

Whispering Gallery Modes & Frequency Combs

Two excursions in the world of photonic resonators

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Synoptic on resonators under consideration

Whispering Gallery Mode resonators

Size, shape: Micro-resonator
(μm or nm, 3D object, 2D design)

Material: Dielectric (glass) or metallic

Aim: Guide light along (sharp) curves

Modelizing equations: 3D Maxwell

Reduction: Eliminate one dimension (flat structures). Reduce to 2D scalar potentials.

Theoretical investigation: Find resonances (sourceless).

In particular: Resonances with small imaginary parts and modes concentrated near the boundary of the resonator – called *Whispering Gallery Mode* (WGM).

Frequency Comb resonators

Size, shape: Fiber-resonator
(m or cm, 3D object, 1D design)

Material: Dielectric (glass)

Aim: Convert frequencies from source

Modelizing equations: Maxwell with non-linear material laws

Reduction: Scalar 1D Schrödinger eq. with cubic non-linearity and complex coefficients (Lugiato-Lefever Eqs, aka LLE)

Theoretical investigation: Find stationary solutions for constant driving force.

In particular: Bifurcations from “flat” solutions to stationary solutions with a regular non-trivial frequency distribution – called *frequency comb*.

Synoptic on problems under consideration

Whispering Gallery Mode resonators

- $C \subset \mathbb{R}^2$ bounded with smooth boundary (the cavity of the resonator).
- ε relative permittivity: $\varepsilon \equiv 1$ outside \bar{C} and $\varepsilon = \varepsilon_c$ in \bar{C} , with $\varepsilon_c > 1$ or $\varepsilon_c < 0$

Problem: Find modes u locally in H^1 and frequencies $k \in \mathbb{C}$ such that

$$\begin{cases} -\operatorname{div} \left(\frac{1}{\varepsilon} \nabla u \right) - k^2 u = 0 & \text{in } \mathbb{R}^2 & (1a) \\ \text{outgoing radiation condition} & |x| \rightarrow +\infty & (1b) \end{cases}$$

Issues: Unbounded domain, no compactness, non self-adjoint.

Frequency Comb resonators

- Variables $t > 0$ (time) and $\theta \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (1D periodic space variable)
- Parameters (real) $d \neq 0$ (dispersion), α (detuning), $f > 0$ (driving force),

Problem: Find $\psi = \psi(\theta, t)$ with complex values

$$\partial_t \psi = id \partial_\theta^2 \psi - (1 + i\alpha)\psi + i\psi|\psi|^2 + f \quad (2)$$

Issues: Find initial conditions to converge to non-trivial stationary solutions

Synoptic on projects

Whispering Gallery Mode resonators

Origin: Project about simulation and computation of WGM (dielectric case).

Partners: Members of FOTON Institute (Fonctions Optiques pour les Technologies de l'information) at Lannion.

Funding: The PhD of Zoïs Moitier 2016-2019 (MESR)

Documents: The PhD dissertation by Zoïs and two joint papers [1] & [2].

Pursuing: Sign-changing coefficients (metamaterials) by Zoïs and new partners (two papers [3] & [9] co-authored by Zoïs).

Frequency Comb resonators

Origin: ANR project "Rollmops" about Fabry-Perot resonators.

Partners: Members of LAAS-CNRS (Laboratoire [pour] l'Analyse et l'Architecture des Systèmes) at Toulouse.

Funding: A 10 month post-doc (for Mouhamad Al Sayed Ali) 2022-2023.

Documents: Our WAVES2024 extended abstract and several informal reports.

Pursuing: Investigation of standard LLE (ring resonators) extends to the proposed case of Fabry-Perot resonators

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WGM: Solving the issue of unboundedness

Cavity: $C \subset \mathbb{R}^2$. For R large enough, C is contained in the ball of radius R .

Operator: $P := -\operatorname{div} \frac{1}{\varepsilon} \nabla$ with ε equal to ε_c inside C and 1 outside C .

Transmission problem: equation (1a) written as

$$Pu - k^2 u = 0 \quad \text{in } \mathbb{R}^2 \quad (1a)$$

Expansion at infinity: For $|x| > R$, $u(x)$ solves $-\Delta u - k^2 u = 0$ and expands in polar coordinates using *Hankel functions* of 1st and 2d kind

$$u(x) = \sum_{m \in \mathbb{Z}} (a_m H_m^{(1)}(kr) + b_m H_m^{(2)}(kr)) e^{im\theta}$$

Outgoing radiation condition: (1b)

