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AN EXAMPLE OF ACCURATE MICROLOCAL TUNNELING IN ONE DIMENSION

ANTIDE DURAFFOUR AND NICOLAS RAYMOND

ABSTRACT. We investigate the spectral analysis of a class of pseudo-differential operators in one dimension. Under symmetry assumptions, we prove an asymptotic formula for the splitting of the first two eigenvalues. This article is an example of extension to pseudo-differential operators of the tunneling effect formulas known for the symmetric electric Schrödinger operator.

1. INTRODUCTION

1.1. Motivation. This article is devoted to the spectral analysis of a microlocal version of the Schrödinger operator $-h^2\Delta + V(x)$. This differential operator has drawn a lot of attention since the eighties, especially with the mathematical study of quantum tunnelling. A manifestation of this phenomenon is the effect of symmetries of V on the spectrum of the operator in the semiclassical limit $h \rightarrow 0$ and has generated much interest lately. The most prominent results in this direction go back to the papers by Simon [27, 28, 29, 30] and the famous Helffer-Sjöstrand series of articles [13, 14, 15, 16] establishing tunneling formulas. These articles motivated the study of purely magnetic tunneling effects, see for instance [3, 5]. These recent works have cast a new light on the semiclassical analysis of the magnetic Schrödinger operator by revealing the central role of the microlocal approach, developed for instance in [31] and also in [20, 21, 22, 23, 24], to tackle spectral problems (see [1] where this view point has recently been used). Especially, the core of the strategy is a microlocal dimensional reduction that leads to an effective pseudo-differential operator in one dimension, even though the original operator is differential. These recent advances lead us to explore tunneling effect for pseudo-differential operators. In this context, already available results deal with operators of the form $a(hD_x) + V(x)$ such as the Klein-Gordon operator of symbol $\sqrt{1 + \xi^2} + V(x)$ (see [12, Section 3.2.3]) as well as the Harper operator (with symbol $\cos(\xi) + \cos(x)$) on $L^2(\mathbb{R})$ (see [17, 19, 18]). In the case when the x and ξ dependency are intertwined, there is no general result as accurate as that established in the Helffer-Sjöstrand papers (or in the works on the Witten Laplacian, see, for instance, [11, 25]). The reason for that is the absence of exponentially sharp estimates for the eigenfunctions in general, even though there exist a priori bounds, see for instance [22].

In the present article, we tackle the case of a family of pseudo-differential operators in dimension one, whose form appears in [5] (where the tunneling effect is determined by sub-principal terms). Namely, we consider the pseudo-differential operator \mathcal{L}_h given by

$$\mathcal{L}_h = a(hD_x) + hb^w,$$

where a and b are real valued and belong to the symbol class

$$S(\mathbb{R}^2) = \{p = p(x, \xi) \in \mathcal{C}^\infty(\mathbb{R}^2) : \forall \alpha \in \mathbb{N}^2, \exists C_\alpha > 0 : |\partial^\alpha p(x, \xi)| \leq C_\alpha\},$$

and where p^w denotes the semiclassical Weyl quantization defined by

$$p^w \psi(x) = (\text{Op}_h^w p) \psi(x) = \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{i(x-y)\eta/h} p\left(\frac{x+y}{2}, \eta\right) \psi(y) dy d\eta.$$

The operator \mathcal{L}_h is selfadjoint and bounded, from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, in virtue of the Calderón-Vaillancourt theorem.

1.2. Framework, heuristics and main result. In the whole article, one will work under the following assumption on the principal symbol a .

Assumption 1.1. *The real valued function $a \in S(\mathbb{R}^2)$ depends on ξ only and we simply write $a(x, \xi) = a(\xi)$. It has a unique minimum at $\xi = 0$, assumed to be 0, which is non-degenerate and not attained at infinity. Moreover, the function a has a holomorphic extension to the strip $\Sigma_r = \mathbb{R} + i(-r, r)$, for some $r > 0$.*

Near $\xi = 0$, the symbol a shares the same features as the symbol of the Laplacian ξ^2 . One could also consider a in a slightly more general class containing ξ^2 . To avoid the corresponding technicalities and to keep the proof as transparent as possible, we choose to focus our attention on the bounded case. Let us now describe the type of analytic perturbation that we want to deal with.

Assumption 1.2. *The function b belongs to $S(\mathbb{R}^2)$ and it can be extended to $\mathbb{R} \times \Sigma_r$ holomorphically in the sense that $b(x, \cdot) \in \mathcal{O}(\Sigma_r)$ for all $x \in \mathbb{R}$ and that $b \in S(\mathbb{R} \times \Sigma_r)$.*

We recall that we want to discuss the effect of symmetries on the spectrum of p^w . That is why we make the following assumption.

Assumption 1.3. *The functions a and b are even in the sense that they satisfy $p(-X) = p(X)$ for all $X = (x, \xi) \in \mathbb{R}^2$. Moreover, $x \mapsto b(x, 0)$ is non-negative and attains its minima at exactly two points $x_\ell < 0$ and $x_r = -x_\ell > 0$, which are non-degenerate minima. We assume that $b(x_\ell, 0) = 0$ and we let*

$$b_\infty = \liminf_{|x| \rightarrow +\infty} b(x, 0) > 0.$$

Note that $a(\xi) = \xi^2$ (which doesn't satisfy Assumption 1.1 though) and $b(x, \xi) = V(x)$ satisfy Assumption 1.3 as soon as V is a non-degenerate symmetric double well, which is covered by [13].

Let us now discuss the heuristics that will allow us to guess and formulate a tunneling estimate for \mathcal{L}_h . We notice that

$$\mathcal{L}_h = \text{Op}_h^w(a(\hbar\xi) + hb(x, \hbar\xi)), \quad \text{with } \hbar = h^{\frac{1}{2}},$$

which follows from a dilation in the integral defining the Weyl quantization. Naively, we write the formal expansion

$$a(\hbar\xi) + hb(x, \hbar\xi) = h \left(\frac{a''(0)}{2} \xi^2 + b(x, 0) \right) + \mathcal{O}_\xi(h^{\frac{3}{2}}).$$

If we forget the a priori non uniform remainder, we are reduced to a Schrödinger operator with double well potential. We denote

$$\mathcal{M}_h = -\hbar^2 \frac{a''(0)}{2} \partial_x^2 + b(x, 0),$$

and we can recall the classical tunneling estimate (see, for instance, [9, 10, 26] and the pedagogical paper [4, Theorem 1.2]). The spectral gap between the lowest two eigenvalues satisfies

$$\lambda_2(\mathcal{M}_h) - \lambda_1(\mathcal{M}_h) = (1 + o(1))A\hbar^{\frac{1}{2}}e^{-\frac{S}{\hbar}}, \quad (1.1)$$

where

$$A = 4 \left(\frac{a''(0)}{2} \right)^{\frac{1}{4}} \left(\frac{\kappa}{\pi} \right)^{\frac{1}{2}} \sqrt{V(0)} \exp \left(- \int_{x_\ell}^0 \frac{\partial_s \sqrt{V(s)} - \kappa}{\sqrt{V(s)}} ds \right),$$

and

$$S = \sqrt{\frac{2}{a''(0)}} \int_{x_\ell}^{x_r} \sqrt{b(s, 0)} ds, \quad V(s) = b(s, 0), \quad \kappa = (\sqrt{V})'(x_\ell).$$

Moreover, for some $c > 0$, $\lambda_3(\mathcal{M}_h) - \lambda_2(\mathcal{M}_h) \geq c\hbar$.

Surprisingly, even though $h\mathcal{M}_h$ is a rough approximation of our operator \mathcal{L}_h , the estimate (1.1) provides us with the one term asymptotics of the spectral gap for \mathcal{L}_h . In fact, our analyticity assumptions will allow us to deal with the remainders and describe the eigenfunctions and their exponential decay (see Section 1.3 below).

Here is the main theorem.

Theorem 1.4. *Under Assumptions 1.1, 1.2 and 1.3, we have*

$$\lambda_2(\mathcal{L}_h) - \lambda_1(\mathcal{L}_h) \underset{h \rightarrow 0}{\sim} h(\lambda_2(\mathcal{M}_h) - \lambda_1(\mathcal{M}_h)).$$

Moreover, for some $c > 0$, $\lambda_3(\mathcal{L}_h) - \lambda_2(\mathcal{L}_h) \geq ch^{\frac{3}{2}}$.

1.3. Organization and strategy. We follow the same guidelines as in the "differential" case (see, for instance, the Bourbaki exposé [26] and [7, Chapter 6]). On the one hand, we will see that a corner stone of the analysis is the celebrated Fefferman-Phong inequality (in an exponentially weighted space). This allows us to extend the Agmon estimates, which are of *local* nature, and yields optimal WKB approximations adapted to the pseudo-differential context. On the other hand, the stationary phase theorem reveals an effective Schrödinger operator that makes our heuristics rigorous.

In Section 2, we explain why the bottom of the spectrum of \mathcal{L}_h is discrete, see Lemma 2.1. Then, we start discussing the spectral analysis of the one-well operator $\mathcal{L}_{h,\ell}$ obtained after sealing the right well, see (2.1). The main result in Section 2 is Proposition 2.2, where we describe WKB quasimodes for $\mathcal{L}_{h,\ell}$. The proof of Proposition 2.2 is given in Section 2.2 and it relies on a classical WKB construction for pseudo-differential operators, which is itself based on the stationary phase theorem (see Appendices A & B).

In Section 3, we provide the reader with the one-term asymptotics of the low-lying eigenvalues of $\mathcal{L}_{h,\ell}$, see Proposition 3.1. In particular, we show that these eigenvalues are simple and separated by gaps of order $h^{\frac{3}{2}}$. The proof relies on Proposition 2.2 and on a microlocalization lemma, namely Lemma 3.3, which shows that the eigenfunctions of $\mathcal{L}_{h,\ell}$ are microlocalized near $(x, \xi) = (x_\ell, 0)$. The localization in x is more subtle than the localization in ξ since it originates from the behavior of the subprincipal symbol (which requires the use of the Fefferman-Phong inequality to be analyzed). These microlocalization results can be adapted to the two-well situation and they allow to prove a first estimate of the tunneling phenomenon, see Proposition 3.2. The spectrum of the double well operator is described, modulo

$\mathcal{O}(h^\infty)$, as the union of the spectra associated with the one-well operators. The eigenvalues are distributed by duets, each duet being separated from the others by gaps of order $h^{\frac{3}{2}}$.

In Section 4, we improve the localization result near x_ℓ by establishing optimal Agmon estimates, see Proposition 4.1. This proposition is an elliptic estimate for the conjugated operator $\mathcal{L}_{h,\ell}^\varphi = e^{\frac{\varphi}{\sqrt{h}}} \mathcal{L}_{h,\ell} e^{-\frac{\varphi}{\sqrt{h}}}$. Of crucial use is again the Fefferman-Phong inequality, but this time in the refined symbol class $S_{\delta_1, \delta_2}(\mathbb{R}^2)$, see Appendix A. In Section 4.2, we apply Proposition 4.1 to get optimal exponential decay estimates of the one-well eigenfunctions (see Proposition 4.4) and, most importantly, to get a very accurate approximation of the eigenfunctions by the WKB quasimodes, see Corollary 4.5. This approximation looks quite similar to that in the Schrödinger case (see [4, Proposition 2.7]). However, we emphasize that we are here in a pseudo-differential context (which is even rather degenerate) and that such good approximations are rare.

Section 5 is devoted to the proof of Theorem 1.4. To do so, we follow the method originally described in the Helffer-Sjöstrand papers. We first establish Proposition 5.1 by showing that the space spanned by the WKB quasimodes is a good approximation of the space spanned by the eigenfunctions associated with the first two eigenvalues. The key step is Lemma 5.3.

Then, we study the interaction term, see Proposition 5.5. Here, the analysis deviates from the usual strategy consisting in representing the interaction term by means of an integral running over an interface between the wells (see [4, Section 4.1] in dimension one). This strategy cannot be used in our pseudo-differential context since it is based on an integration by parts. Instead, we directly replace the one-well groundstates appearing in (5.2) by their WKB expansions and we use the stationary phase theorem. We end up with an integral that can be computed explicitly thanks to the transport equations determining the amplitude of the WKB Ansätze¹.

2. THE ONE-WELL OPERATOR AND WKB CONSTRUCTIONS

This section is devoted to the analysis of the left "one-well" operator $\mathcal{L}_{h,\ell}$, obtained by sealing the well on the right. This operator is defined as follows. We consider

$$\mathcal{L}_{h,\ell} := \mathcal{L}_h + k_\ell(x), \quad (2.1)$$

where k_ℓ is a non-negative smooth function with support in $D(x_r, \eta)$ and such that the function $x \mapsto b_\ell(x, 0) = b(x, 0) + k_\ell(x) = V(x) + k_\ell(x)$ has a unique global minimum at x_ℓ .

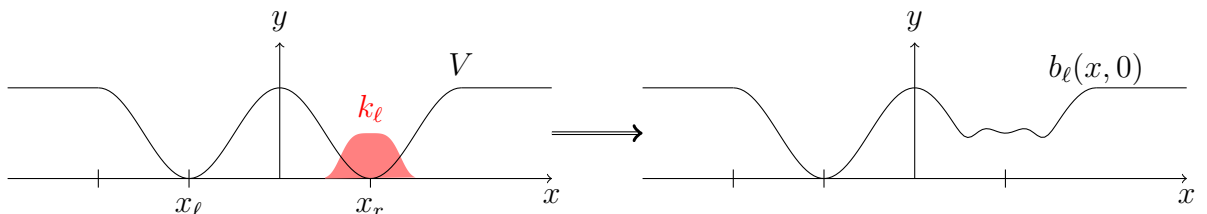


FIGURE 1. Sealing a well

¹Let us mention that it is also possible to follow the approach developed in [17, pp.69-70] and in [12, Section 3.2.3] to bypass the absence of the Green-Riemann formula. However, at least in the one dimensional case, it is more direct to use the transport equation defining the WKB quasimodes.

In a similar way, we define the operator on the right:

$$\mathcal{L}_{h,r} := \mathcal{L}_h + k_\ell(-x).$$

Considering the symmetry operator

$$U : \begin{cases} \psi & \longmapsto \psi(-\cdot) \\ L^2(\mathbb{R}) & \longrightarrow L^2(\mathbb{R}) \end{cases}, \quad (2.2)$$

we observe that $\mathcal{L}_{h,\ell} = U^* \mathcal{L}_{h,r} U$. Thus, by unitary equivalence, all the results that we prove for $\mathcal{L}_{h,\ell}$ will hold for $\mathcal{L}_{h,r}$.

