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MEASURE AND CONTINUOUS VECTOR FIELD AT A BOUNDARY II: GEODESICS AND SUPPORT PROPAGATION

NICOLAS BURQ, BELHASSEN DEHMAN, AND JÉRÔME LE ROUSSEAU

Abstract. Nonnegative measures that are solutions to a transport equation with *continuous* coefficients have been widely studied. Because of the low regularity of the associated vector field, there is no natural flow since nonuniqueness of integral curves is the general rule. It has been known since the works by L. Ambrosio [2] and L. Ambrosio and G. Crippa [3, 4] that such measures can be described as a *superposition* of δ-measures supported on integral curves. In this article, motivated by some *observability* questions for the *wave equation*, we are interested in such transport equations in the case of *domains with boundary*. Associated with a wave equation with $\mathscr{C}^1$-coefficients are bicharacteristics that are integral curves of a continuous Hamiltonian vector field. We first study in details their behaviour in the presence of a boundary and define their natural generalisation that follows the laws of geometric optics. Then, we introduce a natural class of transport equations with a source term on the boundary, and we prove that any *nonnegative* measure satisfying such an equation has a union of maximal generalized bicharacteristics for support. This result is a weak form of the superposition principle in the presence of a boundary. With its companion article [7], this study completes the proof of wave observability generalizing the celebrated result of Bardos, Lebeau, and Rauch [5] in a low regularity framework where coefficients of the wave equation (and associated metric) are $\mathscr{C}^1$ and the boundary and the manifold are $\mathscr{C}^2$.

Contents

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1. Introduction

We are interested in studying the properties of nonnegative measures solutions of ordinary differential equations. If X is a *smooth* vector field, in say $\mathbb{R}^d$, and if a measure μ solves the free transport equation

$$ {}^tX\mu = 0, \tag{1.1}$$

then

(1.2) *the measure μ is invariant along the integral curves of X.*

Such a transport equation arises naturally in many contexts. Our initial motivation lies in the observation of waves. Understanding propagation properties of measures fufilling an equation like (1.1) has become a key point to obtain an observability inequality, that is, an estimate of the energy by a "recording" of the solution in a restricted domain for some time $T > 0$, and, as a corollary, an exact controllability property for the wave equation. We refer to [5, 9] for this topic where microlocal techniques allow one to find the proper geometrical condition for wave observability to hold in connection with measure transport along the geodesic flow. This condition known as the Geometric Control Condition (GCC) roughly states:

every geodesic should reach the observation region in the prescribed time $T > 0$.

In this framework, the vector field that plays the role of X in (1.2) is the Hamiltonian vector field H_p associated with the principal symbol p of the wave operator. It is precisely the vector field that generates *bicharacteristics* in phase-space and geodesics are their projection on the base space.

In the case of a domain with a boundary, generalized geodesics have to be considered. They follow the reflections laws of optics at boundaries and more complex behavior when reaching the boundary tangentially. In a smooth setting, their mathematical study in connection with the propagation of singularities for waves was carried out in the work by R. Melrose and J. Sjöstrand [18]; see also [15, Chapter 24]. Analyzing a transport equation for measures becomes more subtle if the boundary in encountered. In the transport equation we consider there is a source term associated with a second nonnegative measure supported in the boundary. It takes the form

$$ {}^t\mathrm{H}_p\,\mu = -\int_{\varrho\in{}^{\shortparallel}\mathcal{H}_\partial\cup{}^{\shortparallel}\mathcal{G}_\partial} \frac{\delta_{\varrho^+}-\delta_{\varrho^-}}{\langle \xi^+-\xi^-,\mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\,d\nu(\varrho), \tag{1.3} $$

where ν is a nonnegative measure on (the cotangent bundle of) the boundary. All terms in (1.3) are explained in Sections 2 and 3 below. Such an equation was first derived in the work of P. Gérard and É. Leichtnam [14]. In the case of smooth coefficients, the transport of the measure μ can still be well understood along the generalized geodesic flow; see [17] where a slightly different measure is introduced. In our main results below, we consider a generalized version of this equation by adding a term $f\mu$ with f a continuous real function.

In the framework of wave observability, the issue of limited smoothness of coefficients has received little attention in works relying on microlocal techniques as those cited above. The main reason is that using such techniques is known to consume many derivatives. In [6], the derivation of an observability estimate is carried out for waves in the case of rough ($\mathscr{C}^1$) coefficients on a manifold *without* boundary. A transport equation for a nonnegative measure is also at the heart of the argument, yet with a vector field $X = \mathrm{H}_p$ which is only *continuous*. This transport equation takes the simple form of (1.1) since there is no boundary in the setting of [6]. For a regularity as low as Lipschitz, or log-Lipschitz, for the vector field X, the Cauchy-Lipschitz formula and its extensions, yield a unique flow for X and one obtains the same invariance result as in (1.2). If now X is a $\mathscr{C}^0$-vector field, while the Peano theorem yields the existence of integral curves for X, one cannot guarantee the uniqueness of those integral curves and the existence of a flow. This compromises the extension of the transport result in (1.2). Still, a remarkable quantitative result is achieved in the work of L. Ambrosio and G. Crippa [2, 3, 4]: the measure μ solution to (1.1) can be written as a sum of positive measures, each defined as a constant times δ_γ, that is, the measure supported by an integral curve γ of X and invariant along γ. The sum is defined by means of a nonnegative measure on the space of all continuous curves. The result of [4] is thus a *superposition principle*.

The result of [4] relies on a smoothing procedure of the vector field X as in [13] that we have not been able to extend in the presence of a boundary. To prepare the work exposed in the present article, in [6] we proved a weaker version of the superposition principle of [4], that is,

$$\operatorname{supp}\mu \text{ is a union of maximal integral curves of } X,$$

if μ is a solution to the transport equation (1.1). Our proof scheme is much different from that of [4] and is inspired by the microlocal techniques used in [18]. In the present article we extend this latter proof scheme to regions at the boundary. We prove that if a nonnegative mesure μ fulfills a transport equation of the form given in (1.3) then

$$\operatorname{supp}\mu \text{ is a union of maximal generalized bicharacteristics.} \tag{1.4}$$

From this result, an observability estimate for waves is deduced in the companion article [7] in the case of $\mathscr{C}^1$-coefficients for the wave operator under a geometric control condition, using a transport equation of the form of (1.3) also proven in [7]. The main result of the present article, Theorem 3.4 below that states (1.4) in details, is thus the missing link to complete the proof of wave observability in

the case of $\mathscr{C}^1$-coefficients and the presence of a ($\mathscr{C}^2$) boundary thus generalizing the celebrated result of Bardos, Lebeau, and Rauch [5].

Note that the regularity we consider, that is $\mathscr{C}^1$-coefficients for the wave operator, stands as a limit case, as far as the underlying geometry is concerned, since the Hamiltonian vector field H_p that generates bicharacteristics is then only continuous. If considering coefficients with lower regularity, one obtains a Hamiltonian vector field H_p that may not be continuous. Then, the mere existence of bicharacteristics and geodesics is not guaranteed (and the meaning of (1.1) neither)

1.1. **Towards a superposition principle.** With the result on the measure support stated in (1.4) and proven here, a quantitative version of this result in the spirit of the superposition principle of L. Ambrosio and G. Crippa [2, 3, 4] appears now to be a very interesting open question. A first step in this direction is given here: we prove that the Dirac-measure $\delta_{\mathsf{G}_{\bar\gamma}}$ supported on a maximal generalized bicharacteristic $^{\mathsf{G}}\bar\gamma$ obeys a transport equation of the form of (1.3). Not having this property would ruin any hope to obtain a superposition principle, since any sum of such Dirac-measures automatically fulfills an equation of this type. A precise statement and a proof is given in Appendix A.

1.2. **Perspectives in the presence of a flow.** The regularity level we consider here yields a continuous Hamiltonian vector field. As mentionned above, this suffices for generalized bicharacteristics to exist but lack of uniqueness prevents the existence of a flow *in general*. One cannot however exclude that uniqueness holds true and a flow exists in some particular cases. Does the result presented here improve in such cases? We foresee that it does in the following strong form: a measure solution to the transport equation (1.3) is actually transported along the generalized bicharacteristic flow. This is the subject of an ongoing work [8].

A second perspective concerns the case where the Hamiltonian vector field H_p is continuous and moreover lies in the Sobolev class $W^{1,1}$. Then, as H_p is naturally divergence free, this fits the setting of the celebrated article of R.J. DiPerna and P.L. Lions [12]. Then, a flow exists in a weak sense, with flow properties fulfilled almost everywhere. Can one obtain a weak form of transport along such flow for (a large class of) measures solutions to (1.3)? This stands as a very interesting and natural open question. Extension to more general vector fields as in [1] is of great interest also.

1.3. **Outline and notation.** The present article is organized as follows. In Section 2, the geometrical notions necessary to the statement of our results are presented and our main result, that is, a precise formulation of the propagation

of the measure support given in (1.4), is stated in Section 3. Section 4 covers the case of a transport equation in two cases: (1) away from any boundary and (2) a boundary transverse to the vector field. A full treatment of the boundary requires the introduction of additional geometrical notions; this is done in Section 5. The proof of the main result on the propagation of the measure support is carried out in Section 6. A mass property of the boundary measure ν that can be deduced from the transport equation (1.3) is proven in Section 7.

In Appendix A we prove that the linear measure supported by a single generalized bicharacteristic fulfills an equation of the form of (1.3). Appendix B provides a proof of the existence of generalized bicharacteristics. Despite nonuniqueness a continuity property is also given. In Appendix C we present the quasi-normal coordinates that are often used near the boundary.

Often for the sake of concision, the Einstein summation convention for repeated indices are used. In local coordinates $B(x,r)$ denotes the Euclidean open ball centered at x with radius r.

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2. Geometrical setting I

Consider a $\mathscr{C}^2$ compact connected d-dimensional manifold $\mathcal{M}$ with boundary equipped with a $\mathscr{C}^1$-Riemannian metric g. In this section we introduce some geometrical notions that are necessary for the statement of our main results. Many more details are available in Section 5.

A simple example that fits our present setting is that of a bounded open subset Ω of $\mathbb{R}^d$ with a $\mathscr{C}^2$-boundary, that is, with the boundary given locally by $\varphi(x)=0$ with $\varphi\in\mathscr{C}^2(\mathbb{R}^d)$ and $d\varphi\neq 0$. Then $\mathcal{M}=\Omega\cup\partial\Omega$ and one can simply consider the Euclidean metric. In the spirit of this simple example, we consider an open d-dimensional manifold[1] $\tilde{\mathcal{M}}$ such that $\mathcal{M}\subset\tilde{\mathcal{M}}$ and extend the metric g to a neighborhood of $\mathcal{M}$ is a $\mathscr{C}^1$-manner.

[1]The manifold $\tilde{\mathcal{M}}$ can be constructed by embedding $\mathcal{M}$ in $\mathbb{R}^{2d}$ thanks to the Whitney theorem [19].

2.1. **Local coordinates.** Equip a compact neighborhood $\hat{\mathcal{M}}$ of $\mathcal{M}$ in $\tilde{\mathcal{M}}$ with a finite $\mathscr{C}^2$-atlas. A local chart is denoted (O, ϕ) with O an open subset of $\hat{\mathcal{M}}$ and ϕ a one-to-one map from O onto an open subset of $\mathbb{R}^d$. Charts can be chosen so that

$$\begin{aligned} &\phi(O \cap \mathcal{M}) = \phi(O) \cap \{x_d \geq 0\} \text{ is an open subset of } \overline{\mathbb{R}^d_+}, \\ &\phi(O \cap \partial\mathcal{M}) = \phi(O) \cap \{x_d = 0\}, \text{ and } \phi(O \setminus \mathcal{M}) = \phi(O) \cap \{x_d < 0\}, \end{aligned} \tag{2.1}$$

if $O \cap \partial\mathcal{M} \neq \emptyset$. Denote the local coordinates by $x = (x', x_d)$ with $x' \in \mathbb{R}^{d-1}$. Note that $\mathcal{M}$ being compact it contains its boundary $\partial\mathcal{M}$.

In a local chart, the metric g is given by $g_x = g_{ij}(x)dx^i \otimes dx^j$, where $g_{ij} \in \mathscr{C}^1(\phi(O))$. We use the classical notation $(g^{ij}(x))_{i,j}$ for the inverse of $(g_{ij}(x))_{i,j}$. The metric $g_x = (g_{ij}(x))_{i,j}$ provides an inner product on $T_x\mathcal{M}$. The metric $g^*_x = g^{ij}(x)d\xi_i \otimes d\xi_j$ provides an inner product on $T^*_x\mathcal{M}$, denoted $g^*_x(\xi, \tilde{\xi})$, for $\xi, \tilde{\xi} \in T^*_x\mathcal{M}$. Define the associated norm

$$|\xi|_x = g^*_x(\xi, \xi)^{1/2}.$$

In this introductory section, near a boundary point, local coordinates are chosen according to the following proposition as they simplify the exposition of some geometrical notions.

Proposition 2.1 (quasi-normal geodesic coordinates)**.** *Suppose $m^0 \in \partial\mathcal{M}$. There exists a $\mathscr{C}^2$-local chart (O, ϕ) such that $m^0 \in O$, $\phi(m) = (x', z)$, with $x' \in \mathbb{R}^{d-1}$ and $z \in \mathbb{R}$, and*

1. *$\phi(O \cap \mathcal{M}) = \{z \geq 0\} \cap \phi(O)$, $\phi(O \cap \partial\mathcal{M}) = \{z = 0\} \cap \phi(O)$, and $\phi(O \setminus \mathcal{M}) = \{z < 0\} \cap \phi(O)$;*
2. *at the boundary, the representative of the metric has the form*

$$g(x', z = 0) = \sum_{1 \leq i,j \leq d-1} g_{ij}(x', z = 0)dx^i \otimes dx^j + |dz|^2.$$

In other words the matrix of $g = (g_{ij})$ has the block-diagonal form *at the boundary*

$$g(x', z = 0) = \begin{pmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Naturally, the same form holds for $g_x^* = (g^{ij}(x))$ at the boundary. One deduces that

$$g_{jd}(x', z) = z h_{jd}(x', z) \quad \text{and} \quad g_{dd}(x', z) = 1 + z h_{dd}(x', z),$$

for some continuous functions h_{jd}, $j = 1, \dots, d$.

Proposition 2.1 can be found in [10] with a different regularity level. We provide a proof in Appendix C based on that of [10] with a generalization to other levels of regularity.

Remark 2.2. Because of the low regularity of g and $\mathcal{M}$ one *cannot choose* normal geodesic coordinates, that is, local coordinates for which $g_{jd} = g_{dj} = 0$ for $j \neq d$ and $g_{dd} = 1$ near a point m^0 of the boundary. The coordinates that Proposition 2.1 provides only have this property in a neighborhood of m^0 *within* the boundary $\partial\mathcal{M}$.

One sets $\mathcal{L} = \mathbb{R} \times \mathcal{M}$ and $\hat{\mathcal{L}} = \mathbb{R} \times \hat{\mathcal{M}}$. From a local chart (O, ϕ) in the atlas for $\hat{\mathcal{M}}$ one defines a map $\phi_{\mathcal{L}} : (t, m) \mapsto (t, \phi(m))$ from $\mathcal{O} = \mathbb{R} \times O$ onto $\mathbb{R} \times \phi(O)$, yielding a local chart $(\mathcal{O}, \phi_{\mathcal{L}})$ for $\hat{\mathcal{L}}$ and thus a finite atlas.

For $x = \phi(m)$, $m \in O \cap \mathcal{M}$, denote by $v = (v', v^d)$ and $\xi = (\xi', \xi_d)$ the associated coordinates in $T_m\mathcal{M}$ and $T_m^*\mathcal{M}$, with $v', \xi' \in \mathbb{R}^{d-1}$ and $v^d, \xi_d \in \mathbb{R}$. We write $T_x\mathcal{M}$ and $T_x^*\mathcal{M}$ by abuse of notation. In what follows, it will be convenient to write z in place of x_d, in particular for the local coordinates given by Proposition 2.1. Accordingly we shall denote the associated cotangent variable ξ_d by the letter ζ, that is, $\xi = (\xi', \zeta)$. We however do not change the notation for the associated tangent variable v^d. With local charts at the boundary given by Proposition 2.1, if $x \in \partial\mathcal{M}$ and $v \in T_x\partial\mathcal{M}$ then $v = (v', 0)$ and we use the bijective map $(\xi', 0) \mapsto \xi'$ to parameterize $T_x^*\partial\mathcal{M}$.

Also classically set

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x\mathcal{M}, \quad T^*\mathcal{M} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x^*\mathcal{M}$$
$$\big(\text{resp. } T\hat{\mathcal{M}} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x\hat{\mathcal{M}}, \quad T^*\hat{\mathcal{M}} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x^*\hat{\mathcal{M}}\big).$$

With $\mathcal{M}$ containing its boundary $\partial\mathcal{M}$, one sees that $T\mathcal{M}$ (resp. $T^*\mathcal{M}$) contains $\{x\} \times T_x\mathcal{M}$ (resp. $\{x\} \times T_x^*\mathcal{M}$) for $x \in \partial\mathcal{M}$. We denote by $\partial(T^*\mathcal{M})$ the boundary of $T^*\mathcal{M}$ that is the set of (x, ξ) with $x \in \partial\mathcal{M}$. In the local coordinates, $\partial(T^*\mathcal{M})$ is given by $\{z = 0\}$ and $T^*\mathcal{M}$ by $\{z \geq 0\}$.

In the associated local chart on $\mathcal{L}$, the representative of $(t, m) \in \mathcal{L}$ is $(t, x) = (t, x', z)$. We shall use the letter ϱ to denote an element of $T^*\mathcal{L}$, that is,

$\varrho = (t, x; \tau, \xi)$ with $(t, x) \in \mathcal{L}$, $\tau \in \mathbb{R}$ and $\xi \in T^*_x\mathcal{M}$. Classically, we write $T^*\mathcal{L} \setminus 0$ for the set of points $\varrho = (t, x; \tau, \xi)$ with $(\tau, \xi) \neq 0$. The boundary $\partial(T^*\mathcal{L})$ is the set of points $\varrho = (t, x; \tau, \xi)$ such that $x \in \partial\mathcal{M}$. Note that $\partial(T^*\mathcal{L})$ is locally given by $\{z = 0\}$ and $T^*\mathcal{L}$ is locally given by $\{z \geq 0\}$.

2.2. **Wave operator and bicharacteristics.** On the manifold $\mathcal{M}$ consider the elliptic operator $A = A_{\kappa,g} = \kappa^{-1} \operatorname{div}_g(\kappa\nabla_g)$, that is, in local coordinates

$$Af = \kappa^{-1}(\det g)^{-1/2} \sum_{1\leq i,j\leq d} \partial_{x_i}\big(\kappa(\det g)^{1/2} g^{ij}(x)\partial_{x_j} f\big).$$

Its principal symbol is simply $a(x, \xi) = -g^*_x(\xi, \xi) = -g^{ij}_x \xi_i \xi_j = -|\xi|^2_x$. Note that for $\kappa = 1$, one has $A = \Delta_g$, the Laplace-Beltrami operator associated with g on $\mathcal{M}$. Together with A consider the wave operator $P_{\kappa,g} = \partial_t^2 - A_{\kappa,g}$. Its principal symbol in a local chart is given by

$$p(\varrho) = -\tau^2 + |\xi|^2_x.$$

Note that $p(\varrho)$ is smooth in the variables (τ, ξ) and $\mathscr{C}^1$ in x.

For a function f of the variable ϱ, the Hamiltonian vector field H_f is defined by $\mathrm{H}_f(h) = \{f, h\}$, where $\{., .\}$ is the Poisson bracket. In local coordinates one has

$$\begin{aligned} \mathrm{H}_p(\varrho) &= \partial_\tau p(\varrho)\partial_t + \nabla_\xi p(\varrho) \cdot \nabla_x - \nabla_x p(\varrho) \cdot \nabla_\xi \\ &= -2\tau\partial_t + 2g^{ij}(x)\xi_i\partial_{x_j} - \partial_{x_k} g^{ij}(x)\xi_i\xi_j\partial_{\xi_k}. \end{aligned} \tag{2.3}$$

Recall the following definition.

Definition 2.3. Suppose V is an open subset of $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ and $J \subset \mathbb{R}$ is an interval. A $\mathscr{C}^1$-map $\gamma : J \to V \cap \operatorname{Char} p$ is called a bicharacteristic in V if

$$\frac{d}{ds}\gamma(s) = \mathrm{H}_p\big(\gamma(s)\big), \qquad s \in J. \tag{2.4}$$

It is called *maximal* in V if it cannot be extended by another bicharacteristic also valued in V.

Remark 2.4.

(1) If $\gamma(s) = \big(t(s), x(s), \tau(s), \xi(s)\big)$ is a bicharacteristic observe that $\tau(s)$ is constant because of the form of H_p. Since $\gamma(s) \in \operatorname{Char} p$ one has

$$|\xi(s)|_{x(s)} = |\tau(s)|$$

also constant along a bicharacteristic.

(2) With $\tau(s) = \mathrm{Cst} \neq 0$ then $t(s)$ is an affine function of s since $dt/ds = -2\tau(s)$. If $V = T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ and γ is maximal with $S^+ = \sup J < +\infty$ then

$$\lim_{\substack{s \to S^+ \\ s \in J}} \gamma(s) \in \partial(T^*\mathcal{L}).$$

This is part of Lemma 5.24 below.

(3) If needed we also call bicharacteristic a map $\gamma : J \to T^*\hat{\mathcal{L}} \cap \mathrm{Char}\, p$, such that (2.4) holds, using the extension of H_p in $T^*\hat{\mathcal{L}}$. This is then a bicharacteristic above $\hat{\mathcal{L}}$ and we mention it explicitly.

Note that ${}^t\mathrm{H}_p\, f(\varrho) = 2\tau\partial_t f(\varrho) - 2\partial_{x_j}\big(g^{ij}(x)\xi_i f(\varrho)\big) + \partial_{\xi_k}\big(\partial_{x_k} g^{ij}(x)\xi_i\xi_j f(\varrho)\big)$ and deduce

$${}^t\mathrm{H}_p = -\,\mathrm{H}_p\,.$$

Recall also that

$$\mathrm{H}_p\, f(\gamma(s)) = \frac{d}{ds} f(\gamma(s)), \quad \text{if } \gamma \text{ is a bicharacteristic.} \tag{2.5}$$

2.3. **A partition of the cotangent bundle at the boundary.** Denote by ${}^{\|}\partial(T^*\mathcal{L}) \subset \partial(T^*\mathcal{L})$ the bundle of points $\varrho = (\varrho', 0) = (t, x', z = 0, \tau, \xi', 0) \in T^*\mathcal{L}$ for $\varrho' = (t, x', z = 0, \tau, \xi') \in T^*\partial\mathcal{L}$. Identifying ϱ' and $(\varrho', 0)$ as presented above thanks to the chosen local coordinates allows one to indentify ${}^{\|}\partial(T^*\mathcal{L})$ and $T^*\partial\mathcal{L}$.

Denote by $\pi_{\|}$ the map from $\partial(T^*\mathcal{L})$ into ${}^{\|}\partial(T^*\mathcal{L})$ given by

$$\pi_{\|}(t, x', z = 0, \tau, \xi', \zeta) = (t, x', z = 0, \tau, \xi', 0).$$

Definition 2.5 (elliptic, glancing, and hyperbolic regions)**.** One partitions ${}^{\|}\partial(T^*\mathcal{L})$ into three homogeneous regions.

(1) The elliptic region ${}^{\|}\mathcal{E}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p > 0\}$; if $\varrho \in {}^{\|}\mathcal{E}_\partial$ it is called an elliptic point.

(2) The glancing region ${}^{\|}\mathcal{G}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p = 0\}$; if $\varrho \in {}^{\|}\mathcal{G}_\partial$ it is called a glancing point.

(3) The hyperbolic region ${}^{\|}\mathcal{H}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p < 0\}$; if $\varrho \in {}^{\|}\mathcal{H}_\partial$ it is called a hyperbolic point.

Since $p(\varrho) = -\tau^2 + \zeta^2 + g_x(\xi', \xi')_x$ by (2.2) if $\varrho \in \partial(T^*\mathcal{L})$, one has the following properties:

(1) If $\varrho \in {}^{\|}\mathcal{E}_\partial$ then $\pi_{\|}^{-1}\big(\{\varrho\}\big) \cap \mathrm{Char}\, p = \emptyset$.

(2) If $\varrho \in {}^{\|}\mathcal{G}_\partial$ then $\pi_{\|}^{-1}\big(\{\varrho\}\big) \cap \operatorname{Char} p = \{\varrho\}$.
(3) If $\varrho = (t, x', z = 0, \tau, \xi', 0) \in {}^{\|}\mathcal{H}_\partial$ then $\pi_{\|}^{-1}\big(\{\varrho\}\big) \cap \operatorname{Char} p = \{\varrho^-, \varrho^+\}$, where

$$\varrho^\pm = (t, x', z = 0, \tau, \xi^\pm), \quad \text{where } \xi^\pm = (\xi', \zeta^\pm) \text{ with } \zeta^\pm = \pm\sqrt{-p(\varrho)}. \tag{2.6}$$

Associated with the previous partition of ${}^{\|}\partial(T^*\mathcal{L})$ is a partition of $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$. Indeed, if $\varrho \in \operatorname{Char} p \cap \partial(T^*\mathcal{L})$ then $\pi_{\|}(\varrho) \in {}^{\|}\partial(T^*\mathcal{L})$ and $p\big(\pi_{\|}(\varrho)\big) \leq 0$. Note that having $\varrho \in \operatorname{Char} p \cap \partial(T^*\mathcal{L})$ and $p\big(\pi_{\|}(\varrho)\big) = 0$ is equivalent to having $\varrho \in {}^{\|}\mathcal{G}_\partial$.

Definition 2.6 (partition of $\operatorname{Char} p$ at the boundary)**.** One partitions $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$ into two homogeneous regions $\mathcal{G}_\partial$ and $\mathcal{H}_\partial$:

(1) $\mathcal{G}_\partial = {}^{\|}\mathcal{G}_\partial$; $\varrho \in \mathcal{G}_\partial \ \Leftrightarrow\ \varrho \in \operatorname{Char} p$ and $\pi_{\|}(\varrho) = \varrho$.
(2) $\varrho \in \mathcal{H}_\partial$ if $\varrho \in \operatorname{Char} p$ and $\pi_{\|}(\varrho) \in {}^{\|}\mathcal{H}_\partial$. It is also called a hyperbolic point. If $\varrho = (t, x', z = 0, \tau, \xi', \zeta)$ one says that $\varrho \in \mathcal{H}_\partial^+$ if $\zeta > 0$ and $\varrho \in \mathcal{H}_\partial^-$ if $\zeta < 0$.

Thus, if $\varrho \in {}^{\|}\mathcal{H}_\partial$ then $\pi_{\|}^{-1}\big(\{\varrho\}\big) \cap \operatorname{Char} p = \{\varrho^-, \varrho^+\}$ with $\varrho^+ \in \mathcal{H}_\partial^+$ and $\varrho^- \in \mathcal{H}_\partial^-$, with $\varrho^\pm$ as given in (2.6).

Introducing the following involution on $\partial(T^*\mathcal{L})$

$$\Sigma(t, x', z = 0, \tau, \xi', \zeta) = (t, x', z = 0, \tau, \xi', -\zeta), \tag{2.7}$$

one finds that $\Sigma(\varrho^-) = \varrho^+$ if $\varrho \in {}^{\|}\mathcal{H}_\partial$. Thus, Σ is a one-to-on map from $\mathcal{H}_\partial^-$ onto $\mathcal{H}_\partial^+$.

2.4. **Glancing region, gliding vector field, and generalized bicharacteristics.** Recall that we denote by z the variable x_d. One computes

$$\mathrm{H}_p\, z(\varrho) = \mathrm{H}_p\, z(x, \xi) = 2g^{dj}(x)\xi_j.$$

Observe that $\mathrm{H}_p\, z$ is a $\mathscr{C}^1$-function. Note that $\mathrm{H}_p\, z_{|z=0} = 2\zeta$ in the present local coordinates. Hence, ${}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial = \{z = \mathrm{H}_p\, z = p = 0\}$ and $\mathcal{H}_\partial^\pm = \{z = p = 0,\ \mathrm{H}_p\, z \gtrless 0\}$ locally. With (2.5) this means that a bicharacteristic going through a point $\varrho \in \mathcal{H}_\partial$ has a contact of order exactly one with the boundary: it is transverse to $\partial(T^*\mathcal{L})$. A bicharacteristic going through a point $\varrho \in \mathcal{G}_\partial$ has a contact of order greater than or equal to two: it is tangent to $\partial(T^*\mathcal{L})$.

One can further compute $\mathrm{H}_p^2\, z$. It is continuous and gives the following partition of $\mathcal{G}_\partial$.

Definition 2.7 (partition of $\mathcal{G}_\partial$)**.** Introduce

$$\mathcal{G}_\partial^{\mathrm{d}} = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) > 0\},$$
$$\mathcal{G}_\partial^{3} = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) = 0\},$$
$$\mathcal{G}_\partial^{\mathrm{g}} = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) < 0\}.$$

One calls $\mathcal{G}_\partial^{\mathrm{d}}$ the diffractive set, $\mathcal{G}_\partial^{\mathrm{g}}$ the gliding set. One calls $\mathcal{G}_\partial^{3}$ the glancing set of order three: if $\varrho^0 \in \mathcal{G}_\partial^{3}$ a bicharacteristic that goes through ϱ^0 has a contact with the boundary of order greater than or equal to three.

2.7 On ${}^{\|}\partial(T^*\mathcal{L})$ one defines

$$\mathrm{H}_p^{\mathcal{G}}(\varrho) = \Big(\mathrm{H}_p + \frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p}\,\mathrm{H}_z\Big)(\varrho), \tag{2.8}$$

referred to as the gliding vector field. Note that in the present coordinates one has $\mathrm{H}_z^2 p_{|z=0} = 2$. More explainations on $\mathrm{H}_p^{\mathcal{G}}$ are given in Section 5.4. In turn, one defines the following vector field on $T^*\mathcal{L}$

$${}^{\mathsf{G}}X(\varrho) = \begin{cases} \mathrm{H}_p(\varrho) & \text{if } \varrho \in T^*\mathcal{L} \setminus \mathcal{G}_\partial^{\mathrm{g}}, \\ \mathrm{H}_p^{\mathcal{G}}(\varrho) & \text{if } \varrho \in \mathcal{G}_\partial^{\mathrm{g}}, \end{cases} \tag{2.9}$$

that is, ${}^{\mathsf{G}}X = \mathrm{H}_p + \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{g}}}(\mathrm{H}_p^{\mathcal{G}} - \mathrm{H}_p)$.

Definition 2.8 (generalized bicharacteristic)**.** Suppose $J \subset \mathbb{R}$ is an interval, B a discrete subset of J, and

$${}^{\mathsf{G}}\gamma : J \setminus B \to \mathrm{Char}\, p \cap T^*\mathcal{L}.$$

One says that ${}^{\mathsf{G}}\gamma$ is a generalized bicharacteristic if the following properties hold:

(1) For $s \in J \setminus B$, ${}^{\mathsf{G}}\gamma(s) \notin \mathcal{H}_\partial$ and the map ${}^{\mathsf{G}}\gamma$ is differentiable at s with

$$\frac{d}{ds}\,{}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big).$$

(2) If $S \in B$, then ${}^{\mathsf{G}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \in J \setminus B$ sufficiently close to S and moreover
   (a) if $[S-\varepsilon, S] \subset J$ for some $\varepsilon > 0$, then ${}^{\mathsf{G}}\gamma(S^-) = \lim_{s\to S^-} {}^{\mathsf{G}}\gamma(s) \in \mathcal{H}_\partial^-$;
   (b) if $[S, S+\varepsilon] \subset J$ for some $\varepsilon > 0$, then ${}^{\mathsf{G}}\gamma(S^+) = \lim_{s\to S^+} {}^{\mathsf{G}}\gamma(s) \in \mathcal{H}_\partial^+$;
   (c) and if $[S-\varepsilon, S+\varepsilon] \subset J$ for some $\varepsilon > 0$, then ${}^{\mathsf{G}}\gamma(S^+) = \Sigma\big({}^{\mathsf{G}}\gamma(S^-)\big)$.

Recall that $T^*\mathcal{L}$ contains its boundary $\partial(T^*\mathcal{L})$; as a result a generalized bicharacteristic ${}^{\mathsf{G}}\gamma(s)$ may lie in the boundary for s in some interval. Details on generalized bicharacteristics are given in Section 5.6.

When one refers to a (generalized) bicharacteristic one often means the points visited in $T^*\mathcal{L}$ by $s \mapsto {}^{\mathsf{G}}\gamma(s)$ as s varies, that is,

$$\{{}^{\mathsf{G}}\gamma(s);\ s \in J \setminus B\}.$$

Observe however that this set may not be a closed set if $B \neq \emptyset$ as its intersection with $\mathcal{H}_\partial$ is empty. Consequently we rather use its closure to describe the set of reached points.

Definition 2.9 (generalized bicharacteristic)**.** By generalized bicharacteristic one also refers to

$${}^{\mathsf{G}}\bar{\gamma} = \overline{\{{}^{\mathsf{G}}\gamma(s);\ s \in J \setminus B\}} = \{{}^{\mathsf{G}}\gamma(s);\ s \in J \setminus B\} \cup \bigcup_{s\in B} \{{}^{\mathsf{G}}\gamma(s^-), {}^{\mathsf{G}}\gamma(s^+)\}.$$

The following theorem states that for every point of $T^*\mathcal{L}$ one can find a maximal generalized bicharacteristic that goes through this point.

Theorem 2.10. *Suppose $J \setminus B \ni s \mapsto {}^{\mathsf{G}}\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$ is a generalized bicharacteristic. If ${}^{\mathsf{G}}\gamma$ is maximal then $J = \mathbb{R}$. Moreover, $t(\mathbb{R}) = \mathbb{R}$ if $\tau(s) = \mathrm{Cst} \neq 0$.*

If $\varrho^0 \in \operatorname{Char} p \cap T^*\mathcal{L}$ there exists a maximal generalized bicharacteristic $s \mapsto {}^{\mathsf{G}}\gamma(s)$ with $s \in \mathbb{R} \setminus B$ such that ${}^{\mathsf{G}}\gamma(0) = \varrho^0$ if $\varrho^0 \notin \mathcal{H}_\partial$ and ${}^{\mathsf{G}}\gamma(0^\pm) = \varrho^0$ if $\varrho^0 \in \mathcal{H}_\partial^\pm$.

If it does not create confusion, by abuse of notation, we sometimes write ${}^{\mathsf{G}}\gamma(0) = \varrho^0$ even in the case $\varrho^0 \in \mathcal{H}_\partial$ with the understanding that ${}^{\mathsf{G}}\gamma(0^\pm) = \varrho^0$ if $\varrho^0 \in \mathcal{H}_\partial^\pm$.

Note that there is no uniqueness of such a maximal generalized bicharacteristic because of the limited smoothness of ${}^{\mathsf{G}}X$. The result of Theorem 2.10 is classical in the case of smooth coefficients; see [18] or [15, Section 24.3]. Here, in the case of the present limited smoothness a possible proof of Theorem 2.10 follows quite closely the arguments developed in what follows. Instead of duplicating a quite long proof, we chose an argument that allows one to consider Theorem 2.10 as a consequence of our main result, Theorem 3.4 below. We refer to Appendix B.1 for this proof.

Despite the lack of uniqueness, some form of continuity related to all bicharacteristics passing through one point holds. For $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \operatorname{Char} p \cap T^*\mathcal{L}$ and $T > 0$ introduce

$$\Gamma^T(\varrho^0) = \{|t - t^0| \leq T\} \cap \bigcup_{\varrho^0 \in {}^{\mathsf{G}}\bar{\gamma}} {}^{\mathsf{G}}\bar{\gamma},$$

that is, the union of all generalized bicharacteristic that pass through ϱ^0, restricted to the time interval $[t^0 - T, t^0 + T]$.