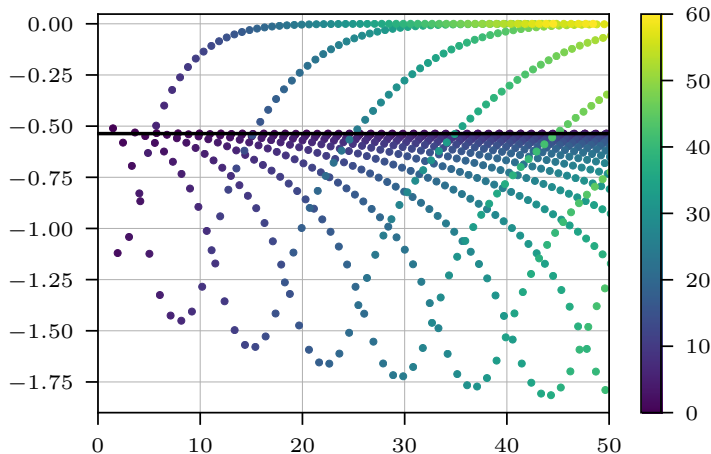
$$b_m = 0, \quad \forall m \in \mathbb{Z} \quad (1b)$$

Complex resonances of P :¹ Solutions k of (1a)-(1b) with mode $u \neq 0$.

Existence: Resonances form a discrete and infinite set in \mathbb{C} with $\operatorname{Im} k < 0$

¹Equivalently resonances are the poles k of the resolvent $k \mapsto (P - k^2)^{-1}$ obtained by extension from the complex upper plane \mathbb{C}^+ (where it is continuous from L^2 to its domain) to the complex lower plane \mathbb{C}^- , as application of [6] (because P is a black-box Hamiltonian).

WGM: Resonances $k \in \mathbb{C}$ when C is a disc and $\varepsilon_c > 1$



Radius of C equal 1 and $\varepsilon_c = 2.25$.

$\text{Re } k$ in abscissa, $\text{Im } k$ in ordinate

Separation of variables \Rightarrow each k associated with one value of m (marked by colorbar)

WGM: Almost self-adjoint and almost bounded

Special resonances: Those k with small $|\operatorname{Im} k|$, i.e. close to real axis.

Special modes: Those u concentrated close to ∂C

... appear by asymptotic analysis for $|m| \rightarrow \infty$ in a framework close to self-adjoint operators [13] “from quasimodes to resonances”.

Radial case: C disc and ε_c radial function. Look for $u = \sum_{m \in \mathbb{Z}} u_m(r) e^{im\theta}$ with

$$-\frac{1}{r} \partial_r \left(\frac{r}{\varepsilon} \partial_r u_m \right) + \frac{m^2}{\varepsilon r^2} u_m = k^2 u_m$$

Setting $\boxed{h = \frac{1}{m}}$, $w_h = u_m$, $\mathcal{H} = \frac{1}{r} \partial_r \left(\frac{r}{\varepsilon} \partial_r \right)$, $E_h = \frac{k^2}{m^2}$, and

$$V(r) = \frac{1}{\varepsilon(r) r^2}$$

we find a **semi-classical Schrödinger equation** in \mathbb{R}^+ with potential V :

$$-h^2 \mathcal{H} w_h + V w_h = E_h w_h, \quad h \rightarrow 0$$

WGM: Potential wells in radial case

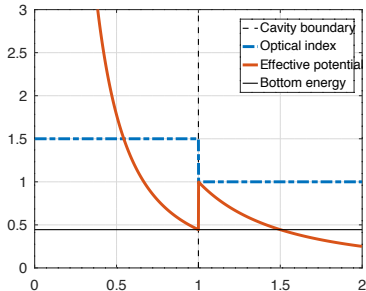
Characteristic quantities: Set optical index $n_c(r) = \sqrt{\varepsilon_c(r)}$ and

$$n_0 = \lim_{r \nearrow R} n_c(r), \quad n_1 = \lim_{r \nearrow R} n'_c(r), \quad n_2 = \lim_{r \nearrow R} n''_c(r)$$

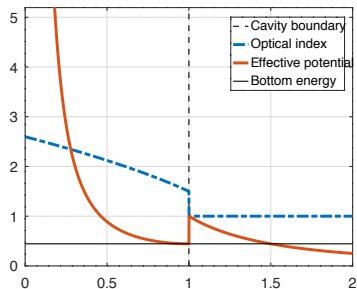
Effective curvature:

$$\tilde{\kappa} = 1 + R \frac{n_1}{n_0}$$

Configurations: **A** by the condition $\tilde{\kappa} > 0$ and **B** by the condition $\tilde{\kappa} = 0$



A



B

WGM: Asymptotics as $m \rightarrow \infty$ in radial case

Theorem

(i) For any integer $j \geq 1$, there exists a quasimode $(\underline{k}_j(m), \underline{u}_j(m))$:

Configuration A $\underline{k}_j(m)$ expands in integer powers of $m^{-1/3}$

$$\underline{k}_j(m) = \frac{m}{Rn_0} \left[1 + \frac{a_j}{2} \left(\frac{2}{m}\right)^{\frac{2}{3}} - \frac{1}{2n_0 n'_0} \left(\frac{2}{m}\right) + \mathcal{O}\left(\frac{1}{m}\right)^{\frac{4}{3}} \right]$$

