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Abel-type series and definite integrals involving special functions

Robert Reynolds

ABSTRACT. This is a precis of power series evaluations in terms of integrals over finite intervals. The definite integrals and summands involve special functions. The power series involve an Abel-type summand also used by Dobiński in combinatorial identities. The definite integrals involve the application of the Lagrange theorem to Brychkov and other integrals.

1. Introduction

In this work we look at an extension of Abel's series using the Lagrange inversion theorem given by equation (18) in [1] and the method in section (7) in [2]. Series in terms of integrals over finite intervals are present in current literature in [3], [4], [5] and [DLMF,10.22] to name a few with many other works available in other books and online. In this work we use Abel's theorem to derive a double series in terms of an integral over a finite interval, where the summand is similar to the summand involving series for combinatorial identities like in [1]. These type of combinatorial identities are used in minimal spanning trees as in [6] and vector-multiplicative coalescent processes [12], with many more applications in works available online. Integrals over a finite interval are used in evaluating Bessel functions and its sub-functions like cylinder functions and Struve functions

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[DLMF,10.22.26]. This work involves integrals over finite intervals where the integrand is the product of special functions [DLMF,10.22.35]. Similar work using the Lagrange theorem involving integrals were conducted by Bateman [7], Sack [8], Conway [9], Heragy et al. [10] and Russell [11] page 168. These previous works focussed mainly indefinite integrals while this current work focusses on definite integrals. Our main goal is to expand on the concepts of definite integrals and series to tabulate new results involving double series in terms of integrals over a finite interval. The layout of this work is simple where we derive a theorem with special cases in terms of examples that follow. We have provided references during the evaluations to assist readers in referencing evaluations.

2. Preliminaries

Some of the functions used are Pochhammer symbol, [DLMF,5.3(iii)], the exponential integral function, [DLMF,6.11.2], the n^{th} harmonic number [DLMF, 25.16.5], gamma function, [DLMF, 5.2.1], Hurwitz-Lerch transcendent, [DLMF,25.14], generalized hypergeometric function [DLMF,16.2.i], modified Bessel functions, [DLMF,10.25], Kelvin function, [DLMF,10.64], q-binomial coefficient, [DLMF,26.9.E4], q-Pochhammer symbol, [Wolfram Research (2008),1], the sine integral function $\text{Si}(z)$, [DLMF,6.2(ii)], the error function $\text{erf}(z)$, [DLMF,7.2(i)], the imaginary error function $\text{erf}(iz)/i$, [DLMF,7.2(i)].

3. Table of results

In the proceeding work, series and integrals are derived and restrictions are given.

THEOREM 3.1. *Extended Abel theorem. The reciprocal pochhammer symbol series representation.*

$$(3.1) \quad \frac{e^{ax}x}{(m-x)(1-bx)(1-m+x)_m}$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(1+n)}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$ where $f(x) = \frac{x e^{x(a+bn)}}{(x-m)(-m+x+1)_m}$

EXAMPLE 3.2.

$$(3.2) \quad \frac{e^{ax} (m + amx - (a+b+abm)x^2 + abx^3 + x(-m+x)(-1+bx)(-H_x + H_{-m+x}))}{(m-x)^2(-1+bx)^2(1-m+x)_m}$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n (-1+bx) \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{x \Gamma(m) \Gamma(n)}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

EXAMPLE 3.3.

$$(3.3) \quad e^{bpx} x^{-p} \int_0^x \frac{e^{(a-bp)t} t^p}{(t-m)(1-m+t)_m} dt$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m+1} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m) n! (n+p)}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.4.

$$(3.4) \quad \int_0^x \frac{e^{at+bp(-t+x)} t^p x^{-p} (b(t-x) + \log(\frac{x}{t}))}{(-m+t)(1-m+t)_m} dt$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m+1} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(n+p)^2 n! \Gamma(m)}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.5.

$$(3.5) \quad \int_0^x \frac{(\frac{t}{x})^p \Gamma(m) \sinh(at) \sinh(bp(t-x))}{(t-m)(1-m+t)_m} dt$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{e^{-j(a+bn)+i(j+m)\pi} \binom{-1+m}{-1+j}}{4(n+p)n!} \left(-e^{2aj} (a-bn)^{1+n} (e^{bx} x)^n E_{-n}(j(a-bn)) \right)$$

$$\begin{aligned}
& +(-a-bn)^{1+n} (e^{bx}x)^n E_{-n}(-j(a+bn)) + e^{2bjn} (e^{-bx}x)^n (e^{2aj}(a+bn)^{1+n} E_{-n}(j(a+bn)) \\
& \quad -(-a+bn)^{1+n} E_{-n}(-aj+bjn))
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.6.

$$\begin{aligned}
(3.6) \quad & \int_0^x \frac{e^{bp(-t+x)} t^p x^{-p} \cosh(at)}{(-m+t)(1-m+t)_m} dt \\
& = \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{e^{-aj+bjn+i(j+m)\pi} (e^{-bx}x)^n \binom{-1+m}{-1+j}}{2(n+p)n! \Gamma(m)} (-e^{2aj}(a+bn)^{1+n} E_{-n}(j(a+bn)) \\
& \quad -(-a+bn)^{1+n} E_{-n}(-aj+bjn))
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.7.

$$\begin{aligned}
(3.7) \quad & \int_0^x \frac{t^p x^{-p} \cosh(at) \cosh(bp(t-x))}{(-m+t)(1-m+t)_m} dt \\
& = - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{e^{-aj-bjn+i(j+m)\pi} \binom{-1+m}{-1+j}}{4(n+p)n! \Gamma(m)} \left(e^{2aj}(a-bn)^{1+n} (e^{bx}x)^n E_{-n}(j(a-bn)) \right. \\
& \quad +(-a-bn)^{1+n} (e^{bx}x)^n E_{-n}(-j(a+bn)) + e^{2bjn} (e^{-bx}x)^n (e^{2aj}(a+bn)^{1+n} E_{-n}(j(a+bn)) \\
& \quad \left. +(-a+bn)^{1+n} E_{-n}(-aj+bjn)) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.8.

$$\begin{aligned}
(3.8) \quad & \int_0^x \frac{t^p x^{-p} \cosh(at) \sinh(bp(t-x))}{(-m+t)(1-m+t)_m} dt \\
& = \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{e^{-aj-bjn+i(j+m)\pi} \binom{-1+m}{-1+j}}{4(n+p)n! \Gamma(m)} \left(-e^{2aj}(a-bn)^{1+n} (e^{bx}x)^n E_{-n}(j(a-bn)) \right. \\
& \quad -(-a-bn)^{1+n} (e^{bx}x)^n E_{-n}(-j(a+bn)) + e^{2bjn} (e^{-bx}x)^n (e^{2aj}(a+bn)^{1+n} E_{-n}(j(a+bn)) \\
& \quad \left. +(-a+bn)^{1+n} E_{-n}(-aj+bjn)) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, 0 < Re(p) < 1, |Re(x)| < 1$.

EXAMPLE 3.9.

$$(3.9) \quad \int_0^x \frac{e^{(a-b)t+bt+bx} t}{(m-t)(e^{bx}t + e^{bt}x)(1-m+t)_m} dt$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) \Phi(-1, 1, 1+n)}{n! \Gamma(m)}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

EXAMPLE 3.10.

$$(3.10) \quad \int_0^x \frac{e^{(a-b)t+bt-\frac{bx}{2}} \left(e^{\frac{bx}{2}} \sqrt{t} - e^{\frac{bt}{2}} \sqrt{x} \tan^{-1} \left(\frac{e^{-\frac{1}{2}(bt)+\frac{bx}{2}} \sqrt{t}}{\sqrt{x}} \right) \right)}{\sqrt{t}(-m+t)(1-m+t)_m} dt$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j}}{2(-1+2n)n! \Gamma(m)}$$

$$\times E_{-n}(j(a+bn))(-4+\pi+2\Phi(-1, 1, 1+n))$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

EXAMPLE 3.11.

$$(3.11) \quad \int_0^x \frac{e^{(a-b)t+bx} t {}_2F_1 \left(\alpha, \beta; \gamma; \frac{e^{-bt+bx}t}{x} \right)}{(-m+t)x(1-m+t)_m} dt$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)}$$

$$\times {}_3F_2(1+n, \alpha, \beta; 2+n, \gamma; 1)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1$.

EXAMPLE 3.12.

$$(3.12) \quad \int_0^x \frac{e^{(a-b)t+bx} t {}_2F_2 \left(\alpha, \beta; \gamma, \delta; \frac{e^{-bt+bx}t}{x} \right)}{(-m+t)x(1-m+t)_m} dt$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)}$$

$$\times {}_3F_3(1+n, \alpha, \beta; 2+n, \gamma, \delta; 1)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1, |Re(\delta)| < 1$.

EXAMPLE 3.13.

$$(3.13) \quad \int_0^x \frac{e^{(a-b)t+bx} {}_2F_2\left(\alpha, \beta; \gamma, \delta; \frac{e^{-bt+bx}t}{x}\right)}{(-m+t)x(1-m+t)_m} dt$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)}$$

$$\times {}_3F_3(1+n, \alpha, \beta; 2+n, \gamma, \delta; 1)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1, |Re(\delta)| < 1$.

EXAMPLE 3.14.

$$(3.14) \quad \int_0^x \frac{e^{(a-b)t+bx} {}_1F_2\left(\alpha; \beta, \gamma; \frac{e^{-bt+bx}t}{x}\right)}{(-m+t)x(1-m+t)_m} dt$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)}$$

$$\times {}_2F_3(1+n, \alpha; 2+n, \beta, \gamma; 1)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1$.

EXAMPLE 3.15.

$$(3.15) \quad \int_0^x \frac{e^{at-bt+bx} {}_1F_1\left(\alpha; \beta; \frac{e^{b(-t+x)}t}{x}\right)}{(-m+t)x(1-m+t)_m} dt$$

$$= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)}$$

$$\times {}_2F_2(1+n, \alpha; 2+n, \beta; 1)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1$.

EXAMPLE 3.16.

$$\begin{aligned}
(3.16) \quad & \int_0^x \frac{e^{at-bt+bx} \left(\sqrt{\pi} - e^{\frac{e^{b(t-x)}x}{2t}} \sqrt{\frac{e^{b(t-x)}x}{t}} K_{\frac{1}{2}+l} \left(\frac{e^{b(t-x)}x}{2t} \right) \right) (-1-m+t)!}{\sqrt{\pi}x(-1+t)!} dt \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{1+j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{n! \Gamma(m)} \\
&\quad \times \left(-\frac{1}{1+n} + \frac{(2l)! {}_2F_2(-l, -1-l-n; -2l, -l-n; 1)}{(1+l+n)l!} \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, l \in \mathbb{Z}_+$.

EXAMPLE 3.17.

$$\begin{aligned}
(3.17) \quad & \int_0^x \frac{e^{(a-b)t+bx} t \left(\frac{e^{-\frac{1}{2}(bt)+\frac{bx}{2}} \sqrt{t}}{\sqrt{x}} \right)^{-\alpha} J_{\alpha} \left(\frac{e^{\frac{1}{2}(-bt+bx)} \sqrt{t}}{\sqrt{x}} \right) \Gamma(-m+t)}{x\Gamma(1+t)} dt \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} 2^{-\alpha} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(2+n)\Gamma(1+\alpha)} \\
&\quad \times {}_1F_2 \left(1+n; 2+n, 1+\alpha; -\frac{1}{4} \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

EXAMPLE 3.18.

$$\begin{aligned}
(3.18) \quad & \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \Gamma(m) \Gamma(-m+t) {}_3F_3 \left(1, 1, 1; 1+\frac{u}{2}, \frac{3}{2}+\frac{u}{2}, 2+v; \frac{e^{-2bt+2bx}t^2}{x^2} \right)}{x\Gamma(1+t)\Gamma(2+u)\Gamma(2+v)} dt \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n! (1+u)! (1+v)!} \\
&\quad \times {}_4F_4 \left(1, 1, 1, \frac{1}{2} + \frac{n}{2}; \frac{3}{2} + \frac{n}{2}, 1 + \frac{u}{2}, \frac{3}{2} + \frac{u}{2}, 2+v; 1 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1, |Re(v)| < 1$.

EXAMPLE 3.19.

$$\begin{aligned}
(3.19) \quad & \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \left(\frac{e^{-bt+bx} t}{x} \right)^{-v} I_v \left(\frac{2e^{\frac{1}{2}(-2bt+2bx)} t}{x} \right) \Gamma(m)}{(m-t)x(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(2+n)\Gamma(1+v)} \\
&\quad \times {}_1F_2 \left(\frac{1}{2} + \frac{n}{2}; \frac{3}{2} + \frac{n}{2}, 1+v; 1 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

EXAMPLE 3.20.

$$\begin{aligned}
(3.20) \quad & - \int_0^x \frac{(-1)^{-m} e^{at+bt-bx+b(-t+x)} \left(\frac{e^{-bt+bx} t}{x} \right)^{-v} \Gamma(m) \mathbf{H}_v \left(\frac{2e^{\frac{1}{2}(-2bt+2bx)} t}{x} \right)}{(-m+t)(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{2(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\sqrt{\pi} \Gamma(2+n) \Gamma(\frac{1}{2}(3+2v))} \\
&\quad \times {}_2F_3 \left(1, \frac{1}{2} + \frac{n}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2}, \frac{3}{2} + v; -1 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(v)| < 1$.

EXAMPLE 3.21.

$$\begin{aligned}
(3.21) \quad & \int_0^x \frac{(-1)^{-m} e^{at+bt-bx+b(-t+x)} \left(\frac{e^{-bt+bx} t}{x} \right)^{-v} \Gamma(m) \mathbf{L}_v \left(\frac{2e^{\frac{1}{2}(-2bt+2bx)} t}{x} \right)}{(m-t)(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{2(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n) \sqrt{\pi} n! \Gamma(\frac{1}{2}(3+2v))} \\
&\quad \times {}_2F_3 \left(1, \frac{1}{2} + \frac{n}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2}, \frac{3}{2} + v; 1 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(v)| < 1$.

EXAMPLE 3.22. See [DLMF, 10.8.3]

$$(3.22) \quad \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \left(\frac{e^{b(-t+x)} t}{x} \right)^{-u-v} J_u \left(\frac{2e^{b(-t+x)} t}{x} \right) J_v \left(\frac{2e^{b(-t+x)} t}{x} \right) \Gamma(m)}{(m-t)x(1-m+t)_m} dt$$

$$\begin{aligned}
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n! \Gamma(1+u) \Gamma(1+v)} \\
&\times {}_3F_4 \left(\frac{1}{2} + \frac{n}{2}, \frac{1}{2} + \frac{u}{2} + \frac{v}{2}, 1 + \frac{u}{2} + \frac{v}{2}; \frac{3}{2} + \frac{n}{2}, 1+u, 1+v, 1+u+v; -4 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1, |Re(v)| < 1$.

EXAMPLE 3.23.

(3.23)

$$\begin{aligned}
&\int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \left(\frac{e^{b(-t+x)} t}{x} \right)^{-u-v} I_u \left(\frac{2e^{b(-t+x)} t}{x} \right) I_v \left(\frac{2e^{b(-t+x)} t}{x} \right) \Gamma(m)}{(m-t)x(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n! \Gamma(1+u) \Gamma(1+v)} \\
&\times {}_3F_4 \left(\frac{1}{2} + \frac{n}{2}, \frac{1}{2} + \frac{u}{2} + \frac{v}{2}, 1 + \frac{u}{2} + \frac{v}{2}; \frac{3}{2} + \frac{n}{2}, 1+u, 1+v, 1+u+v; 4 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1, |Re(v)| < 1$.

EXAMPLE 3.24. See [DLMF,10.53.1]

(3.24)

$$\begin{aligned}
&\int_0^x \frac{(-1)^{-m} e^{at-bt+bx+b(-t+x)} t^2 \left(\frac{e^{-bt+bx} t}{x} \right)^{-u} I_{-1+u} \left(\frac{2e^{\frac{1}{2}(-2bt+2bx)} t}{x} \right) \Gamma(m) \Gamma(u)}{(m-t)x^2(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) {}_1F_2 \left(\frac{1}{2} + \frac{n}{2}; \frac{3}{2} + \frac{n}{2}, u; 1 \right)}{(1+n)n!}
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1$.

EXAMPLE 3.25.

$$\begin{aligned}
(3.25) \quad &\int_0^x \frac{e^{-im\pi+at+3b(-t+x)} t^2 x \Gamma(m) \Gamma(-m+t) \sin(2\pi u)}{(e^{4b(-t+x)} t^4 + x^4 - 2e^{2b(-t+x)} t^2 x^2 \cos(2\pi u)) \Gamma(t)} dt \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{ie^{aj+bjn+i\pi(j-2u)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{4n!} \\
&\quad \times \left(\Phi \left(e^{-2i\pi u}, 1, \frac{3+n}{2} \right) - e^{4i\pi u} \Phi \left(e^{2i\pi u}, 1, \frac{3+n}{2} \right) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1$.

EXAMPLE 3.26.

$$\begin{aligned}
 (3.26) \quad & \int_0^x \frac{(-1)^{-m} e^{at+2bt-2bx+3b(-t+x)} t \Gamma(m)}{(m-t)x(1-m+t)_m} \\
 & \times \left(-1 + \text{bei}_0 \left(\frac{2e^{\frac{1}{4}(-2bt+2\pi u+2bx)} \sqrt{t}}{\sqrt{x}} \right)^2 + \text{ber}_0 \left(\frac{2e^{\frac{1}{4}(-2bt+2\pi u+2bx)} \sqrt{t}}{\sqrt{x}} \right)^2 \right) dt \\
 & = \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)(3+n)n!} \\
 & \quad \times \left(-3 - n - e^{2\pi u} {}_1F_4 \left(\frac{3}{2} + \frac{n}{2}; \frac{3}{2}, 2, 2, \frac{5}{2} + \frac{n}{2}; \frac{1}{4} e^{2\pi u} \right) \right. \\
 & \quad \left. + 3\text{bei}_0 \left(2e^{\frac{\pi u}{2}} \right)^2 + n\text{bei}_0 \left(2e^{\frac{\pi u}{2}} \right)^2 + 3\text{ber}_0 \left(2e^{\frac{\pi u}{2}} \right)^2 + n\text{ber}_0 \left(2e^{\frac{\pi u}{2}} \right)^2 \right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(u)| < 1$.

EXAMPLE 3.27.

$$\begin{aligned}
 (3.27) \quad & \int_0^x \left(\sum_{p=0}^{\infty} \frac{(-1)^{-m+p} e^{at+b(1+2(\frac{1}{2}+p)^2)(-t+x)} \left(\frac{t}{x} \right)^{1+2(\frac{1}{2}+p)^2} \Gamma(m) \sin((1+2p)z)}{(m-t)(1-m+t)_m} \right) dt \\
 & = - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{i(-1)^j e^{aj+bjn-iz} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{2\sqrt{2}\sqrt{-1-n}(3+2n)n!} \\
 & \quad \times \left(- \left(\left(1 + \sqrt{2}\sqrt{-1-n} \right) {}_2F_1 \left(1, \frac{1}{2} - \frac{\sqrt{-1-n}}{\sqrt{2}}; \frac{3}{2} - \frac{\sqrt{-1-n}}{\sqrt{2}}; -e^{-2iz} \right) \right) \right. \\
 & \quad \left. + e^{2iz} \left(1 + \sqrt{2}\sqrt{-1-n} \right) {}_2F_1 \left(1, \frac{1}{2} - \frac{\sqrt{-1-n}}{\sqrt{2}}; \frac{3}{2} - \frac{\sqrt{-1-n}}{\sqrt{2}}; -e^{2iz} \right) \right. \\
 & \quad \left. - \left(-1 + \sqrt{2}\sqrt{-1-n} \right) \left({}_2F_1 \left(1, \frac{1}{2} + \frac{\sqrt{-1-n}}{\sqrt{2}}; \frac{3}{2} + \frac{\sqrt{-1-n}}{\sqrt{2}}; -e^{-2iz} \right) \right. \right. \\
 & \quad \left. \left. - e^{2iz} {}_2F_1 \left(1, \frac{1}{2} + \frac{\sqrt{-1-n}}{\sqrt{2}}; \frac{3}{2} + \frac{\sqrt{-1-n}}{\sqrt{2}}; -e^{2iz} \right) \right) \right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(z)| < 1$.

EXAMPLE 3.28. Dougall-Dixon definite integral form. See Eq. (17.1) in [13].

(3.28)

$$\begin{aligned}
& \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \Gamma(m) {}_3F_2 \left(-\alpha, -\beta, \delta; 1+\alpha+\delta, 1+\beta+\delta; -\frac{e^{-2bt+2bx}t^2}{x^2} \right)}{(m-t)x(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n!} \\
&\quad \times {}_4F_3 \left(\frac{1}{2} + \frac{n}{2}, -\alpha, -\beta, \delta; \frac{3}{2} + \frac{n}{2}, 1+\alpha+\delta, 1+\beta+\delta; -1 \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1, |Re(\delta)| < 1$.

EXAMPLE 3.29.

$$\begin{aligned}
(3.29) \quad & - \int_0^x \frac{(-1)^{-m} e^{at+2bt+b(-t+x)} tx \left(\frac{e^{-2bt}(-e^{2bx}t^2 + e^{2bt}x^2)}{x^2} \right)^{\alpha} \Gamma(m)}{(-m+t)(-e^{2bx}t^2 + e^{2bt}x^2)(1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) \Gamma\left(\frac{3+n}{2}\right) \Gamma(\alpha)}{\Gamma(2+n) \Gamma\left(\frac{1}{2}(1+n+2\alpha)\right)}
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1$

EXAMPLE 3.30.

$$\begin{aligned}
(3.30) \quad & - \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \Gamma(m)}{(-m+t)x \sqrt{\frac{e^{-2bt}(-e^{2bx}t^2 + e^{2bt}x^2)}{x^2}} (1-m+t)_m} dt \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} \sqrt{\pi} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) \Gamma\left(\frac{3+n}{2}\right)}{\Gamma\left(\frac{2+n}{2}\right) \Gamma(2+n)}
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1$

EXAMPLE 3.31. Dougall's formulae see (1.1) in [14]

$$\begin{aligned}
(3.31) \quad & \int_0^x \frac{(-1)^{-m} e^{at+b(t+x)} \left(1 - \frac{e^{2b(-t+x)}t^2}{x^2} \right)^{\alpha} x \alpha \Gamma(m) \Gamma(-m+t)}{(-e^{2bx}t^2 + e^{2bt}x^2) \Gamma(t)} dt \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) \Gamma\left(\frac{3+n}{2}\right) \Gamma(1+\alpha)}{\Gamma(2+n) \Gamma\left(\frac{1+n}{2} + \alpha\right)}
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1$

EXAMPLE 3.32.

$$\begin{aligned}
 (3.32) \quad & \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \Gamma(m) {}_2F_1\left(\alpha, \beta; 1+\alpha-\beta; \frac{e^{-2bt+2bx} t^2}{x^2}\right)}{(m-t)x(1-m+t)_m} dt \\
 &= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n!} \\
 &\quad \times {}_3F_2\left(\frac{1}{2} + \frac{n}{2}, \alpha, \beta; \frac{3}{2} + \frac{n}{2}, 1+\alpha-\beta; 1\right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1$

EXAMPLE 3.33. The q -Pochhammer symbol.

$$\begin{aligned}
 (3.33) \quad & \int_0^x \frac{(-1)^{-m} e^{at+b(-t+x)} t \Gamma(m) \left(-\frac{e^{2b(-t+x)} t^2}{x^2}; q\right)_l}{(m-t)x(1-m+t)_m} dt \\
 &= \sum_{p=0}^l \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} q^{\frac{1}{2}(-1+p)p} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) \binom{l}{p}_q}{(1+n+2p)n!}
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(q)| < 1$.

EXAMPLE 3.34. Ramanujan Mock theta functions see [Wolfram,15]

$$\begin{aligned}
 (3.34) \quad & - \int_0^x \frac{(-1)^{-m} \exp\left(at - 3bt + \frac{e^{2b(-t+x)} t^2}{x^2} + 3bx\right) (1+q)t^3 \left(\frac{e^{2b(-t+x)} t^2}{x^2}\right)^q}{(m-t)x^3(1-m+t)_m} \\
 &\quad \times \Gamma(m) \left(\Gamma(-1-q) - \Gamma\left(-1-q, \frac{e^{2b(-t+x)} t^2}{x^2}\right)\right) dt \\
 &= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^j e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx} x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{(1+n)n!} \\
 &\quad \times {}_2F_2\left(1, \frac{1}{2} + \frac{n}{2}; \frac{3}{2} + \frac{n}{2}, -q; 1\right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(q)| < 1$.

EXAMPLE 3.35.

$$(3.35) \quad \int_{-1}^z \frac{e^{(a-b)x} x^2}{(x-m)(1-bx)(1-m+x)_m} dx$$

$$\begin{aligned}
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{aj+b(-1+j)n} (a+bn)^{1+n} \binom{-1+m}{-1+j}}{\Gamma(m)\Gamma(1+n)} \\
&\times \left(-(-e^b)^n E_{-1-n}(-b(1+n)) + e^{bn(1+z)} z^2 (e^{-bz} z)^n E_{-1-n}(b(1+n)z) \right) E_{-n}(j(a+bn))
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1$.

EXAMPLE 3.36.

$$\begin{aligned}
(3.36) \quad &\int_{-1}^z \frac{e^{(a-bs)x} x^{1+s}}{(m-x)(1-bx)(1-m+x)_m} dx \\
&= - \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)-b(n+s)} (a+bn)^{1+n} \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(1+n)} \\
&\times \left((-e^b)^{n+s} E_{-n-s}(-b(n+s)) + e^{b(n+s)(1+z)} z (e^{-bz} z)^{n+s} E_{-n-s}(b(n+s)z) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(s)| > 1$.

EXAMPLE 3.37.

$$\begin{aligned}
(3.37) \quad &\int_{-1}^z \frac{e^{ax-bx} x^2 \log(x)}{(m-x)(1-bx)(1-m+x)_m} dx \\
&= \sum_{j=1}^m \sum_{n=0}^{\infty} \frac{(-1)^{j+m} e^{aj} (a+bn)^{1+n} \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(1+n)} \\
&\times \left(e^{b(-1+j)n} (-e^b)^n \left(-\frac{i(-b(1+n))^{-n} \pi \Gamma(2+n, 0, -b(1+n))}{b^2(1+n)^2} \right. \right. \\
&\quad \left. \left. + \frac{{}_2F_2(2+n, 2+n; 3+n, 3+n; b(1+n))}{(2+n)^2} \right) \right. \\
&\quad \left. + e^{bn(j+z)} z^2 (e^{-bz} z)^n \left(-\frac{{}_2F_2(2+n, 2+n; 3+n, 3+n; -b(1+n)z)}{(2+n)^2} \right. \right. \\
&\quad \left. \left. + (b(1+n)z)^{-2-n} \Gamma(2+n, 0, b(1+n)z) \log(z) \right) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1$.

EXAMPLE 3.38. Extended Adams integral. Eq. (6.706) in [15].

$$(3.38) \quad \int_0^1 \frac{\Gamma(m) e^{(a-b)x^\beta} (1-x)^{-1+\delta} x^{-1+\alpha+2\beta}}{(-m+x^\beta)(-1+bx^\beta)(1-m+x^\beta)_m} dx$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{j+m} e^{j(a+bn)} (-b(1+n))^l (a+bn)^{1+n} \binom{-1+m}{-1+j} E_{-n}(j(a+bn)) (-1+\delta)!}{(\alpha + (1+l+n)\beta)! \Gamma(1+n) (1+\alpha + (1+l+n)\beta)_{-1+\delta}}$$

where $|Re(a)| < 1, |Re(b)| < 1/Re(\delta) < 1/(2\pi), |Re(\alpha)| < 1, |Re(\beta)| < 1$.

EXAMPLE 3.39.

$$(3.39) \quad \frac{(\cosh^h(x) - 2^{-h} e^{-hx}) \Gamma(-m+x)}{(-1+bx)\Gamma(x)}$$

$$= \sum_{j=1}^m \sum_{n=0}^{\infty} \sum_{l=1}^h \frac{(-1)^{j+m} 2^{-h} e^{j(-h+2l+bn)} (-h+2l+bn)^{1+n} (e^{-bx}x)^n \binom{h}{l} \binom{-1+m}{-1+j}}{\Gamma(m)\Gamma(1+n)}$$

$$\times E_{-n}(j(-h+2l+bn))$$

where $|Re(b)| < 1, |Re(x)| < 1, Im(b) \neq 0$

THEOREM 3.40.

$$(3.40) \quad \frac{e^{ax}}{(1-bx)(-c^2+x^2)} = - \sum_{n=0}^{\infty} \frac{(-1)^{-n} c^{-2-n} e^{-c(a+bn)} (e^{-bx}x)^n}{2n!}$$

$$\times \left(\Gamma(1+n, -c(a+bn)) + (-1)^n e^{2c(a+bn)} \Gamma(1+n, c(a+bn)) \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(x)| < 1$ where $f(x) = \frac{e^{x(a+bn)}}{x^2-c^2}$

THEOREM 3.41. *The reciprocal Pochhammer symbol.*

$$(3.41) \quad \frac{1}{(1+x)_p} = \sum_{n=0}^{\infty} \sum_{j=1}^p \binom{-1+p}{-1+j} \frac{(-1)^j (-j)^{-1-n}}{(-1+p)!} x^n$$

where $|Re(x)| < 1$ where $f(x) = \frac{1}{(x+1)_p}$.