Proposition 2.11. *Suppose $\varrho^0 \in \operatorname{Char} p \cap T^*\mathcal{L} \setminus 0$ and $T > 0$.*

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall \varrho^1, \varrho \in \operatorname{Char} p \cap T^*\mathcal{L},$$
$$\operatorname{dist}(\varrho^1, \varrho^0) \leq \delta \text{ and } \varrho \in \Gamma^T(\varrho^1) \quad \Rightarrow \quad \operatorname{dist}(\varrho, \Gamma^T(\varrho^0) \leq \varepsilon.$$

A proof is given in Appendix B.2.

3. Main result and open questions

Our main result concerns the description of the support of a measure density[2] μ defined on $\mathscr{U}$ an open subset of $T^*\hat{\mathcal{L}}$. Recall that $\hat{\mathcal{L}}$ is a local extension of $\mathcal{L}$; see Section 2.

A first assumption made on μ is the following.

Assumption 3.1. *The measure μ is nonnegative and supported in $\mathscr{U} \cap \operatorname{Char} p \cap T^*\mathcal{L} \setminus 0$.*

In particular, μ vanishes in a neighborhood of $(\tau, \xi) = 0$ and in $\mathscr{U} \setminus T^*\mathcal{L}$.

A second assumed property is the following one.

Assumption 3.2. *One has, in the sense of distributions,*

$$ {}^t\mathrm{H}_p\, \mu = f\mu - \int_{\varrho \in {}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x \rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}}\, d\nu(\varrho) \qquad \text{in } \mathscr{U}, \tag{3.1}$$

with f a continuous real function on $T^*\hat{\mathcal{L}}$ and ν a nonnegative measure on ${}^{\|}\partial(T^*\mathcal{L})$ and where $\varrho^\pm$ and $\xi^\pm$ are as given in (2.6).

Here, n_x stands for the unitary inward pointing normal vector in the sense of the metric; it is recalled at the end of Section 5.1.

[2]The word 'density' is omitted in what follows, yet μ has the density property and thus acts on continuous functions on $\hat{\mathcal{L}}$.

Remark 3.3. If $\varrho \in {}^{\|}\mathcal{G}_\partial$ then ϱ^- and ϱ^+ coincide with ϱ and $\xi^+ = \xi^-$. The value of the integrand in (3.1) thus requires some explanation in this case. In fact, first consider $\varrho^0 = (\varrho^{0\prime}, 0) \in {}^{\|}\mathcal{H}_\partial$ with $\varrho^{0\prime} = (t^0, x^{0\prime}, z = 0, \tau^0, \xi^{0\prime})$. Then $\varrho^{0,\pm} \neq \varrho^0$ and (2.6) give $\xi^{0,+} - \xi^{0,-} = 2\zeta^+ dz$, yielding $\langle \xi^{0,+} - \xi^{0,-}, \mathsf{n}_{x^0} \rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}} = 2\zeta^+$ since $\mathsf{n}_x = \partial_z$ in the considered coordinates. With a $\mathscr{C}^1$-test function $q(\varrho)$ one has

$$\langle \delta_{\varrho^{0,+}} - \delta_{\varrho^{0,-}}, q \rangle = q\big(\varrho^{0\prime}, \zeta^+\big) - q\big(\varrho^{0\prime}, -\zeta^+\big).$$

The integrand is thus

$$\frac{q\big(\varrho^{0\prime}, \zeta^+\big) - q\big(\varrho^{0\prime}, -\zeta^+\big)}{2\zeta^+}.$$

If now a sequence $(\varrho^{(n)})_n \subset {}^{\|}\mathcal{H}_\partial$ converges to $\varrho \in {}^{\|}\mathcal{G}_\partial$ then

$$\frac{\langle \delta_{\varrho^{(n),+}} - \delta_{\varrho^{(n),-}}, q \rangle}{\langle \xi^{(n),+} - \xi^{(n),-}, \mathsf{n}_x \rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}}} \to \partial_\zeta q(\varrho). \tag{3.2}$$

The integrand in (3.1) for $\varrho \in {}^{\|}\mathcal{G}_\partial$ is thus to be understood as the derivative with respect to the variable ζ at $\zeta = 0$. Note that this interpretation is very coordinate dependent. We give a more geometrical interpretation using more intrinsic coordinates in Section 5.7.

Our main result states that if a point lies in the support of μ solution to (3.1), then there exists a maximal generalized bicharacteristic initiated at this point contained in $\operatorname{supp}\mu$.

Theorem 3.4. *Suppose μ is a measure density that fulfills Assumptions 3.1 and 3.2. Suppose $\varrho^0 \in \operatorname{supp}\mu$. There exists a maximal generalized bicharacteristic $s \mapsto {}^{\mathsf{G}}\gamma(s)$ with $s \in J \setminus B$, with B a discrete subset of $\mathbb{R}$, such that*

$$\varrho^0 \in {}^{\mathsf{G}}\bar{\gamma} \subset \operatorname{supp}\mu,$$

with ${}^{\mathsf{G}}\gamma$ and ${}^{\mathsf{G}}\bar{\gamma}$ as in Definitions 2.8 and 2.9. In other words, the support of μ is a union of maximal generalized bicharacteristics in $\mathscr{U}$.

If $\mathscr{U} = T^*\hat{\mathcal{L}}$ then $J = \mathbb{R}$ for each maximal generalized bicharacteristic.

As mentionned in the introductory section, this is an extension of the superposition principle of L. Ambrosio and G. Crippa [2, 3, 4], yet in a nonquantitative form: here, we only describe the geometry of the support of the measure and not the measure itself. A very natural open question is the following:

Is there an extension of the quantitative superposition principle of [4] *for a nonnegative measure that fulfills both Assumptions 3.1 and 3.2?*

In other words, can one write such a mesure μ as a "sum" of positive measures, each defined as a constant times $\delta_{\mathsf{G}_{\bar\gamma}}$, where ${}^{\mathsf{G}}\bar\gamma$ is a generalized bicharacteristic as in Definition 2.8?

A first natural question is the following: for a generalized bicharacteristic ${}^{\mathsf{G}}\bar\gamma$, is the measure $\delta_{\mathsf{G}_{\bar\gamma}}$ well defined and does it fulfill a transport equation of the form of (3.1)? Despite the fact that ${}^{\mathsf{G}}\gamma(s)$ can have an infinite number of points of discontinutity, for $s \in B$, that can accumulate, one can answer positively these question. This is done in the beginning of Appendix A.

Note that the nonquantitative superposition principle of Theorem 3.4 suffices for the purpose of the companion article [7] towards the derivation of observability estimate for the wave equation in the case of $\mathscr{C}^1$-coefficients and a $\mathscr{C}^2$-boundary.

An important consequence of Assumptions 3.1 and 3.2 is also the following property of the measure ν.

Proposition 3.5. *There exists $C > 0$ such that $|\tau| \geq C > 0$ in $\operatorname{supp}\nu \cap ({}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial)$. One has $\langle \nu, \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}} \cup \mathcal{G}_\partial^3}\rangle = 0$ and $\langle \mu, \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}}\rangle = 0$, that is, the measure ν has no mass on $\mathcal{G}_\partial^{\mathrm{d}} \cup \mathcal{G}_\partial^3$ and the measure μ has no mass on $\mathcal{G}_\partial^{\mathrm{d}}$.*

This proposition is due to N. Burq and P. Gérard [9] in the case of smooth coefficients. The proof requires refinements in the present low regularity setting. It is given in Section 7.

4. Transport equation, measure support propagation away from or across boundaries

4.1. **Support propagation away from boundaries.** Suppose Ω is an open subset of a $\mathscr{C}^2$ d-dimensional manifold. Denote by ${}^1\mathscr{D}'(\Omega)$ and ${}^1\mathscr{D}'^{,0}(\Omega)$ the spaces of density distributions and density Radon measures on Ω.

Consider a continuous vector field X and a continuous real function f on Ω and suppose μ is a nonnegative measure density on Ω. Assume that μ is such that ${}^tX\mu = f\mu$ in the sense of distributions, that is,

$$\langle {}^tX\mu, a\rangle_{{}^1\mathscr{D}'(\Omega),\mathscr{C}_c^\infty(\Omega)} = \langle \mu, Xa\rangle_{{}^1\mathscr{D}'^{,0}(\Omega),\mathscr{C}_c^0(\Omega)} = \langle \mu, fa\rangle_{{}^1\mathscr{D}'^{,0}(\Omega),\mathscr{C}_c^0(\Omega)}, \tag{4.1}$$

for $a \in \mathscr{C}_c^\infty(\Omega)$. If f vanishes and X is moreover Lipschitz, one concludes that μ is invariant along the flow that X generates. However, if X is not Lipschitz, there is no such flow in general. Yet, integral curves do exist by the Cauchy-Peano theorem.

Away from any boundary a precise statement associated with (1.4) is given in the following theorem.

Theorem 4.1. *On Ω, suppose X is a continuous vector field, f is a continuous real function, and μ is a nonnegative density measure that is solution to ${}^tX\mu = f\mu$ in the sense of distributions. Then, the support of μ is a union of maximally extended integral curves of the vector field X.*

In other words, if $m^0 \in \Omega$ is in $\operatorname{supp}\mu$, then there exist an interval I in $\mathbb{R}$ with $0 \in I$ and a $\mathscr{C}^1$ curve $\gamma : I \to \Omega$ that cannot be extended such that $\gamma(0) = m^0$ and

$$\frac{d}{ds}\gamma(s) = X(\gamma(s)), \qquad s \in I,$$

and $\gamma(I) \subset \operatorname{supp}\mu$.

This theorem and its proof can be found in [6]. We decided to reproduce the argument here as it increases the readability of the present article for the following two reasons :

(1) the proof of Theorem 4.1 is much simpler than the argument we develop below to understand the structure of the support of μ at a boundary if fulfilling the more general equation (3.1);
(2) the techniques used at the boundary, despite their high level of technicality, are in the same spirit as those in the proof of Theorem 4.1. In particular, some of the cutoff functions introduced in the proof of Theorem 4.1 are used further in the article.

The strategy of the proof of Theorem 4.1 is very much inspired by the Melrose and Sjöstrand approach to the propagation of singularities [18] and relies on careful choices of test functions allowing one to construct sequences of points in the support of the measure relying on nonnegativity[3]. Then, a limiting procedure leads to the conclusion, in the spirit of the classical proof of the Cauchy-Peano theorem.

Theorem 4.1 is stated on an open subset of a smooth manifold. Yet, its result is of local nature. Using a local chart one may assume that Ω is an open subset of $\mathbb{R}^d$ instead without any loss of generality.

The proof of Theorem 4.1 is made of two steps that are stated in the following propositions.

Proposition 4.2. *Suppose X is a $\mathscr{C}^0$-vector field on Ω an open subset of $\mathbb{R}^d$. For a closed set F of Ω, the following two properties are equivalent.*

[3] of the measure in our case and of some operators for Melrose and Sjöstrand, via the Gårding inequality.

(1) *The set F is a union of maximally extended integral curves of the vector field X.*
(2) *For any compact $K \subset \Omega$ where the vector field X does not vanish,*

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall x \in K \cap F,\ \forall \delta \in [-\delta_0, \delta_0], \quad B\big(x + \delta X(x), \delta\varepsilon\big) \cap F \neq \emptyset.$$

Proposition 4.3. *On Ω an open subset of $\mathbb{R}^d$, suppose X is a $\mathscr{C}^0$-vector field and f is a continuous real function. Consider a nonnegative measure μ on Ω solution to ${}^tX\mu = f\mu$ in the sense of distributions. Then, the closed set $F = \operatorname{supp}\mu$ satisfies the second property in Proposition 4.2.*

Proof of Proposition 4.2. First, we prove that Property (1) implies Property (2) and consider a compact set K of $\mathbb{R}^d$ such that $K \subset \Omega$ and $K \cap F \neq \emptyset$.

There exists $\eta > 0$ such that $K \subset K_\eta \subset \Omega$ with $K_\eta = \{x \in \Omega;\ \operatorname{dist}(x, K) \leq \eta\}$. One has $\|X\| \leq C_0$ on K_η for some $C_0 > 0$. Suppose $x \in K$ and $\gamma(s)$ is a maximal integral curve defined on an interval $]a, b[$, $a, b \in \overline{\mathbb{R}}$ and such that $0 \in]a, b[$ and $\gamma(0) = x$. If $b < \infty$ then there exists $s^1 \in]0, b[$ such that $\gamma(s^1) \notin K_\eta$. Since $\gamma(s) \in K_\eta$ if $s < \eta/C_0$, one finds that $b \geq \eta/C_0$. Similarly, one has $|a| \geq \eta/C_0$. Consequently, there exists $S > 0$ such that any maximal integral curve $\gamma(s)$ of the vector field X with $\gamma(0) \in K$ is defined for $s \in I = (-S, S)$.

Pick $x \in K \cap F$. According to the Property (1), there exists

$$\gamma : I \to F \text{ such that } \dot\gamma(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$

By uniform continuity of the vector field X in a compact neighborhood of K one has

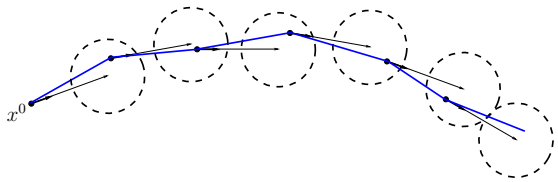
$$\gamma(s) = \gamma(0) + \int_0^s \dot\gamma(s)ds = \gamma(0) + \int_0^s X(\gamma(s))ds = x + sX(x) + r(s),$$

for $s \in (-S, S)$, where $\lim_{s\to 0} \|r(s)\|/s = 0$, *uniformly* with respect to x. One deduces that for any $\varepsilon > 0$ there exists $0 < \delta_0 < S$ such that $\|r(s)\| < s\varepsilon$ for any $s \in (-\delta_0, \delta_0)$, which implies

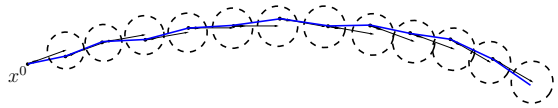
$$F \ni \gamma(s) \in B\big(x + sX(x), s\varepsilon\big).$$

Second, we prove that Property (2) implies Property (1). It suffices to prove that for any $x \in F$ there exist an open interval $I \ni 0$ and an integral curve

$$\gamma : I \to F \text{ such that } \dot\gamma(s) = X(\gamma(s)) \text{ and } \gamma(0) = x.$$



(a) Iterative construction of the curve γ_n.



(b) Convergence of γ_n as n increases.

FIGURE 1. Construction and convergence of the sequence $(\gamma_n)_n$.

Then, the standard continuation argument shows that this local integral curve included in F can be extended to a maximal integral curve also included in F.

If $X(x) = 0$, then the trivial integral curve $\gamma(s) = x$, $s \in \mathbb{R}$, is included in F. As a consequence, one assumes $X(x) \neq 0$ and one picks a compact neighborhood K of x containing $B(x, \eta)$ with $\eta > 0$ and where, for some $0 < c_K < C_K$,

$$c_K \leq \|X(y)\| \leq C_K, \quad y \in K.$$

Let $n \in \mathbb{N}^*$. Set $x_{n,0} = x$ and $\varepsilon = 1/n$ and apply Property (2). One deduces that there exist $0 < \delta_n \leq 1/n$ and a point

$$x_{n,1} \in F \cap B\big(x_{n,0} + \delta_n X(x_{n,0}), \delta_n/n\big).$$

If $x_{n,1} \in K$ one can perform this construction again, yet starting from $x_{n,1}$ instead of $x_{n,0}$. If a sequence of points $x_{n,0}, x_{n,1}, \ldots, x_{n,L^+}$ is obtained in this manner one has

$$x_{n,\ell+1} \in F \cap B\big(x_{n,\ell} + \delta_n X(x_{n,\ell}), \delta_n/n\big), \qquad \ell = 0, \ldots, L^+ - 1. \tag{4.2}$$

One can carry on the construction as long as $x_{n,L^+} \in K$. The same construction for $\ell \leq 0$ can be performed, with the property

$$x_{n,\ell-1} \in F \cap B\big(x_{n,\ell} - \delta_n X(x_{n,\ell}), \delta_n/n\big), \qquad |\ell| = 0, \ldots, L^- - 1. \tag{4.3}$$

Having $\|X\| \leq C_K$ on K and $B(x, \eta) \subset K$ ensures that one can construct the sequence at least for

$$L^+ = L^- = L_n = \Big\lfloor \frac{\eta}{\delta_n(C_K + 1)} \Big\rfloor + 1 \leq \Big\lfloor \frac{\eta}{\delta_n(C_K + 1/n)} \Big\rfloor + 1,$$

where $\lfloor . \rfloor$ denotes the floor function. With the constructed points $x_{n,\ell}$, $|\ell| \leq L_n$, define the following continuous curve $\gamma_n(s)$ for $|s| \leq L_n \delta_n$:

$$\gamma_n(s) = x_{n,\ell} + (s - \ell\delta_n)\frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} \text{ for } s \in [\ell\delta_n, (\ell+1)\delta_n) \text{ and } |\ell| \leq L_n - 1.$$

This curve and its construction is illustrated in Figure 1(a). Note that $\gamma_n(s)$ remains in a compact set, uniformly with respect to n. In this compact set X is uniformly continuous.

Set $S = \eta/(C_K + 1)$. Since $S \leq L_n\delta_n$, in fact, we only consider the function $\gamma_n(s)$ for $|s| \leq S$ in what follows. Note that since $x_{n,\ell} \in F$ for $|\ell| \leq L_n$ then one has

$$\text{dist}\left(\gamma_n(s), F\right) \leq \delta_n(C_K + 1/n), \qquad |s| \leq S. \tag{4.4}$$

From (4.2), for $\ell \geq 0$ and $s \in (\ell\delta_n, (\ell+1)\delta_n)$, one has

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell+1} - x_{n,\ell}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

Similarly, from (4.3), for $\ell \leq 0$ and $s \in ((\ell-1)\delta_n, \ell\delta_n)$, one has

$$\dot{\gamma}_n(s) = \frac{x_{n,\ell} - x_{n,\ell-1}}{\delta_n} = X(x_{n,\ell}) + \mathcal{O}(1/n).$$

In any case, using the uniform continuity of the vector field X, one finds

$$\dot{\gamma}_n(s) = X(\gamma_n(s)) + e_n(s),$$

where the error $|e_n|$ goes to zero *uniformly* with respect to $|s| \leq S$ as $n \to +\infty$.

Since the curve γ_n is continuous, one finds

$$\gamma_n(s) = \gamma_n(0) + \int_0^s \dot{\gamma}_n(\sigma)d\sigma = x + \int_0^s X(\gamma_n(\sigma))d\sigma + \int_0^s e_n(\sigma)\, d\sigma. \tag{4.5}$$

Let now n grow to infinity. With (4.5), the family of curves $(s \mapsto \gamma_n(s), |s| \leq S)_{n\in\mathbb{N}^*}$ is equicontinuous and pointwise bounded; by the Arzelà-Ascoli theorem one can extract a subsequence $(s \mapsto \gamma_{n_p})_{p\in\mathbb{N}}$ that converges uniformly to a curve $\gamma(s), |s| \leq S$. Convergence is illustrated in Figure 1(b). Passing to the limit $n_p \to +\infty$ in (4.5) one finds that $\gamma(s)$ is solution to

$$\gamma(s) = x + \int_0^s X(\gamma(\sigma))d\sigma.$$

From (4.4), for any $|s| \leq S$, there exists $(y_p)_p \subset F$ such that $\lim_{p\to+\infty} y_p = \gamma(s)$. Since F is closed one concludes that $\gamma(s) \in F$. □

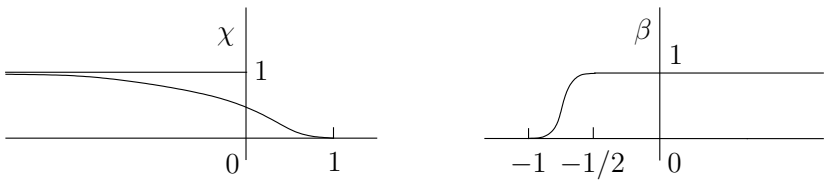


FIGURE 2. The two localization functions χ (a) and β (b) used to build the test function q.

Positivity argument and proof of Proposition 4.3. Consider a compact set K where the vector field X does not vanish. By continuity of the vector field there exist $0 < c_K \leq C_K$ such that $0 < c_K \leq \|X(x)\| \leq C_K$, for all $x \in K$.

Consider $x^0 \in K \cap \mathrm{supp}(\mu)$. By performing a rotation and a dilation of coefficient $\|X(x^0)\| \in [c_K, C_K]$, one can assume that $X(x^0) = (1, 0, \ldots, 0) \in \mathbb{R}^d$. One writes $x = (x_1, x')$ with $x' \in \mathbb{R}^{d-1}$.

Let $\chi \in \mathscr{C}^\infty(\mathbb{R})$ be given by

$$\chi(s) = \mathbf{1}_{s<1} \exp(1/(s-1)), \tag{4.6}$$

and $\beta \in \mathscr{C}^\infty(\mathbb{R})$ be such that

$$\beta \equiv 0 \text{ on }]-\infty, -1], \quad \beta' > 0 \text{ on }]-1, -1/2[, \quad \beta \equiv 1 \text{ on } [-1/2, +\infty[. \tag{4.7}$$

These two functions are represented in Figure 2. Then set

$$q = e^{Ax_1}(\chi \circ v)(\beta \circ w), \quad g = e^{Ax_1}(\chi' \circ v)(\beta \circ w)Xv, \quad h = e^{Ax_1}(\chi \circ v)(\beta' \circ w)Xw, \tag{4.8}$$

with $A > 0$ meant to be chosen sufficiently large below and

$$v(x) = 1/2 - \delta^{-1}(x_1 - x_1^0) + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2$$
$$\text{and} \quad w(x) = 2\varepsilon^{-1}\big(1 - \delta^{-1}(x_1 - x_1^0)\big),$$

for $\varepsilon > 0$ and $\delta > 0$ both meant to be chosen small in what follows. One has

$$Xq = g + h + A(Xx_1)q.$$

The function q is compactly supported. Indeed, in the support of $\beta \circ w$ one has $w \geq -1$ implying

$$x_1 - x_1^0 \leq \delta(1 + \varepsilon/2),$$

while on the support of $\chi \circ v$ one has $v \leq 1$ which gives

$$-1/2 + 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq \delta^{-1}(x_1 - x_1^0).$$

On the supports of q and $(\chi' \circ v)(\beta \circ w)$ one thus finds

$$-\delta/2 \leq x_1 - x_1^0 \leq \delta(1 + \varepsilon/2) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq 3/2 + \varepsilon/2. \tag{4.9}$$

Similarly, on the support of $\beta' \circ w$ one has $-1 \leq w \leq -1/2$ yielding

$$\delta(1 + \varepsilon/4) \leq x_1 - x_1^0 \leq \delta(1 + \varepsilon/2),$$

which implies that on the support of h one has

$$\delta(1 + \varepsilon/4) \leq x_1 - x_1^0 \leq \delta(1 + \varepsilon/2) \quad \text{and} \quad 8(\varepsilon\delta)^{-2}\|x' - x^{0\prime}\|^2 \leq 3/2 + \varepsilon/2. \tag{4.10}$$

In particular, in the case $\varepsilon \leq 1$, one finds

$$\operatorname{supp} h \subset B\big(x^0 + \delta X(x^0), \varepsilon\delta\big). \tag{4.11}$$

These estimations of the supports of q and h are illustrated in Figure 3.

Lemma 4.4. *For any $0 < \varepsilon \leq 1$ there exists $\delta_0 > 0$ such that for any $x^0 \in K$ and $0 < \delta \leq \delta_0$*

1. *the function g is nonnegative and is positive in a neighborhood of x^0.*
2. *$Xx_1 \geq 1/2$ in $\operatorname{supp} q$.*

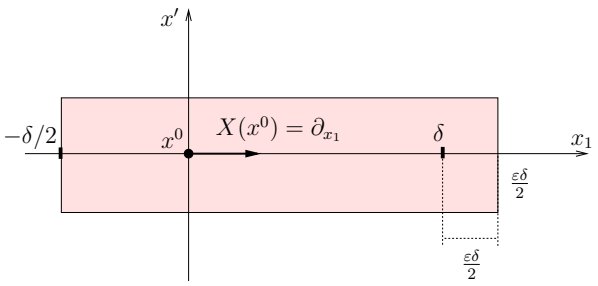
Proof. Consider $0 < \varepsilon \leq 1$. One has $g = (\chi' \circ v)(\beta \circ w)Xv$. Since $\beta \geq 0$ and $\chi' < 0$ it suffices to prove that $Xv(x) \leq 0$ for x in the support of $(\chi' \circ v)(\beta \circ w)$ for $\delta > 0$ chosen sufficiently small, uniformly with respect to $x^0 \in K$.

Write

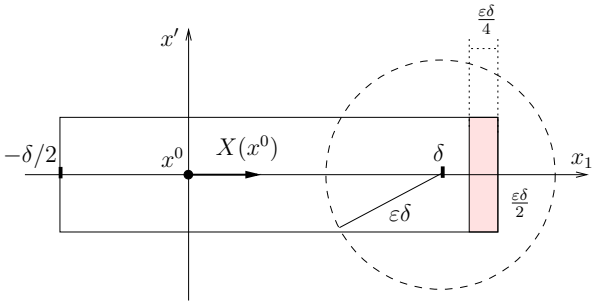
$$X(x) - X(x^0) = \alpha^1(x, x^0)\partial_{x_1} + \alpha'(x, x^0) \cdot \nabla_{x'}, \tag{4.12}$$

with $\alpha^1(x, x^0) \in \mathbb{R}$ and $\alpha'(x, x^0) \in \mathbb{R}^{d-1}$. By (4.9), for $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$ one has $\|x - x^0\| \lesssim \delta$. From the uniform continuity of X in any compact set one concludes that

$$|\alpha^1(x, x^0)| + \|\alpha'(x, x^0)\| = o(1) \quad \text{as } \delta \to 0^+, \tag{4.13}$$



(a) Support of q.



(b) Support of h.

FIGURE 3. Estimation of the test function supports in the case $\varepsilon \leq 1$.

uniformly[4] with respect to $x^0 \in K$ and $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$. Using that $X(x^0) = \partial_{x_1}$ and the form of v given above, one writes

$$\begin{aligned} Xv(x) &= \big(X(x)v\big)(x) = \big(\partial_{x_1} v + \big(X(x) - X(x^0)\big)v\big)(x) \\ &= -\delta^{-1}\Big(1 + \alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})\Big). \end{aligned}$$

Using again (4.9), one thus finds for $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$

$$\big|\alpha^1(x, x^0) - 16\varepsilon^{-1}(\varepsilon\delta)^{-1}\alpha'(x, x^0) \cdot (x' - x^{0\prime})\big| \lesssim \big|\alpha^1(x, x^0)\big| + \varepsilon^{-1}\|\alpha'(x, x^0)\|.$$

With ε fixed above and with (4.13) one finds that $Xv(x) \sim -\delta^{-1}$ as $\delta \to 0^+$ uniformly with respect to $x^0 \in K$ and $x \in \operatorname{supp}(\chi' \circ v)(\beta \circ w)$.

One also has $g(x^0) = -\delta^{-1}\chi'(1/2)\beta(2\varepsilon^{-1}) > 0$ and thus g is positive in a neighborhood of x^0, concluding the first part.

[4]Observe that the change of variables made above for $X(x^0) = (1, 0, \ldots, 0)$ does not affect uniformity since the dilation is made by a factor in $[c_K, C_K]$.

With (4.12) one has $Xx_1 = 1+\alpha^1(x,x^0)$ and thus $Xx_1 \geq 1/2$ if $\|x-x^0\| \leq \eta$, with $\eta > 0$ sufficiently small, uniformly in $x^0 \in K$. The estimate of $\operatorname{supp} q$ in (4.9) gives $\|x-x^0\| \leq \eta$ for δ_0 chosen sufficiently small. This gives the second part. □

We are now in a position to conclude the proof of Proposition 4.3. Note that it suffices to prove the result for $0 < \varepsilon \leq 1$. Choose $\delta_0 > 0$ as given by Lemma 4.4. With (4.1), for $0 < \delta \leq \delta_0$, one has

$$0 = \big\langle \mu, (X-f)q \big\rangle = \langle \mu, g\rangle + \langle \mu, h\rangle + \big\langle \mu, (A(Xx_1)-f)q \big\rangle.$$

By Lemma 4.4, $(A(Xx_1)-f)q \geq 0$ for $A \geq 2\sup_K |f|$, implying $\big\langle \mu, (A(Xx_1)-f)q\big\rangle \geq 0$. By Lemma 4.4, $g \geq 0$ and g is positive in a neighborhood of x^0. As $x^0 \in \operatorname{supp}\mu$ one finds $\langle \mu, g\rangle > 0$. Consequently, $\langle \mu, h\rangle \neq 0$. By the support estimate for h given in (4.11) the conclusion follows: $\operatorname{supp}\mu \cap B\big(x^0+\delta X(x^0), \varepsilon\delta\big) \neq \emptyset$. □

4.2. **Support propagation at a boundary in the transverse case.** As above consider X a $\mathscr{C}^0$-vector field in an open subset Ω of a $\mathscr{C}^2$ d-dimensional manifold. Suppose $\mathcal{I}$ is a $\mathscr{C}^1$-hypersurface and $x^1 \in \mathcal{I} \cap \Omega$. In a local chart $(\mathcal{O}, \phi)$ with $\mathcal{O} \subset \Omega$ neighborhood of x^1, $\mathcal{I}$ is given by $\varphi(x) = 0$ with $d\varphi(x^1) \neq 0$ for some $\mathscr{C}^1$-function φ. Assume that X is transverse to $\mathcal{I}$ at x^1, meaning that $X\varphi(x^1) = d\varphi(x^1)\big(X(x^1)\big) \neq 0$. This property remains true in a bounded neighborhood $V \Subset \mathcal{O}$ of x^1. Set

$$V^+ = V \cap \{\varphi > 0\}, \quad V^- = V \cap \{\varphi < 0\}.$$

Consider a nonnegative measure μ that is solution in V to the following transport equation with a single-layer potential

$$ {}^tX\mu = f\mu + \tilde{\mu} \otimes \delta_{\mathcal{I}}, \tag{4.14}$$

where $\tilde{\mu}$ is a measure on $\mathcal{I}$. Recall that $\delta_{\mathcal{I}} = |\nabla\varphi|\,\varphi^*\delta$ (see e.g. [16, Theorem 6.1.5]).

The main result of this section is the following proposition.

Theorem 4.5. *Suppose X is transverse to the hypersurface $\mathcal{I}$ in V and μ is a nonnegative measure that vanishes in V^- and is solution to (4.14). If $x^* \in \mathcal{I} \cap \operatorname{supp}\mu$, then there exists an integral curve of X, $s \mapsto \gamma(s)$, such that $\gamma(0) = x^*$ and*

1. *if $X\varphi > 0$, then there exists $S > 0$ such that $\{\gamma(s)\}_{s\in[0,S[} \subset \operatorname{supp}\mu$;*
2. *if $X\varphi < 0$, then there exists $S > 0$ such that $\{\gamma(s)\}_{s\in]-S,0]} \subset \operatorname{supp}\mu$.*

In other words, a half integral curve of X initiated at x is *locally* contained in $\operatorname{supp}\mu$. This half integral curve is naturally located in $\overline{V^+}$. The theorem is based on the following proposition whose proof is given below.

Proposition 4.6. *Suppose $\alpha = \pm 1$ and $K \subset V$ is a compact set such that $\alpha X\varphi > 0$ on $K \cap \mathcal{I}$. Then,*

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall x^0 \in K \cap \overline{V^+} \cap \operatorname{supp}\mu,\ \forall \delta \in [0, \delta_0],$$
$$B\big(x^0 + \delta\alpha X(x^0), \delta\varepsilon\big) \cap \operatorname{supp}\mu \neq \emptyset.$$

Remark 4.7. In the case X is $\mathscr{C}^1$ one can use its flow to find coordinates $x = (x', x_d)$ such that the hypersuface $\mathcal{I}$ is locally given by $\{x_d = 0\}$ and $X = \partial_{x_d}$. Then the result of Proposition 4.6 becomes obvious. Here, we give a more involved proof that applies to the present case of a continuous vector field.

Proof of Theorem 4.5. On the one hand, Proposition 4.6 is the counterpart of Proposition 4.3, and it can be used to adapt the proof of Theorem 4.1 and obtain a proof of Theorem 4.5. On the other hand, one can use the result of Proposition 4.6 in conjunction with the result of Theorem 4.1 to obtain a little shorter proof of Theorem 4.5. We choose this second strategy here.

We treat the case $X\varphi > 0$ here; the other case can be treated similarly. Suppose $x^0 \in \mathcal{I} \cap \operatorname{supp}\mu$. Consider a bounded neighborhood W^0 of x^0 where $X\varphi \geq C_0 > 0$. Consider $n \in \mathbb{N}$ and set $\delta = \varepsilon = 1/(n+1)$. By Proposition 4.6 there exists $x_n^1 \in B(x^0 + X(x^0)/n, 1/n^2)$ such that $x_n^1 \in \operatorname{supp}\mu$. For n chosen sufficiently large, one has $x_n^1 \in V^+$. In V^+ one has ${}^tX\mu = f\mu$. One then applies Theorem 4.1: there exists a *maximal* integral curve $\tilde{\gamma}_n :]S_1^n, S_2^n[\to V^+$ of the vector field X, with $S_1^n < 0 < S_2^n$ with $\tilde{\gamma}_n(0) = x_n^1$, that lies in $\operatorname{supp}\mu$. Since X is bounded in V^+ (recall that $\overline{V}$ is chosen compact), one finds that there exists $S > 0$ such that $S < S_2^n$ for n chosen sufficiently large.

Set $s_n^1 = \|x_n^1 - x^0\|/\|X(x^0)\|$. One has $s_n^1 \to 0$ as $n \to \infty$. Then define $\gamma_n : [0, S] \to V^+$ by

$$\gamma_n(s) = \begin{cases} x^0 + s\frac{x_n^1 - x^0}{s_n^1} & \text{if } 0 \leq s \leq s_n^1, \\ \tilde{\gamma}_n(s - s_n^1) & \text{if } s_n^1 \leq s \leq S. \end{cases}$$

We then follow the arguments in the proof of Proposition 4.2. One has

$$\dot{\gamma}_n(s) = X(\gamma_n(s)) + e_n(s),$$

for $s \in [0, S]$ where the error $|e_n|$ goes to zero *uniformly* with respect to $|s| \leq S$ as $n \to +\infty$. Naturally, one has $e_n(s) = 0$ for $s \in [s_n^1, S]$. Equation 4.5 is also valid and the Arzelà-Ascoli theorem applies leading to a limit curve that fulfills the sought requirements. □

Proof of Proposition 4.6. We prove the result in the case $X\varphi > 0$, that is $\alpha = 1$, on $K \cap \mathcal{I}$. The proof in the case $X\varphi < 0$ can be written *mutatis mutandis*.

In $K \cap \mathcal{I}$, having $X\varphi > 0$ simply means having X pointing towards $\{\varphi > 0\}$. In fact, in $K \cap \mathcal{I}$ one has $X\varphi \geq C_0 > 0$ for some $C_0 > 0$. Hence, in a bounded open neighborhood W_1 in $\mathbb{R}^d$ of $K \cap \mathcal{I}$ one has $X\varphi \geq C_0/2$. Introduce also W_2 a bounded open neighborhood of $\overline{W_1}$ where $X\varphi \geq C_0/4$.

Since $K \setminus W_1$ is compact, with the result of Proposition 4.3, it suffices to consider the case $x^0 \in K \cap W_1 \cap \operatorname{supp} \mu$. Suppose x^0 is such a point. In the compact set $\overline{W_1} \cap K$ one has $c_1 \leq \|X\| \leq C_1$. As in the proof of Proposition 4.3, performing a rotation and a dilation of coefficient $\|X(x)\| \in [c_1, C_1]$, one can assume that $X(x^0) = (1, 0, \dots, 0) \in \mathbb{R}^d$. By abuse of notation, we still use the letter φ for the function used to define the hypersurface $\mathcal{I}$.

From the equation (4.14) satisfied by μ one has

$$\langle ({}^tX - f)\mu, \varphi q \rangle_{\mathscr{D}'^1(\mathbb{R}^d), \mathscr{C}^1_c(\mathbb{R}^d)} = 0, \qquad q \in \mathscr{C}^1_c(\mathbb{R}^d), \tag{4.15}$$

since $\varphi \delta_{\mathcal{I}} = 0$.