$0 < a_1 < a_2 < \dots$ are the zeros of the reverse Airy function, $n'_0 = \sqrt{n_0^2 - 1}$

Configuration B $\underline{k}_j(m)$ expands in integer powers of $m^{-1/2}$

$$\underline{k}_j(m) = \frac{m}{Rn_0} \left[1 + \frac{(4j-1)\sqrt{\mu}}{2} \left(\frac{1}{m}\right) + \mathcal{O}\left(\frac{1}{m}\right)^{\frac{3}{2}} \right]$$

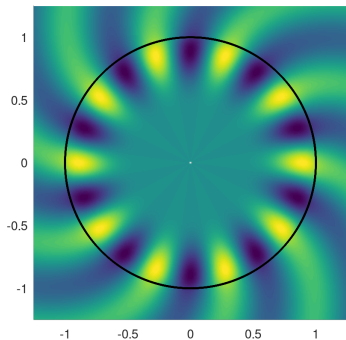
in which $\mu = 2 - R^2 n_2 / n_0$

(ii) Quasi-resonances $\underline{k}_j(m)$ are close to true resonances $k_j(m)$ modulo a super-algebraic error $\mathcal{O}(m^{-\infty})$

WGM: Real part for modes $j = 1$ in configuration A

Radius of C equal 1 and $\varepsilon_c = 2.25$.

$m = 10, j = 1, k = 8.802 - 0.21122i$

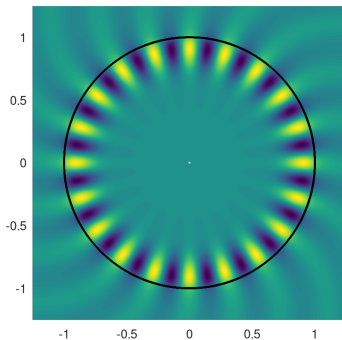


$m = 10$

$$\tilde{k}_1(10) = 8.802 - 0.211 i$$

$$\underline{k}_1(10) = 8.9345 \pm \mathcal{O}(0.4)$$

$m = 20, j = 1, k = 16.2966 - 0.0176475i$



$m = 20$

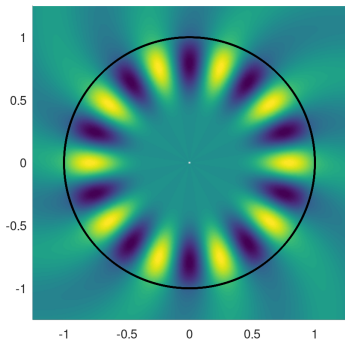
$$\tilde{k}_1(20) = 16.297 - 0.018 i$$

$$\underline{k}_1(20) = 16.294 \pm \mathcal{O}(0.3)$$

WGM: Real part for modes $j = 1$ in configuration B

Radius of C equal 1 and $\varepsilon_c = 2.25(1 + 2(R - r))$

$m = 10, j = 1, k = 7.8138 - 0.026126i$

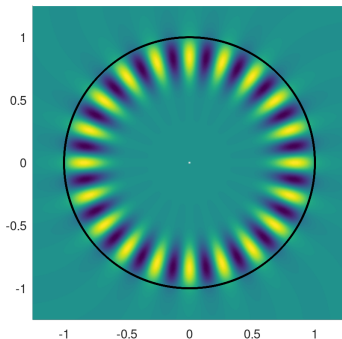


$m = 10$

$$\tilde{k}_1(10) = 7.8138 - 0.0261 i$$

$$\underline{k}_1(10) = 8.3987 \pm \mathcal{O}(0.3)$$

$m = 20, j = 1, k = 14.6373 - 0.000407259i$



$m = 20$

$$\tilde{k}_1(20) = 14.637 - 0.0004 i$$

$$\underline{k}_1(20) = 15.065 \pm \mathcal{O}(0.2)$$

WGM : Meta- or metallic materials with constant $\varepsilon_c < 0$



More difficult to analyze because P is not bounded from below

Negative eigenvalues $z = k^2$ with $\text{Im } k > 0$ with WGM structure as $m \rightarrow \infty$

Plasmonic resonances when $\varepsilon_c < -1$ (result from sign-change)



Forbidden value $\varepsilon_c = -1$ (then the transmission problem is not elliptic)

Plasmonic eigenvalues when $-1 < \varepsilon_c < 0$ (result from sign-change)



Absence of other WGM resonances (in proofs / computations)

Theorem – when $\varepsilon_c < -1$, in radial case

(i) Exists a quasimode $(\underline{k}(m), \underline{u}(m))$ expanding in integer powers of m^{-1}

