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# Scalable Command Governor via Semi-Ellipsoidal Set for Linear Systems with Time-Varying and/or Soft Constraints

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## Abstract

This paper provides a solution for a command governor (CG) that employs a semi-ellipsoidal set. The motivation is driven by the need to address the shortcomings of polyhedral and ellipsoidal sets widely used in CG. In particular for many applications, polyhedral sets are too complex to construct while ellipsoidal sets are too conservative. Furthermore, both types of sets are generally designed for systems with time-invariant and hard constraints. The contributions of the paper are: i) we provide new convex conditions to construct an invariant and constraint-admissible semi-ellipsoidal set for discrete-time linear systems with both time-invariant/time-varying and/or hard/soft constraints; ii) we propose a computationally efficient procedure to solve the online optimization problem associated with the newly introduced semi-ellipsoidal set in CG. Three numerical examples with comparison to earlier solutions from the literature illustrate the effectiveness of the proposed approach.

*Key words:* Command Governor, Linear System, State and Input Constraint, Invariant Set, ADMM

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## 1 Introduction

This paper is concerned with the tracking problem of discrete-time linear time-invariant systems, subject to possibly time-varying and/or soft constraints on both input and state. This is a problem that has been extensively studied over the last decades, and a number of different solutions are available as, for example, those based on model predictive control (MPC) [14], or command governor (CG) [5].

In MPC [14], an optimization problem is solved at each time instant to find the optimal input that drives the predicted plant output to the desired reference. MPC provides a natural way to take the time-varying and/or soft constraints into account. However, it is not trivial to guarantee recursive feasibility and asymptotic stability for MPC with time-varying and/or soft constraints.

CG provides a useful alternative to MPC [5]. The main idea of CG is to use optimization to find the best *applied reference* or equivalently the *command* based on the current state and the current value of the *desired reference*. CG may be attractive to practitioner concerned with on-line computational effort and/or interested in preserving an existing well-designed nominal controller.

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In CG, the state and the command are imposed to lie in the set  $\Omega$ , which is invariant and constraint-admissible. The set  $\Omega$  is constructed offline, and is used to guarantee the constraint satisfaction. In CG, ellipsoidal [17] and polyhedral sets [6] are widely considered for  $\Omega$ . The popularity of ellipsoidal sets stems from their computational efficiency via linear matrix inequality (LMI) formulation, their fixed complexity with respect to the state space dimension. However, it is well known [1], [16] that ellipsoids provide a conservative approximation of the domain of attraction (DoA). In contrast, polyhedral sets can provide a less conservative solution. However, constructing a polyhedral invariant set is computationally demanding, especially for CG, where the extended closed-loop system matrix is of high dimension, and has eigenvalues on the unit circle.

The set  $\Omega$  is computed using iterative procedures and/or convex optimization. In general, there is no closed-form relationship between  $\Omega$  and the state/input constraint sets. Consequently, considering time-varying and/or soft constraints in CG is not straightforward. As a result, most of CG schemes in the literature assume that the constraints are time-invariant and hard. There are only two papers that we are aware of, and that consider time-varying or soft constraints: [11], [12]. In [12], it was shown that the use of strictly contractive sets instead of invariant ones can handle time-varying constraints, if the constraints vary slowly enough. In [11], slack variables were

considered to relax the constraints. However, the introduction of slack variables along with the command in the state extension makes the problem of constructing  $\Omega$  very complex.

In [13], [19] an interesting class of invariant and constraint-admissible sets is considered for nominal linear systems with input and state constraint. The set, which is called semi-ellipsoidal set, is formed as the intersection of ellipsoidal and polyhedral sets. However, the construction of the set is based on a non-convex optimization problem.

In this paper, we follow the works in [13], [19]. The main aim is to propose new CG schemes that employ invariant and constraint-admissible semi-ellipsoidal sets. The main contributions of the paper are:

- (1) To the best of the author's knowledge, this is the first time a semi-ellipsoidal set is used in the context of a CG.
- (2) We provide new convex conditions to construct an invariant and constraint-admissible semi-ellipsoidal set for CGs with linear feedback. The new set is formed explicitly by the state and input constraint sets. Consequently, it can be easily extended to deal with time-varying and/or soft constraints.
- (3) As shown in [4] for CGs, using saturated feedback instead of linear one can enlarge the DoA, and hence improve the transient time. We provide convex conditions to construct invariant and constraint-admissible semi-ellipsoidal sets for CGs with saturated feedback. We consider both time-invariant/time-varying and/or hard/soft constraints.
- (4) Moving from an ellipsoidal and polyhedral set to a semi-ellipsoidal set introduces new types of constraints. Hence, the new optimization problem is no longer a quadratically constrained quadratic program [17] or a quadratic program [6]. We develop a tailored algorithm specifically for the new optimization problem.

It is worth noticing that the set  $\Omega$  is not required to be invariant in CG, but only to be a subset of the maximal admissible set [6]. This may simplify the offline construction of  $\Omega$  and the online optimization problem. The price to be paid is a potential loss of performance, see, e.g., [20].

The paper is organized as follows. Section 2 covers the problem formulation and earlier works on CG. Section 3, Section 4 are, respectively, dedicated to the construction of an invariant and constraint-admissible semi-ellipsoidal set with linear feedback, with saturated feedback for time-invariant and hard constraints. Then in Section 5, we extend the results in Section 3 and Section 4 to cope with time-varying and/or soft constraints. Section 6 is concerned with the online optimization

problem. Three numerical examples with comparison to earlier solutions from the literature are evaluated in Section 7. Finally, Section 8 concludes the article.

**Notation:** We denote by  $\mathbb{R}$  the set of real numbers, by  $\mathbb{R}^n$  the set of real  $n \times 1$  vectors, by  $\mathbb{R}^{n \times m}$  the set of real  $n \times m$  matrices. We use  $\mathbf{0}, \mathbf{I}$ , respectively, to denote the zero matrix, and the identity matrix of appropriate dimension. We denote a positive definite matrix  $P$  by  $P \succ 0$ . For a given  $P \succ 0$ ,  $\mathcal{E}(P)$  represents the following ellipsoid

$$\mathcal{E}(P) := \{x \in \mathbb{R}^{n_x} : x^T P^{-1} x \leq 1\} \quad (1)$$

For a given integer  $n$ , we use  $\overline{1, n}$  to denote the set  $1, 2, \dots, n$ . For symmetric matrices, the symbol  $(*)$  denotes each of its symmetric block.

## 2 Problem Formulation and Earlier Works on Command Governor

### 2.1 Problem Formulation

We consider the following discrete-time linear time-invariant system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \quad (2)$$

where  $x(k) \in \mathbb{R}^{n_x}$ ,  $u(k) \in \mathbb{R}^{n_u}$ , and  $y(k) \in \mathbb{R}^{n_y}$  are, respectively, the state, the control input, and the output.  $A \in \mathbb{R}^{n_x \times n_x}$ ,  $B \in \mathbb{R}^{n_x \times n_u}$ ,  $C \in \mathbb{R}^{n_y \times n_x}$ .

The state  $x(k)$  and the input  $u(k)$  are subject to the constraints  $x(k) \in X$ ,  $u(k) \in U$ , where  $X, U$  are polyhedral sets with

$$\begin{cases} X := \{x \in \mathbb{R}^{n_x} : \underline{g}_i \leq f_i^T x \leq \bar{g}_i, \forall i = \overline{1, l_x}\} \\ U := \{u \in \mathbb{R}^{n_u} : \underline{u}_j \leq u_j \leq \bar{u}_j, \forall j = \overline{1, n_u}\} \end{cases} \quad (3)$$

where  $f_i \in \mathbb{R}^{n_x}$ ,  $\underline{g}_i \in \mathbb{R}$ ,  $\bar{g}_i \in \mathbb{R}$ ,  $\underline{u}_j \in \mathbb{R}$ ,  $\bar{u}_j \in \mathbb{R}$ ,  $\forall i = \overline{1, l_x}, \forall j = \overline{1, n_u}$ . At the moment for simplicity, we consider the case where the constraints (3) are time-invariant and hard.

**Control Objectives:** For a given reference signal  $r(k) \in \mathbb{R}^{n_y}$ , the objective of the paper is to calculate a CG based control action  $u(k) = \mathcal{U}(x(k), r(k))$  such that: ii) the output  $y(k)$  follows as close as possible to  $r(k)$ ; ii) the state and input constraints (3) are fulfilled.

We recall some earlier works on CG in the next section.