Consider $0 < \varepsilon \leq 1$ (observe that this case is sufficient for the conclusion to hold). For $\delta > 0$, one uses the function φq as a test function, with q as defined in (4.8). With (4.15) one finds

$$\begin{aligned} 0 &= \langle ({}^tX - f)\mu, \varphi q \rangle_{\mathscr{D}'^1(\mathbb{R}^d), \mathscr{C}^1_c(\mathbb{R}^d)} = \langle \mu, X(\varphi q) - f q \rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} \\ &= \langle \mu, \varphi g \rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} + \langle \mu, \varphi h \rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} + \big\langle \mu, (X\varphi) q \big\rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} \\ &\quad + \big\langle \mu, (A(Xx_1) - f)\varphi q \big\rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)}, \end{aligned} \tag{4.16}$$

recalling that $Xq = g + h + A(Xx_1)q$ with g and h given in (4.8).

In W_2 one has $X\varphi \geq C_0/4$. If $\delta_0 > 0$ is chosen sufficiently small, one has $\operatorname{supp} q \subset W_2$ uniformly with respect to $x^0 \in W_1$ for $0 < \delta \leq \delta_0$. As $q \geq 0$ one finds $(X\varphi)q \geq 0$. Moreover $\big((X\varphi)q\big)(x^0) > 0$ and since $x^0 \in \operatorname{supp} \mu$ this yields

$$\big\langle \mu, (X\varphi)q \big\rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} > 0.$$

By Lemma 4.4, for δ_0 chosen sufficiently small and A sufficiently large, one has $g \geq 0$ if $0 < \delta \leq \delta_0$ and $Ax_1 - f \geq 0$. As $\varphi \geq 0$ in $\operatorname{supp} \mu$ one thus finds

$$\langle \mu, \varphi g \rangle + \big\langle \mu, (A(Xx_1) - f)\varphi q \big\rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} \geq 0, \quad \text{for } 0 < \delta \leq \delta_0. \tag{4.17}$$

From (4.16)–(4.17) one obtains $\langle \mu, \varphi h \rangle_{\mathscr{D}'^0(\mathbb{R}^d), \mathscr{C}^0_c(\mathbb{R}^d)} < 0$. By the support estimate for h given in (4.11) the conclusion follows. □

The following lemma is not of direct use in this section but it is used in another section below.

Lemma 4.8. *Suppose μ is a nonnegative measure solution to (4.14) that vanishes in V^-. Suppose $x \in \mathcal{I}$. Then $x \in \operatorname{supp}\mu$ if and only if $x \in \operatorname{supp}(\tilde{\mu} \otimes \delta_{\mathcal{I}})$*

Proof. If $x \notin \operatorname{supp}\mu$ then μ vanishes in a neighborhood of x and from (4.14) one has $\tilde{\mu} \otimes \delta_{\mathcal{I}}$ vanishing in that neighborhood. This gives $x \notin \operatorname{supp}(\tilde{\mu} \otimes \delta_{\mathcal{I}})$.

Suppose now that $x \notin \operatorname{supp}(\tilde{\mu} \otimes \delta_{\mathcal{I}})$. Then, from (4.14), the equation fulfilled by μ is ${}^tX\mu = f\mu$ locally near x. By Theorem 4.1, if $x \in \operatorname{supp}\mu$ then there exist $S > 0$ and a integral curve $\gamma(s)$ of X such that $\gamma(0) = x$ and $\{\gamma(s)\}_{s\in]-S,S[} \subset \operatorname{supp}\mu$. Yet, as X is transverse to $\mathcal{I}$ here, half of the integral curve lies in V^- where μ vanishes; this gives a contradiction. Hence, if $x \notin \operatorname{supp}(\tilde{\mu} \otimes \delta_{\mathcal{I}})$ then $x \notin \operatorname{supp}\mu$. □

5. Geometrical setting II

Here, we carry on with the introduction of the geometrical notions to be used in the subsequent sections. In Section 2 we used the quasi-normal geodesic coordinates of Proposition 2.1 to obtain a 'straight path' towards the necessary notions for the statement of the main result in Theorem 3.4: glancing and hyperbolic regions, hamiltonian vector field and gliding vector field, bicharacteristics and generalized bicharacteristics. In the present section we provide additional results and notions. Yet, we do not rely on quasi-normal geodesic coordinates for the following two reasons: (1) the simplifications provided by such coordinates at the boundary hide some of the geometrical properties, and more important, (2) we wish to 'push' the definition of the glancing and hyperbolic regions and gliding vector field away from the boundary to ease arguments in the proof of Theorem 3.4: extending ${}^{\|}\partial(T^*\mathcal{L})$ away from $\partial(T^*\mathcal{L})$ we obtain a foliation of $T^*\mathcal{L}$. Since the advantageous structure of quasi-normal geodesic coordinates is lost away from the boundary it is better to work in arbitary coordinates from the beginning. Note that due to the considered low regularity of the coefficients, the foliation we introduce is not a geometrical object in the sense that it depends on the chosen coordinates and on an extension of the conormal vector field. Yet, this foliation is only used in a single local chart in what follows.

In Section 2, $T^*_x\partial\mathcal{L}$ was identified with the set of conormal vectors $\xi = (\xi', 0)$. This is not natural in general. In fact, if $m^0 \in \partial\mathcal{M}$ set $x^0 = \phi(m^0)$, for a local chart (O, ϕ). One has $x^0 = (x^{0\prime}, 0)$. The injection $\partial\mathcal{M} \to \mathcal{M}$ yields a natural injection of $T_{m^0}\partial\mathcal{M}$ into $T_{m^0}\mathcal{M}$ and the *surjection* of $T^*_{m^0}\mathcal{M}$ into $T^*_{m^0}\partial\mathcal{M}$ by duality that take the form $v' \mapsto (v', 0)$ and $(\xi', \zeta) \mapsto \xi'$ respectively in the considered local coordinates for $v' \in T_x\partial\mathcal{M}$ and $(\xi', \zeta) \in T^*_x\mathcal{M}$. As most often

done, $T_x\partial\mathcal{M}$ is naturally viewed as a linear subspace of $T_x\mathcal{M}$ and in the chosen coordinates, $v' \in T_x\partial\mathcal{M}$ identifies with $v = (v', 0) \in T_x\partial\mathcal{M}$. However, $T_x^*\partial\mathcal{M}$ is identified with the set of covectors orthogonal to the unit vector field n_x at the boundary. In general such covectors do not take the form $(\xi', 0)$; yet, they do in quasi-normal geodesic coordinates.

Note that if a notation appearing in Section 2 is used in what follows, say ${}^{\|}\partial(T^*\mathcal{L})$, it denotes the same object.

Remark 5.1. In what follows, since the metric g is also defined in $\hat{\mathcal{M}}\setminus\mathcal{M}$ we also consider bicharacteristics that leave or enter $T^*\mathcal{L}$. To avoid possible confusion we write $\operatorname{Char} p \cap T^*\mathcal{L}$ or $\operatorname{Char} p \cap \{z \geq 0\}$ if only considering characteristic points in the cotangent bundle $T^*\mathcal{L}$ and not the extension made outside $\mathcal{M}$ and $\mathcal{L}$.

5.1. **Musical isomorphisms, normal and conormal vectors.** Consider a point $(x, v) \in T\mathcal{M}$. As is done classicaly, denote by $v^\flat$, and $(x, v)^\flat$ by extension, the unique element of $T_x^*\mathcal{M}$ such that

$$\langle v^\flat, u\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = g_x(v, u), \qquad u \in T_x\mathcal{M}.$$

In local coordinates, this reads $(v^\flat)_i = g_{ij}(x)v^j$, $1 \leq i \leq d$. One thus obtains a map $\flat : v \mapsto v^\flat$ from $T_x\mathcal{M}$ into $T_x^*\mathcal{M}$, and by extension from $T\mathcal{M}$ into $T^*\mathcal{M}$. With the invertibility of $g_x = (g_{ij}(x))_{i,j}$ one readily sees that $\flat$ is an isomorphism. Moreover, one has $g_x^*(v^\flat, v^\flat) = g_x(v, v)$, meaning that $\flat$ is an isometry.

The inverse isometry is denoted by $\sharp$. One has $\sharp : \xi \mapsto \xi^\sharp$ from $T_x^*\mathcal{M}$ onto $T_x\mathcal{M}$, and by extension from $T^*\mathcal{M}$ onto $T\mathcal{M}$. One has

$$\langle \xi^\sharp, \omega\rangle_{T_x\mathcal{M}, T_x^*\mathcal{M}} = g_x^*(\xi, \omega), \qquad \omega \in T_x^*\mathcal{M},$$

and in local coordinates

$$(\xi^\sharp)^i = g^{ij}(x)\xi_j, \quad 1 \leq i \leq d.$$

One can also write, for $\xi \in T_x^*\mathcal{M}$ and $v \in T_x\mathcal{M}$

$$\langle \xi, v\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = g_x(\xi^\sharp, v) = g_x^*(\xi, v^\flat) = \langle \xi^\sharp, v^\flat\rangle_{T_x\mathcal{M}, T_x^*\mathcal{M}}.$$

From (2.3) one finds

$$\mathrm{H}_p(\varrho) = -2\tau\partial_t + 2(\xi^\sharp)^j\partial_{x_j} - \partial_{x_k}g^{ij}(x)\xi_i\xi_j\partial_{\xi_k}.$$

For $x = (x', 0) \in \partial\mathcal{M}$, denote by $\mathsf{n}_x \in T_x\mathcal{M}$ the unitary *inward* [5] pointing normal vector to $\partial\mathcal{M}$, meaning that $g_x(\mathsf{n}_x, \mathsf{n}_x) = 1$ and $g_x(\mathsf{n}_x, v) = 0$ for all

[5] Here, we choose the inward direction for n_x to be consistent with having $z > 0$ if $(x', z, \xi) \in T^*\mathcal{M}$ as far as sign are concerned.

$v \in T_x\partial\mathcal{M}$ and $\mathsf{n}_x^d > 0$. Set $\mathsf{n}_x^* = \mathsf{n}_x^\flat \in T_x^*\mathcal{M}$. One has $g_x^*(\mathsf{n}_x^*, \mathsf{n}_x^*) = 1$ and

$$\langle \mathsf{n}_x^*, v\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = 0, \quad v \in T_x\partial\mathcal{M}.$$

In a local chart at the boundary as in (2.1) this gives $\mathsf{n}_x^* = \big(0, \dots, 0, (g^{dd}(x))^{-1/2}\big)$, that is, $\mathsf{n}_x^* = (g^{dd}(x))^{-1/2} dx_d$. One deduces

$$\mathsf{n}_x^i = \big((\mathsf{n}_x^*)^\sharp\big)^i = g^{ij}(x)(\mathsf{n}_x^*)_j = (g^{dd})^{-1/2} g^{id}(x).$$

Note that n_x and n_x^* have $\mathscr{C}^1$ regularity.

Remark 5.2. We insist on the fact that n_x and n_x^* are here defined on the boundary only. A natural extension away from the boundary would use normal geodesic coordinates that are not available here (see Remark 2.2) or a local geodesic flow but the latter may not exist due to the potential lack of uniqueness of geodesics here.

5.2. **Partition of the cotangent bundle revisited.** Consider a local chart at the boundary as in (2.1). As above, denote $z = x_d$ with $x = (x', z)$ and, accordingly, the associated cotangent variables read $\xi = (\xi', \zeta)$. For $x \in \partial\mathcal{M}$, denote by ${}^{\|}T_x^*\mathcal{M}$ the orthogonal of n_x^* in the sense of g_x^*, that is,

$${}^{\|}T_x^*\mathcal{M} = \{\xi \in T_x^*\mathcal{M};\ g_x^*(\xi, \mathsf{n}_x^*) = 0\}.$$

One has ${}^{\|}T_x^*\mathcal{M} = \flat(T_x\partial\mathcal{M})$ and $T_x^*\mathcal{M} = {}^{\|}T_x^*\mathcal{M} \oplus \operatorname{span}(\mathsf{n}_x^*)$. Denote by $\pi_{\|}$ the orthogonal projection onto ${}^{\|}T_x^*\mathcal{M}$. For $\xi \in T_x^*\mathcal{M}$ set ${}^{\|}\xi = \pi_{\|}(\xi)$ that reads

$$ {}^{\|}\xi = \xi - g_x^*\big(\xi, \mathsf{n}_x^*\big)\mathsf{n}_x^*. \tag{5.1}$$

If $\varrho = (t, x, \tau, \xi) \in T^*\mathcal{L}$ the following computations in the considered local coordinates are useful

$$\begin{aligned} &\mathrm{H}_z^2\, p(\varrho) = \mathrm{H}_z^2\, p(x) = 2g^{dd}(x) \neq 0, \\ &\mathrm{H}_p\, z(\varrho) = \mathrm{H}_p\, z(x, \xi) = 2g^{dj}(x)\xi_j = \big(2\,\mathrm{H}_z^2\, p(x)\big)^{1/2} g_x^*(\xi, \mathsf{n}_x^*), \end{aligned} \tag{5.2}$$

yielding, for $\varrho \in \partial(T^*\mathcal{L})$, that is, $x \in \partial\mathcal{M}$,

$${}^{\|}\xi = \xi - \alpha\,\mathrm{H}_p\, z(x, \xi)\mathsf{n}_x^*, \qquad \text{with } \alpha(x) = \big(2\,\mathrm{H}_z^2\, p(x)\big)^{-1/2},$$

as $\mathsf{n}_x^* = \big(0, \dots, 0, 2\alpha(x)\big)$. In local coordinates this gives

$$ {}^{\|}\xi = \Big(\xi_1, \dots, \xi_{d-1}, \zeta - \frac{\mathrm{H}_p\, z}{\mathrm{H}_z^2\, p}(x, \xi)\Big) = \Big(\xi_1, \dots, \xi_{d-1}, -\frac{2}{\mathrm{H}_z^2\, p}(x) \sum_{j=1}^{d-1} g^{jd}(x)\xi_j\Big). \tag{5.3}$$

Above was mentionned the surjection $T^*_x\mathcal{M}$ into $T^*_x\partial\mathcal{M}$. Consider the map

$$\begin{aligned} {}^{\|}T^*_x\mathcal{M} &\to T^*_x\partial\mathcal{M} \\ (\xi',\zeta) &\mapsto \xi', \end{aligned}$$

One finds that it is an isomorphism, giving a geometrical identification of $T^*_x\partial\mathcal{M}$ as a subspace of $T_x\mathcal{M}$. However, we keep the notation ${}^{\|}T^*_x\mathcal{M}$ to avoid any possible confusion.

One also denotes by Σ the orthogonal symmetry with respect to ${}^{\|}T^*_x\mathcal{M}$, that is,

$$\begin{aligned} \Sigma(\xi) &= \xi - 2g^*_x\big(\xi,\mathsf{n}^*_x\big)\mathsf{n}^*_x \\ &= \xi - 2\alpha\,\mathrm{H}_p\,z(x,\xi)\mathsf{n}^*_x = {}^{\|}\xi - \alpha\,\mathrm{H}_p\,z(x,\xi)\mathsf{n}^*_x. \end{aligned}$$

In local coordinates this gives $\Sigma(\xi) = \Big(\xi_1,\dots,\xi_{d-1},\zeta - 2\frac{\mathrm{H}_p\,z}{\mathrm{H}^2_z\,p}(x,\xi)\Big)$. One has $\Sigma(\xi)+\xi = 2\,{}^{\|}\xi$.

Accordingly, for $x\in\partial\mathcal{M}$ set

$${}^{\|}T^*_{t,x}\mathcal{L} = \{(\tau,\xi)\in T^*_{t,x}\mathcal{L};\ \xi\in {}^{\|}T^*_x\mathcal{M}\},$$

and

$${}^{\|}\partial(T^*\mathcal{M}) = \bigcup_{x\in\partial\mathcal{M}}\{x\}\times {}^{\|}T^*_x\mathcal{M},\quad {}^{\|}\partial(T^*\mathcal{L}) = \bigcup_{(t,x)\in\partial\mathcal{L}}\{(t,x)\}\times {}^{\|}T^*_{t,x}\mathcal{L},$$

and for $\varrho = (t,x,\tau,\xi)\in\partial\mathcal{L}$ one writes $\pi_{\|}(\varrho) = {}^{\|}\varrho = (t,x,\tau,{}^{\|}\xi)$ and $\Sigma(\varrho) = (t,x,\tau,\Sigma(\xi))$.

Naturally, for $\varrho\in\partial(T^*\mathcal{L})$ one has $\pi_{\|}(\varrho) = \pi_{\|}(\Sigma(\varrho)) = {}^{\|}\varrho$ and

$$\varrho\in {}^{\|}\partial(T^*\mathcal{L}) \quad\Leftrightarrow\quad {}^{\|}\varrho = \varrho \quad\Leftrightarrow\quad \Sigma(\varrho) = \varrho.$$

From what is written above for $x\in\partial\mathcal{M}$ one has

$$\begin{aligned} \xi &= {}^{\|}\xi + (\alpha\,\mathrm{H}_p\,z)(\varrho)\,\mathsf{n}^*_x = {}^{\|}\xi + \alpha(x)\,\mathrm{H}_p\,z(x,\xi)\,\mathsf{n}^*_x, \\ \Sigma(\xi) &= {}^{\|}\xi + (\alpha\,\mathrm{H}_p\,z)(\Sigma(\varrho))\,\mathsf{n}^*_x = {}^{\|}\xi - \alpha(x)\,\mathrm{H}_p\,z(x,\xi)\,\mathsf{n}^*_x \end{aligned} \tag{5.4}$$

yielding

$$\mathrm{H}_p\,z(\Sigma(\varrho)) = -\,\mathrm{H}_p\,z(\varrho),\qquad \varrho\in\partial(T^*\mathcal{L}). \tag{5.5}$$

From (5.2) one also has the following characterization of ${}^{\|}\partial(T^*\mathcal{L})$

$$ {}^{\|}\partial(T^*\mathcal{L}) = \{z = \mathrm{H}_p\, z = 0\}. \tag{5.6}$$

Note that for any $\mathscr{C}^2$-function ϕ with $d\phi \neq 0$ such that $\partial\mathcal{M} = \{\phi = 0\}$ locally, one finds

$$ z = \mathrm{H}_p\, z = 0 \quad \Leftrightarrow \quad \phi = \mathrm{H}_p\, \phi = 0,$$

meaning that the use of (5.6) made below is not coordinate dependent.

One can observe that ${}^{\|}\partial(T^*\mathcal{L})$ is a symplectic submanifold of $T^*\mathcal{L}$.

Note that one has

$$ \partial_\zeta\, \mathrm{H}_p\, z = -\,\mathrm{H}_z\, \mathrm{H}_p\, z = \mathrm{H}_z\, \mathrm{H}_z\, p = \mathrm{H}_z^2\, p, \tag{5.7}$$

and

$$ \mathrm{H}_z\, {}^{\|}\xi = -\partial_\zeta {}^{\|}\xi = 0. \tag{5.8}$$

Suppose $\varrho \in \partial(T^*\mathcal{L})$. Since $\pi_\|$ maps $\partial(T^*\mathcal{L})$ into ${}^{\|}\partial(T^*\mathcal{L})$, then $d\pi_\|(\varrho)$, its differential at ϱ, maps $T_\varrho\partial(T^*\mathcal{L})$ into $T_{{}^{\|}\varrho}\,{}^{\|}\partial(T^*\mathcal{L})$.

Lemma 5.3. *Suppose $\varrho \in \partial(T^*\mathcal{L})$. One has* $\ker(d\pi_\|(\varrho)) = \mathrm{span}(\mathrm{H}_z)(\varrho)$.

In local coordinates one has $\mathrm{H}_z = -\partial_\zeta = (0, \dots, 0, -1) \in \mathbb{R}^{2d+2}$ at $\varrho \in \partial(T^*\mathcal{L})$.

Proof. Consider $v \in T_\varrho\partial(T^*\mathcal{L})$, that is, $v \in \mathrm{span}\{\partial_t, \partial_{x_i}, \partial_\tau, \partial_{\xi_j}\}$, $i = 1, \dots, d-1$, $j = 1, \dots, d$. From the form of $\pi_\|$ given in (5.3) one has

$$ d\pi_\|(\varrho)(v) = v - d(\mathrm{H}_p\, z/\,\mathrm{H}_z^2\, p)(\varrho)(v)\partial_\zeta$$

If one has $d\pi_\|(\varrho)(v) = 0$ then one sees that $v \in \mathrm{span}(\mathrm{H}_z)$.

Conversely, if $v = \partial_\zeta$ one has $d\pi_\|(\varrho)(v) = \big(1 - d(\mathrm{H}_p\, z/\,\mathrm{H}_z^2\, p)(\varrho)(\partial_\zeta)\big)\partial_\zeta$ and, using that $\mathrm{H}_z^2\, p(\varrho)$ is independent of ξ,

$$ d(\mathrm{H}_p\, z/\,\mathrm{H}_z^2\, p)(\varrho)(\partial_\zeta) = \partial_\zeta(\mathrm{H}_p\, z/\,\mathrm{H}_z^2\, p)(\varrho) = \frac{\partial_\zeta\, \mathrm{H}_p\, z(\varrho)}{\mathrm{H}_z^2\, p(x)} = \frac{\mathrm{H}_z^2\, p}{\mathrm{H}_z^2\, p}(x) = 1,$$

by (5.7), implying $d\pi_\|(\varrho)(v) = 0$. □

If $x \in \partial\mathcal{M}$ observe with what precedes that the maps

$$(x,\xi) \mapsto (x, {}^{\|}\xi, \mathrm{H}_p\, z(x,\xi)), \quad \text{and} \quad \varrho \mapsto ({}^{\|}\varrho, \mathrm{H}_p\, z(\varrho)), \tag{5.9}$$

yield natural parametrizations of $\partial(T^*\mathcal{M})$ and $\partial(T^*\mathcal{L})$ that is used in what follows. Observe however that these coordinates are only $\mathscr{C}^1$.

Definition 5.4 (outward, inward, and tangentially pointing vectors)**.** Consider $\varrho = (t,x,\tau,\xi) \in T^*\mathcal{L}$ with $x \in \partial\mathcal{M}$. One says that

(1) ϱ points strictly outward if $\mathrm{H}_p\, z(\varrho) < 0$;
(2) ϱ points tangentially if $\mathrm{H}_p\, z(\varrho) = 0$, that is, $\varrho \in {}^{\|}T^*\mathcal{L}$;
(3) ϱ points strictly inward if $\mathrm{H}_p\, z(\varrho) > 0$.

Set $v = \xi^\sharp$. One has $v^d = g^{dj}\xi_j = \mathrm{H}_p\, z(\varrho)/2 = \alpha^{-1} g_x^*(\xi, \mathsf{n}_x^*)/2$. The terminology of Definition 5.4 is thus related to the sign of v^d (and not that of ζ) and, as we shall see below, to the behavior of bicharacteristic that goes through ϱ if moreover $\varrho \in \operatorname{Char} p$; see Lemma 5.12. With (5.5) one has the following properties.

Lemma 5.5. *Consider $\varrho \in \partial(T^*\mathcal{L})$. One has*

(1) *ϱ strictly points inward if and only if $\Sigma(\varrho)$ strictly points outward;*
(2) *ϱ points tangentially if and only if $\Sigma(\varrho) = \varrho$.*

As in Section 2.3 the vector bundle ${}^{\|}\partial(T^*\mathcal{L})$ is written as the union of the three bundles ${}^{\|}\mathcal{E}_\partial$, ${}^{\|}\mathcal{G}_\partial$, ${}^{\|}\mathcal{H}_\partial$. Their definition is identical to that given in Definition 2.5, which we reproduce here to ease reading.

Definition 5.6 (elliptic, glancing, and hyperbolic regions)**.** One partitions ${}^{\|}T^*\mathcal{L}$ into three homogeneous regions.

(1) The elliptic region ${}^{\|}\mathcal{E}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p > 0\}$; if $\varrho \in {}^{\|}\mathcal{E}_\partial$ it is called an elliptic point.
(2) The glancing region ${}^{\|}\mathcal{G}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p = 0\}$; if $\varrho \in {}^{\|}\mathcal{G}_\partial$ it is called a glancing point.
(3) The hyperbolic region ${}^{\|}\mathcal{H}_\partial = {}^{\|}\partial(T^*\mathcal{L}) \cap \{p < 0\}$; if $\varrho \in {}^{\|}\mathcal{H}_\partial$ it is called a hyperbolic point.

Lemma 5.7. *The sets ${}^{\|}\mathcal{E}_\partial$, ${}^{\|}\mathcal{G}_\partial$ and ${}^{\|}\mathcal{H}_\partial$ are also characterized by*

$$\varrho \in {}^{\|}\mathcal{E}_\partial \quad \Leftrightarrow \quad \varrho \in {}^{\|}\partial(T^*\mathcal{L}) \quad \textit{and} \quad \pi_{\|}^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \emptyset,$$
$$\varrho \in {}^{\|}\mathcal{G}_\partial \quad \Leftrightarrow \quad \varrho \in {}^{\|}\partial(T^*\mathcal{L}) \quad \textit{and} \quad \pi_{\|}^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \{\varrho\},$$
$$\varrho \in {}^{\|}\mathcal{H}_\partial \quad \Leftrightarrow \quad \varrho \in {}^{\|}\partial(T^*\mathcal{L}) \quad \textit{and} \quad \pi_{\|}^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \{\varrho^-, \varrho^+\},$$

where, in the last case, $\varrho^\pm = (t, x, \tau, \xi^\pm)$ *if* $\varrho = (t, x, \tau, \xi)$ *with*

$$\xi^+ = \xi + \lambda \mathsf{n}_x^* \quad \textit{and} \quad \xi^- = \xi - \lambda \mathsf{n}_x^*, \tag{5.10}$$

with $\lambda = \big(-p(\varrho)\big)^{1/2} = \big(\tau^2 - |\xi|_x^2\big)^{1/2}$.

For $\varrho \in {}^{\|}\mathcal{H}_\partial$ the notation $\varrho^\pm$ is used in what follows with the definition given in this lemma. By (5.4) one has

$$\alpha(x)\, \mathrm{H}_p\, z(\varrho^+) = \lambda > 0 \quad \text{and} \quad \alpha(x)\, \mathrm{H}_p\, z(\varrho^-) = -\lambda < 0,$$

with $\alpha(x) = \big(2\, \mathrm{H}_z^2\, p(x)\big)^{-1/2}$, that is, ϱ^+ points *inward* and ϱ^- points *outward* in the sense given in Definition 5.4.

Proof of Lemma 5.7. If $\varrho = (t, x; \tau, \xi) \in \partial(T^*\mathcal{L})$, then $\tilde\varrho \in \pi_{\|}^{-1}(\{\varrho\})$ reads $\tilde\varrho = (t, x; \tau, \tilde\xi)$ with $\tilde\xi = \xi + \lambda \mathsf{n}_x^*$ for some $\lambda \in \mathbb{R}$. And one has $|\tilde\xi|_x^2 = |\xi|_x^2 + \lambda^2$ and $p(\tilde\varrho) = p(\varrho) + \lambda^2$. If $\varrho \in {}^{\|}\mathcal{E}$ one has $p(\varrho) > 0$ and thus no choice of $\lambda \in \mathbb{R}$ can yield $p(\tilde\varrho) = 0$. If $\varrho \in {}^{\|}\mathcal{G}$ one has $p(\varrho) = 0$ and thus the only choice of $\lambda \in \mathbb{R}$ to have $p(\tilde\varrho) = 0$ is $\lambda = 0$. If $\varrho \in {}^{\|}\mathcal{H}$ one has $p(\varrho) < 0$ and thus one has $p(\tilde\varrho) = 0$ if and only if $\lambda = \pm\big(-p(\varrho)\big)^{1/2}$. □

The following definition is counterpart to Definition 2.6 yet not coordinate dependent.

Definition 5.8 (partition of $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$)**.** Set

$$\mathcal{G}_\partial = \{\varrho \in \partial(T^*\mathcal{L});\ p(\varrho) = 0 \text{ and } \mathrm{H}_p\, z(\varrho) = 0\},$$

and $\mathcal{H}_\partial = \mathcal{H}_\partial^+ \cup \mathcal{H}_\partial^-$ with

$$\mathcal{H}_\partial^\pm = \{\varrho \in \partial(T^*\mathcal{L});\ p(\varrho) = 0 \text{ and } \mathrm{H}_p\, z(\varrho) \gtrless 0\},$$

that is, the set of characteristic points at the boundary that point tangentially ($\mathcal{G}_\partial$), strictly inward ($\mathcal{H}_\partial^+$), and outward ($\mathcal{H}_\partial^-$). Recall that $\partial(T^*\mathcal{L})$ is locally $\{z = 0\}$.

Together $\mathcal{G}_\partial$ and $\mathcal{H}_\partial$ (resp. $\mathcal{G}_\partial$, $\mathcal{H}_\partial^+$, and $\mathcal{H}_\partial^-$) form a partition of $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$.

The index ∂ in Definition 5.8 expresses that only boundary points are considered, that is $z = 0$. At places we use extensions of these sets away from the boundary. Yet, this is done in local charts only and not in a geometrically invariant way; see Section 5.3.

The sets $\mathcal{H}_\partial^\pm$ are connected and open in $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$. The set $\mathcal{G}_\partial$ is a connected and closed subset of $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$.

We now make the connexion between ${}^\parallel\mathcal{G}_\partial$ and $\mathcal{G}_\partial$ on the one hand, and ${}^\parallel\mathcal{H}_\partial$ and $\mathcal{H}_\partial$ on the other hand. One has ${}^\parallel\mathcal{G}_\partial = \operatorname{Char} p \cap {}^\parallel\partial(T^*\mathcal{L})$. Since ${}^\parallel\partial(T^*\mathcal{L})$ is charaterized by $\mathrm{H}_p z = 0$ and $z = 0$, see (5.6), one finds that ${}^\parallel\mathcal{G}_\partial = \mathcal{G}_\partial$ in fact.

Consider $\varrho \in {}^\parallel\mathcal{H}_\partial$ and $\varrho^\pm$ as given by Lemma 5.7. One has $\varrho^\pm$ in $\operatorname{Char} p$ and

$$\alpha(x)\,\mathrm{H}_p z(\varrho^\pm) = g_x^*(\xi^\pm, \mathsf{n}_x^*) = \pm\lambda \tag{5.11}$$

with $\lambda > 0$ as in Lemma 5.7. Thus $\varrho^+ \in \mathcal{H}_\partial^+$ and $\varrho^- \in \mathcal{H}_\partial^-$. One also has $\pi_\parallel(\varrho^\pm) = \varrho$.

One has thus obtained the following proposition.

Proposition 5.9. *One has $\pi_\parallel(\mathcal{G}_\partial) = \mathcal{G}_\partial = {}^\parallel\mathcal{G}_\partial$ and $\pi_\parallel(\mathcal{H}_\partial) = {}^\parallel\mathcal{H}_\partial$ and, conversely,*

$$\pi_\parallel^{-1}({}^\parallel\mathcal{G}_\partial) \cap \operatorname{Char} p = \mathcal{G}_\partial = {}^\parallel\mathcal{G}_\partial, \qquad \pi_\parallel^{-1}({}^\parallel\mathcal{H}_\partial) \cap \operatorname{Char} p = \mathcal{H}_\partial.$$

One has $\Sigma(\mathcal{G}_\partial) = \mathcal{G}_\partial$ and $\Sigma(\mathcal{H}_\partial^+) = \mathcal{H}_\partial^-$.

1. *If $\varrho \in \mathcal{G}_\partial = {}^\parallel\mathcal{G}_\partial$ then $\pi^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \{\varrho\}$ and $\Sigma(\varrho) = \varrho$.*
2. *If $\varrho \in {}^\parallel\mathcal{H}_\partial$ then $\pi^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \{\varrho^+, \varrho^-\}$ with $\varrho^+ \in \mathcal{H}_\partial^+$ and $\varrho^- \in \mathcal{H}_\partial^-$, as given in Lemma 5.7, and $\Sigma(\varrho^+) = \varrho^-$. Conversely, if $\varrho \in \mathcal{H}_\partial^\pm$ then $\Sigma(\varrho) \in \mathcal{H}_\partial^\mp$ and $\pi_\parallel(\varrho) = \pi_\parallel\big(\Sigma(\varrho)\big) \in {}^\parallel\mathcal{H}_\partial$.*

By extension, if $\varrho \in \mathcal{H}_\partial$ one also says that ϱ is a hyperbolic point. The set $\mathcal{H}_\partial$ is also called the hyperbolic region.

Remark 5.10. The function $\mathrm{H}_p z$ is key in the description of the regions $\mathcal{G}_\partial$ and $\mathcal{H}_\partial^\pm$. Having p only $\mathscr{C}^1$ one may think that $\mathrm{H}_p z$ is only $\mathscr{C}^0$. However, above we computed

$$\mathrm{H}_p z(\varrho) = 2(\xi^\sharp)^d = 2g^{dj}(x)\xi_j. \tag{5.12}$$

One thus sees that $\mathrm{H}_p z$ is in fact a $\mathscr{C}^1$-function of ϱ.

If $\varrho = (t, x, \tau, \xi) \in \operatorname{Char} p \cap \partial(T^*\mathcal{L}) = \mathcal{H}_\partial^\pm \cup \mathcal{G}_\partial$ with what we wrote above one has

$$\xi = {}^\parallel\xi + \alpha\,\mathrm{H}_p z(\varrho)\mathsf{n}_x^*, \quad \text{with} \quad \alpha\,\mathrm{H}_p z(\varrho) = \begin{cases} \big(-p({}^\parallel\varrho)\big)^{1/2} > 0 & \text{if } \varrho \in \mathcal{H}_\partial^+, \\ 0 & \text{if } \varrho \in \mathcal{G}_\partial, \\ -\big(-p({}^\parallel\varrho)\big)^{1/2} < 0 & \text{if } \varrho \in \mathcal{H}_\partial^-. \end{cases} \tag{5.13}$$

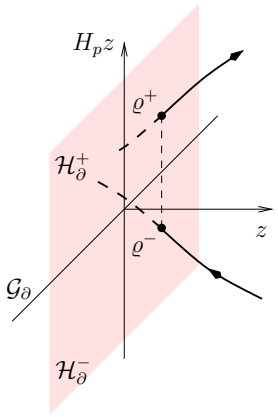


FIGURE 4. Two bicharacteristics; one going through $\varrho^+ \in \mathcal{H}^+_\partial$ and one through $\varrho^- \in \mathcal{H}^-_\partial$. Here $\varrho^+ = \Sigma(\varrho^-)$.

Remark 5.11. Note that $\overline{\mathcal{H}^\pm_\partial} = \mathcal{H}^\pm_\partial \cup \mathcal{G}_\partial$, implying $\overline{\mathcal{H}_\partial} = \mathcal{H}_\partial \cup \mathcal{G}_\partial = \operatorname{Char} p \cap \partial(T^*\mathcal{L})$. If $(\varrho^n)_{n\in\mathbb{N}} \subset {}^{\|}\mathcal{H}_\partial$ converges to $\varrho \in \mathcal{G}_\partial$, then $(\varrho^n)^\pm \to \varrho$ as $n \to +\infty$.

The notions of Definition 5.4 and the description given in Proposition 5.9 are of importance because of the following result, using that for a bicharacteristic $\gamma(s)$, the value of $\mathrm{H}_p f(\gamma(s))$, is equal to the derivative of $s \mapsto f(\gamma(s))$.

Lemma 5.12. *Consider $\varrho \in \operatorname{Char} p$. Denote by $\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$ a bicharacteristic with $x(s) = (x'(s), z(s))$ such that $\gamma(0) = \varrho$.*

(1) *If $\varrho \in \mathcal{G}_\partial$ then $\frac{d}{ds} z_{|s=0} = (\mathrm{H}_p\, z)(\gamma(0)) = 0$;*
(2) *If $\varrho \in \mathcal{H}^\pm_\partial$ then $\frac{d}{ds} z_{|s=0} = (\mathrm{H}_p\, z)(\gamma(0)) \gtrless 0$.*

Thus, for $\varrho \in \mathcal{G}_\partial$, a glancing point at the boundary, any bicharacteristic that goes through ϱ is tangent to $\partial(T^*\mathcal{L})$. For $\varrho \in \mathcal{H}_\partial$, any bicharacteristic that goes through ϱ is transverse to $\partial(T^*\mathcal{L})$, either entering $T^*\mathcal{L}$ if $\varrho \in \mathcal{H}^+_\partial$, or exiting $T^*\mathcal{L}$ if $\varrho \in \mathcal{H}^-_\partial$. This is illustrated in Figure 4. However, the geometry of a bicharacteristic that goes through a glancing point needs to be further described. This is the subject of Section 5.4.

