Singular viscoelastic perturbation to soft lubrication – Supplementary Material –

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RIGID CASE

Notations are all defined in the main text. In the absence of wall deformation, the fluid flow is governed by the incompressible Stokes equation in the lubrication approximation, which reads:

$$U_{ZZ} = P_X \ . \tag{1}$$

The fluid gap profile can be approximated by a parabola, as:

$$H(X,T) \simeq \Delta(T) + [X - X_{\rm G}(T)]^2 \ . \tag{2}$$

Invoking no-slip boundary conditions, $U(X, Z = 0, T) = 0$ and $U(X, Z = H, T) = \dot{X}_{\rm G} + \dot{\Theta}$, the Stokes equation can be solved, leading to the horizontal velocity profile:

$$U = \frac{P_X}{2} Z \left[Z - \Delta - (X - X_{\rm G})^2 \right] + \frac{(\dot{X}_{\rm G} + \dot{\Theta}) Z}{\Delta + (X - X_{\rm G})^2} \ . \tag{3}$$

Volume conservation reads:

$$\partial_T H + \partial_X \int_0^H {\rm d}Z \, U = 0 \ . \tag{4}$$

Integrating the latter, and assuming a vanishing lubrication pressure P at $X \to \pm\infty$, gives:

$$P = -\frac{3\dot{\Delta} + 2(\dot{\Theta} - \dot{X}_{\rm G})(X - X_{\rm G})}{\left[\Delta + (X - X_{\rm G})^2\right]^2} \ . \tag{5}$$

The pressure-induced drag force (per unit length) along Z is given by:

$$D_{p,\perp} = \int_{-\infty}^{\infty} {\rm d}X \, P = -\frac{3\pi}{2} \frac{\dot{\Delta}}{\Delta^{3/2}} \ . \tag{6}$$

The pressure-induced drag force (per unit length) along X is given by:

$$D_{p,\parallel} = -\sqrt{2\epsilon} \int_{-\infty}^{\infty} {\rm d}X \, (X - X_{\rm G}) P = \pi \sqrt{2\epsilon} \frac{\dot{\Theta} - \dot{X}_{\rm G}}{\Delta^{1/2}} \ . \tag{7}$$

The shear-induced drag force (per unit length) along X is given by:

$$D_{\sigma,\parallel} = -\sqrt{\frac{\epsilon}{2}} \int_{-\infty}^{\infty} {\rm d}X \, U_Z|_{Z=H} = -\pi \sqrt{2\epsilon} \frac{\dot{\Theta}}{\Delta^{1/2}} \ . \tag{8}$$

ELASTIC CORRECTION

The fluid gap profile becomes:

$$H(X,T) \simeq \Delta(T) + [X - X_{\rm G}(T)]^2 + \kappa P(X,T) \ . \tag{9}$$

The new no-slip boundary conditions are: $U(X, Z = -\kappa P, T) = 0$ and $U(X, Z = H - \kappa P, T) = \dot{X}_{\rm G} + \dot{\Theta}$. Volume conservation now reads:

$$\partial_T H + \partial_X \int_{-\kappa P}^{H - \kappa P} {\rm d}Z \, U = 0 \ , \tag{10}$$

which leads to the Reynolds equation:

$$12\dot{\Delta} - 24(X - X_{\rm G})\dot{X}_{\rm G} + 12\kappa\dot{P} = \left[H^3 P_X - 6(\dot{X}_{\rm G} + \dot{\Theta}) H \right]_X \ . \tag{11}$$

We now invoke a perturbation analysis at first order in κ, as:

$$\begin{aligned} P &\simeq P^{(00)} + \kappa P^{(10)} \ , \\ D_{p,\perp} &\simeq D_{p,\perp}^{(00)} + \kappa D_{p,\perp}^{(10)} \ , \\ D_{p,\parallel} &\simeq D_{p,\parallel}^{(00)} + \kappa D_{p,\parallel}^{(10)} \ , \\ D_{\sigma,\parallel} &\simeq D_{\sigma,\parallel}^{(00)} + \kappa D_{\sigma,\parallel}^{(10)} \ . \end{aligned} \tag{12}$$

The first-order correction to the Reynolds equation then reads:

$$\begin{aligned} &\left\{ \left[\Delta + (X - X_{\rm G})^2\right]^3 P_X^{(10)} + 3 \left[\Delta + (X - X_{\rm G})^2\right]^2 P^{(00)} P_X^{(00)} \right. \\ &\left. -6 \left(\dot{X}_{\rm G} + \dot{\Theta} \right) P^{(00)} \right\}_X = 12 P_T^{(00)} \ . \end{aligned} \tag{13}$$