## 2.2 Earlier Works on Command Governor

Any steady state  $(x_s, u_s, y_s)$  of the system (2) satisfies the following equation

$$\begin{bmatrix} A - \mathbf{I} & B \\ C & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} y_s \quad (4)$$

For simplicity, in the rest of the paper we assume that  $\begin{bmatrix} A - \mathbf{I} & B \\ C & \mathbf{0} \end{bmatrix}$  is invertible. In this case we can express the solution of (4) as

$$x_s = L_x v, u_s = L_u v \quad (5)$$

where  $v = y_s$  is the so-called command,  $L_x \in \mathbb{R}^{n_x \times n_y}$ ,  $L_u \in \mathbb{R}^{n_u \times n_y}$ .

The CG is an add-on scheme for enforcing the state and input constraints. It is based on an assumption that an asymptotically stabilizing control law  $u(k) = Kx(k)$  is available. This control law is designed to satisfy some performance specifications for small signals near the origin. In the CG, the control input is of the form

$$u(k) = K(x(k) - x_s) + u_s = Kx(k) + Lv \quad (6)$$

where  $L = L_u - KL_x$ .

By substituting (6) into (2), and by considering  $v(k+1) = v(k)$ , we can express the closed-loop system as

$$x_e(k+1) = A_e x_e(k) \quad (7)$$

where  $x_e(k) = [x(k)^T v(k)^T]^T$ , and

$$A_e := \begin{bmatrix} A + BK & BL \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Using (6), the constraints (3) become  $x_e \in X_e$  with

$$X_e := \left\{ x \in \mathbb{R}^{n_x}, v \in \mathbb{R}^{n_y} : \begin{array}{l} x \in X \\ Kx + Lv \in U \end{array} \right\} \quad (8)$$

We say that a set  $\Omega \subset \mathbb{R}^{n_x+n_y}$  is invariant for (7) if  $\forall x_e(k) \in \Omega$ , one has  $x_e(k+1) \in \Omega$ . In addition, if  $\Omega \subseteq X_e$ , then  $\Omega$  is constraint-admissible with respect to (8).

For a given state  $x(k)$  and a given reference signal  $r(k)$  at time instant  $k$ , the CG applies the control law  $u(k) = Kx(k) + Lv^*(k)$  to the system (2), where  $v^*(k)$  is the

solution of the following optimization problem

$$\begin{aligned} \min_v & \{(v - r(k))^T Q (v - r(k))\} \\ \text{s.t.} & \left[ x(k)^T v^T \right]^T \in \Omega \end{aligned} \quad (9)$$

where  $Q \succ 0$  is a weighting matrix, and  $\Omega$  is any invariant and constraint-admissible set for (7), (8).

The basic idea of CG is the following. If there is no constraint violation, then  $v^*(k) = r(k)$  is the solution of (9). Hence, the CG does not interfere with the operation of the system. If a potential for constraint violation exists, the CG seeks the closest admissible command  $v^*(k)$  to  $r(k)$ . In the extreme case thanks to the robust invariance of  $\Omega$ ,  $v(k+1) = v^*(k)$  remains a feasible solution of (9). This implies that the CG temporarily isolates the system from further variations of the reference to assure the constraint satisfaction.

Assuming feasibility at the initial condition, the CG (6), (9) guarantees recursive feasibility, i.e., if (9) is feasible at time  $k$ , then it is feasible at time  $k+1$ . If  $r(k) = r$  remains constant and  $r$  is reachable, then  $v^*(k)$  converges to  $r$  in a *finite time*. If  $r$  is not reachable, then  $v^*(k)$  will converge to the closest feasible value to  $r$  in a *finite time*. The finite-settling-time convergence is a desirable property. It shows that after transients caused by large changes in  $r(k)$ , the CG (6), (9) becomes inactive and nominal closed-loop system performance is recovered.

As written in Introduction, in the CG literature, only ellipsoidal and polyhedral sets are used for  $\Omega$ . The aim of this paper is to propose new CG schemes that are based on semi-ellipsoidal sets. We will first consider a semi-ellipsoidal set with a linear feedback. We then extend our approach to the case of nonlinear saturated feedback.

## 3 Semi-Ellipsoidal Set with Linear Feedback

In this section, we will provide a convex procedure to obtain an invariant and constraint-admissible semi-ellipsoidal set for (7), (8). The main idea is to exploit the particular structure of  $A_e, E_e$ . Starting from the original  $x$ -space, we show how to obtain invariance and constraint-admissibility conditions for semi-ellipsoidal set in the extended  $x_e$ -space via the translation and the scaling operators.

### 3.1 Translation

For a given  $P \succ 0$ , consider the sets  $\Omega_x(P)$  and  $\Omega_v(P)$  with

$$\Omega_x(P) := \left\{ \begin{bmatrix} x \\ v \end{bmatrix} : \left| (x - L_x v)^T P^{-1} (x - L_x v) \leq 1 \right. \right\} \quad (10)$$

and,  $\forall i = \overline{1, l_x}, \forall j = \overline{1, n_u}$

$$\Omega_v(P) := \left\{ v : \begin{cases} -f_i^T L_x v \leq -\underline{g}_i - \sqrt{f_i^T P f_i} \\ f_i^T L_x v \leq \bar{g}_i - \sqrt{f_i^T P f_i} \\ -L_{u,j} v \leq -\underline{u}_j - \sqrt{K_j P K_j^T} \\ L_{u,j} v \leq \bar{u}_j - \sqrt{K_j P K_j^T} \end{cases} \right\} \quad (11)$$

where  $L_{u,j}$  and  $K_j$  are, respectively, the  $j$ -th row of  $L_u$  and  $K$ .

In the  $(x, v)$ -space,  $\Omega_x(P)$  is an unbounded ellipsoid. In the  $x$ -space,  $\Omega_x(P)$  is the translation of  $\mathcal{E}(P)$  from the origin to the point  $L_x v$ . In other words,  $\Omega_x(P)$  is an ellipsoid centered at  $L_x v$ . Define

$$\Omega_e(P) := \Omega_x(P) \cap \Omega_v(P) \quad (12)$$

Clearly,  $\Omega_e(P)$  is a semi-ellipsoidal set as it is the intersection of the ellipsoid  $\Omega_x(P)$  and the polyhedral set  $\Omega_v(P)$ .

**Remark 1:** There is another useful geometric interpretation of  $\Omega_e(P)$ , namely,  $\Omega_e(P)$  is a parameterized ellipsoid  $\Omega_x(P)$  with its center constrained within the set  $\Omega_v(P)$ .  $\square$

**Theorem 1:** The set  $\Omega_e(P)$  is invariant and constraint-admissible for (7), (8) if and only if  $P$  satisfies the following condition

$$\begin{bmatrix} P & A_c P \\ (*) & P \end{bmatrix} \succeq 0, \quad (13)$$

**Proof:** We divide the proof into two parts: invariance proof and constraint admissibility proof.

**Invariance Proof:** One needs to show that  $\forall x_e(k) \in \Omega_e(P)$ , one has  $x_e(k+1) \in \Omega_e(P)$ . As  $v(k+1) = v(k)$ , it is clear that  $x_e(k+1) \in \Omega_v(P)$  if  $x_e(k) \in \Omega_v(P)$ .

It remains to show that  $x_e(k+1) \in \Omega_x(P)$ . Using (6), (7), one gets

$$x(k+1) = A_c x(k) + B(u_s(k) - K x_s(k)) \quad (14)$$

where  $x_s(k) = L_x v(k)$ ,  $u_s(k) = L_u v(k)$ . Using (4), (5) one gets

$$x_s(k+1) = A x_s(k) + B u_s(k)$$

where  $x_s(k+1) = L_x v(k+1)$ . Thus, with (14)

$$x(k+1) - x_s(k+1) = A_c(x(k) - x_s(k))$$

or, equivalently

$$\zeta(k+1) = A_c \zeta(k) \quad (15)$$

where  $\zeta(k) = x(k) - L_x v(k)$ . It is well known [2] that condition (13) is necessary and sufficient for the invariance of  $\mathcal{E}(P)$  with respect to (15). Hence,  $\forall x(k), \forall v(k)$  such that  $(x(k) - L_x v(k))^T P^{-1} (x(k) - L_x v(k)) \leq 1$ , one has  $(x(k+1) - L_x v(k+1))^T P^{-1} (x(k+1) - L_x v(k+1)) \leq 1$ . It follows that  $x_e(k+1) \in \Omega_x(P), \forall x_e(k) \in \Omega_x(P)$ .

