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► To cite this version:

Elias Zikkos. On exact systems $t \alpha \cdot e^{2\pi i n t}$ $n \in \mathbb{Z}$ in $L^2(0, 1)$ which are not Schauder Bases and their generalizations. 2024. ⟨hal-04633820⟩

HAL Id: hal-04633820

<https://hal.science/hal-04633820v1>

Preprint submitted on 3 Jul 2024

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On exact systems $\{t^\alpha \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A}$ in $L^2(0, 1)$ which are not Schauder Bases and their generalizations

Elias Zikkos
Khalifa University of Science and Technology
Abu Dhabi, United Arab Emirates
email address: elias.zikkos@ku.ac.ae and eliaszikkos@yahoo.com

June 20, 2024

Abstract

Let $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ be an exponential Schauder Basis for $L^2(0, 1)$, for $\lambda_n \in \mathbb{R}$, and let $\{r_n(t)\}_{n \in \mathbb{Z}}$ be its dual Schauder Basis. Let A be a non-empty subset of the integers containing exactly M elements. We prove that for $\alpha > 0$ the weighted system

$$\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$$

is exact in the space $L^2(0, 1)$, that is, it is complete and minimal in $L^2(0, 1)$, if and only if

$$M - \frac{1}{2} \leq \alpha < M + \frac{1}{2}.$$

We also show that such a system is not a Riesz Basis for $L^2(0, 1)$. In particular, the weighted trigonometric system $\{t^\alpha \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A}$ is exact in $L^2(0, 1)$, if and only if $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$, but it is not a Schauder Basis for $L^2(0, 1)$.

Keywords: Weighted Exact Systems, Completeness and Minimality, Schauder Bases, Riesz Bases, Vandermonde matrices.

AMS 2010 Mathematics Subject Classification. 42C30, 42A65

1 Introduction and our Result

In [2] Heil and Yoon investigated the properties of systems of weighted exponentials of the form $\{g(t) \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A}$ where $g \in L^2(0, 1) \setminus \{0\}$ and A is a non-empty finite subset of the integers containing exactly M elements. Amongst their results, they prove in [2, Theorem 4] that the system

$$\{t^J \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A},$$

where J is a positive integer, is complete and minimal in $L^2(0, 1)$ if and only if $J = M$.

They also raise the interesting question [2, page 100] whether similar results hold for the system

$$\{t^\alpha \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A} \tag{1.1}$$

where α is a real positive number, not necessarily an integer.

Motivated by the above and the work of Shukurov [6], we provide the following answer: we prove that when the cardinality $|A|$ of the set A is equal to M , then the system (1.1) is complete and minimal in $L^2(0, 1)$ if and only if

$$M - \frac{1}{2} \leq \alpha < M + \frac{1}{2}. \tag{1.2}$$

We also show that such a complete and minimal system is not a Schauder Basis for $L^2(0, 1)$.

Example 1.1. The system $\{t^\alpha \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A}$ is complete and minimal in $L^2(0, 1)$ if

$$|A| = 1, \quad \text{and} \quad \frac{1}{2} \leq \alpha < \frac{3}{2},$$

$$|A| = 2, \quad \text{and} \quad \frac{3}{2} \leq \alpha < \frac{5}{2},$$

$$|A| = 3, \quad \text{and} \quad \frac{5}{2} \leq \alpha < \frac{7}{2},$$

and so on, but no such system is a Schauder Basis for $L^2(0, 1)$.

We point out that the condition (1.2) is necessary and sufficient in order for more general systems of the form $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$ to be complete and minimal in $L^2(0, 1)$, where $\{r_n\}_{n \in \mathbb{Z}}$ is the dual Schauder basis for an exponential Schauder basis $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ for $L^2(0, 1)$. That is the content of our result which reads as follows.