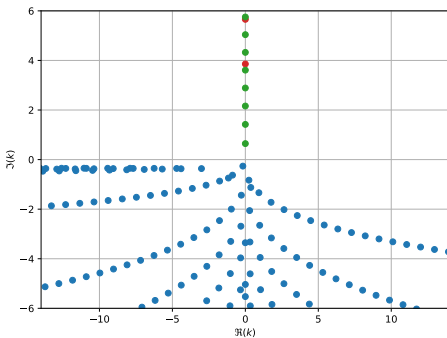
$$\underline{k}(m) = \frac{m}{R\eta_0} \sqrt{\eta_0^2 - 1} \left[1 + \frac{1 - \eta_0^2}{2\eta_0} \left(\frac{1}{m}\right) + \mathcal{O}\left(\frac{1}{m}\right)^2 \right]$$

in which $\eta_0^2 = -\varepsilon_c$

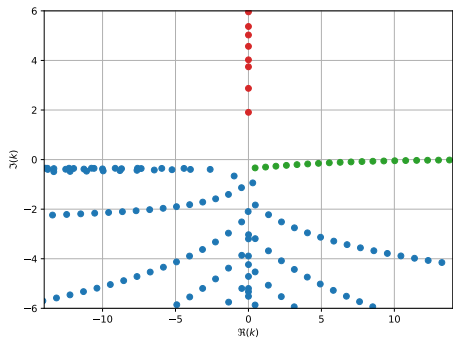
(ii) Quasi-resonances $\underline{k}(m)$ are close to true resonances $k(m)$ modulo a super-algebraic error $\mathcal{O}(m^{-\infty})$

WGM : Spectrum and resonances for $\epsilon_c < 0$ in k value

C disc with radius $R = 1$.



$$\epsilon_c = -0.667$$



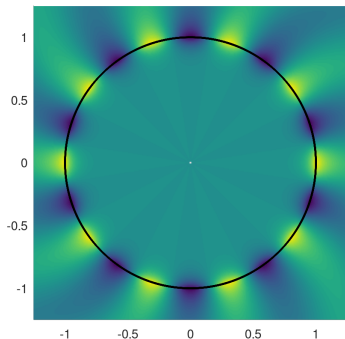
$$\epsilon_c = -2.25$$

● WGM eigen-freq. ● Plasmonic eigen-freq. or resonances ● Other resonances

WGM: Real part for plasmonic resonance modes

C disc with radius $R = 1$ and $\varepsilon_c = -2.25$.

$m = 10, j = 0, k = 7.031 - 0.093817i$

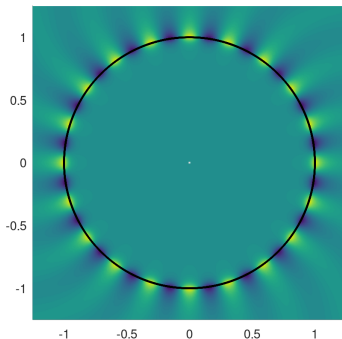


$m = 10$

$$\tilde{k}(10) = 7.031 - 0.094i$$

$$\underline{k}(10) = 7.1430 \pm \mathcal{O}(0.1)$$

$m = 20, j = 0, k = 14.5267 - 0.0142203i$



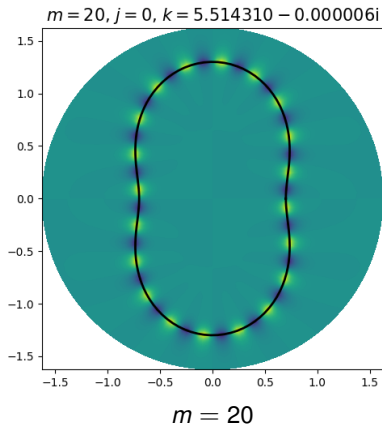
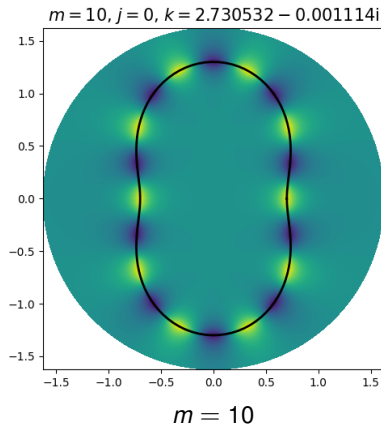
$m = 20$