EXAMPLE 3.42.

$$(3.42) \quad \int_0^1 \frac{x^{-1+s}}{(1+x)_p} dx = \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{(-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j}}{(n+s)(-1+p)!}$$

where $|Re(s)| < 1$.

EXAMPLE 3.43.

(3.43)

$$\int_0^1 \frac{e^{sx^2}}{(1+x)_p} dx = \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{(-1)^j (-j)^{-1-n} (-s)^{-\frac{1}{2}-\frac{n}{2}} \binom{-1+p}{-1+j} \left(\Gamma\left(\frac{1+n}{2}\right) - \Gamma\left(\frac{1+n}{2}, -s\right) \right)}{2(-1+p)!}$$

where $|Re(s)| < 1$.

EXAMPLE 3.44.

$$\begin{aligned} (3.44) \quad & \int_0^1 \frac{e^{ax} \log(bx)}{(1+x)_p} dx \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{(-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j}}{(-1+p)!} \left(-\frac{{}_2F_2(1+n, 1+n; 2+n, 2+n; a)}{(1+n)^2} \right. \\ & \quad \left. + (-a)^{-1-n} (\Gamma(1+n) - \Gamma(1+n, -a)) \log(b) \right) \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 3.45.

(3.45)

$$\int_0^1 \frac{\sin(bx)}{(1+x)_p} dx = \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{(-1)^j b (-j)^{-1-n} \binom{-1+p}{-1+j} {}_1F_2\left(1 + \frac{n}{2}, \frac{3}{2}, 2 + \frac{n}{2}; -\frac{b^2}{4}\right)}{(2+n)(-1+p)!}$$

where $|Re(b)| < 1$.

EXAMPLE 3.46.

(3.46)

$$\int_0^1 \frac{1}{(1+x)_p (1-zx^a)^s} dx = \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{{}_2F_1\left(\frac{1+n}{a}, s; \frac{1+a+n}{a}; z\right) \binom{-1+p}{-1+j} (-1)^j (-j)^{-1-n}}{(1+n)(-1+p)!}$$

where $|Re(a)| < 1, |Re(z)| < 1, |Re(s)| < 1$.

EXAMPLE 3.47.

$$\begin{aligned} (3.47) \quad & \int_0^1 \frac{(1-x)^b x^a (1+x)^b (1-t^2 x^2)^c \Gamma(1+x)}{\Gamma(1+p+x)} dx \\ &= - \sum_{n=0}^{\infty} \sum_{j=1}^p \frac{(-1)^j (-j)^{-n} \Gamma(1+b) \Gamma\left(\frac{1}{2}(1+a+n)\right)}{2\Gamma(1+j) \Gamma\left(\frac{1}{2}(3+a+2b+n)\right) \Gamma(1-j+p)} \end{aligned}$$

$$\times {}_2F_1\left(-c, \frac{1}{2}(1+a+n); \frac{1}{2}(3+a+2b+n); t^2\right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(t)| < 1$.

EXAMPLE 3.48.

$$(3.48) \quad \int_0^1 \sqrt{\frac{16x(1-x)}{\pi(x+1)(1-t^2x^2)}} dx \\ = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\frac{1}{4}(3+2n)\right)}{\Gamma\left(\frac{1}{4}(9+2n)\right)} {}_2F_1\left(\frac{1}{2}, \frac{1}{4}(3+2n); \frac{1}{4}(9+2n); t^2\right)$$

where $|Re(t)| < 1$.

EXAMPLE 3.49.

$$(3.49) \quad \int_0^1 \sqrt{\frac{16x(1-x)}{\pi(x+1)(x+2)^2(1-t^2x^2)}} dx \\ = \sum_{j=1}^2 \sum_{n=0}^{\infty} \frac{(-1)^{j+1}(-j)^{-n} \Gamma\left(\frac{1}{2}\left(\frac{3}{2}+n\right)\right) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{3}{2}+n\right); \frac{1}{2}\left(\frac{9}{2}+n\right); t^2\right)}{\Gamma(3-j)\Gamma(1+j)\Gamma\left(\frac{1}{2}\left(\frac{9}{2}+n\right)\right)}$$

where $|Re(t)| < 1$.

EXAMPLE 3.50.

$$(3.50) \quad \int_0^1 \sqrt{\frac{4x(1-t^2x^2)}{\pi(1-x)(1+x)^3}} dx = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{1}{2}\left(\frac{3}{2}+n\right)\right) {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}\left(\frac{3}{2}+n\right); \frac{1}{2}\left(\frac{5}{2}+n\right); t^2\right)}{\Gamma\left(\frac{1}{2}\left(\frac{5}{2}+n\right)\right)}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(t)| < 1$.

EXAMPLE 3.51.

$$(3.51) \quad \int_0^1 \sqrt{\frac{4x}{\pi(1-x)(1+x)^3(1-t^2x^2)}} dx \\ = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left(\frac{1}{2}\left(\frac{3}{2}+n\right)\right) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}\left(\frac{3}{2}+n\right); \frac{1}{2}\left(\frac{5}{2}+n\right); t^2\right)}{\Gamma\left(\frac{1}{2}\left(\frac{5}{2}+n\right)\right)}$$

where $|Re(t)| < 1$.

4. Integrals over finite intervals

In this section we look at definite integrals over finite intervals involving Special functions and constants. We also table some errata for interested readers.

EXAMPLE 4.1.

$$(4.1) \quad \int_0^\phi \frac{x^a (1 - x^2 z^2)^b}{(1+x)_p} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j (-j)^{-1-n} \phi^{1+a+n} \binom{-1+p}{-1+j} {}_2F_1\left(-b, \frac{1}{2}(1+a+n); \frac{1}{2}(3+a+n); z^2 \phi^2\right)}{(1+a+n)(-1+p)!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\phi)| < 1$.

EXAMPLE 4.2.

$$(4.2) \quad \int_0^1 \frac{(1-x^2)^{-\frac{1}{2}+v} \cosh(xz)}{(1+x)_p} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} (-j)^{-n} \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(\frac{1}{2}+v\right) {}_1F_2\left(\frac{1}{2} + \frac{n}{2}; \frac{1}{2}, 1 + \frac{n}{2} + v; \frac{z^2}{4}\right)}{2\Gamma(1+j)\Gamma(1-j+p)\Gamma\left(1 + \frac{n}{2} + v\right)}$$

where $Re(v) > 0$.

EXAMPLE 4.3.

$$(4.3) \quad \int_0^1 \frac{(1-x)^{-1-b-v} x^{a+b+v} (2-x+xz)^v \Gamma(1+x)}{\Gamma(1+p+x)} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} 2^v (-j)^{-n} \Gamma(-b-v) \Gamma(1+a+b+n+v)}{\Gamma(1+j) \Gamma(1+a+n) \Gamma(1-j+p)}$$

$$\times {}_2F_1\left(-v, 1+a+b+n+v; 1+a+n; \frac{1-z}{2}\right)$$

where $Re(a) > 0, 3 < Re(b) < 4, 0 < Im(b) < 1, Re(z) < 1, -6 < Re(v) < -4$.

EXAMPLE 4.4.

$$(4.4) \quad \int_0^1 \frac{x^{-1+a} (1-ux)^{-b} (1-vx)^{-c}}{(1+x)_p} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j (-j)^{-1-n} F_1(a+n; b, c; 1+a+n; u, v) \binom{-1+p}{-1+j}}{(a+n)(-1+p)!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(u)| < 1, |Re(v)| < 1$.

EXAMPLE 4.5.

(4.5)

$$\int_0^1 \frac{e^{xz} (1-x)^v x^{-1-v}}{(1+x)_p} dx = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} (-j)^{-n} \Gamma(n-v) \Gamma(1+v) {}_1F_1(n-v; 1+n; z)}{\Gamma(1+j) \Gamma(1+n) \Gamma(1-j+p)}$$

where $-1 < Re(v) < 0$.

EXAMPLE 4.6. From Eq. (4.1.1.1) in [16].

$$(4.6) \quad \int_0^a \frac{x^{s-1} (a-x)^{t-1} (1-bx(a-x))^v}{(1+x)_p} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{\left((-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j} \right) a^{-1+n+s+t} B(n+s, t)}{(-1+p)!}$$

$$\times {}_3F_2 \left(n+s, t, -v; \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b}{4} \right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.7. From Eq. (4.1.2.2) in [16].

$$(4.7) \quad \int_0^a \frac{e^{b(a-x)x} (a-x)^{-1+t} x^{-1+s}}{(1+x)_p} dx$$

$$= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{-1+n+s+t} (-j)^{-1-n} B(n+s, t) \binom{-1+p}{-1+j}}{(-1+p)!}$$

$$\times {}_2F_2 \left(n+s, t; \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b}{4} \right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.8. From Eq. (4.1.3.1) in [16].

$$(4.8) \quad \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \sinh \left(b \sqrt{(a-x)x} \right)}{(1+x)_p} dx$$

$$\begin{aligned}
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{n+s+t} b (-j)^{-1-n} B\left(\frac{1}{2} + n + s, \frac{1}{2} + t\right) \binom{-1+p}{-1+j}}{(-1+p)!} \\
&\quad \times {}_2F_3\left(\frac{1}{2} + n + s, \frac{1}{2} + t; \frac{3}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{16}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.9. From Eq. (4.1.3.18) in [16].

$$\begin{aligned}
(4.9) \quad &\int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \cosh\left(b\sqrt{(a-x)x}\right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{-1+n+s+t} (-j)^{-1-n} B(n+s, t) \binom{-1+p}{-1+j}}{(-1+p)!} \\
&\quad \times {}_2F_3\left(n+s, t; \frac{1}{2}, \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{16}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.10. From Eq. (4.1.5.98) in [16].

$$\begin{aligned}
(4.10) \quad &\int_0^a \frac{x^{-1+s} \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) \sin(bx)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{1+n+s} b (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{1}{2}(1+n+s)\right)}{(n+s)(1+n+s)(-1+p)! \Gamma\left(\frac{n+s}{2}\right)} \\
&\quad \times {}_2F_3\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -\frac{1}{4} a^2 b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.11. Eq. (4.1.5.98) in [16]. Errata. The quoted equation is in error.

$$\begin{aligned}
(4.11) \quad &\int_0^a x^{s-1} \sin(bx) \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) dx \\
&= \frac{(\sqrt{\pi} a^{s+1} b \Gamma\left(\frac{s+1}{2}\right)) {}_2F_3\left(\frac{s+1}{2}, \frac{s+1}{2}; \frac{3}{2}, \frac{s}{2} + 1, \frac{s+3}{2}; -\frac{1}{4} (a^2 b^2)\right)}{s(s+1) \Gamma\left(\frac{s}{2}\right)}
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.12. From Eq. (4.1.5.99) in [16].

$$\begin{aligned}
 (4.12) \quad & \int_0^a \frac{x^{-1+s} \cos(bx) \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{n+s} (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{n+s}{2}\right)}{2(n+s)(-1+p)! \Gamma\left(\frac{1}{2}(1+n+s)\right)} \\
 &\quad \times {}_2F_3\left(\frac{n}{2} + \frac{s}{2}, \frac{n}{2} + \frac{s}{2}; \frac{1}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; -\frac{1}{4}a^2b^2\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.13. From Eq. (4.1.5.140) in [16].

$$\begin{aligned}
 (4.13) \quad & \int_0^a \frac{x^{-1+s} \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) \log(bx + \sqrt{1+b^2x^2})}{\sqrt{1+b^2x^2}(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{1+n+s} b (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{1}{2}(1+n+s)\right)}{2(1+n+s)(-1+p)! \Gamma\left(1 + \frac{n+s}{2}\right)} \\
 &\quad \times {}_4F_3\left(1, 1, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -a^2b^2\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.14. From Eq. (4.1.6.63) in [16].

$$\begin{aligned}
 (4.14) \quad & \int_0^1 \frac{x^{-1+s} \cos^{-1}(x) \cosh(ax)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{1}{2}(1+n+s)\right)}{(n+s)^2(-1+p)! \Gamma\left(\frac{n+s}{2}\right)} \\
 &\quad \times {}_2F_3\left(\frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{1}{2}, 1 + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{a^2}{4}\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.15. From Eq. (4.1.6.66) in [16]. pp. 163.

$$(4.15) \quad \int_0^1 \frac{x^{-1+s} \cos^{-1}(x) \sin(ax)}{(1+x)_p} dx$$

$$\begin{aligned}
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} 2^{-2-n-s} a(-j)^{-n} \pi \Gamma(1+n+s)}{\Gamma(1+j) \Gamma(1-j+p) \Gamma\left(\frac{3}{2} + \frac{n}{2} + \frac{s}{2}\right)^2} \\
&\times {}_2F_3\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -\frac{a^2}{4}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.16. From Eq. (4.1.6.90) in [16].

$$\begin{aligned}
(4.16) \quad &\int_0^1 \frac{x^{-1+s} \cos^{-1}(x) \log(ax + \sqrt{1+a^2x^2})}{\sqrt{1+a^2x^2}(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} 2^{-2-n-s} a(-j)^{-n} \pi \Gamma(1+n+s)}{\Gamma(1+j) \Gamma(1-j+p) \Gamma\left(\frac{3}{2} + \frac{n}{2} + \frac{s}{2}\right)^2} \\
&\times {}_4F_3\left(1, 1, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -a^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.17. From Eq. (4.1.6.126) in [16].

$$\begin{aligned}
(4.17) \quad &\int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \tan^{-1}\left(b\sqrt{(a-x)x}\right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} a^{n+s+t} b(-j)^{-n} \Gamma\left(\frac{1}{2} + n + s\right) \Gamma\left(\frac{1}{2} + t\right)}{\Gamma(1+j) \Gamma(1-j+p) \Gamma(1+n+s+t)} \\
&\times {}_4F_3\left(\frac{1}{2}, 1, \frac{1}{2} + n + s, \frac{1}{2} + t; \frac{3}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; -\frac{1}{4}a^2b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.18. From Eq. (4.1.6.144) in [16]

$$\begin{aligned}
(4.18) \quad &\int_0^a \frac{x^{-1+s} \tan^{-1}(bx) \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \left(-\frac{(-1)^j a^{1+n+s} b(-j)^{-n} \sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}\right)}{(n+s)^2 \Gamma(1+j) \Gamma(1-j+p) \Gamma\left(\frac{n}{2} + \frac{s}{2}\right)} \right. \\
&\quad \left. \times {}_3F_2\left(\frac{1}{2}, 1, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}; -a^2b^2\right) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(-1)^j 2^{-1+n+s} a^{1+n+s} b (-j)^{-n} \Gamma\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}\right)^2}{(n+s)\Gamma(1+j)\Gamma(1-j+p)\Gamma(2+n+s)} \\
& \times {}_3F_2\left(1, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; 1 + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -a^2 b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.19. From Eq. (4.1.6.148) in [16].

$$\begin{aligned}
(4.19) \quad & \int_0^a \frac{x^{-1+s} (a^2 - x^2)^{-1+t} \sin^{-1}(bx)^2}{(1+x)_p} dx \\
& = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} a^{n+s+2t} b^2 (-j)^{-n} \Gamma\left(1 + \frac{n}{2} + \frac{s}{2}\right) \Gamma(t)}{2\Gamma(1+j)\Gamma(1-j+p)\Gamma\left(1 + \frac{n}{2} + \frac{s}{2} + t\right)} \\
& \quad \times {}_4F_3\left(1, 1, 1, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 2, 1 + \frac{n}{2} + \frac{s}{2} + t; a^2 b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.20. From Eq. (4.1.6.151) in [16]. Errata. The quoted equation is in error.

$$\begin{aligned}
(4.20) \quad & \int_0^a x^{s-1} (a-x)^{t-1} \sin^{-1}\left(b\sqrt{x(a-x)}\right)^2 dx \\
& = a^{s+t+1} b^2 B(s+1, t+1) {}_5F_4\left(1, 1, 1, s+1, t+1; \frac{3}{2}, 3, \frac{s+t}{2} + 1, \frac{1}{3}(s+t+3); \frac{1}{4}(ab)^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.21. From Eq. (4.1.6.151) in [16].

$$\begin{aligned}
(4.21) \quad & \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \sin^{-1}\left(b\sqrt{(a-x)x}\right)^2}{(1+x)_p} dx \\
& = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{j+1} a^{1+n+s+t} b^2 (-j)^{-n} \Gamma(1+n+s) \Gamma(1+t)}{\Gamma(1+j)\Gamma(1-j+p)\Gamma(2+n+s+t)} \\
& \quad \times {}_5F_4\left(1, 1, 1, 1+n+s, 1+t; \frac{3}{2}, 3, 1 + \frac{n}{3} + \frac{s}{3} + \frac{t}{3}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{4}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.22. From Eq. (4.1.6.163) in [16].

$$\begin{aligned}
 (4.22) \quad & \int_0^a \frac{x^{-1+s} \sin^{-1}(bx)^2 \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \left(-\frac{(-1)^j a^{2+n+s} b^2 (-j)^{-n} \sqrt{\pi} \Gamma\left(\frac{n}{2} + \frac{s}{2}\right)}{4\Gamma(1+j)\Gamma(1-j+p)\Gamma\left(\frac{3}{2} + \frac{n}{2} + \frac{s}{2}\right)} \right. \\
 &\quad \times {}_4F_3\left(1, 1, 1, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 2, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; a^2 b^2\right) \\
 &\quad + \frac{(-1)^j 2^{n+s} a^{2+n+s} b^2 (-j)^{-n} \Gamma\left(\frac{n}{2} + \frac{s}{2}\right) \Gamma\left(1 + \frac{n}{2} + \frac{s}{2}\right)}{\Gamma(1+j)\Gamma(1-j+p)\Gamma(3+n+s)} \\
 &\quad \times {}_4F_3\left(1, 1, 1 + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}, 2 + \frac{n}{2} + \frac{s}{2}; a^2 b^2\right) \Bigg)
 \end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(b)| < 1$.

EXAMPLE 4.23. From Eq. (4.2.1.1) in [16].

$$\begin{aligned}
 (4.23) \quad & \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \text{Li}_2(b(a-x)x)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j a^{1+n+s+t} b (-j)^{-1-n} B(1+n+s, 1+t) \binom{-1+p}{-1+j}}{(-1+p)!} \\
 &\quad \times {}_5F_4\left(1, 1, 1, 1+n+s, 1+t; 2, 2, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b}{4}\right)
 \end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(b)| < 1$.

EXAMPLE 4.24. From Eq. (4.3.1.1) in [16].

$$\begin{aligned}
 (4.24) \quad & \int_0^a \frac{x^{-1+s} \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) \text{Si}(bx)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \left(-\frac{(-1)^j a^{1+n+s} b (-j)^{-n} \sqrt{\pi} \Gamma\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}\right)}{(n+s)^2 \Gamma(1+j)\Gamma(1-j+p)\Gamma\left(\frac{n}{2} + \frac{s}{2}\right)} \right. \\
 &\quad \times {}_2F_3\left(\frac{1}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}; -\frac{1}{4} a^2 b^2\right) \\
 &\quad + \frac{(-1)^j 2^{-1+n+s} a^{1+n+s} b (-j)^{-n} \Gamma\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}\right)^2}{(n+s)\Gamma(1+j)\Gamma(1-j+p)\Gamma(2+n+s)} \\
 &\quad \times {}_2F_3\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -\frac{1}{4} a^2 b^2\right) \Bigg)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.25. From Eq. (4.3.4.1) in [16].

$$\begin{aligned}
 (4.25) \quad & \int_0^1 \frac{x^{-1+s} \cos^{-1}(x) \text{Si}(ax)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \left(-\frac{(-1)^j 2^{-2-n-s} a(-j)^{-n} \pi(1+n+s) \Gamma(n+s)}{\Gamma(1+j) \Gamma(1-j+p) \Gamma\left(\frac{3}{2} + \frac{n}{2} + \frac{s}{2}\right)^2} \right. \\
 &\quad \times {}_2F_3\left(\frac{1}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -\frac{a^2}{4}\right) \\
 &\quad + \frac{(-1)^j 2^{-2-n-s} a(-j)^{-n} \pi \Gamma(n+s)}{\Gamma(1+j) \Gamma(1-j+p) \Gamma\left(\frac{3}{2} + \frac{n}{2} + \frac{s}{2}\right)^2} \\
 &\quad \times {}_2F_3\left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -\frac{a^2}{4}\right) \Bigg)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.26. From Eq. (4.4.1.1) in [16].

$$\begin{aligned}
 (4.26) \quad & \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \text{erf}\left(b\sqrt{(a-x)x}\right)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{2(-1)^j a^{n+s+t} b(-j)^{-1-n} B\left(\frac{1}{2} + n + s, \frac{1}{2} + t\right) \frac{(-1+p)}{(-1+j)}}{\sqrt{\pi}(-1+p)!}
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.27. From Eq. (4.4.2.1) in [16].

$$\begin{aligned}
 (4.27) \quad & \int_0^a \frac{e^{b^2(a-x)x} (a-x)^{-1+t} x^{-1+s} \text{erf}\left(b\sqrt{(a-x)x}\right)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{2(-1)^j a^{n+s+t} b(-j)^{-1-n} B\left(\frac{1}{2} + n + s, \frac{1}{2} + t\right) \frac{(-1+p)}{(-1+j)}}{\sqrt{\pi}(-1+p)!} \\
 &\quad \times {}_3F_3\left(1, \frac{1}{2} + n + s, \frac{1}{2} + t; \frac{3}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{4}\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.28. From Eq. (4.4.4.1) in [16].

$$\begin{aligned}
(4.28) \quad & \int_0^a \frac{x^{-1+s} \operatorname{erf}(bx) \log \left(\frac{a+\sqrt{a^2-x^2}}{x} \right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \left(-\frac{(-1)^j a^{1+n+s} b (-j)^{-n} \Gamma \left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2} \right)}{(n+s) \Gamma(1+j) \Gamma(1-j+p) \Gamma \left(1 + \frac{n}{2} + \frac{s}{2} \right)} \right. \\
&\quad \times {}_2F_2 \left(\frac{1}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, 1 + \frac{n}{2} + \frac{s}{2}; -a^2 b^2 \right) \\
&\quad + \frac{(-1)^j 2^{n+s} a^{1+n+s} b (-j)^{-n} \Gamma \left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2} \right)^2}{\sqrt{\pi} (n+s) \Gamma(1+j) \Gamma(1-j+p) \Gamma(2+n+s)} \\
&\quad \left. \times {}_2F_2 \left(\frac{1}{2} + \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; 1 + \frac{n}{2} + \frac{s}{2}, \frac{3}{2} + \frac{n}{2} + \frac{s}{2}; -a^2 b^2 \right) \right)
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(b)| < 1$.

EXAMPLE 4.29. From Eq. (4.4.6.8) in [16]. Errata. The quoted equation is in error.

$$\begin{aligned}
(4.29) \quad & \int_0^a x^{s-1} (a-x)^{t-1} \operatorname{erf} \left(b \sqrt[4]{x(a-x)} \right) \operatorname{erfi} \left(b \sqrt[4]{x(a-x)} \right) dx \\
&= \frac{4a^{s+t} b^2 B \left(s + \frac{1}{2}, t + \frac{1}{2} \right) {}_4F_5 \left(\frac{1}{4}, \frac{1}{8}, s + \frac{1}{2}, t + \frac{1}{2}; \frac{3}{4}, \frac{3}{2}, \frac{5}{4}, \frac{1}{2}(s+t+1), \frac{s+t}{2} + 1; \frac{1}{16}(ab)^2 \right)}{\pi}
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(b)| < 1$.

EXAMPLE 4.30. From Eq. (4.4.6.8) in [16].

$$\begin{aligned}
(4.30) \quad & \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \operatorname{erf} \left(b \sqrt[4]{(a-x)x} \right) \operatorname{erfi} \left(b \sqrt[4]{(a-x)x} \right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{4(-1)^j a^{n+s+t} b^2 (-j)^{-1-n} B \left(\frac{1}{2} + n + s, \frac{1}{2} + t \right) \binom{-1+p}{-1+j}}{\pi(-1+p)!} \\
&\quad \times {}_4F_5 \left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2} + n + s, \frac{1}{2} + t; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{16} \right)
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(b)| < 1$.

EXAMPLE 4.31. From Eq. (4.7.4.3) in [16].

$$(4.31) \quad \int_0^a \frac{x^{-1+s} J_v(bx) \log \left(\frac{a+\sqrt{a^2-x^2}}{x} \right)}{(1+x)_p} dx$$

$$\begin{aligned}
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-1-v} a^{n+s+v} b^v (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{1}{2}(n+s+v)\right)}{(n+s+v)(-1+p)! \Gamma(1+v) \Gamma\left(\frac{1}{2}(1+n+s+v)\right)} \\
&\times {}_2F_3\left(\frac{n}{2} + \frac{s}{2} + \frac{v}{2}, \frac{n}{2} + \frac{s}{2} + \frac{v}{2}; \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1+v; -\frac{1}{4}a^2b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.32. From Eq. (4.7.5.1) in [16].

$$\begin{aligned}
(4.32) \quad &\int_0^1 \frac{x^{-1+s} \cos^{-1}(x) J_v(ax)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-v} a^v (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma\left(\frac{1}{2}(1+n+s+v)\right)}{(n+s+v)^2 (-1+p)! \Gamma(1+v) \Gamma\left(\frac{1}{2}(n+s+v)\right)} \\
&\times {}_2F_3\left(\frac{n}{2} + \frac{s}{2} + \frac{v}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}; 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1+v; -\frac{a^2}{4}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.33. From Eq. (4.9.1.1) in [16]. Errata. The quoted equation is in error.

$$\begin{aligned}
(4.33) \quad &\int_0^a x^{s-1} (a-x)^{t-1} I_v\left(b\sqrt{x(a-x)}\right) dx \\
&= \frac{B\left(s + \frac{v}{2}, t + \frac{v}{2}\right) a^{s+t+v-1} \left(\frac{b}{2}\right)^v {}_2F_3\left(s + \frac{v}{6}, t + \frac{3v}{5}; v+1, \frac{1}{2}(s+t+v), \frac{1}{2}(s+t+v+1); \frac{1}{16}(ab)^2\right)}{\Gamma(v+1)}
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.34. From Eq. (4.9.1.1) in [16].

$$\begin{aligned}
(4.34) \quad &\int_0^a \frac{(a-x)^{-1+t} x^{-1+s} I_v\left(b\sqrt{(a-x)x}\right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-v} a^{-1+n+s+t+v} b^v (-j)^{-1-n} B\left(n+s+\frac{v}{2}, t+\frac{v}{2}\right) \binom{-1+p}{-1+j}}{(-1+p)! \Gamma(1+v)} \\
&\times {}_2F_3\left(n+s+\frac{v}{6}, t+\frac{3v}{5}; \frac{n}{2} + \frac{s}{2} + \frac{t}{2} + \frac{v}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2} + \frac{v}{2}, 1+v; \frac{a^2b^2}{16}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.35. From Eq. (4.9.2.1) in [16].

$$\begin{aligned}
 (4.35) \quad & \int_0^a \frac{e^{b(a-x)x}(a-x)^{-1+t}x^{-1+s}I_v(b(a-x)x)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-v} a^{-1+n+s+t+2v} b^v (-j)^{-1-n} B(n+s+v, t+v) \binom{-1+p}{-1+j}}{(-1+p)! \Gamma(1+v)} \\
 &\times {}_3F_3 \left(\frac{1}{2} + v, n+s+v, t+v; \frac{n}{2} + \frac{s}{2} + \frac{t}{2} + v, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2} + v, 1+2v; \frac{a^2 b}{2} \right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.36. From Eq. (4.9.4.1) in [16].

$$\begin{aligned}
 (4.36) \quad & \int_0^a \frac{x^{-1+s} I_v(bx) \log \left(\frac{a+\sqrt{a^2-x^2}}{x} \right)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-1-v} a^{n+s+v} b^v (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma \left(\frac{1}{2}(n+s+v) \right)}{(n+s+v)(-1+p)! \Gamma(1+v) \Gamma \left(\frac{1}{2}(1+n+s+v) \right)} \\
 &\times {}_2F_3 \left(\frac{n}{2} + \frac{s}{2} + \frac{v}{2}, \frac{n}{2} + \frac{s}{2} + \frac{v}{2}; \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1+v; \frac{a^2 b^2}{4} \right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.37. From Eq. (4.9.5.1) in [16].

$$\begin{aligned}
 (4.37) \quad & \int_0^1 \frac{x^{-1+s} \cos^{-1}(x) I_v(ax)}{(1+x)_p} dx \\
 &= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-v} a^v (-j)^{-1-n} \sqrt{\pi} \binom{-1+p}{-1+j} \Gamma \left(\frac{1}{2}(1+n+s+v) \right)}{(n+s+v)^2 (-1+p)! \Gamma(1+v) \Gamma \left(\frac{1}{2}(n+s+v) \right)} \\
 &\times {}_2F_3 \left(\frac{n}{2} + \frac{s}{2} + \frac{v}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}; 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1 + \frac{n}{2} + \frac{s}{2} + \frac{v}{2}, 1+v; \frac{a^2}{4} \right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 4.38. From Eq. (4.11.1.1) in [16].

$$(4.38) \quad \int_0^a \frac{(a-x)^{-1+t} x^{-1+s} \mathbf{H}_v \left(b\sqrt{(a-x)x} \right)}{(1+x)_p} dx$$

$$\begin{aligned}
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j 2^{-v} a^{n+s+t+v} b^{1+v} (-j)^{-1-n} \binom{-1+p}{-1+j} \Gamma\left(n+s+\frac{1+v}{2}\right) \Gamma\left(t+\frac{1+v}{2}\right)}{\sqrt{\pi}(-1+p)! \Gamma\left(\frac{3}{2}+v\right) \Gamma(1+n+s+t+v)} \\
&\quad \times {}_3F_4\left(1, \frac{1}{2}+n+s+\frac{v}{2}, \frac{1}{2}+t+\frac{v}{2}; \frac{3}{2}, \frac{1}{2}+\frac{n}{2}+\frac{s}{2}+\frac{t}{2}+\frac{v}{2}, 1+\frac{n}{2}+\frac{s}{2}\right. \\
&\quad \left.+\frac{t}{2}+\frac{v}{2}, \frac{3}{2}+v; -\frac{1}{16}a^2b^2\right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(s)| < 1, |Re(t)| < 1, |Re(v)| < 1$.