Theorem 1.1. Let $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ be an exponential Schauder Basis for $L^2(0, 1)$, for $\lambda_n \in \mathbb{R}$, and let $\{r_n\}_{n \in \mathbb{Z}}$ be its Dual Schauder Basis for $L^2(0, 1)$. Let $A \subset \mathbb{Z}$ so that $|A| = M$ for some positive integer M . Then the system

$$\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A} \tag{1.3}$$

is complete and minimal in the space $L^2(0, 1)$, if and only if $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$.

However, for such values of α the system (1.3) is not a Riesz Basis for $L^2(0, 1)$ and neither a frame. In particular, the weighted trigonometric system $\{t^\alpha \cdot e^{2\pi int}\}_{n \in \mathbb{Z} \setminus A}$ is complete and minimal in the space $L^2(0, 1)$, if and only if $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$, but it is not even a Schauder Basis for $L^2(0, 1)$.

We point out that under the assumption that the elements of the set A are consecutive integers, the non-basicality of the system (1.1) follows from the work of Kazarian [4, Corollary 1]. In such a case, a complete and minimal system (1.1) is a Markushevich Basis [5, Theorem 1].

Remark 1.1. The proof of Theorem 1.1 depends on the invertibility of Vandermonde matrices.

2 Preliminaries

For the terminology presented below, one may consult the books of Young [7], Heil [3], and Christensen [1].

Let \mathcal{H} be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. Let $F := \{f_n\}_{n \in J}$ be a countable family of vectors in \mathcal{H} with $J \subset \mathbb{Z}$. We say that F is complete if the closed span of F in \mathcal{H} is equal to \mathcal{H} and we say that F is minimal if each element f_n does not belong to the closed span of the remaining vectors of F in \mathcal{H} . If F is both complete and minimal then it is called **exact**. It is known that the minimality of F is equivalent to the existence of a biorthogonal sequence $\{g_n\}_{n \in J}$ in \mathcal{H} , that is

$$\langle f_n, g_m \rangle = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}.$$

This biorthogonal family $\{g_n\}_{n \in J}$ is unique if F is exact, but $\{g_n\}_{n \in J}$ itself does not have to be exact. If both F and $\{g_n\}_{n \in J}$ are exact, we say that they are Markushevich bases for \mathcal{H} .

An exact system F is a Schauder Basis for \mathcal{H} , if for any $f \in \mathcal{H}$, there exist unique numbers $d_{f,n}$ so that

$$f = \sum_{n \in J} d_{f,n} f_n,$$

with convergence in the $L^2(0, 1)$ norm. In this case, the unique biorthogonal family $\{g_n\}_{n \in J}$ is also a Schauder Basis for \mathcal{H} , called the dual Schauder basis of F , and for any $f \in \mathcal{H}$ we have

$$f = \sum_{n \in J} \langle f, g_n \rangle f_n = \sum_{n \in J} \langle f, f_n \rangle g_n,$$

with convergence in the $L^2(0, 1)$ norm ([3, Theorem 5.12 and Corollary 5.22] and [2, relation (1)]). A Schauder basis may converge conditionally, whereas a Riesz Basis for \mathcal{H} is a bounded Schauder basis that converges unconditionally ([3, Theorem 7.11]). We recall that $\{f_n\}_{n \in J}$ is a Riesz basis for \mathcal{H} if $f_n = V(e_n)$ for all $n \in J$ where $\{e_n\}_{n \in J}$ is an orthonormal basis for \mathcal{H} and V is a bounded bijective operator from \mathcal{H} onto \mathcal{H} .

3 Proof of Theorem 1.1

First we will show that the system (1.3) is complete in $L^2(0, 1)$ if $\alpha \geq M - \frac{1}{2}$ and then show that it is minimal if $\alpha < M + \frac{1}{2}$. We then prove that for $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$ the exact system (1.3) is not a Riesz basis for $L^2(0, 1)$, while the exact system (1.1) is not even a Schauder Basis for $L^2(0, 1)$.