We conclude this section by noting that the glancing set ${}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial$ is a submanifold away from 0 (in the cotangent variable). The singularity at 0 comes from its natural conic structure.

Proposition 5.13 (submanifold property of $\mathcal{G}_\partial$)**.** *The set ${}^{\|}\partial(T^*\mathcal{L})$ is a $\mathscr{C}^1$-submanifolds of $T^*\mathcal{L}$ of codimension two respectively. Away from $(\tau, \xi) = (0, 0)$,*

the set $\mathcal{G}_\partial$ is a $\mathscr{C}^1$-submanifold of $T^*\mathcal{L}$ of codimension three defined by $p = z = \mathrm{H}_p\, z = 0$ (and thus a $\mathscr{C}^1$-submanifold of ${}^{\|}\partial(T^*\mathcal{L})$ of codimension one).

Proof. The first result is clear since ${}^{\|}\partial(T^*\mathcal{M})$ is the orthogonal of n_x^* in $\{z = 0\}$, the metric is $\mathscr{C}^1$, and n_x^* and dz are linearly independent.

The second result amounts to proving that p, z, and $\mathrm{H}_p\, z$ have differentials of rank three at a point $\varrho = (t, x, \tau, \xi) \in \mathcal{G}_\partial$ with $(\tau, \xi) \neq (0, 0)$. Let us consider such a point.

Observe that if $p(\varrho) = 0$ then $(\tau, \xi) = (0, 0) \ \Leftrightarrow\ \tau = 0 \ \Leftrightarrow \xi = 0$. One thus has $\tau \neq 0$. Assume that $\alpha dz(\varrho) + \beta dp(\varrho) + \gamma d(\mathrm{H}_p\, z)(\varrho) = 0$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. As $\partial_\tau z = \partial_\tau\, \mathrm{H}_p\, z = 0$, and $\partial_\tau p = 2\tau \neq 0$ one obtains $\beta = 0$. Since $\partial_\zeta z = 0$ and $\partial_\zeta\, \mathrm{H}_p\, z = \mathrm{H}_z^2\, p \neq 0$ one also has $\gamma = 0$. Finally, as $dz \neq 0$ one finds $\alpha = 0$. □

5.3. **Some local extensions away from the boundary.** Above, the glancing set $\mathcal{G}_\partial$ and the hyperbolic sets $\mathcal{H}_\partial$ are defined at the boundary, in some geometrical fashion, that is, independently of the chosen local coordinates. In Section 6 it is convenient to "push" the notions of glancing set and the hyperbolic sets away from $\partial(T^*\mathcal{L})$, that is $\{z = 0\}$. Yet, as mentionned above there is no geometrical way to extend n_x^* away from $\partial\mathcal{M}$ in a $\mathscr{C}^1$ geometrical fashion. Still, the construction of generalized bicharacteristics performed in Section 6 only relies on local arguments. Here, we thus extend the formentionned notions away from the boundary, yet only in a fixed local chart.

Consider a local chart $\mathcal{C} = (O, \phi)$ at the boundary as in (2.1) where the boundary is given by $\{z = 0\}$.

Extend n_x^* to be equal to $\mathsf{n}_x^{*,\mathcal{C}} = \big(0, \dots, 0, (g^{dd}(x))^{-1/2}\big)$, that is, $\mathsf{n}_x^{*,\mathcal{C}} = (g^{dd}(x))^{-1/2} dx_d \in T_x^*\mathcal{M}$ above all points x of the chart. (The use of the notation $T_x^*\mathcal{M}$ is quite abusive but we have now been sufficiently clear that the extension is not geometrical by any means.)

For any x in the chart one can set

$$ {}^{\|}T_x^*\mathcal{M} = \big(\mathsf{n}_x^{*,\mathcal{C}}\big)^\perp $$

and for $\xi \in T_x^*\mathcal{M}$ one can define $\pi_{\|}$ and ${}^{\|}\xi$ as in (5.1). For $\varrho = (t, x, \tau, \xi) \in T^*\mathcal{L}$ one defines ${}^{\|}\varrho = (t, x, \tau, {}^{\|}\xi)$ and the definition of ${}^{\|}T^*_{(t,x)}\mathcal{L}$ follows similarly as above. Then Set

$$ {}^{\|}T^*\mathcal{M} = \bigcup_{x \in \phi(O)} \{x\} \times {}^{\|}T_x^*\mathcal{M}, \quad {}^{\|}T^*\mathcal{L} = \bigcup_{(t,x) \in I \times \phi(O)} \{(t, x)\} \times {}^{\|}T^*_{(t,x)}\mathcal{L}. $$

One has ${}^{\|}T^*\mathcal{M} \cap \{z = 0\} = {}^{\|}\partial(T^*\mathcal{M})$ and ${}^{\|}T^*\mathcal{L} \cap \{z = 0\} = {}^{\|}\partial(T^*\mathcal{L})$. In the local chart, one can then define

$$ {}^{\|}\mathcal{E} = {}^{\|}T^*\mathcal{L} \cap \{p > 0\}, \quad {}^{\|}\mathcal{G} = {}^{\|}T^*\mathcal{L} \cap \{p = 0\}, \quad {}^{\|}\mathcal{H} = {}^{\|}T^*\mathcal{L} \cap \{p < 0\}, $$

thus extending the elliptic, glancing and hyperbolic regions away from the boundary. The characterization result of Lemma 5.7 extends *mutatis mutandis*. If one sets

$$ \mathcal{G} = \{\varrho \in T^*\mathcal{L};\ p(\varrho) = 0,\ \text{and}\ \mathrm{H}_p\, z(\varrho) = 0\}, $$

and $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$ with

$$ \mathcal{H}^\pm = \{\varrho \in T^*\mathcal{L};\ p(\varrho) = 0,\ \text{and}\ \mathrm{H}_p\, z(\varrho) \gtrless 0\}. $$

One has

$$ \operatorname{Char} p \cap T^*\mathcal{L} = \mathcal{G} \cup \mathcal{H} = \mathcal{G} \cup \mathcal{H}^+ \cup \mathcal{H}^-, $$

and

$$ \mathcal{G}_\partial = \mathcal{G} \cap \partial(T^*\mathcal{L}), \quad \mathcal{H}_\partial = \mathcal{H} \cap \partial(T^*\mathcal{L}), \quad \text{and} \quad \mathcal{H}_\partial^\pm = \mathcal{H}^\pm \cap \partial(T^*\mathcal{L}). $$

If $\varrho \in {}^{\|}\mathcal{H}_\partial$ then $\pi_{\|}^{-1}(\{\varrho\}) \cap \operatorname{Char} p = \{\varrho^-, \varrho^+\}$ and formulae (5.10) extend: with $\varrho = (t, x, \tau, \xi)$ one has $\varrho^\pm = (t, x, \tau, \xi^\pm)$ with

$$ \xi^+ = \xi + \lambda \mathsf{n}_x^* \quad \text{and} \quad \xi^- = \xi - \lambda \mathsf{n}_x^*, \tag{5.14} $$

with $\lambda = \big(-p(\varrho)\big)^{1/2} = \big(\tau^2 - |\xi|_x^2\big)^{1/2}$.

The result of Proposition 5.9 also extends:

$$ \pi_{\|}^{-1}({}^{\|}\mathcal{G}) \cap \operatorname{Char} p = \mathcal{G} = {}^{\|}\mathcal{G}, \qquad \pi_{\|}^{-1}({}^{\|}\mathcal{H}) \cap \operatorname{Char} p = \mathcal{H}, $$

and so does (5.13): if $\varrho = (t, x, \tau, \xi) \in \operatorname{Char} p \cap T^*\mathcal{L}$ one has

(5.15)

$$ \xi = {}^{\|}\xi + \alpha\, \mathrm{H}_p\, z(\varrho) \mathsf{n}_x^*, \quad \text{with}\ \ \alpha\, \mathrm{H}_p\, z(\varrho) = \begin{cases} \big(-p({}^{\|}\varrho)\big)^{1/2} > 0 & \text{if } \varrho \in \mathcal{H}^+, \\ 0 & \text{if } \varrho \in \mathcal{G}, \\ -\big(-p({}^{\|}\varrho)\big)^{1/2} < 0 & \text{if } \varrho \in \mathcal{H}^-. \end{cases} $$

In (5.9), a natural parametrizations of $\partial(T^*\mathcal{M})$ and $\partial(T^*\mathcal{L})$ is mentionned. It extends to $T^*\mathcal{M}$ and $T^*\mathcal{L}$ with what is introduced above:

$$\begin{array}{ll} T^*\mathcal{M} & \to {}^{\|}T^*\mathcal{M}\times\mathbb{R} \\ (x,\xi) & \mapsto (x, {}^{\|}\xi, \mathrm{H}_p\, z(x,\xi)) \end{array} \quad \text{and} \quad \begin{array}{ll} T^*\mathcal{L} & \to {}^{\|}T^*\mathcal{L}\times\mathbb{R} \\ \varrho & \mapsto ({}^{\|}\varrho, \mathrm{H}_p\, z(\varrho)), \end{array} \tag{5.16}$$

With ${}^{\|}T^*\mathcal{M}$ given by $\{\mathrm{H}_p\, z = 0\}\cap T^*\mathcal{M}$ and ${}^{\|}T^*\mathcal{L}$ given by $\{\mathrm{H}_p\, z = 0\}\cap T^*\mathcal{L}$, these two maps are in fact $\mathscr{C}^1$ local diffeomorphisms by the first part of the following proposition obtained by adapting the proof of Proposition 5.13.

Proposition 5.14. *The set ${}^{\|}T^*\mathcal{L}$ is a $\mathscr{C}^1$-submanifold of $T^*\mathcal{L}$ of codimension one, in a local chart. Away from $(\tau,\xi)=(0,0)$, the set $\mathcal{G}$ is a $\mathscr{C}^1$-submanifold of $T^*\mathcal{L}$ of codimension two defined by $\mathrm{H}_p\, z = p = 0$ (and thus a $\mathscr{C}^1$-submanifold of ${}^{\|}T^*\mathcal{L}$ of codimension one) in a local chart.*

A use we make of the parametrizations given in (5.16) is through the following lemma.

Lemma 5.15. *Suppose $s\mapsto\varrho(s)\in T^*\mathcal{L}$. Assume that $s\mapsto{}^{\|}\varrho(s)$ and $s\mapsto \mathrm{H}_p\, z\big(\varrho(s)\big)$ are both differentiable at $s=s_0$. Then $s\mapsto\varrho(s)$ is also differentiable at $s=s_0$.*

Assume moreover that $\varrho(s_0)\in{}^{\|}T^*\mathcal{L}$ and $\frac{d}{ds}\mathrm{H}_p\, z\big(\varrho(s)\big)_{|s=s_0}=0$. Then, $\frac{d}{ds}\varrho(s)_{|s=s_0}=\frac{d}{ds}{}^{\|}\varrho(s)_{|s=s_0}$.

Proof. Write $\varrho(s)=\big(t(s),x(s),\tau(s),\xi(s)\big)$. The first part is a consequence of Proposition 5.14. With (5.15) one has

$$\begin{aligned}\frac{d}{ds}\xi(s)_{|s=s_0} = \frac{d}{ds}{}^{\|}\xi(s)_{|s=s_0} + \frac{d}{ds}\,\mathrm{H}_p\, z\big(\varrho(s)\big)_{|s=s_0}\alpha(x(s_0))n^*_{x(s_0)} \\ + \mathrm{H}_p\, z\big(\varrho(s_0)\big)\frac{d}{ds}\big(\alpha(x(s))n^*_{x(s)}\big)_{|s=s_0}.\end{aligned}$$

If $\varrho(s_0)\in{}^{\|}T^*\mathcal{L}$ then $\mathrm{H}_p\, z\big(\varrho(s_0)\big)=0$, which yields the second result. □

In the local extension, $\pi_{\|}$ maps $T^*\mathcal{L}$ into ${}^{\|}T^*\mathcal{L}$. If $\varrho\in T^*\mathcal{L}$ then, $d\pi_{\|}(\varrho)$, its differential at ϱ maps linearly $T_\varrho T^*\mathcal{L}$ into $T_{{}^{\|}\varrho}{}^{\|}T^*\mathcal{L}$. With a proof similar to that of Lemma 5.3 one has the following result.

Lemma 5.16. *Consider $\varrho\in T^*\mathcal{L}$. One has* $\ker(d\pi_{\|}(\varrho))=\mathrm{span}(\mathrm{H}_z)(\varrho)$.

Proof. The only difference concerns the choice of $v\in T_\varrho T^*\mathcal{L}$ whereas one chose $v\in T_\varrho\partial(T^*\mathcal{L})$ in the proof Lemma 5.3. Thus one has $v\in\mathrm{span}\{\partial_t,\partial_{x_i},\partial_\tau,\partial_{\xi_j}\}$, $i=1,\dots,d$, $j=1,\dots,d$. The remainder of the proof is unchanged. □

In Section 6 below, we denote by $v \in \mathbb{R}^{2d+1}$ local coordinates for ${}^{\|}T^*\mathcal{L}$ and use the variable ϑ to denote the value of $\mathrm{H}_p z$. Then, according to (5.16), (v, ϑ) gives $\mathscr{C}^1$-coordinates on $T^*\mathcal{L}$. One has

$$\partial_\zeta = \sum_{j=1}^{2d+1} (\partial_\zeta v_j)\partial_{v_j} + (\partial_\zeta \vartheta)\partial_\vartheta.$$

Note that (5.7) and (5.8) also hold here. One thus has $\partial_\zeta v_j = 0$, yielding

$$\partial_\zeta = -\,\mathrm{H}_z = \mathrm{H}_z^2\, p\, \partial_\vartheta. \tag{5.17}$$

5.4. **Partition of the glancing region $\mathcal{G}_\partial$ and gliding vector field.** By Definition 5.8 and Proposition 5.13, the glancing region $\mathcal{G}_\partial$ is the $\mathscr{C}^1$-submanifold of $T^*\mathcal{L}$ locally given by $z = \mathrm{H}_p z = p = 0$. By (5.12) $\mathrm{H}_p z$ is a $\mathscr{C}^1$-function. As in Section 2.4 it makes sense to compute $\mathrm{H}_p^2 z$ yielding a $\mathscr{C}^0$-function and we recall the partition of $\mathcal{G}_\partial = \mathcal{G}_\partial^{\mathrm{d}} \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ made in Definition 2.7:

- $\mathcal{G}_\partial^{\mathrm{d}} = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) > 0\}$, the diffractive set;
- $\mathcal{G}_\partial^3 = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) = 0\}$, the glancing sets of order three, meaning that a bicharacteristics that goes through a point of $\mathcal{G}_\partial^3$ has a contact with the boundary of order at least equal to three;
- $\mathcal{G}_\partial^{\mathrm{g}} = \{\varrho \in \mathcal{G}_\partial; \mathrm{H}_p^2 z(\varrho) < 0\}$, the gliding set.

As $\mathrm{H}_p^2 z$ is a $\mathscr{C}^0$-function, the set $\mathcal{G}_\partial^3$ defined by $z = \mathrm{H}_p z = \mathrm{H}_p^2 z = p = 0$ is a $\mathscr{C}^0$-submanifolds of $T^*\mathcal{L}$.

Lemma 5.17. *Consider $\varrho \in \mathcal{G}_\partial^{\mathrm{d}}$. Denote by $\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$ a bicharacteristic above $\hat{\mathcal{L}}$ with $x(s) = (x'(s), z(s))$ such that $\gamma(0) = \varrho$. Then, $\frac{d}{ds} z_{|s=0} = 0$ and $\frac{d^2}{ds^2} z_{|s=0} > 0$, meaning that for some $S > 0$, $\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \in]-S, S[\setminus\{0\}$.*

Here, $\gamma(s)$ is not a bicharacteristic in the sense of Definition 2.3 as it encounters a point in $\partial(T^*\mathcal{L})$ at $s = 0$. It is understood as in Remark 2.4-(3). The behavior of a bicharacteristic above $\hat{\mathcal{L}}$ going through a point of $\mathcal{G}_\partial^{\mathrm{d}}$ is illustrated in Figure 5.

Proof. The result follows as $\frac{d}{ds} z_{|s=0} = \mathrm{H}_p z(\varrho) = 0$ and $\frac{d^2}{ds^2} z_{|s=0} = \mathrm{H}_p^2 z(\varrho) > 0$. □

The map $\pi_\|$ maps $\partial(T^*\mathcal{L})$ onto ${}^{\|}\partial(T^*\mathcal{L})$ and acts as a projection onto ${}^{\|}\partial(T^*\mathcal{M})$ in the cotangent ξ variable; see the beginning of Section 5.2. For $\varrho \in$

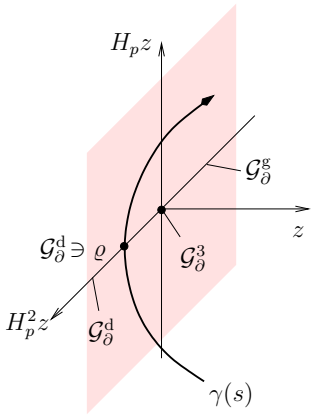


FIGURE 5. A bicharacteristic going through a diffractive point ($\mathcal{G}_\partial^{\mathrm{d}}$).

$\partial(T^*\mathcal{L})$ the differential $d\pi_\parallel(\varrho)$ maps $T_\varrho\partial(T^*\mathcal{L})$ into $T_{\parallel\varrho}{}^{\parallel}\partial(T^*\mathcal{L})$. For $\varrho \in {}^{\parallel}\partial(T^*\mathcal{L})$ one has $\varrho = \pi_\parallel(\varrho)$ and $\mathrm{H}_p(\varrho) \in T_\varrho\partial(T^*\mathcal{L})$ since $\mathrm{H}_p\, z(\varrho) = 0$. One may thus set

$$\mathrm{H}_p^{\mathcal{G}}(\varrho) = d\pi_\parallel(\varrho)\big(\mathrm{H}_p(\varrho)\big) \in T_\varrho{}^{\parallel}\partial(T^*\mathcal{L}). \tag{5.18}$$

Lemma 5.18. *Consider $\varrho \in {}^{\parallel}\partial(T^*\mathcal{L})$. One has*

$$\mathrm{H}_p^{\mathcal{G}}(\varrho) = \Big(\mathrm{H}_p + \frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p}\,\mathrm{H}_z\Big)(\varrho). \tag{5.19}$$

As already seen above $\mathrm{H}_z^2 p$ does not vanish; this makes formula (5.19) sensible. One calls $\mathrm{H}_p^{\mathcal{G}}$ the *gliding* vector field, here defined above ${}^{\parallel}\partial(T^*\mathcal{L})$. Below we extend its definition at every point of $T^*\mathcal{L}$; yet this definition is only local and depends on the considered coordinates.

This definition of $\mathrm{H}_p^{\mathcal{G}}$ is more satisfactory than that given in Section 2.4. There no detail was given; we only intended to be able to state our main result, that is, Theorem 3.4.

Proof. Since $\ker(d\pi_\parallel(\varrho)) = \mathrm{span}(\mathrm{H}_z)$ by Lemma 5.3 one has $\mathrm{H}_p^{\mathcal{G}}(\varrho) = \mathrm{H}_p(\varrho) + \lambda\,\mathrm{H}_z(\varrho)$ for some $\lambda \in \mathbb{R}$. Since $\mathrm{H}_p^{\mathcal{G}}(\varrho) \in T_\varrho{}^{\parallel}\partial(T^*\mathcal{L})$ and ${}^{\parallel}\partial(T^*\mathcal{L})$ is defined by $z = \mathrm{H}_p\, z = 0$ one has

$$0 = \mathrm{H}_p^{\mathcal{G}}(\mathrm{H}_p\, z)(\varrho) = \big(\mathrm{H}_p^2 z + \lambda\,\mathrm{H}_z\,\mathrm{H}_p\, z\big)(\varrho) = \big(\mathrm{H}_p^2 z - \lambda\,\mathrm{H}_z^2 p\big)(\varrho).$$

Since $\mathrm{H}_z^2 p \neq 0$ one finds that $\lambda = (\mathrm{H}_p^2 z/\,\mathrm{H}_z^2 p)(\varrho)$, hence the given formula for $\mathrm{H}_p^{\mathcal{G}}(\varrho)$. □

Lemma 5.19. *If $\varrho \in \mathcal{G}_\partial$ then $\mathrm{H}_p^{\mathcal{G}}(\varrho) \in T_\varrho \mathcal{G}_\partial$.*

Note that the tangent space $T_\varrho \mathcal{G}_\partial$ makes sense since $\mathcal{G}_\partial$ is $\mathscr{C}^1$-manifold by Proposition 5.13.

For $\varrho \in \mathcal{G}_\partial$, observe that $\mathrm{H}_p(\varrho)$ and $\mathrm{H}_p^{\mathcal{G}}(\varrho)$ coincide if and only if $\varrho \in \mathcal{G}_\partial^3$, that is $\mathrm{H}_p^2 z = 0$.

Proof. Since $\varrho \in {}^{\|}\partial(T^*\mathcal{L})$ one has $\mathrm{H}_p^{\mathcal{G}}(\varrho) \in T_\varrho {}^{\|}\partial(T^*\mathcal{L})$ by the definition of $\mathrm{H}_p^{\mathcal{G}}$ in (5.18). With ${}^{\|}\partial(T^*\mathcal{L})$ given by $z = \mathrm{H}_p z = 0$ on has $\mathrm{H}_p^{\mathcal{G}} z(\varrho) = \mathrm{H}_p^{\mathcal{G}} \mathrm{H}_p z(\varrho) = 0$.

On ${}^{\|}\partial(T^*\mathcal{L})$, with (5.19) one computes

$$\mathrm{H}_p^{\mathcal{G}} p = \frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p} \mathrm{H}_z p = -\frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p} \mathrm{H}_p z = 0.$$

Hence, the result since $\mathcal{G}_\partial$ is given by $z = \mathrm{H}_p z = p = 0$. □

A fairly important remark is the following one.

Remark 5.20. Observe that $\mathrm{H}_p(\varrho) \neq 0$ if $\tau \neq 0$, where as above $\varrho = (t, x, \tau, \xi) \in T^*\mathcal{L}$. In fact $\mathrm{H}_p(\varrho)t = -2\tau$. If $\varrho \in \operatorname{Char} p$, one has

$$\tau \neq 0 \quad \Leftrightarrow \quad \xi \neq 0 \quad \Leftrightarrow \quad (\tau, \xi) \neq (0, 0).$$

Hence, if $\varrho \in \operatorname{Char} p$ is such that $(\tau, \xi) \neq (0, 0)$ then $\mathrm{H}_p(\varrho) \neq 0$. Considering the form of $\mathrm{H}_p^{\mathcal{G}}$ given above one also has $\mathrm{H}_p^{\mathcal{G}}(\varrho)t = -2\tau$. One thus finds $\mathrm{H}_p^{\mathcal{G}}(\varrho) \neq 0$ if $\varrho \in \operatorname{Char} p$ is such that $(\tau, \xi) \neq (0, 0)$.

In the same framework as in Section 5.3, in a local chart, we extend the definition of $\mathrm{H}_p^{\mathcal{G}}$ away from ${}^{\|}\partial(T^*\mathcal{L})$ by setting

$$\mathrm{H}_p^{\mathcal{G}} = \mathrm{H}_p + \Big(\frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p} - \frac{\mathrm{H}_p \mathrm{H}_z^2 p}{(\mathrm{H}_z^2 p)^2} \mathrm{H}_p z \Big) \mathrm{H}_z \,. \tag{5.20}$$

On ${}^{\|}\partial(T^*\mathcal{L})$ where $\mathrm{H}_p z = 0$ formula (5.20) coincides with (5.19). Observe that

$$\mathrm{H}_p^{\mathcal{G}} z = \mathrm{H}_p z. \tag{5.21}$$

The reason for formula (5.20) is as follows. Consider a bicharacteristic $\gamma(s) \in \operatorname{Char} p \cap T^*\mathcal{L}$ and set ${}^{\|}\gamma(s) = \pi_{\|}(\gamma(s)) \in {}^{\|}T^*\mathcal{L}$. If $\gamma(s) = (t(s), x(s), \tau, \xi(s))$ then ${}^{\|}\gamma(s) = (t(s), x(s), \tau, {}^{\|}\xi(s))$ and one has

$$\frac{d}{ds}{}^{\|}\gamma(s) = d\pi_{\|}(\gamma(s))\big(\frac{d}{ds}\gamma(s)\big) = d\pi_{\|}(\gamma(s))\big(\mathrm{H}_p(\gamma(s))\big) \in T_{{}^{\|}\gamma(s)}{}^{\|}T^*\mathcal{L}.$$

If $\gamma(s) \in \mathcal{G}_\partial \subset {}^{\|}\partial(T^*\mathcal{L})$ then ${}^{\|}\gamma(s) = \gamma(s)$ and $d\pi_{\|}(\gamma(s))\big(\mathrm{H}_p(\gamma(s))\big) = \mathrm{H}_p^{\mathcal{G}}\big(\gamma(s)\big) \in T_{\gamma(s)}{}^{\|}\partial(T^*\mathcal{L})$ by the definition of $\mathrm{H}_p^{\mathcal{G}}$ in ${}^{\|}\partial(T^*\mathcal{L})$ introduced in (5.18).

More generally, for $\gamma(s) \in {}^{\|}T^*\mathcal{L}$ one has $\gamma(s) = {}^{\|}\gamma(s)$ and thus one has $d\pi_{\|}(\gamma(s))\big(\mathrm{H}_p(\gamma(s))\big) \in T_{\gamma(s)}{}^{\|}T^*\mathcal{L}$. The proof of Lemma 5.18 applies, with Lemma 5.3 replaced by Lemma 5.16, and yields

$$d\pi_{\|}(\gamma(s))\big(\mathrm{H}_p(\gamma(s))\big)\big(\gamma(s)\big) = \Big(\mathrm{H}_p + \frac{\mathrm{H}_p^2 z}{\mathrm{H}_z^2 p}\,\mathrm{H}_z\Big)\big(\gamma(s)\big),$$

which coincides with (5.20) since $\mathrm{H}_p z = 0$ on ${}^{\|}T^*\mathcal{L}$.

Yet, if $\gamma(s) \in \mathcal{H}$, with $\mathcal{H}$ locally defined as in Section 5.3, then $\gamma(s) \neq {}^{\|}\gamma(s)$ implying that $d\pi_{\|}(\gamma(s))\big(\mathrm{H}_p(\gamma(s))\big) \notin T_{\gamma(s)}T^*\mathcal{L}$. Local coordinates can be of some help however. One has ${}^{\|}\xi(s) = \xi(s) - (\alpha\,\mathrm{H}_p\,z)\big(\gamma(s)\big)\mathsf{n}_x^*$ yielding

$${}^{\|}\gamma(s) = \gamma(s) - (\alpha\,\mathrm{H}_p\,z)\big(\gamma(s)\big)\mathsf{n}_x^*,$$

here identifying n_x^* with $(0,0,0,\mathsf{n}_x^*)$ in the (t,x,τ,ξ) variables. Recalling that the ζ-componant of $(\alpha\,\mathrm{H}_p\,z)\mathsf{n}_x^*$ is $\mathrm{H}_p\,z/\,\mathrm{H}_z^2\,p$ while other componants are zero one obtains

$$\begin{aligned}
\frac{d}{ds}{}^{\|}\gamma(s) &= \frac{d}{ds}\gamma(s) - \frac{d}{ds}\big((\mathrm{H}_p\,z/\,\mathrm{H}_z^2\,p)\big(\gamma(s)\big)\big)\partial_\zeta \\
&= \mathrm{H}_p\big(\gamma(s)\big) + \frac{\mathrm{H}_p\,\mathrm{H}_p\,z}{\mathrm{H}_z^2\,p}\big(\gamma(s)\big)\,\mathrm{H}_z - \frac{\mathrm{H}_p\,\mathrm{H}_z^2\,p}{(\mathrm{H}_z^2\,p)^2}\,\mathrm{H}_p\,z\big(\gamma(s)\big)\,\mathrm{H}_z \\
&= \mathrm{H}_p\big(\gamma(s)\big) + \Big(\frac{\mathrm{H}_p^2\,z}{\mathrm{H}_z^2\,p} - \frac{\mathrm{H}_p\,\mathrm{H}_z^2\,p}{(\mathrm{H}_z^2\,p)^2}\,\mathrm{H}_p\,z\Big)\big(\gamma(s)\big)\,\mathrm{H}_z\,.
\end{aligned}$$

In the local coordinates considered here it is thus not difficult to identify a tangent vector at $\gamma(s)$ with a tangent vector at ${}^{\|}\gamma(s)$. With this identification one may write

$$\frac{d}{ds}{}^{\|}\gamma(s) = \mathrm{H}_p^{\mathcal{G}}\big(\gamma(s)\big), \tag{5.22}$$

with the understanding that $\mathrm{H}_p^{\mathcal{G}}\big(\gamma(s)\big)$ is computed at $\gamma(s)$ according to (5.20) and viewed as a tangent vector at ${}^{\|}\gamma(s)$. We use this notation in what follows, with the necessary care since $\mathrm{H}_p^{\mathcal{G}}\big(\gamma(s)\big)$ may not be equal to $\mathrm{H}_p^{\mathcal{G}}\big({}^{\|}\gamma(s)\big)$. In fact the following result holds.

Lemma 5.21. *One has* $\mathrm{H}_p^{\mathcal{G}}(\varrho) = \mathrm{H}_p^{\mathcal{G}}\big({}^{\|}\varrho\big)$ *if and only if* ${}^{\|}\varrho = \varrho$, *that is,* $\varrho \in {}^{\|}T^*\mathcal{L}$.

Proof. Suppose $\mathrm{H}_p^{\mathcal{G}}(\varrho) = \mathrm{H}_p^{\mathcal{G}}\left({}^{\|}\varrho\right)$. With (5.21) one has $\mathrm{H}_p\, z(\varrho) = \mathrm{H}_p^{\mathcal{G}}\, z(\varrho) = \mathrm{H}_p^{\mathcal{G}}\, z\left({}^{\|}\varrho\right) = \mathrm{H}_p\, z\left({}^{\|}\varrho\right) = 0$. Hence the conclusion. □

If one is not inclined to perform this abuse of notation one should stick with

$$\frac{d}{ds}{}^{\|}\gamma(s) = d\pi_{\|}\big(\gamma(s)\big)\Big(\mathrm{H}_p\left(\gamma(s)\right)\Big). \tag{5.23}$$

Note that neither (5.22) nor (5.23) is to be viewed as a differential equation for ${}^{\|}\gamma(s)$, since the right-hand-side is a function of $\gamma(s)$ and not ${}^{\|}\gamma(s)$.

Remark 5.22. Observe that Remark 5.20 applies to the local extension of $\mathrm{H}_p^{\mathcal{G}}$ away from the boundary: if $\varrho \in \operatorname{Char} p \cap T^*\mathcal{L}$ is such that $(\tau, \xi) \neq (0, 0)$ then $\mathrm{H}_p^{\mathcal{G}}(\varrho) \neq 0$.

5.5. **The compressed cotangent bundle.** The symmetry Σ introduced in Section 5.2 acts as an involution on $\partial(T^*\mathcal{L})$ that leaves ${}^{\|}\partial(T^*\mathcal{L})$ invariant. For $(t, x) \in \partial\mathcal{L}$, one sets

$${}^{c}T^*_{(t,x)}\mathcal{L} = {}^{T^*_{(t,x)}\mathcal{L}}\big/_{\Sigma},$$

called the *compressed* cotangent space at (t, x), and

$${}^{c}T^*\mathcal{L} = \bigcup_{(t,x)\in\partial\mathcal{L}} \{(t,x)\} \times {}^{c}T^*_{(t,x)}\mathcal{L} \;\cup \bigcup_{(t,x)\in\mathcal{L}\setminus\partial\mathcal{L}} \{(t,x)\} \times T^*_{(t,x)}\mathcal{L},$$

called the *compressed* cotangent bundle. The quotient with respect to Σ is thus only performed above the boudary $\partial\mathcal{L}$, that is $\{z = 0\}$ in local coordinates. This allows one to identify a point $\varrho \in \mathcal{H}_\partial^\pm$ with $\Sigma(\varrho) \in \mathcal{H}_\partial^\mp$. This turns out to be usefull in the construction of generalized bicharacteristics in what follows, allowing such bicharacteristics to be continuous across hyperbolic points.

Denote by ${}^{c}\phi$ the associated quotient map $T^*\mathcal{L} \to {}^{c}T^*\mathcal{L}$. One has ${}^{c}\phi(\mathcal{H}_\partial^+) = {}^{c}\phi(\mathcal{H}_\partial^-)$. It acts as the identity on $T^*\mathcal{L}\backslash\partial(T^*\mathcal{L})$ and on ${}^{\|}\partial(T^*\mathcal{L})$. For $\varrho \in T^*\mathcal{L}\backslash\mathcal{H}_\partial$ one thus writes ${}^{c}\phi(\varrho) = \varrho$ by abuse of notation. Set

$${}^{c}\mathcal{H}_\partial = {}^{c}\phi(\mathcal{H}_\partial) = {}^{c}\phi(\mathcal{H}_\partial^+) = {}^{c}\phi(\mathcal{H}_\partial^-).$$

One can endow ${}^{c}T^*\mathcal{L}$ with a natural metric inherited from that of $T^*\mathcal{L}$. In fact, given two points ${}^{c}\varrho^0, {}^{c}\varrho^1 \in {}^{c}T^*\mathcal{L}$ consider a path $\gamma(s) = [0, 1] \setminus B \to T^*\mathcal{L}$, with $B = \{s_1, \dots, s_k\} \subset [0, 1]$, with $s_1 < s_2 < \cdots < s_k$, such that the limits

$$\gamma(s_n^-) = \lim_{s\to s_n^-} \gamma(s) \quad \text{and} \quad \gamma(s_n^+) = \lim_{s\to s_n^+} \gamma(s), \qquad n = 1, \dots, k,$$

exist and $\gamma(s_n^-), \gamma(s_n^+) \in \partial(T^*\mathcal{L}) \setminus {}^{\|}\partial(T^*\mathcal{L})$ with moreover

$$\Sigma(\gamma(s_n^-)) = \gamma(s_n^+),$$

and

$${}^c\phi(\gamma(0)) = {}^c\varrho^0, \quad {}^c\phi(\gamma(1)) = {}^c\varrho^1.$$

One sets

$$\text{length}(\gamma) = \ell_{[0,s_1]}\gamma + \sum_{n=1}^{k-1} \ell_{[s_n,s_{n+1}]}\gamma + \ell_{[s_k,1]}\gamma,$$

where $\ell_{[a,b]}\gamma$ stands for the length of γ for $s \in [a,b]$. One then defines

$$ {}^c\text{dist}({}^c\varrho^0, {}^c\varrho^1) = \inf \text{length}(\gamma), \tag{5.24}$$

where the infimum is computed over all paths fulfilling the above conditions.

The quotient map is continuous: for some $C > 0$ one has

$$ {}^c\text{dist}\left({}^c\phi(\varrho^0), {}^c\phi(\varrho^1)\right) \leq C\|\varrho^0 - \varrho^1\|, \quad \varrho^0, \varrho^1 \in T^*\mathcal{L}. \tag{5.25}$$

With the metric structure now given on ${}^cT^*\mathcal{L}$ note that the path ${}^c\gamma : [0,1] \to {}^cT^*\mathcal{L}$ given by ${}^c\gamma(s) = {}^c\phi(\gamma(s))$ is continuous if γ is as described previously.

The following lemma follows from what is above.

