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GENUS THEORY, GOVERNING FIELD, RAMIFICATION AND FROBENIUS

by

Roslan Ibara Ngiza Mfumu & Christian Maire

Abstract. — In this work we develop, through a governing field, genus theory for a number field K with tame ramification in T and splitting in S , where T and S are finite disjoint sets of primes of K . This approach extends that initiated by the second author in the case of the class group. It allows expressing the S - T genus number of a cyclic extension L/K of degree p in terms of the rank of a matrix constructed from the Frobenius elements of the primes ramified in L/K , in the Galois group of the underlying governing extension. For quadratic extensions L/\mathbb{Q} , the matrices in question are constructed from the Legendre symbols between the primes ramified in L/\mathbb{Q} and the primes in S .

1. Introduction

Let K be a number field, and let S and T be two finite and disjoint sets of places of K . We assume that T contains only finite places. Let K_T^S denote the maximal abelian extension of K , totally decomposed at all places in S (or even S -split), unramified outside of T , and with at most tame ramification at the places $v \in T$ (or even T -tamely ramified). This is a finite extension, and the Artin map allows us to identify the Galois group $\text{Gal}(K_T^S/K)$ with the S -ray class group of K modulo $\mathfrak{m} := \prod_{v \in T} v$, which we denote by $\text{Cl}_{K,\mathfrak{m}}^S$. For more details, see Section §1.1.1.

Now let L/K be an extension of number fields ramified at the set Σ . The genus theory provides information about the class group $\text{Cl}_{L,\mathfrak{m}_L}^{S_L}$ in terms of Σ and the behavior of the S -units of L in L/K . See Theorem 2.1.

The first remarkable result in genus theory dates back to Gauss, concerning the 2-Sylow subgroup of the class group of quadratic extensions of \mathbb{Q} (see [8, Chapter 1, §1] and [4, Chapter IV, §4, Exercise 4.2.10]). The phenomenon described by Gauss has been studied, developed, and generalized by many authors, including the works of Hasse [6], Leopoldt [9], Furuta [3], and others. For more details, see [4, III.4].

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The introduction of the sets T and S was initiated by Jaulent [7], Federer [1], and others. A very good overview of all this can be found in [7, Chapter II, 2.4, Chapter III, 2.1].

The work presented here is inspired by [11]. We develop the S - T genus theory via a governing extension denoted by F_T^S/K , where the usual ramification conditions are interpreted through relations between Frobenius elements. As a consequence, and similar to [11, Theorem 1.3], questions in genus theory can be translated into questions about the behavior of Frobenius elements in a governing field, for which the Chebotarev density theorem becomes central.

When the base field K is given and the Galois group of L/K is a fixed abelian group, Frei, Loughran, and Newton [2] studied the asymptotic behavior of the genus number of L/K (for the class group) with respect to the discriminant of L . It would be interesting to revisit their results in light of our work.

Before presenting our results, let us begin by specifying the context.

1.1. The context. —

1.1.1. Ray class groups. — Let K be a number field, T a finite set of finite places of K , and S a finite set of places of K , disjoint from T . Let us denote $S = S_0 \cup S_\infty$, where S_0 contains only non-archimedean places and S_∞ contains archimedean places, which we assume to be contained in the set $Pl_{K,\infty}^{re}$ of real places of K .

For a place v of K , let ι_v denote the embedding of K into its completion K_v .

Set

- $I_{K,T}$ the group of nonzero fractional ideals of K prime to T ,
- $\mathfrak{m} = \prod_{v \in T} v$ the ray modulus of K associated to T ,
- $P_{K,\mathfrak{m}}^{S_\infty}$ the subgroup of principal ideals (x) of $I_{K,T}$, $x \equiv 1 \pmod{\mathfrak{m}}$, and $\iota_v(x) > 0$ for all $v \in pl_{K,\infty}^{re} \setminus S_\infty$,
- $\langle S_0 \rangle$ the subgroup of $I_{K,T}$ built over places of S_0 ,
- $R_{K,\mathfrak{m}}^S$ the subgroup $P_{K,\mathfrak{m}}^{S_\infty} \langle S_0 \rangle$ of $I_{K,T}$.

Let $\text{Cl}_{K,\mathfrak{m}}^S$ be the S -ray-class group modulo \mathfrak{m} , *i.e.*

$$\text{Cl}_{K,\mathfrak{m}}^S := I_{K,T} / R_{K,\mathfrak{m}}^S.$$

By class field theory, $\text{Cl}_{K,\mathfrak{m}}^S$ is isomorphic to the Galois group of K_T^S/K , where K_T^S/K is the maximal abelian extension K , T -tamely ramified and S -split, see [4, Chapter II, §5].

1.1.2. Genus fields and genus numbers. — Let p be a prime number and let L/K be a cyclic extension of degree p . Denote by Σ the set of ramification of L/K ; for an infinite place v , we will speak of decomposition versus non-decomposition, but not of ramification.

Let T_L (respectively S_L) denote the places of L lying above those of T (resp. of S), and consider $\text{Cl}_{L,\mathfrak{m}_L}^S$, where $\mathfrak{m}_L := \prod_{w \in T_L} w$.

Let M/K be the maximal abelian extension K contained in L_T^S/K .

$$\begin{array}{ccccc} L & \text{---} & & \text{---} & M & \text{---} & L_T^S \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ K & \text{---} & & & K_T^S & & \end{array}$$

The field M is the S - T genus field associated with L/K , and the quantity $(g_T^S)^* = [M : L]$ is the S - T genus number. Set $g_T^S = [M : K_T^S]$: this is the quantity g_T^S that we are studying. It can be observed that it is easy to pass from g_T^S to $(g_T^S)^*$ as soon as $\#\text{Cl}_{K,m}^S$ is known, and the knowledge of g_T^S provides information about Cl_{L,m_L}^S . Of course, the genus theory makes sense when the field L is not contained in K_T^S , because otherwise $M = K_T^S$. The extension M/K being abelian, its Galois group can be approached through class field theory, which allows expressing $[M : K]$ in terms of the ramification in L/K and the units of K , thus leading to a non-trivial lower bound for $\#\text{Cl}_{L,m_L}^S$.

Since L/K is cyclic of degree p , the Galois group $\text{Gal}(M/K_T^S)$ is abelian of exponent p (see Theorem 2.1). Thus, when $p > 2$, the infinite places play no role. Consequently, we assume that $S_\infty = \text{Pl}_{K,\infty}^{re}$ in this case.

1.1.3. Governing fields. — We continue with a fixed prime number p . We then assume that for $v \in T$, we have $N_v = 1 \pmod{p}$, where N_v is the cardinality of the residue field of the completion K_v of K at v . Note that without this condition, the p -part contributed by the place v of T in $\text{Cl}_{K,m}^S$, would be trivial.