3.1 If $\alpha \geq M - \frac{1}{2}$, the system (1.3) is Complete

Since $\{r_n(t)\}_{n \in \mathbb{Z}}$ is the dual Schauder Basis of $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ for $L^2(0, 1)$, then for any $f \in L^2(0, 1)$ we have

$$f(t) = \sum_{n \in \mathbb{Z}} \langle f, r_n \rangle e^{i\lambda_n t}$$

with the series converging in the $L^2(0, 1)$ norm.

For notation purposes, let $t_\alpha(t) := t^\alpha$. Now, if the system (1.3) is not complete in $L^2(0, 1)$, then there exists a non-zero function $h(t) \in L^2(0, 1)$ so that

$$\langle h, t_\alpha r_n \rangle = 0, \quad \text{for all } n \in \mathbb{Z} \setminus A.$$

Since $|t^\alpha| \leq 1$ for $t \in [0, 1]$, then ht_α belongs to $L^2(0, 1)$. Combining the above shows that

$$h(t) \cdot t^\alpha = \sum_{n \in A} \langle ht_\alpha, r_n \rangle e^{i\lambda_n t},$$

almost everywhere on $[0, 1]$, and such that at least one of the coefficients $\langle ht_\alpha, r_n \rangle$ for $n \in A$ is not equal to zero.

Rewrite now the set $\{\lambda_i : i \in A\}$ as $\{\Lambda_1, \Lambda_2, \dots, \Lambda_M\}$ such that $\Lambda_1 < \Lambda_2 < \dots < \Lambda_M$. Then

$$h(t) \cdot t^\alpha = \sum_{j=1}^M a_j e^{i\Lambda_j t}, \quad a_j \in \mathbb{C}.$$

The analytic extension to the complex plane of the above right hand side is the entire function

$$H_\alpha(z) := \sum_{j=1}^M a_j e^{i\Lambda_j z}, \quad (3.1)$$

so $H_\alpha(t) = h(t) \cdot t^\alpha$ for $t \in [0, 1]$. We will prove below that H_α cannot vanish M times or more at the point $z = 0$. Suppose it does: then

$$0 = H_\alpha(0) = H'_\alpha(0) = H''_\alpha(0) = \dots = H_\alpha^{(M-1)}(0) \quad (3.2)$$

where $H_\alpha^{(k)}(z)$ is the k^{th} derivative of H_α with respect to the variable z . From (3.1) and (3.2), we get

$$\begin{aligned} 0 &= a_1 + a_2 + a_3 + a_4 + \dots + a_M \\ &= a_1 \cdot \Lambda_1 + a_2 \cdot \Lambda_2 + a_3 \cdot \Lambda_3 + a_4 \cdot \Lambda_4 + \dots + a_M \cdot \Lambda_M \\ &= a_1 \cdot \Lambda_1^2 + a_2 \cdot \Lambda_2^2 + a_3 \cdot \Lambda_3^2 + a_4 \cdot \Lambda_4^2 + \dots + a_M \cdot \Lambda_M^2 \end{aligned}$$

and so on, until the last equation ($0 = H_\alpha^{(M-1)}(0)$)

$$0 = a_1 \cdot \Lambda_1^{M-1} + a_2 \cdot \Lambda_2^{M-1} + a_3 \cdot \Lambda_3^{M-1} + a_4 \cdot \Lambda_4^{M-1} + \dots + a_M \cdot \Lambda_M^{M-1}.$$

We can write this system of equations as

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & \Lambda_4^2 & \cdots & \Lambda_M^2 \\ \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 & \Lambda_4^3 & \cdots & \Lambda_M^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_1^{M-1} & \Lambda_2^{M-1} & \Lambda_3^{M-1} & \Lambda_4^{M-1} & \cdots & \Lambda_M^{M-1} \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_M \end{pmatrix}.$$