Lemma 5.23. *Suppose J is an interval, ${}^c\gamma : J \to {}^cT^*\mathcal{L}$ is continuous, and $B \subset J$ is a discrete set, that is, made of isolated points, such that ${}^c\gamma(s) \in {}^c\mathcal{H}_\partial$ if and only if $s \in B$. Then, there exists a unique map $\gamma : J \setminus B \to T^*\mathcal{L}$ such that ${}^c\phi(\gamma(s)) = {}^c\gamma(s)$ if $s \in J \setminus B$ and γ is continuous away from points in B. Moreover, for $S \in B$ the limits*

$$\gamma(S^-) = \lim_{s \to S^-} \gamma(s) \in \mathcal{H}_\partial^- \quad \text{and} \quad \gamma(S^+) = \lim_{s \to S^+} \gamma(s) \in \mathcal{H}_\partial^+$$

exist and $\Sigma(\gamma(S^-)) = \gamma(S^+)$.

In addition, if $\tilde{J} \subset J$ is an interval such that $\gamma(s)$ lies in one local chart for $s \in \tilde{J}$, then $s \mapsto {}^{\|}\gamma(s) = \pi_{\|}(\gamma)(s)$ can be defined for $s \in \tilde{J} \setminus B$ (see Section 5.3) and moreover extended to the whole interval $\tilde{J}$ in a continuous manner.

5.6. **Broken and generalized bicharacteristics.** Consider $\gamma : J \to T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ a maximal bicharacteristic, with $J =]S_0, S_1[$ (see Definition 2.3). Set $\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$. Recall that $\tau(s) = |\xi(s)| = \text{Cst}$ and consider the only interesting case of a nonzero value for $\tau(s)$. Concerning the potential limit at $s = S_0^+$ and $s = S_1^-$ one has the following result.

Lemma 5.24. *Assume $S_0^+ < +\infty$ (resp. $S_0^- > -\infty$). The limit of $\gamma(s)$ as $s \to S_0^+$ (resp. $s \to S_1^-$) exists and*

$$\lim_{s\to S_0^+} \gamma(s) \in \mathcal{H}_\partial^+ \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{d}} \qquad (\text{resp. } \lim_{s\to S_1^-} \gamma(s) \in \mathcal{H}_\partial^- \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{d}}).$$

In other words, the limit lies in $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$ yet away from $\mathcal{G}_\partial^{\mathrm{g}}$.

Proof. Along a bicharacteristic the value of τ is constant since $\mathrm{H}_p \tau = 0$. As the bicharacteristic lies in $\operatorname{Char} p$ where $|\xi|_x^2 = \tau^2$, one finds that ξ remains bounded. Thus $\gamma(s)$ lies in a compact set. Since H_p is continuous and thus bounded there, one finds that $\gamma(s)$ has a single accumulation point as $s \to S_1^-$. Hence, $\varrho^1 = \lim_{s\to S_1^-} \gamma(s)$ exists and belongs to $\partial(T^*\mathcal{L}) \cap \operatorname{Char} p$. One sets $\gamma(S_1) = \varrho^1$. Assume that $\varrho^1 \in \mathcal{G}_\partial^{\mathrm{g}}$. In local coordinates as in (2.1) one has $\gamma(s) = \big(t(s), x'(s), z(s), \tau(s), \xi'(s), \zeta(s)\big)$ for $s \in [S_1 - \varepsilon, S_1]$ for some $\varepsilon > 0$. Naturally, one has $z(s) \geq 0$. Moreover, $z(S_1) = \mathrm{H}_p z(\varrho^1) = 0$ and $\mathrm{H}_p^2 z(\varrho^1) < 0$. Since $\mathrm{H}_p z(\gamma(s)) = \frac{d}{ds} z(s)$ and $\mathrm{H}_p^2 z\big(\gamma(s)\big) = \frac{d^2}{ds^2} z(s)$ along a bicharacteristic one concludes that $z(s) < 0$ for $s \in [S_1 - \varepsilon', S_1]$ for some $\varepsilon' > 0$, a contradiction. □

The following property is elementary yet important in the generalization of the notion of bicharacteristics. It states the existence of some interval of uniform minimal size for all bicharacteristics originating from a neighborhood of a hyperbolic point.

Lemma 5.25. *Consider $\varrho^0 \in \mathcal{H}_\partial^-$ (resp. $\mathcal{H}_\partial^+$). There exists $S_0 > 0$ and an open neighborhood V^0 of ϱ^0 in $\mathcal{H}_\partial^-$ (resp. $\mathcal{H}_\partial^+$) such that for any maximal bicharacteristic $\gamma(s)$ in $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ defined on $]s_1, s_2[$, with $s_1 < s_2$ and $\lim_{s\to s_2^-} \gamma(s) \in V^0$ (resp. $\lim_{s\to s_1^+} \gamma(s) \in V^0$), one has $s_2 - s_1 \geq S_0$.*

Proof. In the proof we only treat the case $\varrho^0 \in \mathcal{H}_\partial^-$. The case $\varrho^0 \in \mathcal{H}_\partial^+$ can be treated similarly. Near ϱ^0 use local coordinates as in (2.1).

One has $\mathrm{H}_p z(\varrho^0) < 0$. Thus, there exists a $2(d+1)$-dimensional ball $\mathcal{B}_2$ of radius $2R$ centered at ϱ^0 such that $\mathrm{H}_p z < -C_0$ and $\| \mathrm{H}_p \| \leq C_1$ in $V = \mathcal{B}_2 \cap \{z \geq 0\}$ for some $C_0 > 0$ and $C_1 > 0$. Set $V^0 = \mathcal{B}_1 \cap \mathcal{H}_\partial \subset \mathcal{H}_\partial^-$ with $\mathcal{B}_1$ the ball of radius R centered at ϱ^0.

Suppose γ is a maximal bicharacteristic defined on $]s_1, s_2[$ as in the statement, with $\varrho^1 = \lim_{s\to s_2^-} \gamma(s) \in V^0$. Set $\gamma(s_2) = \varrho^1$ by continuous extension.

By continuity $\gamma(s) \in V$ for $s \in [s_2 - \delta, s_2]$ for some $\delta > 0$. If $\gamma(s) \in V$ one has

$$\frac{d}{ds} z(s) = \mathrm{H}_p\, z(\gamma(s)) \leq -C_0.$$

Thus, $z(s) > 0$ for $s < s_2$ and $\gamma([s, s_2]) \subset V$ implying, first, that the *maximal* bicharacteristic cannot cease to exist if $\gamma(s)$ remains in $V \setminus \partial\mathcal{L}$, and second, that $\gamma(s)$ can only exit V through $\partial\mathcal{B}_2 \cap \{z > 0\}$. Set $S^- = \inf\{s_1 \leq S < s_2;\ \gamma([S, s_2]) \subset V\}$. One has $s_2 - s_1 \geq s_2 - S^- \geq R/C_1$. □

A first generalization of bicharacteristics is that of broken bicharacteristics. It is composed of a sequence (finite or not) of bicharacteristics that are connected at hyperbolic points at the boundary and follow the optics law of reflection[6] at such points.

Definition 5.26 (broken bicharacteristic)**.** Suppose $J \subset \mathbb{R}$ is an interval, B a discrete subset of J, and

$$^{\mathsf{B}}\gamma : J \setminus B \to \mathrm{Char}\, p \cap T^*\mathcal{L}.$$

One says that $^{\mathsf{B}}\gamma$ is a broken bicharacteristic if the following properties hold:

(1) for $s \in J \setminus B$, $^{\mathsf{B}}\gamma(s) \notin \mathcal{H}_\partial \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ and the map $^{\mathsf{B}}\gamma$ is differentiable at s with

$$\frac{d}{ds}\, ^{\mathsf{B}}\gamma(s) = \mathrm{H}_p\big(^{\mathsf{B}}\gamma(s)\big).$$

(2) If $S \in B$, then $^{\mathsf{B}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \in J \setminus B$ sufficiently close to S and moreover
   (a) if $[S - \varepsilon, S] \subset J$ for some $\varepsilon > 0$, then $^{\mathsf{B}}\gamma(S^-) = \lim_{s\to S^-} {}^{\mathsf{B}}\gamma(s) \in \mathcal{H}_\partial^-$;
   (b) if $[S, S + \varepsilon] \subset J$ for some $\varepsilon > 0$, then $^{\mathsf{B}}\gamma(S^+) = \lim_{s\to S^+} {}^{\mathsf{B}}\gamma(s) \in \mathcal{H}_\partial^+$;
   (c) and if $[S - \varepsilon, S + \varepsilon] \subset J$ for some $\varepsilon > 0$, then $^{\mathsf{B}}\gamma(S^+) = \Sigma\big(^{\mathsf{B}}\gamma(S^-)\big)$.

Figure 6 sketches what a broken bicharacteristic may look like.

Remark 5.27.

[6]The optics law of reflection, often called Descartes' law or Snell's law, goes back in fact to the works of Euclid (ca. 300 BC) and Hero of Alexandria (ca. 60 AD).

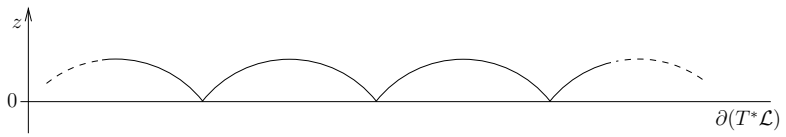


FIGURE 6. A broken bicharacteristic in a neighborhood of part of the boundary.

(1) Near a point $S \in J \setminus B$, $s \mapsto {}^{\mathsf{B}}\gamma(s)$ is $\mathscr{C}^1$. At a point $S \in B$, $s \mapsto {}^{\mathsf{B}}\gamma(s)$ is discontinuous. However, if one sets ${}^{\|}\varrho(s) = \pi_{\|}\big({}^{\mathsf{B}}\gamma(s)\big)$ in local coordinates (see Section 5.3) one sees that ${}^{\|}\varrho(s)$ is $\mathscr{C}^0$. In particular the z-component is continuous and vanishes at $s = S^{\pm}$ for $S \in B$.

If one considers the map ${}^{c}\phi$ introduced in Section 5.5 one sees that $s \mapsto {}^{c}\phi({}^{\mathsf{B}}\gamma(s))$ takes values in the compressed cotangent bundle and can be extended to the whole interval J as a continuous function. At this stage, the map $s \mapsto {}^{c}\phi({}^{\mathsf{B}}\gamma(s))$ is however not needed for the understanding of the behavior of broken bicharacteristics.

(2) Similarly to what is observed in Remark 2.4-(1) one has, for $s \in J \setminus B$,

$$|\xi(s)|_{x(s)} = |\tau(s)|$$

constant along a broken bicharacteristic.

(3) Observe that one allows a broken bicharacteristic to reach points in $\mathcal{G}^{\mathrm{d}}_{\partial}$; there, the tangent vector $\frac{d}{ds}{}^{\mathsf{B}}\gamma(s)$ is also given by $\mathrm{H}_p\big({}^{\mathsf{B}}\gamma(s)\big)$.

(4) Points of B are naturally isolated because of Lemma 5.25. In fact, points of B can only accumulate at the boundary of J as stated in the following lemma.

Lemma 5.28. *Suppose ${}^{\mathsf{B}}\gamma : J \setminus B \to T^*\mathcal{L}$ is a broken bicharacteristic. One has*

(1) $\overline{B} \setminus B \subset \partial J \setminus J$;

(2) *if $S \in \overline{B} \setminus B$ then the limit*

$$\lim_{\substack{s \to S \\ s \in J \setminus B}} {}^{\mathsf{B}}\gamma(s) \quad \textit{exists}$$

and belongs to $\mathcal{G}^3_{\partial} \cup \mathcal{G}^{\mathrm{g}}_{\partial}$.

(3) *If $S \in \partial J \setminus \overline{B}$ then the limit*

$$\lim_{\substack{s \to S \\ s \in J \setminus B}} {}^{\mathsf{B}}\gamma(s) \quad \textit{exists}$$

and does not belong to $\mathcal{G}^{\mathrm{g}}_{\partial}$.

Proof. Consider $S \in \overline{B} \setminus B$. Since dx/ds is bounded, $x(s)$ has a limit as $s \to S$. One may thus use a single local chart for s sufficiently close to S. This allows one to use the local extension of $\mathrm{H}_p^{\mathcal{G}}$ away from $\partial(T^*\mathcal{L})$ as well as the extention of $\pi_{\|}$ and ${}^{\|}T^*\mathcal{L}$ given in Section 5.3. As in the beginning of the proof Lemma 5.24, one sees that ${}^{\mathsf{B}}\gamma(s)$ lies in a compact set for s near S. Thus $s \mapsto {}^{\mathsf{B}}\gamma(s)$ has at least one accumulation point as $J \setminus B \ni s \to S$. We prove that this accumulation point is unique.

Parameterize ${}^{\mathsf{B}}\gamma(s)$ by ${}^{\|}\varrho(s) = \pi_{\|}({}^{\mathsf{B}}\gamma(s)) = (t(s), x'(s), z(s), \tau(s), {}^{\|}\xi(s))$ and $\mathrm{H}_p\, z({}^{\mathsf{B}}\gamma(s))$; see (5.9). By Remark 5.27-(1) ${}^{\|}\varrho(s)$ is a continuous function for $s \in J$ and with (5.22)–(5.23) one has

$$\frac{d}{ds}{}^{\|}\varrho(s) = \mathrm{H}_p^{\mathcal{G}}({}^{\mathsf{B}}\gamma(s)), \quad \text{for } s \in J \setminus B.$$

As $\mathrm{H}_p^{\mathcal{G}}({}^{\mathsf{B}}\gamma(s))$ is bounded as $J \setminus B \ni s \to S$, then $s \mapsto {}^{\|}\varrho(s)$ has a single accumulation point as $J \setminus B \ni s \to S$, denoted by ${}^{\|}\varrho^0 = (t^0, x'^0, z^0, \tau^0, {}^{\|}\xi^0)$. By the continuity of ${}^{\|}\varrho(s)$ the same holds if one allows $J \ni s \to S$.

Consider $(s_n)_n \subset B$ such that $s_n \to S$. Since $\big(\varrho(s_n^{\pm})\big)_n \subset \mathcal{H}_\partial$, then $\big({}^{\|}\varrho(s_n^{\pm})\big)_n \subset {}^{\|}\mathcal{H}_\partial$, and one finds ${}^{\|}\varrho^0 \in \overline{{}^{\|}\mathcal{H}_\partial} = {}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial$. In particular $z^0 = 0$.

As ${}^{\mathsf{B}}\gamma(s) \in \operatorname{Char} p$, with Lemma 5.7 and (5.11) (and its extension (5.14) in Section 5.3), the function $\mathrm{H}_p({}^{\mathsf{B}}\gamma(s))$ is such that

$$\begin{cases} \alpha(x(s))\, \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big) = \lambda(s) \text{ or } -\lambda(s) & \text{if } \ s \in J \setminus B, \\ \alpha(x(s))\, \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s^{\pm})\big) = \pm\lambda(s) & \text{if } \ s \in B. \end{cases}$$

with $\lambda(s) = \big(-p({}^{\|}\varrho(s))\big)^{1/2}$. If $J \setminus B \ni s \to S$, since ${}^{\|}\varrho(s) \to {}^{\|}\varrho^0$, one finds that $\lambda(s)$ has a limit, and thus $\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)$ has *at most* two accumulation points. The resulting potential accumulation points for ${}^{\mathsf{B}}\gamma(s)$, as $J \setminus B \ni s \to S$, lie in $\pi_{\|}^{-1}(\{{}^{\|}\varrho^0\}) \cap \operatorname{Char} p$ and

- either there are two distinct accumulation points in $\mathcal{H}_\partial$, image of one another by the symmetry map Σ,
- or there is a single accumulation point in $\mathcal{G}_\partial$.

(See Lemma 5.7 and Proposition 5.9.) Our claim is that ${}^{\|}\varrho^0 \in \mathcal{G}_\partial$ and thus ${}^{\mathsf{B}}\gamma(s)$ has a single accumulation point $\varrho^0 = {}^{\|}\varrho^0$, meaning that ${}^{\mathsf{B}}\gamma(s)$ has a limit as $J \setminus B \ni s \to S$. In particular $\mathrm{H}_p\, z(\varrho^0) = 0$.

Proceed by contradiction. Suppose there are two distinct accumulation points $\varrho^{1,-} \in \mathcal{H}_\partial^-$ and $\varrho^{1,+} \in \mathcal{H}_\partial^+$ with $\Sigma(\varrho^{1,-}) = \varrho^{1,+}$. Consider the sequence $(s_n)_n$ introduced above. The sequence ${}^{\mathsf{B}}\gamma(s_n^-)$ has an accumulation point that

lies in $\overline{\mathcal{H}_\partial^-}$; since it is then accumulation point of ${}^{\mathsf{B}}\gamma(s)$ this point is necessarily $\varrho^{1,-}$. Hence, ${}^{\mathsf{B}}\gamma(s_n^-)$ has a unique accumulation point, meaning it converges to $\varrho^{1,-}$. By Lemma 5.25 there exists a neighborhood V^1 of $\varrho^{1,-}$ in $\mathcal{H}_\partial^-$, such that any maximal bicharacteristic in $T^*\mathcal{L}$ initiated at a point of V^1 exists for s in an interval of minimal length $\ell^0 > 0$. This is in contradiction with having ${}^{\mathsf{B}}\gamma(s_n^-) \to \varrho^1$ and $|s_{n+1} - s_n| \to 0$. This proves our claim.

Assume now that $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}}$. Then, in addition to having $\mathrm{H}_p\, z(\varrho^0) = 0$, one has $\mathrm{H}_p^2\, z(\varrho^0) > 0$. One considers an open neighborhood W of ϱ^0 where $\mathrm{H}_p^2\, z > 0$. There exists $\delta > 0$ such that ${}^{\mathsf{B}}\gamma(s) \in W$ for $s \in J \cap]S-\delta, S+\delta[\setminus B$ and ${}^{\mathsf{B}}\gamma(s^\pm) \in W$ for $s \in B \cap]S-\delta, S+\delta[$. Consider $r \in B \cap]S-\delta, S[$. The following argument is similar if one starts from $r \in B \cap]S, S+\delta[$. Set $R = \inf B \cap]r, S]$. Since points in B are isolated (see Remark 5.27-(4)), one has $r < R$. From the continuity of $z(s)$ one has $z(R) = 0$. Yet, let us consider the broken bicharacteristic ${}^{\mathsf{B}}\gamma(s)$ for $s \in]r, R[\subset J$. One has $z(r^+) = 0$ and $z'(r^+) = \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(r^+)\big) > 0$ since ${}^{\mathsf{B}}\gamma(r^+) \in \mathcal{H}_\partial^+$. Since $]r, R[\subset J$ and ${}^{\mathsf{B}}\gamma(s) \in W$ for $s \in]r, R[$ one has $z''(s) = \mathrm{H}_p^2\, z\big({}^{\mathsf{B}}\gamma(s)\big) > 0$ for $s \in]r, R[$. Thus $z(s)$ increases on $]r, R[$ and $z(R) > 0$. A contradiction with $z(R) = 0$. Thus ϱ^0 cannot be in $\mathcal{G}_\partial^{\mathrm{d}}$.

In conclusion, for $S \in \overline{B} \setminus B$, we have found

$$\varrho^0 = \lim_{J \setminus B \ni s \to S} {}^{\mathsf{B}}\gamma(s) \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}.$$

With the definition of a broken bicharacteristic that takes values in $T^*\mathcal{L} \setminus (\mathcal{H}_\partial \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$ one finds that $S \notin J$. Thus $S \in \partial J \setminus J$.

Finally, consider the case of a point $S \in \partial J \setminus \overline{B}$. Thus there exists $\varepsilon > 0$ such that $B \cap]S-\varepsilon, S+\varepsilon[= \emptyset$. With the same argument as in the begining of Lemma 5.24 one finds that

$$\varrho^0 = \lim_{\substack{s \to S \\ s \in J \setminus B}} {}^{\mathsf{B}}\gamma(s) \quad \text{exists.}$$

If $\varrho^0 \in \partial(T^*\mathcal{L})$ then arguing as in the proof of Lemma 5.24 one finds that $\varrho^0 \notin \mathcal{G}_\partial^{\mathrm{g}}$. □

Figure 7 illustrates the behavior of a broken bicharacteristic ${}^{\mathsf{B}}\gamma(s)$ as $s \to S$ with S an accumulation of B. For a broken bicharacteristic defined on $J \setminus B$ with $J =]S_0, S[$ if $\lim_{s \to S^-} {}^{\mathsf{B}}\gamma \in \partial(T^*\mathcal{L})$ and $S \notin \overline{B}$ this means that the broken bicharacteristic is in fact a simple bicharacteristic in a neighborhood of S^- and Lemma 5.24 applies (this is also covered by the third item of Lemma 5.28). If $S \in \overline{B}$ then by Lemma 5.28 one has $\lim_{s \to S^-} {}^{\mathsf{B}}\gamma(s) \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$. The following

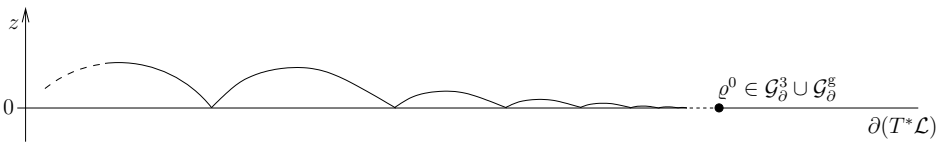


FIGURE 7. A broken bicharacteristic in the case B has an accumulation point.

lemma yields a finer understanding of the behavior of the broken bicharacteristic as it reaches such a limit point.

Lemma 5.29. *Consider a local chart at the boundary (O, ϕ) as in (2.1) and W a bounded open subset of $T^*\mathcal{L}$ that lies above O. Suppose $\varrho^0 \in \big(\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^g\big) \cap W$. Let also $S > 0$, $J =]0, S[$, B a discrete subset of J, and ${}^{\mathsf{B}}\gamma : J \setminus B \to T^*\mathcal{L}$ be a broken bicharacteristic. Assume that $lim_{s \to S^-} {}^{\mathsf{B}}\gamma(s)$ exists in $T^*\mathcal{L}$ and that this limit is ϱ^0. If one sets ${}^{\mathsf{B}}\gamma(S) = \varrho^0$, then, ${}^{\mathsf{B}}\gamma$ is differentiable at $s = S^-$ and*

$$\frac{d}{ds}{}^{\mathsf{B}}\gamma(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0).$$

In fact, there exists some $C > 0$ uniform with respect to ϱ^0, such that

$$0 \le z\big({}^{\mathsf{B}}\gamma(s)\big) \le \int_s^S \big|\,\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(\sigma)\big)\big|\, d\sigma, \tag{5.26}$$

and

$$\big|\,\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)\big| \le C \int_s^S \big|\big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(\sigma)\big) - \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(S)\big)\big|\, d\sigma, \tag{5.27}$$

for $s \in J \setminus B$ such that ${}^{\mathsf{B}}\gamma(s) \in W$.

Proof. Consider $S_0 \in]0, S[$ such that ${}^{\mathsf{B}}\gamma(s) \in W$ for $s \in [S_0, S] \setminus B$. Set ${}^{\|}\varrho(s) = (t(s), x'(s), z(s), \tau(s), {}^{\|}\xi(s)) = \pi_{\|}\big({}^{\mathsf{B}}\gamma(s)\big)$ and ${}^{\|}\varrho^0 = \pi_{\|}\big(\varrho^0\big)$ (with the local extension introduced in Section 5.3). Along the broken bicharacteristic, $s \mapsto {}^{\|}\varrho(s)$ is continuous and piecewise $\mathscr{C}^1$. One has *piecewisely* on $[S_0, S[$

$$\frac{d}{ds}{}^{\|}\varrho(s) = \mathrm{H}_p^{\mathcal{G}}\big({}^{\mathsf{B}}\gamma(s)\big),$$

by (5.22)–(5.23) since a broken bicharacteristic is a regular bicharacteristic away from $s \in B$. For $s, s' \in [S_0, S[$, one thus has ${}^{\|}\varrho(s') - {}^{\|}\varrho(s) = \int_s^{s'} \mathrm{H}_p^{\mathcal{G}}\big({}^{\mathsf{B}}\gamma(\sigma)\big) d\sigma$.

One has ${}^{\|}\varrho(s') \to {}^{\|}\varrho^0 = {}^{\|}\varrho(S)$ as $s' \to S^-$. Dominated convergence yields

$$ {}^{\|}\varrho(S) - {}^{\|}\varrho(s) = \int_s^S \mathrm{H}_p^{\mathcal{G}}\big({}^{\mathsf{B}}\gamma(\sigma)\big)d\sigma. \tag{5.28}$$

As ${}^{\mathsf{B}}\gamma(s) \to \varrho^0$ one has $\mathrm{H}_p^{\mathcal{G}}\big({}^{\mathsf{B}}\gamma(s)\big) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0) + o(1)$ as $s \to S^-$, yielding

$$ {}^{\|}\varrho(S) - {}^{\|}\varrho(s) = (S-s)\,\mathrm{H}_p^{\mathcal{G}}(\varrho^0) + o(S-s). $$

This implies that ${}^{\|}\varrho(s)$ is differentiable at $s = S^-$ and $\frac{d}{ds}{}^{\|}\varrho(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0)$.

Since the ∂_z component of $\mathrm{H}_p^{\mathcal{G}}$ is $(\mathrm{H}_p\, z)\partial_z$ and $z(S) = 0$, with (5.28) one has

$$ -z(s) = \int_s^S \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(\sigma)\big)d\sigma, $$

yielding (5.26).

If one proves

$$ \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big) = o(1)(s-S), \quad \text{for } s \to S^- \text{ in } J \setminus B, \tag{5.29}$$

then one concludes that ${}^{\mathsf{B}}\gamma$ is differentiable at $s = S^-$ by Lemma 5.15 with $\frac{d}{ds}{}^{\mathsf{B}}\gamma(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0)$.

We now prove (5.29) by proving the estimate for $\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)$ in (5.27). Consider the function

$$ g(s) = \frac{1}{2}\big(\mathrm{H}_p\, z\big)^2\big({}^{\mathsf{B}}\gamma(s)\big) - \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(S)\big)z(s). \tag{5.30}$$

Since $\big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(S)\big) \le 0$ and $z(s) \ge 0$ one has $g(s) \ge 0$. Despite having $\big(\mathrm{H}_p\, z\big)\big({}^{\mathsf{B}}\gamma(s)\big)$ discontinuous across any point of B, observe that $g(s)$ can be extended to $[S_0, S]$ as a continuous function. One has $g(S) = 0$. Moreover, between two points of B one has

$$ \frac{d}{ds}g(s) = \big(\mathrm{H}_p\, z\big)\big({}^{\mathsf{B}}\gamma(s)\big)\varepsilon(s,S), \quad \text{with } \varepsilon(s,S) = \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(s)\big) - \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{B}}\gamma(S)\big). $$

One finds

$$ \Big|\frac{d}{ds}g(s)\Big| \le \big|\big(\mathrm{H}_p\, z\big)\big({}^{\mathsf{B}}\gamma(s)\big)\big|\,|\varepsilon(s,S)| \lesssim g^{1/2}(s)|\varepsilon(s,S)|, \qquad s \in [S_0, S[\setminus B. $$

Classically, with $a > 0$, one replaces g by $g^a = g + a > 0$ for which one has $|\frac{d}{ds} g^a(s)| \lesssim g^a(s)^{1/2} |\varepsilon(s, S)|$ leading to

$$\Big|\frac{d}{ds}\big(g^a(s)^{1/2}\big)\Big| \lesssim |\varepsilon(s, S)|, \qquad s \in [S_0, S[\backslash B.$$

Consider $S_0 \leq s \leq s' < S$. Note that $B \cap [s, s']$ is finite since B does not have any accumulation point in $]0, S[$ by Lemma 5.28. Note also that $\varepsilon(s, S)$ is bounded on $[S_0, S]$. As a result $(g^a)^{1/2}$ is Lipschitz and thus absolutely continuous on $[s, s']$ and one finds

$$\big|g^a(s)^{1/2} - g^a(s')^{1/2}\big| \leq \int_s^{s'} \Big|\frac{d}{ds}\big(g^a(s)^{1/2}\big)\Big|\, d\sigma \lesssim \int_s^{s'} |\varepsilon(\sigma, S)|\, d\sigma.$$

Letting $a \to 0^+$ one obtains

$$\big|g(s)^{1/2} - g(s')^{1/2}\big| \lesssim \int_s^{s'} |\varepsilon(\sigma, S)|\, d\sigma.$$

Since $s \mapsto \varepsilon(s, S)$ is integrable on $[S_0, S]$ and $g(s') \to 0$ as $s' \to S^-$, one finally obtains

$$0 \leq |\,\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)\big| \lesssim g^{1/2}(s) \lesssim \int_s^S |\varepsilon(\sigma, S)|\, d\sigma.$$

This is estimate (5.27). Observe that $\varepsilon(s, S) = o(1)$ as $s \to S^-$, because of the continuity of the function $\mathrm{H}_p^2 z$. Hence, the estimate for $\mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)$ in (5.29) follows. □

Remark 5.30. If $\mathcal{M}$ is a $\mathscr{C}^3$-manifold and g a $\mathscr{C}^2$-metric then $\mathrm{H}_p\, z$ is $\mathscr{C}^2$ and one can prove that a broken bicharacteristic ${}^{\mathsf{B}}\gamma(s)$ as above cannot reach a point in $\mathcal{G}_\partial^{\mathrm{g}}$ as a limit as $s \to S^-$ even if $S \in \overline{B}$. The proof goes by contradiction as follows using the setting of the proof of Lemma 5.29. Consider $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{g}}$ and a broken bicharacteristic that converges to ϱ^0 as $s \to S^-$. Consider V a bounded neighborhood of ϱ^0 where $\mathrm{H}_p^2 z \leq -C_0 < 0$ and where ${}^{\mathsf{B}}\gamma(s)$ lies for $s \in [S - \varepsilon, S[$ for some $\varepsilon > 0$. With $p \in \mathscr{C}^2$ one finds that $\mathrm{H}_p^2 z \in \mathscr{C}^1$. Set $g(s) = \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big)^2/2 - z(s)\, \mathrm{H}_p^2\, z\big({}^{\mathsf{B}}\gamma(s)\big)$. One has g continuous and nonnegative, and moreover $g(s) = 0$ if and only if $z(s) = \mathrm{H}_p\, z\big({}^{\mathsf{B}}\gamma(s)\big) = 0$. Since moreover $\mathrm{H}_p^2\, z\big({}^{\mathsf{B}}\gamma(s)\big) < 0$ one finds that $g(s) = 0$ if and only if ${}^{\mathsf{B}}\gamma(s) \in \mathcal{G}_\partial^{\mathrm{g}}$. Away from the points of B one finds $\frac{d}{ds} g(s) = -z(s)\, \mathrm{H}_p^3\, z\big({}^{\mathsf{B}}\gamma(s)\big)$. Since $\mathrm{H}_p^3\, z$ is continuous and thus bounded in V one obtains

$$\Big|\frac{d}{ds} g(s)\Big| \lesssim z(s) \lesssim -z(s)\, \mathrm{H}_p^2\, z\big({}^{\mathsf{B}}\gamma(s)\big) \lesssim g(s).$$

Since g is absolutely continuous on $[s, s']$ for $s < s' < S$ it follows that

$$|g(s') - g(s)| \leq \int_s^{s'} \left|\frac{d}{d\sigma} g(\sigma)\right| d\sigma \lesssim \int_s^{s'} g(\sigma) d\sigma.$$

Letting $s' \to S^-$ one has $g(s') \to 0$ and thus $0 \leq g(s) \lesssim \int_s^S g(\sigma) d\sigma$. The Grönwall inequality yields $g \equiv 0$, meaning that ${}^{\mathsf{B}}\gamma(s) \in \mathcal{G}_\partial^{\mathrm{g}}$ for $s \in [S-\varepsilon, S]$; a contradiction since we have considered a broken bicharacteristic on $[S-\varepsilon, S[$.

Here, we have not been able to exclude that a broken bicharacteristic reaches $\mathcal{G}_\partial^{\mathrm{g}}$ in a limiting process. The proof that we have just recalled does not apply due to the lack of smoothness of $\mathrm{H}_p^2 z$. This question remains open. Observe however that in Lemma 5.29 we obtain that $\frac{d}{ds}{}^{\mathsf{B}}\gamma(S) = \mathrm{H}_p^{\mathcal{G}}\left({}^{\mathsf{B}}\gamma(S)\right)$ in both cases $\varrho^0 \in \mathcal{G}_\partial^3$ and $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{g}}$. This suffices for the analysis we carry out in what follows.

Note also that no thorough study of nonuniqueness issues at boundary has been carried out for $\mathscr{C}^k$ coefficients, $k \geq 2$, up to our knowledge.

The following proposition is a consequence of Lemma 5.29 and generalizes part of it.

Proposition 5.31. *Consider $\varrho^0 \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$. Let also $S > 0$, $J = [0, S[$, B a discrete subset of J, and $\gamma : J \setminus B \to T^*\mathcal{L} \cap \operatorname{Char} p$ be a map such that*

$$\lim_{\substack{s \to S^- \\ s \notin B}} \gamma(s) = \varrho^0.$$

One sets $\gamma(S) = \varrho^0$. Assume moreover that

1. *if $s_0 \in B$ then $\gamma(s)$ is a broken bicharacteristic for $s \in [s_0-\delta, s_0[\cup]s_0, s_0+\delta]$ for some $\delta > 0$;*
2. *if $s_0 \in [0, S[\setminus B$ and $\gamma(s_0) \notin \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ then $\gamma(s)$ is a broken bicharacteristic for $s \in [s_0 - \delta, s_0 + \delta] \setminus B$ for some $\delta > 0$;*
3. *${}^{\|}\gamma(s) = \pi_{\|}\big(\gamma(s)\big)$ is differentiable at $s = S^-$ and*

$$\frac{d}{ds}{}^{\|}\gamma(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0). \tag{5.31}$$

Then, $\gamma(s)$ is differentiable at $s = S^-$ and

$$\frac{d}{ds}\gamma(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0). \tag{5.32}$$

Proof. For s near S^-, use a single chart as in the proof of Lemma 5.29. Write $\gamma(s) = \big(t(s), x'(s), z(s), \tau(s), \xi(s)\big)$. One has ${}^{\|}\gamma(s) = \big(t(s), x'(s), z(s), \tau(s), {}^{\|}\xi(s)\big)$.

With (5.31) and Lemma 5.15, if one proves that $s \mapsto \mathrm{H}_p\, z(s)$ is differentiable at $s = S^-$ and $\frac{d}{ds}\big(\mathrm{H}_p\, z(\gamma(s))\big)_{|s=S^-} = 0$ then $s \mapsto \gamma(s)$ is differentiable at $s = S^-$ one obtains (5.32).

Below we prove that

$$\mathrm{H}_p\, z\big(\gamma(s)\big) = o(S-s) \quad \text{as } s \to S^- \text{ in } [0, S[\backslash B. \tag{5.33}$$

Consider $s \in [0, S[\backslash B$. One says that $s \in B_0$ if $\gamma(s) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$; then $z(s) = 0$ and $\mathrm{H}_p\, z\big(\gamma(s)\big) = 0$. One thus only needs to prove (5.33) for $s \to S^-$ in $[0, S[\backslash (B \cup B_0)$.

Consider $\varepsilon > 0$. By continuity of $\mathrm{H}^2_p\, z$, there exists $0 < S_\varepsilon < S$ such that

$$\big|\big(\mathrm{H}^2_p\, z\big)\big(\gamma(s)\big) - \big(\mathrm{H}^2_p\, z\big)\big(\gamma(S)\big)\big| \le \varepsilon,$$

if $s \in [S_\varepsilon, S]\backslash B$. Consider $s \in [S_\varepsilon, S[\backslash (B \cup B_0)$; by assumption γ is locally a broken bicharacteristic. Define s_1 as the supremum of the connected component of s in $[S_\varepsilon, S[\backslash B_0$. Note that s_1 can be equal to S. By continuity one has $\gamma(s_1) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$ and on the interval $[s, s_1]$ one faces the situation described in Lemma 5.29. One thus has

$$\begin{aligned} 0 \le \big|\mathrm{H}_p\, z\big(\gamma(s)\big)\big| &\le C \int_s^{s_1} \big|\big(\mathrm{H}^2_p\, z\big)\big(\gamma(\sigma)\big) - \big(\mathrm{H}^2_p\, z\big)\big(\gamma(s_1)\big)\big| d\sigma \\ &\le 2C\varepsilon(s_1 - s) \lesssim (S-s)\varepsilon, \end{aligned}$$

meaning (5.33) holds. □

The notion of broken bicharacteristic is not sufficient to understand the propagation of the support of measures as points in $\mathcal{G}^3_\partial$ and $\mathcal{G}^{\mathrm{g}}_\partial$ are not considered. Moreover, a sequence of broken bicharacteristics may converge to a curve that is not a broken bicharacteristic. This leads to the introduction of generalized bicharacteristics as in Definition 2.8. With the notion of broken bicharacteristics introduced above one may also write the definition of generalized bicharacteristics as follows.