The following lemma, which is proved in Section 2.1, shows that the one-well operators have necessarily discrete spectrum below the threshold $b_\infty h$.

Lemma 2.1. *Let $C \in (0, b_\infty)$. There exists $h_0 > 0$ such that, for all $h \in (0, h_0)$, the spectrum of $\mathcal{L}_{h,\ell}$ below Ch is discrete.*

The non-emptiness of the discrete spectrum follows from WKB constructions. We let

$$\Phi_\ell : x \in \mathbb{R} \longmapsto \sqrt{\frac{2}{a''(0)}} \left| \int_{x_\ell}^x \sqrt{b_\ell(s, 0)} ds \right|, \quad (2.3)$$

which is a $\mathcal{C}^\infty(\mathbb{R})$ function since $s \longmapsto \operatorname{sgn}(s - x_\ell) \sqrt{b_\ell(s, 0)}$ is smooth.

Proposition 2.2 (WKB quasimodes). *Let $n \in \mathbb{N}^*$. There exist*

- (i) *a smooth function $u_n(\cdot, h)$ and a family $(u_{n,j})_{j \in \mathbb{N}}$ of elements of $S(\mathbb{R})$ satisfying (in the sense of asymptotic expansions in $S(\mathbb{R})$ as defined in Appendix A)*

$$u_n(x, h) \underset{h \rightarrow 0}{\sim} \sum_{j \geq 0} u_{n,j}(x) h^{\frac{j}{2}},$$

where $u_{n,0}$ solves the transport equation

$$\left(i\Phi'_\ell(x) \partial_\xi b_\ell(x, 0) + \frac{a''(0)}{2} \Phi''_\ell(x) + a''(0) \Phi'_\ell(x) \partial_x - (2n - 1)c_0 \right) u_{n,0} = 0,$$

with $c_0 = \sqrt{\frac{a''(0) \partial_{x,x}^2 b(x_\ell, 0)}{4}} > 0$;

- (ii) *a real number $\lambda_n^{\text{WKB}}(h)$ and a family of real number $(\lambda_{n,j})_{j \in \mathbb{N}}$ satisfying*

$$\lambda_n^{\text{WKB}}(h) \underset{h \rightarrow 0}{\sim} \sum_{j \geq 3} \lambda_{n,j} h^{\frac{j}{2}}, \quad \text{with} \quad \lambda_{n,3} = (2n - 1)c_0;$$

such that the following holds.

Considering a smooth function χ with compact support² that equals 1 in a neighbourhood of x_ℓ and letting

$$\Psi_{\ell,n}^{\text{WKB}} = h^{-\frac{1}{8}} \chi u_{n,h} e^{-\frac{\Phi_\ell}{\sqrt{h}}}, \quad (2.4)$$

we have

$$\|(\mathcal{L}_{h,\ell} - \lambda_n^{\text{WKB}}(h)) \Psi_{\ell,n}^{\text{WKB}}\| = \mathcal{O}(h^\infty) \|\Psi_{\ell,n}^{\text{WKB}}\|.$$

In particular, we have

$$\lambda_n(\mathcal{L}_{h,\ell}) \leq (2n - 1)c_0 h^{\frac{3}{2}} + o(h^{\frac{3}{2}}).$$

²From now on, the space of such functions will be denoted $\mathcal{C}_0^\infty(\mathbb{R})$

2.1. Proof of Lemma 2.1. For all $z \leq C$, we have

$$\mathcal{L}_{h,\ell} - zh = [a(\xi) + h(b_\ell(x, \xi) - z)]^w.$$

Thanks to the Taylor formula,

$$b_\ell(x, \xi) = b_\ell(x, 0) + \xi R(x, \xi), \quad \text{with} \quad R(x, \xi) = \int_0^1 \partial_\xi b_\ell(x, t\xi) dt,$$

so that

$$a(\xi) + h(b_\ell(x, \xi) - z) = a(\xi) + h\xi R(x, \xi) + h(b_\ell(x, 0) - z).$$

Let us consider a partition of the unity $\chi_1(\xi) + \chi_2(\xi) = 1$ with $\chi_1 = 1$ near $\xi = 0$. Moreover, the support of χ_1 is chosen so that

$$M := \|\xi R(x, \xi)\|_{L^\infty(\text{supp}\chi_1)} + C < b_\infty. \quad (2.5)$$

Then, we write

$$\begin{aligned} a(\xi) + h(b_\ell(x, \xi) - z) &= [a(\xi) + h\xi R(x, \xi) + h(b_\ell(x, 0) - z)]\chi_1(\xi) \\ &\quad + [a(\xi) + h\xi R(x, \xi) + h(b_\ell(x, 0) - z)]\chi_2(\xi). \end{aligned}$$

There exist $\epsilon, h_0 > 0$ such that for all $h \in (0, h_0)$, all $(x, \xi) \in \mathbb{R}^2$,

$$[a(\xi) + h\xi R(x, \xi) + h(b_\ell(x, 0) - z)]\chi_2(\xi) \geq \epsilon\chi_2(\xi),$$

and

$$[a(\xi) + h\xi R(x, \xi) + h(b_\ell(x, 0) - z)]\chi_1(\xi) \geq h(b_\ell(x, 0) - M)\chi_1(\xi).$$

Thanks to (2.5), there exist $\epsilon_0 > 0$ and $W \in \mathcal{C}_0^\infty(\mathbb{R})$ such that

$$b_\ell(x, 0) - M + W(x) \geq \epsilon_0.$$

Therefore,

$$a(\xi) + h(b_\ell(x, \xi) - z) + hW(x)\chi_1(\xi) \geq \min(\epsilon, \epsilon_0 h) = \epsilon_0 h,$$

as soon as h is small enough. With the Fefferman-Phong inequality (see Appendix A), we deduce that, in the sense of quadratic forms and for h small enough,

$$\mathcal{L}_{h,\ell} - hz + h\mathcal{K}_h \geq \frac{\epsilon_0 h}{2}, \quad \mathcal{K}_h = (W(x)\chi_1(\xi))^w.$$

Thus, the selfadjoint operator $\mathcal{L}_{h,\ell} - hz + h\mathcal{K}_h$ is bijective. Since \mathcal{K}_h is compact, it follows that $\mathcal{L}_{h,\ell} - hz$ is Fredholm with index 0. The discreteness of the spectrum below Ch follows from the analytic Fredholm theory.

Remark 2.3. The proof of Lemma 2.1 can easily be adapted to the double well operator \mathcal{L}_h . Therefore, it indeed makes sense to study the spectral gap.

2.2. Proof of Proposition 2.2.

2.2.1. *WKB expansions.* Proposition 2.2 is essentially a consequence of the following, stronger one.

Proposition 2.4 (WKB constructions). *Let $n \in \mathbb{N}^*$. There exist a family of smooth functions $(u_{n,j})_{j \geq 0}$ and a family of real numbers $(\lambda_{n,j})_{j \geq 3}$ such that the following holds. Let $J \geq 1$ and $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ equal to 1 on a segment I containing x_ℓ . Letting*

$$u_{n,h}^{[J]} = \sum_{j=0}^J h^{\frac{j}{2}} u_{n,j}, \quad \lambda_n^{[J]}(h) = \sum_{j=3}^{J+3} \lambda_{n,j} h^{\frac{j}{2}},$$

we have

$$\left\| e^{\frac{\Phi_\ell}{\sqrt{h}}} (\mathcal{L}_{h,\ell} - \lambda_n^{[J]}(h)) (\chi u_{n,h}^{[J]} e^{-\frac{\Phi_\ell}{\sqrt{h}}}) \right\|_{L^\infty(I)} = \mathcal{O}(h^{\frac{J+4}{2}}). \quad (2.6)$$

Explicitely, $\lambda_{n,3} = (2n-1)c_0$ and we can take

$$u_{1,0}(x) = \left(\frac{\Phi_\ell''(x_\ell)}{\pi} \right)^{1/4} \exp \left(- \int_{x_\ell}^x \left(\frac{\Phi_\ell''(s) - \Phi_\ell''(x_\ell)}{2\Phi_\ell'(s)} + \frac{i}{a''(0)} \partial_\xi b(s,0) \right) ds \right). \quad (2.7)$$

Proof. In the following, we drop the reference to n . Setting $\mathcal{L}_{h,\ell}^{\Phi_\ell} := e^{\frac{\Phi_\ell}{\sqrt{h}}} \mathcal{L}_{h,\ell} e^{-\frac{\Phi_\ell}{\sqrt{h}}}$, we want to find functions u_j and numbers λ_j such that

$$(\mathcal{L}_{h,\ell}^{\Phi_\ell} - \lambda^{[J]}(h))(\chi u_h^{[J]}) = \mathcal{O}_{L^\infty(I)}(h^{\frac{J+4}{2}}). \quad (2.8)$$

Thanks to Lemma B.2, the action of $\mathcal{L}_{h,\ell}^{\Phi_\ell}$ on $\mathcal{C}_0^\infty(\mathbb{R})$ is that of a series of differential operators: there exists a family of differential operators $(P_\gamma(x, D_x))_{0 \leq \gamma \leq J+3}$ such that

$$\mathcal{L}_{h,\ell}^{\Phi_\ell}(\chi u_h^{[J]}) = \sum_{0 \leq \gamma \leq J+3} h^{\gamma/2} P_\gamma(x, D_x)(\chi u_h^{[J]}) + \mathcal{O}_{L^\infty(\mathbb{R})}(h^{\frac{J+4}{2}}),$$

where the first few operators are explicitly given by

$$\begin{cases} P_0(x, D_x) = a(0) = 0, \\ P_1(x, D_x) = i\Phi_\ell'(x)a'(0) = 0, \\ P_2(x, D_x) = -\frac{a''(0)}{2}\Phi_\ell'(x)^2 + b_\ell(x,0) = 0, \\ P_3(x, D_x) = i\Phi_\ell'(x)\partial_\xi b_\ell(x,0) + \frac{a''(0)}{2}\Phi_\ell''(x) + a''(0)\Phi_\ell'(x)\partial_x. \end{cases} \quad (2.9)$$

Therefore $(\mathcal{L}_{h,\ell}^{\Phi_\ell} - \lambda^{[J]}(h))(\chi u_h^{[J]}) = \mathcal{O}_{L^\infty(I)}(h^{\frac{J+4}{2}})$ if and only if $\lambda_0 = \lambda_1 = \lambda_2 = 0$ and the (λ_j, u_j) satisfy the transport equations

$$\begin{cases} (P_3(x, D_x) - \lambda_3)u_0 = 0, \\ (P_3(x, D_x) - \lambda_3)u_j = b_j + \lambda_{j+3}u_0, \quad j \geq 1, \end{cases} \quad (2.10)$$

where $b_j := -P_{j+3}(x, D_x)u_0 - \sum_{\gamma=4}^{j+2} (P_\gamma(x, D_x) - \lambda_\gamma)u_{j-\gamma+3}$.

Let us start by solving the first transport equation, which can be written as

$$\left(ir(\lambda_3, x) + \Phi_\ell''(x) \left(\frac{1}{2} - \frac{\lambda_3}{a''(0)\Phi_\ell''(x_\ell)} \right) + \Phi_\ell'(x)\partial_x \right) u_0 = 0, \quad (2.11)$$

where

$$r(\lambda_3, x) = \frac{1}{a''(0)} \left(\Phi_\ell'(x)\partial_\xi b_\ell(x,0) - i\lambda_3 \frac{\Phi_\ell''(x) - \Phi_\ell''(x_\ell)}{\Phi_\ell''(x_\ell)} \right).$$

It satisfies that $x \mapsto \frac{r(\lambda_3, x)}{\Phi'_\ell(x)}$ is in $\mathcal{C}^\infty(\mathbb{R})$. Since Φ'_ℓ vanishes linearly at x_ℓ , for every $\lambda_3 \in \mathbb{R}$, the differential equation $(P_3(x, D_x) - \lambda_3)u_0 = 0$ can be solved on $]x_\ell, +\infty[$ and on $] - \infty, x_\ell[$. In both cases, the space of solutions is spanned by the function

$$x \mapsto \Phi'_\ell(x)^{\frac{\lambda_3}{a''(0)\Phi''_\ell(x_\ell)} - \frac{1}{2}} \exp\left(-i \int_{x_\ell}^x \frac{r(\lambda_3, s)}{\Phi'_\ell(s)} ds\right). \quad (2.12)$$

Therefore, there is a smooth solution on \mathbb{R} solution if and only if there exists $n \in \mathbb{N}_{\geq 1}$ such that

$$\frac{\lambda_3}{a''(0)\Phi''_\ell(x_\ell)} - \frac{1}{2} = n - 1. \quad (2.13)$$

From now on, we take $\lambda_3 = \lambda_{n,3} = (2n - 1)c_0$. With this choice, (2.12) becomes

$$x \mapsto \Phi'_\ell(x)^{n-1} \exp\left(\frac{-i}{a''(0)} \int_{x_\ell}^x \partial_\xi b(s, 0) ds\right) \exp\left(- (2n - 1) \int_{x_\ell}^x \frac{\Phi''_\ell - \Phi''_\ell(x_\ell)}{2\Phi'_\ell} ds\right). \quad (2.14)$$

The other transport equations in (2.10) become

$$(ir(\lambda_{n,3}, x) - (n - 1)\Phi''_\ell(x) + \Phi'_\ell(x)\partial_x) u_{n,j} = b_{n,j} + \lambda_{n,j+3}u_{n,0}. \quad (2.15)$$

Letting $\tilde{u}_{n,j} := u_{n,j} \exp\left(i \int_{x_\ell}^x \frac{r(\lambda_{n,3}, s)}{\Phi'_\ell(s)} ds\right)$ and $\tilde{b}_{n,j} := b_{n,j} \exp\left(i \int_{x_\ell}^x \frac{r(\lambda_{n,3}, s)}{\Phi'_\ell(s)} ds\right)$, (2.15) becomes

$$(-(n - 1)\Phi''_\ell(x) + \Phi'_\ell(x)\partial_x) \tilde{u}_{n,j} = \tilde{b}_{n,j} + \lambda_{n,j+3}\tilde{u}_{n,0}. \quad (2.16)$$

Here again, by the theory of ODEs, the solutions in $(-\infty, x_\ell)$ and $(x_\ell, +\infty)$ form one dimensional vector spaces. To prove that there exists global solutions it suffices to prove that there exists a \mathcal{C}^∞ solution well defined near x_ℓ .