EXAMPLE 4.39. From Eq. (4.18.2.8) in [16]..

$$\begin{aligned}
(4.39) \quad &\int_0^a \frac{x^{-1+s} L_{\alpha}^{\lambda}(bx) \log\left(\frac{\sqrt{a}+\sqrt{a-x}}{\sqrt{x}}\right)}{(1+x)_p} dx \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{\left((-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j}\right) (a^{n+s} \sqrt{\pi} \Gamma(n+s))}{(-1+p)! (2(n+s)\alpha)! \Gamma\left(\frac{1}{2}+n+s\right)} \\
&\quad \times {}_3F_3\left(n+s, n+s, -\alpha; \frac{1}{2}+n+s, 1+n+s, 1+\lambda; ab\right) (1+\lambda)_{\alpha}
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(\alpha)| < 1, |Re(\lambda)| < 1$.

5. Table of series involving Special functions

EXAMPLE 5.1. From Eq. (5.7.1.3) in [16].

$$(5.1) \quad \sum_{l=0}^h \frac{x^l \binom{h}{l} P_l(z)}{(1+x)_p} = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{\left((-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j}\right) x^n (1+x^2+2xz)^{h/2} P_h\left(\frac{1+xz}{\sqrt{1+x^2+2xz}}\right)}{(-1+p)!}$$

where $|Re(z)| < 1, |Re(x)| < 1$.

EXAMPLE 5.2. From Eq. (5.9.1.1) in [16].

$$(5.2) \quad \sum_{l=0}^h \frac{x^l H_{2l}(z)}{(h-l)! (2l)! (1+x)_p} = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^j (-j)^{-1-n} (-1+x)^h x^n \binom{-1+p}{-1+j} H_{2h}\left(\sqrt{\frac{x}{-1+x}} z\right)}{(2h)! (-1+p)!}$$

where $|Re(z)| < 1, |Re(x)| < 1$.

EXAMPLE 5.3. Eq. (5.9.1.1) in [16]. Errata. The quoted equation is in error.

$$(5.3) \quad \sum_{k=0}^n \frac{t^k H_{2k}(z)}{(n-k)!(2k)!} = \frac{(t-1)^n H_{2n}\left(z\sqrt{\frac{t}{t-1}}\right)}{(2n)!}$$

where $|Re(z)| < 1, |Re(t)| < 1$.

EXAMPLE 5.4. From Eq. (6.17.1.6) in [16].

$$(5.4) \quad \sum_{k=0}^{\infty} \frac{t^k {}_3F_2(-k, 1+k, z; 1, 1; 1)}{k!(1+t)_p} = \frac{\Gamma(1+t) {}_1F_1(1-z; 1; t) {}_1F_1(z; 1; -t)}{\Gamma(1+p+t)}$$

where $|Re(z)| < 1, |Re(t)| < 1$.

EXAMPLE 5.5. From Eq. (6.17.1.10) in [16].

$$(5.5) \quad \sum_{k=0}^{\infty} \frac{kt^k {}_1F_2\left(\frac{1}{2}+k; \frac{3}{2}, \frac{3}{2}+k; z\right)}{(1+2k)!(1+t)_p} \\ = \frac{\sqrt{\frac{t}{z}}\Gamma(1+t) \left(-2\sqrt{z} \cosh(2\sqrt{z}) \sinh(\sqrt{t}) + \sqrt{t} \cosh(\sqrt{t}) \sinh(2\sqrt{z})\right)}{4(t-4z)(p+t)!}$$

where $|Re(z)| < 1, |Re(t)| < 1$.

EXAMPLE 5.6. From [Dieckman]

$$(5.6) \quad \int_0^a \frac{\cos(bx)}{(1+\sin(bx))_p} dx = \frac{\sin(ab)}{b} \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{(-1)^{-1+j-n} \binom{p-1}{j-1} \sin^n(ab)}{j^{n+1}(p-1)!(n+1)}$$

where $Im(b) \leq 0$.

EXAMPLE 5.7. From [Dieckman]

$$(5.7) \quad \int_0^x \frac{1}{(1+\log(1+ty))_p} dy \\ = \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{\left((-1)^j (-j)^{-1-n} \binom{-1+p}{-1+j}\right) ((-1)^n (\Gamma(n+1, -\log(1+tx)) - \Gamma(n+1)))}{(-1+p)! t}$$

where $|Im(x)| < 1$.

EXAMPLE 5.8. From [Dieckman]

$$\begin{aligned}
(5.8) \quad & \int_0^x \frac{y}{(1+ty)(1+\log(1+ty))_p} dy \\
&= \sum_{j=1}^p \sum_{n=0}^{\infty} \frac{\left((-1)^j(-j)^{-1-n} \binom{-1+p}{-1+j}\right) \left((-1)^n(\Gamma(n+1, -\log(1+tx)) - \Gamma(n+1)) - \frac{\log^{n+1}(1+tx)}{n+1}\right)}{(-1+p)! t^2}
\end{aligned}$$

where $|Im(x)| < 1$.

6. Special cases

In this section we evaluate special cases using the previously derived examples.

EXAMPLE 6.1.

$$(6.1) \quad \frac{4(\sqrt{e} - e - \text{Ei}(\frac{1}{2}) + \text{Ei}(1))}{e} = \sum_{n=0}^{\infty} \frac{2^{-n} e^{n/2} (1+n)^{1+n} E_{-n}(1+n)}{(2+n)n!}$$

EXAMPLE 6.2.

$$(6.2) \quad \frac{44}{9} - \frac{8(4e^3 + 9\text{Ei}(\frac{3}{2}) - 9\text{Ei}(3))}{9e^{3/2}} = \sum_{n=0}^{\infty} \frac{2^{-n} e^{n/2} \Gamma(1+n, n)}{(3+n)n!}$$

EXAMPLE 6.3.

$$(6.3) \quad \frac{8(-7e^{3/2} + 6e^2 + 8\text{Ei}(\frac{3}{2}) - 8\text{Ei}(2))}{e^{5/4}} = - \sum_{n=0}^{\infty} \frac{4^{-n} e^{1+\frac{3n}{4}} (1+n)^{1+n} E_{-n}(1+n)}{(3+n)n!}$$

EXAMPLE 6.4.

$$(6.4) \quad - \frac{-4 - 3x + e^{-3+3x} (4e^3 - 9\text{Ei}(3) + 9\text{Ei}(3-3x))}{9x^3} = \sum_{n=0}^{\infty} \frac{(e^{1-x} x)^n \Gamma(1+n, n)}{(3+n)n!}$$

EXAMPLE 6.5.

$$(6.5) \quad \frac{e^x \log\left(\frac{1}{1-x}\right)}{x} = \sum_{n=0}^{\infty} \frac{e^{1+n} (e^{-x} x)^n \Gamma(1+n, 1+n)}{\Gamma(2+n)}$$

where $|Re(x)| < 1, |Im(x)| < 1$.

EXAMPLE 6.6.

$$\begin{aligned}
(6.6) \quad & \frac{1}{12} \left(\pi^2 - 6 \left(\log^2(2) + 2\text{Li}_2 \left(\frac{1}{4} \right) \right) \right) \\
&= \sum_{n=0}^{\infty} \frac{e^{1+n} (1+n)^{-2-n} \left(\Gamma \left(1+n, \frac{1+n}{4} \right) - \Gamma \left(1+n, \frac{1+n}{2} \right) \right) \Gamma(1+n, 1+n)}{\Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.7.

$$(6.7) \quad \frac{e^x (i\pi - \text{Chi}(-1+x) + \text{Ei}(-1) + \text{Shi}(1-x))}{x} = \sum_{n=0}^{\infty} \frac{e^{1+n} (2+n)^{1+n} (e^{-x}x)^n E_{-n}(2+n)}{(1+n)n!}$$

where $0 < \text{Re}(x) < 1, 0 < \text{Im}(x) < 1$.

EXAMPLE 6.8.

$$(6.8) \quad \frac{e^x (i\pi - \text{Chi}(-1+x) + \text{Ei}(-1) + \text{Shi}(1-x))}{x} = \sum_{n=0}^{\infty} \frac{e^{1+n} (2+n)^{1+n} (e^{-x}x)^n E_{-n}(2+n)}{(1+n)n!}$$

where $0 < \text{Re}(x) < 1, 0 < \text{Im}(x) < 1$.

EXAMPLE 6.9.

$$\begin{aligned}
(6.9) \quad & \frac{e^{\frac{1}{6} + \frac{x}{3}} \left(-i\pi - \text{Chi} \left(\frac{1}{6}(-1+x) \right) + \text{Ei} \left(-\frac{1}{6} \right) + \text{Shi} \left(\frac{1}{6} - \frac{x}{6} \right) \right)}{x} \\
&= \sum_{n=0}^{\infty} \frac{e^{\frac{1}{2} + \frac{n}{3}} (e^{-\frac{x}{3}}x)^n \Gamma \left(1+n, \frac{1}{2} + \frac{n}{3} \right)}{\Gamma(2+n)}
\end{aligned}$$

where $0 < \text{Re}(x) < 1, \text{Im}(x) < 0$.

EXAMPLE 6.10.

$$(6.10) \quad \frac{e^{-1+x} (-i\pi - \text{Ei}(1) - \Gamma(0, -1+x))}{x} = - \sum_{n=0}^{\infty} \frac{(e^{1-x}x)^n \Gamma(1+n, n)}{\Gamma(2+n)}$$

where $0 < \text{Re}(x) < 1, 0 < \text{Im}(x) < 1$.

EXAMPLE 6.11.

(6.11)

$$\int_0^1 \frac{{}_2F_1\left(\alpha, \beta; \gamma; e^{\frac{1}{2}(-1+t)}t\right)}{1+t} dt = \sum_{n=0}^{\infty} \frac{(-1)^n e^{1+n} \Gamma\left(1+n, \frac{1+n}{2}\right) {}_3F_2(1+n, \alpha, \beta; 2+n, \gamma; 1)}{\Gamma(2+n)}$$

where $|Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1$.

EXAMPLE 6.12.

$$(6.12) \quad \int_0^1 \frac{e^{\frac{1}{2}(-3-t)} J_0\left(2e^{\frac{1}{2}(-1+t)}t\right)^2}{1+t} dt \\ = \sum_{n=0}^{\infty} \frac{(-e)^n \left(1 + \frac{n}{2}\right)^{1+n} E_{-n}\left(1 + \frac{n}{2}\right) {}_2F_3\left(\frac{1}{2}, \frac{1}{2} + \frac{n}{2}; 1, 1, \frac{3}{2} + \frac{n}{2}; -4\right)}{(1+n)n!}$$

EXAMPLE 6.13.

$$(6.13) \quad \int_0^1 \frac{{}_2F_1\left(\alpha, \beta; 1 + \alpha - \beta; e^{\frac{2}{3}(-1+t)}t^2\right)}{1+t} dt \\ = \sum_{n=0}^{\infty} \frac{(-1)^n e^{\frac{2(1+n)}{3}} \Gamma\left(1+n, \frac{1+n}{3}\right) {}_3F_2\left(\frac{1}{2} + \frac{n}{2}, \alpha, \beta; \frac{3}{2} + \frac{n}{2}, 1 + \alpha - \beta; 1\right)}{\Gamma(2+n)}$$

where $|Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(\gamma)| < 1$.

EXAMPLE 6.14.

(6.14)

$$\int_0^1 \frac{\left(-e^{\frac{2}{3}(-1+t)}t^2; q\right)_l}{1+t} dt = \sum_{p=0}^l \sum_{n=0}^{\infty} \frac{(-1)^n 3^{-1-n} e^{\frac{2(1+n)}{3}} (1+n)^{1+n} q^{\frac{1}{2}(-1+p)p} E_{-n}\left(\frac{1+n}{3}\right) \binom{l}{p}_q}{(1+n+2p)n!}$$

where $|Re(q)| < 1$.

EXAMPLE 6.15.

(6.15)

$$\int_0^1 \frac{e^{\frac{2t}{3} + e^{\frac{2}{3}(-1+t)}t^2} (1+q)t^2 \left(e^{\frac{2}{3}(-1+t)}t^2\right)^q \left(\Gamma(-1-q) - \Gamma\left(-1-q, e^{\frac{2}{3}(-1+t)}t^2\right)\right)}{1+t} dt \\ = - \sum_{n=0}^{\infty} \frac{(-1)^n e^{\frac{2(2+n)}{3}} \Gamma\left(1+n, \frac{1+n}{3}\right) {}_2F_2\left(1, \frac{1}{2} + \frac{n}{2}; \frac{3}{2} + \frac{n}{2}, -q; 1\right)}{\Gamma(2+n)}$$

where $|Re(q)| < 1$.

EXAMPLE 6.16.

$$(6.16) \quad \int_0^1 \frac{e^{(a-b)x^\beta} (1-x)^{-1+\delta} x^{-1+\alpha+\beta}}{(-1+x^\beta)(-1+bx^\beta)} dx$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{e^{a+bn} (-b(1+n))^l \Gamma(\alpha + (1+l+n)\beta) \Gamma(\delta) \Gamma(1+n, a+bn)}{\Gamma(1+l) \Gamma(1+n) \Gamma(\alpha + (1+l+n)\beta + \delta)}$$

where $|Re(a)| < 1, |Re(b)| < 1/Re(\delta), |Re(\alpha)| < 1, |Re(\beta)| < 1$.

EXAMPLE 6.17.

$$(6.17) \quad \frac{\sqrt{\pi} \Gamma(-1+\delta) (16\delta {}_2F_1(-\frac{1}{2}, 1; \frac{1}{2} + \delta; \frac{1}{8}) + 9 {}_2F_1(\frac{1}{2}, 1; \frac{1}{2} + \delta; \frac{1}{8}) - 16\delta {}_2F_1(\frac{1}{2}, 1; \frac{1}{2} + \delta; \frac{1}{8}))}{14\Gamma(\frac{1}{2} + \delta)}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{8^{-l} e^{\frac{1+n}{8}} (-1-n)^l \Gamma(\frac{3}{2} + l + n) \Gamma(\delta) \Gamma(1+n, \frac{1+n}{8})}{\Gamma(1+l) \Gamma(1+n) \Gamma(\frac{3}{2} + l + n + \delta)}$$

where $Re(\delta) > \pi$.

EXAMPLE 6.18.

$$(6.18) \quad \frac{\sqrt{3}\pi \Gamma(-1+\delta) {}_2F_1(1, -1+\delta; \frac{1}{3} + \delta; -1)}{\Gamma(-\frac{4}{3}) \Gamma(\frac{1}{3} + \delta)}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{2^{-l} e^{\frac{1+n}{2}} (-1-n)^l \Gamma(\frac{4}{3} + l + n) \Gamma(\delta) \Gamma(1+n, \frac{1+n}{2})}{\Gamma(1+l) \Gamma(1+n) \Gamma(\frac{4}{3} + l + n + \delta)}$$

where $Re(\delta) > \pi$.

EXAMPLE 6.19.

$$(6.19) \quad \frac{\sqrt{\pi} \Gamma(-1+\delta) {}_2F_1(1, -1+\delta; \frac{1}{2} + \delta; -1)}{\Gamma(\frac{1}{2} + \delta)}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{2^{-l} e^{\frac{1+n}{2}} (-1-n)^l \Gamma(\frac{3}{2} + l + n) \Gamma(\delta) \Gamma(1+n, \frac{1+n}{2})}{\Gamma(1+l) \Gamma(1+n) \Gamma(\frac{3}{2} + l + n + \delta)}$$

where $Re(\delta) > 2\pi$.

EXAMPLE 6.20.

$$\begin{aligned}
(6.20) \quad & \frac{\pi \Gamma(-1+\delta) {}_2F_1\left(1, -1+\delta; \frac{1}{4}+\delta; -1\right)}{\sqrt{2} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}+\delta\right)} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{2^{-l} e^{\frac{1+n}{2}} (-1-n)^l \Gamma\left(\frac{5}{4}+l+n\right) \Gamma(\delta) \Gamma\left(1+n, \frac{1+n}{2}\right)}{\Gamma(1+l) \Gamma(1+n) \Gamma\left(\frac{5}{4}+l+n+\delta\right)}
\end{aligned}$$

where $Re(\delta) > 2\pi$.

EXAMPLE 6.21.

$$\begin{aligned}
(6.21) \quad & -\frac{1}{18}(-1)^{-\delta} \left(\frac{3 \times 2^{3+2\delta} {}_2F_1\left(1-2\delta; -\delta, -\delta; 2-2\delta; \frac{3}{2}, \frac{1}{2}\right)}{-1+2\delta} \right. \\
& \quad \left. + \frac{11 \times 4^{1+\delta} {}_2F_1\left(-2\delta; -\delta, -\delta; 1-2\delta; \frac{3}{2}, \frac{1}{2}\right)}{\delta} \right. \\
& \quad - \frac{6\pi \Gamma(-1+\delta) \left(3 \cdot 4^\delta e^{i\pi\delta} \csc(\pi\delta) - 8 \csc(2\pi\delta) {}_2F_1(1-2\delta, -\delta; 2-\delta; 3)\right)}{\Gamma(-\delta) \Gamma(2\delta)} \\
& \quad + \frac{2^\delta e^{i\pi\delta} \left(44 {}_2F_1\left(1-\delta, 1+\delta; 2+\delta; \frac{1}{2}\right) - 9 {}_2F_1\left(2-\delta, 1+\delta; 2+\delta; \frac{1}{2}\right)\right)}{1+\delta} \\
& \quad \left. - \frac{\pi^2 \csc(\pi\delta) \left(45(-4)^\delta \csc(\pi\delta) + 44 \csc(2\pi\delta) {}_2F_1(-2\delta, -\delta; 1-\delta; 3)\right)}{\Gamma(1-\delta)^2 \Gamma(2\delta)} \right) \\
&= \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{2+j} 2^{-l} e^{j\left(\frac{1}{2}+\frac{n}{2}\right)} (-1-n)^l \left(\frac{1}{2}+\frac{n}{2}\right)^{1+n} \left(-\frac{1}{-1+j}\right) E_{-n}\left(j\left(\frac{1}{2}+\frac{n}{2}\right)\right) (-1+\delta)!}{\left(\frac{1}{2}+\frac{1}{2}(1+l+n)\right)! \Gamma(1+n) \left(\frac{3}{2}+\frac{1}{2}(1+l+n)\right)_{-1+\delta}}
\end{aligned}$$

where $2\pi < Re(\delta) < 5\pi, |Im(\delta)| > 1$.

EXAMPLE 6.22.

$$\begin{aligned}
(6.22) \quad & \int_0^1 \frac{4e^{\frac{x^2}{4}} (1-x)^4 x^{3/2}}{(1+x)(4-x^2)} dx \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{15 \times 2^{9-2l} e^{\frac{2+n}{4}} (-1-n)^l \Gamma\left(1+n, \frac{2+n}{4}\right)}{(5+4l+4n)(7+4l+4n)(9+4l+4n)(11+4l+4n)(13+4l+4n)(15+4l+4n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.23.

$$\begin{aligned}
(6.23) \quad & \int_0^1 \frac{4e^{\frac{x^3}{4}} (1-x)^4 x^3}{(1+x+x^2)(4-x^3)} dx \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{5 \cdot 2^{3-2l} e^{\frac{2+n}{4}} (-1-n)^l \Gamma\left(1+n, \frac{2+n}{4}\right)}{3(2+l+n)(3+l+n)(4+3l+3n)(5+3l+3n)(7+3l+3n)(8+3l+3n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.24.

$$\begin{aligned}
(6.24) \quad & \int_0^1 \frac{e^{\frac{x^2}{4}} (-1+x)^4 x^{3/2}}{(1+x)(-4+x^2)} dx \\
&= - \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{15 \times 2^{7-2l} e^{\frac{2+n}{4}} (-1-n)^l \Gamma(1+n, \frac{2+n}{4})}{(5+4l+4n)(7+4l+4n)(9+4l+4n)(11+4l+4n)(13+4l+4n)(15+4l+4n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.25.

$$\begin{aligned}
(6.25) \quad & \int_0^1 \frac{e^{\frac{x^2}{4}} (-1+x)^4 x^{3/2}}{(1+x)(-4+x^2)(-2+x^2)} dx \\
&= - \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{15(-1)^j 2^{5-2l-2n} e^{\frac{1}{4}j(2+n)} (-1-n)^l (2+n)^{1+n} \binom{1}{-1+j} E_{-n}(\frac{1}{4}j(2+n))}{(5+4l+4n)(7+4l+4n)(9+4l+4n)(11+4l+4n)(13+4l+4n)(15+4l+4n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.26.

$$\begin{aligned}
(6.26) \quad & \int_0^1 \frac{e^{\frac{x^2}{12}} (-1+x)^3 x^{3/2}}{(1+x)(-4+x^2)(-2+x^2)} dx \\
&= \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j 4^{2-l} e^{\frac{1}{12}j(4+3n)} (-1-n)^l \left(\frac{1}{3} + \frac{n}{4}\right)^n (4+3n) \binom{1}{-1+j} E_{-n}(\frac{1}{12}j(4+3n))}{(5+4l+4n)(7+4l+4n)(9+4l+4n)(11+4l+4n)(13+4l+4n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.27.

$$\begin{aligned}
(6.27) \quad & \int_0^1 \frac{e^{\frac{x^2}{12}} (-1+x)^2 x^{3/2}}{(1+x)(-4+x^2)(-3+x^2)(-2+x^2)} dx \\
&= \sum_{j=1}^3 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j 4^{-l} e^{\frac{1}{12}j(4+3n)} (-1-n)^l \left(\frac{1}{3} + \frac{n}{4}\right)^n (4+3n) \binom{2}{-1+j} E_{-n}(\frac{1}{12}j(4+3n))}{(5+4l+4n)(7+4l+4n)(9+4l+4n)(11+4l+4n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.28.

$$\begin{aligned}
(6.28) \quad & \int_0^1 \frac{e^{\frac{x^3}{3}} (-1+x)^2 x^{5/2}}{(1+x+x^2)(-6+x^3)(-2+x^3)} dx \\
&= \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} - \frac{(-1)^j 2^{3-l-n} 3^{-2-l-n} e^{\frac{1}{6}j(3+n)} (-1-n)^l (3+n)^{1+n} \binom{1}{-1+j} E_{-n}(\frac{1}{6}j(3+n))}{(3+2l+2n)(7+6l+6n)(11+6l+6n)(13+6l+6n)l! \Gamma(1+n)}
\end{aligned}$$

EXAMPLE 6.29.

$$(6.29) \quad \int_0^1 \frac{e^{\frac{x^4}{3}} (-1+x)^2 x^{16/5}}{(1+x+x^2+x^3)(-6+x^4)(-2+x^4)} dx$$

$$= - \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j 6^{-1-l-n} e^{\frac{1}{6}j(3+n)} (-1-n)^l (3+n)^{1+n} \binom{1}{-1+j} E_{-n} \left(\frac{1}{6}j(3+n) \right)}{\left(\frac{31}{5} + 4l + 4n \right) \left(\frac{36}{5} + 4l + 4n \right) \left(\frac{1}{5} + 4(1+l+n) \right) \left(\frac{6}{5} + 4(1+l+n) \right) l! \Gamma(1+n)}$$

EXAMPLE 6.30.

$$(6.30) \quad \int_0^1 \frac{e^{\frac{x^2}{3}} (-1+x)^2 x^{9/5}}{(1+x)(-6+x^2)(-2+x^2)} dx$$

$$= - \sum_{j=1}^2 \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^j 6^{-1-l-n} e^{\frac{1}{6}j(3+n)} (-1-n)^l (3+n)^{1+n} \binom{1}{-1+j} E_{-n} \left(\frac{1}{6}j(3+n) \right)}{\left(\frac{24}{5} + 2l + 2n \right) \left(\frac{29}{5} + 2l + 2n \right) \left(\frac{4}{5} + 2(1+l+n) \right) \left(\frac{9}{5} + 2(1+l+n) \right) l! \Gamma(1+n)}$$

THEOREM 6.31. *Stirling numbers of the second kind.*

$$(6.31) \quad 1 + (-1 + ce^x)^m = \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{x^n (-1)^{m+p} c^p p^n \binom{m}{p}}{n!}$$

where $|Re(x)| < 1, |Re(c)| < 1$ where $f(x) = (ce^{bx} - 1)^m$

EXAMPLE 6.32.

$$(6.32) \quad \int_0^v e^{ax^2} (1 + (-1 + e^x z)^m) dx$$

$$= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n v^{1+n} (-av^2)^{-\frac{1}{2} - \frac{n}{2}} z^p \binom{m}{p} (\Gamma(\frac{1+n}{2}) - \Gamma(\frac{1+n}{2}, -av^2))}{2n!}$$

where $|Re(a)| < 1, |Re(z)| < 1, |Re(v)| < 1$.

EXAMPLE 6.33.

$$(6.33) \quad \int_0^v e^{az^2} (1 + (-1 + e^x z)^m) dz$$

$$= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n v^{1+p} (-av^2)^{-\frac{1}{2} - \frac{p}{2}} x^n \binom{m}{p} (\Gamma(\frac{1+p}{2}) - \Gamma(\frac{1+p}{2}, -av^2))}{2n!}$$

where $|Re(a)| < 1, |Re(x)| < 1, |Re(v)| < 1$.

EXAMPLE 6.34. From Eq. (4.25.5.1) in [16].

$$(6.34) \quad \int_0^a (a-x)^{-1+t} x^{-1+s} (1 + (-1 + e^x z)^m) {}_1F_1(b; s; wx) {}_1F_1(c; t; w(a-x)) dx$$

$$= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} a^{-1+n+s+t} p^n z^p B(n+s, t) \binom{m}{p} {}_1F_1(b+c; n+s+t; aw)}{n!}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(w)| < 1, |Re(t)| < 1, |Re(z)| < 1$.