$$\tilde{k}(20) = 14.527 - 0.014i$$

$$\underline{k}(20) = 14.597 \pm \mathcal{O}(0.05)$$

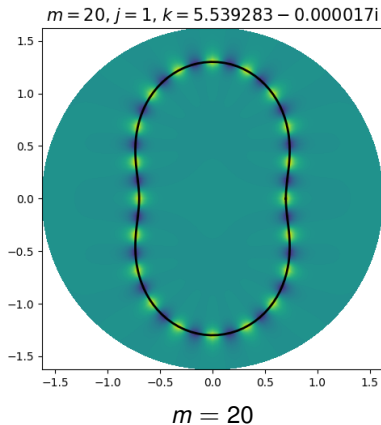
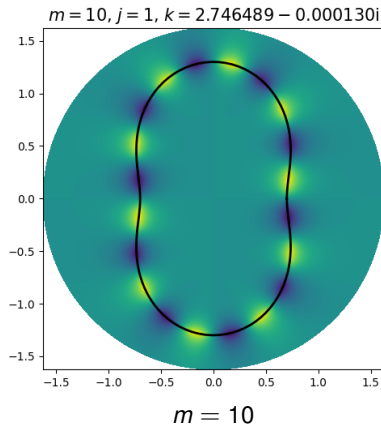
WGM: Real part for plasmonic resonance modes

C “peanut” and $\varepsilon_c = -1.1$.



WGM : Real part for plasmonic resonance modes

C “peanut” and $\varepsilon_c = -1.1$.



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FC: Looking for stationary solutions of LLE

Space domain: 1D torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \ni \theta$ (1D periodic space variable)

Parameters: (real constants)

$d \neq 0$ (dispersion), α (detuning), $f > 0$ (driving force),

Solve Lugiato-Lefever eqs: Find $\psi = \psi(\theta, t)$ with complex values

$$\partial_t \psi = id \partial_\theta^2 \psi - (1 + i\alpha)\psi + i\psi|\psi|^2 + f \quad (2)$$

Problem: Identify stationary solutions

Flat solutions: Solutions $\psi = \psi(\theta, t) = a \in \mathbb{C}$ (constant in time and space) easy to determine analytically, but do not lead to frequency combs

Looking for non-flat stationary solutions by running our symmetric split step code

Rules of evolution of the full equation (2) are difficult, not to say impossible, to describe, except that the dynamics is bounded [10]

Choosing initial conditions ψ_0 to complete eq. (2) as $\psi(\theta, 0) = \psi_0$ seems unphysical and, in general does not entail convergence to non-flat solution...

... **unless you are lucky:** then **attractors** can be observed (distinct ψ_0 yield convergence to same limit as $t \rightarrow \infty$)

FC : Bifurcation from flat solutions

Depending on α and f , existence of 1, 2, or 3 flat solutions $a \in \mathbb{C}$

Disambiguation parameter: $\rho > 0$ (with α chosen)

$\forall \rho > 0, \exists f = \underline{f}(\rho) > 0$ unique and $\exists a = \underline{a}(\rho) \in \mathbb{C}$ unique

such that²

$$\boxed{|a|^2 = \rho} \quad \text{and LLE holds} \quad (1 + i\alpha)a - ia\rho = f$$

Bifurcation analysis is used in literature to solve

$$id \partial_{\theta}^2 \psi - (1 + i\alpha)\psi + i\psi|\psi|^2 + f = 0 \quad (3)$$

resorting in general to central manifold theorem, and concentrating on special cases, example [11], in which the “famous value” $\alpha = \frac{41}{30}$ appears

Issue of multiplicity: If $\psi(\theta)$ solves (3), then for all $\theta_0 \in \mathbb{T}$, the function $\theta \mapsto \psi(\theta - \theta_0)$ is also solution

Restriction to “synchronized” solutions represented by even functions on $(-\pi, \pi)$ is recommended by [10] and allows to kill generic multiplicity and to apply the basic bifurcation theorem by Crandall and Rabinowitz [5]

² $\underline{f}(\rho) = \sqrt{\rho(1+(\alpha-\rho)^2)}$ and $\underline{a}(\rho) = \sqrt{\rho} \frac{1-i(\alpha-\rho)}{|1-i(\alpha-\rho)|}$

FC : Bifurcations for dummies

Theorem of Crandall and Rabinowitz [1971]

Consider equation (for \mathcal{F} smooth enough)

$$\mathcal{F}(u, \lambda) = 0, \quad \text{for } u \in H, \quad \lambda \in \Lambda$$

with H Hilbert space and Λ interval, such that one has the trivial solution

$$\mathcal{F}(0, \lambda) = 0 \quad \forall \lambda \in \Lambda$$

Then

(i) Eq. $\mathcal{F}(u, \lambda) = 0$ possesses one (and only one) branch of non-trivial solutions starting from $(0, \lambda_0)$ if conditions $\boxed{(\mathcal{H}0)\text{--}(\mathcal{H}2)}$ are satisfied

($\mathcal{H}0$) Non-invertibility: The derivative $(\partial_u \mathcal{F})_{(0, \lambda_0)}$ is a non-invertible Fredholm operator with index 0

($\mathcal{H}1$) Simplicity: $\ker(\partial_u \mathcal{F})_{(0, \lambda_0)}$ has dimension 1, let u_0 be a generator