Let E_T^S be the group of S -units of K congruent to 1 $\pmod{\mathfrak{m}}$, that is,

$$E_T^S = \{x \in K^\times \mid x \equiv 1 \pmod{\mathfrak{m}}, v(x) = 0 \forall v \notin S\}.$$

Here, let us clarify the meaning $v(x) = 0$. If $v \in S_0$, we identify the place v with its valuation; if $v \in \text{Pl}_{K,\infty}^{re}$, $v(x) = 0$ means $\iota_v(x) > 0$; for v archimedean, $v \notin \text{Pl}_{K,\infty}^{re}$, we always have $v(x) = 0$.

Let $K' = K(\zeta_p)$, where ζ_p is a primitive p th root of unity.

The governing field F_T^S associated with the triplet (K, T, S) is defined as

$$F_T^S := K'(\sqrt[p]{E_T^S}).$$

We then define $\Gamma_T^S := \text{Gal}(F_T^S/K')$: it is an abelian p -elementary group.

When $T = \emptyset$ and $S = \text{Pl}_{K,\infty}^{re}$, by Dirichlet's theorem the p -rank of $\Gamma_\emptyset^S := \Gamma_\emptyset^S$ is $r_1 + r_2 - 1 + \delta_{K,p} + \#S_0$, where (r_1, r_2) is the signature of K , and where $\delta_{K,p} = 1$ if $\zeta_p \in K$, and 0 otherwise.

1.2. Our result. — To simplify the presentation of our result, we assume that the set Σ does not contain any places above p ; in other words, the extension L/K is tamely ramified. To further simplify the presentation, we also assume that the places in S split in L/K .

For each place $v \in \Sigma$, we choose a place w of K' above v and set $\sigma_v := \sigma_w$, the Frobenius element associated with w in $\Gamma_T^S := \text{Gal}(F_T^S/K')$; of course, this element depends on the choice of w , but we will see that the conditions involving it are independent of this choice.

Let $m = \#\Sigma$, and let $\{e_{v_1}, \dots, e_{v_m}\}$ be a basis of $(\mathbb{F}_p)^m$ indexed by the places v of Σ .

We then consider the linear map $\Theta_{\Sigma,T}^S$ defined by

$$\begin{aligned} \Theta_{\Sigma,T}^S : (\mathbb{F}_p)^m &\longrightarrow \Gamma_T^S \\ e_v &\longmapsto \sigma_v. \end{aligned}$$

We have the following result

Theorem 1.1. — *Under the previous conditions, we have*

$$g_T^S = \#\ker(\Theta_{\Sigma, T}^S).$$

The essence of our work is to translate the ramification conditions through Frobenius elements in a governing field. Therefore, if we ensure that the Frobenius elements associated with the places of T form a linearly independent set in $\Gamma^S := \text{Gal}(F^S/K')$, then we can express quite easily the Galois group Γ_T^S .

Set $H_T := \sum_{v \in T} \mathbb{F}_p \sigma_v \subset \Gamma^S$.

Proposition 1.2. — *Suppose that the family $\{\sigma_v, v \in T\}$ forms a linear independent set over \mathbb{F}_p in Γ^S . Then $\Gamma_T^S \simeq \Gamma^S/H_T$.*

The condition of linear independence has an interpretation. Indeed, according to the Gras-Munnier theorem (see [4, Chapter V, Corollary 2.4.2], [5]), a relation between the Frobenius elements $\sigma_v, v \in T$, implies the existence of a cyclic extension of degree p of K , T -ramified and S -split, and consequently contributes "trivially" to g_T^S . Thus, the condition of linear independence forces the context to avoid this situation.

Theorem 1.1 becomes interesting when we have a good understanding of the governing field F^S , especially when we know about the units. Typically, this occurs for $K = \mathbb{Q}$, but also, as noted in [11, §3.5.3], for $p = 3$ and for base field $K = \mathbb{Q}(\zeta_3)$.

By introducing S -places, the units become S -units, and when the field K is principal, the governing field is relatively easy to describe. A remarkable situation arises when $p = 2$ and L/\mathbb{Q} is a quadratic extension. The quantity g_T^S corresponds to the kernel of a matrix constructed using Legendre symbols. Let's develop a very specific situation.

Let $\Sigma = \{p_1, \dots, p_m\}$ and $S_0 = \{\ell_1, \dots, \ell_{s'}\}$. We assume that $\Sigma \cap (\{S_0\} \cup \{2\}) = \emptyset$. When S_∞ contains the unique infinite place v_∞ , let's set $\ell_0 = -1$ (we identify ℓ_0 with v_∞). Set $s := \#S$.

Here, we don't impose any condition on the behavior of $v \in S$ in L/\mathbb{Q} .

Let $A = (a_{i,j})$ be the matrix of size $s \times m$ defined by

$$a_{i,j} = \left(\frac{\ell_j}{p_i} \right),$$

where $\left(\frac{\ell_i}{p_j} \right) \in \mathbb{F}_2$ is the additive Legendre symbol.

Next, consider the diagonal matrix $D = (d_{i,j})$ size $s \times s$ defined by

$$d_{j,j} = \begin{cases} 1 & \text{if } \ell_j \text{ is inert in } L/\mathbb{Q}, \\ 0 & \text{if } \ell_j \text{ splits in } L/\mathbb{Q}. \end{cases}$$

Finally, let $M = (AD)$ be the matrix of size $(m + s) \times s$.

Corollary 1.3. — *Under the previous conditions, we have:*

$$g_\emptyset^S = \#\ker(A).$$

The rest of our work consists of four sections. In Section 2, we introduce and develop the elements of genus theory that are useful for our results. Section 3 is dedicated to the governing field. It is also in this section that we prove Proposition 1.2. Section 4 focuses on the main theorem and its proof. In the final section, we provide two applications.

2. Elements of genus theory

2.1. S - T genus formula. — For this part, we refer, for example, to [4, Chapter IV, §4], [7, Chapter III, §2], or [10].

Let's go back to the framework of Section §1.1. Let T and S be two finite disjoint sets of places of K , non-archimedean for T and arbitrary for $S = S_0 \cup S_\infty$.

Let L/K be a cyclic extension of degree p .

We denote by

$$E_T^S \cap \mathcal{N}_{L/K} := E_T^S \cap N_{L/K}(U_{L,m}^S),$$

the elements of E_T^S that are locally norms everywhere in L/K .

The following theorem can be formulated in a more general context (see [4, Chapter IV]), but we will focus on the case of cyclic extension of degree p .

Theorem 2.1. — *Let L/K be a cyclic extension of degree p with ramification set Σ . Then $\text{Gal}(M/K_T^S)$ is an abelian group of exponent p . In particular, g_T^S is a power of p , and*

$$\log_p(g_T^S) = \#S^{ns} + \#\Sigma \setminus \Sigma \cap (S \cup T) - \log_p(E_T^S : E_T^S \cap \mathcal{N}_{L/K}),$$

where S^{ns} denotes the set of places of S that are not split in L/K .