7. Table involving complete elliptic integrals

7.1. Preliminaries. These formulae are listed on page 672 in [16], where $0 < Re(k) < 1$.

$$(7.1) \quad D(k) = \int_0^{\pi/2} \frac{\sin^2(t) dt}{\sqrt{1 - k^2 \sin^2(t)}}; \text{ is the complete elliptic integral}$$

$$(7.2) \quad K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}; \text{ is the complete elliptic integral of the first kind}$$

$$(7.3) \quad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(t)} dt; \text{ is the complete elliptic integral of the second kind}$$

EXAMPLE 7.1. From Eq. (4.23.2.1) in [16]. The complete integral and complete integral of the first kind.

$$(7.4) \quad \int_0^1 \frac{x^{-\frac{5}{2}+s} (1 + (-1 + e^x z)^m) K\left(-\frac{1-x}{x}\right) \left((-1 + a^2 x) E\left(\frac{a^2 x}{-1+a^2 x}\right) + K\left(\frac{a^2 x}{-1+a^2 x}\right) \right)}{a^2 \sqrt{1 - a^2 x}} dx$$

$$= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s)^2 {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n+s, n+s; 2, \frac{1}{2} + n+s, \frac{1}{2} + n+s; a^2\right)}{8n! \Gamma\left(\frac{1}{2} + n+s\right)^2}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.2. From Eq. (4.23.2.1) in [16]. The complete integral and complete integral of the second kind.

$$(7.5) \quad \int_0^1 \frac{x^{-2+s} (1 + (-1 + e^x z)^m) E(1-x) \left((-1 + a^2 x) E\left(\frac{a^2 x}{-1+a^2 x}\right) + K\left(\frac{a^2 x}{-1+a^2 x}\right) \right)}{a^2 \sqrt{1 - a^2 x}} dx$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s) \Gamma(1+n+s)}{8n! \Gamma\left(\frac{1}{2}+n+s\right) \Gamma\left(\frac{3}{2}+n+s\right)} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n+s, 1+n+s; 2, \frac{1}{2}+n+s, \frac{3}{2}+n+s; a^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

7.2. Table involving the complete elliptic integral $D(z)$.

EXAMPLE 7.3. From Eq. (4.23.1.1) in [16].

$$\begin{aligned}
(7.6) \quad &\int_0^a \frac{(a-x)^{-2+t} x^{-2+s}}{b^2 \sqrt{1-b^2(a-x)x}} (1 + (-1 + e^x z)^m) \\
&\times \left((-1 + b^2(a-x)x) E\left(\frac{b^2(a-x)x}{-1+b^2(a-x)x}\right) + K\left(\frac{b^2(a-x)x}{-1+b^2(a-x)x}\right) \right) dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} a^{-1+n+s+t} p^n \pi z^p B(n+s, t) \binom{m}{p}}{4n!} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n+s, t; 2, \frac{n}{2} + \frac{s}{2} + \frac{t}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2} + \frac{t}{2}; \frac{a^2 b^2}{4}\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.4. From Eq. (4.23.1.14) in [16].

$$\begin{aligned}
(7.7) \quad &\int_0^a \frac{x^{-3+s} (1 + (-1 + e^x z)^m) \left((-1 + b^2 x^2) E\left(\frac{b^2 x^2}{-1+b^2 x^2}\right) + K\left(\frac{b^2 x^2}{-1+b^2 x^2}\right) \right) \log\left(\frac{a+\sqrt{a^2-x^2}}{x}\right)}{b^2 \sqrt{1-b^2 x^2}} dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} a^{n+s} p^n \pi^{3/2} z^p \binom{m}{p} \Gamma\left(\frac{n+s}{2}\right)}{8(n+s) n! \Gamma\left(\frac{1}{2}(1+n+s)\right)} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, \frac{n}{2} + \frac{s}{2}, \frac{n}{2} + \frac{s}{2}; 2, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; a^2 b^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.5. From Eq. (4.23.1.17) in [16].

$$\begin{aligned}
(7.8) \quad & \int_0^1 \frac{x^{-3+s} (1 + (-1 + e^x z)^m) \cos^{-1}(x) \left((-1 + a^2 x^2) E\left(\frac{a^2 x^2}{-1 + a^2 x^2}\right) + K\left(\frac{a^2 x^2}{-1 + a^2 x^2}\right) \right)}{a^2 \sqrt{1 - a^2 x^2}} dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^{3/2} z^p \binom{m}{p} \Gamma\left(\frac{1}{2}(1 + n + s)\right)}{4(n + s)^2 n! \Gamma\left(\frac{n+s}{2}\right)} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, \frac{n}{2} + \frac{s}{2}, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; 2, 1 + \frac{n}{2} + \frac{s}{2}, 1 + \frac{n}{2} + \frac{s}{2}; a^2\right)
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(z)| < 1$.

7.3. Table involving the product of the complete elliptic integral $E(z)$.

EXAMPLE 7.6. From Eq. (4.22.14.2) in [16].

$$\begin{aligned}
(7.9) \quad & \int_0^1 x^{-1+s} (1 + (-1 + e^x z)^m) E(1 - x) E(a^2 x) dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n + s) \Gamma(1 + n + s)}{4n! \Gamma\left(\frac{1}{2} + n + s\right) \Gamma\left(\frac{3}{2} + n + s\right)} \\
&\quad \times {}_4F_3\left(-\frac{1}{2}, \frac{1}{2}, n + s, 1 + n + s; 1, \frac{1}{2} + n + s, \frac{3}{2} + n + s; a^2\right)
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(z)| < 1$.

EXAMPLE 7.7. From Eq. (4.22.14.7) in [16].

$$\begin{aligned}
(7.10) \quad & \int_0^1 \frac{x^{-1+s} (1 + (-1 + e^x z)^m) E(1 - x) E(a^2 x)}{1 - a^2 x} dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n + s) \Gamma(1 + n + s)}{4n! \Gamma\left(\frac{1}{2} + n + s\right) \Gamma\left(\frac{3}{2} + n + s\right)} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n + s, 1 + n + s; 1, \frac{1}{2} + n + s, \frac{3}{2} + n + s; a^2\right)
\end{aligned}$$

where $|Re(s)| < 1$, $|Re(a)| < 1$, $|Re(z)| < 1$.

7.4. Integrals containing products of $E(z)$ and $K(z)$, the complete elliptic integral of the second kind and complete elliptic integral of the first kind respectively.

EXAMPLE 7.8. From Eq. (4.22.13.2) in [16].

$$\begin{aligned}
 (7.11) \quad & \int_0^1 \frac{x^{-1+s} (1 + (-1 + e^x z)^m) E(1-x) K\left(\frac{a^2 x}{-1+a^2 x}\right)}{\sqrt{1-a^2 x}} dx \\
 &= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 (n+s) z^p \binom{m}{p} \Gamma(n+s)^2}{2(1+2(n+s)) n! \Gamma\left(\frac{1}{2} + n+s\right)^2} \\
 &\quad \times {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, n+s, 1+n+s; 1, \frac{1}{2} + n+s, \frac{3}{2} + n+s; a^2\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.9. From Eq. (4.22.13.3) in [16].

$$\begin{aligned}
 (7.12) \quad & \int_0^1 x^{-\frac{3}{2}+s} (1 + (-1 + e^x z)^m) E(a^2 x) K\left(-\frac{1-x}{x}\right) dx \\
 &= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s)^2}{4n! \Gamma\left(\frac{1}{2} + n+s\right)^2} \\
 &\quad \times {}_4F_3\left(-\frac{1}{2}, \frac{1}{2}, n+s, n+s; 1, \frac{1}{2} + n+s, \frac{1}{2} + n+s; a^2\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.10. From Eq. (4.22.13.9) in [16].

$$\begin{aligned}
 (7.13) \quad & \int_0^1 \frac{x^{-\frac{3}{2}+s} (1 + (-1 + e^x z)^m) E(a^2 x) K\left(-\frac{1-x}{x}\right)}{1-a^2 x} dx \\
 &= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s)^2}{4n! \Gamma\left(\frac{1}{2} + n+s\right)^2} \\
 &\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n+s, n+s; 1, \frac{1}{2} + n+s, \frac{1}{2} + n+s; a^2\right)
 \end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.11. From Eq. (4.22.13.11) in [16].

$$(7.14) \quad \int_0^1 \frac{x^{-\frac{3}{2}+s} (1 + (-1 + e^x z)^m) E\left(\frac{a^2 x}{1+a^2 x}\right) K\left(-\frac{1-x}{x}\right)}{\sqrt{1+a^2 x}} dx$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s)^2}{4n! \Gamma\left(\frac{1}{2} + n + s\right)^2} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{3}{2}, n+s, n+s; 1, \frac{1}{2} + n + s, \frac{1}{2} + n + s; -a^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.12. From Eq. (4.22.13.13) in [16].

$$\begin{aligned}
(7.15) \quad &\int_0^1 \frac{x^{-1+s} (1 + (-1 + e^x z)^m) E(1-x) K\left(\frac{a^2 x}{(1+a^2 x)\left(-1 + \frac{a^2 x}{1+a^2 x}\right)}\right)}{\sqrt{1+a^2 x} \sqrt{1 - \frac{a^2 x}{1+a^2 x}}} dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s) \Gamma(1+n+s)}{4n! \Gamma\left(\frac{1}{2} + n + s\right) \Gamma\left(\frac{3}{2} + n + s\right)} \\
&\quad \times {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, n+s, 1+n+s; 1, \frac{1}{2} + n + s, \frac{3}{2} + n + s; -a^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

EXAMPLE 7.13. From Eq. (4.22.13.19) in [16].

$$\begin{aligned}
(7.16) \quad &\int_0^1 x^{-\frac{3}{2}+s} \sqrt{1+a^2 x} (1 + (-1 + e^x z)^m) E\left(\frac{a^2 x}{1+a^2 x}\right) K\left(-\frac{1-x}{x}\right) dx \\
&= \sum_{n=0}^{\infty} \sum_{p=1}^m \frac{(-1)^{m+p} p^n \pi^2 z^p \binom{m}{p} \Gamma(n+s)^2}{4n! \Gamma\left(\frac{1}{2} + n + s\right)^2} \\
&\quad \times {}_4F_3\left(-\frac{1}{2}, \frac{1}{2}, n+s, n+s; 1, \frac{1}{2} + n + s, \frac{1}{2} + n + s; -a^2\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(b)| < 1, |Re(a)| < 1, |Re(z)| < 1$.

THEOREM 7.14.

$$\begin{aligned}
(7.17) \quad &\frac{e^{ax}}{(-1+bx)(-c^4+x^4)} = \sum_{n=0}^{\infty} \frac{(e^{-bx}x)^n}{4c^4 n!} \left((-c)^{-n} e^{-c(a+bn)} \Gamma(1+n, -c(a+bn)) \right. \\
&\quad + (-ic)^{-n} e^{-ic(a+bn)} \Gamma(1+n, -ic(a+bn)) + (ic)^{-n} e^{ic(a+bn)} \Gamma(1+n, ic(a+bn)) \\
&\quad \left. + c^{-n} e^{c(a+bn)} \Gamma(1+n, c(a+bn)) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| > 1, |Re(x)| < 1$ where $f(x) = \frac{e^{x(a+bn)}}{c^4 - x^4}$

EXAMPLE 7.15. Use the method in section (7) in [2] and eq. (7.17).

$$(7.18) \quad 4c^4 e^{bpx} x^{-p} \int_0^x \frac{e^{(a-bp)t} t^{-1+p}}{c^4 - t^4} dt = \sum_{n=0}^{\infty} \frac{(-1)^{-n} i^{-n} c^{-n} e^{(-1-i)c(a+bn)} (e^{-bx} x)^n}{(n+p)n!} \\ \times \left(i^n e^{ic(a+bn)} \Gamma(1+n, -c(a+bn)) + e^{c(a+bn)} \Gamma(1+n, -ic(a+bn)) \right) \\ + (-1)^n e^{(1+2i)c(a+bn)} \Gamma(1+n, ic(a+bn)) + (-1)^n i^n e^{(2+i)c(a+bn)} \Gamma(1+n, c(a+bn)) \Big)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| > 1, |Re(p)| < 1, |Re(x)| < 1$.

EXAMPLE 7.16. From [Wolfram,07.23.07.0003.01]

$$(7.19) \quad \int_0^{\infty} \frac{e^{(a-p)x} x^{-1+b} {}_1F_1(a; c; xz)}{(-1+bx)(-c^4+x^4)} dx \\ = \sum_{n=0}^{\infty} \frac{1}{4c^4 n!} \left((bn+p)^{-b-n} \Gamma(b+n) \left((-1)^{-n} c^{-n} e^{-c(a+bn)} \Gamma(1+n, -c(a+bn)) \right) \right. \\ \left. + (-1)^{-n} i^{-n} c^{-n} e^{-ic(a+bn)} \Gamma(1+n, -ic(a+bn)) + i^{-n} c^{-n} e^{ic(a+bn)} \Gamma(1+n, ic(a+bn)) \right. \\ \left. + c^{-n} e^{c(a+bn)} \Gamma(1+n, c(a+bn)) \right) {}_2F_1 \left(a, b+n; c; \frac{z}{bn+p} \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| > \pi, |Re(p)| > \pi, |Re(z)| > \pi$.

THEOREM 7.17.

$$(7.20) \quad \frac{(-d+x)^{1+b} \tan^{-1}(u-xz)}{(1-a)x-d} = \sum_{n=0}^{\infty} \frac{(-d)^{b+(-1+a)n} (x(-d+x)^{-a})^n \Gamma(1+b+an)}{2(1+u^2) \Gamma(1+n) \Gamma(2+b+(-1+a)n)} \\ \times (2(1+b+(-1+a)n) (1+u^2) \tan^{-1}(u) \\ + (-d)^n dnz \left((1-iu) \left(\frac{i-u}{dz} \right)^n \left(\frac{z}{-i+u} \right)^n {}_3F_2 \left(1, 1, 1-n; 2, 2+b-n+an; \frac{dz}{-i+u} \right) \right. \\ \left. + (1+iu) \left(\frac{-i+u}{dz} \right)^n \left(\frac{z}{i+u} \right)^n {}_3F_2 \left(1, 1, 1-n; 2, 2+b-n+an; \frac{dz}{i+u} \right) \right) \Big)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(d)| < 1, |Re(u)| < 1, |Re(z)| < 1, |Re(x)| < 1$ where
 $f(x) = \tan^{-1}(u - xz)(x - d)^{an+b}$

EXAMPLE 7.18.

$$\begin{aligned}
 (7.21) \quad & \int_{-\phi}^{\phi} \frac{(-d+x)^{1+b} \tan^{-1}(u-xz)}{-d+(1-a)x} dx = - \sum_{n=0}^{\infty} \frac{(-d)^{-1+b+(-1+a)n} \phi \Gamma(1+b+an)}{2(1+u^2) \Gamma(2+n) \Gamma(2+b+(-1+a)n)} \\
 & \times \left((-(-d-\phi)^{-a} \phi)^n (d+\phi) {}_2F_1 \left(1, 2+n-an; 2+n; -\frac{\phi}{d} \right) \right. \\
 & \left. + (d-\phi) (\phi(-d+\phi)^{-a})^n {}_2F_1 \left(1, 2+n-an; 2+n; \frac{\phi}{d} \right) \right) \\
 & (2(1+b+(-1+a)n) (1+u^2) \tan^{-1}(u) \\
 & + (-d)^n dnz \left((1-iu) \left(\frac{i-u}{dz} \right)^n \left(\frac{z}{-i+u} \right)^n {}_3F_2 \left(1, 1, 1-n; 2, 2+b-n+an; \frac{dz}{-i+u} \right) \right. \\
 & \left. + (1+iu) \left(-\frac{i+u}{dz} \right)^n \left(\frac{z}{i+u} \right)^n {}_3F_2 \left(1, 1, 1-n; 2, 2+b-n+an; \frac{dz}{i+u} \right) \right) \Bigg)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(d)| < 1, |Re(u)| < 1, |Re(z)| < 1, |Re(\phi)| < 1$.

THEOREM 7.19.

$$\begin{aligned}
 (7.22) \quad & \frac{e^{ax}}{(1-bx)(1+cx)^2(1+dx)^2} \\
 & = \sum_{n=0}^{\infty} \frac{(e^{-bx}x)^n}{(c-d)^3} \times \left(\frac{(-c)^n c(c-d) e^{-\frac{a+bn}{c}} (a+bn) \Gamma(n, -\frac{a+bn}{c})}{\Gamma(n)} \right. \\
 & \quad + \frac{(-c)^{2+n} e^{-\frac{a+bn}{c}} (c(1+n) - d(3+n)) \Gamma(1+n, -\frac{a+bn}{c})}{\Gamma(1+n)} \\
 & \quad + (-d)^{2+n} e^{-\frac{a+bn}{d}} \left(\frac{(c-d)(a+bn) \Gamma(n, -\frac{a+bn}{d})}{d \Gamma(n)} \right. \\
 & \quad \left. \left. + \frac{(-d(1+n) + c(3+n)) \Gamma(1+n, -\frac{a+bn}{d})}{\Gamma(1+n)} \right) \right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(c)| < 1, |Re(d)| < 1, |Re(x)| < 1$ where $f(x) =$
 $\frac{e^{x(a+bn)}}{(cx+1)^2(dx+1)^2}$

THEOREM 7.20. *Series representation for quotient rational functions.*

$$\begin{aligned}
 (7.23) \quad & \frac{(1+x^m z)^{1+a}}{1+x^m(z-bmz)} - \frac{3+e^{im\pi}+4\lfloor \frac{1}{2}(-1+m) \rfloor}{2m} \\
 &= \sum_{n=1}^{\infty} \frac{(-z)^{n/m} \left(x(1+x^m z)^{-b} \right)^n \Gamma(-a+(-b+\frac{1}{m})n)}{2n\Gamma(\frac{n}{m})\Gamma(-a-bn)} \\
 &\quad \times \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{\pi(n+2n\lfloor \frac{1}{2}(-1+m) \rfloor)}{m}\right) \right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(x)| < 1, |Im(a)| < 1, |Im(b)| < 1, |Im(z)| < 1, |Re(x)| < 1, m \in \mathbb{Z}_+$ where $f(x) = (zx^m + 1)^{a+bn}$

EXAMPLE 7.21. From eq. (7.23) and the method in section (7) in [2].

$$\begin{aligned}
 (7.24) \quad & \sum_{n=1}^{\infty} \frac{(-z)^{n/m} \left(x(1+x^m z)^{-b} \right)^n \Gamma(-a+(-b+\frac{1}{m})n)}{2n(n+u)\Gamma(\frac{n}{m})\Gamma(-a-bn)} \\
 &\quad \times \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{\pi(n+2n\lfloor \frac{1}{2}(-1+m) \rfloor)}{m}\right) \right) \\
 &= -\frac{3+e^{im\pi}+4\lfloor \frac{1}{2}(-1+m) \rfloor - 2m(1+x^m z)^{bu} {}_2F_1\left(\frac{u}{m}, -a+bu; \frac{m+u}{m}; -x^m z\right)}{2mu}
 \end{aligned}$$

where $u \in \mathbb{C}$.

EXAMPLE 7.22.

$$\begin{aligned}
 (7.25) \quad & z^a \left(2z(-x^m z)^{-1-a} (B_{-x^m z}(1+a, 1+a) - B_{-x^m z}(2+a, 1+a)) \right. \\
 & \left. - \frac{z(3+e^{im\pi}+4\lfloor \frac{1}{2}(-1+m) \rfloor)}{(1+a)m} + \frac{(-1+bm)x^m z^2(3+e^{im\pi}+4\lfloor \frac{1}{2}(-1+m) \rfloor)}{(2+a)m} \right) \\
 &= \sum_{n=1}^{\infty} \frac{m(-z)^{n/m} z^{1+a} (1+x^m z)^{bn} \left(x(1+x^m z)^{-b} \right)^n \Gamma(-a+(-b+\frac{1}{m})n)}{n(m+am+n)\Gamma(\frac{n}{m})\Gamma(-a-bn)} \\
 &\quad \times \left(bm {}_2F_1\left(bn, 1+a+\frac{n}{m}; 2+a+\frac{n}{m}; -x^m z\right) \right. \\
 &\quad \left. + (1-bm) {}_2F_1\left(-1+bn, 1+a+\frac{n}{m}; 2+a+\frac{n}{m}; -x^m z\right) \right) \\
 &\quad \times \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{\pi(n+2n\lfloor \frac{1}{2}(-1+m) \rfloor)}{m}\right) \right)
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(x)| < 1$.

EXAMPLE 7.23.

(7.26)

$$\begin{aligned} \log(1+x^m z) &= \sum_{n=1}^{\infty} \frac{x^n (-z)^{n/m} (-1+(-1+bm)x^m z)^2 \Gamma(-a+(-b+\frac{1}{m})n)}{2n\Gamma(\frac{n}{m})\Gamma(-a-bn)(1+x^m z)^{1+a+bn}} \\ &\quad \times \left(\psi^{(0)}(-a-bn) - \psi^{(0)}\left(-a+\left(-b+\frac{1}{m}\right)n\right) \right) \\ &\quad \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{\pi(n+2n\lfloor\frac{1}{2}(-1+m)\rfloor)}{m}\right) \right) \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(x)| < 1$.

EXAMPLE 7.24.

$$\begin{aligned} (7.27) \quad \tanh^{-1}(x^m z) &= - \sum_{n=1}^{\infty} \frac{x^n}{4n\Gamma(\frac{n}{m})\Gamma(-a-bn)} \\ &\quad \times \left(-(-z)^{n/m} (1+x^m z)^{-1-a-bn} (-1+(-1+bm)x^m z)^2 \right. \\ &\quad \left. + z^{n/m} (1-x^m z)^{-1-a-bn} (1+(-1+bm)x^m z)^2 \right) \Gamma\left(-a+\left(-b+\frac{1}{m}\right)n\right) \\ &\quad \left(\psi^{(0)}(-a-bn) - \psi^{(0)}\left(-a+\left(-b+\frac{1}{m}\right)n\right) \right) \\ &\quad \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{\pi(n+2n\lfloor\frac{1}{2}(-1+m)\rfloor)}{m}\right) \right) \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(x)| < 1$.

EXAMPLE 7.25.

$$\begin{aligned} (7.28) \quad & \frac{z^s \left(-\frac{x^m z {}_2F_1(1,1+s;2+s;-x^m z)}{1+s} + \log(1+x^m z) \right)}{s} \\ &= - \sum_{n=1}^{\infty} \frac{mx^n (-z)^{n/m} z^s (1+x^m z)^{-a-bn}}{2n(n+ms)(n+m(1-a-bn+s))\Gamma(\frac{n}{m})\Gamma(1-a-bn)} \\ &\quad \times \Gamma\left(-a+\left(-b+\frac{1}{m}\right)n\right) \left((n+ms)(a-2abm+b^2m^2+bn-b^2mn+b^2m^2s+ \right. \\ &\quad \left. (-1+bm)^2(a+bn)x^m z) - m(a^2-a(1+2bms)+b((-1+bm)n+bm^2s(1+s))) \right) \end{aligned}$$

$$\begin{aligned}
& (1+x^m z) {}_2F_1\left(1, 1-a-bn+\frac{n}{m}+s; 1+\frac{n}{m}+s; -x^m z\right) \\
& \left(\psi^{(0)}(-a-bn) - \psi^{(0)}\left(-a+\left(-b+\frac{1}{m}\right)n\right)\right) \\
& \left((-1)^n + (-1)^{m+n} + 2 \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{n\pi(1+2\lfloor\frac{1}{2}(-1+m)\rfloor)}{m}\right)\right)
\end{aligned}$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(x)| < 1$.

8. Table of extended Bierens de Haan definite integrals

In this section we apply the method in [18] to the stated equation in [17] to derive extended definite integrals in terms of the Hurwitz-Lerch zeta function.

THEOREM 8.1. *From (35)7 in [17]*

$$\begin{aligned}
(8.1) \quad & \int_0^{\frac{\pi}{4}} \cot(x) \left(\cos^{-1+m}(2x) \log^k(a \cos(2x)) - \log^k(a \sec(2x)) \sec^m(2x) \right) dx \\
& = -\frac{1}{2}i\pi \left(2^{1+k} e^{2im\pi} (i\pi)^k \Phi\left(e^{2im\pi}, -k, 1 - \frac{i \log(a)}{2\pi}\right) + \log^k(a) \right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1$.

THEOREM 8.2. *From (35)8 in [17]*

$$\begin{aligned}
(8.2) \quad & \int_0^{\frac{\pi}{4}} \left(\cos^{-1+m}(2x) \log^k(a \cos(2x)) + \log^k(a \sec(2x)) \sec^m(2x) \right) \tan(x) dx \\
& = -i(2i)^k e^{im\pi} \pi^{1+k} \Phi\left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi}\right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1$.

THEOREM 8.3. *From (35)9 in [17]*

$$\begin{aligned}
(8.3) \quad & \int_0^{\frac{\pi}{4}} \left(\cot^m(x) \log^k(a \cot(x)) + \log^k(a \tan(x)) \tan^m(x) \right) dx \\
& = e^{\frac{1}{2}i(k+m)\pi} \pi^{1+k} \Phi\left(-e^{im\pi}, -k, \frac{1}{2} - \frac{i \log(a)}{\pi}\right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1$.

THEOREM 8.4. *From (35)10 in [17]*

$$\begin{aligned}
 (8.4) \quad & \int_0^{\frac{\pi}{4}} \sec(2x) \tan(x) \left(-\cot^m(x) \log^k(a \cot(x)) + \log^k(a \tan(x)) \tan^{-2+m}(x) \right) dx \\
 &= -\frac{1}{2} i \pi \left(2e^{im\pi} (i\pi)^k \Phi \left(e^{im\pi}, -k, 1 - \frac{i \log(a)}{\pi} \right) + \log^k(a) \right)
 \end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.5. *From (35)15 in [17]*

$$\begin{aligned}
 (8.5) \quad & \int_0^{\frac{\pi}{4}} \tan(x) \left(-\cot^m(x) \log^k(a \cot(x)) + \log^k(a \tan(x)) \tan^m(x) \right) dx \\
 &= \frac{a^{-m} (-m)^{-k} \Gamma(1+k, -m \log(a))}{m} + e^{\frac{im\pi}{2}} (i\pi)^{1+k} \Phi \left(e^{im\pi}, -k, \frac{1}{2} - \frac{i \log(a)}{\pi} \right)
 \end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.6. *From (35)18 in [17]*

$$\begin{aligned}
 (8.6) \quad & \int_0^{\frac{\pi}{4}} \cot \left(\frac{\pi}{4} + x \right) \left(\csc^m(2x) \log^k(a \csc(2x)) + \log^k(a \sin(2x)) \sin^{-1+m}(2x) \right) dx \\
 &= -i(2i)^k e^{im\pi} \pi^{1+k} \Phi \left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi} \right)
 \end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.7. *From (35)19 in [17]*

$$\begin{aligned}
 (8.7) \quad & - \int_0^{\frac{\pi}{4}} \cot \left(\frac{\pi}{4} + x \right) \left(\csc^m(2x) \log^k(a \csc(2x)) - \log^k(a \sin(2x)) \sin^m(2x) \right) dx \\
 &= \frac{1}{2} \left(\frac{a^{-m} (-m)^{-k} \Gamma(1+k, -m \log(a))}{m} + e^{im\pi} (2i\pi)^{1+k} \Phi \left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi} \right) \right)
 \end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.8. *From (35)22 in [17]*

$$\begin{aligned}
(8.8) \quad & \int_0^{\frac{\pi}{4}} \left(\cos^m(2x) \log^k(a \cos(2x)) - \log^k(a \sec(2x)) \sec^m(2x) \right) \tan(x) dx \\
&= \frac{1}{2} \left(\frac{a^{-m}(-m)^{-k} \Gamma(1+k, -m \log(a))}{m} + e^{im\pi} (2i\pi)^{1+k} \Phi \left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi} \right) \right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.9. *From (35)23 in [17]. Errata.*

$$(8.9) \quad \int_0^{\frac{\pi}{4}} \frac{(\tan^p(x) - \cot^p(x)) \tan(x)}{\cos(2x)} dx = -\frac{1}{p} + \frac{1}{2} \pi \cot \left(\frac{p\pi}{2} \right)$$

where $0 < Re(p) < \infty$.

THEOREM 8.10. *From (35)23 in [17]*

$$\begin{aligned}
(8.10) \quad & \int_0^{\frac{\pi}{4}} -\sec(2x) \tan(x) \left(\cot^m(x) \log^k(a \cot(x)) - \log^k(a \tan(x)) \tan^m(x) \right) dx \\
&= a^{-m} (-m)^{-1-k} \Gamma(1+k, -m \log(a)) \\
&\quad - \frac{1}{2} i \pi \left(2e^{im\pi} (i\pi)^k \Phi \left(e^{im\pi}, -k, 1 - \frac{i \log(a)}{\pi} \right) + \log^k(a) \right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.11. *From (35)24 in [17]*

$$\begin{aligned}
(8.11) \quad & \int_0^{\frac{\pi}{4}} \log^k \left(\frac{a \sin(x)}{\cos(x) - \sin(x)} \right) \sec^3(x) (\cos(x) - \sin(x))^{1-m} \sin^m(x) dx \\
&= \frac{e^{\frac{1}{2}i(k+2m)\pi}}{4\pi} \left(-2k\pi (-2^{1+k} m \pi^k + (2\pi)^k) \Phi \left(e^{2im\pi}, 1-k, \frac{\pi - i \log(a)}{2\pi} \right) \right. \\
&\quad \left. - i(2\pi)^k \left((-1+k)k \Phi \left(e^{2im\pi}, 2-k, \frac{\pi - i \log(a)}{2\pi} \right) \right. \right. \\
&\quad \left. \left. - 4(-1+m)m\pi^2 \Phi \left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi} \right) \right) \right)
\end{aligned}$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.12. *From (35)25 in [17]*

$$(8.12) \quad \int_0^{\frac{\pi}{4}} \sec(2x) \left(-\cot^{2m}(x) \log^k(a \cot^2(x)) + \log^k(a \tan^2(x)) \tan^{2m}(x) \right) dx$$

$$= 2^k e^{2im\pi} (i\pi)^{1+k} \Phi \left(-e^{2im\pi}, -k, 1 - \frac{i \log(a)}{2\pi} \right) - \frac{1}{2} i\pi \log^k(a)$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.13. *From (35)26 in [17]*

$$(8.13) \quad \int_0^{\frac{\pi}{4}} \left(\cos^{-1+m}(2x) \cot(x) \log^k(a \cos(2x)) - \cos^{-m}(2x) \cot(x) \log^k(a \sec(2x)) \right) dx$$

$$= -\frac{1}{2} i\pi \left(2^{1+k} e^{2im\pi} (i\pi)^k \Phi \left(e^{2im\pi}, -k, 1 - \frac{i \log(a)}{2\pi} \right) + \log^k(a) \right)$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.14. *From (35)27 in [17]*

$$(8.14) \quad \int_0^{\frac{\pi}{4}} \csc(2x) \log^k(a(-1 + \cot(x))) (\cos(x) - \sin(x))^m \sin^{-m}(x) dx$$

$$= 2^k e^{im\pi} (i\pi)^{1+k} \Phi \left(e^{2im\pi}, -k, \frac{\pi - i \log(a)}{2\pi} \right)$$

where $Re(a) < 0, 0 < Re(m) < 1, Im(m) \neq 0$.