The Taylor formula ensures the existence of $(\alpha_{n,j}^k)_{0 \leq k \leq n-1} \in \mathbb{R}^n$ and of $r_{n,j} \in \mathcal{C}^\infty(\text{Neigh}(x_\ell))$ such that, near x_ℓ ,

$$\tilde{b}_{n,j}(x) = (\alpha_{n,j}^0 + \alpha_{n,j}^1(\Phi'_\ell(x))^2 + \dots + \alpha_{n,j}^{n-1}(\Phi'_\ell(x))^{n-1} + r_{n,j}(x)(\Phi'_\ell(x))^n) \Phi''_\ell(x).$$

Setting $\tilde{v}_{n,j} = \sum_{k=0}^{n-2} \frac{\alpha_{n,j}^k}{k - (n-1)} (\Phi'_\ell)^k$ and choosing $\lambda_{n,j+3} = \alpha_{n,j}^{n-1}$, we get

$$\left(- (n - 1)\Phi''_\ell + \Phi'_\ell \frac{d}{dx}\right) (\tilde{u}_{n,j} - \tilde{v}_{n,j}) = (\Phi'_\ell)^n \tilde{r}_{n,j} \text{ with } \tilde{r}_{n,j} \in \mathcal{C}^\infty(\text{Neigh}(x_\ell)). \quad (2.17)$$

This last equation has smooth solutions of the form $(\Phi'_\ell)^n \tilde{s}_{n,j}$ in a neighbourhood of x_ℓ . This proves that the equations (2.10) have smooth solutions on \mathbb{R} and concludes the proof. \square

2.2.2. *Proof of Proposition 2.2.* For all $n \in \mathbb{N}_{\geq 1}$, the Borel lemma provides us with

- a smooth function $u_n(\cdot, h)$ having the asymptotic expansion $\sum_{j \geq 0} u_{n,j}(x) h^{\frac{j}{2}}$,
- a number $\lambda_n^{\text{WKB}}(h)$ having the asymptotic expansion $\sum_{j \geq 3} \lambda_{n,j} h^{\frac{j}{2}}$,

with the $u_{n,j}$ and $\lambda_{n,j}$ given by Proposition 2.4. Let us recall that $\Psi_{\ell,n}^{\text{WKB}}$ is defined in (2.4). We first deal with $n = 1$,

$$\|\Psi_{\ell,1}^{\text{WKB}}\|^2 = h^{-\frac{1}{4}} \int_{\mathbb{R}} \chi(x)^2 |u_{1,h}(x)|^2 e^{-\frac{2\Phi_\ell(x)}{\sqrt{h}}} dx.$$

We recall that, on the support of χ , we have $u_{1,h} = u_{1,0} + \mathcal{O}(h^{\frac{1}{2}})$ and we apply the Laplace method to get

$$\|\Psi_\ell^{\text{WKB}}\|^2 = \sqrt{\frac{\pi}{\Phi_\ell''(x_\ell)}} |u_{1,0}(x_\ell)|^2 + \mathcal{O}(h^{\frac{1}{2}}) = 1 + \mathcal{O}(h^{\frac{1}{2}}),$$

where we used (2.7) (especially the normalization constant). Thanks to Proposition 2.4, we obtain

$$\|(\mathcal{L}_{h,\ell} - \lambda_1^{\text{WKB}}(h)) \Psi_\ell^{\text{WKB}}\|_{L^2(I)} = \mathcal{O}(h^\infty).$$

Then, we have

$$\begin{aligned} \|(\mathcal{L}_{h,\ell} - \lambda_1^{\text{WKB}}(h)) \Psi_\ell^{\text{WKB}}\|_{L^2(\mathbb{C}I)} &= \left\| e^{-\frac{\Phi_\ell}{\sqrt{h}}} (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \lambda_1^{\text{WKB}}(h)) (h^{-\frac{1}{8}} \chi u_{1,h}) \right\|_{L^2(\mathbb{C}I)} \\ &\leq \left\| e^{-\frac{\Phi_\ell}{\sqrt{h}}} \right\|_{L^\infty(\mathbb{C}I)} \left\| (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \lambda_1^{\text{WKB}}(h)) (h^{-\frac{1}{8}} \chi u_{1,h}) \right\|_{L^2(\mathbb{C}I)} \\ &= \mathcal{O}(h^\infty), \end{aligned}$$

where we used $\min_{\mathbb{C}I} \Phi_\ell > 0$ and Lemma B.2 for the last estimate. By using the spectral theorem, this establishes Proposition 2.2 when $n = 1$.

For $n \geq 2$, the estimate follows in the same way except for the estimate of the L^2 -norm where we use Lemma B.3 (to deal with the fact that $u_{n,0}$ vanishes at x_ℓ), which gives $\mathcal{O}(h^\infty) = \mathcal{O}(h^\infty) \|\Psi_{\ell,n}^{\text{WKB}}\|$.

3. FIRST TUNNELING ESTIMATE

This section is devoted to establishing the following two propositions.

Proposition 3.1 (Microlocal harmonic approximation). *We have, for all $n \geq 1$,*

$$\lambda_n(\mathcal{L}_{h,\ell}) = (2n - 1)c_0 h^{\frac{3}{2}} + o(h^{\frac{3}{2}}).$$

Proposition 3.2 (Rough tunneling estimate). *We have*

$$\lambda_2(\mathcal{L}_h) - \lambda_1(\mathcal{L}_h) = \mathcal{O}(h^\infty), \quad \lambda_1(\mathcal{L}_h) = \lambda_1(\mathcal{L}_{h,\ell}) + \mathcal{O}(h^\infty),$$

and for all $c \in (0, c_0)$, letting h small enough, $\lambda_3(\mathcal{L}_h) - \lambda_2(\mathcal{L}_h) \geq ch^{\frac{3}{2}}$.

3.1. Microlocal harmonic approximation.

3.1.1. Microlocalization.

Lemma 3.3 (Localization of eigenfunctions). *Let $n \in \mathbb{N}_{\geq 1}$ and consider the n -th normalized eigenfunction $\psi_{h,n}$ of $\mathcal{L}_{h,\ell}$. Let $\delta \in (0, \frac{1}{6})$ and $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\rho = 1$ in a neighbourhood of 0. We denote $\rho_\delta : \mathbb{R} \ni s \mapsto \rho(h^{-\delta}s)$. Then, we have*

$$\psi_{h,n} = \rho_\delta(\xi)^w \psi_{h,n} + \mathcal{O}(h^\infty) \psi_{h,n}, \quad \psi_{h,n} = \rho_\delta(x - x_\ell) \psi_{h,n} + \mathcal{O}(h^\infty) \psi_{h,n}.$$

Proof. In order to lighten the notations we will write, only in this proof, $\lambda_n(h) = \lambda_n(\mathcal{L}_{h,\ell})$ and recall that $\lambda_n(h) = \mathcal{O}(h^{\frac{3}{2}})$. We consider the auxiliary function $\tilde{\rho} \in \mathcal{C}_0^\infty(\mathbb{R})$ equal to 1 near 0 and such that $(1 - \rho)\tilde{\rho} = 0$.

First, let us consider $\delta \in (0, \frac{1}{2})$ and prove the microlocalization in ξ . It suffices to find some pseudo-differential operator Q_h solving

$$Q_h(\mathcal{L}_{h,\ell} - \lambda_n(h)) = 1 - \rho_\delta(\xi)^w + \mathcal{O}(h^\infty)$$

and evaluate the previous expression at $\psi_{h,n}$. By Taylor expanding a near 0 at first order, we have $c > 0$ such that

$$a(\xi) + hb(x, \xi) + \tilde{\rho}_\delta(\xi) - \lambda_n(h) \geq ch^{2\delta}.$$

Using then the Fefferman-Phong inequality in $S_\delta(\mathbb{R}^2)$ this proves that $\mathcal{L}_{h,\ell} - \lambda_n(h) + \tilde{\rho}_\delta(\xi)^w$ is bijective (for h small enough). Thanks to the pseudo-differential calculus, we know that $(\mathcal{L}_{h,\ell} - \lambda_n(h) + \tilde{\rho}_\delta(\xi)^w)^{-1}$ is a pseudo-differential operator, of symbol $q_h \in h^{-2\delta}S_{3\delta}$ and of norm $\mathcal{O}(h^{-2\delta})$. We obtain

$$q_h^w(\mathcal{L}_{h,\ell} - \lambda_n(h)) = 1 - q_h^w \tilde{\rho}_\delta(\xi)^w. \quad (3.1)$$

Finally, notice that $(1 - \rho_\delta)\tilde{\rho}_\delta = 0$ and $\|q_h^w\| = \mathcal{O}(h^{-2\delta})$. Multiplying pseudo-differential operators with disjoint supports $(1 - \rho_\delta(\xi)^w)q_h^w \tilde{\rho}_\delta(\xi)^w = \mathcal{O}(h^\infty)$ thus

$$(1 - \rho_\delta(\xi)^w)q_h^w(\mathcal{L}_{h,\ell} - \lambda) = 1 - \rho_\delta(\xi)^w - (1 - \rho_\delta(\xi)^w)q_h^w \tilde{\rho}_\delta(\xi)^w = 1 - \rho_\delta(\xi)^w + \mathcal{O}(h^\infty). \quad (3.2)$$

To conclude it suffices to use $(1 - \rho_\delta(\xi)^w)q_h^w(\mathcal{L}_{h,\ell} - \lambda_n(h))\psi_{h,n} = 0$ and the previous equality.

Let us now turn to the localization in x . We solve once again, letting $\delta \in [0, \frac{1}{4})$,

$$Q_h(\mathcal{L}_{h,\ell} - \lambda_n(h)) = 1 - \rho_\delta(x - x_\ell) + \mathcal{O}(h^\infty).$$

Notice that by Taylor expanding b to the first order and using Young's inequality

$$a(\xi) + h(b(x, \xi) + \tilde{\rho}_\delta(x - x_\ell)) - \lambda_n(h) \gtrsim h^{1+2\delta}.$$

Using the Fefferman-Phong inequality in $S_\delta(\mathbb{R}^2)$, there exists $c > 0$ such that, in terms of quadratic forms,

$$\mathcal{L}_{h,\ell} + h\tilde{\rho}_\delta(x - x_\ell) - \lambda_n(h) \gtrsim h^{1+2\delta} - h^{3-4\delta} \geq ch^{1+2\delta}.$$

This proves that $\mathcal{L}_{h,\ell} + h\tilde{\rho}_\delta(x - x_\ell) - \lambda_n(h)$ is invertible. We then write

$$1 = (\mathcal{L}_{h,\ell} + h\tilde{\rho}_\delta(x - x_\ell) - \lambda_n(h))^{-1}(\mathcal{L}_{h,\ell} - \lambda) + h(\mathcal{L}_{h,\ell} + h\tilde{\rho}_\delta(x - x_\ell) - \lambda_n(h))^{-1}\tilde{\rho}_\delta(x - x_\ell). \quad (3.3)$$

By the pseudo-differential calculus we know that the operator $(\mathcal{L}_{h,\ell} + h\tilde{\rho}_\delta(x - x_\ell) - \lambda_n(h))^{-1}$ is a pseudo-differential operator, we denote q_h its symbol belonging to $h^{-1-2\delta}S_{3\delta}(\mathbb{R}^2)$ and obtain

$$q_h^w(\mathcal{L}_{h,\ell} - \lambda_n(h)) = 1 - hq_h^w \tilde{\rho}_\delta(x - x_\ell). \quad (3.4)$$

Finally, since $(1 - \rho_\delta)\tilde{\rho}_\delta = 0$ and $\|q_h^w\| = \mathcal{O}(h^{-1-2\delta})$, by multiplication of pseudo-differential operators with disjoint supports $(1 - \rho_\delta(x - x_\ell))q_h^w \tilde{\rho}_\delta(x - x_\ell) = \mathcal{O}(h^\infty)$ thus

$$(1 - \rho_\delta(x - x_\ell))q_h^w(\mathcal{L}_{h,\ell} - \lambda_n(h)) = 1 - \rho_\delta(x - x_\ell) + \mathcal{O}(h^\infty). \quad (3.5)$$

□

3.1.2. *Proof of Proposition 3.1.* We only have to prove the lower bound. For that purpose, let us consider an orthonormal family of eigenfunctions $(\psi_j)_{1 \leq j \leq n}$ associated with the eigenvalues $(\lambda_j(\mathcal{L}_{h,\ell}))_{1 \leq j \leq n}$ and set

$$E = \text{span}_{1 \leq j \leq n} \psi_j.$$

We have, for all $\psi \in E$,

$$\langle \mathcal{L}_{h,\ell} \psi, \psi \rangle \leq \lambda_n(\mathcal{L}_{h,\ell}) \|\psi\|^2.$$

Thanks to the Calderón-Vaillancourt theorem and Lemma 3.3, we find that

$$\langle \mathcal{L}_{h,\ell} \psi, \psi \rangle = \langle (a(\xi) + hb_\ell(x, \xi))^w \psi, \psi \rangle = \langle (a(\xi) + hb_\ell(x, \xi))^w \rho_\delta(\xi)^w \psi, \rho_\delta(\xi)^w \psi \rangle + \mathcal{O}(h^\infty) \|\psi\|^2.$$

Then, by using the composition theorems for pseudo-differential operators in the class $S_\delta(\mathbb{R}^2)$ (see, for instance, [32, Theorems 4.17, 4.18 & 4.24]) and support considerations on ρ'_δ , we get

$$\langle \mathcal{L}_{h,\ell} \psi, \psi \rangle = \langle [a(\xi) + hb_\ell(x, \xi)] \rho_\delta^2(\xi)^w \psi, \psi \rangle + \mathcal{O}(h^\infty) \|\psi\|^2.$$

By the Taylor formula, we have

$$hb_\ell(x, \xi) = hb_\ell(x, 0) + h\xi r(x, \xi), \quad r \in S(\mathbb{R}^2),$$

and, by using the Young inequality $2h|\xi r(x, \xi)| \leq h^\delta \xi^2 + h^{2-\delta} |r(x, \xi)|^2$, we get

$$[a(\xi) + hb_\ell(x, \xi)] \rho_\delta^2 \geq \left[\frac{a''(0)}{2} \xi^2 (1 - \tilde{C}h^\delta) + hb_\ell(x, 0) - Ch^{2-\delta} \right] \rho_\delta^2.$$

The Fefferman-Phong inequality in $S_\delta(\mathbb{R})$ (recalled in Appendix A.2) yields

$$\langle \mathcal{L}_{h,\ell} \psi, \psi \rangle \geq Q_h^{\text{elec}}(\rho_\delta^w \psi) - Ch^{2-4\delta} \|\psi\|^2,$$

where

$$\forall \varphi \in H^2(\mathbb{R}), \quad Q_h^{\text{elec}}(\varphi) = \frac{a''(0)}{2} (1 - \tilde{C}h^\delta) \|hD_x \varphi\|^2 + h \int_{\mathbb{R}} b(x, 0) |\varphi|^2 dx.$$

Hence, for all $\psi \in E$,

$$Q_h^{\text{elec}}(\rho_\delta^w \psi) \leq (\lambda_n(\mathcal{L}_{h,\ell}) + Ch^{2-4\delta}) \|\rho_\delta^w \psi\|^2.$$

Due to Lemma 3.3, we have $\dim(\rho_\delta^w E) = n$ and the min-max theorem implies that

$$\lambda_n(Q_h^{\text{elec}}) \leq \lambda_n(\mathcal{L}_{h,\ell}) + Ch^{2-4\delta}.$$

By choosing $\delta \in (0, \frac{1}{8})$ and using the harmonic approximation

$$(2n-1)c_0 h^{3/2} + o(h^{3/2}) \leq \lambda_n(Q_h^{\text{elec}}) \leq \lambda_n(\mathcal{L}_{h,\ell}) + o(h^{3/2}),$$

the lower bound follows.