From the previous section, one has:

$$\begin{aligned} P^{(00)} &= -\frac{3\dot{\Delta} + 2(\dot{\Theta} - \dot{X}_{\rm G})(X - X_{\rm G})}{\left[\Delta + [X - X_{\rm G}]^2\right]^2} \ , \\ D_{p,\perp}^{(00)} &= -\frac{3\pi}{2} \frac{\dot{\Delta}}{\Delta^{3/2}} \ , \\ D_{p,\parallel}^{(00)} &= \pi \sqrt{2\epsilon} \frac{\dot{\Theta} - \dot{X}_{\rm G}}{\Delta^{1/2}} \ , \\ D_{\sigma,\parallel}^{(00)} &= -\pi \sqrt{2\epsilon} \frac{\dot{\Theta}}{\Delta^{1/2}} \ . \end{aligned} \tag{14}$$

Solving Eq. (13) with vanishing $P^{(10)}$ at $X \to \pm\infty$ leads to:

$$
\begin{aligned}
D_{p,\perp}^{(10)} &= \frac{45\pi\ddot{\Delta}}{16\Delta^{7/2}} - \frac{63\pi\dot{\Delta}^2}{8\Delta^{9/2}} + \frac{3\pi(\dot{\Theta}-\dot{X}_{\rm G})^2}{8\Delta^{7/2}} \ , \\
D_{p,\parallel}^{(10)} &= \sqrt{\frac{\epsilon}{2}}\left[\frac{23\pi\dot{\Delta}(\dot{\Theta}-\dot{X}_{\rm G})}{8\Delta^{7/2}} + \frac{\pi(\ddot{X}_{\rm G}-\ddot{\Theta})}{2\Delta^{5/2}}\right] \ , \\
D_{\sigma,\parallel}^{(10)} &= \sqrt{\frac{\epsilon}{2}}\left[\frac{\pi(\ddot{\Theta}-\ddot{X}_{\rm G})}{4\Delta^{5/2}} + \frac{\pi\dot{\Delta}\dot{X}_{\rm G}}{2\Delta^{7/2}} - \frac{19\pi\dot{\Delta}\dot{\Theta}}{8\Delta^{7/2}}\right] \ .
\end{aligned}
\tag{15}
$$

VISCOELASTIC CORRECTION

The fluid gap profile becomes:

$$
H(X,T) \simeq \Delta(T) + [X - X_{\rm G}(T)]^2 + \kappa P(X,T) - \beta\dot{P}(X,T) \ . \tag{16}
$$

The new no-slip boundary conditions are: $U(X, Z = -\kappa P + \beta\dot{P}, T) = 0$ and $U(X, Z = H - \kappa P + \beta\dot{P}, T) = \dot{X}_{\rm G} + \dot{\Theta}$. Volume conservation now reads:

$$
\partial_T H + \partial_X \int_{-\kappa P + \beta\dot{P}}^{H - \kappa P + \beta\dot{P}} {\rm d}Z\, U = 0 \ , \tag{17}
$$

which leads to the Reynolds equation:

$$
12\dot{\Delta} - 24(X - X_{\rm G})\dot{X}_{\rm G} + 12\kappa\dot{P} - 12\beta\ddot{P} = \left[H^3 P_X - 6(\dot{X}_{\rm G} + \dot{\Theta})H\right]_X \ . \tag{18}
$$

We now invoke a perturbation analysis at first order in α, as:

$$
\begin{aligned}
P &\simeq P^{(00)} + \kappa\left[P^{(10)} - \alpha P^{(11)}\right] \ , \\
D_{p,\perp} &\simeq D_{p,\perp}^{(00)} + \kappa\left[D_{p,\perp}^{(10)} - \alpha D_{p,\perp}^{(11)}\right] \ , \\
D_{p,\parallel} &\simeq D_{p,\parallel}^{(00)} + \kappa\left[D_{p,\parallel}^{(10)} - \alpha D_{p,\parallel}^{(11)}\right] \ , \\
D_{\sigma,\parallel} &\simeq D_{\sigma,\parallel}^{(00)} + \kappa\left[D_{\sigma,\parallel}^{(10)} - \alpha D_{\sigma,\parallel}^{(11)}\right] \ .
\end{aligned}
\tag{19}
$$