THEOREM 8.15. *From (35)31 in [17]. Errata.*

$$(8.15) \quad \int_0^{\frac{\pi}{4}} \sin(2x) (\cot^{-p}(x) + \tan^{-p}(x)) dx = \frac{1}{2} p\pi \csc \left(\frac{p\pi}{2} \right)$$

where $0 < Re(p) < 1$.

THEOREM 8.16. *From (35)31 in [17].*

$$(8.16) \quad \int_0^{\frac{\pi}{4}} \sin(2x) \left(\cot^{-m}(x) \log^k(a \tan(x)) + \log^k(a \cot(x)) \tan^{-m}(x) \right) dx$$

$$= -e^{\frac{1}{2}i(k+m)\pi} \pi^k \left(k\Phi \left(e^{im\pi}, 1 - k, \frac{1}{2} - \frac{i \log(a)}{\pi} \right) + im\pi \Phi \left(e^{im\pi}, -k, \frac{1}{2} - \frac{i \log(a)}{\pi} \right) \right)$$

where $Re(a) < 0, 0 < Re(m) < 1$.

THEOREM 8.17. *From eq. (24)8 in [17]. Extended Jürgensen definite integral with examples.*

$$(8.17) \quad \int_q^\infty \frac{(x-q)^{m-1} \log^k(a(x-q))}{x-r} dx \\ = (2\pi i)^{k+1} (r-q)^{m-1} \Phi \left(e^{2im\pi}, -k, -\frac{i(i\pi + \log(a) + \log(q-r))}{2\pi} \right)$$

where $Re(q) > Re(r), Re(m) \geq 0, Re(a) < 0$.

EXAMPLE 8.18.

$$(8.18) \quad \int_q^\infty \frac{\log^k(a(-q+x))}{\sqrt{-q+x}(-r+x)} dx \\ = \frac{2^{1+2k} (i\pi)^{1+k} \left(-\zeta \left(-k, \frac{3\pi-i\log(a)-i\log(q-r)}{4\pi} \right) + \zeta \left(-k, -\frac{i(i\pi+\log(a)+\log(q-r))}{4\pi} \right) \right)}{\sqrt{-q+r}}$$

EXAMPLE 8.19.

$$(8.19) \quad \int_q^\infty \frac{\log(\log(a(-q+x)))}{\sqrt{-q+x}(-r+x)} dx = -\frac{(2i\pi) \log \left(\frac{(1-i)\Gamma\left(\frac{\pi-i\log(a)-i\log(q-r)}{4\pi}\right)}{2(\sqrt{2\pi}\Gamma\left(\frac{3\pi-i\log(a)-i\log(q-r)}{4\pi}\right))} \right)}{\sqrt{-q+r}}$$

where $Re(q) > Re(r)$.

EXAMPLE 8.20.

$$(8.20) \quad \int_q^\infty \frac{\log(\log(q-x))}{\sqrt{-q+x}(-r+x)} dx = -\frac{2i\pi \log \left(\frac{(\frac{1}{2}-\frac{i}{2})\Gamma\left(\frac{2\pi-i\log(q-r)}{4\pi}\right)}{\sqrt{2\pi}\Gamma\left(\frac{4\pi-i\log(q-r)}{4\pi}\right)} \right)}{\sqrt{-q+r}}$$

where $Re(q) > Re(r)$.

EXAMPLE 8.21.

$$(8.21) \quad \int_1^\infty \frac{\log(\log(1-x))}{\sqrt{-1+x}(-r+x)} dx = -\frac{2i\pi \log \left(\frac{(\frac{1}{2}-\frac{i}{2})\Gamma\left(\frac{2\pi-i\log(1-r)}{4\pi}\right)}{\sqrt{2\pi}\Gamma\left(\frac{4\pi-i\log(1-r)}{4\pi}\right)} \right)}{\sqrt{-1+r}}$$

where $Re(r) < 1$.

EXAMPLE 8.22.

$$(8.22) \quad \int_1^\infty \frac{\log(\log(1-x))}{\sqrt{-1+xx}} dx = -2\pi \log\left(\frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}}\right)$$

EXAMPLE 8.23.

$$(8.23) \quad \int_1^\infty \frac{\log(\log(1-x))}{\sqrt{-1+x}(1+x)} dx = -\sqrt{2}\pi \log\left(\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\Gamma\left(\frac{2\pi-i\log(2)}{4\pi}\right)}{\sqrt{2\pi}\Gamma\left(\frac{4\pi-i\log(2)}{4\pi}\right)}\right)$$

EXAMPLE 8.24.

$$(8.24) \quad \int_q^\infty \frac{\log(\log(a(-q+x)))}{(-q+x)^{3/4}(-r+x)} dx$$

$$= -\frac{(1-i)\pi \log(2i\pi)}{(-q+r)^{3/4}} - \frac{2i\pi \left(i \log\left(\frac{\Gamma\left(\frac{3\pi-i\log(a)-i\log(q-r)}{8\pi}\right)}{2\Gamma\left(\frac{7\pi-i\log(a)-i\log(q-r)}{8\pi}\right)}\right) + \log\left(\frac{\Gamma\left(-\frac{i(i\pi+\log(a)+\log(q-r))}{8\pi}\right)}{2\Gamma\left(\frac{5\pi-i\log(a)-i\log(q-r)}{8\pi}\right)}\right) \right)}{(-q+r)^{3/4}}$$

EXAMPLE 8.25. Double integral involving the elliptic integral of the first kind, Catalan's constant and π .

$$(8.25) \quad -\int_q^\infty \int_0^\phi \frac{1}{\sqrt{-q+x} \log^2\left(\frac{-q+x}{q+q^2 \sin^2(t)}\right) (x+q^2 \sin^2(t))} dt dx = \frac{2CF(\phi|-q)}{\pi\sqrt{q}}$$

where $q \in \mathbb{C}, \phi \in \mathbb{C}$.

EXAMPLE 8.26.

$$(8.26) \quad \int_q^\infty \frac{\log\left(\frac{q-x}{q-r}\right) \log\left(\log\left(\frac{q-x}{q-r}\right)\right)}{\sqrt{-q+x}(-r+x)} dx = \frac{4\pi^2 \log\left(-\frac{(-1)^{7/8} A^3}{2^{7/12} \sqrt[4]{e\pi}}\right)}{\sqrt{-q+r}}$$

EXAMPLE 8.27. Apéry's constant.

$$(8.27) \quad \int_q^\infty \frac{\log^2\left(\frac{q-x}{q-r}\right) \log\left(\log\left(\frac{q-x}{q-r}\right)\right)}{\sqrt{-q+x}(-r+x)} dx = \frac{14i\pi\zeta(3)}{\sqrt{-q+r}}$$

where $Re(q) > Re(r)$.

EXAMPLE 8.28.

$$(8.28) \quad \int_q^\infty \frac{\log\left(\frac{-q+x}{q-r}\right) \log\left(\log\left(\frac{-q+x}{q-r}\right)\right)}{\sqrt{-q+x}(-r+x)} dx = \frac{4C\pi}{\sqrt{-q+r}}$$

where $Re(q) > Re(r)$.

EXAMPLE 8.29. Similarity of two definite integrals.

$$(8.29) \quad \int_q^\infty \frac{\log\left(\frac{-q+x}{q-r}\right) \log\left(\log\left(\frac{-q+x}{q-r}\right)\right)}{\sqrt{-q+x}(-r+x)} dx = - \int_q^\infty \frac{2i\pi^2}{\sqrt{x-q}(r-x) \log^2\left(\frac{x-q}{q-r}\right)} dx$$

where $Im(r) > 0$.

EXAMPLE 8.30.

$$(8.30) \quad \int_q^\infty \frac{(x-q)^{m-1} \log^k\left(\frac{q-x}{q-r}\right)}{x-r} dx = e^{-2im\pi} (2i\pi)^{1+k} (r-q)^{m-1} \text{Li}_{-k}(e^{2im\pi})$$

where $Re(q) > Re(r)$.

EXAMPLE 8.31.

$$(8.31) \quad \int_q^\infty \frac{(-q+x)^{-1+m}}{(-r+x) \log\left(\frac{q-x}{q-r}\right)} dx = e^{-2im\pi} (r-q)^{-1+m} \log\left(\frac{1}{2}(i + \cot(m\pi))\right)$$

where $Re(q) > Re(r)$.

9. Extended Grobner integral in terms of a quadruple series involving special functions with examples

In 1950, Wolfgang Gröbner published one of his most famous and highly cited volume on definite integrals. In this section we use the method in [18] and apply it to the cited equation to derive extended forms of this integral. Special cases in terms of special functions and fundamental constants.

THEOREM 9.1. *From Eq. (311)12c in [19].*

$$\begin{aligned}
(9.1) \quad & \int_{-\infty}^{\infty} (-1)^{1+p} \alpha^{mx} (b + c\alpha^{nx})^{-p} (-1+p)! \log^k(a\alpha^x) dx \\
&= \sum_{q=0}^{\infty} \sum_{j=0}^{p-1} \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{|\log(\alpha)|} (-1)^{-j+l+p} b^{\frac{m}{n}-p} c^{-\frac{m}{n}} e^{\frac{im\pi}{n}} m^{-1-q} \left(2 + \frac{m}{n}\right)^{-h+j-l} \left(\frac{i}{n}\right)^{-h+k-q} \\
&\quad n^{-1-h} p^l (2\pi)^{-h+k-q} \binom{-1}{q} \binom{j}{l} \binom{j-l}{h} \\
&\quad \left(n\Phi \left(e^{\frac{2im\pi}{n}}, 1+h-k+q, \frac{\pi - in \log(a) - in \log\left(\left(\frac{b}{c}\right)^{1/n}\right)}{2\pi} \right) \right. \\
&\quad \left. (-h+k-q)_{1+h+q} + 2i(m+n-np)\pi \right. \\
&\quad \left. \Phi \left(e^{\frac{2im\pi}{n}}, h-k+q, \frac{\pi - in \log(a) - in \log\left(\left(\frac{b}{c}\right)^{1/n}\right)}{2\pi} \right) (1-h+k-q)_{h+q} \right) S_{-1+p}^{(j)}
\end{aligned}$$

where $Re(a) < 0, Re(m) < Re(n)$.

EXAMPLE 9.2. The case when $p = 1$.

$$\begin{aligned}
(9.2) \quad & \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log^k(a\alpha^x)}{b + c\alpha^{nx}} dx \\
&= \frac{b^{-1+\frac{m}{n}} c^{-\frac{m}{n}} e^{\frac{im\pi}{n}} \left(\frac{i}{n}\right)^{-1+k} (2\pi)^{1+k} \Phi \left(e^{\frac{2im\pi}{n}}, -k, \frac{\pi - in (\log(a) + \log\left(\left(\frac{b}{c}\right)^{1/n}\right))}{2\pi} \right)}{n^2 |\log(\alpha)|}
\end{aligned}$$

where $Re(a) < 0, Re(m) < Re(n)$.

EXAMPLE 9.3.

$$\begin{aligned}
(9.3) \quad & \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log^k(a\alpha^x)}{b + c\alpha^{2mx}} dx \\
&= \frac{1}{\sqrt{b}\sqrt{cm^2}|\log(\alpha)|} i \left(\frac{i}{m}\right)^{-1+k} \pi^{1+k} \left(2^k \zeta \left(-k, \frac{\pi - 2im (\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right))}{4\pi} \right) \right. \\
&\quad \left. - 2^k \zeta \left(-k, \frac{1}{2} \left(1 + \frac{\pi - 2im (\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right))}{2\pi} \right) \right) \right)
\end{aligned}$$

where $Re(a) < 0, Re(m) < Re(n)$.

EXAMPLE 9.4.

$$\begin{aligned}
(9.4) \quad & \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log(\log(a\alpha^x))}{b + c\alpha^{2mx}} dx \\
&= \frac{\pi}{2\sqrt{b}\sqrt{cm}|\log(\alpha)|} \left(\log\left(\frac{2i\pi}{m}\right) - 2\log\left(-3\pi - 2im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)\right) \right. \\
&\quad \left. + 2\log\left(-\pi - 2im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)\right) \right. \\
&\quad \left. - 2\log\Gamma\left(-\frac{3}{4} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)}{2\pi}\right) \right. \\
&\quad \left. + 2\log\Gamma\left(-\frac{1}{4} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)}{2\pi}\right) \right)
\end{aligned}$$

where $Re(a) < 0, Re(m) < Re(n)$.

EXAMPLE 9.5.

$$(9.5) \quad \gamma = 4\log\Gamma\left(-\frac{3}{4}\right) - 4\log\Gamma\left(-\frac{1}{4}\right) - \frac{\gamma_1\left(\frac{1}{4}\right)}{\pi} + \frac{\gamma_1\left(\frac{3}{4}\right)}{\pi} - 2\log\left(\frac{2\pi}{9}\right) + \log\left(\frac{\pi^3}{\Gamma\left(\frac{1}{4}\right)^4}\right)$$

EXAMPLE 9.6.

$$\begin{aligned}
(9.6) \quad & \frac{\pi \log(-\alpha) - \pi \log(\alpha) - 2\left(\gamma_1\left(\frac{1}{4}\right) - \gamma_1\left(\frac{3}{4}\right) + \pi(\gamma + \log(4))\right)}{2\alpha} \\
&= \frac{\pi\left(-4\log\Gamma\left(-\frac{3}{4}\right) + 4\log\Gamma\left(-\frac{1}{4}\right) + i\pi + 2\log\left(\frac{2\pi}{9}\right)\right)}{2|\alpha|}
\end{aligned}$$

where $Re(\alpha) < 1, Re(m) < 1$.

EXAMPLE 9.7.

$$\begin{aligned}
(9.7) \quad & \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log(\log(a\alpha^x))}{1 + \alpha^{2mx}} dx \\
&= \frac{\pi}{2m|\log(\alpha)|} \left(\log\left(\frac{2i\pi}{m}\right) - 2\log(-3\pi - 2im\log(a)) \right)
\end{aligned}$$

$$+2\log(-\pi - 2im\log(a)) + 2\log\Gamma\left(-\frac{\pi + 2im\log(a)}{4\pi}\right) - 2\log\Gamma\left(-\frac{3}{4} - \frac{im\log(a)}{2\pi}\right))$$

where $Re(a) < 0, Re(m) < Re(n)$.

EXAMPLE 9.8.

$$(9.8) \quad -\int_{-\infty}^{\infty} \frac{2^{-\frac{4x}{3}}}{\left(1 + 2^{-\frac{3x}{2}}\right)^2 \log(-2^{-x})} dx$$

$$= \sum_{q=0}^{\infty} \sum_{j=0}^1 \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{\log(2)} (-1)^{2-j+l} 2^{-3-2h+j-4q} 3^{1+2h-2j+2l+2q} 13^{-h+j-l} e^{\frac{8i\pi}{9}} (i\pi)^{-1-h-q}$$

$$\begin{aligned} & \binom{-1}{q} \binom{j}{l} \binom{j-l}{h} \left(\frac{3}{2} \Phi\left(e^{-\frac{1}{9}(2i\pi)}, 2+h+q, \frac{5}{4}\right) (-1-h-q)_{1+h+q} \right. \\ & \quad \left. - \frac{1}{3} i\pi \Phi\left(e^{-\frac{1}{9}(2i\pi)}, 1+h+q, \frac{5}{4}\right) (-h-q)_{h+q} \right) S_1^{(j)} \end{aligned}$$

EXAMPLE 9.9.

$$(9.9) \quad -\int_{-\infty}^{\infty} \frac{2^{-\frac{4x}{3}}}{(1 + 2^{-x})^2 \log^2(-2^{-x})} dx$$

$$= \sum_{q=0}^{\infty} \sum_{j=0}^1 \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{\log(2)} (-1)^{2-j+l} 2^{-4-2h+j-3q} 3^{1+h-j+l+q} 5^{-h+j-l} e^{-\frac{1}{3}(2i\pi)} (i\pi)^{-2-h-q}$$

$$\begin{aligned} & \binom{-1}{q} \binom{j}{l} \binom{j-l}{h} \left(\frac{2}{3} i e^{-\frac{1}{3}(2i\pi)} \pi (-1-h-q)_{h+q} \text{Li}_{2+h+q}\left(e^{\frac{2i\pi}{3}}\right) \right. \\ & \quad \left. + e^{-\frac{1}{3}(2i\pi)} (-2-h-q)_{1+h+q} \text{Li}_{3+h+q}\left(e^{\frac{2i\pi}{3}}\right) \right) S_1^{(j)} \end{aligned}$$

EXAMPLE 9.10.

$$(9.10) \quad \int_{-\infty}^{\infty} (-1)^{1+p} \alpha^{mx} (b + c\alpha^{2mx})^{-p} (-1+p)! \log^k(a\alpha^x) dx$$

$$= \sum_{q=0}^{\infty} \sum_{j=0}^{p-1} \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{\sqrt{c} |\log(\alpha)|} i (-1)^{-j+l+p} 2^{-1-j+l} 5^{-h+j-l} b^{\frac{1}{2}-p} \left(\frac{i}{m}\right)^{-h+k-q}$$

$$m^{-2-h-q} p^l \pi^{-h+k-q} \binom{-1}{q} \binom{j}{l} \binom{j-l}{h}$$

$$\left(2m \left(2^{-1-h+k-q} \zeta \left(1+h-k+q, \frac{\pi - 2im\log(a) - 2im\log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)}{4\pi} \right) \right) \right)$$

$$\begin{aligned}
& -2^{-1-h+k-q}\zeta\left(1+h-k+q,\frac{1}{2}\left(1+\frac{\pi-2im\log(a)-2im\log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)}{2\pi}\right)\right)\right) \\
& (-h+k-q)_{1+h+q}+2i(3m-2mp)\pi\left(2^{-h+k-q}\zeta\left(h-k+q,\frac{\pi-2im\log(a)-2im\log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)}{4\pi}\right)\right) \\
& -2^{-h+k-q}\zeta\left(h-k+q,\frac{1}{2}\left(1+\frac{\pi-2im\log(a)-2im\log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)}{2\pi}\right)\right)\right)(1-h+k-q)_{h+q}\Big)S_{-1+p}^{(j)}
\end{aligned}$$

EXAMPLE 9.11.

$$\begin{aligned}
(9.11) \quad & \int_{-\infty}^{\infty} \frac{(-1)^p \alpha^{mx} (1+\alpha^x)^{-p} (-1+p)!}{\log(-\alpha^x)} dx \\
& = \sum_{q=0}^{\infty} \sum_{j=0}^{p-1} \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{|\log(\alpha)|} (-1)^{-j+l+p} i^{-h-q} e^{-im\pi} m^{-1-q} (2+m)^{-h+j-l} p^l (2\pi)^{-1-h-q} \binom{-1}{q} \\
& \quad \binom{j}{l} \binom{j-l}{h} (-2(1+m-p)\pi(-h-q)_{h+q} \text{Li}_{1+h+q}(e^{2im\pi}) \\
& \quad + i(-1-h-q)_{1+h+q} \text{Li}_{2+h+q}(e^{2im\pi})) S_{-1+p}^{(j)}
\end{aligned}$$

EXAMPLE 9.12.

$$\begin{aligned}
(9.12) \quad & \int_{-\infty}^{\infty} \frac{(-1)^p 2^{-\frac{3x}{2}} (1+2^{-x})^{-p} (-1+p)!}{\log(-2^{-x})} dx \\
& = \sum_{q=0}^{\infty} \sum_{j=0}^{-1+p} \sum_{l=0}^j \sum_{h=0}^{j-l} \frac{1}{\log(2)} i (-1)^{-j+l+p} i^{-h-q} 2^{-j+l} 3^{-1-q} 7^{-h+j-l} p^l \pi^{-1-h-q} \binom{-1}{q} \binom{j}{l} \binom{j-l}{h} \\
& \quad S_{-1+p}^{(j)} \left(2(1-2^{-h-q}) \left(\frac{5}{2} - p \right) \pi(-h-q)_{h+q} \zeta(1+h+q) - i(1-2^{-1-h-q}) \right. \\
& \quad \left. (-1-h-q)_{1+h+q} \zeta(2+h+q) \right)
\end{aligned}$$

EXAMPLE 9.13.

$$(9.13) \quad \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log(\log(a\alpha^x))}{b+c\alpha^{4mx}} dx$$

$$\begin{aligned}
& \pi \left(\log \left(\frac{i\pi}{2m} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)} \right) \right) \right) \right) \\
& = \frac{\phantom{\pi \left(\log \left(\frac{i\pi}{2m} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{im \left(\log(a) + \log \left(\left(\frac{b}{c} \right)^{\frac{1}{4m}} \right) \right)}{2\pi} \right)} \right) \right) \right)}}{2\sqrt{2}b^{3/4}\sqrt[4]{cm}|\log(\alpha)|}
\end{aligned}$$

EXAMPLE 9.14.

$$\begin{aligned}
(9.14) \quad & \int_{-\infty}^{\infty} \frac{\alpha^x \log(\log(a\alpha^x))}{1 + \alpha^{4x}} dx \\
& = \frac{\pi \left(\log \left(\frac{i\pi}{2} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{i \log(a)}{2\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{i \log(a)}{2\pi} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{i \log(a)}{2\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{i \log(a)}{2\pi} \right)} \right) \right) \right)}{2\sqrt{2}|\log(\alpha)|}
\end{aligned}$$

EXAMPLE 9.15.

$$\begin{aligned}
(9.15) \quad & \int_{-\infty}^{\infty} \frac{\alpha^x \log(\log(\alpha^x))}{1 + \alpha^{4x}} dx = \frac{\pi \left(\log \left(\frac{i\pi}{2} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} \right)}{2\Gamma \left(\frac{5}{8} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} \right)}{2\Gamma \left(\frac{7}{8} \right)} \right) \right) \right)}{2\sqrt{2}|\log(\alpha)|}
\end{aligned}$$

EXAMPLE 9.16.

$$\begin{aligned}
(9.16) \quad & \int_{-\infty}^{\infty} \frac{\alpha^{mx} \log(\log(\alpha^x))}{b + 2\alpha^{4mx}} dx \\
& = \frac{\pi \left(\log \left(\frac{i\pi}{2m} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{im \log \left(2^{-\frac{1}{4m}} b^{\frac{1}{4m}} \right)}{2\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{im \log \left(2^{-\frac{1}{4m}} b^{\frac{1}{4m}} \right)}{2\pi} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{im \log \left(2^{-\frac{1}{4m}} b^{\frac{1}{4m}} \right)}{2\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{im \log \left(2^{-\frac{1}{4m}} b^{\frac{1}{4m}} \right)}{2\pi} \right)} \right) \right) \right) \right)}{2 \times 2^{3/4}b^{3/4}m|\log(\alpha)|}
\end{aligned}$$

EXAMPLE 9.17.

$$\begin{aligned}
(9.17) \quad & \int_{-\infty}^{\infty} \frac{\alpha^x \log(\log(a\alpha^x))}{1 + \alpha^{4x}} dx \\
& = \frac{\pi \left(\log \left(\frac{i\pi}{2} \right) - (1-i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{i \log(a)}{2\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{i \log(a)}{2\pi} \right)} \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{i \log(a)}{2\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{i \log(a)}{2\pi} \right)} \right) \right) \right)}{2\sqrt{2}|\log(\alpha)|}
\end{aligned}$$

EXAMPLE 9.18.

$$\begin{aligned}
(9.18) \quad & \int_{-\infty}^{\infty} \alpha^{mx} \left(\frac{1}{b + c\alpha^{2mx}} + \frac{1}{b + c\alpha^{4mx}} \right) \log(\log(a\alpha^x)) \, dx \\
&= \frac{\pi}{4b^{3/4}\sqrt{cm}|\log(\alpha)|} \left(\sqrt{2}\sqrt[4]{c} \left(\log\left(\frac{i\pi}{2m}\right) - (1-i) \left(\log\left(\frac{\Gamma\left(\frac{1}{8} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{4m}}\right)}\right)}{2\pi}\right)}{2\Gamma\left(\frac{5}{8} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{4m}}\right)}\right)}{2\pi}\right)}\right) \right. \right. \\
&\quad \left. \left. + i \log\left(\frac{\Gamma\left(\frac{3}{8} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{4m}}\right)}\right)}{2\pi}\right)}{2\Gamma\left(\frac{7}{8} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{4m}}\right)}\right)}{2\pi}\right)}\right) \right) \right) \right) \\
&\quad + 2\sqrt[4]{b} \left(\log\left(\frac{2i\pi}{m}\right) - 2\log\left(-3\pi - 2im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)\right) \right) \\
&\quad + 2\log\left(-\pi - 2im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)\right) \\
&\quad - 2\log\Gamma\left(-\frac{3}{4} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)}{2\pi}\right) \\
&\quad \left. + 2\log\Gamma\left(-\frac{1}{4} - \frac{im\left(\log(a) + \log\left(\left(\frac{b}{c}\right)^{\frac{1}{2m}}\right)\right)}{2\pi}\right) \right) \right)
\end{aligned}$$

EXAMPLE 9.19.

$$\begin{aligned}
(9.19) \quad & \int_{-\infty}^{\infty} \frac{\log(\log(ie^x)) (1 + (1-2i)\cosh(x) + (1+2i)\sinh(x))}{2(\cosh(\frac{x}{2}) + i\sinh(\frac{3x}{2}))} \, dx \\
&= \frac{(\frac{1}{4} + \frac{i}{4})}{\sqrt{2}} \pi \left(\left((-1-i) + \sqrt{2} \right) \csc\left(\frac{\pi}{8}\right) \left(\log(i\pi) + (1-i) \log\left(\frac{2\Gamma(\frac{11}{16})}{\Gamma(\frac{3}{16})}\right) \right. \right. \\
&\quad \left. \left. + (1+i) \log\left(\frac{2\Gamma(\frac{15}{16})}{\Gamma(\frac{7}{16})}\right) \right) + 4i \left(\log(9) - \log(4i\pi) + 2\log\Gamma\left(-\frac{3}{4}\right) - 2\log\Gamma\left(-\frac{1}{4}\right) \right) \right)
\end{aligned}$$

EXAMPLE 9.20.

$$(9.20) \quad \int_{-\infty}^{\infty} \frac{\alpha^{mx}}{(1 + \alpha^{2mx}) \log(a\alpha^x)} \, dx = \frac{i \left(\psi^{(0)}\left(\frac{\pi - 2im \log(a)}{4\pi}\right) - \psi^{(0)}\left(\frac{3}{4} - \frac{im \log(a)}{2\pi}\right) \right)}{2|\log(\alpha)|}$$

EXAMPLE 9.21.

(9.21)

$$\int_{-\infty}^{\infty} \frac{\alpha^{x/2} \log(-\alpha^x) \log(\log(-\alpha^x))}{1 + \alpha^x} dx = \frac{-\pi^3 - 8i\pi^2 \log\left(\frac{A^3}{\sqrt[3]{2}\sqrt[4]{e}}\right) + 2i\pi^2 \log(2\pi)}{2|\log(\alpha)|}$$

where $Re(\alpha) > 0$.

EXAMPLE 9.22.

$$(9.22) \quad \int_{-\infty}^{\infty} \frac{\alpha^{x/2} \log^2(-\alpha^x) \log(\log(-\alpha^x))}{1 + \alpha^x} dx = \frac{14\pi\zeta(3)}{|\log(\alpha)|}$$

where $Re(\alpha) > 0$.

EXAMPLE 9.23.

$$(9.23) \quad \int_{-\infty}^{\infty} \frac{\alpha^{x/2} \log(\alpha^x) \log(\log(\alpha^x))}{1 + \alpha^x} dx = -\frac{4iC\pi}{|\log(\alpha)|}$$

where $Re(\alpha) > 0$.