3.2. Proof of Proposition 3.2.

3.2.1. *Localization.* Let us state a localization result for the eigenfunctions of the double well operator, which follows from the same arguments as in the proof of Lemma 3.3.

Lemma 3.4 (Localization of \mathcal{L}_h). *Let $M > 0$ and $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$ and ρ is equal to 1 in a small neighbourhood of 0. Letting $\delta \in (0, \frac{1}{8})$ we set, for all $x \in \mathbb{R}$,*

$$\underline{\rho}_\delta(x) = \rho(h^{-\delta}(x - x_\ell)) + \rho(h^{-\delta}(x - x_r)).$$

For all $n \geq 1$, we have

$$\underline{\rho}_\delta \psi_{h,n} = \psi_{h,n} + \mathcal{O}(h^\infty). \quad (3.6)$$

Recalling the notation introduced at beginning of Section 2, we consider the tensored operator

$$\mathcal{L}_h^{\text{mod}} = \mathcal{L}_{h,\ell} \oplus \mathcal{L}_{h,r} \text{ acting on } L^2(\mathbb{R}) \oplus L^2(\mathbb{R}).$$

Its low-lying spectrum is made of eigenvalues of multiplicity two:

$$\forall n \in \mathbb{N}_{\geq 1}, \quad \lambda_{2n-1}(\mathcal{L}_h^{\text{mod}}) = \lambda_{2n}(\mathcal{L}_h^{\text{mod}}) = \lambda_n(\mathcal{L}_{h,\ell}). \quad (3.7)$$

3.2.2. *End of the Proof of Proposition 3.2.* Let us consider the spaces

$$F_{h,n} = \text{span}(\psi_{n,\ell}, \psi_{n,r}),$$

which is of dimension two thanks to Lemma 3.3. Then, for all $\psi_h \in F_h$, we have

$$(\mathcal{L}_h - \lambda_n(\mathcal{L}_{h,\ell}))\psi_h = \mathcal{O}(h^\infty)\|\psi_h\|,$$

where we used the $\mathcal{O}(h^\infty)$ -orthogonality of $(\psi_{n,\ell}, \psi_{n,r})$. From the spectral theorem, we deduce that there are at least two eigenvalues of \mathcal{L}_h that are $\mathcal{O}(h^\infty)$ -close to $\lambda_n(\mathcal{L}_{h,\ell})$. Thus,

$$\forall n \in \mathbb{N}_{\geq 1}, \quad \lambda_n(\mathcal{L}_h) \leq \lambda_n(\mathcal{L}_h^{\text{mod}}) + \mathcal{O}(h^\infty). \quad (3.8)$$

For all $\psi \in L^2(\mathbb{R})$, we let

$$Q(\psi) := \langle \mathcal{L}_h \psi, \psi \rangle, \quad Q_\star(\psi) := \langle \mathcal{L}_{h,\star} \psi, \psi \rangle \text{ for } \star = \ell, r.$$

For all $(\psi, \tilde{\psi}) \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, we let

$$Q_\oplus(\psi, \tilde{\psi}) := Q_\ell(\psi) + Q_r(\tilde{\psi}).$$

Let us then consider an orthonormal family of eigenfunctions $(\psi_j)_{1 \leq j \leq n}$ associated with the eigenvalues $(\lambda_j(\mathcal{L}_h))_{1 \leq j \leq n}$ and set

$$E = \text{span}_{1 \leq j \leq n} \psi_j.$$

We have, for all $\psi \in E$,

$$Q(\psi) \leq \lambda_n(\mathcal{L}_h)\|\psi\|^2. \quad (3.9)$$

We consider $\chi_\ell \in \mathcal{C}_0^\infty(\mathbb{R})$ supported outside $(x_r - \eta, x_r + \eta)$ and satisfying $\chi_\ell = 1$ near x_ℓ . This allows to define by symmetry $\chi_r = \chi_\ell(\cdot - x_\ell + x_r)$ and $\chi = \chi_\ell + \chi_r$. Thanks to Lemma 3.3, we have, for all $\psi \in E$,

$$Q(\psi) \geq Q(\chi\psi) + \mathcal{O}(h^\infty)\|\psi\|^2,$$

and also $\chi_\ell \mathcal{L}_h \chi_r \psi = \mathcal{O}(h^\infty)\|\psi\|$. We infer that

$$Q(\psi) \geq Q(\chi\psi) + \mathcal{O}(h^\infty)\|\psi\|^2 \geq Q_\oplus(\chi_\ell \psi, \chi_r \psi) + \mathcal{O}(h^\infty)\|\psi\|^2.$$

Combining this last estimate with (3.9), we get

$$\lambda_n(\mathcal{L}_h) \geq \max_{\psi \in E \setminus \{0\}} \frac{Q_\oplus(\chi_\ell \psi, \chi_r \psi)}{\|\psi\|^2} + \mathcal{O}(h^\infty),$$

so that, again by Lemma 3.3, we have

$$\lambda_n(\mathcal{L}_h) \geq \max_{\psi \in E \setminus \{0\}} \frac{Q_{\oplus}(\chi_{\ell}\psi, \chi_r\psi)}{(1 + \mathcal{O}(h^{\infty}))(\|\chi_{\ell}\psi\|^2 + \|\chi_r\psi\|^2)} + \mathcal{O}(h^{\infty}).$$

By noticing that $E \ni \psi \mapsto (\chi_{\ell}\psi, \chi_r\psi)$ is injective, we deduce that

$$\lambda_n(\mathcal{L}_h) \geq (1 + \mathcal{O}(h^{\infty})) \min_{\substack{F \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \\ \dim F = n}} \max_{(\psi, \tilde{\psi}) \in F \setminus \{0\}} \frac{Q_{\oplus}(\psi, \tilde{\psi})}{\|\psi\|^2 + \|\tilde{\psi}\|^2} + \mathcal{O}(h^{\infty}).$$

By the min-max theorem, it follows that

$$\lambda_n(\mathcal{L}_h) \geq (1 + \mathcal{O}(h^{\infty}))\lambda_n(\mathcal{L}_{h,\text{mod}}) + \mathcal{O}(h^{\infty}).$$

Recalling (3.8), (3.7), and Proposition 3.1, this ends the proof.

4. EXPONENTIAL ESTIMATES OF THE EIGENFUNCTIONS

Exponential decay estimates in the context of Schrödinger operators usually follow from the famous Agmon estimates ([2, 13]). These estimates can be interpreted as elliptic estimates for a conjugated operator (by a suitable exponential).

Let $\varphi \in \mathcal{C}^{\infty}(\mathbb{R})$ such that $\varphi' = \varphi_1 + h^{\frac{1}{4}}\varphi_2$ where $\varphi_1 \in S(\mathbb{R})$, $\varphi_2 \in S_{\frac{1}{4}}(\mathbb{R})$. We will work with the conjugated operator:

$$\mathcal{L}_{h,\ell}^{\varphi} = e^{\frac{\varphi}{\sqrt{h}}} \mathcal{L}_{h,\ell} e^{-\frac{\varphi}{\sqrt{h}}}. \quad (4.1)$$

4.1. A functional inequality.

Proposition 4.1. *Let $C_0, R > 0$ and $\varphi \in \mathcal{C}^{\infty}(\mathbb{R})$ such that $\varphi' = \varphi_1 + h^{\frac{1}{4}}\varphi_2$ where $\varphi_1 \in S(\mathbb{R})$, $\varphi_2 \in S_{\frac{1}{4}}(\mathbb{R})$ satisfy $\varphi_1(x_{\ell}) = \varphi_2(x_{\ell}) = 0$ and*

$$\forall x \in \mathfrak{C}B(x_{\ell}, Rh^{1/4}), \quad b_{\ell}(x, 0) - \frac{a''(0)}{2}\varphi'(x)^2 \geq C_0 h^{\frac{1}{2}}. \quad (4.2)$$

We also assume that there exists $C > 0$ such that

$$\forall x \in B(x_{\ell}, Rh^{\frac{1}{4}}) \quad |\varphi(x)| \leq Ch^{\frac{1}{2}}. \quad (4.3)$$

Then, for all $M < C_0$, there exist $c, h_0 > 0$ such that, for all $h \in (0, h_0)$ and $\lambda < Mh^{\frac{3}{2}}$, for all $v \in L^2(\mathbb{R})$,

$$h^{\frac{3}{2}}\|v\|_{L^2(\mathfrak{C}B(x_{\ell}, Rh^{\frac{1}{4}}))}^2 \leq c\|(\mathcal{L}_{h,\ell}^{\varphi} - \lambda)v\| \|v\| + ch^{\frac{3}{2}}\|v\|_{L^2(B(x_{\ell}, Rh^{\frac{1}{4}}))}^2. \quad (4.4)$$

4.1.1. *An elliptic estimate.* The proof of Proposition 4.1 is mainly a consequence of the following coercivity estimate, which is based on quadratic form manipulations like in the Schrödinger case.

Lemma 4.2. *There exist $h_0, \epsilon, C > 0$ such that, for all $h \in (0, h_0)$, all $v \in L^2(\mathbb{R})$ and all $\lambda \in \mathbb{R}$, we have*

$$\text{Re} \langle (\mathcal{L}_{h,\ell}^{\varphi} - \lambda)v, v \rangle \geq \langle (hb_{\ell}(x, 0) - h\frac{a''(0)}{2}\varphi'(x)^2 - \lambda)v, v \rangle + Ch^{\frac{3}{2}+\epsilon}\|v\|_{L^2(\mathbb{R})}^2.$$

Proof. Thanks to Lemma A.2, we have

$$\mathcal{L}_{h,\ell}^\varphi = p_\varphi^w + \mathcal{O}_{S_{1/4}(\mathbb{R})}(h^2), \quad p_\varphi(x, \xi) = a(\xi + ih^{\frac{1}{2}}\varphi') + hb_\ell(x, \xi + ih^{\frac{1}{2}}\varphi'). \quad (4.5)$$

Since φ' is bounded and by the Taylor formula, we find r_h belonging to $S(\mathbb{R}^2)$ such that

$$\operatorname{Re}(p_\varphi) = a(\xi) + hb_\ell(x, \xi) - h\varphi'(x)^2 \frac{a''(\xi)}{2} + h^2 r_h(x, \xi).$$

We have

$$\operatorname{Re}(p_\varphi) = a(\xi) + hb_\ell(x, 0) - h\varphi'(x)^2 \frac{a''(0)}{2} + \underbrace{h(r_0(\xi)\varphi'(x)^2 + \xi r_1(x, \xi))}_{=R_h} + h^2 r_h(x, \xi), \quad (4.6)$$

with $r_0(\xi) = \frac{a''(\xi)}{2} - \frac{a''(0)}{2}$ and $r_1 \in S(\mathbb{R}^2)$ given by the Taylor Formula. By using that $\varphi' \in L^\infty(\mathbb{R})$, there exists $C > 0$ such that, for all $(x, \xi) \in \mathbb{R}^2$, we have $|R_h(x, \xi)| \leq C\sqrt{a(\xi)}$. Using Young's inequality with $\epsilon \in (0, \frac{1}{6})$, we get

$$Ch\sqrt{a(\xi)} \leq \frac{C}{2}(h^\epsilon a(\xi) + h^{2-\epsilon}),$$

so that

$$hR_h(x, \xi) + h^2 r_h(x, \xi) + Ch^\epsilon (a(\xi) + h^{2-2\epsilon}) \geq 0.$$

The symbol in the left-hand side belongs to $S_{\frac{1}{4},0}(\mathbb{R}^2)$ defined in (A.1). We can factor out h^ϵ and then apply the Fefferman-Phong inequality in the class $S_{\frac{1}{4},0}(\mathbb{R}^2)$ (see Theorem A.1 for details). This gives, in the sense of quadratic forms,

$$[hR_h(x, \xi) + h^2 r_h(x, \xi) + Ch^\epsilon (a(\xi) + h^{2-2\epsilon})]^w \geq -Ch^{\frac{3}{2}+\epsilon}.$$

Combining this with (4.6) and (4.5), we obtain, for all $v \in L^2(\mathbb{R})$,

$$\begin{aligned} \operatorname{Re} \langle (\mathcal{L}_{h,\ell}^\varphi - \lambda)v, v \rangle &= \langle (\operatorname{Re} \mathcal{L}_{h,\ell}^\varphi - \lambda)v, v \rangle \\ &\geq \langle (1 - Ch^\epsilon)a^w v, v \rangle + \langle (hb_\ell(x, 0) - h\frac{a''(0)}{2}\varphi'(x)^2 - \lambda)v, v \rangle - Ch^{\frac{3}{2}+\epsilon}\|v\|^2. \end{aligned}$$

Since a is non-negative, the result follows. \square

4.1.2. *Proof of Proposition 4.1.* Applying Lemma 4.2 and using the condition on λ , we get

$$h \int_{\mathbb{R}} \left(b_\ell(x, 0) - \varphi'(x)^2 \frac{a''(0)}{2} - (M + ch^\epsilon)h^{\frac{1}{2}} \right) |v|^2 dx \leq \operatorname{Re} \langle (\mathcal{L}_{h,\ell}^\varphi - \lambda)v, v \rangle.$$

Then using the $\varphi' = \varphi_1 + h^{\frac{1}{4}}\varphi_2$ where $\varphi_1 \in S(\mathbb{R})$, $\varphi_2 \in S_{\frac{1}{4}}(\mathbb{R})$ satisfy $\varphi_1(x_\ell) = \varphi_2(x_\ell) = 0$, we infer that $|\varphi'|^2 = \mathcal{O}(1)h^{\frac{1}{2}}$ on $\{|x - x_\ell| \leq Rh^{\frac{1}{4}}\}$. This gives

$$\begin{aligned} h \int_{|x-x_\ell| \geq Rh^{\frac{1}{4}}} \left(b_\ell(x, 0) - \varphi'(x)^2 \frac{a''(0)}{2} - (M + ch^\epsilon)h^{\frac{1}{2}} \right) |v|^2 dx \\ \leq \operatorname{Re} \langle (\mathcal{L}_{h,\ell}^\varphi - \lambda)v, v \rangle + ch^{\frac{3}{2}}\|v\|_{L^2(B(x_\ell, Rh^{\frac{1}{4}}))}^2. \end{aligned}$$

Then, (4.2) provides us with

$$h^{\frac{3}{2}} \int_{|x-x_\ell| \geq Rh^{\frac{1}{4}}} (C_0 - (M + ch^\epsilon)) |v|^2 dx \leq \operatorname{Re} \langle (\mathcal{L}_{h,\ell}^\varphi - \lambda)v, v \rangle + ch^{\frac{3}{2}}\|v\|_{L^2(B(x_\ell, Rh^{\frac{1}{4}}))}^2.$$

It remains to use the Cauchy-Schwarz inequality and the conclusion follows.