The above square matrix is an invertible Vandermonde matrix since the Λ_i 's are all different. Clearly now we have $a_j = 0$ for all $j = 1, 2, 3, \dots, M$, thus $H_\alpha(z)$ is the Zero function. But $H_\alpha(t) = h(t) \cdot t^\alpha$ for $t \in [0, 1]$, thus $h(t)$ is identically equal to zero, a contradiction. Therefore $H_\alpha(z)$ vanishes at the point $z = 0$ at most $M - 1$ times, hence $H_\alpha(z) = z^k \cdot g(z)$ for some integer k in the set $\{0, 1, \dots, M - 1\}$ and some entire function g such that $g(0) \neq 0$. Since $H_\alpha(t) = h(t) \cdot t^\alpha$ for $t \in [0, 1]$, we have

$$h(t) \cdot t^\alpha = t^k \cdot g(t), \quad \text{for some } k \in \{0, 1, 2, \dots, M - 1\}, \quad g \in C[0, 1] \quad g(0) \neq 0.$$

Thus

$$|h(t)|^2 = |g(t)|^2 \cdot t^{2k-2\alpha}.$$

Since $g(0) \neq 0$, then there are $\epsilon > 0$ and $\delta > 0$, so that $|g(t)| > \epsilon$ for all $t \in [0, \delta]$. So, for any $0 < \rho < \delta$, one has $|h(t)|^2 > \epsilon^2 \cdot t^{2k-2\alpha}$ for all $t \in [\rho, \delta]$. Therefore,

$$\int_\rho^\delta |h(t)|^2 dt > \epsilon^2 \cdot \int_\rho^\delta t^{2k-2\alpha} dt.$$

Since $k \leq M - 1$ and $\alpha \geq M - \frac{1}{2}$, then $2k - 2\alpha \leq -1$ and it follows that

$$\int_\rho^\delta t^{2k-2\alpha} dt \rightarrow \infty \quad \text{as} \quad \rho \rightarrow 0^+.$$

Hence $\int_0^1 |h(t)|^2 dt = \infty$ which contradicts the assumption that $h \in L^2(0, 1)$. Therefore, if $\alpha \geq M - \frac{1}{2}$ then the system (1.3) is complete in $L^2(0, 1)$.

3.2 If $\alpha < M + \frac{1}{2}$, the system (1.3) is Minimal

Like before, the set $\{\lambda_i : i \in A\}$ is rewritten as $\{\Lambda_1, \Lambda_2, \dots, \Lambda_M\}$ such that $\Lambda_1 < \Lambda_2 < \dots < \Lambda_M$.

We will show that for each $n \in \mathbb{Z} \setminus A$, there exist real numbers $a_{n,j}$ for $j = 1, 2, \dots, M$, so that the entire function

$$f_n(z) := e^{i\lambda_n z} + a_{n,1}e^{i\Lambda_1 z} + a_{n,2}e^{i\Lambda_2 z} + \dots + a_{n,M}e^{i\Lambda_M z} \quad (3.3)$$

vanishes at the point $z = 0$ at least M times and

$$f_n(t) = t^M \cdot g_n(t) \quad \text{for some continuous function } g_n \text{ on } [0, 1]. \quad (3.4)$$

The minimality of (1.3) follows from (3.3) and (3.4). Indeed, from (3.4) we have

$$\frac{f_n(t)}{t^\alpha} = \frac{f_n(t)}{t^M} \cdot \frac{t^M}{t^\alpha} = g_n(t) \cdot \frac{t^M}{t^\alpha}.$$

So

$$\left| \frac{f_n(t)}{t^\alpha} \right|^2 = |g_n(t)|^2 \cdot t^{2M-2\alpha}.$$

Since g_n is continuous on $[0, 1]$ and $\alpha < M + \frac{1}{2}$, thus $2M - 2\alpha > -1$, it follows that $f_n(t)/t^\alpha$ belongs to $L^2(0, 1)$. Then, for all positive integers n, m which are not in the set A , we have

$$\left\langle \frac{f_n}{t^\alpha}, t_\alpha r_m \right\rangle = \int_0^1 \left(e^{i\lambda_n t} + a_{n,1}e^{i\Lambda_1 t} + a_{n,2}e^{i\Lambda_2 t} + \dots + a_{n,M}e^{i\Lambda_M t} \right) \cdot \overline{r_m} dt = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, the family $\{\frac{f_n(t)}{t^\alpha}\}_{n \in \mathbb{Z} \setminus A}$ is biorthogonal to the sequence $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$. This means that the system (1.3) is minimal.

