mathbf{w} \mathbf{n})$  is effectively zero by compensation. In fact, in discrete mechanics, the inertia  $\boldsymbol{\kappa}$  represents the curvature of the inertial potential  $\phi_i = |\mathbf{v}|^2/2$ . As with the spatial curvature of an interface, the two vectors  $\mathbf{t}$  and  $\mathbf{m} = \mathbf{t} \times \mathbf{n}$  are orthogonal and rotating these two vectors with axes orthogonal to  $\mathbf{n}$  does not change the local mean curvature of the interface. Thus, whatever the vectors  $\mathbf{t}$  and  $\mathbf{m}$  and whatever the position of the centre of rotation of the motion  $\mathbf{w}$ , the mean curvature of the inertial potential is zero. For example, this experiment involving the injection of a fluid into a cylindrical cavity in a space station in geostationary orbit is not influenced by the distance separating it from the Earth. The potential field  $\phi(r)$  defined by the velocity component  $\mathbf{u}(r)$  is axisymmetric and the current lines are radial lines centred on the origin. Figure 3c illustrates the results recorded by an observer linked to the experiment.

Noting that the global solution  $\mathbf{V}$  contains the solution  $\mathbf{W}$  of a rigid rotational flow, it is possible to remove it *a priori* to eliminate the problem posed by the incoherence of the Navier-Stokes equations relating to this velocity field. In practice, this separation is not possible in more complex problems, such as turbulence. The discrete law of motion intrinsically eliminates all uniform translation and rotation solutions and is not equivalent at all. The latter equation only takes accelerated motion into account, which is an essential difference from classical theories.

The reasons for the disagreement between the result obtained from the Navier-Stokes equations on pressure (28) and from discrete mechanics (35) are given below. Note already the inconsistency of the solution deduced from the Navier-Stokes equations where the pressure depends on the polar angle  $\theta$  whereas the problem is axially symmetric.

#### 4. Taylor-Green vortex

The flow described in this section is much more complex, involving the decay of energy by a vortex-scale cascade in a turbulent motion known as a Taylor-Green vortex; it is an emblematic case used in turbulence physics and to validate simulation codes based on the Navier-Stokes equations. Unlike the previous analytical case, the conclusions drawn from the comparison with discrete mechanics are more difficult to interpret. However, apart from discretisation errors, all the results obtained by different authors from the Navier-Stokes equations are identical and, similarly, the solution obtained from discrete mechanics is perfectly convergent. If there are any notable differences, they are due solely to the physical model adopted.

The case of the Taylor-Green vortex defined by the velocity field  $V = U(x, y, z) \mathbf{e}_x + V(x, y, z) \mathbf{e}_y$ , the initial 3D Taylor-Green vortex in the domain  $L^3 = [0, 2\pi]^3$ :

$$\begin{cases} U = v_0 \sin\left(\frac{x}{L}\right) \cos\left(\frac{y}{L}\right) \cos\left(\frac{z}{L}\right), \\ V = -v_0 \cos\left(\frac{x}{L}\right) \sin\left(\frac{y}{L}\right) \left(\cos\frac{z}{L}\right), \\ W = 0, \end{cases} \quad (37)$$

where  $v_0$  is a constant chosen here equal to unity. The equilibrium pressure field deduced from the equation of motion is written as:

$$p = \frac{\rho v_0^2}{16} \left( \cos\left(\frac{2x}{L}\right) + \cos\left(\frac{2y}{L}\right) \right) \left( \cos\left(\frac{2z}{L}\right) + 2 \right) + p_0. \quad (38)$$