($\mathcal{H}2$) Transversality: The second derivative $(\partial_u \partial_\lambda \mathcal{F})_{(0, \lambda_0)}$ applied to u_0 does not belong to the range of $(\partial_u \mathcal{F})_{(0, \lambda_0)}$

(ii) The branch of non-trivial solutions (u, λ) can be smoothly parametrized as

$$u = \Phi(\tau), \quad \lambda = \mu(\tau), \quad \tau \in (-\eta, \eta) \quad (\text{with } \eta > 0)$$

FC : How to apply C&R theorem to LLE

Set

$$\mathcal{G}(\psi, f) = \text{id } \psi'' - (1 + i\alpha)\psi + i\psi|\psi|^2 + f$$

Recast equation $\mathcal{G}(\psi, f) = 0$ in the form $\mathcal{F}(u, \rho) = 0$ in which $u = 0$ correspond to flat solutions (here we use disambiguation functions $\rho \mapsto \underline{f}(\rho)$ and $\rho \mapsto \underline{a}(\rho)$):

$$\mathcal{F}(u, \rho) := \underline{a}(\rho)^{-1} \mathcal{G}(\underline{a}(\rho)(1 + u), \underline{f}(\rho))$$

Note that, for any $\rho > 0$, $\mathcal{F}(u, 0) = \underline{a}(\rho)^{-1} \mathcal{G}(\underline{a}(\rho), \underline{f}(\rho)) = 0$

Multiplicative form $a(1 + u)$ borrowed from [11] (instead of additive $a + u$ as [10]) allows for simple expansion in **linear**, **quadratic** and **cubic** terms:

$$\mathcal{F}(u, \rho) = \underbrace{\begin{pmatrix} -1 & d\mathcal{D} + \alpha - \rho \\ -d\mathcal{D} - \alpha + 3\rho & -1 \end{pmatrix}}_{\mathcal{L}(\rho)} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \rho \begin{pmatrix} -2u_1 u_2 \\ 3u_1^2 + u_2^2 \end{pmatrix} + \rho \begin{pmatrix} -u_1^2 u_2 - u_2^3 \\ u_1^3 + u_1 u_2^2 \end{pmatrix}$$

where $\mathcal{D}v = -v''$, and $u = u_1 + iu_2$. We have

$$\partial_u \mathcal{F}_{(0, \rho_0)} = \mathcal{L}(\rho_0) \quad \text{and} \quad \partial_u \partial_\rho \mathcal{F}_{(0, \rho_0)} = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \quad \forall \rho_0$$

FC : C&R theorem for synchronized solutions of LLE

The operator $\mathcal{D}v = -v''$ defined on $H_{\text{per}}^2(\mathbb{T})$ (periodic) has the eigen-basis

$$\cos n\theta \quad (n \geq 0) \quad \text{and} \quad \sin n\theta \quad (n \geq 1) \quad n \text{ integer}$$

Synchronized solutions are even in $(-\pi, \pi)$. They are generated by $\{\cos n\theta\}_{n \in \mathbb{N}}$. Hence the 2×2 block **diagonalization** of the linear part

$$\mathcal{L}(\rho) = [\hat{\mathcal{L}}_n(\rho)]_{n \in \mathbb{N}} \quad \text{with} \quad \hat{\mathcal{L}}_n(\rho) = \begin{pmatrix} -1 & d n^2 + \alpha - \rho \\ -d n^2 - \alpha + 3\rho & -1 \end{pmatrix}$$

Non-invertibility condition ($\mathcal{H}0$) holds at ρ_0 if $\exists k \in \mathbb{N}^*$ and $\exists \varepsilon \in \{\pm 1\}$ s.t.

$$dk^2 = 2\rho_0 - \alpha + \varepsilon \sqrt{\rho_0^2 - 1}$$

Transversality condition ($\mathcal{H}2$) holds if

$$\varepsilon = 1 \quad \text{or} \quad \rho_0 \neq \frac{2}{\sqrt{3}} \quad (\simeq 1.155) \quad \text{when} \quad \varepsilon = -1$$

Simplicity condition ($\mathcal{H}1$) is generically satisfied where ($\mathcal{H}0$) holds

FC : Taylor expansion at onsets of bifurcations

Non-trivial branch $\tau \mapsto (u = \Phi(\tau), \rho = \mu(\tau))$ exists by C&R theorem at (almost) any $\rho_0 \geq 1$, any $\alpha \in \mathbb{R}$ and any $\varepsilon \in \{\pm 1\}$ since $\exists k \in \mathbb{N}^*$ and $d \in \mathbb{R}$ with

$$dk^2 = 2\rho_0 - \alpha + \varepsilon\sqrt{\rho_0^2 - 1}$$

Taylor expansion at $\tau = 0$

$$\begin{cases} u(\rho(\tau)) = \tau\Phi^{(1)} + \tau^2\Phi^{(2)} + \dots \\ \rho(\tau) = \rho_0 + \tau\mu^{(1)} + \tau^2\mu^{(2)} + \dots \end{cases} \quad \tau \rightarrow 0$$