So, it remains to verify the existence of the numbers $a_{n,j}$ so that (3.4) is true. Now, if a trigonometric polynomial f_n as in (3.3) vanishes at the point $z = 0$ at least M times, then

$$0 = f_n(0) = f_n'(0) = f_n''(0) = \dots = f_n^{(M-1)}(0) \quad (3.5)$$

where $f_n^{(k)}(z)$ is the k^{th} derivative of f_n with respect to z . From (3.3) and (3.5) we get

$$\begin{aligned} 0 &= 1 + a_{n,1} + a_{n,2} + a_{n,3} + \dots + a_{n,M} \\ &= \lambda_n + a_{n,1} \cdot \Lambda_1 + a_{n,2} \cdot \Lambda_2 + a_{n,3} \cdot \Lambda_3 + \dots + a_{n,M} \cdot \Lambda_M \\ &= \lambda_n^2 + a_{n,1} \cdot \Lambda_1^2 + a_{n,2} \cdot \Lambda_2^2 + a_{n,3} \cdot \Lambda_3^2 + \dots + a_{n,M} \cdot \Lambda_M^2 \end{aligned}$$

and so on, until the last equation ($0 = f_n^{(M-1)}(0)$)

$$0 = \lambda_n^{M-1} + a_{n,1} \cdot \Lambda_1^{M-1} + a_{n,2} \cdot \Lambda_2^{M-1} + a_{n,3} \cdot \Lambda_3^{M-1} + \dots + a_{n,M} \cdot \Lambda_M^{M-1}.$$

We rewrite this system of equations as

$$\begin{pmatrix} -1 \\ -\lambda_n \\ -\lambda_n^2 \\ \vdots \\ -\lambda_n^{M-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \dots & \Lambda_M \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & \dots & \Lambda_M^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_1^{M-1} & \Lambda_2^{M-1} & \Lambda_3^{M-1} & \dots & \Lambda_M^{M-1} \end{pmatrix} \cdot \begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,M} \end{pmatrix}. \quad (3.6)$$

Since the Λ_i 's are all different, then the above square Vandermonde matrix is invertible. Therefore, real numbers $a_{n,j}$, $j = 1, 2, \dots, M$ do exist so that f_n vanishes at $z = 0$ at least M times. Thus

$$f_n(z) = z^M \cdot g_n(z) \quad \text{for some entire function } g_n(z),$$

hence (3.4) holds.

3.3 The system (1.3) is Exact if and only if $M - \frac{1}{2} \leq \alpha < M + \frac{1}{2}$

If $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$, then the system (1.3) is both complete and minimal in $L^2(0, 1)$, hence it is exact.

Assume now that the system (1.3) is exact: we will show that $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$.

Suppose first that it is exact but $\alpha < M - \frac{1}{2}$, say $M - \frac{3}{2} \leq \alpha < M - \frac{1}{2}$. Then, if $B \subset \mathbb{Z}$ such that $B = A \setminus \{\tau\}$ where τ is an integer not in the set A , then the cardinality of B equals $M - 1$. Hence the system $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus B}$ is exact. So the system $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$ will have excess equal to -1 instead of zero. Similarly for other positive values of α when $\alpha < M - \frac{3}{2}$. So α cannot be less than $M - \frac{1}{2}$.

Suppose now that it is exact but $\alpha \geq M + \frac{1}{2}$, say $M + \frac{1}{2} \leq \alpha < M + \frac{3}{2}$. Then, if $C \subset \mathbb{Z}$ such that $C = A \cup \{\tau\}$ where τ is an integer not in the set A , then the cardinality of C equals $M + 1$. Hence the system $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus C}$ is exact. So the system $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$ will have excess equal to $+1$ instead of zero. Similarly for other positive values of α when $\alpha \geq M + \frac{3}{2}$. So α cannot be greater than or equal to $M + \frac{1}{2}$.