From this initial condition imposed for the solution of the Navier-Stokes equations with periodic boundary conditions in the three directions of space, the observation of the velocity and pressure fields successively shows the birth of large-scale structures which then lead to a decrease in the scales of the turbulence due to the cascade of energy created by the finest structures. The flow typology does not change much from a Reynolds number of  $Re = 1000$ , at least for the largest vortex structures; a large number of articles, including [35, 3, 31, 11, 43, 13, 33, 34, 1, 32] describe the results obtained from the point of view of the phenomenology of turbulence or the numerical aspects linked to the resolution of the Navier-Stokes equations. Time integration is performed over a time interval equal to  $T = [0, 20]$ , sufficient to observe all the phases of turbulence decay due to viscous effects. From the velocity and pressure fields, statistics are calculated as a function of time, such as the mean turbulent kinetic energy  $E_k$  and the enstrophy  $\Omega$ . Figure 4 shows the result of the mean kinetic energy scaled by the unit value obtained as a function of time for a Reynolds number of  $Re = 1600$  for the Navier-Stokes (N.S.) model.

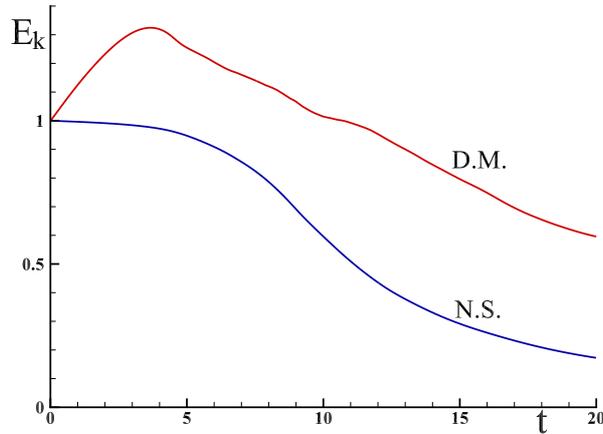


Figure 4: Taylor-Green vortex at  $Re = 1600$ ; simulations provides the evolution of dimensionless values of kinetic energy  $E_k$  for the Navier-Stokes model (N.S.) and for the discrete mechanics model (D.M.).

The kinetic energy decreases very slowly in the first instants, approximately until a time  $t \approx 5s$ , when the structures generated by the energy cascade are small enough to generate perceptible viscous effects. This decay of  $E_k$  over time is observed by all authors without exception; the kinetic energy eventually decreases exponentially beyond a time equal to  $t \approx 15$ .

This case is taken up by the discrete mechanics model [9] with the same parameters. The vortex structures observed differ very little from those obtained previously, but the evolution of the energies, in particular the mean kinetic energy  $E_k$  calculated from those obtained on each segment  $\Gamma$  of the primal geometric structure in Figure (1) (D.M. model) is very significantly different. The kinetic energy first increases up to a time  $t = 4.5$ , then decreases slightly before showing a change in the nature of the flow at  $t_c = 5s$ . Before this transition, the flow can be considered inviscid, and the simulations in this case,  $\nu = 0$ , show a behaviour very close to the case  $Re = 1600$ . The change in nature at this time  $t_c$  is concomitant with the appearance of a very localised vortex bursting phenomenon in certain zones of the flow. Previously, for  $t < t_c$  the motions were similar to the vortex-stretching phenomenon where each vortex undergoes elongation and possibly compression. For  $t > t_c$ , interactions between vortices of different sizes favour viscous dissipation and the kinetic energy decreases according to a linear law as a function of time, then according to an exponential law for  $t > 15$ . All these phenomena are described in detail in reference [9]. The energy spectrum  $E_k(k)$  as a function of the wavenumber is consistent with Kolmogorov's theory where the energy decreases with a slope equal to  $-5/3$  in the inertial equilibrium zone. These results are qualitatively confirmed for the temporal behaviour of a

single filament; in particular, we find the stretching and bursting phases of the vortex in the Taylor-Green case.