Solving Eq. (18) at first order in α, with vanishing $P^{(11)}$ at $X \to \pm\infty$, leads to:

$$
\begin{aligned}
D_{p,\perp} = &-\frac{3\pi}{2}\frac{\dot{\Delta}}{\Delta^{3/2}} + \kappa\left[\frac{45\pi\ddot{\Delta}}{16\Delta^{7/2}} - \frac{63\pi\dot{\Delta}^2}{8\Delta^{9/2}} + \frac{3\pi\left(\dot{\Theta}-\dot{X}_{\rm G}\right)^2}{8\Delta^{7/2}}\right] - \beta\left[\frac{45\pi\dddot{\Delta}}{16\Delta^{7/2}} - \frac{567\pi\dot{\Delta}}{32\Delta^{9/2}}\left(\ddot{\Delta} - \frac{\dot{\Delta}^2}{\Delta}\right)\right. \\
&\left. + \frac{21\pi\dot{\Delta}}{32\Delta^{9/2}}\left(-6\dot{X}_{\rm G}^2 - \dot{\Theta}^2 + 7\dot{X}_{\rm G}\dot{\Theta}\right) + \frac{3\pi\ddot{\Theta}}{2\Delta^{7/2}}\left(\frac{\dot{\Theta}}{4} - \frac{7\dot{X}_{\rm G}}{8}\right) + \frac{3\pi\ddot{X}_{\rm G}}{2\Delta^{7/2}}\left(\frac{3\dot{X}_{\rm G}}{2} - \frac{7\dot{\Theta}}{8}\right)\right] \ ,
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
D_{p,\parallel} = &\ \pi\sqrt{2\epsilon}\frac{\dot{\Theta}-\dot{X}_{\rm G}}{\Delta^{1/2}} + \kappa\sqrt{\frac{\epsilon}{2}}\left[\frac{23\pi\dot{\Delta}(\dot{\Theta}-\dot{X}_{\rm G})}{8\Delta^{7/2}} + \frac{\pi(\ddot{X}_{\rm G}-\ddot{\Theta})}{2\Delta^{5/2}}\right] - \beta\sqrt{\frac{\epsilon}{2}}\left[-\frac{21\pi\dot{\Delta}^2}{4\Delta^{9/2}}\left(\dot{\Theta} - \frac{7\dot{X}_{\rm G}}{4}\right)\right. \\
&\left. - \frac{9\pi\dot{X}_{\rm G}}{8\Delta^{7/2}}(\dot{\Theta}-\dot{X}_{\rm G})^2 + \frac{\pi\ddot{\Delta}}{16\Delta^{7/2}}(43\dot{\Theta} - 73\dot{X}_{\rm G}) + \frac{\pi\dot{\Delta}}{16\Delta^{7/2}}(23\ddot{\Theta} - 53\ddot{X}_{\rm G}) + \frac{\pi}{2\Delta^{5/2}}(\dddot{X}_{\rm G} - \dddot{\Theta})\right] \ ,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
D_{\sigma,\parallel} = &-\pi\sqrt{2\epsilon}\frac{\dot{\Theta}}{\Delta^{1/2}} + \kappa\sqrt{\frac{\epsilon}{2}}\left[\frac{\pi(\ddot{\Theta}-\ddot{X}_{\rm G})}{4\Delta^{5/2}} + \frac{\pi\dot{\Delta}\dot{X}_{\rm G}}{2\Delta^{7/2}} - \frac{19\pi\dot{\Delta}\dot{\Theta}}{8\Delta^{7/2}}\right] - \beta\sqrt{\frac{\epsilon}{2}}\left[\frac{21\pi\dot{\Delta}^2}{32\Delta^{9/2}}(7\dot{\Theta} - 5\dot{X}_{\rm G})\right. \\
&\left. + \frac{9\pi\dot{X}_{\rm G}}{16\Delta^{7/2}}(\dot{\Theta}-\dot{X}_{\rm G})^2 - \frac{\pi\ddot{\Delta}}{4\Delta^{7/2}}\left(11\dot{\Theta} - \frac{29}{4}\dot{X}_{\rm G}\right) - \frac{\pi\dot{\Delta}}{4\Delta^{7/2}}\left(\ddot{\Theta} - \frac{19}{4}\ddot{X}_{\rm G}\right) - \frac{\pi}{4\Delta^{5/2}}(\dddot{X}_{\rm G} - \dddot{\Theta})\right] \ .
\end{aligned}
\tag{22}
$$

NUMERICAL INTEGRATION

In order to solve Eqs. (11), (12), and (13) of the main text, we use the `DifferentialEquations.jl` library from the open-source software for Scientific Ma-

chine Learning (SciML) collection in Julia. This choice was motivated by the strong variable coupling in our problem, which requires the use of the mass-matrix formalism. Moreover, stiffness was expected due to the third-order time derivative terms. Therefore, we employed the Rosembrock method of fifth order (`Rodas5P`). It is an A-stable method which is compatible with the mass-matrix formalism and has a stable adaptative time step. However, there were still some need for fine tuning the solver's tolerances, mainly around `1e-8` for both absolute and relative tolerances. Finally, we noted that the solver fails to simulate the sedimentation behaviour at long times. This is likely due to Δ flipping sign when approaching zero, because of the limitations on the smallest step size available.

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