Thus, the study of g_T^S is closely related to the quantity $E_T^S \cap \mathcal{N}_{L/K}$. This will be done through the governing field F_T^S . To achieve this, Proposition 2.4 from the upcoming section is central.

We set $\Sigma' := \Sigma \setminus \Sigma \cap (S \cup T)$.

2.2. Genus fields and ray class fields. — Let L/K be an abelian extension. For $v \in Pl_K$, we denote by $D_v := D_v(L/K)$ the decomposition group of v in L/K and by $I_v := I_v(L/K)$ its inertia group. It is worth mentioning that for an archimedean place v , we do not speak of ramification but rather of non-decomposition.

We now make the choice of a place $w|v$, and we set $L_v := L_w$. Recall that the local reciprocity symbol provides the following isomorphisms:

$$\frac{K_v^\times}{N_{L_v/K_v} L_v^\times} \simeq D_v, \quad \frac{U_v}{N_{L_v/K_v} U_{L_v}} \simeq I_v.$$

Here, $U_v \subset K_v^\times$ denotes the group of local units at v , and N_{L_v/K_v} denotes the norm map of the local extension L_v/K_v . For a real infinite place v , we adopt the convention $U_v = (\mathbb{R}^\times)^2$, and for a complex place $U_v = \mathbb{C}^\times$.

Thus,

- for places $v \in \Sigma' := \Sigma \setminus \Sigma \cap (S \cup T)$, the Artin map induces a surjective morphism from U_v to I_v , with kernel $W_v := N_{L_v/K_v} U_{L_v}$,
- for places $v \in S$, the Artin map induces a surjective morphism from K_v^\times to D_v , with kernel $W_v := N_{L_v/K_v} L_v^\times$. Note that $W_v = K_v^\times$ if and only if v splits in L/K .

Set

$$W = \prod_{v \in (S \cup \Sigma) \setminus (T \cap \Sigma)} W_v = \prod_{v \in \Sigma'} W_v \prod_{v \in S} W_v.$$

Remark 2.2. — Observe that if L/K is cyclic of degree p , then for any place $v \in \Sigma' \cup S$ we have $\iota_v((E_T^S)^p) \subset W_v$.

Definition 2.3. — We denote by $K_{\Sigma,S,T}$ the abelian extension of K corresponding, via the global Artin map, to the idèle group V :

$$V := W \left(\prod_{v \notin \Sigma' \cup S \cup T} U_v \right) \left(\prod_{v \in T} U_v^1 \right) = \left(\prod_{v \in \Sigma' \cup S} W_v \right) \left(\prod_{v \notin \Sigma' \cup S \cup T} U_v \right) \left(\prod_{v \in T} U_v^1 \right).$$

Here, U_v^1 is the subgroup of principal units of U_v .

The following proposition is central.

Proposition 2.4. — We have $M = K_{\Sigma,S,T}$. Moreover

$$\text{Gal}(K_{\Sigma,S,T}/K_T^S) \simeq \frac{U_{K,\Sigma'}^S}{\nu(E_T^S)W},$$

where $U_{K,\Sigma'}^S = \prod_{v \in S} K_v^\times \prod_{v \in \Sigma'} U_v$, and where $\nu : E_T^S \rightarrow U_{K,\Sigma'}^S$ is the diagonal embedding.

Proof. — Let's note that:

- a finite place $v \notin \Sigma' \cup S \cup T$ of K is unramified in M/K ,
- a place $v \in T$ is tamely ramified in M/K .

Therefore, the global Artin symbol for the extension M/K is trivial on

$$\left(\prod_{v \notin \Sigma' \cup S \cup T} U_v \right) \left(\prod_{v \in T} U_v^1 \right).$$

Now let's look at the part W .

For $v \in S$, since v splits totally in L_S^T/L and thus in M/L , then $M_v = L_v$. Consequently, every element ε of W_v is also locally norm in M/K . In other words, the local symbol at v in the extension M/K vanishes on W_v .

For $v \in \Sigma'$, let $\varepsilon \in W_v$. Then, by definition of W_v , there exists $z \in U_v$ such that $\varepsilon = N_{L_v/K_v}(z)$. But since the extension M_v/K_v is unramified at v , the element z is norm in M_v/L_v , and thus ε is norm in M_v/K_v . In other words, here too, the local symbol at v in the extension M/K vanishes on W_v .

In conclusion, the global Artin symbol for the extension M/K is trivial on V . Therefore, by maximality of $K_{\Sigma,S,T}$, we have $M \subset K_{\Sigma,S,T}$.

Let's show the reverse inclusion. For that, observe that $K_{\Sigma,S,T}/L$ is an abelian extension such that:

- every place $v \in T$ is tamely ramified (possibly unramified);
- for every place $v \in S$, the following commutative diagram holds:

$$\begin{array}{ccc} K_v^\times/W_v & \twoheadrightarrow & D_v(K_{\Sigma,S,T}/K) \\ & \searrow \simeq & \downarrow \\ & & D_v(L/K) \end{array}$$

showing that $D_v(K_{\Sigma,S,T}/L)$ is trivial, hence $K_{\Sigma,S,T}/L$ is decomposed at every place $v \in S$;

- similarly, every place $v \in \Sigma'$ is unramified in $K_{\Sigma,S,T}/L$.

Thus, $K_{\Sigma,S,T}$ is contained in L_S^T , and by maximality of M , we deduce that $K_{\Sigma,S,T} \subset M$. Consequently, $M = K_{\Sigma,S,T}$.

In summary, if we denote by \mathcal{I}_K the idèle group of K , and by $\mathcal{U}_{K,T}^S$ the idèle subgroup given by

$$\mathcal{U}_{K,T}^S := \prod_{v \in S} K_v^\times \prod_{v \in T} U_v^1 \prod_{v \notin T \cup S} U_v,$$

we have

$$\text{Gal}(K_{\Sigma,S,T}/K) \simeq \mathcal{I}_K / VK^\times \text{ and } \text{Gal}(K_T^S/K) \simeq \mathcal{I}_K / \mathcal{U}_{K,T}^S K^\times.$$

Therefore,

$$\text{Gal}(K_{\Sigma,S,T}/K_T^S) \simeq \mathcal{U}_{K,T}^S K^\times / VK^\times \simeq \mathcal{U}_{K,T}^S / (VK^\times) \cap \mathcal{U}_{K,T}^S \simeq \mathcal{U}_{K,T}^S / VE_T^S.$$

We conclude by noticing that $\mathcal{U}_{K,T}^S / V \simeq U_{K,\Sigma'}^S / W$. \square

3. Governing fields

Set $K' = K(\mu_p)$. We fix a generator ζ_p of μ_p . If B is an \mathbb{F}_p -module, let $B^\vee := \text{hom}(B, \mu_p)$. By Kummer duality, recall that for a subgroup of A of K'^\times , one has $A(K'^\times)^p / (K'^\times)^p \simeq \text{Gal}(K'(\sqrt[p]{A})/K')^\vee$. Moreover, if $A \subset K^\times$, then

$\text{Gal}(K'(\sqrt[p]{A})/K')^\vee \simeq A(K'^\times)^p / (K'^\times)^p \simeq A/A \cap (K'^\times)^p \simeq A/A \cap (K^\times)^p \simeq A(K^\times)^p / (K^\times)^p$, because $[K' : K]$ is coprime to p .