**Constraint Admissibility Proof:** One needs to show that  $\Omega_e(P) \subseteq X_e$ , or equivalently  $x \in X, Kx + Lv \in U, \forall x_e \in \Omega_e(P)$ . First we will show that  $x \in X, \forall x_e \in \Omega_e(P)$ . For a given  $v \in \mathbb{R}^{n_v}$ , consider the following optimization problem,  $i \in \overline{1, l_x}$

$$\begin{aligned} \min_x \{ & f_i^T x \} \\ \text{s.t. } & (x - L_x v)^T P^{-1} (x - L_x v) \leq 1 \end{aligned} \quad (16)$$

As the cost is linear, the constraint of (16) is always active. The Lagrangian is

$$\mathcal{L}(x, \lambda) = f_i^T x + \frac{\lambda}{2} ((x - L_x v)^T P^{-1} (x - L_x v) - 1)$$

where  $\lambda > 0$  is the Lagrange multiplier. One has

$$\frac{\partial \mathcal{L}}{\partial x} = f_i + \lambda^* P^{-1} (x^* - L_x v) = 0$$

Therefore  $x^* = L_x v - \frac{1}{\lambda^*} P f_i$ . As the constraint in (16) is active, one obtains

$$(L_x v - \frac{1}{\lambda^*} P f_i - L_x v)^T P^{-1} (L_x v - \frac{1}{\lambda^*} P f_i - L_x v) = 1$$

or equivalently,  $\lambda^* = \sqrt{f_i^T P f_i}$ . Hence

$$x^* = L_x v - \frac{1}{\sqrt{f_i^T P f_i}} P f_i$$

It follows that

$$\min_x \{ f_i^T x \} = f_i^T L_x v - \sqrt{f_i^T P f_i}$$

Using (11),  $\forall v \in \Omega_v(P)$  one has  $f_i^T L_x v - \sqrt{f_i^T P f_i} \geq \underline{g}_i, \forall i = \overline{1, l_x}$ . Therefore,  $\forall i = \overline{1, l_x}$

$$\min_x \{ f_i^T x \} \geq \underline{g}_i, \forall x_e \in \Omega_e(P) \quad (17)$$

Analogously, it can be shown that

$$\max_x \{ f_i^T x \} \leq \bar{g}_i, \forall x_e \in \Omega_e(P) \quad (18)$$

Combining (17), (18), one gets  $x \in X, \forall x_e \in \Omega_e(P)$ .

Now we show that  $Kx + Lv \in U, \forall x_e \in \Omega_e(P)$ . Using similar arguments like those for the state constraints, it can be shown that for a given  $v, \forall x_e \in \Omega_x(P)$

$$\min_x \{K_j x + L_j v\} = L_{u,j} v - \sqrt{K_j P K_j^T}$$

where  $L_j$  is the  $j$ -th row of  $L$ . Using (11),  $\forall v \in \Omega_v(P)$  one has

$$L_{u,j} v - \sqrt{K_j P K_j^T} \geq \underline{u}_j$$

Hence,  $\forall j = \overline{1, n_u}$

$$\min_x \{K_j x + L_j v\} \geq \underline{u}_j, \forall x_e \in \Omega_e(P) \quad (19)$$

Analogously, one obtains

$$\max_x \{K_j x + L_j v\} \leq \bar{u}_j, \forall x_e \in \Omega_e(P) \quad (20)$$

Using (19), (20), one gets  $x_e \in \Omega_v \forall x_e \in \Omega_e(P)$ . The proof is complete.  $\square$

Once invariance and constraint-admissibility conditions for  $\Omega_e(P)$  are obtained, our next step is to maximize the size of  $\Omega_e(P)$ . As  $\Omega_e(P)$  can be considered as a parameterized ellipsoid, our idea is to optimize  $\Omega_x(P)$  or equivalently  $P$  for a particular fixed value of  $v = v_f, v_f \in \Omega_v(P)$ . Using (11), since  $f_i^T P f_i \geq 0, K_j P K_j^T \geq 0$  we obtain a necessary condition of  $v_f$  for guaranteeing that  $P$  exists

$$\begin{cases} \underline{g}_i \leq f_i^T L_x v_f \leq \bar{g}_i, \forall i = \overline{1, l_x}, \\ \underline{u}_j \leq L_{u,j}^T v_f \leq \bar{u}_j, \forall j = \overline{1, n_u} \end{cases} \quad (21)$$

Using (11), for a fixed  $v_f$  satisfying (21), the constraints on  $P$  are

$$\begin{cases} f_i^T P f_i \leq (f_i^T L_x v_f - \underline{g}_i)^2, \forall i = \overline{1, l_x} \\ f_i^T P f_i \leq (f_i^T L_x v_f - \bar{g}_i)^2, \forall i = \overline{1, l_x} \\ K_j P K_j^T \leq (L_{u,j} v_f - \underline{u}_j)^2, \forall j = \overline{1, n_u} \\ K_j P K_j^T \leq (L_{u,j} v_f - \bar{u}_j)^2, \forall j = \overline{1, n_u} \end{cases} \quad (22)$$

The problem of optimizing  $P$  can be formulated as

$$\begin{aligned} & \min_P \{f(P)\} \\ & \text{s.t. (13), (22)} \end{aligned} \quad (23)$$

where  $f(P)$  can be any convex function, such as  $-\log \det(P), -\text{tr}(P)$ , e.t.c.

**Remark 2:** A simple choice for  $v_f$  is  $v_f = \mathbf{0}$ , if  $X, U$  contain the origin in their interior.  $\square$

Problem (23) is a convex semi-definite program (SDP). We can solve it efficiently using free available LMI parser such as YALMIP [15] or CVX [8].

### 3.2 Scaling

In this section we show how to use the scaling operator to enlarge the semi-ellipsoidal set obtained in Section 3.1. We assume that  $\Omega_e$  is available as a result from Section 3.1. For a given scalar  $\alpha \geq 0$ , define

$$\Omega_x(\alpha, P) := \left\{ \begin{bmatrix} x \\ v \end{bmatrix} : \left| (x - L_x v)^T P^{-1} (x - L_x v) \leq \alpha^2 \right. \right\} \quad (24)$$

and,  $\forall i = \overline{1, l_x}, \forall j = \overline{1, n_u}$

$$\Omega_v(\alpha, P) := \left\{ v : \begin{cases} -f_i^T L_x v \leq -\underline{g}_i - \alpha \sqrt{f_i^T P f_i} \\ f_i^T L_x v \leq \bar{g}_i - \alpha \sqrt{f_i^T P f_i} \\ -L_{u,j} v \leq -\underline{u}_j - \alpha \sqrt{K_j P K_j^T} \\ L_{u,j} v \leq \bar{u}_j - \alpha \sqrt{K_j P K_j^T} \end{cases} \right\} \quad (25)$$

Define also

$$\Omega_e(\alpha, P) := \Omega_x(\alpha, P) \cap \Omega_v(\alpha, P) \quad (26)$$

It is clear that  $\Omega_e(\alpha, P)$  is a semi-ellipsoidal set.

Note that  $\Omega_x(\alpha, P)$  is an unbounded ellipsoid in the  $(x, v)$ -space for any  $\alpha \geq 0$ . Note also that  $\Omega_x(\alpha, P)$  is the scaling of  $\Omega_x(P)$  with the scaling factor  $\alpha$ , i.e.,  $\forall [x^T \ v^T]^T \in \Omega_x(P)$ , one has  $[\alpha x^T \ \alpha v^T]^T \in \Omega_x(\alpha, P)$ . However, this is not the case for  $\Omega_v(\alpha, P)$ , i.e.,  $\Omega_v(\alpha, P)$  is not a scaling of  $\Omega_v(P)$ .

**Theorem 2:** Given any  $\alpha \geq 0$  such that  $\Omega_v(\alpha, P)$  is non-empty. If  $\Omega_e(P)$  is invariant for (7), then  $\Omega_e(\alpha, P)$  is invariant and constraint-admissible for (7), (8).

**Proof:** Similar to the proof of Theorem 1, we divide the proof of Theorem 2 into two parts: invariance proof and constraint-admissibility proof.

**Invariance Proof:** Clearly, for any  $\alpha$  such that  $\Omega_v(\alpha, P)$  is non-empty, one has  $x_e(k+1) \in \Omega_v(\alpha, P), \forall x_e(k) \in \Omega_v(\alpha, P)$  as  $v(k+1) = v(k)$ .