The evolution of the mean kinetic energy is linked to the way in which the statistics are carried out. In classical mechanics, it is defined as the average of the quantity  $\rho |\mathbf{V}|^2/2$  over the physical domain  $\Omega$ ; considering the inviscid fluid, the Navier-Stokes equations multiplied scalar by the velocity can be used to obtain its evolution over time:

$$\frac{d}{dt} \int_{\Omega} \rho \frac{|\mathbf{V}|^2}{2} dv = - \int_{\partial\Omega} p \mathbf{V} \cdot \mathbf{n} ds = 0, \quad (39)$$

if the boundary conditions satisfy the condition  $\mathbf{V} \cdot \mathbf{n} = 0$  or if they are periodic. Thus, in the absence of viscous effects, the mean kinetic energy should remain constant over time. This is indeed what is observed in the many simulations based on these equations. It is therefore not a result of post-processing but an essential feature of the physical model. It should be noted that the boundary term disappears because of the weak formulation adopted. From a physical point of view, this amounts to considering that the distant boundary conditions influence the characteristics of the turbulence, which is essentially a local phenomenon.

In discrete mechanics, the kinetic energy is defined locally on each segment  $\Gamma$  of the primal structure,  $e_k = (\mathbf{v} \cdot \mathbf{v}/2) \mathbf{t}$ . The law of motion multiplied by the velocity  $\mathbf{v}$  allows us to write:

$$\frac{d}{dt} \int_{\Gamma} \frac{|\mathbf{v}|^2}{2} dl = - \int_{\Gamma} \mathbf{v} \cdot \nabla \phi dl, \quad (40)$$

where  $\mathbf{v} \cdot \mathbf{v}$  as well as  $\mathbf{v} \cdot \nabla \phi$  are defined on the same segment because  $\mathbf{v}$  and  $\nabla \phi$  are themselves associated with this support, they are scalars linked to the oriented segment. The integral to the second member of the relation (40) has, except in special cases, no reason to be zero. The mean value  $E_k(t)$  obtained over all the segments from the local kinetic energies  $e_k$  is perfectly identifiable as the kinetic energy of the flow, a non-conservative quantity. Only the total energy characterised by the intrinsic acceleration  $\gamma$  is conserved, the kinetic and potential energies are not *a priori*.

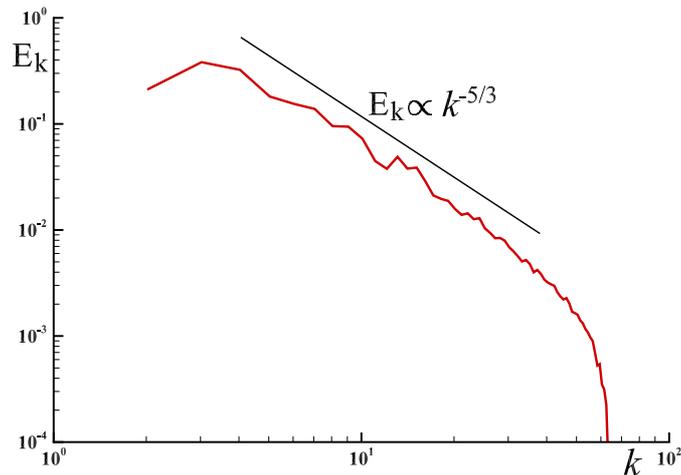


Figure 5: Energy spectrum as a function of the wave number  $E_k(k)$ .

Equation (17) is a local law of conservation of acceleration, i.e. energy per unit mass and length. The acceleration, velocity, gradient of scalar potential  $-\nabla \phi$  and viscous term  $-\nabla \otimes (\nu \nabla \times \mathbf{v})$  are projected onto the segment  $\Gamma$  of Figure 1; the intrinsic acceleration becomes the simple sum of the contributions relating to the scalar potential and viscosity. In the absence of viscous

effects, Bernoulli's theorem predicts, for a steady flow, that the pressure difference (potential) between two points is equal to the difference in the quantity  $|\mathbf{v}|^2/2$ ; this result applies between the two vertices  $a$  and  $b$  of the segment  $\Gamma$ . Unfortunately the interactions in the dynamic regime do not allow us *a priori* to translate all the inertial interactions on the local kinetic energy  $e_k$  in the zone of production of turbulence; its integration on all the segments  $\Gamma$  leads to a behaviour of the turbulent kinetic energy in  $E_k \propto -t^2$  in this production zone. Furthermore, if the characteristic length is  $L$ , we can find the overall energy  $E_k$  for the viscous zone,