3.1. Frobenius. — For any place v of K , let's define

$$\mathcal{E}_{T,v}^S = \{\varepsilon \in E_T^S, \varepsilon \in (K_v^\times)^p\}.$$

This group of S -units fits into the exact sequence

$$1 \longrightarrow \mathcal{E}_{T,v}^S (K^\times)^p / (K^\times)^p \longrightarrow E_T^S (K^\times)^p / (K^\times)^p \longrightarrow i_v(E_T^S) \longrightarrow 1,$$

where $i_v : E_T^S \longrightarrow K_v^\times / (K_v^\times)^p$ is induced by the embedding ι_v of K into K_v . By Kummer duality we have

$$i_v(E_T^S)^\vee \simeq (E_T^S (K^\times)^p / \mathcal{E}_{T,v}^S (K^\times)^p)^\vee \simeq \text{Gal}(K'(\sqrt[p]{E_T^S})/K'(\sqrt[p]{\mathcal{E}_{T,v}^S})).$$

This latter Galois group is easily to interpret:

Lemma 3.1. — *One has $\text{Gal}(K'(\sqrt[p]{E_T^S})/K'(\sqrt[p]{\mathcal{E}_{T,v}^S})) = D_v(F_T^S/K')$.*

(We will see later that it does not depend on the choice of a place $w|v$ of K' .)

Proof. — Let's denote by N the subfield of F_T^S/K' corresponding, via Galois theory, to $D_v(F_T^S/K')$. Clearly, $K'(\sqrt[p]{\mathcal{E}_{T,v}^S}) \subset N$. For the reverse inclusion, note that if there exists an intermediate subfield N' of degree p over N , then, as $\text{Gal}(K'(\sqrt[p]{E_T^S})/K'(\sqrt[p]{\mathcal{E}_{T,v}^S}))$ is an abelian p -elementary group, N' arises from the compositum with a cyclic extension N_0/K' of degree p : there exists $x \in E_T^S$ such that $N_0 = K'(\sqrt[p]{x})$. Now, since v splits in N/K' , it follows that $x \in (K'_v)^p$, hence $x \in K_v^p$ because $[K'_v : K_v]$ is coprime to p ; thus $N_0 \subset K'(\sqrt[p]{\mathcal{E}_{T,v}^S})$, which leads to a contradiction. \square

When v is unramified in F_T^S/K' , the Galois group of $K'(\sqrt[p]{E_T^S})/K'(\sqrt[p]{\mathcal{E}_{T,v}^S})$ is generated by the Frobenius element associated to the choice of a place $w|v$ of K' .

For now on, let's fix $w|v$ and set $\sigma_v := \sigma_w$, where σ_w is the Frobenius at w in $\text{Gal}(K'(\sqrt[p]{\mathcal{E}^S})/K')$.

Next, let D_v be the decomposition group of v in the extension F_T^S/K' .

Let

$$\Phi_v : (E_T^S(K^\times)^p / \mathcal{E}_{T,v}^S(K^\times)^p)^\vee \longrightarrow \text{Gal}(F_T^S/K'(\sqrt[p]{\mathcal{E}_{T,v}^S})) = D_v$$

be the isomorphism arising from Kummer duality. Recall how Φ_v is defined: for $\chi \in (E_T^S(K^\times)^p / \mathcal{E}_{T,v}^S(K^\times)^p)^\vee$, we associate the element $g_\chi := \Phi_v(\chi)$ defined as follows:

$$g_\chi(\sqrt[p]{\varepsilon}) = \chi(\varepsilon) \cdot \sqrt[p]{\varepsilon},$$

for any $\varepsilon \in E_T^S$.

For $v \in \Sigma' \cup S$, consider the local map φ_v also derived from Kummer duality:

$$\varphi_v : (A_v/W_v)^\vee \hookrightarrow (A_v/A_v^p)^\vee \twoheadrightarrow i_v(E_T^S)^\vee \xrightarrow{\sim} D_v,$$

where $A_v = U_v$ (respectively $A_v = K_v^\times$) for $v \in \Sigma'$ (resp. $v \in S$).

When $(A_v/W_v)^\vee$ is non-trivial, it is generated by a certain character $\chi_v = \chi_w$. Now observe that if we choose another place $w'|v$ of K' , then $w' = hw$ for some $h \in \text{Gal}(K'/K)$. Let $\chi_{w'} := \chi_{hw} := \chi_w(h^{-1}(\cdot))$; this is a non-trivial character of $(A_{w'}/W_{w'})^\vee$.

Lemma 3.2. — *Set $g_w := \varphi_w(\chi_w)$ and $g_{w'} := \varphi_{w'}(\chi_{w'})$. Then $\langle g_w \rangle = \langle g_{w'} \rangle$.*

Proof. — This is a consequence of Kummer theory where we have $g_{w'} = g_w^a$ for some $a \in \mathbb{F}_p^\times$ (see, for example, [4, Chapter I, §6, Theorem 6.2]). \square

Thus, all the subsequent results do not depend on the choice of $w|v$.

We will now describe φ_v more precisely.

(i) This is the most important case. Let $v \in \Sigma'$. Recall that $U_v/W_v \simeq \mathbb{Z}/p$, hence $U_v^p \subset W_v$. There exists a non-trivial element χ_v of $(U_v/W_v)^\vee$ such that

$$\langle \chi_v \rangle = (U_v/W_v)^\vee \hookrightarrow (U_v/U_v^p)^\vee.$$

Then $\varphi_v(\chi_v)$ is an element $g_v := g_{\chi_v}$ of D_v , defined by

$$g_v(\sqrt[p]{\varepsilon}) = \chi_v(i_v(\varepsilon)) \cdot \sqrt[p]{\varepsilon},$$

for all $\varepsilon \in E_T^S$.

Let $Pl_{K,p} = \{v \in Pl_K, v|p\}$ be the set of p -adic places of K .

Observe that if $v \notin Pl_{K,p} \cup S_0$, then v is unramified in F_T^S/K' , and $U_v^p = W_v$. In particular, D_v is a cyclic group generated by the Frobenius σ_v at v . Thus

$$\varphi_v : \langle \chi_v \rangle \twoheadrightarrow \langle \sigma_v \rangle.$$

By taking a suitable power of χ_v , we obtain $\varphi_v(\chi_v) = \sigma_v$.