Sequence of equations obtained by inserting this into $\mathcal{F}(u, \rho) = 0$

$$\begin{aligned} \mathcal{L}(\rho_0)[\Phi^{(1)}] &= 0, & \ell = 1 \\ \mathcal{L}(\rho_0)[\Phi^{(\ell)}] + \mu^{(\ell-1)}\partial_u\partial_\rho\mathcal{F}_{(0,\rho_0)}[\Phi^{(1)}] &= f^{(\ell)}, & \ell = 2, 3, \dots \end{aligned}$$

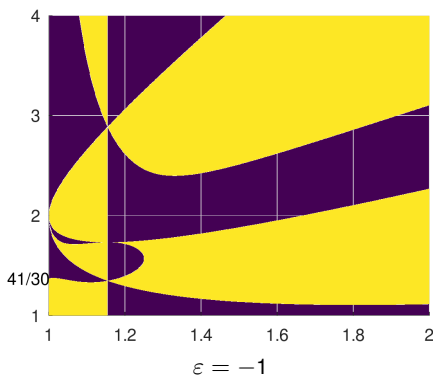
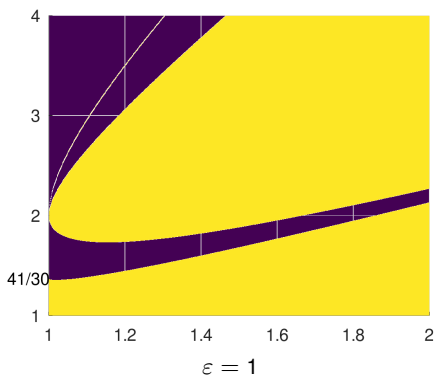
where $f^{(\ell)}$ involves previous terms. **Transversality condition ensures solvability.** The first terms are

$$\Phi^{(1)} = \left(\rho_0 + \varepsilon\sqrt{\rho_0^2 - 1} \right) \cos k\theta, \quad \mu^{(1)} = 0$$

and we get an explicit formula for $\mu^{(2)}$ **as a function of $(\rho_0, \alpha, \varepsilon)$** . The sign of $\mu^{(2)}$ determines whether the bifurcation is super- or sub-critical.

FC: Sign maps of $\mu^{(2)}$

$\rho \in [1, 2]$ (abscissa), $\alpha \in [1, 4]$ (ordinate)



- Lowest sign change on the axis $\rho = 1$ occurs at $\alpha = 41/30$.
- Sign change across the line $\rho = 2/\sqrt{3}$ (for $\varepsilon = -1$) where transversality is not satisfied

FC : Frequency combs near onsets of bifurcation

Truncated expansions: For $N \in \mathbb{N}$ set

$$\rho^{[M]}(\tau) = \rho_0 + \sum_{\ell=1}^N \tau^\ell \mu^{(\ell)} \quad \text{and} \quad u^{[M]}(\tau; \theta) = \sum_{\ell=1}^N \tau^\ell (\Phi_1^{(\ell)}(\theta) + i \Phi_2^{(\ell)}(\theta))$$

and, using the disambiguation functions $a \mapsto \underline{a}(\rho)$ and $\rho \mapsto \underline{f}(\rho)$, set

$$\psi^{[M]}(\tau) := \underline{a}(\rho^{[M]}(\tau)) (1 + u^{[M]}(\tau)) \quad \text{and} \quad f^{[M]}(\tau) = \underline{f}(\rho^{[M]}(\tau))$$

Approximate branches of solutions of stationary LLE in the sense that

$$\mathcal{G}(\psi^{[M]}(\tau), f^{[M]}(\tau)) = \mathcal{O}(\tau^{N+1}) \quad \tau \rightarrow 0.$$

Form of $\psi^{[M]}(\tau, \theta)$

$$\psi^{[M]}(\tau, \theta) = \Psi_0(\tau) + \Psi_1(\tau) \cos k\theta + \Psi_2(\tau) \cos 2k\theta + \dots + \Psi_N(\tau) \cos Nk\theta$$