It remains to prove that  $x_e(k+1) \in \Omega_x(\alpha, P), \forall x_e(k) \in \Omega_e(\alpha, P)$ . Note that if  $\alpha = 0$ , then  $x_e(k) \in \Omega_e(\alpha, P)$  if and only if  $x_e(k) = \mathbf{0}$ . Hence  $x_e(k+1) = 0 \in \Omega_e(\alpha, P)$ . If  $\alpha > 0$ , one has

$$\frac{x_e(k+1)}{\alpha} = A_e \frac{x_e(k)}{\alpha}$$

Using the two facts that: i)  $\frac{x_e(k)}{\alpha} \in \Omega_e(P), \forall x_e(k) \in \Omega_e(\alpha, P), \forall \alpha > 0$ ; ii)  $\Omega_e(P)$  is invariant for (7), one gets  $\frac{x_e(k+1)}{\alpha} \in \Omega_e(P)$ . It follows that  $x_e(k+1) \in \Omega_x(\alpha, P)$ .

**Constraint Admissibility Proof:** The main idea of the proof is to show that  $x \in X$  and  $Kx + Lv \in U$ ,  $\forall x_e \in \Omega_e(\alpha, P)$ . The proof is omitted here, as it follows the same arguments as the one of Theorem 1.  $\square$

Using Theorem 2, the following Algorithm can be used to construct an invariant and constraint-admissible semi-ellipsoidal set  $\Omega_e(\alpha, P)$ .

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Algorithm 1:  $\Omega_e(\alpha, P)$  Construction

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- 1: Select  $v_f$ .
  - 2: Obtain  $P$  by solving (23).
  - 3: Construct  $\Omega_v(\alpha, P)$ .
  - 4:  $\Omega_e(\alpha, P) := \Omega_x(\alpha, P) \cap \Omega_v(\alpha, P)$ .
- 

#### 4 Semi-Ellipsoidal Set with Saturated Feedback

The aim of this section is to provide a way to construct an invariant and constraint-admissible semi-ellipsoidal set for CG using the following saturated control law

$$u(k) = \text{sat}(Kx(k) + Lv(k)) \quad (27)$$

where the saturation function  $\text{sat}(Kx + Lv)$  is defined as

$$\text{sat}(Kx + Lv) = [\text{sat}(K_1x + L_1v) \dots \text{sat}(K_{n_u}x + L_{n_u}v)]^T$$

$$\text{sat}(K_jx + L_jv) = \begin{cases} \underline{u}_j, & \text{if } K_jx + L_jv \leq \underline{u}_j, \\ u_j, & \text{if } \underline{u}_j \leq K_jx + L_jv \leq \bar{u}_j, \\ \bar{u}_j, & \text{if } \bar{u}_j \leq K_jx + L_jv \end{cases}$$

As noticed in [4], the motivation of using the nonlinear saturated control law (27) instead of the linear one (6) is that the associated DoA of (27) can be significantly larger than that of (6). As a result, the transient behavior of the closed-loop system can thus be improved.

It is worth noticing that in the linear feedback case (6), Theorem 1 provides a foundation to construct a semi-ellipsoidal invariant and constraint-admissible set. Theorem 2 is built upon Theorem 1. Hence, due to space limitations, the focus in this section is on extending Theorem 1 to handle the saturated control law (27).

In the following we recall the linear differential inclusion (LDI) modeling framework in [9]. We will use it to model the saturation nonlinearity. Define  $\mathcal{S}$  as the set of  $n_u \times n_u$  diagonal matrix whose diagonal elements are either 0 or 1. For example, if  $n_u = 2$  then

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

There are  $2^{n_u}$  elements in  $\mathcal{S}$ . Define  $S_m, m = \overline{1, 2^{n_u}}$  as an element in  $\mathcal{S}$ . Define also  $S_m^- = \mathbf{I} - S_m$ . Clearly,  $S_m^- \in \mathcal{S}$ .

**Lemma 1:** For  $\xi \in \mathbb{R}^{n_u}$  such that  $\underline{u}_j \leq \xi_j \leq \bar{u}_j, \forall j = \overline{1, n_u}$ , one has,

$$\text{sat}(u) \in \text{Co}(S_m u + S_m^- \xi), \forall m = \overline{1, 2^{n_u}} \quad (28)$$

where  $\text{Co}(\cdot)$  denotes the convex hull operator.

For example, if  $n_u = 2$ , we have

$$\begin{bmatrix} \text{sat}(u_1) \\ \text{sat}(u_2) \end{bmatrix} \in \text{Co} \left( \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ \xi_2 \end{bmatrix}, \begin{bmatrix} \xi_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \right)$$

By substituting (27) into (2), and by considering  $v(k+1) = v(k)$ , one gets

$$\begin{bmatrix} x(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} Ax(k) + B\text{sat}(Kx(k) + Lv(k)) \\ v(k) \end{bmatrix} \quad (29)$$

Given matrices  $P_s \in \mathbb{R}^{n_x \times n_x}, H \in \mathbb{R}^{n_u \times n_x}$ , define  $\Omega_v(P_s, H)$  as  $\forall i = \overline{1, l_x}, \forall j = \overline{1, n_u}$

$$\Omega_v(P_s, H) := \left\{ v : \begin{cases} -f_i^T L_x v \leq -\underline{g}_i - \sqrt{f_i^T P_s f_i} \\ f_i^T L_x v \leq \bar{g}_i - \sqrt{f_i^T P_s f_i} \\ -L_{u,j} v \leq -\underline{u}_j - \sqrt{H_j P_s H_j^T} \\ L_{u,j} v \leq \bar{u}_j - \sqrt{H_j P_s H_j^T} \end{cases} \right\} \quad (30)$$

where  $H_j$  is the  $j$ -th row of  $H$ . Define also

$$\Omega_e(P_s, H) = \Omega_x(P_s) \cap \Omega_v(P_s, H) \quad (31)$$

where  $\Omega_x(P_s)$  is the parameterized ellipsoid (10) with  $P = P_s$ . Clearly,  $\Omega_e(P_s, H)$  is a semi-ellipsoidal set. We have the following result.

**Theorem 3:** If there exist  $P_s \succ 0, H \in \mathbb{R}^{n_u \times n_x}$  such that

$$\begin{bmatrix} P_s & A_m P_s + B_m Y \\ (*) & P_s \end{bmatrix} \succeq 0, \forall m = \overline{1, 2^{n_u}}, \quad (32)$$

and that the set  $\Omega_v(P_s, H)$  is non-empty, where  $A_m = A + BS_m K, B_m = BS_m^-, \forall m = \overline{1, 2^{n_u}}, Y = HP_s$ , then the set  $\Omega_e(P_s, H)$  is invariant and constraint-admissible for (29), (8).

**Proof:** As the proofs that  $\Omega_v(P_s, H)$  is invariant, and that  $\Omega_e(P_s, H)$  is constraint-admissible follow the same arguments as the ones of the proofs of Theorem 1, we skip these proofs here. It remains to show that  $\Omega_x(P_s)$  is robustly invariant. Define  $N = L_u - HL_x$ . Using Lemma 1, for system (2) under the saturated control law (27), one has,  $\forall x, \forall v : \underline{u}_j \leq H_j x + N_j v \leq \bar{u}_j, j = \overline{1, n_u}$ ,

$$x(k+1) \in \text{Co}(A_m(x(k) - L_x v(k)) + B_u s(k)) \quad (33)$$

where  $\mathbf{A}_m = A + BS_m K + BS_m^- H, \forall m = \overline{1, 2^{n_u}}, u_s(k) = L_u u(k)$ . Using (4), (5) one gets

$$x_s(k+1) = Ax_s(k) + Bu_s(k)$$

with  $x_s(k+1) = L_x v(k+1), x_s(k) = L_x v(k)$ . Thus, with (33)

$$x(k+1) - x_s(k+1) \in \text{Co}(\mathbf{A}_m(x(k) - x_s(k))) \quad (34)$$

Recall that  $Y = HP_s$ . Rewrite (32) as

$$\begin{bmatrix} P_s & \mathbf{A}_m P_s \\ (*) & P_s \end{bmatrix} \succeq 0, \forall m = \overline{1, 2^{n_u}} \quad (35)$$

It is well known [2] that conditions (35) assure the invariance of  $\mathcal{E}(P_s)$  for system (34), i.e.,  $\forall x(k), \forall v(k)$  such that  $(x(k) - L_x v(k))^T P_s^{-1} (x(k) - L_x v(k)) \leq 1$ , one has

$$(x(k+1) - L_x v(k+1))^T P_s^{-1} (x(k+1) - L_x v(k+1)) \leq 1$$

Hence  $\Omega_x(P_s)$  is invariant for (29).  $\square$

**Remark 3:** It is clear that if  $H = K$ , then  $\Omega_e(P_s, H) = \Omega_e(P_s)$ . In this case, one also has

$$A + BS_m K + BS_m^- K = A + BK, \forall m = \overline{1, 2^{n_u}}$$

It follows that (32) becomes (13) with  $P_s = P, Y = KP_s$ . Consequently, compared to the linear control law (6), the use of the saturated one (27) in conjunction with the LDI framework introduces  $H$  as an additional degree of freedom.  $\square$

We use  $H$  and  $P_s$  as decision variables to maximize the size of  $\Omega_e(P_s, H)$ . For this purpose, we select a particular value of  $v = v_f, v_f \in \Omega_v(P_s, H)$ . Using (30) and as  $f_i^T P_s f_i \geq 0, H_j P_s H_j^T \geq 0, \forall i = \overline{1, l_x}, \forall j = \overline{1, n_u}$ , it follows that (21) provides a necessary condition of  $v_f$  for the existence of  $P_s$  and  $H$ .