$$\left\{ \begin{array}{l} \frac{de_k}{dt} = -\mathbf{v} \cdot \nabla \phi \quad \Rightarrow \quad E_k \propto -t^2, \\ \frac{de_k}{dt} = -\mathbf{v} \cdot \nabla \otimes (\nu \nabla \times \mathbf{v}) \quad \Rightarrow \quad E_k \propto \exp(-(\nu/L^2) t), \end{array} \right. \quad (41)$$

which shows, at the first instants, that the global kinetic energy increases, which compensates for the decrease in global pressure observed in the simulation. On the other hand, by removing the pressure contribution, the evolution of  $e_k$  for the dissipation zone can be obtained by local integration over  $\Gamma$ ; this leads to a time evolution of the type  $e_k \propto \exp(-\nu t/dh^2)$  for the dissipation zone where  $dh$  is the length of this segment. In this case, the interactions are linear and the local  $e_k$  and global  $E_k$  kinetic energies evolve identically in time, although the values of  $e_k$  obviously depend on the position of  $\Gamma$  in space. The behaviour observed in the results of the numerical simulation in the physical domain on the global kinetic energy  $E_k$  is illustrated in Figure 4. In the transfer zone, the simulation gives an evolution of the type  $E_k(t) \propto -t$  in time and a decay of  $E_k(k) \propto k^{-5/3}$  in the spectral domain [19]. This result obtained by Kolmogorov from the dimensional analysis reproduces in Figure 5 the energy cascade described by Richardson for the self-similar part in which the energy diffuses mainly from the larger scales to the smaller ones. While the behaviour of the mean kinetic energy in  $E_k(t) \propto -t^2$  in the production zone differs from that obtained from the Navier-Stokes model, the energy decay corresponds well to Kolmogorov's theory in the inertial zone. Solutions of the discrete law of motion are strong solutions, whereas solutions of the Navier-Stokes equations are often sought as weak solutions. In the case of moderate constraints, the solutions coincide, but for turbulent flows, the events that characterise the transitions between the different regimes are local and have no connection with the limits.

The N.S. and D.M. models diverge significantly on the evolution of the kinetic energy in the first instants, whereas the analysis of the instantaneous velocity fields shows that they are very close. This is the only problem to date that has led to a divergence between the two models. All the other numerous analytical solutions or those derived from numerical simulations give the same results, apart from a few methodological errors, including for compressible or two-phase flows. Limiting the analysis of the behaviour of  $E_k$  for the moment to times less than  $t = t_c$ , we find that it remains constant for N.S. and increases for D.M. when the flow can be considered as non-viscous. It is not possible *a priori* to decide directly one way or the other.

So what is the right solution? The answer to this question is not only characterised by changes in kinetic energy, because kinetic energy is not a conserved quantity, only total energy is. The pressure and velocity fields (37) and (38) injected into the Navier-Stokes equation correspond to a solution that reflects the initial equilibrium. In the absence of viscous effects, or at the first instants of the simulation when they are negligible, the sum of the kinetic and pressure energies must be constant over time and therefore, if the pressure energy decreases, which is the case for the simulation carried out with this equation, the kinetic energy must increase; this argument is intrinsic and does not refer to the discrete model. In fact, none of the authors cited mention this behaviour and, moreover, none are interested in the pressure energy to know whether the total energy is conserved. The pressure seems to be considered as an adjustment variable, a

simple Lagrangian, to satisfy the Navier-Stokes equations. The form of inertia used in classical mechanics does not seem to correctly account for the exchange between kinetic and pressure energies in rotating flows. Inertia in its form  $\mathbf{V} \cdot \nabla \mathbf{V}$  or in its equivalent form  $\nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V}$  inhibits the natural transformation of energies introduced by initial conditions alone. The drop in pressure at the first instants is not compensated for by the increase in kinetic energy. In conclusion, we can say that in this case, the Navier-Stokes equations do not conserve total energy.