In this way we conclude that if the system (1.3) is exact, then $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$.

3.4 The exact system (1.3) is not a Riesz Basis for $L^2(0, 1)$

Suppose that for $\alpha \in [M - \frac{1}{2}, M + \frac{1}{2})$ the exact system $\{t^\alpha \cdot r_n(t)\}_{n \in \mathbb{Z} \setminus A}$ is a Riesz Basis for $L^2(0, 1)$ and hence a Schauder Basis as well. Let f_n be as in (3.3): since the family $\{f_n/t^\alpha\}_{n \in \mathbb{Z} \setminus A}$ is biorthogonal

to the Schauder Basis (1.3), then $\{f_n/t_\alpha\}_{n \in \mathbb{Z} \setminus A}$ is the dual Schauder basis, hence for any $f \in L^2(0, 1)$, we have

$$f(t) = \sum_{n \in \mathbb{Z} \setminus A} \langle f, \frac{f_n}{t_\alpha} \rangle \cdot t^\alpha r_n(t),$$

with convergence in the $L^2(0, 1)$ norm. This implies that $|\langle f, \frac{f_n}{t_\alpha} \rangle| \cdot \|t_\alpha r_n\| \rightarrow 0$ as $n \rightarrow \infty$.

In particular, for each integer m belonging to the set A which has been excluded from \mathbb{Z} , we have

$$t^\alpha \cdot r_m(t) = \sum_{n \in \mathbb{Z} \setminus A} \langle t_\alpha r_m, \frac{f_n}{t_\alpha} \rangle \cdot t^\alpha r_n(t) \quad (3.7)$$

with convergence in the $L^2(0, 1)$ norm. Fix such an $m \in A$. Then

$$|\langle t_\alpha r_m, \frac{f_n}{t_\alpha} \rangle| \cdot \|t_\alpha r_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (3.8)$$

In (3.3) we have for every $n \in \mathbb{Z} \setminus A$, $f_n(t) = e^{i\lambda_n t} + a_{n,1}e^{i\Lambda_1 t} + a_{n,2}e^{i\Lambda_2 t} + \dots + a_{n,M}e^{i\Lambda_M t}$. Since $m \in A$ and $\{\lambda_i : i \in A\}$ is rewritten as $\{\Lambda_1, \Lambda_2, \dots, \Lambda_M\}$ such that $\Lambda_1 < \Lambda_2 < \dots < \Lambda_M$, then $\lambda_m = \Lambda_j$ for some $j \in \{1, 2, \dots, M\}$, and $e^{i\lambda_m t} = e^{i\Lambda_j t}$. Since the family $\{r_n\}_{n \in \mathbb{Z}}$ is biorthogonal to the family $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$, we get

$$\langle t_\alpha r_m, \frac{f_n}{t_\alpha} \rangle = \overline{a_{n,j}} \quad \text{for all } n \in \mathbb{Z} \setminus A.$$

Replacing into (3.8) we have

$$|a_{n,j}| \cdot \|t_\alpha r_n\| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (3.9)$$

Now, if the system (1.3) is a Riesz basis for $L^2(0, 1)$, then there are positive constants p and q so that $p < \|t_\alpha r_n\| < q$ for all $n \in \mathbb{Z} \setminus A$ ([1, Lemma 3.6.9]). We will show that there is a positive number δ , independent of $n \in \mathbb{Z} \setminus A$ and $j \in \{1, 2, \dots, M\}$, so that

$$\delta \cdot |\lambda_n|^{M-1} \leq |a_{n,j}| \quad \text{finally for all } n \in \mathbb{Z} \setminus A, \quad \text{and } j \in \{1, 2, \dots, M\}. \quad (3.10)$$

But then, combining the above with $p < \|t_\alpha r_n\| < q$ for all $n \in \mathbb{Z} \setminus A$, gives

$$p \cdot \delta \cdot |\lambda_n|^{M-1} \leq |a_{n,j}| \cdot \|t_\alpha r_n\| \quad \text{for all } n \in \mathbb{Z} \setminus A, \quad \text{and } j \in \{1, 2, \dots, M\}.$$

However, $M-1 \geq 0$ and $|\lambda_n| \rightarrow \infty$ as $n \rightarrow \infty$, thus the above shows that (3.9) is not true. Hence the exact system (1.3) is not a Riesz Basis for $L^2(0, 1)$.