Definition 5.32 (generalized bicharacteristic)**.** Suppose $J \subset \mathbb{R}$ is an interval, B a discrete subset of J, and

$${}^{\mathsf{G}}\gamma : J \setminus B \to \mathrm{Char}\, p \cap T^*\mathcal{L}.$$

One says that ${}^{\mathsf{G}}\gamma$ is a generalized bicharacteristic if the following properties hold:

(1) for $s \in J \setminus B$, ${}^{\mathsf{G}}\gamma(s) \notin \mathcal{H}_\partial$ and the map ${}^{\mathsf{G}}\gamma$ is differentiable at s with

$$\frac{d}{ds}{}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big).$$

(2) if $s^0 \in B$, then ${}^{\mathsf{G}}\gamma$ is a broken bicharacteristic on an interval $[s^0 - \varepsilon, s^0 + \varepsilon] \setminus \{s^0\}$ for some $\varepsilon > 0$.

Recall the definition of the vector field ${}^{\mathsf{G}}X$ in (2.9).

Remark 5.33.

(1) From Lemma 5.18 one sees that ${}^{\mathsf{G}}X$ is continuous at points of $T^*\mathcal{L} \setminus \mathcal{G}^{\mathrm{g}}_\partial$, in particular at points in $\mathcal{G}^3_\partial$. It is however discontinuous at points of $\mathcal{G}^{\mathrm{g}}_\partial$. In fact if $\varrho \in \mathcal{G}^{\mathrm{g}}_\partial$ then ${}^{\mathsf{G}}X(\varrho) = \mathrm{H}^{\mathcal{G}}_p(\varrho) \neq \mathrm{H}_p(\varrho)$ since $\mathrm{H}^2_p z < 0$ and in any neighborhood of ϱ there are points in $\mathcal{H}_\partial$ where ${}^{\mathsf{G}}X = \mathrm{H}_p$. Yet, note that restricted to $\mathcal{G}_\partial$ the vector field ${}^{\mathsf{G}}X$ is continuous.

(2) Similarly to what is observed in Remarks 2.4-(1) and 5.27 one has, for $s \in J \setminus B$,

$$|\xi(s)|_{x(s)} = |\tau(s)|$$

constant along a generalized bicharacteristic.

Lemma 5.34. *Suppose $s^0 \in J \setminus \overline{B}$ and ${}^{\mathsf{G}}\gamma(s^0) \in \mathcal{G}^{\mathrm{g}}_\partial$. Then, ${}^{\mathsf{G}}\gamma(s) \in \mathcal{G}^{\mathrm{g}}_\partial$ for s in a neighborhood of s^0.*

Proof. Proceed by contradiction and assume that there exists a sequence $s_n \in J \setminus \overline{B}$ that converges to s^0 such that ${}^{\mathsf{G}}\gamma(s_n) \notin \mathcal{G}^{\mathrm{g}}_\partial$. If a subsequence s_{n_k} is such that ${}^{\mathsf{G}}\gamma(s_{n_k}) \in \mathcal{G}^3_\partial$, then

$$0 = \mathrm{H}^2_p z\big({}^{\mathsf{G}}\gamma(s_{n_k})\big) \underset{k\to+\infty}{\longrightarrow} \mathrm{H}^2_p z\big({}^{\mathsf{G}}\gamma(s^0)\big) < 0,$$

a contradiction. Thus, there exists a subsequence s_{n_k} with ${}^{\mathsf{G}}\gamma(s_{n_k}) \notin \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$. Then, the generalized bicharacteristic is a broken bicharacteristic in a neighborhood of s_{n_k}. Consider the part of the generalized bicharacteristic that is a maximal broken bicharacteristic. It ceases to exist at a point between s^0 and s_{n_k}. There, by Lemma 5.29 it reaches a point in $\mathcal{G}^3_\partial$ (here one is away from hyperbolic points). One is thus back to assuming that there is a subsequence r_n that converges to s^0 with ${}^{\mathsf{G}}\gamma(r_n) \in \mathcal{G}^3_\partial$ leading to a contradiction. □

If $s \mapsto {}^{\mathsf{G}}\gamma(s)$ is a generalized bicharacteristic it is obviously discontinuous across points $s \in B$ and continuous otherwise. The following lemma states $\mathscr{C}^1$-regularity away from points in $\overline{B}$.

Lemma 5.35. *If $s^0 \in J \setminus \overline{B}$ then $s \mapsto {}^{\mathsf{G}}\gamma(s)$ is $\mathscr{C}^1$ in a neighborhood of s^0.*

Proof. There exists a neighborhood W of s^0 with $W \cap \overline{B} = \emptyset$. If one proves that ${}^{\mathsf{G}}\gamma(s)$ is $\mathscr{C}^1$ at s^0 the same holds for any point in W, hence the result.

First, assume that ${}^{\mathsf{G}}\gamma(s^0) \notin \mathcal{G}^{\mathrm{g}}_{\partial}$. At such point the vector fields ${}^{\mathsf{G}}X$ is continuous; see the first part of Remark 5.33. It follows that ${}^{\mathsf{G}}\gamma(s)$ is $\mathscr{C}^1$ at s^0. Second, assume that ${}^{\mathsf{G}}\gamma(s^0) \in \mathcal{G}^{\mathrm{g}}_{\partial}$. Then, ${}^{\mathsf{G}}\gamma(s) \in \mathcal{G}^{\mathrm{g}}_{\partial}$ for s in a neighborhood of s^0 by Lemma 5.34. There, one has ${}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big) = \mathrm{H}^{\mathcal{G}}_p\big({}^{\mathsf{G}}\gamma(s)\big)$. Since $\mathrm{H}^{\mathcal{G}}_p$ is continuous this yields the result. □

Remark 5.36.

(1) For $s^0 \in \overline{B} \setminus B$, the definition only states the differentiability of ${}^{\mathsf{G}}\gamma(s)$ at $s = s^0$. In particular, the definition does not imply that the derivative of ${}^{\mathsf{G}}\gamma$ is continuous near such a point.

(2) In a local chart with the notation of Section 5.3, if one sets ${}^{\|}\varrho(s) = \pi_{\|}\big({}^{\mathsf{G}}\gamma(s)\big)$ one sees that ${}^{\|}\varrho(s)$ is $\mathscr{C}^0$. In particular the z-component is continuous and vanishes at $s = S^{\pm}$ for $S \in B$. If one considers the map ${}^{c}\phi$ introduced in Section 5.5 one sees that $s \mapsto {}^{c}\phi({}^{\mathsf{G}}\gamma(s))$ takes values in the compressed cotangent bundle and can be extended to the whole interval J as a continuous function. This aspect is used in the construction of a generalized bicharacteristic in what follows.

Lemma 5.37. *One has the following equivalence*

$$s \in \overline{B} \setminus B \quad \Leftrightarrow \quad \exists (s_n)_n \subset B, \text{ with } s_{n+1} \notin \{s_k; k \leq n\} \text{ such that } s_n \to s. \tag{5.34}$$

Moreover, $\overline{B} \setminus B$ is a closed set.

Proof. The "⇒" part of (5.34) is straightforward. The "⇐" part is a consequence of B being a discrete set. Assume now that $s \in \overline{\overline{B} \setminus B}$. Then there exists $(s_n)_n \subset \overline{B} \setminus B$ such that $s_n \to s$. One can construct a sequence $(s'_n)_n \subset B$ with $s'_{n+1} \notin \{s'_k; k \leq n\}$ such that $s'_n \to s$. One has $s \in \overline{B} \setminus B$ by (5.34). □

Lemma 5.38. *Suppose ${}^{\mathsf{G}}\gamma : J \setminus B \to T^*\mathcal{L}$ is a generalized bicharacteristic.*

(1) *If $s \in J \cap \overline{B} \setminus B$ then ${}^{\mathsf{G}}\gamma(s) \in \mathcal{G}^3_{\partial} \cup \mathcal{G}^{\mathrm{g}}_{\partial}$.*

(2) *Set $S = \sup J$ and suppose that $S < +\infty$. Then, the limit*

$$\varrho^0 = \lim_{\substack{s \to S \\ s \in J \setminus B}} {}^{\mathsf{G}}\gamma(s) \tag{5.35}$$

exists. If one sets ${}^{\mathsf{G}}\gamma(S) = \varrho^0$ then ${}^{\mathsf{G}}\gamma(s)$ is differentiable at $s = S^-$ and

$$\frac{d}{ds}{}^{\mathsf{G}}\gamma(S^-) = {}^{\mathsf{G}}X(\varrho^0). \tag{5.36}$$

Then, Part (1) applies if $S \in \overline{B} \setminus B$. A similar result holds for $S = \inf J$.

Proof. **Part (1):** $s \in J \cap \overline{B} \setminus B$. The argument of the proof of Lemma 5.28 can be applied.

Part (2): $S = \sup J < +\infty$. The result is clear if $S \in J \setminus B$ using the first part. Suppose $S \notin J \setminus B$. First, consider the case $S \notin \overline{B}$. Then one has ${}^{\mathsf{G}}\gamma(s) \in \operatorname{Char} p \cap T^*\mathcal{L} \setminus \mathcal{H}_\partial$ on an interval of the form $[S-\varepsilon, S[$ for some $\varepsilon > 0$, and

$$\frac{d}{ds}{}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big), \quad s \in [S-\varepsilon, S[.$$

Since ${}^{\mathsf{G}}\gamma(s)$ remains bounded, then ${}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big)$ remains bounded yielding the existence of the limit

$$\varrho^0 = \lim_{\substack{s \to S \\ s \in [S-\varepsilon, S[}} {}^{\mathsf{G}}\gamma(s).$$

On the one hand if $\varrho^0 \in T^*\mathcal{L} \setminus \mathcal{H}_\partial$, then $s \mapsto {}^{\mathsf{G}}X\big({}^{\mathsf{G}}\gamma(s)\big)$ is continuous on $[S-\varepsilon, S[$ and has a limit at $s = S^-$; the proof follows the arguments of Lemma 5.35. This implies that ${}^{\mathsf{G}}\gamma(s)$ is differentiable at $s = S^-$ and (5.36) holds. On the other hand if $\varrho^0 \in \mathcal{H}_\partial$ then $\varrho^0 \in \mathcal{H}_\partial^-$, meaning that locally the generalized bicharacteristic is a regular bicharacteristic reaching a hyperbolic point. This again gives that ${}^{\mathsf{G}}\gamma(s)$ is differentiable at $s = S^-$ and (5.36) holds.

Second, consider the case $S \in \overline{B}$. The argument of the proof of Lemma 5.28 can be applied *mutatis mutandis* yielding the existence of the limit ϱ^0 in (5.35). Then, the proof of Lemma 5.29 can be applied with some slight modifications. One finds with the same arguments that

$$\frac{d}{ds}{}^{\|}\varrho(S^-) = \mathrm{H}_p^{\mathcal{G}}(\varrho^0).$$

One introduces the same function g as in (5.30). At points in $J \setminus B$ the function g is differentiable, even at point in $\overline{B} \setminus B$. In fact, if $s^1 \in J \setminus \overline{B}$ on finds

$$\frac{d}{ds}g(s^1) = \big(\mathrm{H}_p\, z\big)\big({}^{\mathsf{G}}\gamma(s^1)\big)\varepsilon(s^1, S), \tag{5.37}$$
$$\text{with} \quad \varepsilon(s^1, S) = \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{G}}\gamma(s^1)\big) - \big(\mathrm{H}_p^2\, z\big)\big({}^{\mathsf{G}}\gamma(S)\big).$$

as in the proof of Lemma 5.29. At a point $s^1 \in J \cap \overline{B} \setminus B$ one has ${}^{\mathsf{G}}\gamma(s^1) \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ by the first part of the lemma. Hence, $g(s^1) = \frac{d}{ds}g(s^1) = 0$ since $z(s^1) = 0$,

$\frac{d}{ds}z(s^1) = \mathrm{H}_p^{\mathcal{G}}\, z(s^1) = \mathrm{H}_p\, z(s^1) = 0$, and $\frac{d}{ds}\,\mathrm{H}_p\, z(s^1) = \mathrm{H}_p^{\mathcal{G}}\,\mathrm{H}_p\, z(s^1) = 0$ (recall that $\mathrm{H}_p^{\mathcal{G}}\,\mathrm{H}_p\, z = 0$ on ${}^{\|}\partial(T^*\mathcal{L})$ since $\mathrm{H}_p^{\mathcal{G}}$ is tangent to ${}^{\|}\partial(T^*\mathcal{L})$). Hence (5.37) holds in $J \setminus B$. Setting $g^a = g + a$ for $a > 0$ one finds

$$\Big|\frac{d}{ds}\big(g^a(s)^{1/2}\big)\Big| \lesssim |\varepsilon(s,S)|, \qquad s \in J \setminus B.$$

Since B is at most countable, one finds that $(g^a)^{1/2}$ is Lipschitz thus absolutely continuous on J. The remainder of the proof of Lemma 5.29 then applies. One finds that ${}^{\mathsf{G}}\gamma(s)$ is differentiable at $s = S^-$ and (5.36) holds. □

For a generalized bicharacteristic the notation ${}^{\mathsf{G}}\bar\gamma$ was introduced in Definition (2.9). If one considers the map ${}^{c}\phi$ associated with the compressed cotangent bundle introduced in Section 5.5 observe that

$${}^{c}\phi({}^{\mathsf{G}}\bar\gamma) = \overline{{}^{c}\phi\{{}^{\mathsf{G}}\gamma(s);\ s \in J \setminus B\}} \quad \text{and} \quad {}^{\mathsf{G}}\bar\gamma = {}^{c}\phi^{-1}\big({}^{c}\phi({}^{\mathsf{G}}\bar\gamma)\big).$$

5.7. **On the measure equation.** The measure equation of Assumption 3.2, is one of the two hypothesis in our main result, Theorem 3.4:

$${}^{t}\mathrm{H}_p\,\mu = f\mu - \int_{\varrho \in {}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}}\, d\nu(\varrho) \qquad \text{in } \mathscr{U}.$$

With what precedes one sees that this equation is of geometrical nature.

In Section 3 we gave an interpretation of the integrand for $\varrho \in {}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial$ using that $\mathcal{G}_\partial \cup \mathcal{H}_\partial = \overline{\mathcal{H}_\partial}$. Here a similar interpretation can be done yet using the geometrical $\mathscr{C}^1$-parametrization 5.9 of the cotangent bundle.

We thus use the parametrization $({}^{\|}\varrho, \mathrm{H}_p\, z(\varrho))$ of $T^*\mathcal{L}$ and denote by ϑ the variable $\mathrm{H}_p\, z(\varrho)$. If $\varrho = (t,x,\tau,\xi) \in {}^{\|}\mathcal{H}$ then with Lemma 5.7 one has $\varrho^\pm = (t,x,\tau,\xi^\pm) \in \mathcal{H}^\pm$ with

$$\xi^+ = \xi + \alpha(x)\vartheta^+\mathsf{n}_x^* \quad \text{and} \quad \xi^- = \xi + \alpha(x)\vartheta^-\mathsf{n}_x^*$$

and $\vartheta^+ > 0$ and $\vartheta^- = -\vartheta^+$. Then

$$\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = \alpha(x)(\vartheta^+ - \vartheta^-) = 2\alpha(x)\vartheta^+. \tag{5.38}$$

If $q = q({}^{\|}\varrho, \vartheta)$ is a $\mathscr{C}^1$-test function then, for $\varrho \in {}^{\|}\mathcal{H}_\partial$ one has

$$\frac{\langle \delta_{\varrho^+} - \delta_{\varrho^-}, q\rangle}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}} = \frac{q(\varrho, \vartheta^+) - q(\varrho, -\vartheta^+)}{2\alpha(x)\vartheta^+}.$$

If now a sequence $(\varrho^{(n)})_n \subset {}^{\|}\mathcal{H}_\partial$ converges to $\varrho = {}^{\|}\varrho \in {}^{\|}\mathcal{G}_\partial$ then

$$\frac{\langle \delta_{\varrho^{(n)+}} - \delta_{\varrho^{(n)-}}, q\rangle}{\langle \xi^{(n)+} - \xi^{(n)-}, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}} \to \frac{1}{\alpha(x)}\partial_\vartheta q(\varrho,0). \tag{5.39}$$

Up to the factor $1/\alpha$, the integrand in (3.1) for $\varrho \in {}^{\|}\mathcal{G}_\partial$ is thus to be understood as the derivative with respect to the variable ϑ at $\vartheta = 0$. If one considers the quasi-normal geometric coordinates given by Proposition 2.1 that are used in Section 2 then one has $\vartheta = 2\zeta$ and $\alpha = 1/2$ since $\alpha = (2\,\mathrm{H}_z^2\,p)^{-1/2}$ and $\mathrm{H}_z^2\,p = 2$ in those coordinates. One thus recovers the observation made in Remark 3.3.

Remark 5.39. Observe that since $\Sigma(\varrho^+) = \varrho^-$, under Assumption 3.2 one finds

$$\Sigma_*({}^t\mathrm{H}_p\,\mu - f\mu) = -({}^t\mathrm{H}_p\,\mu - f\mu). \tag{5.40}$$

6. Propagation of the measure support

This section is devoted the proof of our main result, Theorem 3.4.

In what follows, we consider all possible situations for ϱ^0: $\varrho^0 \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$, that is, a point away from the boundary (Section 6.1), $\varrho^0 \in \mathcal{H}_\partial$, that is, a hyperbolic point at the boundary (Section 6.2), $\varrho^0 \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$, that is, an order-3-glancing or a gliding point (Section 6.3.1), and $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}}$, that is a diffractive point (Section 6.3.2). In all situations, we prove a local result. These results put together yield the proof of Theorem 3.4; see Section 6.4. Working locally allows one to use a single chart. This permits to use the extension away from the boundary of notions only geometrically meaningfull at the boundary ${}^{\|}T_\varrho^*\mathcal{L}$, $\mathcal{G}$, $\mathcal{H}$, $\mathrm{H}_p^{\mathcal{G}}$, etc.; see Section 5.3. We make use of these extension without mentionning it in what follows.

6.1. Away from the boundary.

With Assumption 3.2, away from the boundary, one has ${}^t\mathrm{H}_p = f\mu$ locally. Then, Theorem 4.1 gives the following result.

Proposition 6.1. *Suppose $\varrho^0 \in \operatorname{supp}\mu \cap \big(T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})\big)$. If V is an open neighborhood of ϱ^0 in $T^*\mathcal{L}$ such that $V \cap \partial(T^*\mathcal{L}) = \emptyset$ then there exists a maximally extended bicharacteristic $\gamma(s)$ in V, for $s \in J$, an open interval of $\mathbb{R}$, with $0 \in J$, such that*

$$\gamma(0) = \varrho^0 \quad \textit{and} \quad \{\gamma(s);\ s \in J\} \subset \operatorname{supp}\mu.$$

6.2. **Hyperbolic points.** Consider $\varrho^0 \in \mathcal{H}_\partial^\pm$. Consider a local chart as in (2.1). As $\mathcal{H}_\partial^\pm$ is an open subset of $\partial(T^*\mathcal{L}) \cap \operatorname{Char} p$ one can choose V^0 an open subset of $T^*\mathbb{R}^{1+d}$ such that $\varrho^0 \in V^0$ and

$$\big(V^0 \cup \Sigma(V^0)\big) \cap \partial(T^*\mathcal{L}) \cap \operatorname{Char} p \subset \mathcal{H}_\partial^\pm.$$

If $q \in \mathscr{C}_c^1(\mathbb{R}^{2d+2})$ is such that $\operatorname{supp} q \subset V^0$, then, from (3.1) one deduces

$$\big\langle {}^t\mathrm{H}_p\,\mu, q\big\rangle = \big\langle \mu, fq\big\rangle \mp \int_{\varrho \in {}^\parallel\mathcal{H}_\partial \cup {}^\parallel\mathcal{G}_\partial} \frac{q(\varrho^\pm)}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}} d\nu(\varrho),$$

and $\xi^+ - \xi^- \neq 0$ on the support of $\varrho \mapsto q(\varrho^\pm)$ that does not meet ${}^\parallel\mathcal{G}_\partial$. One thus finds that the measure μ is solution in V^0 to an equation of the form

$${}^t\mathrm{H}_p\,\mu = f\mu + \tilde\mu \otimes \delta_{z=0},$$

where $\tilde\mu$ is a measure on $\{z = 0\} = \partial(T^*\mathcal{L})$ in V^0. The following lemma is the result of Lemma 4.8 translated into the present setting.

Lemma 6.2. *Suppose $\varrho \in \mathcal{H}_\partial$. Then, $\varrho \in \operatorname{supp}\mu$ if and only if $\varrho \in \operatorname{supp}(\tilde\mu \otimes \delta_{z=0})$.*

Similarly, one has ${}^t\mathrm{H}_p\,\mu = f\mu + \hat\mu \otimes \delta_{z=0}$ in $\Sigma(V^0)$, with $\hat\mu$ a measure on $\{z = 0\} \cap \Sigma(V^0)$. From (5.40) one has

$$\Sigma_*\big(\tilde\mu \otimes \delta_{z=0}\big) = -\hat\mu \otimes \delta_{z=0} \quad \text{in} \quad \Sigma(V^0). \tag{6.1}$$

Lemma 6.2 and (6.1) give the following proposition.

Proposition 6.3. *Suppose $\varrho \in \mathcal{H}_\partial$. Then, $\varrho \in \operatorname{supp}\mu$ if and only if $\Sigma(\varrho) \in \operatorname{supp}\mu$.*

By Lemma 5.12, for $\varrho \in \mathcal{H}_\partial^+$ if $\gamma(s)$ is a bicharacteristic that goes through ϱ at $s = 0$, then $\{\gamma(s)\}_{s\in]0,S[}$ lies in $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ while $\{\gamma(s)\}_{s\in]-S,0[}$ lies in the complement of $T^*\mathcal{L}$, for some $S > 0$. For $\varrho \in \mathcal{H}_\partial^-$, this is the opposite.

Definition 6.4. For $\varrho \in \mathcal{H}_\partial^+$ (resp. $\mathcal{H}_\partial^-$) and $\gamma(s)$ a bicharacteristic that goes through ϱ at $s = 0$ call $\gamma(s)$, for $s \geq 0$ (resp. $s \leq 0$), a *half* bicharacteristic initiated at ϱ.

For $\varrho \in \mathcal{H}_\partial$ a half bicharacteristic is locally contained in $T^*\mathcal{L}$, that is, in $\{z \geq 0\}$.

Definition 6.5. Suppose F is a closed set of $T^*\mathcal{L}$ and $\varrho \in F \cap \mathcal{H}_\partial^+$ (resp. $F \cap \mathcal{H}_\partial^-$). One says that a half bicharacteristic $\gamma(s)$ initiated at ϱ is locally contained in F if for some $S > 0$ one has $\{\gamma(s)\}_{s\in[0,S[} \subset F$ (resp. $\{\gamma(s)\}_{s\in]-S,0]} \subset F$).

With the notion introduced in Definitions 6.4 and 6.5 we now state the following result.

Proposition 6.6. *Suppose $\varrho^0 \in \mathcal{H}_\partial^\pm$ and V^0 is open subset of $T^*\mathbb{R}^{1+d}$ such that $V^0 \cap \partial(T^*\mathcal{L}) \cap \operatorname{Char} p \subset \mathcal{H}_\partial^\pm$. If $\varrho^0 \in \operatorname{supp}\mu$, then there exists a half bicharacteristic initiated at ϱ^0 that is locally contained in $\operatorname{supp}\mu$. Moreover, this half bicharacteristic can be chosen maximally extended in V^0.*

Proof. Applying Theorem 4.5 to ϱ^0 gives the existence of a half bicharacteristic $\{\gamma(s)\}_{s\in[0,S[}$, with $S > 0$, initiated at ϱ^0 contained in $\operatorname{supp}\mu$. Since $\gamma(S/2) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$, combining with the result of Proposition 6.1 one can obtain a maximally extended such half bicharacteristic in V^0. □

From Propositions 6.3 and 6.6 one deduces the following result.

Corollary 6.7. *Suppose $\varrho^0 \in \mathcal{H}_\partial$. There is no half bicharacteristic initiated at ϱ^0 locally contained in $\operatorname{supp}\mu$ if and only if there is no half bicharacteristics initiated at $\Sigma(\varrho^0)$ locally contained in $\operatorname{supp}\mu$.*

Equivalently, this result reads as follows.

Corollary 6.7′. *Suppose $\varrho^0 \in \mathcal{H}$. If there exists a half bicharacteristic initiated at ϱ^0 locally contained in $\operatorname{supp}\mu$ then $\{\varrho^0, \Sigma(\varrho^0)\} \subset \operatorname{supp}\mu$ and there exists a half bicharacteristic initiated at $\Sigma(\varrho^0)$ locally contained in $\operatorname{supp}\mu$.*

In other words, if a bicharacteristic locally contained in $\operatorname{supp}\mu$ hits the boundary at a hyperbolic point, then the support of μ is transported along at least one generalized bicharacteristic through that hyperbolic point and its image by Σ.

6.3. **Glancing points.** In Section 6.3.1, we provide a propagation tool for general glancing points to be used for points in $\mathcal{G}_\partial^{\mathrm{g}}$ and $\mathcal{G}_\partial^{3}$. For points in $\mathcal{G}_\partial^{\mathrm{d}}$, that is, diffractive points, we provide a more useful propagation tool in Section 6.3.2.

Lemma 6.8. *Suppose $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \mathcal{G}_\partial$, that is, $\mathrm{H}_p\, z(\varrho^0) = 0$, and V^0 is a bounded neighborhood of ϱ^0 in $T^*\mathcal{L}$ that lies in a local chart. There exists $C_0 > 0$ such that*

$$|\,\mathrm{H}_p\, z(\varrho)| \le C_0 \|{}^{\|}\varrho - {}^{\|}\varrho^0\|^{1/2}, \qquad \varrho \in \operatorname{Char} p \cap V^0.$$

Proof. With (5.15), if $\varrho = (t, x, \tau, \xi) \in \operatorname{Char} p \cap V^0$ one has

$$\alpha(x)^2\, \mathrm{H}_p\, z(\varrho)^2 = \lambda^2 = \tau^2 - |{}^{\|}\xi|_x^2 = -p({}^{\|}\varrho).$$

As $\varrho^0 \in \mathcal{G}_\partial$, one has $p(\varrho^0) = p({}^\parallel\varrho^0) = 0$, yielding

$$0 \le \alpha(x)^2 \,\mathrm{H}_p\, z(\varrho)^2 = p({}^\parallel\varrho^0) - p({}^\parallel\varrho) \lesssim \|{}^\parallel\varrho - {}^\parallel\varrho^0\|,$$

since V^0 is bounded and p is $\mathscr{C}^1$. Since $\alpha(x)^{-1} = (2\,\mathrm{H}_z^2\, p)^{1/2}$ is bounded on V^0 the result follows. □

6.3.1. *General glancing points.* We prove the following Proposition.

Proposition 6.9. *Suppose K is a compact set of $T^*\mathcal{L}\backslash 0$ that lies in a local chart. There exists $C_0 > 0$ such that*

$$\forall \varepsilon > 0,\ \exists \delta_0 > 0,\ \forall \varrho^0 \in K \cap \mathcal{G}_\partial \cap \operatorname{supp}\mu,\ \forall \delta \in]0, \delta_0],\ \exists \varrho \in \operatorname{supp}\mu \tag{6.2}$$
$$\text{such that } {}^\parallel\varrho \in B\big(\varrho^0 + \delta\,\mathrm{H}_p^{\mathcal{G}}(\varrho^0), C_0\delta\varepsilon\big),$$

and $|\,\mathrm{H}_p\, z(\varrho)| \le C_0\delta^{1/2}$.

The notation $\varrho^0 + \delta\,\mathrm{H}_p^{\mathcal{G}}(\varrho^0)$ is here to be understood in the local coordinates where such computation makes sense.

Proof. We make some preliminary remarks:

(1) Since $\mathrm{H}_p\, z(\varrho^0) = 0$, the estimation of $|\,\mathrm{H}_p\, z(\varrho)|$ follows from Lemma 6.8 as one has $\|{}^\parallel\varrho - {}^\parallel\varrho^0\| \lesssim \delta$. We thus prove the existence of ϱ such that (6.2) holds.
(2) It suffices to consider $0 < \varepsilon \le 1$.

Consider $\varrho^0 \in K \cap \mathcal{G}_\partial$. Then, one has ${}^\parallel\varrho^0 = \varrho^0$. On $K \cap \operatorname{supp}\mu$ one has $0 < c_K \le \|\,\mathrm{H}_p^{\mathcal{G}}\,\| \le C_K$ by Remark 5.20 since $\operatorname{supp}\mu \subset \operatorname{Char} p$ by Assumption 3.1. One has $\mathrm{H}_p^{\mathcal{G}}(\varrho^0) \in T_{\varrho^0}\mathcal{G}_\partial \subset T_{\varrho^0}{}^\parallel T^*\mathcal{L}$. On ${}^\parallel T^*\mathcal{L}$, by performing a rotation and a dilation of scale factor $\|\,\mathrm{H}_p^{\mathcal{G}}(\varrho^0)\| \in [c_K, C_K]$, one can assume that $\mathrm{H}_p^{\mathcal{G}}(\varrho^0) = (1, 0, \dots, 0)$. Since $\mathrm{H}_p^{\mathcal{G}}\, z(\varrho^0) = \mathrm{H}_p\, z(\varrho^0) = 0$, the above transformation can be chosen not affecting the z variable. One may thus use $u \in \mathbb{R}^{2d}$ such that (u, z) are new coordinates on ${}^\parallel T^*\mathcal{L}$ and $\mathrm{H}_p^{\mathcal{G}}(\varrho^0) = \partial_{u_1}$, for $u = (u_1, u')$ with $u_1 \in \mathbb{R}$ and $u' \in \mathbb{R}^{2d-1}$. With $\vartheta = \mathrm{H}_p\, z(\varrho)$, one can use (u, z, ϑ) as local coordinates on $T^*\mathcal{L}$. Since $\varrho^0 \in \mathcal{G}_\partial$ the coordinates of ϱ^0 are of the form $(u_1^0, u^{0\prime}, z = 0, \vartheta = 0)$.

Consider the functions χ and β as in (4.6)–(4.7) and also a convex function $j \in \mathscr{C}^\infty(\mathbb{R})$ such that

$$j \equiv 0 \text{ on }]-\infty, 1/2], \quad j' > 0 \text{ on }]1/2, +\infty[, \quad j(s) > s \text{ for } s \ge 1.$$

Observe that these properties imply

$$\big(a \ge 1 \text{ and } j(s) \le a\big) \ \Rightarrow s \le a. \tag{6.3}$$

A possible choice is simply $j(s) = \alpha s \mathbf{1}_{2s-1>0} e^{1/(1-2s)}$ with $\alpha > e$. Pick also $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$ nonnegative such that

$$\psi \equiv 1 \text{ in } [-R, R], \quad \psi \equiv 0 \text{ in } \mathbb{R} \setminus [-R-1, R+1],$$

for some $R > 0$ to be set below. Then set

$$q(u, z, \vartheta) = e^{Au_1} (\chi \circ v)(u, z)\, (\beta \circ w)(u)\, \psi(\vartheta),$$

with

$$v(u, z) = 1/2 - \delta^{-1}(u_1 - u_1^0) + 8(\varepsilon\delta)^{-2} \|u' - u^{0\prime}\|^2 + j\big(2(\delta\varepsilon)^{-1} z\big),$$

and

$$w(u) = 2\varepsilon^{-1}\big(1 - \delta^{-1}(u_1 - u_1^0)\big).$$

Arguing as in the proof of Proposition 4.3 one finds

$$-\delta/2 \leq u_1 - u_1^0 \leq \delta(1 + \varepsilon/2), \tag{6.4}$$

and

$$8(\varepsilon\delta)^{-2} \|u' - u^{0\prime}\|^2 + j\big(2(\delta\varepsilon)^{-1} z\big) \leq 3/2 + \varepsilon/2,$$

in $\operatorname{supp} q$. The second inequality implies

$$8\|u' - u^{0\prime}\|^2 \leq (\varepsilon\delta)^2\big(3/2 + \varepsilon/2\big) \leq 2(\varepsilon\delta)^2, \tag{6.5}$$

and

$$z \leq \frac{\delta\varepsilon}{2}\big(3/2 + \varepsilon/2\big) \leq \delta\varepsilon, \tag{6.6}$$

using (6.3). One thus finds that $\operatorname{supp} q \cap \{z \geq 0\}$ is compact. Since $\operatorname{supp} \mu \subset \{z \geq 0\}$ by Assumption 3.1, then the action of ${}^t\mathrm{H}_p\, \mu - f\mu$ on the test function q makes sense.

In $\operatorname{Char} p$, with $|u - u^0|$ and z bounded as above, $|\vartheta|$ is also bounded by means of Lemma 6.8. Thus, for $R > 0$ chosen sufficiently large one finds that

$$\psi \equiv 1 \text{ in a neighborhood of the } \vartheta\text{-projection of } \operatorname{supp} q \cap \operatorname{Char} p. \tag{6.7}$$

Applying the measure equation (3.1) of Assumption 3.2 to q, one first considers the contribution of the r.h.s. of (3.1) associated with hyperbolic points. Consider $\varrho \in {}^{\|}\mathcal{H}_\partial$. One has $\{\varrho^+, \varrho^-\} = \pi_{\|}^{-1}\{\varrho\} \cap \operatorname{Char} p$, and the two points only

differ by their ϑ-component: $\mathrm{H}_p\, z(\varrho^+) = -\mathrm{H}_p\, z(\varrho^-)$; see Lemma 5.7, Proposition 5.9 and (5.13). With the form of ψ and the value of R chosen above, one has

$$q\big(\varrho^+\big) = q\big(\varrho^-\big)$$

by (6.7). Hence, the contribution of ${}^{\|}\mathcal{H}_\partial$ to the integral in (3.1) vanishes with the present choice of test function.

Second, one considers the contribution of the r.h.s. of (3.1) associated with glancing points. Consider $\varrho \in {}^{\|}\mathcal{G}_\partial$. The contribution to the integrand is described in (5.39), that is, a differentiation of the test function with respect to ϑ. Since

$$\partial_\vartheta q(u,z,\vartheta) = e^{Au_1}(\chi\circ v)(u,z)\,(\beta\circ w)(u)\,\psi'(\vartheta)$$

with (6.7) one finds that that $\partial_\vartheta q$ vanishes in ${}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial \subset \operatorname{Char} p$. One concludes that

$$\langle {}^t\mathrm{H}_p\,\mu, q\rangle = \langle \mu, \mathrm{H}_p\, q\rangle = \langle \mu, fq\rangle. \tag{6.8}$$

One has $\mathrm{H}_p\, q = g + h + j + A(\mathrm{H}_p\, u_1)q$, with

$$\begin{aligned}
g(u,z,\vartheta) &= e^{Au_1}(\chi'\circ v)(u,z)\,(\beta\circ w)(u)\,\psi(\vartheta)\,\mathrm{H}_p\, v(u,z,\vartheta),\\
h(u,z,\vartheta) &= e^{Au_1}(\chi\circ v)(u,z)\,(\beta'\circ w)(u)\,\psi(\vartheta)\,\mathrm{H}_p\, w(u,z,\vartheta),\\
j(u,z,\vartheta) &= e^{Au_1}(\chi\circ v)(u,z)\,(\beta\circ w)(u)\;\mathrm{H}_p\,\psi(u,z,\vartheta).
\end{aligned}$$

By (6.7), as $\operatorname{supp}\mu \subset \operatorname{Char} p$ by Assumption 3.1, one has

$$\langle \mu, j\rangle = 0. \tag{6.9}$$

The support properties (6.4)–(6.6) naturally hold also for $\operatorname{supp}\big(g\big)$ and $\operatorname{supp} h$. Moreover, arguing as in the proof of Proposition 4.3, for $0 < \varepsilon \le 1$, one has

$$\varrho = (u,z,\vartheta) \in \operatorname{supp} h \quad\Rightarrow\quad u \in B\big(u^0 + \delta\,\mathrm{H}_p^{\mathcal{G}}(\varrho^0), \varepsilon\delta\big) \;\text{ and }\; z \le \varepsilon\delta. \tag{6.10}$$

Recall that $\mathrm{H}_p^{\mathcal{G}}(\varrho^0) = \partial_{u_1}$.