4.2. Consequences. Let us now analyze the exponential decay of the eigenfunctions of $\mathcal{L}_{h,\ell}$. We recall that Φ_ℓ is given by (2.3)

$$\Phi_\ell : x \in \mathbb{R} \mapsto \sqrt{\frac{2}{a''(0)}} \left| \int_{x_\ell}^x \sqrt{b_\ell(s,0)} ds \right|. \quad (4.7)$$

The behavior at infinity of Φ_ℓ will not be important in the analysis. That is why we consider a bounded version of it, denoted by $\tilde{\Phi}_\ell$. We take $A > 0$ such that, for all $x \notin [-A, A]$, $\Phi_\ell(x) > \Phi_\ell(x_r)$ and define the following.

Lemma 4.3. *There exists a function $\tilde{\Phi}_\ell \in S(\mathbb{R})$ such that:*

- $\tilde{\Phi}_\ell = \Phi_\ell$ on $[-A, A]$,
- $\tilde{\Phi}_\ell$ is constant on $\mathbb{C}[-2A, 2A]$,
- $\tilde{\Phi}_\ell(\pm 2A) < \Phi_\ell(\pm 2A)$,
- for all $x \in \mathbb{R}$, $(x - x_\ell)\tilde{\Phi}'_\ell(x) \geq 0$,
- $|\tilde{\Phi}'_\ell| \leq |\Phi'_\ell|$ thus $\tilde{\Phi}_\ell \leq \Phi_\ell$.

Proof. We consider $\chi_0 \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi_0 = 1$ on $[-A, A]$, $\chi_0 = 0$ on $\mathbb{R} \setminus [-2A, 2A]$ and $0 \leq \chi_0 \leq 1$, and define

$$\forall x \in \mathbb{R}, \quad \tilde{\Phi}_\ell(x) = \int_{x_\ell}^x \chi_0(s) \Phi'_\ell(s) ds.$$

It is then straightforward that $\tilde{\Phi}_\ell$ satisfies the desired requirements. \square

Due to Proposition 3.1, there exist $h_0, M > 0$ such that, for all $h \in (0, h_0)$,

$$\lambda_1(\mathcal{L}_{h,\ell}) \leq Mh^{\frac{3}{2}}. \quad (4.8)$$

In the following, we let $\mu(h) = \lambda_1(\mathcal{L}_{h,\ell})$.

Corollary 4.4 (Agmon estimates). *Let $\varepsilon \in (0, 1)$ and ψ_h a normalized eigenfunction of $\mathcal{L}_{h,\ell}$ associated with the eigenvalue $\mu(h)$. There exist $C, h_0 > 0$ such that, for all $h \in (0, h_0)$,*

$$\|e^{(1-\varepsilon)\tilde{\Phi}_\ell/\sqrt{h}}\psi_h\| \leq C\|\psi_h\|.$$

Proof. We let $\varphi = \sqrt{1 - \varepsilon\tilde{\Phi}_\ell}$. It satisfies the assumptions of Proposition 4.1 with $C_0 > M$ and $\varphi_2 = 0$. Indeed, we have

$$b_\ell(x,0) - \frac{a''(0)}{2}\varphi'(x)^2 \geq b_\ell(x,0) - \frac{a''(0)}{2}(\Phi'_\ell)^2 = \varepsilon b_\ell(x,0),$$

and there remains to use the quadratic behavior of $b_\ell(\cdot, 0)$ near x_ℓ and to choose R large enough to get (4.2).

Then, we consider the eigenvalue equation

$$(\mathcal{L}_{h,\ell} - \mu(h))\psi_h = 0,$$

which is equivalent to

$$(\mathcal{L}_{h,\ell}^\varphi - \mu(h))v = 0, \quad v = e^{\frac{\varphi}{\sqrt{h}}}\psi_h,$$

where we recall the notation (4.1). Applying Proposition 4.1 to $v = e^{\frac{\varphi}{\sqrt{h}}}\psi_h$, we find that

$$\|v\| \leq C \|v\|_{L^2(B(x_\ell, Rh^{\frac{1}{4}}))}.$$

Thus, by using that φ/\sqrt{h} is bounded on $B(x_\ell, Rh^{\frac{1}{4}})$, we get some constant $C > 0$ such that

$$\|e^{\varphi/\sqrt{h}}\psi_h\| \leq C \|\psi_h\|.$$

□

Let us now turn to the WKB approximation of ψ_h . More precisely, we want to have an exponentially sharp approximation of ψ_h on the interval $K = [-A, x_r - \eta]$, for $\eta > 0$ small enough. This will follow from Proposition 4.1. We consider $\underline{\chi}_\ell \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\underline{\chi}_\ell \geq 0$, $\underline{\chi}_\ell = 1$ on $[-2A, 2A]$ and $\underline{\chi}_\ell = 0$ on $(-\infty, -3A) \cup (3A, +\infty)$.

We recall that our WKB quasimode is $\Psi_\ell^{\text{WKB}} := h^{-\frac{1}{8}} \underline{\chi}_\ell e^{-\frac{\Phi_\ell}{\sqrt{h}}} u_{1,h}$, see Proposition 2.2.

Corollary 4.5 (WKB approximation). *Let $\Pi_{h,\ell}$ the projection on the groundstate of $\mathcal{L}_{h,\ell}$. Then, we have*

$$\left\| e^{\frac{\Phi_\ell}{\sqrt{h}}} (\Psi_\ell^{\text{WKB}} - \Pi_{h,\ell} \Psi_\ell^{\text{WKB}}) \right\|_{L^2(K)} = \mathcal{O}(h^\infty).$$

Proof. The idea to use Proposition 4.1 with a weight φ that is a refined version of $\sqrt{1 - \epsilon \tilde{\Phi}_\ell}$ close to x_ℓ . For that purpose, for $R > 0$, $N \in \mathbb{N}$, we let

$$\varphi_h(x) := \tilde{\Phi}_\ell(x) - Nh^{\frac{1}{2}} \int_{x_\ell}^x \rho_{R,h}(s) \frac{\tilde{\Phi}'_\ell(s)}{\tilde{\Phi}_\ell(s)} ds,$$

where the function $\rho_{R,h}$ is given by

$$\rho_{R,h}(s) := \rho \left(\frac{s - x_\ell}{Rh^{\frac{1}{4}}} \right),$$

with $\rho \in \mathcal{C}_0^\infty(\mathbb{R})$ satisfying $\rho \equiv 1$ on $\mathbb{C}B(0, 1)$, $\rho \geq 0$ and $\text{supp } \rho \in \mathbb{C}B(0, \frac{1}{2})$. We notice that

$$\varphi'_h = \tilde{\Phi}'_\ell - h^{\frac{1}{4}} N \rho_{R,h} \frac{h^{\frac{1}{4}} \tilde{\Phi}'_\ell}{\tilde{\Phi}_\ell},$$

where $\tilde{\Phi}'_\ell \in S(\mathbb{R})$ and $N \rho_{R,h} \frac{h^{\frac{1}{4}} \tilde{\Phi}'_\ell}{\tilde{\Phi}_\ell} \in S_{\frac{1}{4}}(\mathbb{R})$.

Let us establish (4.2) with $C_0 > M$ (M being given in (4.8)). Away from a small neighbourhood of x_ℓ denoted by $\text{Neigh}(x_\ell)$, (independent of h), we have $\min_{\mathbb{C}\text{Neigh}(x_\ell)} \tilde{\Phi}_\ell > 0$ and this implies, for all $x \in \mathbb{C}\text{Neigh}(x_\ell)$,

$$(\varphi'_h(x))^2 \leq \tilde{\Phi}_\ell'^2(x) \left(1 - h^{\frac{1}{2}} \frac{N}{\min_{\mathbb{C}\text{Neigh}(x_\ell)} \tilde{\Phi}_\ell} \right)^2 = \frac{2b_\ell(x, 0)}{a''(0)} \left(1 - 2h^{\frac{1}{2}} \frac{N}{\min_{\mathbb{C}\text{Neigh}(x_\ell)} \tilde{\Phi}_\ell} + \mathcal{O}(h) \right). \quad (4.9)$$

Thus (4.2) is satisfied outside $\text{Neigh}(x_\ell)$. Then, we choose $R = R(N)$ so that $\Phi_\ell(x_\ell \pm \frac{1}{2}Rh^{1/4}) \geq Nh^{\frac{1}{2}}$. Thus $\frac{Nh^{\frac{1}{2}}}{\Phi_\ell} \leq 1$ on $\{\rho_R = 1\}$. Therefore, for all $x \in \mathbb{C}B(x_\ell, Rh^{1/4}) \cap \text{Neigh}(x_\ell)$,

$$\begin{aligned} b_\ell(x, 0) - \frac{a''(0)}{2} \varphi'_h(x)^2 &\geq \frac{a''(0)}{2} \Phi_\ell'^2 \left(1 - \left(1 - \frac{Nh^{\frac{1}{2}}}{\Phi_\ell} \right)^2 \right) \\ &\geq Nh^{\frac{1}{2}} \frac{a''(0)}{2} c \left(2 - \frac{Nh^{\frac{1}{2}}}{\Phi_\ell} \right) \\ &\geq Nh^{\frac{1}{2}} \frac{a''(0)}{2} c, \end{aligned}$$

where we used that $\frac{(\Phi'_\ell)^2}{\Phi_\ell}$ is continuous near x_ℓ and bounded from below by a constant $c > 0$. Taking N large enough the condition (4.2) is satisfied with $C_0 > M$.

When getting further from x_ℓ , Φ_ℓ increases, thus we have for h small enough and $x \in B(x_\ell, Rh^{\frac{1}{4}})$

$$-Nh^{\frac{1}{2}} \ln 4 \sim -Nh^{\frac{1}{2}} \ln \left(\frac{\max \Phi_\ell(\pm Rh^{\frac{1}{4}})}{\min \Phi_\ell(\pm \frac{1}{2}Rh^{\frac{1}{4}})} \right) \leq \varphi_h(x) \leq \Phi_\ell(x) \leq \max \Phi_\ell(\pm Rh^{\frac{1}{4}}) = \mathcal{O}(h^{\frac{1}{2}}).$$

This proves that the condition (4.3) is fulfilled.

Let us consider $v = e^{\frac{\varphi_h}{\sqrt{h}}} (\Psi_\ell^{\text{WKB}} - \Pi_h \Psi_\ell^{\text{WKB}})$, which satisfies

$$(\mathcal{L}_{h,\ell}^{\varphi_h} - \mu(h))v = e^{(\varphi_h - \Phi_\ell)/\sqrt{h}} (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))u_{1,h}.$$

Recalling that $\underline{\chi}_\ell = 1$ on $[-2A, 2A]$, by Proposition 2.4, the Borel lemma and using $\varphi_h \leq \Phi_\ell$ on $[-2A, 2A]$, we get

$$\|(\mathcal{L}_{h,\ell}^{\varphi_h} - \mu(h))v\|_{L^2(-2A, 2A)} = \mathcal{O}(h^\infty). \quad (4.10)$$

Moreover, by using Lemma 4.3, especially because

$$\forall s \in \mathbb{C}(-2A, 2A), \quad \tilde{\Phi}_\ell(s) - \Phi_\ell(s) < \max_{\pm} \left(\tilde{\Phi}_\ell(\pm 2A) - \Phi_\ell(\pm 2A) \right) < 0,$$

we obtain

$$\begin{aligned} \|(\mathcal{L}_{h,\ell}^{\varphi_h} - \mu(h))v\|_{L^2(\mathbb{C}(-2A, 2A))} &= \|e^{\frac{\varphi_h - \Phi_\ell}{\sqrt{h}}} (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h)) (h^{-\frac{1}{8}} \chi_\ell u_{1,h})\|_{L^2(\mathbb{C}(-2A, 2A))} \\ &\leq |e^{\frac{\tilde{\Phi}_\ell - \Phi_\ell}{\sqrt{h}}}|_{L^\infty(\mathbb{C}(-2A, 2A))} \|\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h)\|_{\mathcal{L}(L^2(\mathbb{R}))} \|h^{-\frac{1}{8}} \chi_\ell u_{1,h}\| \\ &= \mathcal{O}(h^\infty). \end{aligned}$$

This gives

$$\|(\mathcal{L}_{h,\ell}^{\varphi_h} - \lambda_1(\mathcal{L}_{h,\ell}))v\| = \mathcal{O}(h^\infty).$$

We apply Proposition 4.1 to get

$$\|v\| \leq \tilde{C} \|v\|_{L^2(B(x_\ell, Rh^{1/4}))} + \mathcal{O}(h^\infty). \quad (4.11)$$