THEOREM 9.24. From Eq. (431)25a in [19].

$$(9.24) \quad \int_b^{\infty} \frac{(-b^\alpha + x^\alpha)^{-m} \log^k\left(\frac{a}{-b^\alpha + x^\alpha}\right)}{x} dx \\ = -\frac{b^{-m\alpha} e^{im\pi} (2i\pi)^{1+k} \Phi\left(e^{2im\pi}, -k, -\frac{i(i\pi + \log(a) + \log(b^{-\alpha}))}{2\pi}\right)}{\alpha}$$

where $0 < Re(m) < 1, 0 < Im(m) < 1, -1 < Im(b) < 0$.

THEOREM 9.25. From Eq. (151)12 in [19].

$$(9.25) \quad \int_0^1 \frac{x^{-1-m} \left(x^n \log^k\left(\frac{a}{x}\right) - x^{2m} \log^k(ax)\right)}{-1 + x^n} dx \\ = -\frac{i\pi \left(2^{1+k} e^{\frac{2im\pi}{n}} \left(\frac{i}{n}\right)^k \pi^k \Phi\left(e^{\frac{2im\pi}{n}}, -k, 1 - \frac{in \log(a)}{2\pi}\right) + \log^k(a)\right)}{n}$$

where $0 < Re(m) < 1$.

EXAMPLE 9.26.

(9.26)

$$\int_0^1 \frac{x^{-1-m} (x^n - (-1)^k x^{2m}) \log^k(x)}{x^n - 1} dx = (-1)^{k+1} \left(\frac{2\pi i}{n} \right)^{k+1} \text{Li}_{-k} \left(e^{\frac{2im\pi}{n}} \right)$$

where $\text{Re}(k) > 0, 0 < \text{Re}(m) < 1$.

EXAMPLE 9.27.

(9.27)

$$\int_0^1 \frac{x^{m-1} \log(x)}{(x^{2m} - 1)(a^2 \pi^2 + \log^2(x))} dx = \frac{1}{4am} - \frac{1}{4} \psi^{(0)} \left(1 + \frac{am}{2} \right) + \frac{1}{4} \psi^{(0)} \left(\frac{1}{2}(1 + am) \right)$$

where $0 < \text{Re}(m) < 1$.

EXAMPLE 9.28.

$$\begin{aligned} (9.28) \quad & \int_0^1 \frac{x^{m-1}}{x^{2m} - 1} \left(\frac{1}{(a\pi i - \log(x))^{n+1}} - \frac{1}{(a\pi i + \log(x))^{n+1}} \right) dx \\ &= \frac{2^{-1-n} a^{-1-n} (i\pi)^{-n} (2^n (-1)^{(n)} + a^{1+n} m^{1+n} (-\psi^{(n)}(1 + \frac{am}{2}) + \psi^{(n)}(\frac{1}{2}(1 + am))))}{mn!} \end{aligned}$$

where $0 < \text{Re}(m) < 1, n \in \mathbb{Z}_+$.

10. Table of series and definite logarithmic integrals

In this section the Lagrange theorem is used to derive a series representation for the logarithm function. Algebraic and calculus techniques are then applied to evaluate the series and integrals in terms of special functions that follow.

THEOREM 10.1.

(10.1)

$$\log(1 + x^m) = \sum_{n=1}^{\infty} \frac{(-1)^{-n} x^n \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m} \right) \right)}{2n}$$

where $| \text{Re}(x) | < 1, m \in \mathbb{Z}_+$ where $f(x) = \log(1 + x^m)$.

EXAMPLE 10.2.

$$\begin{aligned}
 (10.2) \quad & \int_0^\infty x^{-1+s} \log(1 + a^2 e^{-2mx} - 2ae^{-mx} \cos(t)) \, dx \\
 &= \Gamma(s) \sum_{n=1}^\infty (-1)^{-\frac{(1+m)n}{m}} a^{n/m} n^{-1-s} \cos\left(\frac{nt}{m}\right) \\
 &\quad \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)
 \end{aligned}$$

where $0 < \operatorname{Re}(s) < 1, \operatorname{Im}(a) \leq 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.3. From Eq. (6.5.28) in [20]. A note on an Erdeyli integral.

(Errata)

$$(10.3) \quad \int_0^\infty x^{-1+s} \log(1 + a^2 e^{-2x} - 2ae^{-x} \cos(t)) \, dx = -2\Gamma(s) \sum_{n=1}^\infty \frac{a^n \cos(nt)}{n^{s+1}}$$

where $0 < \operatorname{Re}(s) < 1, \operatorname{Im}(a) \leq 0$.

EXAMPLE 10.4. From (4.223.3) in [5]. Errata

$$(10.4) \quad \int_0^\infty \log(1 + e^{-2x} - 2e^{-x} \cos(t)) \, dx = -\operatorname{Li}_2(e^{-it}) - \operatorname{Li}_2(e^{it})$$

where $|\operatorname{Re}(x)| < \pi$.

EXAMPLE 10.5. From Eq. (6.4.23) in [20]

$$\begin{aligned}
 (10.5) \quad & \int_0^1 x^{-1+s} (1 + bx)^{-v} \log(1 + ax^m) \, dx \\
 &= \sum_{n=1}^\infty \frac{(-1)^{-n} a^{n/m} {}_2F_1(n + s, v; 1 + n + s; -b)}{2n(n + s)} \\
 &\quad \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)
 \end{aligned}$$

where $|\operatorname{Re}(a)| < 1, |\operatorname{Re}(b)| < 1, |\operatorname{Re}(x)| < 1, |\operatorname{Re}(v)| > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.6. From Eq. (2.6.9.2) in [3]

$$\begin{aligned}
(10.6) \quad & \frac{b^p \left(-ab^m m {}_2F_1 \left(1, \frac{m+p}{m}; 2 + \frac{p}{m}; -ab^m \right) + (m+p) \log(1+ab^m) \right)}{p(m+p)} \\
& = \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} b^{n+p} \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m} \right) \right)}{2n(n+p)}
\end{aligned}$$

where $|Re(a)| < 1, Re(p) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.7. From Eq. (3.387.1) in [5].

$$\begin{aligned}
(10.7) \quad & \int_0^{\infty} e^{-x} x^b \log(1+ax^m) dx \\
& = \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} \Gamma(1+b+n) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m} \right) \right)}{2n}
\end{aligned}$$

where $|Re(a)| < 1, Re(p) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.8. From Eq. (3.387.2) in [5].

$$\begin{aligned}
(10.8) \quad & \int_{-1}^1 (1-x^2)^{v-1} \log(1+a^2 e^{-2imx} - 2ae^{-imx} \cos(t)) dx \\
& = \sum_{n=1}^{\infty} 2^{-\frac{1}{2}+v} \left(-\frac{1}{a} \right)^{-\frac{n}{m}} e^{-in\pi} n^{-1+\frac{1}{2}(1-2v)} \sqrt{\pi} J_{\frac{1}{2}(-1+2v)}(n) \cos\left(\frac{nt}{m}\right) \Gamma(v) \\
& \quad \left(-1 + e^{im\pi} - 2e^{in\pi} \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m} \right) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(v)| < 1$.

EXAMPLE 10.9. From Eq. (3.387.3) in [5].

$$\begin{aligned}
(10.9) \quad & \int_{-1}^1 (1-x^2)^{v-1} \log(1+a^2 e^{-2imx} - 2ae^{-imx} \cos(t)) dx \\
& = \sum_{n=1}^{\infty} 2^{-\frac{1}{2}+v} \left(-\frac{1}{a} \right)^{-\frac{n}{m}} e^{-in\pi} n^{-1+\frac{1}{2}(1-2v)} \sqrt{\pi} J_{\frac{1}{2}(-1+2v)}(n) \cos\left(\frac{nt}{m}\right) \Gamma(v) \\
& \quad \left(-1 + e^{im\pi} - 2e^{in\pi} \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m} \right) \right)
\end{aligned}$$

where $|Re(a)| < 1, Re(p) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.10. From Eq. (4.293.14) in [5].

$$(10.10) \quad \int_0^1 (1-x)^{-1+v} x^{-1+u} \log(1+ax^m) dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} \Gamma(n+u) \Gamma(v) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)}{2n \Gamma(n+u+v)}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.11.

$$(10.11) \quad \int_0^1 \log^u \left(\frac{1}{x} \right) \log(1+ax^m) dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} (1+n)^{-1-u} \Gamma(1+u) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)}{2n}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.12. From (4.293.13(6)) in [5].

$$(10.12) \quad \int_0^1 \log^u \left(\frac{1}{x} \right) \log(1-x^2) dx = -\Gamma(1+u) \sum_{n=1}^{\infty} \frac{(-1)^{-n} (1+(-1)^n)}{n(1+n)^{u+1}}$$

where $Re(u) > \pi, |Im(u)| > \pi$.

EXAMPLE 10.13.

$$(10.13) \quad \int_0^1 x^{-1+u} (1+bx^c)^{-1+v} \log(1+ax^m) dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} {}_2F_1\left(\frac{n+u}{c}, 1-v; \frac{c+n+u}{c}; -b\right) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)}{2n(n+u)}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.14.

(10.14)

$$\frac{1}{4a^2} \left(4a - 2\pi + a^2\pi + 2\sqrt{1-a^2}\pi - ia^2\pi^2 - 4\sqrt{1-a^2}\sin^{-1}(a) + a^2\pi \log(4) \right)$$

$$\begin{aligned}
& -2a^2\pi \log\left(1 + \sqrt{1-a^2}\right) - 4a^2 \sin^{-1}(a) \log\left(-\frac{-1+ia+\sqrt{1-a^2}}{1+ia+\sqrt{1-a^2}}\right) \\
& -4ia^2\text{Li}_2\left(-ia-\sqrt{1-a^2}\right) + 4ia^2\text{Li}_2\left(ia+\sqrt{1-a^2}\right) \\
& = \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^n \sqrt{\pi} \Gamma\left(\frac{3}{2} + \frac{n}{2}\right)}{n(1+n)\Gamma\left(2 + \frac{n}{2}\right)}
\end{aligned}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.15. From (4.295.30(6)) in [5]. Generalized form.

$$(10.15) \quad \int_0^1 \frac{\log(1-(ax)^2)}{x\sqrt{1+bx^2}} dx = - \sum_{n=1}^{\infty} \frac{(-1)^{-n} (1+(-1)^n) a^n {}_2F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{2+n}{2}; -b\right)}{n^2}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.16. From Eq. (4.314.1(11)) in [5]. Extended Bierens de haan form.

$$\begin{aligned}
(10.16) \quad & \int_0^1 \frac{(x^{-1+p} - x^{-1+q}) \log(1+ax^m)}{\log(x)} dx \\
& = \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} \log\left(\frac{n+p}{n+q}\right) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right)\right)}{2n}
\end{aligned}$$

where $Re(a) < 1, Re(p) > 0, Re(q) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.17.

$$\begin{aligned}
(10.17) \quad & \int_0^1 \frac{\log(1+ax^m)}{x(q^2 + \log^2(x))} dx \\
& = \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^{n/m} \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right)\right)}{4nq} \\
& \quad (2\text{Ci}(nq) \sin(nq) + \cos(nq)(\pi - 2\text{Si}(nq)))
\end{aligned}$$

where $Re(a) < 1, Re(q) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.18. Table (4.316.1) in [5]. Derivation of generalized forms.

$$\begin{aligned}
(10.18) \quad & \int_0^1 \frac{\log^p\left(\frac{1}{x}\right) \log(1+ax^m)}{x} dx \\
&= \frac{\Gamma(1+p)}{2} \sum_{n=1}^{\infty} (-1)^{-n} a^{n/m} n^{-2-p} \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)
\end{aligned}$$

where $Re(a) < 1, Re(p) > 0, m \in \mathbb{Z}_+$.

EXAMPLE 10.19. Table (4.316.2) in [5]. Derivation of generalized forms.

$$\begin{aligned}
(10.19) \quad & \int_0^1 \frac{\log^p\left(\frac{1}{x}\right) \log(1+a^2x^{2m}-2ax^m \cos(t))}{x} dx \\
&= \sum_{n=1}^{\infty} \left(-\frac{1}{a}\right)^{-\frac{n}{m}} e^{-in\pi} n^{-2-p} \cos\left(\frac{nt}{m}\right) \Gamma(1+p) \\
&\quad \left(-1 + \cos(m\pi) + i \sin(m\pi) - 2e^{in\pi} \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(p)| < 1$.

EXAMPLE 10.20. From Eq. (1.5.13) and (1.5.14) in [20]

$$\begin{aligned}
(10.20) \quad & \int_0^{\infty} x^{-1+s} \log(1+a(e^{-bx})^m) dx \\
&= \sum_{n=1}^{\infty} \frac{1}{2} (-1)^{-n} a^{n/m} b^{-s} n^{-1-s} \Gamma(s) \left(-1 + (-1)^m - 2(-1)^n \csc\left(\frac{n\pi}{m}\right) \sin\left(\frac{2n\pi \lfloor \frac{m}{2} \rfloor}{m}\right) \right)
\end{aligned}$$

where $|Re(a)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.21. Mellin transform. Generalized Brychkov form Eq. (2.5.7.7) and (2.5.7.8) in [16].

$$(10.21) \quad \int_0^{\infty} x^{s-1} \log(1+ae^{-bx}) dx = -b^{-s} \Gamma(s) \text{Li}_{1+s}(-a)$$

where $|Re(a)| < 1, |Re(b)| < 1$.

EXAMPLE 10.22.

$$(10.22) \quad \int_0^1 x^{-1+s} \log(1+ae^{-bx^\alpha}) dx = - \sum_{n=1}^{\infty} \frac{(-1)^{-n} a^n (bn)^{-\frac{s}{\alpha}} \left(\Gamma\left(\frac{s}{\alpha}\right) - \Gamma\left(\frac{s}{\alpha}, bn\right) \right)}{n\alpha}$$

where $|Re(a)| < 1, |Re(b)| < 1$.

THEOREM 10.23.

(10.23)

$$\frac{e^{ax}}{(1-bx)(1+x)_m} = \sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} (e^{-bx}x)^n \binom{m}{j} \Gamma(1+n, -j(a+bn))}{m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, m \in \mathbb{Z}_+$, where $f(x) = \frac{e^{x(a+bn)}}{(x+1)_m}$

EXAMPLE 10.24.

$$(10.24) \quad -\frac{x\Gamma(x)^2}{\Gamma(-m+x)\Gamma(1+m+x)} = \frac{(m-x)(1-m+x)_m}{x(1+x)_m} \\ = \frac{\sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} (e^{-bx}x)^n \binom{m}{j} \Gamma(1+n, -j(a+bn))}{m! n!}}{\sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{j+m} e^{j(a+bn)} (a+bn)^{1+n} (e^{-bx}x)^n \binom{-1+m}{-1+j} E_{-n}(j(a+bn))}{\Gamma(m)\Gamma(1+n)}}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.25.

$$(10.25) \quad \int_0^{\infty} \frac{e^{(a-c)x} {}_1F_1(\alpha; \beta; xz)}{(1-bx)(1+x)_m} dx \\ = \sum_{n=0}^{\infty} \sum_{j=1}^m -\frac{(-1)^{j-n} e^{-j(a+bn)} j^{-n} (c+bn)^{-1-n} \binom{m}{j} \Gamma(1+n) \Gamma(1+n, -j(a+bn))}{m! n!} \\ {}_2F_1\left(1+n, \alpha; \beta; \frac{z}{c+bn}\right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(c)| > 2\pi, m \in \mathbb{Z}_+$.

EXAMPLE 10.26.

$$(10.26) \quad \int_{-1}^1 \frac{e^{-x+ax} (1-x^2)^{-\frac{1}{2}+v}}{(1-bx)(1+x)_m} dx \\ = \sum_{n=0}^{\infty} \sum_{j=1}^m -\frac{(-1)^{j-n} e^{-j(a+bn)} j^{-n} \sqrt{\pi} \binom{m}{j} \Gamma\left(\frac{1}{2}+v\right) \Gamma(1+n, -j(a+bn))}{4m! n!} \\ \left(2(1+(-1)^n) \Gamma\left(\frac{1+n}{2}\right) {}_1\tilde{F}_2\left(\frac{1+n}{2}; \frac{1}{2}, 1+\frac{n}{2}+v; \frac{1}{4}(1+bn)^2\right)\right)$$

$$+ (-1 + (-1)^n) (1 + bn) \Gamma \left(1 + \frac{n}{2} \right) {}_1\tilde{F}_2 \left(\frac{2+n}{2}; \frac{3}{2}, \frac{3+n}{2} + v; \frac{1}{4}(1+bn)^2 \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1, |Re(v)| > 3\pi, m \in \mathbb{Z}_+$.

EXAMPLE 10.27. From Eq. (4.21.2.1) in [16] pp.267. The complete elliptic integral of the first kind.

$$(10.27) \quad \int_0^1 \frac{e^{ax} x^{-\frac{3}{2}+s} K\left(-\frac{1-x}{x}\right)}{(1-bx)(1+x)_m} dx$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} \pi \binom{m}{j} \Gamma(n+s)^2 \Gamma(1+n, -j(a+bn))}{2m! n! \Gamma\left(\frac{1}{2} + n + s\right)^2}$$

$$\times {}_2F_2 \left(n+s, n+s; \frac{1}{2} + n + s, \frac{1}{2} + n + s; -bn \right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.28. From Eq. (4.22.2.1) in [16] pp. 283. The complete elliptic integral.

$$(10.28) \quad \int_0^1 \frac{e^{2ax} x^{-1+s} E(1-x)}{(1-bx)(1+x)_m} dx$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} \pi \binom{m}{j} \Gamma(n+s) \Gamma(1+n+s) \Gamma(1+n, -j(a+bn))}{2m! n! \Gamma\left(\frac{1}{2} + n + s\right) \Gamma\left(\frac{3}{2} + n + s\right)}$$

$$\times {}_2F_2 \left(n+s, 1+n+s; \frac{1}{2} + n + s, \frac{3}{2} + n + s; a - bn \right)$$

where $Re(s) > 0, |Re(a)| < 1, |Re(b)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.29.

$$(10.29) \quad \sum_{n=0}^{\infty} \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)-bnx} j^{-n} x^n \binom{m}{j} \Gamma(1+n, -j(a+bn))}{m! (n+1)!} = \frac{e^{bx}}{x} \int_0^x \frac{e^{at-bt}}{(1+t)_m} dt$$

where $|Re(a)| < 1, |Re(b)| < 1, m \in \mathbb{Z}_+$.

EXAMPLE 10.30. From Eq. (2.2.1.6) in [21]. The parabolic cylinder function.

$$\begin{aligned}
(10.30) \quad & \int_0^\infty \frac{e^{ax-x^2\alpha} x^{-1+s}}{(1-bx)(1+x)_m} dx \\
&= \sum_{n=0}^\infty \sum_{j=1}^m \frac{(-1)^{-1+j-n} 2^{\frac{1}{2}(-n-s)} e^{-j(a+bn)+\frac{b^2 n^2}{8\alpha}} j^{-n} \alpha^{\frac{1}{2}(-n-s)} \binom{m}{j} \Gamma(n+s) \Gamma(1+n, -j(a+bn))}{m! n!} \\
&\quad \times D_{-n-s} \left(\frac{bn}{\sqrt{2}\sqrt{\alpha}} \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(s)| < 1, Re(\alpha) > 2\pi$.

EXAMPLE 10.31. From Eq. (2.2.2.7) in [21]. The confluent hypergeometric function.

$$\begin{aligned}
(10.31) \quad & \int_0^\infty \frac{e^{ax} x^{-1+s} (x+\alpha)^{-\rho}}{(1-bx)(1+x)_m} dx \\
&= \sum_{n=0}^\infty \sum_{j=1}^m \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} \alpha^{n+s-\rho} \binom{m}{j} \Gamma(n+s) \Gamma(1+n, -j(a+bn))}{m! n!} \\
&\quad U(n+s, 1+n+s-\rho, bn\alpha)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(s)| < 1, Re(\rho) > 2\pi$.

EXAMPLE 10.32. From Eq. Eq. (3.2.23) in [20]. Exponential Fourier transform.

$$\begin{aligned}
(10.32) \quad & \int_{-\infty}^\infty \frac{e^{iax-x^2\alpha^2} (ix)^v}{(1-ibx)(1+ix)_m} dx \\
&= \sum_{n=0}^\infty \sum_{j=1}^m \frac{(-1)^{-1+j-n} 2^{\frac{1}{2}(-n-v)} e^{-j(a+bn)-\frac{b^2 n^2}{8\alpha^2}} j^{-n} \sqrt{\pi} \alpha^{-1-n-v} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{m! n!} \\
&\quad D_{n+v} \left(\frac{bn}{\sqrt{2}\alpha} \right)
\end{aligned}$$

where $Re(\alpha) > \pi$.

EXAMPLE 10.33. From Eq. (5.2.9.5) in [3].

$$(10.33) \quad \frac{e^{ax} \left(-1 + \frac{-1+e^{bx}}{bx} \right)}{(1-bx)(1+x)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} j^{-n} k^k (e^{-bx} x)^{k+n} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)! m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

THEOREM 10.34. *From Eq. (5.2.9.5) in [3]*

$$(10.34) \quad \frac{e^{ax} \left(-1 + \frac{-1+e^{bx}}{bx} \right)}{(1-bx)(1+x)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} j^{-n} k^k (e^{-bx} x)^{k+n} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)! m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

THEOREM 10.35. *From Eq. (5.2.9.6) in [3]*

$$(10.35) \quad \frac{e^{ax} (-1 + e^{bx}(1-bx))}{bx(1-bx)^2(1+x)_m}$$

$$= - \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} j^{-n} k^{1+k} (e^{-bx} x)^{k+n} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)! m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

THEOREM 10.36. *From Eq. (5.2.9.7) in [3]*

$$(10.36) \quad \frac{e^{ax} \left(-\frac{1}{2} + \frac{1-e^{2bx}-2bx+4be^{bx}x}{4b^2x^2} \right)}{(1-bx)(1+x)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} j^{-n} k^k (e^{-bx} x)^{k+n} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(2+k)! m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

THEOREM 10.37. *From Eq. (5.2.9.8) in [3]*

$$(10.37) \quad \frac{e^{ax} (1 - e^{2bx} + 2be^{bx}x)}{2b^2x^2(1-bx)(1+x)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} j^{-n} k^{1+k} (e^{-bx} x)^{k+n} \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(2+k)! m! n!}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(x)| < 1$.

THEOREM 10.38. *From Eq. (6.8.12.2) in [16].*

$$\begin{aligned}
 (10.38) \quad & \frac{e^{\frac{az}{2}} \left(\alpha^{-1-\frac{v}{2}} I_v(z\sqrt{\alpha}) - \frac{2^{-v} z^v}{\alpha \Gamma(1+v)} \right)}{\left(1 - \frac{bz}{2}\right) \left(1 + \frac{z}{2}\right)_m} \\
 &= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} 2^{-k-n} e^{-j(a+bn)} \left(e^{-\frac{1}{2}(bz)}\right)^n j^{-n} k^{\frac{1}{2}(-k-v)} z^{k+n} (k+\alpha)^{-1+k}}{k! m! n!} \\
 & \quad \times J_{k+v}(\sqrt{k}z) \binom{m}{j} \Gamma(1+n, -j(a+bn))
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.39. *From Eq. (6.8.12.3) in [16].*

$$\begin{aligned}
 (10.39) \quad & \frac{2^{-2-v} e^{\frac{az}{2}} z^{2+v}}{\left(1 - \frac{bz}{2}\right) \Gamma(2+v) \left(1 + \frac{z}{2}\right)_m} \\
 &= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=1}^{\infty} \frac{(-1)^{-1+j-n} 2^{-k-n} e^{-j(a+bn)} \left(e^{-\frac{1}{2}(bz)}\right)^n j^{-n} k^{-1+\frac{k-v}{2}} z^{k+n}}{k! m! n!} \\
 & \quad \times J_{k+v}(\sqrt{k}z) \binom{m}{j} \Gamma(1+n, -j(a+bn))
 \end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.40. *From Eq. (6.14.5.1) in [16].*

$$\begin{aligned}
 (10.40) \quad & \frac{e^{aw+bw} \left(-1 + \left(1 - \frac{bw}{z}\right)^{1+\lambda}\right) z}{bw(1-bw)(1+\lambda)(1+w)_m} \\
 &= - \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} j^{-n} (e^{-bw}w)^{k+n} \left(-\frac{b}{z}\right)^k \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)m! n!}
 \end{aligned}$$

$$L_k^{-k+\lambda}((1+k)z)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.41. *From Eq. (6.14.5.2) in [16].*

$$\begin{aligned}
(10.41) \quad & \frac{e^{aw+bw} I_1(2\sqrt{-bwz})}{(1-bw)\sqrt{-bwz}(1+w)_m} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} b^k e^{-j(a+bn)} (e^{-bw})^n j^{-n} (1+k)^{-1+k} w^n (e^{-bw} w)^k \binom{m}{j}}{k! m! n!} \\
&\quad \Gamma(1+n, -j(a+bn)) L_k \left(\frac{z}{1+k} \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.42. From eq. (6.14.7.1) in [16].

$$\begin{aligned}
(10.42) \quad & - \frac{e^{aw-bw} \left(1 - (-bwz)^{\frac{1-\lambda}{2}} J_{-1+\lambda}(2\sqrt{-bwz}) \Gamma(\lambda) \right) (\lambda)_{1+\eta}}{bw(1-bw)z\eta! (1+w)_m} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} (1+k)^{-1+k} w^n (-be^{bw} w)^k \binom{m}{j}}{m! n! (1+\eta+\lambda)_k} \\
&\quad \Gamma(1+n, -j(a+bn)) L_k^{\eta+\lambda} \left(\frac{z}{1+k} \right) L_{\eta}^{k+\lambda}(-b(1+k)w)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.43. From Eq. (6.14.7.2) in [16].

$$\begin{aligned}
(10.43) \quad & \frac{e^{w+aw} u (1 - {}_1F_2(\eta + \lambda; u, \lambda; -wz)) (\lambda)_{\eta}}{w(1-bw)z\eta! (1+w)_m} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} (1+k)^{-1+k} w^n (e^{-w} w)^k \binom{m}{j}}{m! n! (1+u)_k} \\
&\quad \Gamma(1+n, -j(a+bn)) L_k^u \left(\frac{z}{1+k} \right) L_{\eta}^{k+\lambda}((1+k)w)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.44. From Eq. (6.15.4.1) in [16].

$$\begin{aligned}
(10.44) \quad & \frac{2e^{-w+aw} z \lambda \left(1 - {}_1F_1\left(-\frac{1}{2} - \lambda; -2\lambda; -\frac{2w}{z}\right) \right)}{w(1-bw)(1+2\lambda)(1+w)_m} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} 2^{-k} e^{-j(a+bn)} (e^{-bw})^n j^{-n} w^n \left(\frac{e^w w}{z}\right)^k \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)m! n! (1-\lambda)_k}
\end{aligned}$$

$$C_k^{(-k+\lambda)}(1 + (1+k)z)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.45. From Eq. (6.15.4.2) in [16].

$$(10.45) \quad \frac{e^{-w+aw} (1 - {}_1F_1(-1 + \lambda; -\frac{1}{2}; wz^2))}{2w(1-bw)z^2(-1+\lambda)(1+w)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} (1+k)^{-1+k} w^n (-e^w w)^k \binom{m}{j}}{m! n! (1-\lambda)_k}$$

$$\Gamma(1+n, -j(a+bn)) C_{2k}^{(-k+\lambda)} \left(\frac{z}{\sqrt{1+k}} \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.46. From Eq. (6.15.4.3) in [16].

$$(10.46) \quad \frac{e^{-w+aw} (-1 + {}_1F_1(\lambda; \frac{1}{2}; wz^2))}{w(1-bw)z(1+w)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} (1+k)^{-\frac{1}{2}+k} w^n (-e^w w)^k \binom{m}{j}}{m! n! (1-\lambda)_k}$$

$$\Gamma(1+n, -j(a+bn)) C_{1+2k}^{(-k+\lambda)} \left(\frac{z}{\sqrt{1+k}} \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.47. From eq. (6.15.4.4) in [16].

$$(10.47) \quad \frac{e^{-w+aw} (1 - {}_1F_1(-\frac{1}{2} - \lambda; -\frac{1}{2}; w(1-z)))}{w(1-bw)(-1+z)(1+2\lambda)(1+w)_m}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} w^n (-e^w w)^k (k+z)^k \binom{m}{j} \Gamma(1+n, -j(a+bn))}{(1+k)m! n! (1-\lambda)_k}$$

$$C_{2k}^{(-k+\lambda)} \left(\sqrt{\frac{-1+z}{k+z}} \right)$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.48. (6.15.4.5) in [16].

$$\begin{aligned}
(10.48) \quad & \frac{2e^{-w+aw}\lambda\left(-1+{}_1F_1\left(-\frac{1}{2}-\lambda;\frac{1}{2};w(1-z)\right)\right)}{w(1-bw)\sqrt{-1+z}(1+2\lambda)(1+w)_m} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^m \sum_{k=0}^{\infty} \frac{(-1)^{-1+j-n} e^{-j(a+bn)} (e^{-bw})^n j^{-n} w^n (-e^w w)^k (k+z)^{\frac{1}{2}+k} \binom{m}{j}}{(1+k)m! n! (1-\lambda)_k} \\
&\quad \Gamma(1+n, -j(a+bn)) C_{1+2k}^{(-k+\lambda)} \left(\sqrt{\frac{-1+z}{k+z}} \right)
\end{aligned}$$

where $|Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1, |Re(\alpha)| < 1$.