For any  $v_f$  satisfying (21), using the first two equation of (30), one gets

$$\begin{cases} f_i^T P_s f_i \leq (f_i^T L_x v_f - \underline{g}_i)^2, \forall i = \overline{1, l_x} \\ f_i^T P_s f_i \leq (f_i^T L_x v_f - \bar{g}_i)^2, \end{cases} \quad (36)$$

Using the last two equation of (30), one obtains

$$\begin{cases} H_j P_s H_j^T \leq (L_{u,j} v_f - \underline{u}_j)^2, \forall j = \overline{1, n_u} \\ H_j P_s H_j^T \leq (L_{u,j} v_f - \bar{u}_j)^2, \end{cases}$$

thus, using the Schur complement and  $Y = HP_s$

$$\begin{cases} \begin{bmatrix} (L_{u,j} v_f - \underline{u}_j)^2 & Y_j \\ Y_j^T & P_s \end{bmatrix} \succeq 0, \\ \begin{bmatrix} (L_{u,j} v_f - \bar{u}_j)^2 & Y_j \\ Y_j^T & P_s \end{bmatrix} \succeq 0 \end{cases} \quad \forall j = \overline{1, n_u} \quad (37)$$

where  $Y_j$  is the  $j$ -th row of  $Y, \forall j = \overline{1, n_u}$ .

The problem of optimizing  $P_s, H$  can be formulated as

$$\begin{cases} \min_{P_s, Y} \{f(P_s)\} \\ \text{s.t. (32), (36), (37)} \end{cases} \quad (38)$$

where  $f(\cdot)$  can be any convex function. Note that (38) is a SDP program.

Concerning the scaling operator, using similar arguments as the ones in Section 3.2, it is possible to show that if  $\Omega_e(P_s, H)$  is invariant and constraint-admissible for (29), (8), then so is the set

$$\Omega_e(\alpha, P_s, H) = \Omega_x(\alpha, P_s) \cap \Omega_v(\alpha, P_s, H) \quad (39)$$

where  $\Omega_x(\alpha, P_s)$  is defined in (24) with  $P = P_s, \alpha \geq 0$ , and

$$\Omega_v(\cdot) := \left\{ v : \begin{cases} F_v v \leq g - F_\alpha \alpha \end{cases} \right\} \quad (40)$$

with

$$F_v = \begin{bmatrix} -f_1^T L_x \\ \vdots \\ -f_{l_x}^T L_x \\ f_1^T L_x \\ \vdots \\ f_{l_x}^T L_x \\ -L_{u,1} \\ \vdots \\ -L_{u,n_u} \\ L_{u,1} \\ \vdots \\ L_{u,n_u} \end{bmatrix}, g = \begin{bmatrix} -\underline{g}_1 \\ \dots \\ -\underline{g}_{l_x} \\ \bar{g}_1 \\ \dots \\ \bar{g}_{l_x} \\ -\underline{u}_1 \\ \dots \\ -\underline{u}_{n_u} \\ \bar{u}_1 \\ \dots \\ \bar{u}_{n_u} \end{bmatrix}, F_\alpha = \begin{bmatrix} \sqrt{f_1^T P_s f_1} \\ \vdots \\ \sqrt{f_{l_x}^T P_s f_{l_x}} \\ \sqrt{f_1^T P_s f_1} \\ \vdots \\ \sqrt{f_{l_x}^T P_s f_{l_x}} \\ \sqrt{H_1 P_s H_1^T} \\ \vdots \\ \sqrt{H_{n_u} P_s H_{n_u}^T} \\ \sqrt{H_1 P_s H_1^T} \\ \vdots \\ \sqrt{H_{n_u} P_s H_{n_u}^T} \end{bmatrix} \quad (41)$$



## 5 Semi-Ellipsoidal Set with Time-Varying and/or Soft Constraints

Perhaps the most interesting aspect of the results in Section 3 and in Section 4 is that they can be easily extended to cope with time-varying and/or soft constraints. This is because

- The semi-ellipsoidal set  $\Omega_e(\alpha, P)$  or  $\Omega_e(\alpha, P_s, H)$  is the intersection of the unbounded ellipsoid  $\Omega_x(\alpha, P)$  or  $\Omega_x(\alpha, P_s)$  and the polyhedral set  $\Omega_v(\alpha, P)$  or  $\Omega_v(\alpha, P_s, H)$ .
- The set  $\Omega_v(\alpha, P)$  or  $\Omega_v(\alpha, P_s, H)$  is formed explicitly by the constraints of  $X, U$ .

From this point on we will focus only on semi-ellipsoidal set with saturated feedback since the results in Section 3 are special cases of the results in Section 4. For simplicity of discussion, we consider only the case where the bounds  $\underline{g}_i(k), \bar{g}_i(k)$  are time-varying,  $\forall i = \overline{1, l_x}$ , i.e., the set  $X(k)$  at time instant  $k$  is

$$X(k) = \{x : \underline{g}_i(k) \leq f_i^T x \leq \bar{g}_i(k), i = \overline{1, l_x}\} \quad (42)$$

Clearly if,  $\forall k \geq 0$

$$\underline{g}_i(k+1) \leq \underline{g}_i(k), \bar{g}_i(k+1) \geq \bar{g}_i(k), \forall i = \overline{1, l_x}$$

then  $X(k) \subseteq X(k+1)$ . In this case, it is quite simple to guarantee the recursive feasibility in CG. Otherwise, if  $\underline{g}_i(k), \bar{g}_i(k)$  vary arbitrarily, then there is no control strategy that can guarantee the recursive feasibility. A way to cope with this problem is to relax the state constraints (42) as,  $\forall i = \overline{1, l_x}$

$$X(k) = \{x : \underline{g}_i(k) - \epsilon_i(k) \leq f_i^T x \leq \bar{g}_i(k) + \epsilon_i(k)\} \quad (43)$$

where  $\epsilon_i(k) \geq 0$  are slack variables,  $\forall i = \overline{1, l_x}$ . Note that in addition to guaranteeing the recursive feasibility, the use of  $\epsilon_i(k)$  also allows to trade-off constraint violation against the improvements in tracking performance.

Consider any matrices  $P_s, H$  satisfying (32),  $Y = HP_s$ . Define

$$\epsilon(k) = [\epsilon_1(k) \ \epsilon_2(k) \ \dots \ \epsilon_{l_x}(k)]^T \quad (44)$$

Define also the set  $\Omega_v(\epsilon(k), \alpha, P_s, H)$  as

$$\Omega_v(\cdot) := \left\{ v : \begin{cases} F_v v \leq g(k) - F_\alpha \alpha - F_\epsilon \epsilon(k) \end{cases} \right\} \quad (45)$$

where  $F_v, F_\alpha$  and  $g(k)$  are defined in (41) with time-varying  $\underline{g}_i(k), \bar{g}_i(k), \forall i = \overline{1, l_x}$ , and

$$F_\epsilon = [-1 \dots -1 \ -1 \dots -1 \ 0 \ \dots \ 0 \ 0 \ \dots \ 0]^T$$

Define also

$$\Omega_e(\epsilon(k), \alpha, P_s, H) = \Omega_x(\alpha, P_s) \cap \Omega_v(\epsilon(k), \alpha, P_s, H) \quad (46)$$

The following corollary is a direct consequence of the results in Section 4.