Moreover, by the spectral theorem and then Proposition 2.2, we get

$$\begin{aligned} \|\Psi_\ell^{WKB} - \Pi_h \Psi_\ell^{WKB}\| &\leq (\lambda_2(\mathcal{L}_{h,\ell}) - \lambda_1(\mathcal{L}_{h,\ell}))^{-1} \|(\mathcal{L}_{h,\ell} - \lambda_1(\mathcal{L}_{h,\ell}))\Psi_\ell^{WKB}\| \\ &= (\lambda_2(\mathcal{L}_{h,\ell}) - \lambda_1(\mathcal{L}_{h,\ell}))^{-1} \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty). \end{aligned}$$

By using (4.11) and (4.3), we deduce that

$$\begin{aligned} \|e^{\frac{\varphi_h}{\sqrt{h}}}(\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB})\| &\leq \tilde{C} \|e^{\frac{\varphi_h}{\sqrt{h}}}(\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB})\|_{L^2(B(x_\ell, Rh^{1/4}))} + \mathcal{O}(h^\infty) \\ &\leq \tilde{C} e^C \|\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB}\|_{L^2(B(x_\ell, Rh^{1/4}))} + \mathcal{O}(h^\infty) \\ &= \mathcal{O}(h^\infty). \end{aligned}$$

Then, we find that

$$\begin{aligned} \mathcal{O}(h^\infty) &= \|e^{\frac{\varphi_h}{\sqrt{h}}}(\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB})\| \geq \|e^{\frac{\varphi_h}{\sqrt{h}}}(\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB})\|_{L^2(K)} \\ &\geq \inf_K e^{\frac{\varphi_h - \Phi_\ell}{\sqrt{h}}} \|e^{\frac{\Phi_\ell}{\sqrt{h}}}(\Psi_\ell^{WKB} - \Pi_{h,\ell}\Psi_\ell^{WKB})\|_{L^2(K)}. \end{aligned}$$

There remains to notice that

$$e^{\frac{\varphi_h - \Phi_\ell}{\sqrt{h}}} \geq \left| \frac{|\Phi_\ell|_{L^\infty(K)}}{\Phi_\ell(\pm Rh^{1/4})} \right|^{-N} \underset{h \rightarrow 0}{\sim} \left| \frac{2|\Phi_\ell|_{L^\infty(K)}}{\Phi_\ell''(x_\ell)R^2} \right|^{-N} h^{\frac{N}{2}},$$

which is absorbed by the $\mathcal{O}(h^\infty)$ in the left-hand-side. This concludes the proof. \square

5. THE INTERACTION TERM

In this section, we prove Theorem 1.4. We recall that U is given in (2.2). We consider $\chi_\ell \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi_\ell \geq 0$, $\chi_\ell = 1$ on $[-A, x_r - 2\eta]$ and $\chi_\ell = 0$ on $(-\infty, -2A) \cup (x_\ell - \eta, +\infty)$. We let $\chi_r = U\chi_\ell$, $\psi_{h,r} = U\psi_{h,\ell}$. The function $\psi_{h,r}$ is a groundstate of $\mathcal{L}_{h,r}$. We also set

$$f_{h,\ell} = \chi_\ell \psi_{h,\ell}, \quad f_{h,r} = U f_{h,\ell}.$$

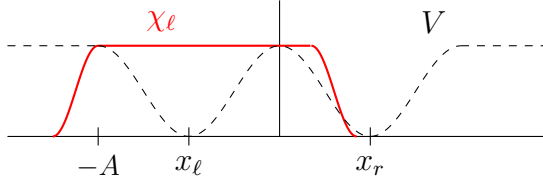


FIGURE 2. The function χ_ℓ

Recalling Proposition 2.2 we also denote $u_{h,\ell} = u_{1,h}$ and $u_{h,r} = U u_{h,\ell}$. Our WKB Ansatz are, with $\underline{\chi}_\ell$ defined in the previous section (and $\underline{\chi}_r = \underline{\chi}_\ell(\cdot)$)

$$\Psi_\ell^{WKB} = h^{-\frac{1}{8}} \underline{\chi}_\ell e^{-\frac{\Phi_\ell}{\sqrt{h}}} u_{h,\ell}, \quad \Psi_r^{WKB} = U \Psi_\ell^{WKB} = h^{-\frac{1}{8}} \underline{\chi}_r e^{-\frac{\Phi_r}{\sqrt{h}}} u_{h,r}.$$

Since $\|\Psi_\ell^{WKB}\| = 1 + \mathcal{O}(h^{\frac{1}{2}})$, we can write

$$\Pi_{h,\ell} \Psi_\ell^{WKB} = c(h) \psi_{h,\ell} \text{ with } |c(h)| = 1 + \mathcal{O}(h^{\frac{1}{2}}), \quad (5.1)$$

and, by symmetry, $\Pi_{h,r} \Psi_r^{WKB} = c(h) \psi_{h,r}$. Let us also denote by $\psi_{h,1}, \psi_{h,2}$ an orthonormalized pair of eigenfunctions of \mathcal{L}_h associated with its first two eigenvalues. We also consider the orthogonal projection $\Pi_h : L^2(\mathbb{R}) \rightarrow \text{span}(\psi_{h,1}, \psi_{h,2})$.

5.1. The interaction formula. The aim of this section is to establish the following proposition.

Proposition 5.1. *We have*

$$\lambda_2(\mathcal{L}_h) - \lambda_1(\mathcal{L}_h) = 2|w_h| + \tilde{\mathcal{O}}(e^{-\frac{2S}{\sqrt{h}}}), \quad (5.2)$$

with $\mu(h) = \lambda_1(\mathcal{L}_{h,\ell})$, $w_h = \langle (\mathcal{L}_h - \mu(h))f_{h,\ell}, f_{h,r} \rangle$ and $S = \int_{x_\ell}^{x_r} \sqrt{b(s,0)} ds$.

We recall that $r_h = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$ means that for every $\gamma > 0$, we have $r_h = \mathcal{O}(e^{-\frac{S-\gamma}{\sqrt{h}}})$ up to adjusting η .

5.1.1. *The space $\text{span}(f_{h,\ell}, f_{h,r})$ is a good approximation of $\text{span}(\psi_{h,1}, \psi_{h,2})$. The groundstates of $\mathcal{L}_{h,\ell}$ and $\mathcal{L}_{h,r}$ are good quasimodes for the double well operator \mathcal{L}_h , with remainder of order $\tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$.*

Lemma 5.2. *For $\star = r, \ell$, we have*

$$(\mathcal{L}_h - \lambda_1(\mathcal{L}_{h,\star}))\psi_{h,\star} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}).$$

Proof. By symmetry, it suffices to prove for $\star = \ell$. We recall that

$$\mathcal{L}_h\psi_{h,\ell} = \lambda_1(\mathcal{L}_{h,\ell})\psi_{h,\ell} - k_\ell\psi_{h,\ell},$$

where k_ℓ has support in $D(x_r, \eta)$.

Thanks to Corollary 4.4, we have

$$\|k_\ell\psi_{h,\ell}\| \leq |k_\ell|_{L^\infty} e^{-\frac{(1-\epsilon)(S+\mathcal{O}(\eta))}{\sqrt{h}}} \left\| e^{(1-\epsilon)\frac{\tilde{\Phi}_\ell}{\sqrt{h}}}\psi_{h,\ell} \right\| \leq C e^{-\frac{(1-\epsilon)(S+\mathcal{O}(\eta))}{\sqrt{h}}} \|\psi_{h,\ell}\|,$$

where we used that $\|\Phi_\ell(x) - S\|_{L^\infty(x_r-\eta, x_r+\eta)} = \mathcal{O}(\eta)$. Up to choosing $\epsilon, \eta > 0$ small enough, we have the desired estimate. \square

In fact, the same estimate holds if we insert the cutoff functions χ_\star .

Lemma 5.3. *For $\star = r, \ell$, we have*

$$(\mathcal{L}_h - \lambda_1(\mathcal{L}_{h,\ell}))f_{h,\star} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}), \quad (\text{Id} - \Pi_h)f_{h,\star} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}).$$

Moreover, we have $\langle f_{h,\ell}, f_{r,h} \rangle = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$ and $\lambda_j(\mathcal{L}_h) = \mu(h) + \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$

Proof. Using again Corollary 4.4,

$$\|(1 - \chi_\ell)\psi_{h,\ell}\| \leq \|e^{-(1-\epsilon)\frac{\tilde{\Phi}_\ell}{\sqrt{h}}}(1 - \chi_\ell)e^{(1-\epsilon)\frac{\tilde{\Phi}_\ell}{\sqrt{h}}}\psi_{h,\ell}\| = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}).$$

Combining this with the boundedness of \mathcal{L}_h , we get

$$(\mathcal{L}_h - \lambda_1(\mathcal{L}_{h,\ell}))(\psi_{h,\ell} - f_{h,\ell}) = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}).$$

Using Lemma 5.2, this gives $(\mathcal{L}_h - \lambda_1(\mathcal{L}_{h,\ell}))f_{h,\star} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$ when $\star = \ell$. The case $\star = r$ follows by symmetry.

Corollary 4.4 shows that $\langle f_{h,\ell}, f_{h,r} \rangle = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}})$ (mainly because $\tilde{\Phi}_\ell + \tilde{\Phi}_r = S$ on $[x_\ell, x_r]$). Therefore, the spectral theorem and Proposition 3.2 show that, for $j = 1, 2$,

$$\lambda_j(\mathcal{L}_h) = \mu(h) + \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}),$$

and also

$$(\text{Id} - \Pi_h)f_{h,\star} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}).$$

□

5.1.2. *Proof of Proposition 5.1.* We let

$$g_\star = \Pi_h f_{h,\star},$$

and

$$G = \begin{pmatrix} \langle g_\ell, g_\ell \rangle & \langle g_\ell, g_r \rangle \\ \langle g_r, g_\ell \rangle & \langle g_r, g_r \rangle \end{pmatrix} = \begin{pmatrix} g_\ell \\ g_r \end{pmatrix} \cdot (g_\ell \ g_r) \geq 0.$$

We consider the matrix of the quadratic form associated with \mathcal{L}_h in the basis (g_ℓ, g_r) ,

$$L = \begin{pmatrix} \langle \mathcal{L}_h g_\ell, g_\ell \rangle & \langle \mathcal{L}_h g_\ell, g_r \rangle \\ \langle \mathcal{L}_h g_r, g_\ell \rangle & \langle \mathcal{L}_h g_r, g_r \rangle \end{pmatrix}.$$

The following proposition is a classical consequence of Lemma 5.3, cf. the concise presentation in [8, Section 4.1]

Proposition 5.4. *The family (g_ℓ, g_r) is asymptotically an orthonormal basis of F , in the sense that*

$$G = \text{Id} + \mathbf{T} + \tilde{\mathcal{O}}(e^{-2S/\sqrt{h}}), \quad \mathbf{T} = \begin{pmatrix} 0 & \langle f_{h,\ell}, f_{h,r} \rangle \\ \langle f_{h,r}, f_{h,\ell} \rangle & 0 \end{pmatrix} = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}). \quad (5.3)$$

In addition, we have

$$L = \begin{pmatrix} \mu(h) & w_h \\ \bar{w}_h & \mu(h) \end{pmatrix} + \mu(h)\mathbf{T} + \tilde{\mathcal{O}}(e^{-2S/\sqrt{h}}), \quad w_h = \langle (\mathcal{L}_h - \mu(h))f_{h,\ell}, f_{h,r} \rangle = \tilde{\mathcal{O}}(e^{-\frac{S}{\sqrt{h}}}). \quad (5.4)$$

Let us consider the new family

$$\begin{pmatrix} \mathfrak{g}_\ell \\ \mathfrak{g}_r \end{pmatrix} = G^{-\frac{1}{2}} \begin{pmatrix} g_\ell \\ g_r \end{pmatrix},$$

which is orthonormal since

$$\begin{pmatrix} \mathfrak{g}_\ell \\ \mathfrak{g}_r \end{pmatrix} \cdot (g_\ell \ g_r) = G^{-\frac{1}{2}} \begin{pmatrix} g_\ell \\ g_r \end{pmatrix} \cdot (g_\ell \ g_r) G^{-\frac{1}{2}} = \text{Id}.$$

The matrix of \mathcal{L}_h in the orthonormal basis $(\mathfrak{g}_\ell, \mathfrak{g}_r)$ is $G^{-\frac{1}{2}}LG^{-\frac{1}{2}}$ and we have

$$\begin{aligned} G^{-\frac{1}{2}}LG^{-\frac{1}{2}} &= \left(1 - \frac{\mathbf{T}}{2}\right) \left(\begin{pmatrix} \mu(h) & w_h \\ \bar{w}_h & \mu(h) \end{pmatrix} + \mu(h)\mathbf{T} \right) \left(1 - \frac{\mathbf{T}}{2}\right) + \tilde{\mathcal{O}}(e^{-2S/\sqrt{h}}) \\ &= \begin{pmatrix} \mu(h) & w_h \\ \bar{w}_h & \mu(h) \end{pmatrix} + \tilde{\mathcal{O}}(e^{-2S/\sqrt{h}}). \end{aligned}$$

The splitting of $\begin{pmatrix} \mu(h) & w_h \\ \bar{w}_h & \mu(h) \end{pmatrix}$ is $2|w_h|$ thus by the min-max theorem for hermitian matrix this proves Proposition 5.1.

5.2. Estimate of the interaction. Theorem 1.4 is a consequence of Proposition 5.1 and the following proposition.

Proposition 5.5. *We have*

$$w_h = (1 + o_{h \rightarrow 0}(1)) 2 \left(\frac{a''(0)}{2} \right)^{\frac{1}{4}} h^{\frac{5}{4}} \sqrt{V(0)} \left(\frac{\kappa}{\pi} \right)^{\frac{1}{2}} \exp \left(- \int_{x_\ell}^0 \frac{\partial_s \sqrt{V(s)} - \kappa}{\sqrt{V(s)}} ds \right) e^{-\frac{S}{\sqrt{h}}}.$$

Proof. From Proposition 5.1 it suffices to compute w_h . We start with the following lemma.

Lemma 5.6. *We have*

$$w_h := \langle (\mathcal{L}_{h,\ell} - \mu) \chi_\ell \psi_{h,\ell}, \chi_r \psi_{h,r} \rangle = e^{-\frac{S}{\sqrt{h}}} \langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r} \rangle + \mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}}).$$

Proof. The main obstacle comes from the fact that $\Phi_\ell(x) + \Phi_r(x) = S$ is not satisfied everywhere between x_ℓ and x_r .