The physical analysis of this Taylor-Green flow and that of the superposition of two elementary flows form the basis of the discussion on the completeness of the Navier-Stokes equations.

## 5. Discussion

The Navier-Stokes equations have an excellent representativity over a wide range of flows. The aim is not to question the solutions obtained when the formal framework is the one used to derive these equations, but to show that they are not complete in the sense that solutions of physical interest escape their acceptance. The most striking example is their inability to translate relativistic motion, since these equations are not invariant under a Lorentz transformation. Attempts to make the Navier-Stokes equations relativistic are hampered by the application of Einstein's equation formalism [15, 42]. Even if the field of application of the theory of relativity is unrelated to that of the Navier-Stokes equations, it will become necessary to establish physical laws that are sufficiently complete to deal with problems of a different nature. The flows encountered in astrophysics require certain properties that are not included in the equations of classical fluid mechanics. The formalism of discrete mechanics solves this problem by abandoning the concepts of classical mechanics, including the notion of mass, to derive a law of motion in the form of a wave equation, whatever its nature - gravitational waves, acoustic waves or light. This law is fundamentally relativistic. This non-relativistic character and some other objections to the Navier-Stokes equations are discussed in detail in [7].

The main issue dealt with here relates to what is sometimes called the objectivity or material frame-indifference, introduced by Truesdell and Noll [40, 36, 37, 38, 39] to express the fact that the stresses in a material do not depend on a global rotation of the physical system. This problem arises mainly in continuum mechanics for solids, where derivatives such as Jaumann's are supposed to solve the problem [21]. According to the authors, there are different interpretations of the objectivity and the material frame-indifference, which are of a different nature according to [29]. In fact, this is a broader problem involving the notion of Galilean invariance; this notion attached to an inertial frame of reference in uniform motion at constant velocity is not sufficient to describe the isotropy of space at the origin of the conservation of angular momentum, in application of Noether's theorem [22]. In physics, rotation symmetry, or invariance by rotation, is the property of a physical theory or system that it is not modified by any rotation of space. When the system is invariant to any rotation of space, it is said to be isotropic, in which case all directions of space are equivalent. This property is not satisfied by the Navier-Stokes equations, as the simple example given above clearly shows. The transformation of the viscous term  $\nabla \cdot \boldsymbol{\sigma}$  where  $\boldsymbol{\sigma}$  is the strain rate tensor into a rotational form  $\nabla \times (\mu \nabla \times \mathbf{V})$  makes it rotationally invariant [7]. But there are still all the other terms that are not invariant by a continuous rotation, including inertia and source terms. Although any rotation about any direction can be decomposed into a combination of rotations about the three axes ( $Ox, Oy, Oz$ ), the existence of preferred directions makes it difficult to demonstrate invariance by global rotation, especially for non-linear terms.

The solution consists of two motions  $\mathbf{U} = a/r \mathbf{e}_r$  and  $\mathbf{W} = \boldsymbol{\omega} r \mathbf{e}_\theta$  reveals a lack of representativeness of the Navier-Stokes equations for pressure, even though it is perfectly admissible