Lemma 6.10. *For any $0 < \varepsilon \le 1$ there exists $\delta_0 > 0$ such that for any $\varrho^0 \in K \cap \mathcal{G}_\partial \cap \operatorname{supp}\mu$ and $0 < \delta \le \delta_0$*

1. *the function g is nonnegative in $\operatorname{supp}\mu$ and is positive in a neighborhood of ϱ^0.*
2. *$\mathrm{H}_p\, u_1 \ge 1/2$ in $\operatorname{supp} q \cap \operatorname{supp}\mu$.*

Consider $\delta_0 > 0$ as given by Lemma 6.10 and $0 < \delta \leq \delta_0$. By (6.8) and (6.9) one finds

$$0 = \langle ({}^t\mathrm{H}_p - f)\mu, q\rangle = \langle \mu, g\rangle + \langle \mu, h\rangle + \big\langle \mu, \big(A(\mathrm{H}_p\, u_1) - f\big)q\big\rangle.$$

One has $(A(\mathrm{H}_p\, u_1) - f)q \geq 0$ in $\mathrm{supp}(\mu)$ for $A \geq 2\sup_K |f|$, implying

$$\big\langle \mu, \big(A(\mathrm{H}_p\, u_1) - f\big)q\big\rangle \geq 0.$$

With $g \geq 0$, $\varrho^0 \in \mathrm{supp}\,\mu$ and $g(\varrho^0) > 0$ one obtains $\langle \mu, h\rangle \neq 0$, meaning that $\mathrm{supp}\,\mu \cap \mathrm{supp}\,h \neq \emptyset$. With (6.10) one concludes the proof of Proposition 6.9. □

The proof of Lemma 6.10 is very close to that of Lemma 4.4. However, some details need to be handled carefully. We thus provide a complete proof

Proof of Lemma 6.10. Consider $0 < \varepsilon \leq 1$. One has

$$g(u, z, \vartheta) = e^{Au_1}(\chi' \circ v)(u, z)\ (\beta \circ w)(u)\, \psi(\vartheta)\, \mathrm{H}_p\, v(u, z, \vartheta).$$

Since $\beta \geq 0$, $\chi' < 0$, and $\psi \geq 0$, it suffices to prove that $\mathrm{H}_p\, v(u, z, \vartheta) \leq 0$ for (u, z, ϑ) in $\mathrm{supp}\, q \cap \mathrm{supp}\,\mu$ for $\delta > 0$ chosen sufficiently small, uniformly with respect to $\varrho^0 \in K$. Since v is independent of ϑ and $\mathrm{H}_z = -\partial_\zeta = -\,\mathrm{H}_z^2\, p\, \partial_\vartheta$ one has $\mathrm{H}_p\, v = \mathrm{H}_p^{\mathcal{G}}\, v$ by (5.17); see (5.20).

If $\varrho \in \mathrm{supp}\, g \cap \mathrm{supp}\,\mu$ then $\|{}^{\|}\varrho - {}^{\|}\varrho^0\| \lesssim \delta$ by (6.4)–(6.6) with in particular $0 \leq z \lesssim \delta$ since $z \geq 0$ in $\mathrm{supp}\,\mu$ by Assumption 3.1. With Lemma 6.8 one has $|\vartheta| \lesssim \|{}^{\|}\varrho - {}^{\|}\varrho^0\|^{1/2}$. Thus $\vartheta = o(1)$ as $\delta \to 0^+$ implying $\|\varrho - \varrho^0\| = o(1)$.

Write

$$\mathrm{H}_p^{\mathcal{G}}(\varrho) - \mathrm{H}_p^{\mathcal{G}}(\varrho^0) \in \alpha^1(\varrho, \varrho^0)\partial_{u_1} + \alpha'(\varrho, \varrho^0) \cdot \nabla_{u'} + \gamma(\varrho, \varrho^0)\partial_z + \mathrm{span}\{\partial_\vartheta\},$$

with $\alpha^1(\varrho, \varrho^0) \in \mathbb{R}$, $\alpha'(\varrho, \varrho^0) \in \mathbb{R}^{2d-1}$, and $\gamma(\varrho, \varrho^0) \in \mathbb{R}$. From the uniform continuity of $\mathrm{H}_p^{\mathcal{G}}$ in any compact set one concludes that

$$|\alpha^1(\varrho, \varrho^0)| + \|\alpha'(\varrho, \varrho^0)\| + |\gamma(\varrho, \varrho^0)| = o(1) \ \text{ as } \delta \to 0^+, \tag{6.11}$$

uniformly with respect to $\varrho^0 \in K \cap \mathcal{G}_\partial \cap \mathrm{supp}\,\mu$ and $\varrho \in \mathrm{supp}\, q \cap \mathrm{supp}\,\mu$. Using that $\mathrm{H}_p^{\mathcal{G}}(\varrho^0) = \partial_{u_1}$ and the form of v, one writes

$$\mathrm{H}_p^{\mathcal{G}}\, v(\varrho) = \big(\partial_{u_1} v + (\mathrm{H}_p^{\mathcal{G}}(\varrho) - \mathrm{H}_p^{\mathcal{G}}(\varrho^0))v\big)(\varrho) = -\delta^{-1} + r$$

with $r = -\delta^{-1}\alpha^1(\varrho, \varrho^0) + 16(\varepsilon\delta)^{-2}\alpha'(\varrho, \varrho^0)\cdot(u' - u^{0\prime}) + 2(\delta\varepsilon)^{-1}\gamma(\varrho, \varrho^0) j'\big(2(\delta\varepsilon)^{-1} z\big)$. With (6.5) and (6.6) one finds

$$|r| \lesssim \delta^{-1}\big|\alpha^1(\varrho, \varrho^0)\big| + (\delta\varepsilon)^{-1}\|\alpha'(\varrho, \varrho^0)\| + (\delta\varepsilon)^{-1}|\gamma(\varrho, \varrho^0)|,$$

using that $|j'(s)| \lesssim 1$ if $s \leq 2$. With ε fixed above and with (6.11) one has $r = \delta^{-1}o(1)$. One thus finds $\mathrm{H}_p^{\mathcal{G}} v(\varrho) \sim -\delta^{-1}$ as $\delta \to 0^+$ uniformly with respect to $\varrho^0 \in K \cap \mathcal{G}_\partial \cap \operatorname{supp}\mu$ and $\varrho \in \operatorname{supp} q \cap \operatorname{supp}\mu$.

One has $g(\varrho^0) = -\delta^{-1}\chi'(1/2)\beta(2\varepsilon^{-1}) > 0$ and thus g is positive in a neighborhood of ϱ^0.

Note that $\mathrm{H}_p\, u_1 = \mathrm{H}_p^{\mathcal{G}}\, u_1 = 1 + \alpha'(\varrho, \varrho^0)$. With (6.11) one finds tht $\mathrm{H}_p\, u_1 \geq 1/2$ for δ sufficiently small, uniformly in with respect to $\varrho^0 \in K \cap \mathcal{G}_\partial \cap \operatorname{supp}\mu$ and $\varrho \in \operatorname{supp} q \cap \operatorname{supp}\mu$. □

6.3.2. *Diffractive points.* For points in $\mathcal{G}_\partial^{\mathrm{d}}$ we rely on the following result.

Proposition 6.11. *Suppose $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}} \cap \operatorname{supp}\mu$. There exist $C > 0$ and $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ there exist $\varrho_p \in \operatorname{supp}\mu \cap \mathcal{H}^-$ and $\varrho_f \in \operatorname{supp}\mu \cap \mathcal{H}^+$ such that*

$$\|^{\|}\varrho_p - {}^{\|}\varrho^0\| + \|^{\|}\varrho_f - {}^{\|}\varrho^0\| \leq C\delta, \quad -C\delta^{1/2} \leq \mathrm{H}_p\, z(\varrho_p) \leq -\delta^2,$$
$$\text{and} \quad \delta^2 \leq \mathrm{H}_p\, z(\varrho_f) \leq C\delta^{1/2}.$$

In the statement, the subscript p stands for *past* and f for *future*, in view of the use we make of these two points in the construction of a bicharacteristic contained in $\operatorname{supp}\mu$ that goes through a point $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}} \cap \operatorname{supp}\mu$ (see Section 6.4).

Proof of Proposition 6.11. If $\varrho^0 = (t^0, x^{0\prime}, z^0, \tau^0, \xi^0)$ one has $z^0 = 0$. Introduce the following two functions

$$\phi_0(\varrho) = \|^{\|}\varrho - {}^{\|}\varrho^0\|^2 \quad \phi(\varrho) = -\mathrm{H}_p\, z(\varrho) + \phi_0(\varrho).$$

Note that ϕ_0 is independent of the variable $\vartheta = \mathrm{H}_p\, z(\varrho)$.

Here, we write the proof of the existence of the point ϱ_f as in the statement of the proposition. For the existence of the point ϱ_p one simply changes $\phi(\varrho)$ into $\mathrm{H}_p\, z(\varrho) + \phi_0(\varrho)$ and the proof follows *mutatis mutandis*.

One has $\mathrm{H}_p\, \phi(\varrho) = -\mathrm{H}_p^2\, z(\varrho) + \mathrm{H}_p\, \phi_0(\varrho)$. Since $|\mathrm{H}_p\, \phi_0(\varrho)| \lesssim \|^{\|}\varrho - {}^{\|}\varrho^0\|$ and since $\mathrm{H}_p^2\, z(\varrho^0) > 0$ as $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}}$, there exist a neighborhood V^0 of ϱ^0 in $T^*\mathcal{L}$ and $C_0 > 0$ such that

$$\mathrm{H}_p\, \phi(\varrho) \leq -C_0 < 0, \qquad \varrho \in V^0. \tag{6.12}$$

Consider $\chi \in \mathscr{C}^\infty(\mathbb{R})$ as given by (4.6) and suppose $\psi \in \mathscr{C}_c^\infty(\mathbb{R})$ nonnegative, with $\operatorname{supp}\psi \subset [-3, 3]$ and such that $\psi \equiv 1$ on a neighborhood of $[-2, 2]$. Introduce the following family of test functions, $\delta > 0$,

$$q(\varrho) = \psi^2(\phi_0(\varrho)/\delta^2)\ \chi(\phi(\varrho)/\delta^2) \exp(-\phi(\varrho)/\delta^2).$$

First, consider the support of q. In $\operatorname{supp} q$ one has

$$\|^{\|}\varrho - {}^{\|}\varrho^0\|^2 \leq 3\delta^2, \qquad \phi(\varrho) \leq \delta^2. \tag{6.13}$$

In $\operatorname{supp} q \cap \operatorname{Char} p$, by Lemma 6.8, from (6.13) one has $|\operatorname{H}_p z(\varrho)| \lesssim \delta^{1/2}$. Consequently, for δ chosen sufficiently small one has $\operatorname{supp} q \cap \operatorname{Char} p \subset V^0$, meaning that

$$\operatorname{H}_p \phi(\varrho) \leq -C_0 < 0 \quad \text{if } \varrho \in \operatorname{supp} q \cap \operatorname{Char} p, \tag{6.14}$$

by (6.12).

Second, compute $\operatorname{H}_p q = h_1 + h_2 + h_3$ with

$$\begin{aligned}
h_1(\varrho) &= 2\delta^{-2}\big(\psi'\psi\big)(\phi_0(\varrho)/\delta^2)\ \chi(\phi(\varrho)/\delta^2)\ \exp(-\phi(\varrho)/\delta^2)\operatorname{H}_p \phi_0(\varrho),\\
h_2(\varrho) &= \delta^{-2}\psi^2(\phi_0(\varrho)/\delta^2)\ \chi'(\phi(\varrho)/\delta^2)\ \exp(-\phi(\varrho)/\delta^2)\operatorname{H}_p \phi(\varrho),\\
h_3(\varrho) &= -\delta^{-2} q(\varrho)\operatorname{H}_p \phi(\varrho).
\end{aligned}$$

Consider the support of h_1. Since $\operatorname{supp}\psi' \cap \mathbb{R}_+ \subset [2,3]$, one has $2\delta^2 \leq \phi_0(\varrho) \leq 3\delta^2$ in $\operatorname{supp} h_1$. As $\phi(\varrho) \leq \delta^2$ by (6.13), one concludes that

$$\operatorname{H}_p z(\varrho) = \phi_0(\varrho) - \phi(\varrho) \geq \delta^2 \quad \text{if } \varrho \in \operatorname{supp}\big(h_1\big).$$

With the functions h_1 and h_2 defined above one has

$$\langle {}^t\operatorname{H}_p \mu - f\mu, q\rangle = \langle \mu, h_1\rangle + \langle \mu, h_2\rangle + \langle \mu, h_3 - fq\rangle. \tag{6.15}$$

Below, we prove the following lemma.

Lemma 6.12. *One has $\langle {}^t\operatorname{H}_p \mu - f\mu, q\rangle \leq 0$, $\langle \mu, h_2\rangle > 0$, and $\langle \mu, h_3 - fq\rangle \geq 0$ for δ chosen sufficiently small.*

From (6.15) and this lemma one concludes that $\langle \mu, h_1\rangle < 0$, meaning that there exists $\varrho_f \in \operatorname{supp}\mu \cap \operatorname{supp} h_1$. The above analysis yields ϱ_f as in the statement of the proposition. □

Proof of Lemma 6.12. First, consider the action of μ on q. With (3.1) in Assumption 3.2 one has

$$\langle {}^t\operatorname{H}_p \mu - f\mu, q\rangle = -\int_{\varrho \in {}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial} \frac{q(\varrho^+) - q(\varrho^-)}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}}\, d\nu(\varrho).$$

Since the measure ν is nonnegative and since $\overline{{}^{\|}\mathcal{H}_\partial} = {}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial$ it suffices to prove that the integrand is nonnegative on ${}^{\|}\mathcal{H}_\partial$ to conclude that $\langle {}^t\operatorname{H}_p \mu - f\mu, q\rangle \leq 0$.

Consider $\varrho \in {}^{\|}\mathcal{H}_\partial$. By Lemma 5.7 one has

$$\langle \xi^+ - \xi^-, \mathsf{n}_x \rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}} = 2\lambda > 0.$$

Next, as $\varrho^\pm \in \mathcal{H}_\partial^\pm$, one has $\mathrm{H}_p\, z(\varrho^+) - \mathrm{H}_p\, z(\varrho^-) > 0$ from Definition 5.8. As $\phi_0(\varrho^+) = \phi_0(\varrho^-) = \phi_0(\varrho)$ one finds

$$\phi(\varrho^+) - \phi(\varrho^-) = -\,\mathrm{H}_p\, z(\varrho^+) + \mathrm{H}_p\, z(\varrho^-) < 0.$$

Using now that $s \mapsto \tilde\chi(s) = \chi(s)\exp(-s)$ is a nonincreasing function, one obtains

$$q(\varrho^+) - q(\varrho^-) = \psi^2(\phi_0(\varrho)/\delta^2)\Big(\tilde\chi(\phi(\varrho^+)/\delta^2) - \tilde\chi(\phi(\varrho^-)/\delta^2)\Big) \geq 0,$$

implying that $\langle {}^t\mathrm{H}_p\, \mu - f\mu, q\rangle \leq 0$.

Second, consider the action of μ on h_2. Since $\chi' \leq 0$ and $\mathrm{H}_p\, \phi \leq 0$ in $\operatorname{supp} q \cap \operatorname{Char} p$ by (6.14) one finds that $h_2 \geq 0$. Consider now the value of h_2 at ϱ^0:

$$\begin{aligned} h_2(\varrho^0) &= \delta^{-2}\psi^2(\phi_0(\varrho^0)/\delta^2)\ \chi'(\phi(\varrho^0)/\delta^2)\ \exp(\phi(\varrho^0)/\delta^2)\ \mathrm{H}_p\, \phi(\varrho^0) \\ &= \delta^{-2}\psi^2(0)\chi'(0)\,\mathrm{H}_p\, \phi(\varrho^0). \end{aligned}$$

Since $\psi(0) = 1$, $\chi'(0) < 0$ and $\mathrm{H}_p\, \phi(\varrho^0) < 0$ by (6.14) one obtains $h_2(\varrho^0) > 0$. Since $\varrho^0 \in \operatorname{supp}\mu$ and μ is nonnegative, one concludes that $\langle \mu, h_2\rangle > 0$.

One has $h_3 - fq = (-\delta^{-2}\,\mathrm{H}_p\, \phi(\varrho) - f)q$. With (6.14) one finds that $-\delta^{-2}\,\mathrm{H}_p\, \phi(\varrho) - f \geq 0$ in $\operatorname{supp}\mu$ for δ chosen sufficiently small. This gives $\langle \mu, h_3 - fq\rangle \geq 0$. □

With Proposition 6.11 one has the following result.

Proposition 6.13. *Suppose $\varrho^0 \in \mathcal{G}_\partial^{\mathrm{d}} \cap \operatorname{supp}\mu$. There exist $S > 0$ and a local bicharacteristic above $\hat{\mathcal{L}}$, $\gamma : [-S, S] \to T^*\hat{\mathcal{L}}$, such that $\gamma(0) = \varrho^0$, $\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \neq 0$, and*

$$\Gamma = \{\gamma(s);\ s \in [-S, S]\} \subset \operatorname{supp}\mu.$$

Proof. In a local chart consider a ball $\mathcal{B}$ of radius R centered at ϱ^0 where $\mathrm{H}_p^2\, z \geq C > 0$. There exist c_0, C_0 both positive such that

$$c_0 \leq \|\,\mathrm{H}_p(\varrho)\| \leq C_0, \quad \varrho \in \mathcal{B}.$$

For $\delta > 0$ chosen small, by Proposition 6.11 there exists $\varrho_\delta \in \mathcal{B} \cap \operatorname{supp}\mu \cap \mathcal{H}^+$ such that

$$\|^{\|}\varrho_\delta - {}^{\|}\varrho^{0\prime}\| \lesssim \delta, \qquad \delta^2 \lesssim \mathrm{H}_p\, z(\varrho) \lesssim \delta^{1/2}.$$

(This point is denoted ϱ_f in Proposition 6.11.) Consequently, for some $C_1 > 0$ one has

$$\|\varrho_\delta - \varrho^0\| \leq C_1 \delta^{1/2}.$$

One either has $\varrho_\delta \in \mathcal{H}^+_\partial$ or $\varrho_\delta \in \mathcal{H}^+ \setminus \partial(T^*\mathcal{L})$. In either case, by Propositions 6.1 and 6.6 there exists a maximally extended bicharacteristic γ_δ in $\mathcal{B} \cap T^*\mathcal{L}$, defined on a interval of the form $]S^-_\delta, S^+_\delta[$ that is moreover contained in $\operatorname{supp}\mu$ and such that $\gamma_\delta(0) = \varrho_\delta$. We only consider this bicharacteristic on the interval $[0, S^+_\delta[$. Since $\mathrm{H}^2_p\, z > 0$ in $\mathcal{B}$, then $\mathrm{H}_p\, z(\gamma_\delta(s))$ increases as $s \in [0, S^+_\delta[$ increases. Since $\mathrm{H}_p\, z(\varrho_\delta) > 0$ one finds that $\mathrm{H}_p\, z$ remains positive and hence z increases along the bicaracteristic. Consequently, if $S^+_\delta < \infty$ then the maximaly extended bicharacteristic leaves $\mathcal{B} \cap T^*\mathcal{L}$ for $s = S^+_\delta$, yet not through the boundary $\{z = 0\}$, that is, $\partial(T^*\mathcal{L})$.

Choose $\delta_0 > 0$ such that $C_1 \delta_0^{1/2} < R/2$. Then, for $0 < \delta \leq \delta_0$ one has $\|\varrho_\delta - \varrho^0\| < R/2$ and $\operatorname{dist}(\varrho_\delta, \partial\mathcal{B}) > R/2$ implying that $S^+_\delta > R/(2C_0)$. Set $S = R/(2C_0)$.

One has

$$\gamma_\delta(s) = \varrho_\delta + \int_0^s \mathrm{H}_p(\gamma_\delta(\sigma))d\sigma, \qquad s \in [0, S].$$

If one lets δ vary in $]0, \delta_0]$, the set of bicharacteristics $\gamma_\delta(s)$ is equicontinuous on $[0, S]$. For $\delta \to 0$, by the Arzelà-Ascoli theorem one can extract a subsequence that converges uniformly to a curve $\gamma(s)$ with $[0, S]$. From the continuity of H_p, one has

$$\gamma(s) = \varrho^0 + \int_0^s \mathrm{H}_p(\gamma(\sigma))d\sigma, \qquad s \in [0, S],$$

that is, a bicharacteristic that goes through ϱ^0 at $s = 0$. Moreover, $\gamma(s) \in \operatorname{supp}\mu$ for $s \in [0, S]$ since $\operatorname{supp}\mu$ is a closed set. Arguing as above one has $z > 0$ along this bicharacteristic if $s > 0$.

The argument can be used *mutatis mutandis* to construct the sought bicharacteristic for $s \in [-S, 0[$. □

6.4. **Final construction of generalized bicharacteristic in the measure support.** All possible cases listed in the beginning of Section 6 need to be considered. Cases are then used sequentially, in various orders, depending on the generalized bicharacteristic that is constructed along the proof.

Case 1: ϱ^0 is away from the boundary. if $\varrho^0 \in \operatorname{supp}\mu \cap \big(T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})\big)$ and if one sets $V = \mathscr{U} \cap T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$, Proposition 6.1 yields a maximal bicharacteristic $\gamma(s)$, with $s \in J =]S_1, S_2[$ that lies in $\operatorname{supp}\mu \cap V$ and such that $\gamma(0) = \varrho^0$.

One says that γ leaves $\mathscr{U}$ at $s = S_2^-$ if $S_2 < \infty$ and the limit point at $s = S_2^-$ is in $\partial\mathscr{U}$ (see Lemma 5.24), with the same notation at $s = S_1^+$. If, on the one hand, either $S_1 = -\infty$ or γ leaves $\mathscr{U}$ at $s = S_1^+$, and, on the other hand, $S_2 = +\infty$ or γ leaves $\mathscr{U}$ at $s = S_2^-$, then one has obtained a maximal generalized bicharacteristic contained in $\operatorname{supp}\mu$ that goes through ϱ^0. If however, for instance $S_2 < +\infty$ and γ does not leave $\mathscr{U}$ at $s = S_2^-$, then $\varrho^1 = \lim_{s \to S_2^-} \gamma(s)$ exists by Lemma 5.24 and moreover $\varrho^1 \in \mathcal{H}_\partial^- \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{d}} \setminus \partial\mathscr{U}$. Since $\operatorname{supp}\mu$ is a closed subset of $\mathscr{U}$ one has $\varrho^1 \in \operatorname{supp}\mu$. Note that if $\varrho^1 \in \mathcal{G}_\partial^{\mathrm{d}}$ one can set $\gamma(S_2) = \varrho^1$ and one has $\frac{d}{ds}\gamma(S_2) = \mathrm{H}_p(\varrho^1)$. Now if $\varrho^1 \in \mathcal{G}_\partial^3$ one can also set $\gamma(S_2) = \varrho^1$ and one has $\frac{d}{ds}\gamma(S_2) = \mathrm{H}_p(\varrho^1) = \mathrm{H}_p^{\mathcal{G}}(\varrho^1)$ by Lemma 5.18 as $\mathrm{H}_p^2\, z(\varrho^1) = 0$.

One may then consider Cases 2, 3 and 4 below to carry on the construction of a generalized bicharacteristic contained in $\operatorname{supp}\mu$, with ϱ^1 being now the point where one initiates this generalized bicharacteristic.

Naturally, the same reasoning is applied in the case $-\infty < S_1$ and γ does not leave $\mathscr{U}$ at $s = S_1^+$. Then $\varrho^1 = \lim_{s \to S_1^+} \gamma(s) \in \operatorname{supp}\mu \cap (\mathcal{H}_\partial^+ \cup \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{d}}) \setminus \partial\mathscr{U}$.

Case 2: ϱ^0 is a hyperbolic point. Consider $\varrho^0 \in \operatorname{supp}\mu \cap \mathcal{H}_\partial$. Then $\Sigma(\varrho^0) \in \operatorname{supp}\mu \cap \mathcal{H}_\partial$ by Proposition 6.3. Set $\varrho^{0\pm} \in \mathcal{H}^\pm$ so as to have $\{\varrho^{0-}, \varrho^{0+}\} = \{\varrho^0, \Sigma(\varrho^0)\}$. By Proposition 6.6, there exists $S > 0$ and a locally defined broken bicharacteristic ${}^{\mathsf{B}}\gamma : [-S, 0[\cup]0, S] \to T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ contained in $\operatorname{supp}\mu$ such that

$$\lim_{s \to 0^+} {}^{\mathsf{B}}\gamma(s) = \varrho^{0+} \quad \text{and} \quad \lim_{s \to 0^-} {}^{\mathsf{B}}\gamma(s) = \varrho^{0-}.$$

Moreover ${}^{\mathsf{B}}\gamma(-S), {}^{\mathsf{B}}\gamma(S) \in \operatorname{supp}\mu \cap T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$, that is, the constructed local broken bicharacteristic yields endpoints away from the boundary. To carry on with the construction of the generalized bicharacteristic, one needs then to consider Case 1.

Case 3: ϱ^0 is a diffractive point. Consider $\varrho^0 \in \operatorname{supp}\mu \cap \mathcal{G}_\partial^{\mathrm{d}}$. One may then apply Proposition 6.13: there exist $S > 0$ and a local bicharacteristic above $\hat{\mathcal{L}}$, $\gamma : [-S, S] \to T^*\mathcal{L}$, contained in $\operatorname{supp}\mu$ such that $\gamma(0) = \varrho^0$, $\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \neq 0$. In particular, $\gamma(-S)$ and $\gamma(S)$ are in $\operatorname{supp}\mu \cap T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$. To carry

on with the construction of the generalized bicharacteristic, one needs then to consider Case 1.

Intermezzo: Construction of a maximal broken bicharacteristic. Observe that the construction proposed up to this point may imply going back and forth between Cases 1, 2 and 3 if no point in $\operatorname{supp}\mu \cap \mathcal{G}^3_\partial$ is reached. One then obtains a broken bicharacteristic that lies in $\operatorname{supp}\mu$. One sees that the existence of such a broken bicharacteristic yields the existence of a maximal broken bicharacteristic contained in $\operatorname{supp}\mu$ by means of classical arguments; see for example [11].

Proposition 6.14. *Suppose $\varrho^0 \in \operatorname{supp}\mu \cap T^*\mathcal{L} \setminus (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$. There exists a maximal broken bicharacteristic $s \mapsto {}^{\mathsf{B}}\gamma(s)$ defined for $s \in J \setminus B$ with $0 \in J =]S_1, S_2[$ and B as in Definition 5.26 and such that*

1. *if $\varrho^0 \notin \mathcal{H}_\partial$ then $0 \notin B$ and ${}^{\mathsf{B}}\gamma(0) = \varrho^0$;*
2. *if $\varrho^0 \in \mathcal{H}^+_\partial$ (resp. $\mathcal{H}^-_\partial$) then $0 \in B$ and ${}^{\mathsf{B}}\gamma(0^+) = \lim_{s\to 0^+} {}^{\mathsf{B}}\gamma(s) = \varrho^0$ (resp. ${}^{\mathsf{B}}\gamma(0^-) = \lim_{s\to 0^-} {}^{\mathsf{B}}\gamma(s) = \varrho^0$);*
3. *for all $s \in J \setminus B$, ${}^{\mathsf{B}}\gamma(s) \in \operatorname{supp}\mu$;*
4. *for all $S \in B$, ${}^{\mathsf{B}}\gamma(S^\pm) = \lim_{s\to S^\pm} {}^{\mathsf{B}}\gamma(s) \in \operatorname{supp}\mu \cap \mathcal{H}^\pm_\partial$.*

If, on the one hand, either $S_1 = -\infty$ or γ leaves $\mathscr{U}$ at $s = S_1^+$, and, on the other hand, $S_2 = +\infty$ or γ leaves $\mathscr{U}$ at $s = S_2^-$, then one has obtained a maximal generalized bicharacteristic contained in $\operatorname{supp}\mu$ that goes through ϱ^0. If $\mathscr{U} = T^*\hat{\mathcal{L}}$ then $J = \mathbb{R}$.

Consider now for instance the case $S_2 < +\infty$ and γ does not leave $\mathscr{U}$ at $s = S_2^-$. If $S_2 \notin \overline{B}$ then ${}^{\mathsf{B}}\gamma$ is a maximal bicharacteristic near S_2 implying that $\varrho^1 = \lim_{s\to S_2^-} {}^{\mathsf{B}}\gamma(s)$ exists and belongs to $\operatorname{supp}\mu \cap (\mathcal{H}^-_\partial \cup \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{d}}_\partial)$ by Lemma 5.24. One can discard having $\varrho^1 \in \mathcal{H}^-_\partial \cup \mathcal{G}^{\mathrm{d}}_\partial$ since Cases 2 and 3 above allow one to further extend the broken bicharacteristic contained in $\operatorname{supp}\mu$ contradicting its maximality. Thus, if $S_2 \notin \overline{B}$ one has $\varrho^1 \in \operatorname{supp}\mu \cap \mathcal{G}^3_\partial$.

If now $S_2 \in \overline{B}$ then $\varrho^1 = \lim_{s\to S_2^-} {}^{\mathsf{B}}\gamma(s)$ exists by Lemma 5.28 with moreover $\varrho^1 \in \operatorname{supp}\mu \cap (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$.

Starting from $\varrho^1 \in \operatorname{supp}\mu \cap (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$ obtained in either cases, one now considers Case 4 below to carry on the construction of a generalized bicharacteristic contained in $\operatorname{supp}\mu$, with ϱ^1 being now the point where one initiates the generalized bicharacteristic.

Naturally, the same reasoning is applied in the case $-\infty < S_1$ and γ does not leave $\mathscr{U}$ at $s = S_1^+$. Then, $\varrho^1 = \lim_{s\to S_1^+} {}^{\mathsf{B}}\gamma(s) \in \operatorname{supp}\mu \cap (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$.

Case 4: ϱ^0 is an order-3-glancing point or a gliding point.

The following lemma gives the existence of local generalized bicharacteristic that goes through a point in $\operatorname{supp}\mu \cap (\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$.

Lemma 6.15. *There exists $S_0 > 0$ such that for any $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \operatorname{supp}\mu \cap (\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$ there is a generalized bicharacteristic ${}^{\mathsf{G}}\gamma : J \setminus B \to T^*\mathcal{L}$ where $J = [-S, S]$, with $S = S_0/|\tau^0|$, and B a discrete subset of J, and such that ${}^{\mathsf{G}}\gamma(0) = \varrho^0$ and ${}^{\mathsf{G}}\bar{\gamma} \subset \operatorname{supp}\mu$.*

We recall that the notation ${}^{\mathsf{G}}\bar{\gamma}$ is intoduced in Definition 2.9. The proof of Lemma 6.15 is quite lengthy. We thus rather first conclude the proof of Theorem 3.4 and below, in Section 6.5, we proceed with the proof of Lemma 6.15.

Conclusion of the construction of a maximal generalized bicharacteristic.

With the four cases treated above, given $\varrho^0 \in \operatorname{supp}\mu \subset \mathscr{U} \cap T^*\mathcal{L}$ there exists a local generalized bicharacteristic that goes through this point (with the understanding of a limit if the point $\varrho^0 \in \mathcal{H}_\partial$) and is contained in $\operatorname{supp}\mu$. This yields the existence of a maximal generalized bicharacteristic with the same properties by means of classical arguments; see for example [11].

Suppose now that $\mathscr{U} = T^*\hat{\mathcal{L}}$ and that ${}^{\mathsf{G}}\gamma(s)$ is such a maximal generalized bicharacteristic defined for $s \in J \setminus B$ with $0 \in J =]S_1, S_2[$ and B as in Definition 5.32. Suppose that $S_2 < +\infty$. Then Lemma 5.38 implies that the limit

$$\varrho^1 = \lim_{\substack{s \to S_2 \\ s \in J \setminus B}} {}^{\mathsf{G}}\gamma(s)$$

exists and is in $\operatorname{supp}\mu$ as it is a closed set and moreover

$$\frac{d}{ds}{}^{\mathsf{G}}\gamma(S_2^-) = {}^{\mathsf{G}}X(\varrho^1).$$

Yet, with the above argument there exists a local generalized bicharacteristic that goes through ϱ^1 allowing one to extend ${}^{\mathsf{G}}\gamma(s)$ for $s > S_2$ contradicting its maximality: thus $S_2 = +\infty$. The same reasonning gives $S_1 = -\infty$. This concludes the proof of Theorem 3.4. □

6.5. **Local construction for a point in $\mathcal{G}_\partial^3$ or $\mathcal{G}_\partial^{\mathrm{g}}$.** In this section we prove Lemma 6.15.

First, we construct locally a continuous curve ${}^{\mathsf{G}}\gamma$ that goes through ϱ^0 and is contained in $\operatorname{supp}\mu$ and, second, we prove that it is a generalized bicharacteristic.

Local setting

For any $x \in \partial\mathcal{M}$ and $R > 0$ consider the closed Riemannian ball $B_g(x, R) = \{\tilde{x} \in \mathcal{M};\ \mathrm{dist}_g(x, \tilde{x}) \leq R\}$. In fact, since $\mathcal{M}$ is compact, one can choose $R_0 > 0$ sufficiently small so that, for any $x \in \partial\mathcal{M}$ there exists a local chart (O, ϕ) of $\mathcal{M}$ such that $B_g(x, R_0) \subset O$.

Suppose now $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$ and (O, ϕ) is a local chart chosen as above, that is, such that $B_g(x^0, R_0) \subset O$. One has $|\xi^0|_{x^0} = |\tau^0|$ since $\varrho^0 \in \mathrm{Char}\, p$. If a generalized bicharacteristic ${}^{\mathsf{G}}\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$ going through ϱ^0 at $s = 0$ is constructed then $\tau(s) = \tau^0$ and thus $|\xi(s)|_{x(s)} = |\tau^0|$ as pointed out in Remark 5.33. Thus consider the following compact set

$$K^0 = \big\{\varrho = (t, x, \tau, \xi) \in T^*\mathcal{L}; x \in B_g(x^0, R_0),\ |\tau| + |\xi|_x \leq 4|\tau^0|\big\}.$$

The constructed generalized bicharacteristic ${}^{\mathsf{G}}\gamma(s)$ will be in K^0 for $|s|$ sufficiently small. Note that "room" is made in K^0 in the cotangent directions for an interative process to be carried out while remaining in K^0.

In the local coordinates associated with (O, ϕ) the hamilitonian vector field H_p is as given in (2.3). The componant acting in the x-directions is given by $v(\varrho) = 2g^{ij}\xi_i\partial_{x_j}$. Observe that the same holds for the gliding vector field $\mathrm{H}^{\mathcal{G}}_p$ by (5.20). Because of the form of K^0 there exists $C_0 > 0$ such that

$$\|v(\varrho)\| \leq C_0|\tau^0| \quad \text{if } \varrho \in K^0.$$

Thus, if for some $s \in \mathbb{R}$ one has $\mathrm{dist}_g(x^0, x(s)) \geq R_0$ then $|s| \geq R_0/(C_0|\tau_0|)$. Thus, set $S_{\max} = S_0/|\tau_0|$ with $S_0 = R_0/(2C_0)$. The generalized bicharacteristic is construced in an iterative process for $s \in [-S_{\max}, S_{\max}]$ and the choice of $S_{\max}$ ensures that one remains in K^0 within that process.