Let us define

$$\Psi_\ell(x) = \text{sgn}(x - x_\ell) \int_{x_\ell}^x \sqrt{b(s, 0)} ds$$

and notice that for $x \in [0, x_r]$, $\Psi_\ell(x) + \Phi_r(x) = \int_{x_\ell}^{x_r} \sqrt{b(s, 0)} ds = S$. We write

$$(\mathcal{L}_{h,\ell} - \mu) \chi_\ell \psi_{h,\ell} = e^{-\frac{\Psi_\ell}{\sqrt{h}}} e^{\frac{\Phi_\ell - \Psi_\ell}{\sqrt{h}}} (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}.$$

and estimate

$$\begin{aligned} & |w_h - e^{-\frac{S}{\sqrt{h}}} \langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r} \rangle| \\ &= |e^{-\frac{S}{\sqrt{h}}} \langle (e^{\frac{\Phi_\ell - \Psi_\ell}{\sqrt{h}}} - 1) (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r} \rangle|, \end{aligned} \quad (5.5)$$

using $\underline{\chi}_\ell \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying $\underline{\chi}_\ell \chi_\ell = \chi_\ell$ (in particular $\underline{\chi}'_\ell \chi_\ell = 0$) and $\underline{\chi}_\ell k_\ell = 0$. Therefore we obtain by the pseudodifferential calculus

$$\|[\mathcal{L}_{h,\ell}^{\Phi_\ell}, \underline{\chi}_\ell] \chi_\ell e^{\frac{\Phi_\ell}{\sqrt{h}}} \psi_{h,\ell}\| = \mathcal{O}(h^\infty) \|\chi_\ell e^{\frac{\Phi_\ell}{\sqrt{h}}} \psi_{h,\ell}\|.$$

Then also using $|\chi_r (e^{\frac{\Phi_\ell - \Psi_\ell}{\sqrt{h}}} - 1)| \leq 1$ et $\chi_r \underline{\chi}_\ell (e^{\frac{\Phi_\ell - \Psi_\ell}{\sqrt{h}}} - 1) = 0$ we deduce,

$$\begin{aligned} & |w_h - e^{-\frac{S}{\sqrt{h}}} \langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r} \rangle| \\ &= |e^{-\frac{S}{\sqrt{h}}} \langle (e^{\frac{\Phi_\ell - \Psi_\ell}{\sqrt{h}}} - 1) \underline{\chi}_\ell (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu) e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r} \rangle| + \mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}}) \| \|e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}\| \|e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r}\| \\ &= \mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}}) \|e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}\| \|e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r}\|. \end{aligned} \quad (5.6)$$

Note then that, for instance thanks to Corollary 4.5, $\|e^{\frac{\Phi_\ell}{\sqrt{h}}} \chi_\ell \psi_{h,\ell}\|$ and $\|e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \psi_{h,r}\|$ are of order $\mathcal{O}(h^r)$ for some $r \in \mathbb{Q}$. We infer that the remainder in (5.6) is $\mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}})$. \square

Support considerations and Corollary 4.5 give

$$\| \chi_\star e^{\frac{\Phi_\star}{\sqrt{h}}} (c(h) \psi_{h,\star} - \Psi_\star^{\text{WKB}}) \| = \mathcal{O}(h^\infty),$$

where $c(h)$ is defined in (5.1). Hence,

$$\begin{aligned} \langle (\mathcal{L}_h - \mu(h))\chi_\ell \psi_{h,\ell}, \chi_r \psi_{h,r} \rangle &= |c(h)|^{-2} e^{-\frac{S}{\sqrt{h}}} \langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))\chi_\ell e^{\frac{\Phi_\ell}{\sqrt{h}}} \Psi_\ell^{\text{WKB}}, e^{\frac{\Phi_r}{\sqrt{h}}} \chi_r \Psi_r^{\text{WKB}} \rangle + \mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}}) \\ &= (1 + o(1)) h^{-\frac{1}{4}} e^{-\frac{S}{\sqrt{h}}} \langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))\chi_\ell u_{\ell,h}, \chi_r u_{r,h} \rangle + \mathcal{O}(h^\infty e^{-\frac{S}{\sqrt{h}}}). \end{aligned} \quad (5.7)$$

Thanks to Lemma B.2 (or recalling the proof of Proposition 2.4),

$$\langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))\chi_\ell u_{\ell,h}, \chi_r u_{r,h} \rangle = h^{\frac{3}{2}} \langle (P_3(x, D_x) - c_0)(\chi_\ell u_\ell), \chi_r u_r \rangle + o(h^{\frac{3}{2}} e^{-\frac{S}{\sqrt{h}}}).$$

Since, by construction, $(P_3(x, D_x) - c_0)u_\ell = 0$ (see (2.10)), we get

$$\langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))\chi_\ell u_{\ell,h}, \chi_r u_{r,h} \rangle = h^{\frac{3}{2}} \langle a''(0)\Phi'_\ell \chi'_\ell u_\ell, \chi_r u_r \rangle + o(h^{\frac{3}{2}} e^{-\frac{S}{\sqrt{h}}}).$$

Noticing that $\chi'_\ell \chi_r = \mathbf{1}_{[x_r-2\eta, x_r-\eta]} \cdot \chi'_\ell$, we get

$$\langle (\mathcal{L}_{h,\ell}^{\Phi_\ell} - \mu(h))\chi_\ell u_{\ell,h}, \chi_r u_{r,h} \rangle = a''(0) h^{\frac{3}{2}} e^{-\frac{S}{\sqrt{h}}} \int_{x_r-2\eta}^{x_r-\eta} \chi'_\ell(x) \Phi'_\ell(x) u_\ell(x) \bar{u}_r(x) dx + o(h^{\frac{3}{2}}) e^{-\frac{S}{\sqrt{h}}}.$$

By definition of χ_ℓ , we have $\int_{x_r-2\eta}^{x_r-\eta} \chi'_\ell(x) dx = 1$. In fact, the function $\Phi'_\ell u_\ell \bar{u}_r$ is constant and equal to $\Phi'_\ell(0) |u_\ell(0)|^2$. Indeed, by (2.10) $b_\ell(x, \xi) = b(x, \xi) + k_\ell(x)$ we obtain $\partial_\xi b_\ell = \partial_\xi b$ and

$$\begin{aligned} \left(i \frac{\Phi'_\ell(x) \partial_\xi b(x, 0)}{a''(0)} + \frac{1}{2} \Phi''_\ell(x) + \Phi'_\ell(x) \partial_x - \lambda_3 \right) u_\ell &= 0, \\ \left(i \frac{\Phi'_r(x) \partial_\xi b(x, 0)}{a''(0)} + \frac{1}{2} \Phi''_r(x) + \Phi'_r(x) \partial_x - \lambda_3 \right) u_r &= 0. \end{aligned} \quad (5.8)$$

Using then the fact that $\Phi_r + \Phi_\ell$ is constant on $(x_\ell + \eta, x_r - \eta)$ gives, denoting $v(x) = \frac{\Phi'_\ell(x) \partial_\xi b(x, 0)}{a''(0)}$,

$$\Phi'_\ell(x) \partial_x u_\ell = (-iv(x) - \frac{1}{2} \Phi''_\ell(x) + \lambda_3) u_\ell \text{ and } \Phi'_\ell(x) \partial_x \bar{u}_r = (iv(x) - \frac{1}{2} \Phi''_\ell(x) - \lambda_3) \bar{u}_r.$$

which implies that $\partial_x (\Phi'_\ell u_\ell \bar{u}_r) = 0$. There remains to use the explicit formula (2.7) and to recall (5.7) to end the proof. \square

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APPENDIX A. PSEUDO-DIFFERENTIAL TOOLS

A.1. Notation. Let us recall some usual notation (see, for instance, [32, Chapter 4]).

For $\delta_1, \delta_2 \in (0, 1)$, we consider

$$S_{\delta_1, \delta_2}(\mathbb{R}^2) := \{q_h \in \mathcal{C}^\infty(\mathbb{R}^2); \forall (\gamma_1, \gamma_2) \in \mathbb{N}^2, \exists C_\gamma > 0, |\partial_x^{\gamma_1} \partial_\xi^{\gamma_2} q_h| \leq C_\gamma h^{-\delta_1 \gamma_1} h^{-\delta_2 \gamma_2}\}. \quad (\text{A.1})$$

We let $S_\delta(\mathbb{R}^2) := S_{\delta, \delta}(\mathbb{R}^2)$ and $S(\mathbb{R}^2) = S_0(\mathbb{R}^2)$.

— We say that a symbol $q_h \in S_\delta(\mathbb{R}^2)$ has asymptotic expansion $q_0 + h^{\frac{1}{2}}q_1 + hq_2 + \dots$ with $q_j \in S_\delta(\mathbb{R}^2)$ for each $j \in \mathbb{N}$, if

$$\forall \gamma \in \mathbb{N}^2, \forall m \in \mathbb{N}, \exists C_{\gamma,m} \in \mathbb{R}_+, \left| \partial^\gamma \left(q_h - \sum_{j=0}^{m-1} h^{\frac{j}{2}} q_j \right) \right| \leq C_{\gamma,m} h^{\frac{m}{2}} h^{-|\gamma|\delta}. \quad (\text{A.2})$$

— We say that $(u_h)_{h \in (0, h_0)}$ a family of elements of³ $S_\delta(\mathbb{R})$, has asymptotic expansion $u_0 + h^{\frac{1}{2}}u_1 + hu_2 + h^{\frac{3}{2}}u_3 + \dots$ with each $u_j \in S_\delta(\mathbb{R})$ if

$$\forall \gamma \in \mathbb{N}, \forall m \in \mathbb{N}, \exists C_{\gamma,m} \in \mathbb{R}_+, \left| \partial^\gamma \left(u_h - \sum_{j=0}^{m-1} h^{\frac{j}{2}} u_j \right) \right| \leq C_{\gamma,m} h^{\frac{m}{2}} h^{-\gamma\delta}. \quad (\text{A.3})$$

— We say that $(\lambda_h)_{h \in (0, h_0)} \in \mathbb{R}^{(0, h_0)}$ has asymptotic expansion $\lambda_0 + h^{\frac{1}{2}}\lambda_1 + h\lambda_2 + h^{\frac{3}{2}}\lambda_3$ if

$$\forall m \in \mathbb{N}, \exists C_m \in \mathbb{R}_+, \left| \lambda_h - \sum_{j=0}^{m-1} h^{\frac{j}{2}} a_j \right| \leq C_m h^{\frac{m}{2}}. \quad (\text{A.4})$$

A.2. Fefferman-Phong inequality and Kuranishi trick. We recall some results of [6] concerning the Fefferman-Phong inequality. The first estimate concerns the class of symbols $S(\mathbb{R}^2)$. Letting $q_h \in S(\mathbb{R}^2)$, there exists $C > 0$ such that

$$\forall u \in L^2(\mathbb{R}), \quad \langle \text{Op}_h^w(q_h)u, u \rangle \geq -Ch^2 \|u\|^2. \quad (\text{A.5})$$

Theorem A.1 (Fefferman-Phong inequality). *Let $\delta_1, \delta_2 \in (0, 1)$ such that $\delta_1 + \delta_2 < 1$. If $q_h \in S_{\delta_1, \delta_2}(\mathbb{R})$ is non-negative, then*

$$\forall u \in L^2(\mathbb{R}) \quad \langle q_h^w u, u \rangle \geq -Ch^{2-2(\delta_1+\delta_2)} \|u\|^2. \quad (\text{A.6})$$

Proof. Let $\alpha \in \mathbb{R}$ and consider the isometric scaling:

$$V_\alpha : \begin{cases} L^2(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}) \\ u & \longmapsto & h^{-\alpha} u(h^{-\alpha} \cdot) \end{cases} \quad (\text{A.7})$$

Notice that $V_\alpha^{-1} = V_\alpha^* = V_{-\alpha}$ thus

$$V_{-\delta_1} \text{Op}_h^w(q_h(x, \xi)) V_{\delta_1} = \text{Op}_h^w(q_h(h^{\delta_1}x, h^{-\delta_1}\xi)) = \text{Op}_{h^{1-(\delta_1+\delta_2)}}(q_h(h^{\delta_1}x, h^{\delta_2}\xi)), \quad (\text{A.8})$$

with $q_h(h^{\delta_1} \cdot, h^{\delta_2} \cdot) \in S(\mathbb{R}^2) = S_{0,0}(\mathbb{R}^2)$. The Fefferman-Phong inequality in $S(\mathbb{R}^2)$ (with the new semiclassical parameter $h^{1-\delta_1-\delta_2}$) yields the result. \square

In the following Lemma, the function φ satisfies the assumptions made at the beginning of Section 4. Its proof is an adaptation of [24, Section 3].