(ii) Let $v \in S_0 \setminus S_0 \cap \Sigma$. Then v is unramified in L/K .

First, note that if v splits in L/K , then $W_v = K_v^\times$ and thus φ_v is the trivial map.

Now, suppose v is inert in L/K . Then $W_v = U_v \langle \pi_v^p \rangle$ and thus

$$K_v^\times / W_v \simeq K_v^\times / U_v \langle \pi_v^p \rangle \simeq \langle \pi_v \rangle / \langle \pi_v^p \rangle.$$

Let χ_v be the generator of $(\langle \pi_v \rangle / \langle \pi_v^p \rangle)^\vee$ defined by $\chi_v(\pi_v^i) = \zeta_p^i$. Then $g_v := \varphi(\chi_v)$ satisfies: for all $\varepsilon \in E_T^S$,

$$g_v(\sqrt[p]{\varepsilon}) = \chi_v(\iota_v(\varepsilon)) \cdot \sqrt[p]{\varepsilon}.$$

Thus $\chi_v(\iota_v(\varepsilon)) = 1$ if and only if the valuation $v(\varepsilon)$ of ε is zero modulo p .

(iii) Let $v \in S_0 \cap \Sigma$. This is analogous to (i), noting that $A_v = K_v^\times$.

(iv) Here $p = 2$ and v is a real place in S .

As in (iii), if v is splitting in L/K , then $W_v = K_v^\times$ and φ_v is the trivial map. Otherwise for $\varepsilon \in E_T^S$

$$g_v(\sqrt{\varepsilon}) = \text{sign}(\iota_v(\varepsilon)) \cdot \sqrt{\varepsilon},$$

where $\text{sign}(\iota_v(\varepsilon))$ is the sign of the embedding $\iota_v(\varepsilon)$ of ε in K_v .

3.2. A restriction. — Let $T = \{v_1, \dots, v_t\}$, and for $i = 1, \dots, t$, let σ_{v_i} be the Frobenius at v_i in Γ^S ; set $H_T := \langle \sigma_v, v \in T \rangle$.

Proposition 3.3. — *Suppose that the set $\{\sigma_{v_1}, \dots, \sigma_{v_t}\}$ forms a \mathbb{F}_p -free family in Γ^S . Then*

$$\Gamma_T^S := \text{Gal}(F_T^S/K') \simeq \Gamma^S/H_T.$$

Proof. — Let's give a proof by induction on the cardinality of T . Recall that for $v \in T$, one has $N_v = 1 \pmod p$.

• Suppose $T = \{v\}$. Let $E_{\{v\}}^S = \{\varepsilon \in E^S, \varepsilon \equiv 1 \pmod{v}\}$. Define $\mathcal{E}_v^S = \{\varepsilon \in E^S, \varepsilon \in (K_v^\times)^p\}$. By Hensel's lemma, we have $E_{\{v\}}^S \subset \mathcal{E}_v^S$. Moreover, $E^S/E_{\{v\}}^S \hookrightarrow \mathbb{F}_v^\times$, where \mathbb{F}_v is the residue field at v . Thus, $E^S/E_{\{v\}}^S$ is cyclic of order dividing p . Since $E_{\{v\}}^S \subset \mathcal{E}_v^S$, it follows

$$(1) \quad \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \frac{E^S(K^\times)^p}{E_{\{v\}}^S(K^\times)^p} \twoheadrightarrow \frac{E^S(K^\times)^p}{\mathcal{E}_v^S(K^\times)^p}.$$

By Lemma 3.1, we have:

$$\left(\frac{E^S(K^\times)^p}{\mathcal{E}_v^S(K^\times)^p} \right)^\vee = \langle \sigma_v \rangle \subset \Gamma^S.$$

Now, since $\sigma_v \neq 0$ by assumption, it follows that $\frac{E^S(K^\times)^p}{\mathcal{E}_v^S(K^\times)^p} \simeq \mathbb{Z}/p\mathbb{Z}$. Thus, from (1) we

have $\frac{E^S(K^\times)^p}{E_{\{v\}}^S(K^\times)^p} = \langle \sigma_v \rangle^\vee$, or equivalently $\text{Gal}(F^S/F_T^S) = \langle \sigma_v \rangle$. This concludes this case.

• Let's suppose $T = T_0 \cup \{v\}$, and that the proposition is true for T_0 . Define $\mathcal{E}_{T_0, v}^S = \{\varepsilon \in E^S, \varepsilon \equiv 1 \pmod{v'}, v' \in T_0, \varepsilon \in U_v^p\}$. By Hensel's lemma, we have $E_T^S \subset \mathcal{E}_{T_0, v}^S$. Similarly as before, $\frac{E_{T_0}^S}{E_T^S} \hookrightarrow \mathbb{F}_v^\times$, implying that $\frac{E_{T_0}^S}{E_T^S(E_{T_0}^S)^p}$ is a cyclic group of order dividing p , so we have

$$(2) \quad \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \frac{E_{T_0}^S}{E_T^S(E_{T_0}^S)^p} \twoheadrightarrow \frac{E_{T_0}^S(K^\times)^p}{E_T^S(K^\times)^p} \twoheadrightarrow \frac{E_{T_0}^S(K^\times)^p}{\mathcal{E}_{T_0, v}^S(K^\times)^p}.$$

Then we define

$$\left(\frac{E_{T_0}^S(K^\times)^p}{\mathcal{E}_{T_0, v}^S(K^\times)^p} \right)^\vee = \langle \bar{\sigma}_v \rangle,$$

where $\bar{\sigma}_v$ is the restriction of the Frobenius $\sigma_v \in \Gamma^S$ to $F_{T_0}^S$. By the induction hypothesis,

$$\left(\frac{E^S(\mathbb{K}^\times)^p}{E_{T_0}^S(\mathbb{K}^\times)^p} \right)^\vee = \langle \sigma_{v'}, v' \in T_0 \rangle.$$

But $\bar{\sigma}_v = 1$ would imply $\sigma_v \in \langle \sigma_{v'}, v' \in T_0 \rangle$ which contradicts the assumption. Therefore the surjection in (2) are isomorphisms, and $\text{Gal}(F^S/F_T^S)$ is generated by the Frobenius elements σ_v and $\sigma_{v'}, v' \in T_0$. This concludes the proof. \square

Remark 3.4. — The Galois group $\text{Gal}(F^S/F_{\{v\}}^S)$ may not be generated by the Frobenius at v . Let's give an example.