physically. This is one of the only simple solutions found that calls these equations into question. The pressure,  $p/\rho = -a^2/2 r^2 + \omega^2 r^2/2 - 2 a \omega \theta + cte$ , shows not only that it grows towards infinity as the radius increases, but also that it depends linearly on the polar angle  $\theta$  even though the problem is axially symmetrical. The cause of this difficulty is directly linked to the form of the inertia and in particular to the Lamb vector [23]. A large number of publications deal with the use of the Lamb vector [20, 24, 25, 17, 18, 12, 28, 27], the most recent reporting difficulties in the numerical calculation of forces for aerodynamic profiles due to the existence of an arbitrary pole. However, the form of inertia including the Lamb vector is legitimate since it is strictly equal to the form  $\mathbf{V} \cdot \nabla \mathbf{V}$ , the discussion is not so much about the Lamb vector itself as about the meaning of inertia in the context of continuum mechanics. This is because the Lamb vector  $\mathcal{L}$  is defined in the orthogonal plane associated with  $\nabla \times \mathbf{V}$ ; it is then calculated at each point within the framework of continuum mechanics where all quantities are defined at a point and projected onto the three axes of an orthonormal reference frame. This sequence of operations during its evaluation results in a loss of precision for its use. In addition, the constraints imposed by the general framework of classical mechanics can lead to inconsistencies. For example, for the inertia vector to be zero  $\nabla(|\mathbf{V}|^2/2) - \mathbf{V} \times \nabla \times \mathbf{V} = 0$ , its three components must be zero simultaneously, which means that the Lamb vector must also be the gradient of a potential; this is the case for Poiseuille flow, for example. When solving the Navier-Stokes equations for cases with analytical solutions, certain solutions are discarded *a priori* for physical reasons, for example if they tend towards infinity in a certain parameter domain. This could be the case for  $\mathbf{W}(r)$  in the simple example given, but the problem posed has the solution presented; if it must be excluded, it is because of the discrete law of motion itself.

So why is there a difference between the 2D and 3D formulations of the result of applying the curl operator to the Navier-Stokes equations? The reason is that the notion of curl reveals a conceptual difficulty within the framework of classical mechanics, where all quantities are assigned to a point. Stokes' theorem requires the definition of an exterior normal  $\mathbf{n}$  for a surface bounded by a well-defined contour; a rotational associated with a single point has, intrinsically, no precise meaning. The Lamb vector, like the curl vector, therefore only exists in two dimensions of space, for a surface whose normal can be defined. A number of theoretical difficulties in continuum mechanics are linked to this ambiguity: the curl operator has an indisputable meaning in discrete space, whereas in continuum mechanics it is simply a mathematical device derived from mathematical analysis. To recover coherence, it is therefore necessary to assign the Lamb vector to a plane whose normal  $\mathbf{n}$  is that of the direction of  $\nabla \times \mathbf{v}$ .

Discrete mechanics solves the shortcomings of conventional Lamb vector mechanics by proposing an inertia  $\boldsymbol{\kappa}$  as the sum of a gradient of the inertia potential  $\phi_i$  and the dual curl of the vector inertia potential  $\boldsymbol{\psi}_i = \phi_i \mathbf{n}$ , a formal Helmholtz-Hodge decomposition. The result is independent of the dimension of the space considered. The inertia vector  $\boldsymbol{\kappa} = -\nabla \phi_i + \nabla \otimes \boldsymbol{\psi}_i$  oriented along  $\mathbf{t}$ , the unit vector of the segment  $\Gamma$ , is independent of the orientation of the basis vectors which define the tangent plane  $(\mathbf{t}, \mathbf{m})$ . This vector  $\boldsymbol{\kappa}$  physically represents the mean curvature of the inertial potential; if it is zero, the flow is non-inertial even if each of its two components  $\boldsymbol{\kappa}$  is not. The intrinsic properties of the Helmholtz-Hodge decomposition and those of discrete mechanics in general lead to the elimination of the artefacts of the equations of classical mechanics by abandoning the traditional notions of a Galilean frame of reference, mass, force and so on. The example of simple motion defined by the superposition of the solutions  $\mathbf{u} = a/r \mathbf{t}$  and  $\mathbf{w} = \omega r \mathbf{t}$  and  $\mathbf{w} = \omega r \mathbf{t} \mathbf{t}$  makes it possible to exclude from the outset the effects of rigid rotation on the solution of the scalar potential  $\phi^o$  (pressure). The experiment carried out in a uniformly rotating system does not depend on the position of the observer. The velocity and pressure fields in Figure (3) clearly show that they depend on the observer's position when the problem is solved using the Navier-Stokes equations.