So it remains to verify (3.10). Returning to the relation (3.6), consider the invertible Vandermonde matrix

$$\Lambda := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \dots & \Lambda_M \\ \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & \dots & \Lambda_M^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Lambda_1^{M-1} & \Lambda_2^{M-1} & \Lambda_3^{M-1} & \dots & \Lambda_M^{M-1} \end{pmatrix}$$

and denote by Λ^{-1} , $\det(\Lambda)$, and $\text{adj}(\Lambda)$, the inverse, determinant and adjoint of Λ respectively. Then we have

$$\begin{pmatrix} -1 \\ -\lambda_n \\ -\lambda_n^2 \\ \vdots \\ -\lambda_n^{M-1} \end{pmatrix} = \Lambda \cdot \begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,M} \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} a_{n,1} \\ a_{n,2} \\ a_{n,3} \\ \vdots \\ a_{n,M} \end{pmatrix} &= \Lambda^{-1} \cdot \begin{pmatrix} -1 \\ -\lambda_n \\ -\lambda_n^2 \\ \vdots \\ -\lambda_n^{M-1} \end{pmatrix} \\ &= \frac{1}{\det(\Lambda)} \cdot \text{adj}(\Lambda) \cdot \begin{pmatrix} -1 \\ -\lambda_n \\ -\lambda_n^2 \\ \vdots \\ -\lambda_n^{M-1} \end{pmatrix}. \end{aligned}$$

We now claim that all the entries of the far right column of $\text{adj}(\Lambda)$ are non-zero real numbers. So we need to check for the cofactors of the entries of the bottom row of Λ . For example, for the entry Λ_2^{M-1} , its cofactor is equal to $(-1)^{2+M}$ times the determinant of the matrix that remains when we delete the second column and the last row. That matrix is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \Lambda_1 & \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M \\ \Lambda_1^2 & \Lambda_3^2 & \Lambda_4^2 & \cdots & \Lambda_M^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda_1^{M-2} & \Lambda_3^{M-2} & \Lambda_4^{M-2} & \cdots & \Lambda_M^{M-2} \end{pmatrix}.$$

Clearly this is again an invertible Vandermonde matrix. The same is true for the cofactors of the other elements of the last row of Λ , thus we conclude that all the entries of the far right column of $\text{adj}(\Lambda)$ are non-zero real numbers. This means that for all $n \in \mathbb{Z} \setminus A$ and all $j = 1, 2, \dots, M$, we have

$$a_{n,j} = \sum_{k=1}^{M-1} c_{j,k} (-\lambda_n)^{k-1} + c_{j,M} (-\lambda_n)^{M-1}$$

where all the $c_{j,k}$ are real numbers, and especially $c_{j,M} \neq 0$ for all $j = 1, 2, \dots, M$. This implies the existence of some positive δ so that (3.10) holds.

3.5 The exact system (1.1) is not a Schauder Basis for $L^2(0, 1)$

If the exponential Schauder basis for $L^2(0, 1)$ is the family $\{e^{i2\pi nt}\}_{n \in \mathbb{Z}}$, then its dual Schauder basis is the same family, thus $r_n = e^{i2\pi nt}$ for all $n \in \mathbb{Z}$. In this case, we have $\|t_\alpha r_n\| = 1$, thus (3.9) becomes

$$|a_{n,j}| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

Clearly this is false due to (3.10), therefore the exact system (1.1) is not a Schauder Basis for $L^2(0, 1)$. The proof of Theorem 1.1 is now finished.

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