Lemma A.2 (First Kuranishi trick). *The operator $\mathcal{L}_{h,\ell}^\varphi$ defined in (4.1) is a pseudo-differential operator of symbol*

$$q_\varphi(x, \xi) = a(\xi + ih^{\frac{1}{2}}\varphi') + hb_\ell(x, \xi + ih^{\frac{1}{2}}\varphi') + \mathcal{O}_{S_{1/4}(\mathbb{R})}(h^2). \quad (\text{A.9})$$

³ $S_\delta(\mathbb{R}) = \{u_h \in \mathcal{C}^\infty(\mathbb{R}) \mid \forall \gamma \in \mathbb{N}, \exists C_\gamma > 0 \mid \partial^\gamma u_h \leq C_\gamma h^{-\delta\gamma}\}$

Proof. For all $u \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} \mathcal{L}_{h,\ell}^\varphi u &:= e^{\varphi/h^{1/2}} \mathcal{L}_{h,\ell} e^{-\varphi/h^{1/2}} u = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} p\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}[(x-y)\xi + h^{1/2}(\varphi(x) - \varphi(y))]} u(y) dy d\xi \\ &= \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R} - ih^{1/2}\theta(x,y)} p\left(\frac{x+y}{2}, \xi + ih^{1/2}\theta(x,y)\right) e^{\frac{i}{h}(x-y)\xi} u(y) dy d\xi, \end{aligned}$$

with $\theta : (x, y) \mapsto \begin{cases} \frac{\varphi(x) - \varphi(y)}{x-y} & \text{if } x \neq y, \\ \varphi'(x) & \text{if } x = y. \end{cases}$. Note that $\theta(x, y) = \int_0^1 \varphi'(y + t(x-y)) dt$,

which proves that $\theta \in S_{\frac{1}{4}}(\mathbb{R}^2)$ and is bounded by $\|\varphi'\|_{L^\infty}$. Thanks to Cauchy's theorem and Riemann-Lebesgue's lemma, we have, for all $u \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} &\int_{\mathbb{R}} \int_{[-R,R]} p\left(\frac{x+y}{2}, \xi + ih^{1/2}\theta(x,y)\right) e^{\frac{i}{h}(x-y)\xi} u(y) d\xi dy \\ &- \int_{\mathbb{R}} \int_{[-R,R] - ih^{1/2}\theta(x,y)} p\left(\frac{x+y}{2}, \xi + ih^{1/2}\theta(x,y)\right) e^{\frac{i}{h}(x-y)\xi} u(y) d\xi dy \xrightarrow{R \rightarrow +\infty} 0. \end{aligned}$$

This eventually proves that, in the sense of oscillatory integrals, we have

$$\mathcal{L}_{h,\ell}^\varphi u = \frac{1}{2\pi h} \iint_{\mathbb{R}^2} p\left(\frac{x+y}{2}, \xi + ih^{\frac{1}{2}}\theta(x,y)\right) e^{\frac{i}{h}(x-y)\xi} u(y) dy d\xi. \quad (\text{A.10})$$

This equality extends to $L^2(\mathbb{R})$ by continuity (by the standard theory). Let us now use this formula to prove that $\mathcal{L}_{h,\ell}^\varphi$ is a pseudo-differential operator. To do so, let us use the so-called Kuranishi trick. The Taylor formula gives

$$\begin{aligned} \varphi(x) &= \varphi\left(\frac{x+y}{2}\right) + \left(\frac{x-y}{2}\right) \varphi'\left(\frac{x+y}{2}\right) + \frac{\left(\frac{x-y}{2}\right)^2}{2!} \varphi^{(2)}\left(\frac{x+y}{2}\right) \\ &\quad + \left(\frac{x-y}{2}\right)^3 \int_0^1 \frac{(1-t)^2}{2!} \varphi^{(3)}\left(\frac{x+y}{2} + t\frac{x-y}{2}\right) dt, \end{aligned}$$

and a similar expression for $\varphi(y)$. Hence, for all $n \in \mathbb{N}$, there is a smooth function $I(x, y)$ such that

$$\theta(x, y) = \varphi'\left(\frac{x+y}{2}\right) + (x-y)^2 I(x, y), \quad (\text{A.11})$$

where $I(x, y)$ is expressed in terms of $\varphi^{(3)}$ only. Since the map $\mathbb{R}^2 \ni (x, y) \mapsto \frac{\varphi(x) - \varphi(y)}{x-y}$ is bounded and φ' as well, we obtain that the map $\mathbb{R}^2 \ni (x, y) \mapsto (x-y)^2 I(x, y)$ is bounded. Therefore by the Taylor expansion at order 1 in h ,

$$p\left(\frac{x+y}{2}, \xi + ih^{\frac{1}{2}}\theta(x,y)\right) = p\left(\frac{x+y}{2}, \xi + ih^{\frac{1}{2}}\varphi'\left(\frac{x+y}{2}\right)\right) + h^{\frac{1}{2}}(x-y)^2 r_h(x, y, \xi)$$

where $r_h(x, y, \xi) \in S_{\frac{1}{4}, \frac{1}{4}, 0}(\mathbb{R}^3)$ (this space being the natural adaptation to \mathbb{R}^3 of Definition A.1). Explicitly,

$$r_h(x, y, \xi) = iI(x, y) \int_0^1 \partial_\xi p\left(\frac{x+y}{2}, \xi + ih^{\frac{1}{2}}\varphi'\left(\frac{x+y}{2}\right) + it h^{\frac{1}{2}}(x-y)^2 I(x, y)\right) dt.$$

For all $u \in \mathcal{S}(\mathbb{R})$, by integration by parts,

$$\begin{aligned} R_h u &= \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{\frac{i}{h}(x-y)\xi} (x-y)^2 r_h(x, y, \xi) u(y) d\xi dy \\ &= \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{\frac{i}{h}(x-y)\xi} (-hD_\xi)^2 r_h(x, y, \xi) u(y) d\xi dy, \end{aligned}$$

where we used that $(x-y)e^{\frac{i}{h}(x-y)\xi} = hD_\xi e^{\frac{i}{h}(x-y)\xi}$. Thanks to the transformation formula [32, Theorems 4.20 & 4.21], there exists $\tilde{r}_h \in S_{\frac{1}{4}}(\mathbb{R}^2)$ such that $R_h = h^2 \text{Op}_h^w(\tilde{r}_h)$. This yields $q_\varphi \in S_{\frac{1}{4}}(\mathbb{R}^2)$ such that

$$\mathcal{L}_{h,\ell}^\varphi = \text{Op}_h^w(q_\varphi(x, \xi)) \text{ with } q_\varphi(x, \xi) = p(x, \xi + ih^{\frac{1}{2}}\varphi') + \mathcal{O}_{S_{\frac{1}{4}}(\mathbb{R}^2)}(h^2). \quad (\text{A.12})$$

□

Lemma A.3 (Second Kuranishi trick). *The operator $\mathcal{L}_{h,\ell}^{\Phi_\ell}$ defined in (4.1) is a pseudo-differential operator of symbol*

$$q_{\Phi_\ell}(x, \xi) = a(\xi + ih^{\frac{1}{2}}\Phi'_\ell) + hb_\ell(x, \xi + ih^{\frac{1}{2}}\Phi'_\ell) + h^2 r_h \quad (\text{A.13})$$

where $r_h \in S(\mathbb{R}^2)$ has a full asymptotic expansion in $S(\mathbb{R}^2)$, $r_h = r_0 + h^{\frac{1}{2}}r_1 + hr_2 + \dots$ with r_0, r_1, \dots independent of h .

The proof of this lemma is an adaptation of the proof of the previous lemma and left to the reader.

APPENDIX B. STATIONARY PHASE AND WKB CONSTRUCTIONS

Lemma B.1. *Let $p_h \in S(\mathbb{R}^2)$ having asymptotic expansion $p_0 + h^{\frac{1}{2}}p_1 + \dots$ and $u_h \in \mathcal{C}_0^\infty(\mathbb{R})$ (independent of h) having asymptotic expansion $u_0 + h^{\frac{1}{2}}u_1 + \dots$ with each $u_k \in \mathcal{C}_0^\infty(\mathbb{R})$. The function $\text{Op}_h^w(p_h)u_h$ belongs to $\mathcal{C}^\infty(\mathbb{R})$ and has an asymptotic expansion in powers of h in $S(\mathbb{R})$ given by:*

$$\text{Op}_h^w(p_h)u_h \sim \sum_{k \in \mathbb{N}} \frac{(hD_y)^k}{k!} (\partial_\xi^k p_h(x + y/2, 0)u_h(x + y)) \Big|_{y=0}. \quad (\text{B.1})$$

Proof. It suffices to prove the lemma when $p_h(x, \xi) = p(x, \xi)$ and $u_h = u$ are independent of h . Note that

$$\begin{aligned} \text{Op}_h^w(p)u &= \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{i(x-y)\xi/h} p\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi \\ &= \frac{1}{2\pi h} \iint_{\mathbb{R}^2} e^{-iy\xi/h} p\left(\frac{2x+y}{2}, \xi\right) u(x+y) dy d\xi. \end{aligned}$$

Thanks to the stationary phase results [32, Theorems 3.17 & 4.17], we get

$$\text{Op}_h^w(p)u \sim \sum_{k \in \mathbb{N}} \frac{(hD_y)^k}{k!} (\partial_\xi^k p(x + y/2, 0)u(y)) \Big|_{y=0}. \quad (\text{B.2})$$

□

Lemma B.2. *There exist differential operators $(P_\gamma(x, D_x))_{0 \leq \gamma \leq J}$ such that,*

$$\mathcal{L}_{h,\ell}^{\Phi_\ell}(\chi u_h^{[J]}) = \sum_{0 \leq \gamma \leq J+3} h^{\gamma/2} P_\gamma(x, D_x)(\chi u_h^{[J]}) + \mathcal{O}_{L^\infty(\mathbb{R})}(h^{\frac{J+4}{2}}).$$

The first differential operators are

$$\begin{cases} P_0(x, D_x) = a(0) = 0, \\ P_1(x, D_x) = i\Phi'_\ell(x)a'(0) = 0, \\ P_2(x, D_x) = -\frac{a''(0)}{2}\Phi'_\ell(x)^2 + b_\ell(x, 0) = 0, \\ P_3(x, D_x) = i\Phi'_\ell(x)\partial_\xi b_\ell(x, 0) + \frac{a''(0)}{2}\Phi''_\ell(x) + a''(0)\Phi'_\ell(x)\partial_x. \end{cases} \quad (\text{B.3})$$

Proof. Thanks to the holomorphy assumptions on a and b , we can use Lemma A.3. Then, applying Lemma B.1, we get

$$\begin{aligned} \mathcal{L}_{h,\ell}^{\Phi_\ell}(\chi_\ell u_h^{[J]}) &= \sum_{j=0}^{\frac{J+3}{2}} \text{Op}_h^w \left(\frac{\xi^j}{j!} \partial_\xi^j q_\Phi(x, 0) \right) (\chi_\ell u_h^{[J]}) + \mathcal{O}_{L^\infty(\mathbb{R})}(h^{\frac{J+4}{2}}) \\ &= \sum_{\gamma=0}^{J+3} h^{\gamma/2} P_\gamma(x, D_x) u_h^{[J]} + \mathcal{O}_{L^\infty(\mathbb{R})}(h^{\frac{J+4}{2}}). \end{aligned}$$

The operators $P_\gamma(x, D_x)$ are differential operators of degree lower than $\gamma/2$ obtained using the expansion of $q_\Phi(x, 0)$ in powers of $h^{\frac{1}{2}}$. Explicitely, for the first few terms ($J = 0$), we can write the Taylor expansion of the symbol in $\xi + ih^{\frac{1}{2}}\Phi'_\ell$ and we get

$$\mathcal{L}_{h,\ell}^{\Phi_\ell}(\chi u_h) = \left(\frac{a''(0)}{2} (hD_x + ih^{\frac{1}{2}}\Phi'_\ell(x))^2 + hb_\ell(x, 0) + ih^{\frac{3}{2}}\partial_\xi b_\ell(x, 0)\Phi'_\ell(x) \right) \chi u_h + \mathcal{O}_{L^\infty(\mathbb{R})}(h^2). \quad \square$$

The following last lemma proves that the norm of our normalized WKB quasimodes is not $\mathcal{O}(h^\infty)$. We consider a symbol u_h with asymptotic expansion $u_0 + h^{\frac{1}{2}}u_1 + \dots$, a cutoff $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\chi = 1$ near x_ℓ .

Lemma B.3 (WKB and Laplace Integral). *Let us suppose that there exists $m \in \mathbb{N}$ such that $u_0^{(m)}(x_\ell) \neq 0$ then $\|\chi u_h e^{-\Phi_\ell/\sqrt{h}}\|_{L^2(\mathbb{R})} \geq ch^{\frac{1}{8} + \frac{m}{4}}$.*

Proof. Near x_ℓ the application $\phi_\ell := \text{sgn}(x - x_\ell)\sqrt{\Phi_\ell}$ is a \mathcal{C}^∞ diffeomorphism satisfying $\phi_\ell^2 = \Phi_\ell$ and we have the approximation $\phi_\ell(x) = \sqrt{\frac{\Phi'_\ell(x_\ell)}{2}}(x - x_\ell) + o(x - x_\ell)$. Therefore, modulo a term of order $\mathcal{O}(e^{-\frac{\epsilon}{\sqrt{h}}})$ for some $\epsilon > 0$ we can write

$$\begin{aligned} \|\chi u_h e^{-\Phi_\ell/\sqrt{h}}\|^2 &= \int_{\text{Neigh}(0)} e^{-\frac{2y^2}{\sqrt{h}}} |u_h \circ \phi_\ell^{-1}(y)|^2 (\phi_\ell^{-1})'(y) dy + \mathcal{O}(e^{-\frac{\epsilon}{\sqrt{h}}}) \\ &= \frac{h^{\frac{1}{4}}}{\sqrt{2}} \int_{h^{-\frac{1}{4}}\text{Neigh}(0)} e^{-y^2} |u_h \circ \phi_\ell^{-1}(h^{\frac{1}{4}}y/\sqrt{2})|^2 (\phi_\ell^{-1})'(h^{\frac{1}{4}}y/\sqrt{2}) dy + \mathcal{O}(e^{-\frac{\epsilon}{\sqrt{h}}}). \end{aligned} \quad (\text{B.4})$$

A Taylor expansion gives for some $r \in \llbracket 0, m \rrbracket$,

$$u_h \circ \phi_\ell^{-1} \left(\frac{h^{\frac{1}{4}}y}{\sqrt{2}} \right) = h^{\frac{r}{4}} \left(\sum_{j=0}^{N-1} h^{\frac{j}{4}} P_j(y) + \mathcal{O}(h^{\frac{N}{4}}) \right), \quad (\text{B.5})$$

where each P_j is a polynomial of degree at most $j + r$ and P_0 is non-zero because of the assumption $u_0^{(m)}(x_\ell) \neq 0$. Therefore, we find other polynomials Q_j such that:

$$|u_h \circ \phi_\ell^{-1}(h^{\frac{1}{4}}y/\sqrt{2})|^2 (\phi_\ell^{-1})'(h^{\frac{1}{4}}y/\sqrt{2}) = h^{\frac{r}{2}} \left(\sum_{j=0}^{N-1} h^{\frac{j}{4}} Q_j + \mathcal{O}(h^{\frac{N}{4}}) \right), \quad (\text{B.6})$$

with $Q_0 = \left(\frac{\Phi_\ell''(x_\ell)}{2} \right)^{-\frac{1}{2}} P_0^2$. This yields

$$\|\chi u_h e^{-\Phi_\ell/\sqrt{h}}\|^2 = h^{\frac{1}{4}} h^{\frac{r}{2}} (\Phi_\ell''(x_\ell))^{-\frac{1}{2}} \int_{h^{-\frac{1}{4}} \text{Neigh}(0)} e^{-y^2} P_0(y)^2 dy + \mathcal{O}(h^{\frac{r+1}{2}}),$$

which concludes the proof. \square

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