So what is the correct interpretation, using a classical Galilean frame of reference, a rotating reference frame or Maxwell's reference frame? The answer seems obvious if we take the point of view of an observer in mechanical equilibrium performing this experiment. For him, the motion is strictly radial and axisymmetric and the scalar potential is also axisymmetric; he cannot perceive that he is in uniform rotational motion. The interpretations of the Navier-Stokes equations for a fixed or moving reference frame (including the fictitious Coriolis and centrifugal forces) are erroneous; the observer cannot perceive the rotational motion. The experiment carried out in a geostationary satellite rotating around the Earth, itself rotating around the Sun, must be interpreted in a local frame of reference that excludes uniform rotational motions. The notion of an inertial reference frame in classical mechanics and the theory of relativity is not sufficient to exclude all non-accelerated motion. Furthermore, in these two contexts, uniform rotation is considered to be accelerated motion, which is not the case in Maxwell's frame of reference. The choice of a fixed frame of reference is an illusion of reality that leads to results that depend on the point of view. Discrete mechanics abandons any attempt to represent distant phenomena by a fixed frame of reference or by a change of frame of reference, especially since these events belong to the past, whereas the equations of mechanics claim to predict the near future.

What then is the impact of the choice of a Maxwell  $\mathbb{M}$  reference frame and of discrete mechanics for fluid mechanics? First of all, it should be noted that the solutions of the Navier-Stokes equations and the law of discrete motion are the same, with two exceptions: (i) the example of two elementary motions and (ii) the decay of turbulence for the Taylor-Green flow. The perception fixed by Maxwell's reference frame profoundly modifies the modelling of the various physical phenomena of classical mechanics closely associated with the frame of reference  $\mathbb{R}^3(x, y, z)$ . The local law of discrete motion (1) has only two unit vectors  $\mathbf{t}$  and  $\mathbf{n}$ , which makes it possible to apply Stokes' theorem and transfer the interactions between translational and rotational motion. Maxwell's frame of reference also makes it possible to replace Lamb's vector by a dual curl vector and to write inertia in the form of a Helmholtz-Hodge decomposition. This decomposition is not only a mathematical device for separating curl-free and divergence-free terms, but it also introduces a symmetry that is compatible with Noether's theorem.

The Taylor-Green flow [9] is analysed in the light of the elements presented on the simple example and on the Lamb vector. The results of the Navier-Stokes equations and discrete mechanics on this turbulent flow show a significant divergence in the evolution of the mean kinetic energy over time. In discrete mechanics, the translational kinetic energy  $|\mathbf{v}|^2/2$  assigned to the vertices of the primal structure and the rotational kinetic energy calculated on a facet and associated with its unit vector,  $|\mathbf{v}|^2/2 \mathbf{n}$ , are nested in such a way as to facilitate their exchange. The transfer of kinetic energies between them is a complex mechanism because the two corresponding accelerations,  $\nabla(|\mathbf{v}|^2/2)$  and  $\nabla \otimes (|\mathbf{v}|^2/2 \mathbf{n})$  are orthogonal and cannot *a priori* directly exchange energy. The phenomenon must necessarily be time-dependent to allow energy transfer. From a physical point of view, the total energy injected into the periodic cavity of the Taylor-Green problem at the initial instant is made up of potential energy in the form of pressure and kinetic energy associated with the vortices. Only the total energy is conserved when the flow is non-viscous; in all cases, at the first instants ( $t < 5$ ) of motion, each of the potential and kinetic energies can vary in time while respecting the conservation of the total energy. While the Navier-Stokes equations predict an almost constant kinetic energy, discrete mechanics assumes an increase in kinetic energy. As the mean pressure decreases in this phase, it is natural for the kinetic energy to increase. The pirouettes performed by an ice-skater rely on her ability to modify her moment of inertia to increase her rotational velocity, represented by  $\nabla \times \mathbf{v}$ , while maintaining her angular momentum. In the Taylor-Green case, the fluid filaments are subjected to the phenomenon of vortex stretching, where the elongation of the filaments is compensated for by faster rotation. The proposed explanation for the divergence between the two models is based

on the difficulty of the Lamb inertia term in transforming translational motion into rotation. The nature of the Lamb vector is indeterminate, whereas the corresponding term in discrete mechanics is the dual curl of the inertial potential. In the latter case, the intrinsic acceleration can be transformed without hindrance from the form of kinetic energy into potential energy, or vice versa.