**Corollary 1:** For any  $\alpha \geq 0, \epsilon(k) \geq \mathbf{0}$  such that  $\Omega_v(\epsilon(k), \alpha, P_s, H)$  is non-empty, the set  $\Omega_e(\epsilon(k), \alpha, P_s, H)$  is invariant for (29) and constraint-admissible for the input constraints (3), and for the state constraints (43).

## 6 Online Optimization Problem

### 6.1 Semi-Ellipsoidal Set Based Command Governor

The CG based on the set  $\Omega_e(\epsilon, \alpha, P_s, H)$  for time-varying and/or soft constraints (43) requires the online solution of the following optimization problem

$$\begin{aligned} \min_{v, \alpha, \epsilon} \quad & \{(v - r(k))^T Q_t (v - r(k)) + \epsilon^T Q_\epsilon \epsilon\} \\ \text{s.t.} \quad & \begin{cases} [x(k)^T \ v^T]^T \in \Omega_e(\epsilon(k), \alpha, P_s, H), \\ \alpha \geq 0, \\ \epsilon \geq \mathbf{0} \end{cases} \end{aligned} \quad (47)$$

where  $x(k)$  is the current state;  $Q_t \succ 0, Q_\epsilon \succ 0$  are weighting matrices. The first term in the cost function penalizes the deviation of the command from the desired reference. The second term penalizes the violation of the constraints. Once the optimal solution  $(v^*(k), \alpha^*(k), \epsilon^*(k))$  of (47) is found, the control action is computed as

$$u(k) = \text{sat}(Kx(k) + Lv^*(k)) \quad (48)$$

It is clear that the CG (47), (48) guarantees recursive feasibility, since one can always select  $\epsilon$  such that  $\Omega_e(\epsilon(k), \alpha, P_s, H) \subseteq \Omega_e(\epsilon(k+1), \alpha, P_s, H)$ . Concerning the finite time convergence for the command  $v$ , consider the case when there exists  $k_s$  such that,  $\forall k \geq k_s$

$$\underline{g}_i(k) = \underline{g}_i(k_s), \bar{g}_i(k) = \bar{g}_i(k_s), r(k) = r(k_s) \quad (49)$$

Denote  $(v^*(k_s), \alpha^*(k_s), \epsilon^*(k_s))$  as the optimal solution of the following problem

$$\begin{aligned} \min_{v, \alpha, \epsilon} \quad & \{(v - r(k_s))^T Q_t (v - r(k_s)) + \epsilon^T Q_\epsilon \epsilon\} \\ \text{s.t.} \quad & \begin{cases} v \in \Omega_v(\epsilon(k), \alpha, P_s, H), \\ \alpha \geq 0, \\ \epsilon \geq \mathbf{0} \end{cases} \end{aligned} \quad (50)$$

Note that the control scheme (47), (48) to satisfy the state constraints in a soft way is an extension of the regular CG theory. Hence, if (49) holds  $\forall k \geq k_s$ , then there

exists a time index  $k_f$  such that  $v^*(k) = v^*(k_s), \alpha^*(k) = \alpha^*(k_s), \epsilon^*(k) = \epsilon^*(k_s), \forall k \geq k_f$ . If  $\exists \alpha \geq 0$  such that  $r(k_s) \in \Omega_v(\mathbf{0}, \alpha, P_s, H)$ , then  $v^*(k_s) = r(k_s), \epsilon^*(k_s) = \mathbf{0}$  is the trivial solution of (50). This means that the command converges to the desired reference in finite time. If  $\nexists \alpha \geq 0$  such that  $r(k_s) \in \Omega_v(\mathbf{0}, \alpha, P_s, H)$ , then  $v^*(k_s)$  may not coincide with  $r(k_s)$ . The optimal solution  $(v^*(k_s), \epsilon^*(k_s))$  of (50) is the trade-off between tracking performance and constraint violation.

Concerning the online optimization problem (47), clearly, it is a convex program. In the following we provide a way to solve efficiently (47). Our solver, which is based on the alternating direction method of multipliers (ADMM), can exploit the particular structure of  $\Omega_e(\cdot)$ . Hence, our solver can solve (47) extremely fast via very simple mathematical operations. In the last decade, ADMM has emerged as an effective algorithm for solving structured convex optimization problems [3]. Our main contribution in this section is to show how to convert (47) into a form that the sub-optimization problems associated with the ADMM can be solved efficiently. We recall the ADMM theory in the next section.

## 6.2 Alternating Direction Method of Multipliers

Consider the following optimization problem

$$\begin{aligned} & \min_{s,z} \{h(s)\} \\ & \text{s.t.} \begin{cases} Ms - z = b \\ z \in Z \end{cases} \end{aligned} \quad (51)$$

where  $h(s)$  is a convex function, the matrix  $M$  and the vector  $b$  are of appropriate dimension,  $Z$  is a convex set. One way to solve (51) is to form the augmented Lagrangian

$$\begin{aligned} \mathcal{L}_\rho(s, z, y) = & h(s) + \eta^T(Ms - z - b) \\ & + \frac{\rho}{2}(Ms - z - b)^T(Ms - z - b) \end{aligned} \quad (52)$$

where  $\eta$  is the Lagrange multiplier,  $\rho > 0$  is a tuning parameter that presents the trade-off between the cost function and the equality constraints.

ADMM works by solving iteratively two sub-problems and then updating the Lagrange multiplier. At iteration  $q$  we carry out the following steps.

(1) **Step 1:** minimize  $\mathcal{L}_\rho(s, z, \eta)$  with respect to  $s$

$$s^{(q+1)} := \arg \min_s \left\{ \mathcal{L}_\rho(s, z^{(q)}, \eta^{(q)}) \right\} \quad (53)$$

(2) **Step 2:** minimize  $\mathcal{L}_\rho(s, z, \eta)$  with respect to  $z$

$$z^{(q+1)} := \arg \min_{z \in Z} \left\{ \mathcal{L}_\rho(s^{(q+1)}, z, \eta^{(q)}) \right\} \quad (54)$$

(3) **Step 3:** update the Lagrange multiplier

$$\eta^{(q+1)} = \eta^{(q)} + \rho(Ms^{(q+1)} - z^{(q+1)} - b) \quad (55)$$

We use the superscript  $(q)$  in (53), (54), (55) to denote the values of variables calculated at iteration  $q$ .

The ADMM is particularly useful when it is possible to solve (53), (54) efficiently. The main contribution of this section is to show how to convert (47) into a form that (53), (54) admit closed-form expressions.

The primal and dual residuals at iteration  $q$  are

$$\zeta_p^{(q)} = Ms^{(q)} - z^{(q)} - b, \quad \zeta_d^{(q)} = z^{(q)} - z^{(q-1)} \quad (56)$$

The algorithm is terminated when the primal and dual residuals satisfy a stopping criterion. A typical criterion is  $\|\zeta_p^{(q)}\|_\infty \leq \zeta_p, \|\zeta_d^{(q)}\|_\infty \leq \zeta_d$ , where  $\zeta_p, \zeta_d$  are given tolerances.

## 6.3 Solving the Online Optimization Problem of the Semi-Ellipsoidal Set Based CG Using ADMM

We reformulate (47) as the optimization problem (51) as follows. Let us take

$$s = \begin{bmatrix} v^T & \alpha & \epsilon^T \end{bmatrix}^T \quad (57)$$

thus, using (47), the cost function  $h(s)$  is

$$h(s) = \frac{1}{2}s^T Qs - r(k)^T q_0^T s \quad (58)$$

with  $Q = \text{diag}(Q_t, 0, Q_\epsilon)$ , and  $q_0 = [Q \ 0 \ \mathbf{0}]^T$ .

Define the following auxiliary variables

$$\begin{cases} z_1 = D_s(x(k) - L_x v), & z_2 = \alpha, \\ z_3 = F_v v + F_\alpha \alpha + F_\epsilon \epsilon, & z_4 = \epsilon \\ z = [z_1^T \ z_2 \ z_3^T \ z_4]^T \end{cases} \quad (59)$$

where  $D_s \in \mathbb{R}^{n_x \times n_x}$  is a square root of  $P_s^{-1}$ , i.e.,  $D_s^T D_s = P_s^{-1}$ . Using (57), (59) one has

$$Ms - z = b_0 x(k) \quad (60)$$

with

$$M = \begin{bmatrix} -D_s L_x & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} \\ F_v & F_\alpha & F_\epsilon \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad b_0 = \begin{bmatrix} -D_s \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Using (46), (47), one obtains the set  $Z$

$$Z = \left\{ z : \begin{cases} z_1^T z_1 \leq z_2^2, z_2 \geq 0, \\ z_3 \leq g(k), z_4 \geq \mathbf{0} \end{cases} \right\} \quad (61)$$

Hence problem (47) is reformulated as (51) with  $h(s), M, b = b_0 x(k)$ , and  $Z$  being given in (58), (60), (61), respectively.