THEOREM 10.49.

$$(10.49) \quad \frac{2x^2}{1-x^2(1+2z)} = \sum_{n=1}^{\infty} \frac{(1+(-1)^n) x^n (1-x^2)^{nz} \Gamma\left(\frac{1}{2}n(1+2z)\right)}{\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz)}$$

where $|Re(x)| < |Re(z)| < 1$ where $f(x) = (x^2 - 1)^{-mn}$.

EXAMPLE 10.50.

$$\begin{aligned}
(10.50) \quad & \int_0^1 \frac{e^{-ax} x^2 (1-x^2)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \left(\frac{(1+(-1)^n) \Gamma\left(\frac{1}{2} + \frac{n}{2}\right) \Gamma(1+b+nz) \Gamma\left(\frac{n}{2} + nz\right) {}_1F_2\left(\frac{1}{2} + \frac{n}{2}; \frac{1}{2}, \frac{3}{2} + b + \frac{n}{2} + nz; \frac{a^2}{4}\right)}{4\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz) \Gamma\left(\frac{3}{2} + b + \frac{n}{2} + nz\right)} \right. \\
&\quad \left. - \frac{(1+(-1)^n) an \Gamma(1+b+nz) \Gamma\left(\frac{n}{2} + nz\right) {}_1F_2\left(1 + \frac{n}{2}; \frac{3}{2}, 2 + b + \frac{n}{2} + nz; \frac{a^2}{4}\right)}{8\Gamma(1+nz) \Gamma\left(2 + b + \frac{n}{2} + nz\right)} \right)
\end{aligned}$$

where $3\pi < Re(b) < Re(1/z), Im(a) \geq 0$.

EXAMPLE 10.51.

$$\begin{aligned}
(10.51) \quad & \int_0^1 \frac{e^{-ax^2} x^2 (1-x^2)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \frac{(1+(-1)^n) \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(n\left(\frac{1}{2} + z\right)\right) \Gamma(1+b+nz) {}_1\tilde{F}_1\left(\frac{1+n}{2}; b + \frac{1}{2}(3+n+2nz); -a\right)}{4\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz)}
\end{aligned}$$

where $3\pi < Re(b) < Re(1/z), Im(a) \geq 0$.

EXAMPLE 10.52.

$$\begin{aligned}
(10.52) \quad & \int_0^1 \frac{2e^{-ax^3}(1-x)^b x^2(1+x)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \left(\frac{3^{-1-b-nz} (1+(-1)^n) \Gamma\left(\frac{1+n}{6}\right) \Gamma\left(\frac{3+n}{6}\right) \Gamma\left(\frac{5+n}{6}\right) \Gamma\left(n\left(\frac{1}{2}+z\right)\right) \Gamma(1+b+nz)}{2\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz) \Gamma\left(\frac{1}{6}(3+2b+n+2nz)\right) \Gamma\left(\frac{1}{6}(5+2b+n+2nz)\right) \Gamma\left(\frac{1}{6}(7+2b+n+2nz)\right)} \right. \\
&\quad \times {}_3F_4\left(\frac{1}{6} + \frac{n}{6}, \frac{1}{2} + \frac{n}{6}, \frac{5}{6} + \frac{n}{6}; \frac{1}{2}, \frac{1}{2} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{5}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{7}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}; \frac{a^2}{4}\right) \\
&\quad - \frac{3^{\frac{3}{2}+\frac{n}{2}} (1+(-1)^n) a \Gamma\left(\frac{4+n}{6}\right) \Gamma\left(\frac{6+n}{6}\right) \Gamma\left(\frac{8+n}{6}\right) \Gamma\left(n\left(\frac{1}{2}+z\right)\right) \Gamma(1+b+nz)}{4\pi \Gamma\left(\frac{n}{2}\right) \Gamma(1+nz) \Gamma\left(3+b+n\left(\frac{1}{2}+z\right)\right)} \\
&\quad \left. \times {}_3F_4\left(\frac{2}{3} + \frac{n}{6}, 1 + \frac{n}{6}, \frac{4}{3} + \frac{n}{6}; \frac{3}{2}, 1 + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{4}{3} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{5}{3} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}; \frac{a^2}{4}\right) \right)
\end{aligned}$$

where $3\pi < Re(b) < Re(1/z), Im(a) \geq 0$.

EXAMPLE 10.53.

$$\begin{aligned}
(10.53) \quad & \int_0^1 \frac{2e^{-ax^4}(1-x)^b x^2(1+x)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \frac{2^{-1+\frac{n}{2}+nz-\frac{1}{2}n(1+2z)} (1+(-1)^n) \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(n\left(\frac{1}{2}+z\right)\right) \Gamma(1+b+nz)}{\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz) \Gamma\left(b+\frac{1}{2}(3+n+2nz)\right)} \\
&\quad {}_2F_2\left(\frac{1}{4} + \frac{n}{4}, \frac{3}{4} + \frac{n}{4}; \frac{3}{4} + \frac{b}{2} + \frac{n}{4} + \frac{nz}{2}, \frac{5}{4} + \frac{b}{2} + \frac{n}{4} + \frac{nz}{2}; -a\right)
\end{aligned}$$

where $3\pi < Re(b) < Re(1/z), Im(a) \geq 0$.

EXAMPLE 10.54.

$$\begin{aligned}
(10.54) \quad & \int_{-1}^1 \frac{2e^{-ax^2} x^2 (1-x^2)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \frac{(1+(-1)^n)^2 \Gamma\left(\frac{1+n}{2}\right) \Gamma\left(n\left(\frac{1}{2}+z\right)\right) \Gamma(1+b+nz) {}_1\tilde{F}_1\left(\frac{1+n}{2}; b+\frac{1}{2}(3+n+2nz); -a\right)}{2\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz)}
\end{aligned}$$

where $3\pi < Re(b) < Re(1/z), Im(a) \geq 0$.

EXAMPLE 10.55.

$$\begin{aligned}
(10.55) \quad & \int_{-1}^1 \frac{2e^{-ax^3}(1-x)^b x^2(1+x)^b}{1-x^2(1+2z)} dx \\
&= \sum_{n=1}^{\infty} \left(\frac{3^{-1-b-nz} (1+(-1)^n)^2 \Gamma\left(\frac{1+n}{6}\right) \Gamma\left(\frac{3+n}{6}\right) \Gamma\left(\frac{5+n}{6}\right) \Gamma\left(n\left(\frac{1}{2}+z\right)\right) \Gamma(1+b+nz)}{2\Gamma\left(\frac{n}{2}\right) \Gamma(1+nz) \Gamma\left(\frac{1}{6}(3+2b+n+2nz)\right) \Gamma\left(\frac{1}{6}(5+2b+n+2nz)\right) \Gamma\left(\frac{1}{6}(7+2b+n+2nz)\right)} \right)
\end{aligned}$$

$$\begin{aligned}
& {}_3F_4 \left(\frac{1}{6} + \frac{n}{6}, \frac{1}{2} + \frac{n}{6}, \frac{5}{6} + \frac{n}{6}; \frac{1}{2}, \frac{1}{2} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{5}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{7}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}; \frac{a^2}{4} \right) \\
& + \frac{3^{\frac{3}{2} + \frac{n}{2}} (-1 + (-1)^n) (1 + (-1)^n) a \Gamma \left(\frac{4+n}{6} \right) \Gamma \left(\frac{6+n}{6} \right) \Gamma \left(\frac{8+n}{6} \right) \Gamma \left(n \left(\frac{1}{2} + z \right) \right) \Gamma(1 + b + nz)}{4\pi \Gamma \left(\frac{n}{2} \right) \Gamma(1 + nz) \Gamma \left(3 + b + n \left(\frac{1}{2} + z \right) \right)} \\
& {}_3F_4 \left(\frac{2}{3} + \frac{n}{6}, 1 + \frac{n}{6}, \frac{4}{3} + \frac{n}{6}; \frac{3}{2}, 1 + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{4}{3} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{5}{3} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}; \frac{a^2}{4} \right)
\end{aligned}$$

where $3\pi < \operatorname{Re}(b) < \operatorname{Re}(1/z)$, $\operatorname{Im}(a) \geq 0$.

EXAMPLE 10.56.

$$\begin{aligned}
(10.56) \quad & \int_{-1}^1 \frac{2e^{-ax^4} (1-x)^b x^2 (1+x)^b}{1-x^2(1+2z)} dx \\
& = \sum_{n=1}^{\infty} \frac{2^{-1+\frac{n}{2}+nz-\frac{1}{2}n(1+2z)} (1+(-1)^n)^2 \Gamma \left(\frac{1+n}{2} \right) \Gamma \left(n \left(\frac{1}{2} + z \right) \right) \Gamma(1 + b + nz)}{\Gamma \left(\frac{n}{2} \right) \Gamma(1 + nz) \Gamma \left(b + \frac{1}{2}(3 + n + 2nz) \right)} \\
& \quad {}_2F_2 \left(\frac{1}{4} + \frac{n}{4}, \frac{3}{4} + \frac{n}{4}; \frac{3}{4} + \frac{b}{2} + \frac{n}{4} + \frac{nz}{2}, \frac{5}{4} + \frac{b}{2} + \frac{n}{4} + \frac{nz}{2}; -a \right)
\end{aligned}$$

where $3\pi < \operatorname{Re}(b) < \operatorname{Re}(1/z)$, $\operatorname{Im}(a) \geq 0$.

EXAMPLE 10.57.

$$\begin{aligned}
(10.57) \quad & \int_{-1}^1 \frac{2e^{-ax^6} (1-x)^b x^2 (1+x)^b}{1-x^2(1+2z)} dx \\
& = \sum_{n=1}^{\infty} \frac{3^{-1-b-nz} (1+(-1)^n)^2 \Gamma \left(\frac{1+n}{6} \right) \Gamma \left(\frac{3+n}{6} \right) \Gamma \left(\frac{5+n}{6} \right) \Gamma \left(n \left(\frac{1}{2} + z \right) \right) \Gamma(1 + b + nz)}{2\Gamma \left(\frac{n}{2} \right) \Gamma(1 + nz) \Gamma \left(\frac{1}{6}(3 + 2b + n + 2nz) \right) \Gamma \left(\frac{1}{6}(5 + 2b + n + 2nz) \right) \Gamma \left(\frac{1}{6}(7 + 2b + n + 2nz) \right)} \\
& \quad {}_3F_3 \left(\frac{1}{6} + \frac{n}{6}, \frac{1}{2} + \frac{n}{6}, \frac{5}{6} + \frac{n}{6}; \frac{1}{2} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{5}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}, \frac{7}{6} + \frac{b}{3} + \frac{n}{6} + \frac{nz}{3}; -a \right)
\end{aligned}$$

where $3\pi < \operatorname{Re}(b) < \operatorname{Re}(1/z)$, $\operatorname{Im}(a) \geq 0$.

EXAMPLE 10.58.

$$\begin{aligned}
(10.58) \quad & \int_0^1 \frac{2x^{1+s} (1-x^2)^b \log(1-ax)}{1-x^2(1+2z)} dx \\
& = \sum_{n=1}^{\infty} \left(- \frac{(1+(-1)^n) a \Gamma \left(\frac{1}{2}(1 + n + s) \right) \Gamma \left(n \left(\frac{1}{2} + z \right) \right) \Gamma(1 + b + nz)}{2\Gamma \left(\frac{n}{2} \right) \Gamma(1 + nz) \Gamma \left(\frac{1}{2}(3 + 2b + n + s + 2nz) \right)} \right. \\
& \quad \left. {}_3F_2 \left(\frac{1}{2}, 1, \frac{1}{2} + \frac{n}{2} + \frac{s}{2}; \frac{3}{2}, \frac{3}{2} + b + \frac{n}{2} + \frac{s}{2} + nz; a^2 \right) \right)
\end{aligned}$$

$$-\frac{(1+(-1)^n)a^2\Gamma\left(\frac{1}{2}(2+n+s)\right)\Gamma\left(n\left(\frac{1}{2}+z\right)\right)\Gamma(1+b+nz)}{4\Gamma\left(\frac{n}{2}\right)\Gamma(1+nz)\Gamma\left(\frac{1}{2}(4+2b+n+s+2nz)\right)} \\ {}_3F_2\left(1, 1, 1+\frac{n}{2}+\frac{s}{2}; 2, 2+b+\frac{n}{2}+\frac{s}{2}+nz; a^2\right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1$.

EXAMPLE 10.59.

$$(10.59) \quad \int_0^1 \frac{\log(1+ax)(1-x^2)^b(2x^2)}{1-x^2(1+2z)} dx \\ = \sum_{n=1}^{\infty} \left(\frac{(1+(-1)^n)an\Gamma\left(n\left(\frac{1}{2}+z\right)\right)\Gamma(1+b+nz){}_3F_2\left(\frac{1}{2}, 1, 1+\frac{n}{2}; \frac{3}{2}, 2+b+\frac{n}{2}+nz; a^2\right)}{4\Gamma(1+nz)\Gamma\left(2+b+n\left(\frac{1}{2}+z\right)\right)} \right. \\ \left. - \frac{(1+(-1)^n)a^2\Gamma\left(\frac{3+n}{2}\right)\Gamma\left(n\left(\frac{1}{2}+z\right)\right)\Gamma(1+b+nz){}_3F_2\left(1, 1, \frac{3}{2}+\frac{n}{2}; 2, \frac{5}{2}+b+\frac{n}{2}+nz; a^2\right)}{4\Gamma\left(\frac{n}{2}\right)\Gamma(1+nz)\Gamma\left(b+\frac{1}{2}(5+n+2nz)\right)} \right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1$.

EXAMPLE 10.60.

$$(10.60) \quad \int_{-1}^1 \frac{2e^{-ax}(1-x)^bx^{1+s}(1+x)^b}{1-x^2(1+2z)} dx \\ = \sum_{n=1}^{\infty} \left(-\frac{(1+(-1)^n)(-1+(-1)^{n+s})\Gamma\left(\frac{n+s}{2}\right)\Gamma\left(n\left(\frac{1}{2}+z\right)\right)\Gamma(1+b+nz)}{2\Gamma\left(\frac{n}{2}\right)\Gamma(1+nz)\Gamma\left(\frac{1}{2}(2+2b+n+s+2nz)\right)} \right. \\ {}_1F_2\left(\frac{n}{2}+\frac{s}{2}; \frac{1}{2}, 1+b+\frac{n}{2}+\frac{s}{2}+nz; \frac{a^2}{4}\right) \\ \left. - \frac{(1+(-1)^n)(1+(-1)^{n+s})a\Gamma\left(\frac{1}{2}(1+n+s)\right)\Gamma\left(n\left(\frac{1}{2}+z\right)\right)\Gamma(1+b+nz)}{2\Gamma\left(\frac{n}{2}\right)\Gamma(1+nz)\Gamma\left(\frac{1}{2}(3+2b+n+s+2nz)\right)} \right. \\ \left. {}_1F_2\left(\frac{1}{2}+\frac{n}{2}+\frac{s}{2}; \frac{3}{2}, \frac{3}{2}+b+\frac{n}{2}+\frac{s}{2}+nz; \frac{a^2}{4}\right) \right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(b)| < 1, |Re(z)| < 1$.

THEOREM 10.61.

$$(10.61) \quad \frac{mpx^p}{1-(1+mp)x^p}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-mn} m (x(-1+x^p)^m)^n \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right) \right)}{2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)} \Gamma\left(\frac{n(1+mp)}{p}\right)$$

where $Re(x) > Re(m)$, where $f(x) = (x^p - 1)^{-mn}$

EXAMPLE 10.62.

$$(10.62) \quad \frac{px^{1+p} {}_2F_1\left(1, 1 + \frac{1}{p}; 2 + \frac{1}{p}; (1+mp)x^p\right)}{1+p}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-mn} x(1-x^p)^{-mn} (x(-1+x^p)^m)^n \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right) \right)}{2(1+n)\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)} \times \Gamma\left(n\left(m + \frac{1}{p}\right)\right) {}_2F_1\left(-mn, \frac{1+n}{p}; 1 + \frac{1+n}{p}; x^p\right)$$

where $Re(x) > Re(m)$.

EXAMPLE 10.63.

$$(10.63) \quad \frac{px^{p+s} {}_2F_1\left(1, \frac{p+s}{p}; 1 + \frac{p+s}{p}; (1+mp)x^p\right)}{p+s}$$

$$= \sum_{n=1}^{\infty} \frac{\left(x^{n+s} (1-x^p)^{-mn} (-1+x^p)^{mn} {}_2F_1\left(-mn, \frac{n+s}{p}; 1 + \frac{n+s}{p}; x^p\right)\right)}{(n+s) \left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)\right)} \times \left((-1)^{-mn} \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right) \right) \Gamma\left(\frac{n(1+mp)}{p}\right) \right)$$

where $Re(x) > Re(m)$.

EXAMPLE 10.64.

$$(10.64) \quad \frac{px^{p+s} (1-x^p)^{-a} (-1+x^p)^a {}_F_1\left(\frac{p+s}{p}; -a, 1; 1 + \frac{p+s}{p}; x^p, (1+mp)x^p\right)}{p+s}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{-mn} x^{n+s} (1-x^p)^{-a-mn} (-1+x^p)^{a+mn} \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right) \right)}{2(n+s)\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)}$$

$$\times \Gamma\left(n\left(m + \frac{1}{p}\right)\right) {}_2F_1\left(-a - mn, \frac{n+s}{p}; 1 + \frac{n+s}{p}; x^p\right)$$

where $Re(x) > Re(m), |Re(s)| < 1, Re(a) > 0$.

EXAMPLE 10.65.

$$\begin{aligned}
 (10.65) \quad & \int_{\alpha}^{\beta} \frac{px^{p+u}(-1+x^p)^s \log(ax)}{1-(1+mp)x^p} dx \\
 &= \sum_{n=1}^{\infty} \frac{(-1)^{-mn} (1-\alpha^p)^{-mn-s} (1-\beta^p)^{-mn-s}}{2(1+n+u)^2 \Gamma(1+mn) \Gamma\left(\frac{n}{p}\right)} \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right) \right) \\
 & \quad \Gamma\left(n\left(m + \frac{1}{p}\right)\right) \left(\alpha^{1+n+u} (-1+\alpha^p)^{mn+s} (1-\beta^p)^{mn+s} \right. \\
 & \quad {}_3F_2\left(-mn-s, \frac{1}{p} + \frac{n}{p} + \frac{u}{p}, \frac{1}{p} + \frac{n}{p} + \frac{u}{p}; 1 + \frac{1}{p} + \frac{n}{p} + \frac{u}{p}, 1 + \frac{1}{p} + \frac{n}{p} + \frac{u}{p}; \alpha^p \right) \\
 & \quad - (1-\alpha^p)^{mn+s} \beta^{1+n+u} (-1+\beta^p)^{mn+s} {}_3F_2\left(-mn-s, \frac{1}{p} + \frac{n}{p} + \frac{u}{p}, \frac{1}{p} + \frac{n}{p} + \frac{u}{p}; \right. \\
 & \quad \left. 1 + \frac{1}{p} + \frac{n}{p} + \frac{u}{p}, 1 + \frac{1}{p} + \frac{n}{p} + \frac{u}{p}; \beta^p \right) \\
 & \quad \left. - (1+n+u) \left(\alpha^{1+n+u} (-1+\alpha^p)^{mn+s} (1-\beta^p)^{mn+s} {}_2F_1\left(-mn-s, \frac{1+n+u}{p}; \frac{1+n+p+u}{p}; \alpha^p \right) \log(a\alpha) \right. \right. \\
 & \quad \left. \left. - (1-\alpha^p)^{mn+s} \beta^{1+n+u} (-1+\beta^p)^{mn+s} {}_2F_1\left(-mn-s, \frac{1+n+u}{p}; \frac{1+n+p+u}{p}; \beta^p \right) \log(a\beta) \right) \right)
 \end{aligned}$$

where $Re(x) > Re(m), |Re(s)| < 1, Re(a) > 0, Re(u) > 0, Re(s) > 0, Re(\alpha) > 0, Re(\beta) > 0$.

EXAMPLE 10.66.

$$\begin{aligned}
 (10.66) \quad & \frac{p}{p+s} \left(-\alpha^{p+s} (1-\alpha^p)^{-a} (-1+\alpha^p)^a {}_F_1\left(\frac{p+s}{p}; -a, 1; 2 + \frac{s}{p}; \alpha^p, (1+mp)\alpha^p \right) \right. \\
 & \quad \left. + \beta^{p+s} (1-\beta^p)^{-a} (-1+\beta^p)^a {}_F_1\left(\frac{p+s}{p}; -a, 1; 2 + \frac{s}{p}; \beta^p, (1+mp)\beta^p \right) \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{(n+s) \left(2\Gamma(1+mn) \Gamma\left(\frac{n}{p}\right) \right)} \left(\alpha^{n+s} (-1+\alpha^p)^{1+a+mn} \right. \\
 & \quad \left. {}_2F_1\left(1, \frac{n+p+ap+mn p+s}{p}; \frac{n+p+s}{p}; \alpha^p \right) \right)
 \end{aligned}$$

$$-\beta^{n+s}(-1+\beta^p)^{1+a+mn} {}_2F_1\left(1, \frac{n+p+ap+mp+s}{p}; \frac{n+p+s}{p}; \beta^p\right) \\ \left((-1)^{-mn} \left(2+(-1)^n+(-1)^{n+p}+4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right)\right) \Gamma\left(\frac{n(1+mp)}{p}\right)\right)$$

where $|Re(s)| < 1, |Re(a)| < 1, |Re(\alpha)| < 1, |Re(\beta)| < 1, |Re(m)| < |Re(\alpha)| < |Re(\beta)|$.

THEOREM 10.67.

$$(10.67) \quad \frac{apx^p}{1-ax^p-ampx^p} \\ = \sum_{n=1}^{\infty} \frac{a^{n/p} (x(1-ax^p)^m)^n \left(2+(-1)^n+(-1)^{n+p}+4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right)\right) \Gamma\left(\frac{n(1+mp)}{p}\right)}{2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)}$$

where $|Re(x)| < |Re(m)| < 1$ where $f(x) = (ax^p - 1)^{-mn}$.

EXAMPLE 10.68.

$$(10.68) \quad \frac{apx^{1+p} {}_2F_1\left(1, 1+\frac{1}{p}; 2+\frac{1}{p}; a(1+mp)x^p\right)}{1+p} \\ = \sum_{n=1}^{\infty} \frac{a^{n/p} \left(x^{1+n} {}_2F_1\left(-mn, \frac{1+n}{p}; 1+\frac{1+n}{p}; ax^p\right)\right)}{(1+n) \left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)\right)} \\ \times \left(2+(-1)^n+(-1)^{n+p}+4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos\left(\frac{2jn\pi}{p}\right)\right) \Gamma\left(\frac{n(1+mp)}{p}\right)$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.69.

$$(10.69) \quad \frac{apx^{1+p+s} F_1\left(\frac{1+p+s}{p}; -b, 1; \frac{1+2p+s}{p}; -cx^p, a(1+mp)x^p\right)}{1+p+s} \\ = \sum_{n=1}^{\infty} \frac{a^{n/p} \left(x^{1+n+s} F_1\left(\frac{1+n+s}{p}; -mn, -b; 1+\frac{1+n+s}{p}; ax^p, -cx^p\right)\right)}{(1+n+s) \left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right)\right)}$$

$$\times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right)$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.70.

$$(10.70) \quad - \frac{px^{-p}(a(1+mp)x^p + \log(1 - a(1+mp)x^p))}{(1+mp)^2}$$

$$= \sum_{n=1}^{\infty} \frac{\left(a^{\frac{n+p}{p}} px^n {}_2F_1 \left(-mn, \frac{n+p}{p}; 2 + \frac{n}{p}; ax^p \right) \right)}{(n+p) \left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right) \right)}$$

$$\times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right)$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.71.

$$(10.71) \quad - \frac{a^{1+s}p(-F_1(1+s; -t, 1; 2+s; -a, a(1+mp)x^p) + {}_2F_1(1+s, -t; 2+s; -a))}{(1+mp)(1+s)}$$

$$= \sum_{n=1}^{\infty} \frac{\left(a^{1+\frac{n}{p}+s} px^n F_1 \left(1 + \frac{n}{p} + s; -t, -mn; 2 + \frac{n}{p} + s; -a, ax^p \right) \right)}{(n+p+ps) \left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right) \right)}$$

$$\times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right)$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.72.

$$(10.72) \quad \frac{px^{-p}(-a(1+mp)x^p + \log(a)(a(1+mp)x^p + \log(1 - a(1+mp)x^p)) + \text{Li}_2(a(1+mp)x^p))}{(1+mp)^2}$$

$$= \sum_{n=1}^{\infty} \frac{\left(a^{\frac{n+p}{p}} px^n \left(\frac{{}_3F_2(-mn, 1+\frac{n}{p}, 1+\frac{n}{p}; 2+\frac{n}{p}, 2+\frac{n}{p}; ax^p)}{n+p} - {}_2F_1 \left(-mn, \frac{n+p}{p}; 2 + \frac{n}{p}; ax^p \right) \log(a) \right) \right)}{\left(2\Gamma(1+mn)\Gamma\left(\frac{n}{p}\right) \right) (n+p)}$$

$$\times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right)$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.73.

(10.73)

$$\begin{aligned} & \frac{a^{1+t} p (-F_1(1+t; -s, 1; 2+t; -az, a(1+mp)x^p) + {}_2F_1(-s, 1+t; 2+t; -az))}{(1+mp)(1+t)} \\ &= \sum_{n=1}^{\infty} - \frac{\left(a^{1+\frac{n}{p}+t} p x^n F_1 \left(1 + \frac{n}{p} + t; -mn, -s; 2 + \frac{n}{p} + t; ax^p, -az \right) \right)}{\left(2\Gamma(1+mn)\Gamma \left(\frac{n}{p} \right) \right) (n+p+pt)} \\ & \quad \times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right) \end{aligned}$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.74.

$$\begin{aligned} (10.74) \quad & \frac{2x^{-p} ((1+mp)x^p - \tanh^{-1}((1+mp)x^p))}{(1+mp)^2} \\ &= - \sum_{n=1}^{\infty} \frac{1}{\left(2\Gamma(1+mn)\Gamma \left(\frac{n}{p} \right) \right) (n+p)} \left(x^n \left((-1)^{n/p} {}_2F_1 \left(-mn, \frac{n+p}{p}; 2 + \frac{n}{p}; -x^p \right) \right. \right. \\ & \quad \left. \left. + \exp \left(\frac{i\pi \left(n - 2(n+p) \left\lceil \frac{\Im(p \log(x))}{2\pi} \right\rceil \right)}{p} \right) x^n (-x^p)^{-\frac{n}{p}} {}_2F_1 \left(-mn, \frac{n+p}{p}; 2 + \frac{n}{p}; x^p \right) \right) \right) \\ & \quad \times \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1+mp)}{p} \right) \end{aligned}$$

where $|Re(x)| < |Re(m)| < 1$.

EXAMPLE 10.75.

(10.75)

$$\frac{x^{-p}}{(1+mp)^2} ((1+mp)x^p (-2 + \log(-t) + \log(t)) - \log(-t) \log(1+x^p+mpx^p))$$

$$\begin{aligned}
& + \log(t) \log(1 - (1 + mp)x^p) - \text{Li}_2(-(1 + mp)x^p) + \text{Li}_2((1 + mp)x^p) \\
& = - \sum_{n=1}^{\infty} \frac{1}{\left(2\Gamma(1 + mn)\Gamma\left(\frac{n}{p}\right)\right)(n + p)^2} \\
& \left(x^n \left((-1)^{n/p} \left(-p {}_3F_2 \left(-mn, 1 + \frac{n}{p}, 1 + \frac{n}{p}; 2 + \frac{n}{p}, 2 + \frac{n}{p}; -x^p \right) \right. \right. \right. \\
& \quad \left. \left. + (n + p) {}_2F_1 \left(-mn, \frac{n + p}{p}; 2 + \frac{n}{p}; -x^p \right) \log(-t) \right) \right. \\
& \quad \left. + \exp \left(- \frac{i\pi \left(-n + 2(n + p) \left\lceil \frac{\Im(p \log(x))}{2\pi} \right\rceil \right)}{p} \right) \right. \\
& \quad \left. x^n (-x^p)^{-\frac{n}{p}} \left(-p {}_3F_2 \left(-mn, 1 + \frac{n}{p}, 1 + \frac{n}{p}; 2 + \frac{n}{p}, 2 + \frac{n}{p}; x^p \right) \right. \right. \\
& \quad \left. \left. + (n + p) {}_2F_1 \left(-mn, \frac{n + p}{p}; 2 + \frac{n}{p}; x^p \right) \log(t) \right) \right) \right) \\
& \left(2 + (-1)^n + (-1)^{n+p} + 4 \sum_{j=1}^{\lfloor \frac{p-1}{2} \rfloor} \cos \left(\frac{2jn\pi}{p} \right) \right) \Gamma \left(\frac{n(1 + mp)}{p} \right)
\end{aligned}$$

where $|Re(x)| < |Re(m)| < 1$.