Take $\mathbb{K} = \mathbb{Q}$ and $p = 2$. Choose $T = \{\ell\}$, where $\ell \equiv 1 \pmod{4}$ is a prime number. Let $S = S_\infty = \{v_\infty\}$. We have $E^S = \langle \pm 1 \rangle$ and $E_{\{\ell\}}^S = \langle 1 \rangle$. Thus

$$F^S = \mathbb{Q}(\sqrt{E^S}) = \mathbb{Q}(\sqrt{-1}), \text{ and } F_{\{\ell\}}^S = \mathbb{Q}(\sqrt{E_{\{\ell\}}^S}) = \mathbb{Q}.$$

However, since ℓ is splitting in $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$, it follows that $\sigma_\ell = 1$. Consequently, $\text{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q})$ is not generated by the Frobenius at ℓ .

When $p = 2$ we can handle the archimedean places in the same way. Set $\bar{S} := S_0 \cup \text{Pl}_{\mathbb{K}, \infty}^{re}$. Observe that $E^{\bar{S}}$ is the group of S_0 -units in the classical sense (with no sign condition).

Proposition 3.5. — Take $p = 2$. Set $H^{S_\infty} = \langle \sigma_v; v \in \text{pl}_{\mathbb{K}, \infty}^{re} \setminus S_\infty \rangle \subset \Gamma^{\bar{S}}$ and identify H_{S_∞} with its restriction to $\Gamma_T^{\bar{S}}$. Then we have

$$\Gamma_T^S := \text{Gal}(\mathbb{K}(\sqrt{E_T^S})/\mathbb{K}) \simeq \Gamma_T^{\bar{S}}/H_{S_\infty}.$$

Moreover, if the set $\{\sigma_v, v \in T\}$ forms a linearly independent family in $\Gamma^{\bar{S}}$, then

$$\text{Gal}(\mathbb{K}(\sqrt{E_T^S})/\mathbb{K}) \simeq \Gamma^{\bar{S}}/(H_{S_\infty} + H_T).$$

Proof. — Similar to Lemma 3.1, we can show that $\mathbb{K}(\sqrt{E_T^S})$ corresponds, by Galois theory, to the subgroup H_{S_∞} of $\Gamma_T^{\bar{S}}$.

As for the second part, from Proposition 3.3 we know that $\Gamma_T^{\bar{S}} \simeq \Gamma^{\bar{S}}/H_T$; thus, we conclude with the first point. \square

4. Main result

We keep the notations from the previous sections. In particular, $\Sigma' = \Sigma \setminus \Sigma \cap (S \cup T)$.

For $v \in \Sigma' \cup S$, let's consider the elements $g_v := \varphi_v(\chi_v) \in \Gamma_T^S$ defined in (i) – (iv) of §3.1. Recall that if v is unramified in F_S^T/K' , then $g_v = \sigma_v$ is the Frobenius of v in Γ_T^S .

Let $\Theta_{\Sigma, T}^S$ be the following linear map:

$$\Theta_{\Sigma, T}^S : \left(\frac{U_{\mathbb{K}, \Sigma'}^S}{W} \right)^\vee \longrightarrow \Gamma_T^S.$$

defined by $\Theta_{\Sigma, T}^S(\chi_v) = g_v$.

Theorem 4.1. — The Artin map induces the following isomorphism:

$$\ker(\Theta_{\Sigma, T}^S) \simeq \text{Gal}(\mathbb{K}_{\Sigma, S, T}/\mathbb{K}_T^S)^\vee.$$

Proof. — First, let's observe that

$$E^S \cap (\mathbb{K}^\times)^p = (E^{\bar{S}})^p,$$

where $E^{\bar{S}}$ is the group of S_0 -units in the classical sense. Then

$$E_T^S(E^{\bar{S}})^p/(E^{\bar{S}})^p \simeq E_T^S(\mathbb{K}^\times)^p/(\mathbb{K}^\times)^p.$$

Now, consider the exact sequence obtained from Proposition 2.4 and Remark 2.2:

$$1 \longrightarrow \nu(E_T^S(E^{\bar{S}})^p/(E^{\bar{S}})^p) \longrightarrow U_{\mathbb{K},\Sigma'}^S/W \longrightarrow \text{Gal}(\mathbb{K}_{\Sigma,S,T}/\mathbb{K}_T^S) \longrightarrow 1.$$

By Kummer duality, we have:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Gal}(\mathbb{K}_{\Sigma \cup S}/\mathbb{K}_T^S)^\vee & \longrightarrow & (U_{\mathbb{K},\Sigma'}^S/W)^\vee & \longrightarrow & \left(\nu(E_T^S(E^{\bar{S}})^p/(E^{\bar{S}})^p) \right)^\vee \longrightarrow 1 \\ & & & & \downarrow \text{dotted} & & \downarrow \\ & & & & \Gamma_T^S & \xleftarrow[\simeq]{\Psi} & \left(E_T^S(E^{\bar{S}})^p/(E^{\bar{S}})^p \right)^\vee \end{array}$$

Now,

$$(U_{\mathbb{K},\Sigma'}^S/W)^\vee \simeq \prod_{v \in \Sigma'} (U_v/W_v)^\vee \prod_{v \in S} (U_v/W_v)^\vee.$$

Then, it suffices to observe that the induced map from $(U_{\mathbb{K},\Sigma'}^S/W)^\vee$ to Γ_T^S corresponds to $\Theta_{\Sigma,T}^S$. Therefore, we finally obtain:

$$\text{Gal}(\mathbb{K}_{\Sigma,S,T}/\mathbb{K}_T^S)^\vee \simeq \ker \left((U_{\mathbb{K},\Sigma'}^S/W)^\vee \xrightarrow{\Theta_{\Sigma,T}^S} \Gamma_T^S \right).$$

Hence the result. □

Therefore, it follows that

Corollary 4.2. — We have $g_T^S = \#\ker(\Theta_{\Sigma,T}^S)$.

Proof. — This is a consequence of Theorem 4.1 and Proposition 2.4. □

It follows that if $v \in S$ splits in L/\mathbb{K} , then the component at v in $\frac{U_{\mathbb{K},\Sigma'}^S}{W}$ is trivial. Set $S = S^{sp} \cup S^{ns}$, where S^{sp} is the set of places in S that split in L/\mathbb{K} , and $S^{ns} = S \setminus S^{sp}$. Let $s^{ns} = \#S^{ns}$ and $m := \#\Sigma$.

Then $\left(\frac{U_{\mathbb{K},\Sigma'}^S}{W} \right)^\vee$ is isomorphic to $(\mathbb{Z}/p)^{s^{ns}+m}$.

Corollary 4.3. — We have $g_T^S = \#\ker(\Theta_{\Sigma,T}^S)$. En particular

$$m + s^{ns} - r_T^S \leq \log_p(g_T^S) \leq m + s^{ns},$$

where r_T^S is the p -rank of E_T^S .