Using (53), (54), (55), we carry out the following steps in each iteration.

**Step 1:** The optimization problem is

$$\min_s \left\{ \frac{1}{2} s^T (\mathcal{Q} + \rho M^T M) s - (\rho M^T z^{(q)} - M^T \eta^{(q)} + q_0 r(k) + \rho M^T b_0 x(k))^T s \right\} \quad (62)$$

The optimization problem (62) is an unconstrained quadratic program. The solution to (62) is given explicitly as

$$s^{(q+1)} = \mathcal{M} \left( \rho M^T z^{(q)} - M^T \eta^{(q)} + q_0 r(k) + \rho M^T b_0 x(k) \right) \quad (63)$$

where  $\mathcal{M} = (\mathcal{Q} + \rho M^T M)^{-1}$ .

**Step 2:** The optimization problem is

$$\begin{cases} \min_z \left\{ z^T z - 2(M s^{(q+1)} + \frac{\eta^{(q)}}{\rho} - b_0 x(k))^T z \right\} \\ \text{s.t. } z \in Z \end{cases} \quad (64)$$

Define

$$\nu = M s^{(q+1)} + \frac{\eta^{(q)}}{\rho} - b_0 x(k) \quad (65)$$

Note that  $\nu$  and  $z$  are of the same dimension. We rewrite  $\nu$  as

$$\nu = [\nu_1^T \ \nu_2^T \ \nu_3^T \ \nu_4^T]^T$$

where  $\nu_i$  has the same dimension of  $z_i$ ,  $\forall i = \overline{1, 4}$ .

The cost function and the constraints of (64) are separable in  $z_i, i = \overline{1, 4}$ . We can carry out their update in parallel. The update of  $(z_1, z_2)$  is the solution of the following problem

$$\begin{cases} \min_{z_1, z_2} \{ z_1^T z_1 + z_2^2 - 2\nu_1^T z_1 - 2\nu_2 z_2 \} \\ \text{s.t. } z_1^T z_1 \leq z_2^2, z_2 \geq 0 \end{cases} \quad (66)$$

By using the method of Lagrange multipliers, the solu-

tion of (66) can be calculated analytically as [18]

$$\begin{cases} \text{If } \nu_1^T \nu_1 \leq \nu_2^2, \nu_2 < 0 \\ \quad z_1^{(q+1)} = \mathbf{0}, z_2^{(q+1)} = 0 \\ \text{If } \nu_1^T \nu_1 \leq \nu_2^2, \nu_2 \geq 0 \\ \quad z_1^{(q+1)} = \nu_1, z_2^{(q+1)} = \nu_2 \\ \text{Otherwise} \\ \quad z_1^{(q+1)} = \frac{\nu_2 + \sqrt{\nu_1^T \nu_1}}{2\sqrt{\nu_1^T \nu_1}} \nu_1, z_2^{(q+1)} = \frac{\nu_2 + \sqrt{\nu_1^T \nu_1}}{2} \end{cases} \quad (67)$$

The update of  $z_3$  is the solution of the optimization problem

$$\begin{cases} \min_{z_3} \{ z_3^T z_3 - 2\nu_3^T z_3 \} \\ \text{s.t. } z_3 \leq g(k) \end{cases} \quad (68)$$

The solution of (68) can be given in the closed-form as

$$z_3^{(q+1)} = \min(\nu_3, g(k)) \quad (69)$$

where the min operator is taken component-wise.

Analogously, the update of  $z_4$  is given as

$$z_4^{(q+1)} = \max(\nu_4, \mathbf{0}) \quad (70)$$

For the given stopping tolerances  $\epsilon_d, \epsilon_p$  and initial point  $(s^{(0)}, z^{(0)}, \eta^{(0)})$ , Algorithm 2 shows the ADMM algorithm applied to the problem (47)

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Algorithm 2: ADMM Based Solver for (47)

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- 1: Update  $s^{(q+1)}$  using (63).
- 2: Update  $z^{(q+1)}$  using (67), (69), (70).
- 3: Update  $\eta^{(q+1)}$  as

$$\eta^{(q+1)} \leftarrow \eta^{(q)} + \rho(M s^{(q+1)} - z^{(q+1)} - b_0 x(k))$$


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**Remark 4:** Using Algorithm 2, the updates of  $s^{(q)}$  and  $z^{(q)}$  are given explicitly with the use of the auxiliary variables (59). The use of other types of auxiliary variables, e.g.,  $z_3 = d_2 - L_x \theta_2$  instead of (59), would lead to a sub-optimization problem of the ADMM, which does not have an explicit solution.  $\square$

## 7 Examples

This section illustrates the potential benefit of the new methods by simulations of three examples system. The CVX toolbox [8] was used to solve SDP optimization problems. For comparison purpose, we denote the standard polyhedral based CG, the semi-ellipsoidal set based CG with linear feedback, and with saturated feedback as method 1, method 2, and method 3, respectively.

### 7.1 Example 1: Time-Invariant and Hard Constraints

This example is taken from [4]. Consider system (2) with

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (71)$$

The gain  $K$  is given as

$$K = \begin{bmatrix} -4.2674 & -3.1426 \end{bmatrix} \quad (72)$$

The input and state constraints are  $-0.2 \leq u \leq 0.2$ ,  $-1 \leq x_1 \leq 2$ .

By solving the SDP problem (23) with  $v_f = \mathbf{0}$ , one gets

$$P = \begin{bmatrix} 0.0118 & -0.0184 \\ -0.0184 & 0.0322 \end{bmatrix}$$

By solving the SDP problem (38) with  $v_f = \mathbf{0}$ , one obtains

$$P_s = \begin{bmatrix} 0.0947 & -0.0708 \\ -0.0708 & 0.1180 \end{bmatrix}, H = [-0.6141 \quad -0.7835] \quad (73)$$

Fig. 1 presents the ellipsoid  $\mathcal{E}(P)$  (dashed magenta), the projection of  $\Omega_e(\alpha, P)$  onto the  $x$ -space (dashed magenta), the projection of  $\Omega_e(\alpha, P_s, H)$  onto the  $x$ -space (solid cyan). For this example, we were able to calculate the maximal invariant and constraint-admissible polyhedral set  $\Omega_p$ . Fig. 1 presents the projection of  $\Omega_p$  onto the  $x$ -space (dash-dot yellow). We can observe that: i) the feasible set of method 2 is slightly smaller than that of method 1; ii) both of them are much smaller than that of method 3.

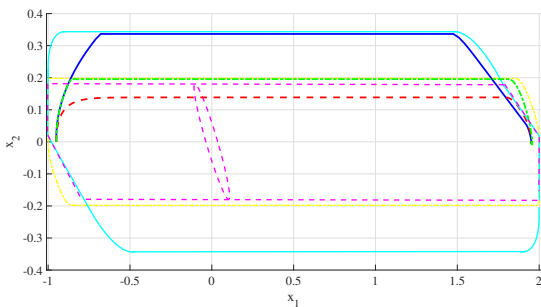
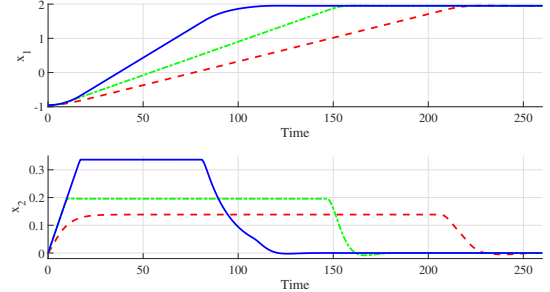


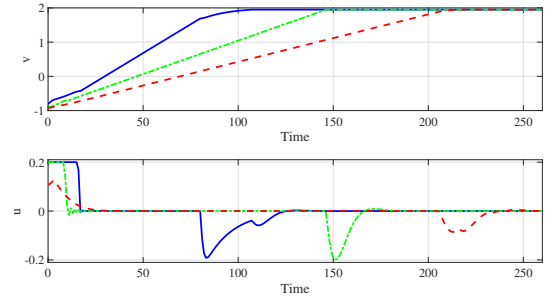
Fig. 1. Feasible sets and phase-space trajectories for example 1.

For the initial condition  $x(0) = [-0.95 \ 0]^T$  and for the reference  $r(k) = 1.95$ , Fig 1 shows the phase-space trajectory of method 1 (dash-dot green), method 2 (dashed red), and method 3.