From a theoretical point of view, the evolution of the mean kinetic energy  $E_k$  resulting from the weak formulation (39) removes the term  $-\mathbf{V} \cdot \nabla p$  from the adopted boundary conditions. The weak formulation of the Navier-Stokes equations is widely used to obtain a numerical solution whose existence and uniqueness can be proved in the context of functional analysis. For very complex problems such as turbulence developed in fluids, it can be risky to use this concept because the stability of vortices subjected to vortex-stretching and vortex-bursting phenomena is no longer directly linked to the boundary conditions of the domain. In discrete mechanics, the evolution of kinetic energy (40) is expressed by a local law  $d\mathbf{e}_k/dt = -\mathbf{v} \cdot \nabla \phi$  established on a single segment  $\Gamma$ . This is a strong solution which does not violate the conservation of total energy per unit mass, i.e. intrinsic acceleration, the sum of potential and kinetic energies; the term  $-\mathbf{v} \cdot \nabla \phi$  is made up of two contributions carried by the same segment; in general, this term is not equal to zero.

A more technical aspect also helps to explain the discrepancies between the results of the Navier-Stokes equations and those of the discrete formulation. The notion of a continuous medium means that quantities defined by partial derivatives in space are assigned to a point. The way in which the operators are reconstructed from these partial derivatives influences the result, not only quantitatively, but also in terms of the physical behaviour of the solution. Abrupt transitions in certain variables or properties are denatured by the spatial averaging required to transform the initial problem into an approximation of the continuous problem. The discrete formulation is characterised by the abandonment of any spatial approximation in order to federate compression and rotation contributions on the same segment of the local Maxwell frame of reference. The four discrete differential operators are used to construct the law of motion on this single segment. It's not quite the same thing to define a rotation by vertex for later use on a face or a cell, or to position the rotation directly on a face. In addition, the potential vector  $\boldsymbol{\psi} = \nu \nabla \times \mathbf{v}$  is uniquely constructed as the energy per unit mass; the quantities  $\nu$  and  $\nabla \times \mathbf{v}$  are inseparable. The influence of these averages may be negligible for certain simple solutions but becomes important for complex turbulent flows where they can introduce a degree of rigidity, particularly in the transfer of translational energy into rotational energy.

All these arguments do not invalidate the Navier-Stokes model itself, since these equations are unanimously accepted to represent real flows over a very wide range of parameters. The alternative discrete model makes it possible to recover, to within numerical errors, the solutions of simulations carried out with the classical model by correcting certain shortcomings of the latter. For example, the discrete law of motion is relativistic and satisfies the results of the theory of special and general relativity. In fact, it is not so much the Navier-Stokes equations that are at issue as the very concept of a continuous medium. The Lamb vector, the calculation of the mean kinetic energy and the obligation to use spatial interpolations are all notions that are made inescapable by the notion of a continuous medium and a global Galilean frame of reference. The Navier-Stokes equations, like other physics equations based on the notion of a continuous medium, involve the same difficulties, but they are amplified by the highly non-linear nature of this equation, which is emblematic of fluid mechanics. The redefinition of inertia in the form of a Helmholtz-Hodge decomposition as the curvature of the Bernoulli potential makes the interactions between translation and rotation phenomena more flexible. In addition, the extension of Galilean invariance to uniform rotation frees the observer from any external interaction.

In summary, two significant results call into question the validity of the complete Navier-

Stokes equations. The first is the superposition of two simple movements, a rotational motion and a radial injection; the solution obtained shows that the pressure field is not axisymmetric. The second example is the Taylor-Green vortex, a flow emblematic of turbulence decay; in this case, it is the total energy that is not conserved for an inviscid flow. This observation on the completeness of the Navier-Stokes equations is not only based on the analysis of these two results, but confirms the objections that have already been made to them. The observed faults are mainly related to inertia and rotation, two main characteristics of turbulent flows.

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