Proof. — It suffices to note that $\dim \Gamma_T^S = \dim E_T^S(\mathbb{K}^\times)^2/(\mathbb{K}^\times)^2 \leq r_T^S$. □

Remark 4.4. — We have $\dim \Gamma_T^S \leq r_{S_0}$, where $r_{S_0} = r_1 + r_2 + |S_0| - \delta_{\mathbb{K},p}$. When the Frobenius elements of the places $v \in T$ are linearly independent in Γ^S , we also have $\dim \Gamma_T^S = \dim \Gamma^S - |T| \leq r_{S_0} - |T|$. (See Proposition 3.3.)

Corollary 4.5 (Theorem 1.1). — *If $S^{ns} = \Sigma \cap Pl_{K,p} = \emptyset$, let $\{e_{v_1}, \dots, e_{v_m}\}$ be a basis of $(\mathbb{F}_p)^m$ indexed by the places v in Σ , and let Θ be the linear map defined by:*

$$\begin{aligned} \Theta : (\mathbb{F}_p)^m &\longrightarrow \Gamma_T^S \\ e_v &\longmapsto \sigma_v. \end{aligned}$$

Then $g_S^T = \#\ker(\Theta)$.

Proof. — In this case, $g_v = \sigma_v$. □

5. Examples

5.1. Quadratic extensions. — Let's take $p = 2$ and $K = \mathbb{Q}$. Let L/\mathbb{Q} be a quadratic extension with set of ramification $\Sigma = \{p_1, \dots, p_m\}$. Set $L = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{Z}$ is square-free.

Let $S_0 = \{\ell_1, \dots, \ell_s\}$. We assume that $\Sigma \cap S = \emptyset$.

We denote ℓ_∞ as the infinite place; then $S_\infty = \{\ell_\infty\}$ or $S_\infty = \emptyset$. In the spirit of Proposition 3.3, we assume $T = \emptyset$.

Let E^S be the group of S -units of \mathbb{Q} . We write $E^S = \langle \ell_0, \dots, \ell_s \rangle$, with $\ell_0 = -1$ or 1 depending on whether $S_\infty = \{\ell_\infty\}$ or not.

In this context, the governing field is written as $F^S = \mathbb{Q}(\sqrt{E^S}) = \mathbb{Q}(\sqrt{\ell_0}, \dots, \sqrt{\ell_s})$. Its Galois group $\Gamma^S := \text{Gal}(F^S/\mathbb{Q})$ is isomorphic to $\prod_{j=0}^s \text{Gal}(\mathbb{Q}(\sqrt{\ell_j})/\mathbb{Q})$. Note that $\text{Gal}(\mathbb{Q}(\sqrt{\ell_0})/\mathbb{Q})$ may be trivial.

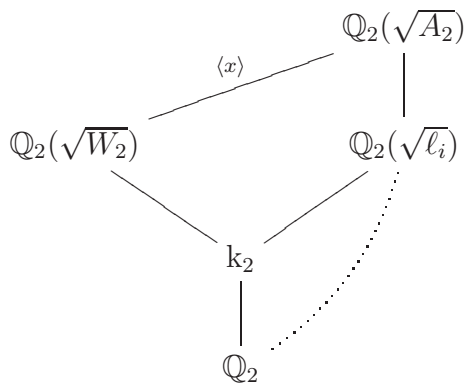
Let's revisit the morphism φ_v defined in Section §3.1 and consider its restriction to $\mathbb{Q}(\sqrt{\ell_j})$: its value is in $\{0, 1\}$. For what follows, the quadratic residue symbol is viewed additively, meaning it takes values in \mathbb{F}_2 .

Lemma 5.1. — *The elements g_ℓ takes the following values:*

- (a) For $\ell \in \Sigma'$ and ℓ odd, $g_\ell = \sigma_\ell$ restricts to $\mathbb{Q}(\sqrt{\ell_j})$ equals $\left(\frac{\ell_j}{\ell}\right)$.
- (b) For $\ell \in S_0^{ns} \setminus S_0^{ns} \cap \Sigma$, g_ℓ restricts to $\mathbb{Q}(\sqrt{\ell_j})$ is trivial if and only if $\ell \neq \ell_j$.
- (c) For $\ell = \ell_\infty$, the element g_{ℓ_∞} restricts to $\mathbb{Q}(\sqrt{\ell_j})$ is trivial unless $\ell_j = \ell_0 = -1$ and L is imaginary.

Proof. — (a) is (i) of §3.1, (b) is (ii) and (c) is (iv). □

It remains to describe g_2 when 2 is ramified in L/\mathbb{Q} . So, suppose $2 \in \Sigma$. We identify g_2 with its restriction to $\text{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$. We have the following extensions



Recall that $A_2 = U_2$ (respectively $A_2 = \mathbb{Q}_2^\times$) if $2 \notin S$ (resp. $2 \in S$).

The desired element g_2 is the image of the reduction of x in $\text{Gal}(\mathbb{Q}_2(\sqrt{\ell_i})/\mathbb{Q}_2) \hookrightarrow \text{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$. Therefore g_2 (restricted) is trivial if and only if $\ell_i \in W_2$ modulo $(A_2)^2$.

Let's take for example $\ell_i = 2$. Observe that for $2 \in S$, then $2 \in W_2$, and thus $g_2 = 0$. On the other hand, if $2 \notin S$, then $W_2 \subset U_2$ and consequently $g_2 = 1$.

In general, everything relies on determining W_2 , which is the conductor at 2 of L/K ; see [4, Chapter II, §1, Exercise 1.6.5] for calculations.

For example, suppose $d \equiv -1$ modulo 8. Then $W_2 = \langle 5 \rangle$. Hence, g_2 restricted to $\text{Gal}(\mathbb{Q}(\sqrt{\ell_i})/\mathbb{Q})$ is trivial if and only if, $\ell_i \equiv 1$ modulo 4.

A particularly noteworthy situation arises when we are only dealing with cases (a) and (b) of Proposition 5.1 and L/\mathbb{Q} is unramified at 2. Let's detail this situation.

Let the canonical basis $\mathcal{B} := \{e_{p_1}, \dots, e_{p_m}, e_{\ell_0}, e_{\ell_1}, \dots, e_{\ell_s}\}$ of \mathbb{F}_2^{m+s+1} be indexed by the places of $\Sigma \cup \{\ell_\infty\} \cup S_0$; here we take $S_\infty = \{\ell_\infty\}$ and set $e_{\ell_0} := e_{\ell_\infty}$.

The map $\Theta := \Theta_\Sigma^S$ on the basis \mathcal{B} , taking values in $\prod_{j=0}^n \text{Gal}(\mathbb{Q}(\sqrt{\ell_j})/\mathbb{Q})$, is defined by

$$\Theta(e_{p_i})|_{\mathbb{Q}(\sqrt{\ell_j})} = \begin{pmatrix} \ell_j \\ p_i \end{pmatrix}, \quad \Theta(e_{\ell_i})|_{\mathbb{Q}(\sqrt{\ell_j})} = \delta_{i,j}^*,$$

where for $0 \leq i \leq s$, $\delta_{i,j} = 0$ when $i \neq j$; for $j > 0$,

$$\delta_{j,j}^* = \begin{cases} 1 & \text{if } \ell_j \text{ is inert in } L/\mathbb{Q} \\ 0 & \text{if } \ell_j \text{ splits in } L/\mathbb{Q} \end{cases}$$

and $\delta_{0,0}^* = 1$ if L is imaginary, 0 otherwise.