Fig. 2 presents the state, the command and the input trajectories as functions of time for method 1 (dash-dot green), for method 2 (dashed red), for method 3 (solid blue). We observe that method 3 has the best performance in terms of transient time, followed by method 1, then by method 2.



(a) State trajectories.



(b) Command and input trajectories

Fig. 2. (a) State trajectories; (b) Command and input trajectories for method 1 (dash-dot green), for method 2 (dashed red), for method 3 (solid blue) for example 1.

Finally, using the function TIC/TOC of MATLAB 2023b, we found that the online computation times for one sampling interval were  $3.1312 \times 10^{-3}$ [sec],  $1.7415 \times 10^{-4}$ [sec], and  $1.5775 \times 10^{-4}$ [sec] for method 1, method 2, and method 3, respectively. We use *quadprog* solver in Matlab to solve the online quadratic program of method 1.

### 7.2 Example 2: Time-Varying and Soft Constraints

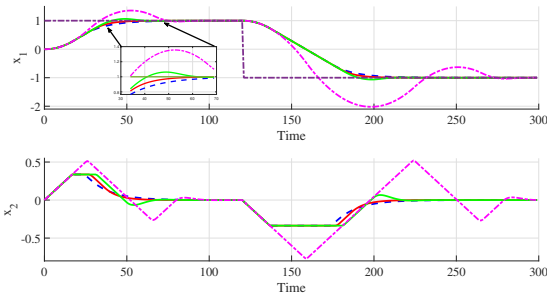
We consider system (2) with  $A, B, C, K$  being given in (71), (72). The input constraints are  $-0.2 \leq u \leq 0.2$ . The objective is to design a CG based control law that tracks a time-varying piecewise constant reference  $r(k)$  while managing overshoot of the output. For this purpose, at time  $k$ , we impose the following constraint on  $x_1$

$$\begin{cases} x_1(k) \leq r(k) + \epsilon(k), & \text{if } x_1(0) \leq r(0) \\ x_1(k) \geq r(k) - \epsilon(k), & \text{if } x_1(0) \geq r(0) \end{cases} \quad (74)$$

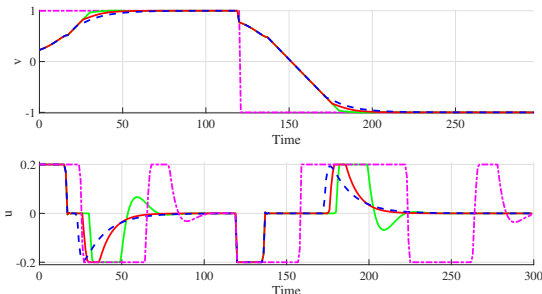
where  $\epsilon \geq 0$ . The reference  $r(k)$  is

$$r(k) = \begin{cases} 1, & \text{if } k \leq 120, \\ -1, & \text{otherwise} \end{cases} \quad (75)$$

The initial condition is  $x(k) = [0 \ 0]^T$ . For simplicity of discussion, we apply only method 3 in this example. We use  $P_s, H$  in (73) for the set  $\Omega_\epsilon(\epsilon(k), \alpha, P_s, H)$ . Three difference simulations consist of three different weights for the optimization problem (47),  $Q_t = 1$  and  $Q_\epsilon = 10, 1, 0.1$ . Fig. 3 presents the reference (dash-dot violet), the state, the command, and the input trajectories for method 3 with  $Q_\epsilon = 10$  (dashed blue),  $Q_\epsilon = 1$  (solid red),  $Q_\epsilon = 0.1$  (solid green). Fig. 3 also presents the simulation results (dash-dot magenta) when  $v(k) = r(k)$  is directly applied to the system, i.e., without using CG. We observe that our approach can handle overshoot of the output.



(a) Reference and state trajectories.



(b) Command and input trajectories

Fig. 3. (a) Reference (dash-dot violet) and state trajectories; (b) Command and input trajectories for method 3 with  $Q_\epsilon = 10$  (dashed blue),  $Q_\epsilon = 1$  (solid red),  $Q_\epsilon = 0.1$  (solid green), and without using CG (dash-dot magenta) for example 2.

### 7.3 Example 3: High-Order System

To demonstrate that our approach can cope with high-order system, we consider the continuous-time helicopter dynamics given in [10]. After discretization of the system

with the sampling time  $T_s = 0.01(\text{sec})$ , one obtains (2) with

$$A = \begin{bmatrix} 1 & 0.01 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.0045 & -0.0000 & 0 & 0 \\ 0 & 0 & 0.9990 & 0.0096 & 0 & 0 \\ 0 & 0 & -0.1909 & 0.9288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9998 & 0.0098 \\ 0 & 0 & 0 & 0 & -0.0301 & 0.9513 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0.001 & 0.1909 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0002 & 0.0486 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The state and input constraints are:  $|x_1| \leq 11$ ,  $|x_5| \leq 1$ ,  $|u_1| \leq 1$  and  $|u_2| \leq 1$ . The feedback gain  $K$  is chosen as an LQ gain with the state and control weighting matrices  $Q_x = \text{diag}\{100, 1, 10, 1, 10, 1\}$ ,  $Q_u = \text{diag}\{0.1, 0.1\}$ .

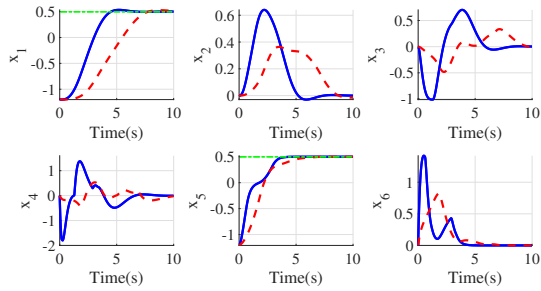
The matrices  $P, P_s, H$  are optimized, respectively, by using (23), (38) with  $f(P) = -\log\det(P)$ ,  $f(P_s) = -\log\det(P_s)$ . Numerical values of  $K, P, P_s, H$  are not reported here, but will be sent to the reader upon request. Using standard procedures in [7], we were not able to construct the polyhedral maximal invariant and constraint-admissible set.

The weighting matrices for the optimization (47) are  $Q_t = \mathbf{I}$ ,  $Q_\epsilon = 0.1\mathbf{I}$ . The reference is  $r(k) = [0.5 \ 0.5]^T$ . The initial condition is  $x(0) = [-1.2 \ 0 \ 0 \ 0 \ -1.2 \ 0]^T$ , which violates the constraints. Fig 4 presents the reference (dash-dot green), the state, the command and the input trajectories as functions of time for method 2 (dashed red), and for method 3 (solid blue).

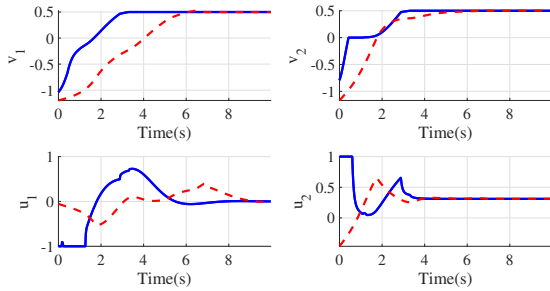
Finally, using the function TIC/TOC of MATLAB 2023b, we found that the online computation times for one sampling interval were  $6.5747 \times 10^{-4}[\text{sec}]$ , and  $5.5552 \times 10^{-4}[\text{sec}]$  for method 2 and method 3, respectively.

## 8 Conclusion

In this paper we provide new command governor schemes that employ a semi-ellipsoidal set. We address both linear and saturated feedback cases, for which we propose new linear matrix inequality conditions to construct an invariant and constraint-admissible semi-ellipsoidal set. Furthermore, we show that these conditions can be extended to handle time-varying and/or soft constraints, providing great flexibility and adaptability. To solve the online optimization problem associated with the introduced semi-ellipsoidal set, we propose a tailored alternating direction method of multipliers based technique.



(a) Reference and state trajectories.



(b) Command and input trajectories

Fig. 4. (a) Reference (dash-dot green) and state trajectories; (b) Command and input trajectories for method 2 (dashed red) and for method 3 (solid blue) for example 3.

The main feature of the new technique is that it requires very simple mathematical operations at each iteration, thus ensuring numerical efficiency for real-time applications. Given their computationally tractable design procedure, the proposed approaches are applicable for medium to large-sized systems. Three numerical examples with comparison to earlier solutions in the literature demonstrate the effectiveness of the new methods.

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