THEOREM 10.76. *From Eq. (22) in [22] and the method in [18].*

$$\begin{aligned}
(10.76) \quad & \int_{-1}^1 e^{-a\pi(m-t+\sqrt{-1+t^2})} \left(\frac{\Gamma(1+k, -a\pi(m-t+\sqrt{-1+t^2}))}{\sqrt{-1+t^2}(-m+t-\sqrt{-1+t^2})^{k+1}} \right. \\
& \left. - \frac{e^{2a\pi\sqrt{-1+t^2}}\Gamma(1+k, a\pi(-m+t+\sqrt{-1+t^2}))}{\sqrt{-1+t^2}(-m+t+\sqrt{-1+t^2})^{k+1}} \right) \sinh(n\sqrt{1-t^2}\cos(nt)) dt \\
& = \frac{(\pi i)(\Gamma(1+k, -m(in+a\pi)) - \Gamma(1+k, im(n+ia\pi)))}{2(e^{am\pi}(-m)^{k+1})}
\end{aligned}$$

where $Re(m) < 0$.

11. Definite integrals in terms of special functions and asymptotic expansions

In this section we apply the method in [18] to the equations cited in the theorems and evaluate the examples that follow. In 1877, Russell [24] published one of

his many papers on definite integrals. In the paper he continued his previous work on this subject by producing a table involving trigonometric integrals.

THEOREM 11.1. *From Eq. (28) in [24].*

$$(11.1) \quad \int_0^1 \frac{u^{-1-m+n} \left(u^{2m} \log^k(au) - \log^k\left(\frac{a}{u}\right) \right)}{-1 + u^{2n}} du \\ = \frac{i\pi \left(-2e^{\frac{im\pi}{n}} \left(\frac{i}{n}\right)^k \pi^k \Phi\left(-e^{\frac{im\pi}{n}}, -k, 1 - \frac{in \log(a)}{\pi}\right) + \log^k(a) \right)}{2n}$$

where $Re(n) > Re(m)$.

EXAMPLE 11.2.

$$(11.2) \quad \int_0^1 \frac{x^{-1-m+v} \left(-(-1)^{-k} + x^{2m} \right) \log^k(x)}{-1 + x^{2v}} dx = \left(\frac{\pi i}{v} \right)^{1+k} \text{Li}_{-k} \left(-e^{\frac{im\pi}{v}} \right)$$

where $Re(v) > Re(m)$.

EXAMPLE 11.3.

$$(11.3) \quad \int_0^1 \frac{x^{-1-m+v} \left(-(-1)^{-k} + x^{2m} \right) \log^k(x)}{-1 + x^{2v}} dx \\ = (2v)^{-k-1} \Gamma(1+k) \left(\zeta \left(1+k, \frac{-m+v}{2v} \right) - i^{2k} \zeta \left(1+k, \frac{m+v}{2v} \right) \right)$$

where $Re(v) > Re(m)$.

EXAMPLE 11.4.

$$(11.4) \quad \int_0^1 \frac{x^{-1-m-s+v} \left(-x^m + x^s \right) (1 + x^{m+s}) \log(\log(x))}{-1 + x^{2v}} dx \\ = \frac{1}{2v} \left(\gamma_1 \left(\frac{-m+v}{2v} \right) - \gamma_1 \left(\frac{m+v}{2v} \right) - \gamma_1 \left(\frac{-s+v}{2v} \right) + \gamma_1 \left(\frac{s+v}{2v} \right) \right. \\ \left. + \pi(\gamma - i\pi + \log(2) + \log(v)) \left(\tan \left(\frac{m\pi}{2v} \right) - \tan \left(\frac{\pi s}{2v} \right) \right) \right)$$

where $Re(v) > Re(m)$.

EXAMPLE 11.5.

$$(11.5) \quad \int_0^1 \frac{u^{-1+n} (u^{-m} + u^m - u^{-v} - u^v)}{(1 - u^{2n}) \log(u)} du = \log \left(\frac{\cos \left(\frac{m\pi}{2n} \right)}{\cos \left(\frac{\pi v}{2n} \right)} \right)$$

where $\operatorname{Re}(n) > \operatorname{Re}(v) > \operatorname{Re}(m)$.

EXAMPLE 11.6.

$$(11.6) \quad \int_0^1 \frac{u^{-1-m+n} (u^{2m} - 1) \log(\log(u))}{1 - u^{2n}} du \\ = \frac{2\gamma_1 \left(\frac{-m+n}{2n} \right) - 2\gamma_1 \left(\frac{m+n}{2n} \right) + \pi \left(2\gamma - i\pi - 2\log \left(\frac{i}{2n} \right) \right) \tan \left(\frac{m\pi}{2n} \right)}{4n}$$

where $\operatorname{Re}(n) > \operatorname{Re}(v) > \operatorname{Re}(m)$.

EXAMPLE 11.7.

$$(11.7) \quad \int_0^1 \frac{\log(\pi^2 + \log^2(u))}{1 + u^2} du = \pi \log \left(\frac{\sqrt{2\pi} \Gamma \left(\frac{5}{4} \right)}{\Gamma \left(\frac{3}{4} \right)} \right)$$

EXAMPLE 11.8.

$$(11.8) \quad \int_0^1 \frac{x^{-1+2v} \log(a^2 \pi^2 + \log^2(x))}{1 + x^{4v}} dx = \frac{\pi}{4v} \log \left(\frac{\pi \Gamma \left(\frac{3}{4} + av \right)^2}{v \Gamma \left(\frac{1}{4} + av \right)^2} \right)$$

where $\operatorname{Im}(a) \leq 0$

EXAMPLE 11.9.

$$(11.9) \quad \int_0^1 \frac{x^{-1+v} \log(x)}{(-1 + x^{2v}) (\pi^2 + 4v^2 \log^2(x))^2} dx = \frac{-1 + 2C}{16\pi^2 v^2}$$

where $\operatorname{Re}(v) \geq 0$.

EXAMPLE 11.10.

$$(11.10) \quad \int_0^1 \frac{x^{-1+v} \left(i\pi \log \left(\frac{\pi - 2iv \log(x)}{\pi + 2iv \log(x)} \right) + 2v \log(x) \left(-i\pi + \log \left(\frac{\pi^2}{4v^2} + \log^2(x) \right) \right) \right)}{-1 + x^{2v}} dx \\ = -\frac{2C\pi}{v} - \frac{\pi^2 \log \left(-\frac{i\pi}{2v} \right)}{2v}$$

where $Re(v) \geq 0$.

THEOREM 11.11. *From Eq. (29) in [24].*

$$\begin{aligned}
 (11.11) \quad & \int_0^1 \frac{x^{-1-m+u} \left(-\log^k \left(\frac{a}{x} \right) + x^{2m} \log^k(ax) \right)}{-1+x^u} dx \\
 & = -a^{-m} (-m)^{-1-k} \Gamma(1+k, -m \log(a)) \\
 & \quad + \frac{i\pi \left(2^{1+k} e^{\frac{2im\pi}{u}} \pi^k \left(\frac{i}{u} \right)^k \Phi \left(e^{\frac{2im\pi}{u}}, -k, 1 - \frac{iu \log(a)}{2\pi} \right) + \log^k(a) \right)}{u}
 \end{aligned}$$

where $Re(u) > Re(m)$.

EXAMPLE 11.12.

$$\begin{aligned}
 (11.12) \quad & \int_0^1 \frac{x^{-1-m-s+u} (-x^m + x^s) ((-1)^k + x^{m+s}) \log^k(x)}{-1+x^u} dx \\
 & = (-1)^k (-m^{-1-k} + s^{-1-k}) \Gamma(1+k) + \frac{(2\pi)^{1+k} \left(\frac{i}{u} \right)^{-1+k} \left(\text{Li}_{-k} \left(e^{\frac{2im\pi}{u}} \right) - \text{Li}_{-k} \left(e^{\frac{2is\pi}{u}} \right) \right)}{u^2}
 \end{aligned}$$

where $Re(u) > Re(s) > Re(m)$.

EXAMPLE 11.13.

$$\begin{aligned}
 (11.13) \quad & \int_0^1 \frac{(-1)^{-k} x^{-1-m-s+u} (-x^m + x^s) (1 + (-1)^k x^{m+s}) \log^k(x)}{-1+x^u} dx \\
 & = (-1)^k m^{-1-k} s^{-1-k} (m^{1+k} - s^{1+k}) \Gamma(1+k) \\
 & \quad + u^{-1-k} \Gamma(1+k) \left(-\zeta \left(1+k, 1 - \frac{m}{u} \right) + \zeta \left(1+k, 1 - \frac{s}{u} \right) \right. \\
 & \quad \left. + i^{2k} \left(\zeta \left(1+k, \frac{m}{u} \right) - \zeta \left(1+k, \frac{s}{u} \right) \right) \right)
 \end{aligned}$$

where $Re(u) > Re(s) > Re(m)$.

EXAMPLE 11.14.

$$(11.14) \quad \int_0^1 \frac{x^{-1-m-s+u} (-x^m + x^s) (-1+x^{m+s})}{(-1+x^u) \log(x)} dx = \log \left(\frac{s \csc \left(\frac{\pi s}{u} \right) \sin \left(\frac{m\pi}{u} \right)}{m} \right)$$

where $Re(u) > Re(s) > Re(m)$.

EXAMPLE 11.15.

$$\begin{aligned}
 (11.15) \quad & \int_0^1 \frac{x^{-1-m-s+u} (-x^m + x^s) (1 + x^{m+s}) \log(\log(x))}{-1 + x^u} dx \\
 &= \frac{u}{msu} ((-(\gamma - i\pi)(m - s)) + s \log(m) - m \log(s)) \\
 &+ m\pi s \cot\left(\frac{m\pi}{u}\right) (-\gamma + i\pi - \log(u)) + m\pi s \cot\left(\frac{\pi s}{u}\right) (\gamma - i\pi + \log(u)) \\
 &+ ms \left(\gamma_1 \left(1 - \frac{m}{u}\right) - \gamma_1 \left(1 - \frac{s}{u}\right) - \gamma_1 \left(\frac{m}{u}\right) + \gamma_1 \left(\frac{s}{u}\right) \right)
 \end{aligned}$$

where $Re(u) > Re(s) > Re(m)$.

EXAMPLE 11.16.

$$(11.16) \quad \int_0^1 \frac{x^{-1+u} (-x^m + x^s)}{-1 + x^u} dx = \frac{-\psi^{(0)}\left(\frac{m+u}{u}\right) + \psi^{(0)}\left(\frac{s+u}{u}\right)}{u}$$

where $Re(u) > Re(s) > Re(m)$.

EXAMPLE 11.17.

$$\begin{aligned}
 (11.17) \quad & \int_0^1 \frac{x^{-1+\frac{3u}{2}} \log(a^2 \pi^2 + \log^2(x))}{1 + x^u} dx \\
 &= \frac{1}{u} \left(\frac{1}{2} i(-4 + \pi) \pi + 2e^{-\frac{1}{2}ia\pi u} \left(E_1 \left(-\frac{1}{2}ia\pi u \right) + e^{ia\pi u} E_1 \left(\frac{1}{2}ia\pi u \right) \right) + 4 \log(ia\pi) \right. \\
 &\quad \left. + 2\pi \log \left(\frac{\sqrt{u}\Gamma\left(\frac{1}{4}(1+au)\right)}{2\sqrt{\pi}\sqrt{i}\Gamma\left(\frac{1}{4}(3+au)\right)} \right) \right)
 \end{aligned}$$

where $Re(a) \geq 0$.

EXAMPLE 11.18.

$$(11.18) \quad \int_0^1 \frac{x^{-1+\frac{u}{2}} (-x^u (\pi - iu \log(x))^2 + (\pi + iu \log(x))^2)}{(-1 + x^u) (\pi^2 + u^2 \log^2(x))^2} dx = -\frac{i(4 - 4C + \pi\Gamma(0, \frac{i\pi}{2}))}{2\pi u}$$

where $Re(u) > 0$.

EXAMPLE 11.19.

$$\begin{aligned}
(11.19) \quad & \int_0^1 \frac{x^{-1+\frac{u}{2}} \left(\log(x) \log \left(\frac{a\pi+i\log(x)}{a\pi-i\log(x)} \right) + ia\pi \log(-a^2\pi^2 - \log^2(x)) \right)}{a^2\pi^2 + \log^2(x)} dx \\
& = e^{-\frac{1}{2}ia\pi u} \left(\Gamma \left(0, -\frac{1}{2}ia\pi u \right) \log(ia\pi) + G_{2,3}^{3,0} \left(-\frac{1}{2}ia\pi u \left| \begin{matrix} 1, 1 \\ 0, 0, 0 \end{matrix} \right. \right) \right. \\
& \quad \left. - e^{ia\pi u} \left(\Gamma \left(0, \frac{1}{2}ia\pi u \right) \log(ia\pi) + G_{2,3}^{3,0} \left(\frac{1}{2}ia\pi u \left| \begin{matrix} 1, 1 \\ 0, 0, 0 \end{matrix} \right. \right) \right) \right)
\end{aligned}$$

where $Re(a) > 0$.

THEOREM 11.20. *From Eq. (7.85) in [26]*

$$\begin{aligned}
(11.20) \quad & \int_{-\infty}^{\infty} \frac{e^{bmx}(c+x)^k}{\cosh^2(bx)} dx \\
& = -2b^{-k-1} e^{\frac{1}{2}i(k+m)\pi} \pi^k \left(k\Phi \left(e^{im\pi}, 1-k, \frac{1}{2} - \frac{ibc}{\pi} \right) + im\pi \Phi \left(e^{im\pi}, -k, \frac{1}{2} - \frac{ibc}{\pi} \right) \right)
\end{aligned}$$

where $|Re(b)| > |Re(m)| > 0$.

EXAMPLE 11.21.

$$(11.21) \quad \int_{-\infty}^{\infty} \frac{\log(c+x) \tanh(bx)}{\cosh(bx)} dx = \frac{i}{b} \left(H_{-\frac{3}{4} - \frac{ibc}{2\pi}} - H_{-\frac{2ibc+\pi}{4\pi}} \right)$$

where $Re(b) > Re(c)$.

THEOREM 11.22. *An integral over the real line.*

$$\begin{aligned}
(11.22) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+m} \log^k(at)}{1+bt^{2q}} dt \\
& = i^{1+k} b^{-\frac{m}{2q}} e^{\frac{im\pi}{2q}} \pi^{1+k} q^{-1-k} \left((-1)^m \Phi \left(e^{\frac{im\pi}{q}}, -k, \frac{\pi - 2iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{2\pi} \right) \right. \\
& \quad \left. - \Phi \left(e^{\frac{im\pi}{q}}, -k, \frac{\pi - 2iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{2\pi} \right) \right)
\end{aligned}$$

where $Re(q) > Re(m)$ and q in any positive integer.

EXAMPLE 11.23.

$$\begin{aligned}
(11.23) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+q} \log^k(at)}{1+bt^{2q}} dt \\
&= \frac{(2i)^k \pi^{1+k} q^{-1-k}}{\sqrt{b}} \left((-1)^q \left(-\zeta \left(-k, \frac{\pi - 2iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right) \right. \right. \\
&\quad \left. \left. + \zeta \left(-k, \frac{1}{4} \left(3 - \frac{2iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right) \right) \right) \\
&\quad \left. + \zeta \left(-k, \frac{\pi - 2iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right) \right. \\
&\quad \left. \left. - \zeta \left(-k, \frac{1}{4} \left(3 - \frac{2iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right) \right) \right) \right)
\end{aligned}$$

where $Re(q) > Re(m)$ and q in any positive integer.

EXAMPLE 11.24.

$$\begin{aligned}
(11.24) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+q} \log(\log(at))}{1+bt^{2q}} dt \\
&= \frac{\pi}{4\sqrt{b}q} \left(i\pi + \log(4) + 2\log(\pi) - 2\log(q) - 4\log \left(-3 - \frac{2iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right. \\
&\quad \left. + 4\log \left(-1 - \frac{2iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) + (-1)^q (-i\pi + \log(64) - 2\log(\pi) + 2\log(q) \right. \\
&\quad \left. + 4\log \left(-\frac{3}{4} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{2\pi} \right) - 4\log \left(-1 - \frac{2iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right) \\
&\quad \left. + 4\log \Gamma \left(-\frac{3}{4} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{2\pi} \right) \right. \\
&\quad \left. - 4\log \Gamma \left(\frac{1}{4} \left(-1 - \frac{2iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right) \right) \\
&\quad \left. - 4\log \Gamma \left(-\frac{3}{4} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{2\pi} \right) \right)
\end{aligned}$$

$$+4\log\Gamma\left(\frac{1}{4}\left(-1-\frac{2iq\left(\log(a)+\log\left(b^{-\frac{1}{2q}}\right)\right)}{\pi}\right)\right)\right)$$

where q in any positive integer.

EXAMPLE 11.25.

$$\begin{aligned} (11.25) \quad & \int_{-\infty}^{\infty} \frac{t \log(\log(t))}{1+t^4} dt \\ &= \frac{1}{8}\pi \left(4i\pi - 4\log(3) - 4(i\pi + \log(3)) - 3\log(4) + \log(64) - 4\log\Gamma\left(-\frac{3}{4}\right) \right. \\ & \quad \left. + 4\log\Gamma\left(-\frac{1}{4}\right) + 4\log\Gamma\left(\frac{1}{4}\right) - 4\log\Gamma\left(\frac{3}{4}\right) \right) \end{aligned}$$

EXAMPLE 11.26.

$$\begin{aligned} (11.26) \quad & \int_{-\infty}^{\infty} \frac{t^2 \log(\log(t))}{1+t^4} dt \\ &= \frac{i\pi}{2\sqrt{2}} \left(\pi - (1+i) \left(\log(5) + i \left(\log(7) - (1-i)\log(2\pi) - \log\left(\Gamma\left(\frac{1}{8}\right)\right) \right. \right. \right. \\ & \quad \left. \left. + i \log\left(3\Gamma\left(\frac{3}{8}\right)\right) + \log\Gamma\left(-\frac{7}{8}\right) + \log\Gamma\left(-\frac{5}{8}\right) \right) \right) \end{aligned}$$

EXAMPLE 11.27.

$$\begin{aligned} (11.27) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+q} \log(\log(t))}{1+t^{2q}} dt \\ &= \frac{\pi}{4q} \left(5i\pi - 4(i\pi + \log(3)) + \log(4) + 2\log(\pi) - 2\log(q) - 4\log\Gamma\left(-\frac{3}{4}\right) + 4\log\Gamma\left(-\frac{1}{4}\right) \right. \\ & \quad \left. + (-1)^q \left(-i\pi + \log(64) - 2\log(\pi) + 4\log\left(-\frac{3}{4} + \frac{q}{2}\right) + 2\log(q) - 4\log(-1+2q) \right. \right. \\ & \quad \left. \left. + 4\log\Gamma\left(-\frac{3}{4} + \frac{q}{2}\right) - 4\log\Gamma\left(\frac{1}{4}(-1+2q)\right) \right) \right) \end{aligned}$$

EXAMPLE 11.28.

$$(11.28) \quad \int_{-\infty}^{\infty} \frac{t^s - t^m}{t(1+t^4)\log(t)} dt = \frac{4}{3} i e^{\frac{1}{8}i\pi(m+s)} \left(3 \sin\left(\frac{1}{8}\pi(m-s)\right) + e^{\frac{1}{4}i\pi(m+s)} \sin\left(\frac{3}{8}\pi(m-s)\right) \right)$$

where $0 < \operatorname{Re}(m) < 1, 0 < \operatorname{Re}(s) < 1$.

EXAMPLE 11.29.

$$\begin{aligned}
(11.29) \quad & \int_{-\infty}^{\infty} \frac{t \log(\log(t))}{1+t^4} dt \\
&= \frac{1}{8} \pi \left(4i\pi - 4\log(3) - 4(i\pi + \log(3)) - 3\log(4) + \log(64) - 4\log\Gamma\left(-\frac{3}{4}\right) \right. \\
&\quad \left. + 4\log\Gamma\left(-\frac{1}{4}\right) + 4\log\Gamma\left(\frac{1}{4}\right) - 4\log\Gamma\left(\frac{3}{4}\right) \right)
\end{aligned}$$

EXAMPLE 11.30.

$$\begin{aligned}
(11.30) \quad & \int_{-\infty}^{\infty} \frac{t^2 \log(\log(t))}{1+t^6} dt \\
&= \frac{1}{12} \pi \left(6i\pi + 4\log\left(\frac{4}{3}\right) - 4\log(3) - 4(i\pi + \log(3)) + \log(4) + 4\log(5) - \log(64) \right. \\
&\quad \left. + 4\log(\pi) - 4\log\Gamma\left(-\frac{3}{4}\right) + 4\log\Gamma\left(-\frac{1}{4}\right) - 4\log\Gamma\left(\frac{3}{4}\right) + 4\log\Gamma\left(\frac{5}{4}\right) \right)
\end{aligned}$$

EXAMPLE 11.31.

$$\begin{aligned}
(11.31) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+\frac{q}{2}} \log^k(at)}{1+bt^{2q}} dt \\
&= \frac{(-1)^{3/4} i^k \pi^{1+k} q^{-1-k}}{\sqrt[4]{b}} \left(i^q \Phi \left(i, -k, \frac{1}{2} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right. \\
&\quad \left. - \Phi \left(i, -k, \frac{1}{2} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{\pi} \right) \right)
\end{aligned}$$

EXAMPLE 11.32.

$$\begin{aligned}
(11.32) \quad & \int_{-\infty}^{\infty} \frac{t^{-1+\frac{q}{2}} \log(\log(at))}{1+bt^{2q}} dt \\
&= \frac{\pi}{2\sqrt{2}\sqrt[4]{b}q} ((-1+i^q)(-i\pi - 2\log(\pi) + 2\log(q)) + (2+2i)i^q \\
&\quad \left(-i \log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)} \right) + \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{iq \left(\log(-a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)} \right) \right)
\end{aligned}$$

$$-(2-2i) \left(\log \left(\frac{\Gamma \left(\frac{1}{8} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)}{2\Gamma \left(\frac{5}{8} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)} \right) \right) + i \log \left(\frac{\Gamma \left(\frac{3}{8} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)}{2\Gamma \left(\frac{7}{8} - \frac{iq \left(\log(a) + \log \left(b^{-\frac{1}{2q}} \right) \right)}{4\pi} \right)} \right) \right) \right)$$

where $Re(b) > 0$ and q is any positive integer.

EXAMPLE 11.33.

$$(11.33) \quad \int_{-\infty}^{\infty} \frac{t \log(\log(t))}{1+t^8} dt$$

$$= \frac{\pi \left((-2+2i) \left(\log \left(\frac{\Gamma(\frac{1}{8})}{2\Gamma(\frac{5}{8})} \right) + i \log \left(\frac{\Gamma(\frac{3}{8})}{2\Gamma(\frac{7}{8})} \right) \right) + (2+2i) \left(-i \log \left(\frac{\Gamma(\frac{9}{8})}{2\Gamma(\frac{13}{8})} \right) + \log \left(\frac{\Gamma(\frac{11}{8})}{2\Gamma(\frac{15}{8})} \right) \right) \right)}{8\sqrt{2}}$$

EXAMPLE 11.34.

$$(11.34) \quad \int_{-\infty}^{\infty} \frac{t^{3/2} \log(\log(t))}{1+t^{10}} dt$$

$$= \frac{\pi}{10\sqrt{2}} \left((-1+i)(-i\pi + 2\log(5) - 2\log(\pi)) - (2-2i) \left(\log \left(\frac{\Gamma(\frac{1}{8})}{2\Gamma(\frac{5}{8})} \right) + i \log \left(\frac{\Gamma(\frac{3}{8})}{2\Gamma(\frac{7}{8})} \right) \right) \right. \\ \left. - (2-2i) \left(-i \log \left(\frac{\Gamma(\frac{11}{8})}{2\Gamma(\frac{15}{8})} \right) + \log \left(\frac{\Gamma(\frac{13}{8})}{2\Gamma(\frac{17}{8})} \right) \right) \right)$$

EXAMPLE 11.35.

$$(11.35) \quad \int_{-\infty}^{\infty} \frac{t^{-1+\frac{q}{2}} \log(\log(t))}{1+t^{2q}} dt$$

$$= \frac{\pi}{2\sqrt{2}q} ((-1+i^q)(-i\pi - 2\log(\pi) + 2\log(q))$$

$$- (2-2i) \left(\log \left(\frac{\Gamma(\frac{1}{8})}{2\Gamma(\frac{5}{8})} \right) + i \log \left(\frac{\Gamma(\frac{3}{8})}{2\Gamma(\frac{7}{8})} \right) \right)$$

$$+ (2+2i)i^q \left(-i \log \left(\frac{\Gamma(\frac{1}{8} + \frac{q}{4})}{2\Gamma(\frac{5}{8} + \frac{q}{4})} \right) + \log \left(\frac{\Gamma(\frac{3}{8} + \frac{q}{4})}{2\Gamma(\frac{7}{8} + \frac{q}{4})} \right) \right))$$

EXAMPLE 11.36.

EXAMPLE 11.37.

12. Asymptotic series

In this section we apply the method in [18] to the equations listed in the theorems. The integral form formulae were originally worked on by E. W. Barnes and Hilding Olof Faxén.

THEOREM 12.1. *From Eq. 24(i) and 25(i) on page 269 and 270 in [23] and [18].*

$$(12.1) \quad \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{\binom{-b}{j} \Gamma(b) \Gamma(n-s) (1-j+k)_j}{(\Gamma(1+n) \Gamma(-s))} \frac{e^{a(-j+k)\pi}}{(m+n)^{b+j}} \\ = \int_0^{\infty} e^{-mx} (1-e^{-x})^s x^{-1+b} (e^{a\pi} + x)^k dx$$

where $\operatorname{Re}(a) > \pi$.

THEOREM 12.2. *From Eq. 14(VI) on page 284 in [23] and [18].*

$$(12.2) \quad \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{a(-j+k)\pi-x} (m+n)^{-b-j} x^{1+n} \binom{-b}{j} \Gamma(b) (1-j+k)_j}{n!} \\ = \int_0^x e^{-z} \left(1 - \frac{z}{x}\right)^{-1+m} \left(-\log\left(1 - \frac{z}{x}\right)\right)^{-1+b} \left(e^{a\pi} + \log\left(1 - \frac{z}{x}\right)\right)^k dz$$

where $\operatorname{Re}(a) > \pi$.

THEOREM 12.3. *From Eq. 42(I) in [23] and [18].*

$$(12.3) \quad \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{m^{-j+n} \binom{n}{j} \Gamma(1+n\alpha) (1-j+k)_j a^{-j+k}}{\Gamma(1+n)} = \int_0^{\infty} e^{-z+ms^{\alpha}} (a+z^{\alpha})^k dz$$

where $0 < \operatorname{Re}(\alpha) < 1$.

THEOREM 12.4. *From Eq. (VI) on page 284 in [23] and [18].*

$$(12.4) \quad \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{m^{-1-j+n} \binom{-1+n}{j} \Gamma(1+nt) (1-j+k)_j \log^{-j+k}(a)}{\Gamma(1+n+nt)} \\ = \int_0^1 e^{-m(-1+z)z^t} z^t (\log(a) - (-1+z)z^t)^k dz$$

where $0 < \operatorname{Re}(t) < 1, -1 < \operatorname{Re}(a) < 1$.

THEOREM 12.5. *From Eq. (5.5.1) in [25] and [18]. Extended Faxén expansion series of an integral form.*

$$(12.5) \quad \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{a(-j+k)\pi} m^{-j+n} \binom{n}{j} \Gamma(n\alpha + \beta) (1-j+k)_j}{\Gamma(1+n)} \\ = \int_0^{\infty} e^{-t+mt^{\alpha}} t^{-1+\beta} (-t + t^{\alpha} + \exp(a\pi))^k dt$$

where $0 < \operatorname{Re}(\alpha) < 1, \operatorname{Re}(a) > \pi$.

THEOREM 12.6. *From Eq. (1) in [27]*

$$(12.6) \quad \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{a^{-j+k} m^{-j+n} \pi^{-j+k} \binom{n}{j} \Gamma\left(\frac{1+n}{\alpha}\right) (1-j+k)_j}{\alpha n!} = \int_0^{\infty} e^{mu-u^{\alpha}} (a\pi+u)^k du$$

where $\operatorname{Re}(\alpha) > 1, \operatorname{Re}(a) > 0$.

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