The matrix Θ is then written as follows

$$\begin{pmatrix} \begin{pmatrix} -1 \\ p_1 \end{pmatrix} & \begin{pmatrix} -1 \\ p_2 \end{pmatrix} & \cdots & \begin{pmatrix} -1 \\ p_m \end{pmatrix} & \delta_{0,0}^* & 0 & \cdots & 0 \\ \begin{pmatrix} \ell_1 \\ p_1 \end{pmatrix} & \begin{pmatrix} \ell_1 \\ p_2 \end{pmatrix} & \cdots & \begin{pmatrix} \ell_1 \\ p_m \end{pmatrix} & 0 & \delta_{1,1}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} \ell_s \\ p_1 \end{pmatrix} & \begin{pmatrix} \ell_s \\ p_2 \end{pmatrix} & \cdots & \begin{pmatrix} \ell_s \\ p_m \end{pmatrix} & 0 & 0 & \cdots & \delta_{s,s}^* \end{pmatrix}.$$

Observe that if we take $S_\infty = \emptyset$, then to obtain the matrix, simply remove the first row and the $(m+1)$ th column of the above matrix.

5.2. A result of existence. — In this subsection, we make explicit Corollary 4.3 by adapting Theorem 1.3 from [11]. To simplify, we only consider extensions that are splitting at infinity. Thus, we assume that $Pl_{\mathbb{K},\infty}^{r,c} \subset S$. Note that for $p > 2$, the archimedean places play no role in the calculations.

Recall that $r_S = \dim E^S = r_1 + r_2 + \#S_0 - 1 + \delta_{\mathbb{K}}$. Let $s = \#S_0$ and define $A_S := A_{S_0} := \prod_{v \in S_0} N_v$.

Theorem 5.2. — *Let \mathbb{K} be a number field, and let $S = S_0 \cup Pl_{\mathbb{K},\infty}^{r,c}$ be a set of places of \mathbb{K} . Let $k, m \geq 1$, such that $m - r_S \leq k \leq m$. Let p be a prime number. Then there exist infinitely many sets Σ of finite places of \mathbb{K} of size m , such that there exists an extension L/\mathbb{K} cyclic of degree p , Σ -(totally) ramified, S -split, and with $g^S := g_\emptyset^S = p^k$.*

Moreover, assuming GRH, when m is fixed, such a set Σ can be chosen such that the absolute norm N_v of each of its elements v is smaller than $p^{2rs+2}(c_1 \log p + c_2 \log A_S)^2$, where c_1 and c_2 are constants depending on K and m .

Proof. — The proof relies on [11, Section 4, Proof of Theorem 1.3], with two modifications due to S . The first modification concerns the "existence" part, and the second concerns the "quantitative" part.

- We need the existence of a cyclic extension L/K of degree p with specific properties. For the first part, we utilize the Gras-Munnier theorem with splitting (see [4, Chapter V, Section 2, Corollary 2.4.2] and [5]).

Let $V_S := \{x \in K^\times, v(x) = 0, \forall v \notin S\}$ be the S -Selmer group of K . Set $\tilde{F}^S := K'(\sqrt[p]{V_S})$. The exact sequence

$$1 \longrightarrow E^S/(E^S)^p \longrightarrow V_S/(K^\times)^p \longrightarrow \text{Cl}_K^S[p] \longrightarrow 1$$

shows that $[\tilde{F}^S : F^S] = O(1)$, uniformly for p and S .

Let k and m be non-negative integers such that $m - r_S \leq k \leq m$. Set $r = m - k \leq r_S$. Let p^l be the degree of \tilde{F}^S/F^S .

Let us take an \mathbb{F}_p -basis $(e_i)_{i=1, \dots, r_S}$ of $\text{Gal}(F^S/K')$ and complete it to an \mathbb{F}_p -basis $(e_i)_{i=1, \dots, r_S+l}$ of $\text{Gal}(\tilde{F}^S/K')$. By Chebotarev's density theorem, let $\Sigma = \{v_1, \dots, v_m\}$ be places of K prime to p such that their Frobenius elements, denoted $\sigma_{v_i} \in \text{Gal}(\tilde{F}_S/K') \subset \text{Gal}(\tilde{F}_S/K)$, satisfy:

- (a) $\sigma_{v_1} = -(e_1 + \dots + e_r)$;
- (b) for $i = 2, \dots, r+1$, $\sigma_{v_i} = e_{i-1}$;
- (c) for $i = r+2, \dots, m$, $\sigma_{v_i} = 0$,

when $r \geq 1$. When $r = 0$, choose the v_i 's such that $\sigma_{v_i} = 0$, $i = 1, \dots, m$.

One has $\sum_{i=1}^m \sigma_{v_i} = 0$: By the Gras-Munnier theorem referenced earlier, there exists a cyclic extension L/K of degree p , Σ -totally ramified and S -split. Furthermore by the choice of the e_i 's and the v_i 's, the map Θ of Corollary 4.5 has rank r . Hence, $\text{Gal}(M/K^S) \simeq (\mathbb{F}_p)^{m-r} = (\mathbb{F}_p)^k$.

- We now need a second modification: it concerns the discriminant $\text{disc}(\tilde{F}^S)$ of \tilde{F}^S . The introduction of S can affect $\text{disc}(\tilde{F}^S)$ of \tilde{F}^S compared to $\text{disc}(\tilde{F}^\emptyset)$ in two different ways. Firstly, the places prime to p contained in S , may become ramified, and for the places w of K' in S lying above p , the valuation of the local conductor \mathfrak{f}_w in a cyclic extension L_w/K'_w of degree p may increase. Let e_w be the index of ramification of w in K'/\mathbb{Q} . Observe now that

- for $w|p$, $w(\mathfrak{f}_w) \leq 1 + \frac{p}{p-1}e_w$,
- for $w \nmid p$, $w(\mathfrak{f}_w) \leq p - 1$.

See for example [12, Chapter III, §6, Remarks & Chapter V, §3, Lemma 3]. If we follow the calculations from [11, proof of Proposition 3.2], we obtain:

$$|\text{disc}(\tilde{F}^S)| \leq (|\text{disc}(K)| \cdot A_S \cdot p^{4[K:\mathbb{Q}]})^{p^{r_S+l+1}}.$$

By using the fact that $[\tilde{F}^S : F^S] = O(1)$, Lemma 4.4 of [10] provides the result. \square

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