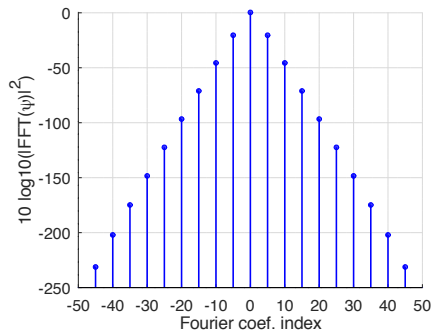
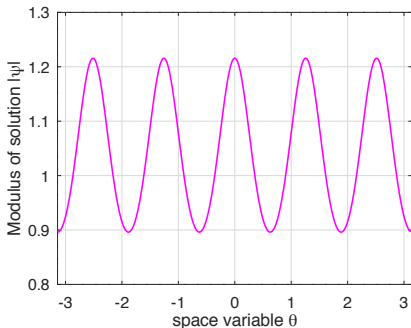
where $\Psi_\ell(\tau) \in \mathbb{C}$ depend only on $(\rho_0, \alpha, \varepsilon)$ and

$$|\Psi_\ell(\tau)| = \mathcal{O}(\tau^\ell) \quad \tau \rightarrow 0$$

FC : Frequency combs, an example

Parameters $\alpha = 1$, $\rho_0 = 1.1$, $\varepsilon = 1$, $\tau = 0.1$, with $k = 5$, $d = 0.06633$

$\rho(\tau) = 1.1229$ with 4 terms in asymptotic formulas



Our symmetric split-step evolution code gave results very close to asymptotic formulas ($4 \cdot 10^{-4}$ relative error) and highlighted an attractor.

Section title

- 1 Synopsis
- 2 Whispering Gallery Mode resonators
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- 4 Conclusion**
- 5 References

My experience with projects

- + Questions arise from physics through contacts with teams outside mathematical community
- After an exploratory calculatory phase, we had to realize that it is not enough to dive into blind computations and that we ignore some essential mathematical topics connected with these questions
- + We have to find our way in a broad literature and become acquainted with new subjects (e.g. scattering resonances, bifurcations). This is exciting but...
- ... takes time, much time. The time scale of projects is shorter than the time scale necessary to unveil new results. This situation is unfavorable to short term positions (typically post-docs)
- + The mere existence of the project is a strong motivation to organize a regular working group
- + - Eventually we succeed in answering relevant and interesting questions, but not exactly the questions raised at the beginning!

Tx

Thank You for Your Attention!

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Bibliography (from extended abstract)

- [1] [S. BALAC](#), [M. DAUGE](#), [Y. DUMEIGE](#), [P. FÉRON](#) & [Z. MOITIER](#), Mathematical analysis of whispering gallery modes [...] *Eur. Phys. J. D*, **74**:221 (2020)
- [2] [S. BALAC](#), [M. DAUGE](#) & [Z. MOITIER](#), Asymptotics for 2D whispering gallery modes [...] *IMA Journal of Applied Mathematics*, **86**, 2 (2021), 1212-1265
- [3] [C. CARVALHO](#) & [Z. MOITIER](#), Scattering resonances [...] sign-changing coefficient, *IMA Journal of Applied Mathematics*, **88**, 6 (2023), 1-43
- [4] [Y. K. CHEMBO](#), Kerr optical frequency combs: theory, applications and perspectives, *Nanophotonics*, **5** (2016), 214–230
- [5] [M. G. CRANDALL](#) & [P. H. RABINOWITZ](#), Bifurcation from simple eigenvalues, *J. Functional Analysis*, **8** (1971), 321-340
- [6] [S. DYATLOV](#) & [M. ZWORSKI](#), *Mathematical theory of scattering resonances*, Graduate Studies in Mathematics, **200** (2019)
- [7] [B. HELFFER](#) & [J. SJÖSTRAND](#), *Résonances en limite semi-classique*, Mémoires de la Société Mathématique de France, **24-25** (1986)
- [8] [P. LALANNE](#), [W. YAN](#), [K. VYNCK](#), [C. SAUVAN](#) & [J. HUGONIN](#), Light Interaction with Photonic and Plasmonic Resonances, *Laser & Photonics Reviews* **12**:1700113 (2018)
- [9] [R. MANDEL](#), [Z. MOITIER](#) & [B. VERFÜRTH](#), Nonlinear Helmholtz equations with sign-changing diffusion coefficient, *Comptes Rendus. Mathématique* **360** (2022) 513-538
- [10] [R. MANDEL](#) & [W. REICHEL](#), A priori bounds and global bifurcation results for frequency combs modeled by the Lugiato-Lefever equation, *SIAM J. Appl. Math.*, **77** (2017), 315-345.
- [11] [T. MIYAJI](#), [I. OHNISHI](#) & [Y. TSUTSUMI](#), Bifurcation analysis to the Lugiato-Lefever [...], *Physica D: Nonlinear Phenomena*, **239** (2010), 2066-2083
- [12] [J. MUSGRAVE](#), [S. HUANG](#) & [M. NIE](#), Microcombs in fiber Fabry–Perot cavities., *APL Photonics*, **8**:121101 (2023)
- [13] [S.-H. TANG](#) & [M. ZWORSKI](#), From quasimodes to resonances, *Mathematical Research Letters*, **5** (1998), pp 261-272