Also, since one remains in K^0, the same local chart (O, ϕ) can be used. We also use local coordinates as (u, z, ϑ) as introduced in the proof of Proposition 6.9. We assume that it can be used in the whole local chart. This can be assumed from the beginning by refining the atlas. Recall that (u, z) provides coordinates for ${}^{\mathbb{I}}T^*\mathcal{L}$ with $u \in \mathbb{R}^{2d}$ and $\vartheta = \mathrm{H}_p z$. Below, we will go back and forth between the (t, x, τ, ξ) and (u, z, ϑ) coordinates. By abuse of notation we write $\varrho = (u, z, \vartheta)$ or $\varrho = (t, x, \tau, \xi)$. Here, our starting point is of the forms $\varrho^0 = (t^0, x^1, \tau^1, \xi^1)$ and $\varrho^0 = (u^0, z = 0, \vartheta = 0)$.

For a point ϱ^ℓ, $\ell \in \mathbb{N}$, constructed below $(t^\ell, x^\ell, \tau^\ell, \xi^\ell)$ and $(u^\ell, z^\ell, \vartheta^\ell)$ refer to its coordinates in the two variable systems.

Construction of ${}^{\mathsf{G}}\boldsymbol{\gamma}$

Consider $n \in \mathbb{N}^*$ and $\varepsilon = 1/n$. Use $\delta_0 > 0$ as given by Proposition 6.9 using $K = K^0$ therein. Set $\delta_n = \min(\delta_0, 1/n)$. In the local coordinates (u, z, ϑ) we construct a *piecewise* continuous curve γ_n initiated at ϱ^0.

Consider $\varrho^1 \in \operatorname{supp}\mu$ as given by Proposition 6.9 with $\delta = \delta_n$ therein. Since $\mathrm{H}_p^{\mathcal{G}}\,\tau = 0$ note that in the (t, x, τ, ξ) coordinates one has

$$|\tau^1 - \tau^0| \leq \varepsilon\delta_n \leq 1/n^2. \tag{6.16}$$

Now, in the (u, z, ϑ) coordinates, one defines the following affine curve

$$\gamma_n(s) = \varrho^0 + \frac{s}{\delta_n}(\varrho^1 - \varrho^0) \quad \text{for } s \in [0, \delta_n] \quad \text{and} \quad S_1 = \delta_n.$$

One then faces two options to further construct γ_n.

(1) If $\varrho^1 \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$, like ϱ^0, one picks a second point ϱ^2 also according Proposition 6.9, yet starting from ϱ^1, and one further constructs γ_n on the interval $[S_1, S_1 + \delta_n]$ in some affine manner as above.

$$\gamma_n(s) = \varrho^1 + \frac{s - S_1}{\delta_n}(\varrho^2 - \varrho^1) \quad \text{for } s \in [S_1, S_1 + \delta_n] \quad \text{and} \quad S_2 = S_1 + \delta_n.$$

One carries on with this iteration yielding points $\varrho^1, \dots, \varrho^k \in \operatorname{supp}\mu \cap (\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$ and $S_\ell = \ell\delta_n$, $\ell = 1, \dots, k$, until either $S_k > S_{\max}$, meaning one is done with the construction of γ_n for $s \in [0, S_{\max}]$, or $\varrho^k \in \operatorname{supp}\mu \cap T^*\mathcal{L} \setminus (\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$, in which case one turns to the second construction option just below.

Observe that $(k-1)\delta_n \leq S_{\max}$ and iterating estimate (6.16) one has

$$|\tau^\ell - \tau^0| \leq \varepsilon\ell\delta_n \leq \varepsilon S_{\max} + \varepsilon\delta_n = S_0/(n|\tau^0|) + 1/n^2, \quad \ell = 1, \dots, k. \tag{6.17}$$

This means that for n chosen sufficiently large one has $\varrho^1, \dots, \varrho^k \in K^0$.

(2) If after one or serveral steps using the above piecewise affine construction $\varrho^\ell \in \operatorname{supp}\mu \cap T^*\mathcal{L} \setminus (\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})$ one constructs a maximal broken bicharacteristic in K^0 initiated at $\varrho^\ell \in K^0$ according to Proposition 6.14. Maximality is only understood in the future here, that is, for $s \geq S_\ell$: this maximal broken bicharacteristic is defined on $[S_\ell, S_{\ell+1}[\setminus B_{[S_\ell, S_{\ell+1}[}$, with $B_{[S_\ell, S_{\ell+1}[}$ a discrete subset of $[S_\ell, S_{\ell+1}[$. It may happen that $S_\ell \in B_{[S_\ell, S_{\ell+1}[}$ if $\varrho^\ell \in \mathcal{H}_\partial$: then the maximal broken bicharacteristic enters $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ through ϱ^ℓ if $\varrho^\ell \in \mathcal{H}_\partial^+$ or $\Sigma(\varrho^\ell)$ if $\varrho^\ell \in \mathcal{H}_\partial^-$ (see Case 2 in Section 6.4).

Two instances may occur: (a) the maximal broken bicharacteristic is such that $S_{\ell+1} > S_{\max}$ and one is done with the construction of γ_n, or (b) it reaches a point $\varrho^{\ell+1} \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ at $s = S_{\ell+1}$. Starting, from this point $\varrho^{\ell+1}$ one reinitiates the construction with the first option above until one reaches $s = S_{\max}^-$. Note that τ remains constant along the

broken bicharacteristic just constructed. Hence, with (6.17) one has the estimate

$$|\tau^{\ell+1} - \tau^0| = |\tau^\ell - \tau^0| \leq \varepsilon \ell \delta_n.$$

Note that γ_n contains at most $\lfloor S_{\max}/\delta_n \rfloor + 1$ affine pieces given by Proposition 6.9 as in item (1) above. In particular, simlarly to (6.17) one finds that

$$|\tau^k - \tau^0| \leq \varepsilon k \delta_n \leq \varepsilon S_{\max} + \varepsilon \delta_n \leq S_0/(n|\tau^0|) + 1/n^2,$$

for any ϱ^k that is an endpoint of an affine piece constructed above. Thus, for n chosen sufficiently large the constructed curve γ_n remains in K^0 as announced above.

With the above construction of the curve γ_n one obtains an alternating sequence of affine pieces and maximal broken bicharacteristics. Following a part of γ_n made by a broken bicharacteristic, one finds an affine part (unless that broken bicharacteristic ends the construction of γ_n). Hence, the number of broken bicharacteristics that constitutes γ_n is also finite. Denote by m_n this number. One has $m_n \leq \lfloor S_{\max}/\delta_n \rfloor + 1$. If broken bicharacteristics compose γ_n, that is, if $m_n \geq 1$, set $M_n = \{1, \ldots, m_n\}$ and for each broken bicharacteristic set $B_{n,j}$, $j \in M_n$, to be the discrete set of points s where the jth broken bicharacteristic is discontinuous, that is, at hyperbolic points. Each point of $B_{n,j}$ is isolated. Yet, if $\#B_{n,j} = \infty$, points of $B_{n,j}$ accumulate to some $s \notin B_{n,j}$. In such case, recall that $\gamma_n(s^-) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$ (see Lemma 5.28). Define $B_n = \cup_{j \in M_n} B_{n,j}$. It contains all the points where γ_n is discontinuous (if any) corresponding to hyperbolic points at the boundary.

The jth broken bicharacteristic is defined for $s \in [S, S'[\backslash B_{n,j}$. Set $\sigma^0_{n,j} = S$. It may happen that $\sigma^0_{n,j} \in B_{n,j}$. Index the (at most countable) other ordered elements of $B_{n,j}$ as follows:

$$\sigma^0_{n,j} < \sigma^1_{n,j} < \sigma^2_{n,j} < \cdots < \sigma^\ell_{n,j} < \cdots,$$

with $1 \leq \ell \leq L_{n,j} = \#B_{n,j}$. If $L_{n,j} = +\infty$ set

$$\sigma^\infty_{n,j} = \sup_{0 \leq \ell \leq L_{n,j}} \sigma^\ell_{n,j}.$$

If $L_{n,j} < \infty$ set also $\sigma^{L_{n,j}+1}_{n,j} = \sigma^\infty_{n,j}$ to be the value s of the endpoint of the jth broken bicharacteristic. Using the index value $L_{n,j} + 1$ can be useful in summations in what follows. With the maximal broken bicharacteristic defined on $[S, S'[\backslash B_{n,j}$ one has $\sigma^\infty_{n,j} = S'$. If $j < m_n$ then $\sigma^\infty_{n,j} < S_{\max}$. Note however that

one may have $\sigma^\infty_{n,m_n} > S_{\max}$ since the m_nth maximal broken bicharacteristic may carry one beyond $s = S_{\max}$. One has

$$\sigma^0_{n,1} < \sigma^1_{n,1} < \cdots < \sigma^\infty_{n,1} < \sigma^0_{n,2} < \cdots < \sigma^\infty_{n,2} < \sigma^0_{n,3} < \cdots \\ \cdots < \sigma^\infty_{n,m_n-1} < \sigma^0_{n,m_n} < \cdots < \sigma^\infty_{n,m_n}.$$

As mentionned above one has

$$\gamma_n\big(\sigma^\infty_{n,j}\big) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial, j = 1, \dots, m_n.$$

For $0 \le s_1 \le s_2 \le S_{\max}$ set

$$M_n^{[s_1,s_2]} = \big\{ j \in M_n;\ s_1 < \sigma^\infty_{n,j} \text{ and } \sigma^0_{n,j} < s_2 \big\},$$

meaning that on the interval $[s_1, s_2]$ one encounters the jth broken bicharacteristics for $j \in M_n^{[s_1,s_2]}$. Note that $M_n^{[s_1,s_2]}$ may be empty.

The curve γ_n is differentiable away from endpoints of affine parts and away from points in B_n. Write $\gamma_n(s) = ({}^{\|}\gamma_n(s), \vartheta_n(s))$ using the notation of Section 5.3, and using the (u, z) coordinates for ${}^{\|}\gamma_n(s) \in {}^{\|}T^*\mathcal{L}$.

Now set

$${}^c\gamma_n(s) = \begin{cases} {}^c\phi\big(\gamma_n(s)\big) = \gamma_n(s) & \text{if } s \notin B_n \\ {}^c\phi\big(\lim_{s' \to s^-} \gamma_n(s)\big) = {}^c\phi\big(\lim_{s' \to s^+} \gamma_n(s)\big) & \text{if } s \in B_n. \end{cases}$$

The map ${}^c\phi$ is defined in Section 5.5. The curve ${}^c\gamma_n(s)$ is continuous with values in the compressed cotangent bundle ${}^cT^*\mathcal{L}$. Our next goal is to prove the equicontinuity of $s \mapsto {}^c\gamma_n(s) = {}^c\phi\big(\gamma_n(s)\big)$ for $n \in \mathbb{N}^*$ chosen and $s \in [0, S_{\max}]$.

On a piece of γ_n given by a maximal bicharacteristic (within a broken bicharacteristic), say on $[s_1, s_2]$, one has

$$\gamma_n(s) = \gamma_n(s_1^+) + \int_{s_1}^{s} \mathrm{H}_p\big(\gamma_n(\sigma)\big)\, d\sigma. \tag{6.18}$$

This yields

$$\big\|{}^{\|}\gamma_n(s_2') - {}^{\|}\gamma_n(s_1')\big\| + \big|\vartheta_n(s_2') - \vartheta_n(s_1')\big| \lesssim s_2' - s_1', \qquad s_1 \le s_1' \le s_2' \le s_2. \tag{6.19}$$

On a piece given by an affine part of the construction, say on $[s_1, s_1 + \delta_n]$, one has, from Proposition 6.9

$$\frac{d}{ds}{}^{\|}\gamma_n(s) = \mathrm{H}_p^{\mathcal{G}}\big(\gamma_n(s_1)\big) + \mathcal{O}(1/n) = \mathrm{H}_p^{\mathcal{G}}\big(\gamma_n(s)\big) + {}^{\|}e_n(s),$$

where the errors $|^{\shortparallel}e_n|$ goes to zero *uniformly* as $n \to +\infty$ by the uniform continuity of $\mathrm{H}_p^{\mathcal{G}}$ in K^0,

$$|\vartheta_n(s)| \lesssim (s - s_1)\delta_n^{1/2},$$

using that $\vartheta_n(s_1) = 0$. One thus finds

$$^{\shortparallel}\gamma_n(s) = {}^{\shortparallel}\gamma_n(s_1) + \int_{s_1}^{s} \mathrm{H}_p^{\mathcal{G}}\left(\gamma_n(\sigma)\right) d\sigma + \int_{s_1}^{s} {}^{\shortparallel}e_n(\sigma)\, d\sigma. \tag{6.20}$$

Since $s \mapsto {}^{\shortparallel}\gamma_n(s)$ is continuous, with (6.18)–(6.20) one obtains

$$\left\| |^{\shortparallel}\gamma_n(s_2) - {}^{\shortparallel}\gamma_n(s_1) \right\| \lesssim s_2 - s_1, \qquad 0 \le s_1 \le s_2 \le S_{\max}, \tag{6.21}$$

uniformly with respect to $n \in \mathbb{N}^*$.

We now proceed with an estimation of the variations of $\vartheta_n(s)$. Note that $s \mapsto \vartheta_n(s)$ is not continuous on the parts made with a broken bicharacteristic as hyperbolic points are encountered. For this reason, we are interested by the equicontinuity of $({}^c\gamma_n)_{n\in\mathbb{N}^*}$ and our goal is to obtain an estimation of ${}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right)$, for $0 \le s_1 \le s_2 \le S_{\max}$. We refer to Section 5.5 for the definition of the distance ${}^c\mathrm{dist}(.,.)$.

A first case to be considered is $M_n^{[s_1,s_2]} = \emptyset$. Then, with (6.21) one has

$${}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right) \lesssim \|\gamma_n(s_2) - \gamma_n(s_1)\| \lesssim s_2 - s_1 + Z_{[s_1,s_2]},$$

where

$$Z_{[s_1,s_2]} = |\vartheta_n(s_2) - \vartheta_n(s_1)|,$$

to be estimated in Lemma 6.16 below.

Next, different cases have to be considered if $M_n^{[s_1,s_2]} \neq \emptyset$. In all cases, a simple yet key observation is

$$\vartheta_n(\sigma_{n,j}^{\infty}) = 0 \quad \text{for } j = 1, \dots, m_n, \tag{6.22}$$

since $\gamma_n(\sigma_{n,j}^{\infty}) \in \mathcal{G}_{\partial}^3 \cup \mathcal{G}_{\partial}^{\mathrm{g}}$.

Given $0 \le s_1 \le s_2 \le S_{\max}$, one has $M_n^{[s_1,s_2]} = \{j_1, \dots, j_N\}$ for some $N \le m_n$ and with $1 \le j_1 < \cdots < j_N$. Set

$$\ell_{n,j_1}^{\inf} = \min\{0 \le \ell \le L_{n,j_1} + 1;\ s_1 \le \sigma_{n,j_1}^{\ell}\},$$
$$\ell_{n,j_N}^{\sup} = \sup\{0 \le \ell \le L_{n,j_N} + 1;\ \sigma_{n,j_N}^{\ell} \le s_2\},$$

and

$$\sigma^{\inf}_{n,j_1} = \sigma^{\ell^{\inf}_{n,j_1}}_{n,j_1} \quad \text{and} \quad \sigma^{\sup}_{n,j_N} = \sup\{\sigma^{\ell}_{n,j_N};\ \sigma^{\ell}_{n,j_N} \le s_2\},$$

Recall that $L_{n,j}$ is possibly infinite.

If $\sigma^{\sup}_{n,j_N} \le s_2 < \sigma^{\infty}_{n,j_N}$ one has the estimate, using (5.24)-(5.25),

$$\begin{aligned}
&{}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right)\\
&\quad\lesssim \left\|\gamma_n(\sigma^{\inf,-}_{n,j_1}) - \gamma_n(s_1^+)\right\| + \sum_{\ell^{\inf}_{n,j_1}\le\ell\le L_{n,j_1}} \left\|\gamma_n(\sigma^{\ell+1,-}_{n,j_1}) - \gamma_n(\sigma^{\ell,+}_{n,j_1})\right\|\\
&\qquad + \left\|]\gamma_n(\sigma^{0,-}_{n,j_N}) - \gamma_n(\sigma^{\infty,+}_{n,j_1})\right\|\\
&\qquad + \sum_{0\le\ell\le\ell^{\sup}_{n,j_N}-1} \left\|\gamma_n(\sigma^{\ell+1,-}_{n,j_N}) - \gamma_n(\sigma^{\ell,+}_{n,j_N})\right\| + \left\|\gamma_n(s_2^-) - \gamma_n(\sigma^{\sup,+}_{n,j_N})\right\|.
\end{aligned}$$

With (6.21) and (6.22) one finds

$$ {}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right) \lesssim s_2 - s_1 + Z_{[s_1,s_2]}, \tag{6.23}$$

with $Z_{[s_1,s_2]}$ given by

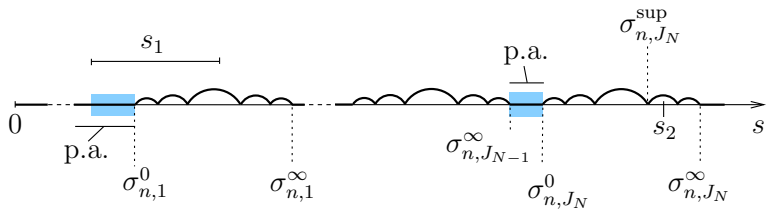
$$\begin{aligned}
Z_{[s_1,s_2]} = {}& \left|\vartheta_n(\sigma^{\inf,-}_{n,j_1}) - \vartheta_n(s_1^+)\right| + \sum_{\ell^{\inf}_{n,j_1}\le\ell\le L_{n,j_1}} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_1}) - \vartheta_n(\sigma^{\ell,+}_{n,j_1})\right|\\
&+ \left|\vartheta_n(\sigma^{0,-}_{n,j_N})\right| + \sum_{0\le\ell\le\ell^{\sup}_{n,j_N}-1} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_N}) - \vartheta_n(\sigma^{\ell,+}_{n,j_N})\right|\\
&+ \left|\vartheta_n(s_2^-) - \vartheta_n(\sigma^{\sup,+}_{n,j_N})\right|.
\end{aligned} \tag{6.24}$$

If $\sigma^{\infty}_{n,j_N} = s_2$, one writes

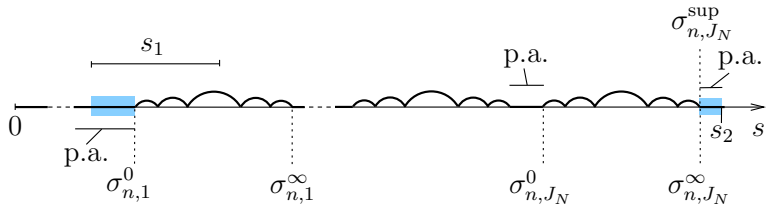
$$\begin{aligned}
&{}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right)\\
&\quad\lesssim \left\|\gamma_n(\sigma^{\inf,-}_{n,j_1}) - \gamma_n(s_1^+)\right\| + \sum_{\ell^{\inf}_{n,j_1}\le\ell\le L_{n,j_1}} \left\|\gamma_n(\sigma^{\ell+1,-}_{n,j_1}) - \gamma_n(\sigma^{\ell,+}_{n,j_1})\right\|\\
&\qquad + \left\|\gamma_n(s_2) - \gamma_n(\sigma^{\infty,+}_{n,j_1})\right\|,
\end{aligned}$$

yielding the same estimate as in (6.23) with now $Z_{[s_1,s_2]}$ given by

$$Z_{[s_1,s_2]} = \left|\vartheta_n(\sigma^{\inf,-}_{n,j_1}) - \vartheta_n(s_1^+)\right| + \sum_{\ell^{\inf}_{n,j_1}\le\ell\le L_{n,j_1}} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_1}) - \vartheta_n(\sigma^{\ell,+}_{n,j_1})\right|. \tag{6.25}$$



(a) Case $s_2 < \sigma^{\infty}_{n,j_N}$; (6.24) applies.



(b) Case $\sigma^{\infty}_{n,j_N} < s_2$; (6.26) applies.

FIGURE 8. Two configuratons of $[s_1, s_2]$ with respect to γ_n; 'p.a.' stands for 'piecewise affine'. A shaded box refers to a location where an estimate making the term $\sqrt{s_2 - s_1}$ appear is used in the proof of Lemma 6.16. Note that discontinuities of broken bicharacteristics at hyperbolic points are not represented here for convenience.

since $\gamma_n(s_2) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$ and thus $\vartheta_n(s_2) = 0$.

If now $\sigma^{\infty}_{n,j_N} < s_2$, one has $\ell^{\sup}_{n,j_N} = L_{n,j_N} + 1$ or equivalently $\sigma^{\sup}_{n,j_N} = \sigma^{L_{n,j_N}+1}_{n,j_N} = \sigma^{\infty}_{n,j_N}$. Arguing as above one has

$$
\begin{aligned}
&{}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right) \\
&\qquad \lesssim \left\|\gamma_n(\sigma^{\mathrm{inf},-}_{n,j_1}) - \gamma_n(s_1^+)\right\| + \sum_{\ell^{\mathrm{inf}}_{n,j_1} \le \ell \le L_{n,j_1}} \left\|\gamma_n(\sigma^{\ell+1,-}_{n,j_1}) - \gamma_n(\sigma^{\ell,+}_{n,j_1})\right\| \\
&\qquad\quad + \left\|\gamma_n(\sigma^{\infty,-}_{n,j_N}) - \gamma_n(\sigma^{\infty,+}_{n,j_1})\right\| + \left\|\gamma_n(s_2^-) - \gamma_n(\sigma^{\infty,+}_{n,j_N})\right\|,
\end{aligned}
$$

yielding the same estimate as in (6.23) with now $Z_{[s_1,s_2]}$ given by

(6.26)
$$
Z_{[s_1,s_2]} = \left|\vartheta_n(\sigma^{\mathrm{inf},-}_{n,j_1}) - \vartheta_n(s_1^+)\right| + \sum_{\ell^{\mathrm{inf}}_{n,j_1} \le \ell \le L_{n,j_1}} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_1}) - \vartheta_n(\sigma^{\ell,+}_{n,j_1})\right| + \left|\vartheta_n(s_2^-)\right|.
$$

Some configurations are illustrated in Figure 8. In each case above one has the following result.

Lemma 6.16. *There exists $C > 0$, independent of $n \in \mathbb{N}^*$, such that*

$$Z_{[s_1,s_2]} \leq C\big(s_2 - s_1 + \sqrt{s_2 - s_1}\big),$$

for all $0 \leq s_1 \leq s_2 \leq S_{\max}$.

A proof is given below. Then the different estimates above yield

$${}^c\mathrm{dist}\left({}^c\gamma_n(s_2), {}^c\gamma_n(s_1)\right) \lesssim s_2 - s_1 + \sqrt{s_2 - s_1},$$

implying the equicontinuity of $({}^c\gamma_n)_{n\in\mathbb{N}^*}$ on $[0, S_{\max}]$. Since the sequence is also pointwise bounded one can extract a subsequence $\left(s \mapsto {}^c\gamma_{n_p}\right)_{p\in\mathbb{N}}$ that converges uniformly to a curve ${}^c\gamma(s)$, $s \in [0, S_{\max}]$ with values in ${}^cT^*\mathcal{L}$ by the Arzelà-Ascoli theorem. One has ${}^c\gamma(0) = {}^c\phi(\varrho^0) = \varrho^0$. Set

$$B = \{s \in [0, S_{\max}];\ {}^c\gamma(s) \in {}^c\phi(\mathcal{H}_\partial)\}.$$

For $s \in [0, S_{\max}] \setminus B$ one defines ${}^{\mathsf{G}}\gamma(s) = {}^c\phi^{-1}({}^c\gamma(s))$. One has ${}^{\mathsf{G}}\gamma(0) = \varrho^0$.

For any $s \in [0, S_{\max}]$, the convergence of $\left({}^c\gamma_{n_p}\right)$ yields a sequence of points of ${}^c\phi\big(\operatorname{supp}\mu\big)$ that converges to ${}^c\gamma(s)$, implying that ${}^c\gamma(s) \in {}^c\phi\big(\operatorname{supp}\mu\big)$ since this set is closed. Thus, for any $s \in [0, S_{\max}] \setminus B$ one has ${}^{\mathsf{G}}\gamma(s) \in \operatorname{supp}\mu$.

It now remains to prove that ${}^{\mathsf{G}}\gamma$ is a generalized bicharacteristic on $[0, S_{\max}]\setminus B$, including that B is a discrete set.

6.5.1. ***Proof that the limit curve is a local generalized bicharacteristic.*** We review (yet again) all possible occurences. Above, we considered a subsequence γ_{n_p} to achieve convergence (by means of the Arzelà-Ascoli theorem). Here, for the sake of simplicity one uses γ_n to denote this subsequence.

(1) Points away from $\partial(T^*\mathcal{L})$. Assume that $\varrho^1 = {}^{\mathsf{G}}\gamma(s^0) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$. Since ϱ^1 is away from $\mathcal{G}_\partial \cup \mathcal{H}_\partial$, for n chosen sufficiently large γ_n is a piece of bicharacteristic in a neighborhood of ϱ^1. Thus, for some $\delta > 0$, one has

$$\gamma_n(s) = \gamma_n(s^0) + \int_{s^0}^{s} \mathrm{H}_p\left(\gamma_n(\sigma)\right) d\sigma, \qquad s \in [s^0 - \delta, s^0 + \delta].$$

As convergence is uniform, passing to the limit $n \to \infty$, one finds

$${}^{\mathsf{G}}\gamma(s) = \varrho^1 + \int_{s^0}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \qquad s \in [s^0 - \delta, s^0 + \delta],$$

meaning that ${}^{\mathsf{G}}\gamma$ is a bicharacteristic in a neighborhood of ϱ^1.

(2) Hyperbolic points. Consider $s^0 \in B$. One has ${}^c\gamma(s^0) \in {}^c\phi(\mathcal{H}_\partial)$. Set $\{\varrho^{1,+}, \varrho^{1,-}\} = {}^c\phi^{-1}\big(\{{}^c\gamma(s^0)\}\big)$ with $\varrho^{1,\pm} \in \mathcal{H}_\partial^\pm$ and $\varrho^{1,\pm} = \Sigma(\varrho^{1,\mp})$.

Note that ${}^c\mathrm{dist}\big({}^c\gamma(s^0), \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})\big) > 0$. Hence, there exist $C > 0$ and $\delta_0 > 0$ such that

$$ {}^c\mathrm{dist}\big({}^c\gamma(s), \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}})\big) \geq C > 0, \qquad s \in [s^0 - \delta_0, s^0 + \delta_0], $$

meaning that for some $n_0 \in \mathbb{N}$ chosen sufficiently large, each term of the sequence γ_n is made of a broken bicharacteristic on the interval $[s^0 - \delta_0, s^0 + \delta_0]$ for $n \geq n_0$.

For each point $\varrho^{1,\pm}$, use an open neighborhood $V^{0,\pm}$ of $\varrho^{1,\pm}$ in $\mathcal{H}_\partial^\pm$ and $S_0^\pm > 0$ as given by Lemma 5.25. Set $0 < \delta_1 < \min(S_0^+/2, S_0^-/2, \delta_0)$.

For $R > 0$ set $\mathcal{B}_R^+$ as the open ball of radius R in the variables (u, ϑ) centered at the point $\varrho^{1,\pm} = (u^1, z^1 = 0, \vartheta^1)$. Set $C_1 = \vartheta^1/2 > 0$. There exist $R_0 > 0$ and $Z > 0$ such that

$$ \mathcal{B}_{2R_0}^+ \subset V^{0,+} \quad \text{and} \quad \Sigma\big(\mathcal{B}_{2R_0}^+\big) \subset V^{0,-}, $$

and

$$ |\vartheta| \geq C_1 \text{ in } W_2^+ \cup W_2^- \quad \text{with } W_2^+ = \mathcal{B}_{2R_0}^+ \times [0, Z[\text{ and } W_2^- = \Sigma\big(\mathcal{B}_{2R_0}^+\big) \times [0, Z[. $$

For $N \in \mathbb{N}$, such that $Z_N = C_1\delta_1/(N+2) \leq Z$, and $k = 1, 2$, further set

$$ W_{k,N}^+ = \mathcal{B}_{kR_0}^+ \times [0, Z_N[\text{ and } W_{k,N}^- = \Sigma\big(B_{kR_0}^+\big) \times [0, Z_N[. $$

Note that $W_{k,N}^\pm \cap \{z = 0\} \subset V^{0,\pm} \subset \mathcal{H}_\partial^\pm$.

In $W_2^+ \cup W_2^-$ one has $|\,\mathrm{H}_p\, z| = |\vartheta| \geq C_1$. For $N_0 \in \mathbb{N}$ chosen sufficiently large and $N \geq N_0$, observe that any bicharacteristic initiated in $W_{1,N}^+$ (resp. $W_{1,N}^-$) exits $W_{2,N}^+$ (resp. $W_{2,N}^-$) at two points: one located in $\mathcal{B}_{2R_0}^+ \times \{0\}$ and one located in $\mathcal{B}_{2R_0}^+ \times \{Z_N\}$ (resp. points located in $\Sigma\big(\mathcal{B}_{2R_0}^+\big) \times \{0, Z_N\}$). Moreover, this occurs within a s-interval of size at most $Z_N/C_1 = \delta_1/(N+2)$. Set $\delta_2 = \delta_1/(N_0+1)$.

For some $n_1 \geq N_0$, if $n \geq n_1$ one has ${}^c\gamma_n(s^0) \in {}^c\phi\big(W_{1,N_0}^+ \cup W_{1,N_0}^-\big)$. From the discussion above, one finds that there exists

$$ s_n \in [s^0 - \delta_2, s^0 + \delta_2] $$

such that ${}^c\gamma_n(s_n) \in {}^c\phi(\mathcal{H}_\partial)$. Since $\gamma_n(s_n^-) \in V^{0,-}$, $\gamma_n(s_n^+) \in V^{0,+}$, and $2\delta_2 < \min(S_0^+, S_0^-)$ one concludes with Lemma 5.25 that s_n is the unique value of $s \in [s^0 - \delta_2, s^0 + \delta_2]$ such that ${}^c\gamma_n(s) \in {}^c\phi(\mathcal{H}_\partial)$.

Similarly, given $N \geq N_0$, for some $n_2 = n_2(N) \geq n_1$, if $n \geq n_2$, one has ${}^c\gamma_n(s^0) \in {}^c\phi\big(W^+_{1,N} \cup W^-_{1,N}\big)$. With the same argument as above there exists

$$s'_n \in [s^0 - \delta_1/(N+1), s^0 + \delta_1/(N+1)] \subset [s^0 - \delta_2, s^0 + \delta_2],$$

such that ${}^c\gamma_n(s'_n) \in {}^c\phi(\mathcal{H}_\partial)$. Naturally, the uniqueness of s_n obtained above yields $s'_n = s_n$. One therefore obtain the convergence of the sequence $(s_n)_{n\geq n_1}$ to s^0.

Consider now $0 < \varepsilon < \delta_2/2$. For some $n_3 \geq n_1$, if $n \geq n_3$ one has $s_n \leq s^0 + \varepsilon$. On the interval $]s_n, s^0 + \delta_2]$, the curve γ_n is a bicharacteristic located in $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ and

$$\gamma_n(s) = \gamma_n(s^0 + \varepsilon) + \int_{s^0+\varepsilon}^{s} \mathrm{H}_p\left(\gamma_n(\sigma)\right) d\sigma, \quad s \in [s^0 + \varepsilon, s^0 + \delta_2].$$

On $]s_n, s^0 + \delta_2]$ one has $\vartheta_n(s) = \mathrm{H}_p\, z\big(\gamma_n(s)\big) > 0$. Yet, locally, one has $\mathrm{H}_p\, z = \vartheta \geq C_1 > 0$ and $z_n(s_n) = 0$, then $z_n(s^0 + \varepsilon) \geq \varepsilon C_1$. On the interval $[s^0 + \varepsilon, s^0 + \delta_2]$ one thus has ${}^c\gamma_n(s) = \gamma_n(s)$. With the uniform limit of $({}^c\gamma_n)_n$ one obtains for ${}^{\mathsf{G}}\gamma(s) = \big(u(s), z(s), \vartheta(s)\big)$ that $z(s) \geq z(s^0 + \varepsilon) \geq C_1\varepsilon$ for $s \in [s^0 + \varepsilon, s^0 + \delta_2]$ and

$${}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}\gamma(s^0 + \varepsilon) + \int_{s^0+\varepsilon}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \quad s \in [s^0 + \varepsilon, s^0 + \delta_2].$$

As $\varepsilon > 0$ is arbitrary this implies that ${}^{\mathsf{G}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $s \in]s^0, s^0 + \delta_2]$. With the continuity of ${}^c\gamma$ one has moreover $\lim_{s\to s^{0,+}} {}^{\mathsf{G}}\gamma(s) = \varrho^{1,+} \in \mathcal{H}^+$ and

$${}^{\mathsf{G}}\gamma(s) = \varrho^{1,+} + \int_{s^0}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \quad s \in]s^0, s^0 + \delta_2].$$

On the interval $]s^0, s^0 + \delta_2]$, the curve ${}^{\mathsf{G}}\gamma(s)$ is thus a piece of bicharacteristic initiated from a point in $\mathcal{H}^+_\partial$, here $\varrho^{1,+}$.

Similarly, one finds ${}^{\mathsf{G}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ for $]s^0 - \delta_2, s^0[$, $\lim_{s\to s^{0,-}} {}^{\mathsf{G}}\gamma(s) = \varrho^{1,-} \in \mathcal{H}^-$, and on the interval $[s^0 - \delta_2, s^0[$, the curve ${}^{\mathsf{G}}\gamma(s)$ is this a piece of bicharacteristic initiated from a point in $\mathcal{H}^-_\partial$, here $\varrho^{1,-}$.

Since, $\varrho^{1,-} = \Sigma\big(\varrho^{1,+}\big)$ this proves that ${}^{\mathsf{G}}\gamma$ fulfills the required conditions at hyperbolic points for broken bicharacteristics and thus generalized bicharacteristics; See Definitions 5.26 and 5.32. In particular $s \in [s^0 - \delta_2, s^0[\cup]s^0, s^0 + \delta_2[$ one has $\frac{d}{ds}{}^{\mathsf{G}}\gamma(s) = \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(s)\right)$.

Moreover, what is above implies that the set B is discrete.

(3) Points in $\mathcal{G}^{\mathrm{d}}_\partial$. Suppose $s^0 \in [0, S_{\max}]$ is such that $\varrho^1 = {}^{\mathsf{G}}\gamma(s^0) \in \mathcal{G}^{\mathrm{d}}_\partial$. One has $\mathrm{dist}({}^{\mathsf{G}}\gamma(s^0), \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial) > 0$. Thus, there exist $C > 0$ and $\delta_0 > 0$ such that

$$\mathrm{dist}\left({}^{\mathsf{G}}\gamma(s), \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial\right) \geq C > 0, \qquad s \in [s^0 - \delta_0, s^0 + \delta_0],$$

meaning that for some $n_0 \in \mathbb{N}$ chosen sufficiently large, each term of the sequence γ_n is made of a broken bicharacteristic on the interval $[s^0-\delta_0, s^0+\delta_0]$ for $n \geq n_0$. One has $\mathrm{H}_p\, z(\varrho^1) = 0$ and $\mathrm{H}_p^2\, z(\varrho^1) > 0$. Thus, there exists a neighborhood V^0 of ϱ^1 where $\mathrm{H}_p^2 z \geq C_2$ for some $C_2 > 0$. For some $n_1 \geq n_0$ and some $0 < \delta_1 < \delta_0$, one has $\gamma_n(s) \in V^0$ for $n \geq n_1$ and $s \in [s^0-\delta_1, s^0+\delta_1]$.

One faces two occurences: either (1) there exists a subsequence $n_p \to \infty$ such that $B_{n_p} \cap [s^0-\delta_1, s^0+\delta_1] = \emptyset$, that is, γ_{n_p} does not encounter any hyperbolic point in this interval or (2) for n_0 chosen sufficiently large one has $B_n \cap [s^0-\delta_1, s^0+\delta_1] \neq \emptyset$ if $n \geq n_0$.

In case (1), one concudes as for a point away from $\partial(T^*\mathcal{L})$: passing to the limit $n_p \to \infty$ one obtains

$$
{}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}\gamma(s^0) + \int_{s^0}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \qquad s \in [s^0-\delta_1, s^0+\delta_1],
$$

meaning that ${}^{\mathsf{G}}\gamma$ is a bicharacteristic in a neighborhood of ϱ^1.

Consider now case (2) and $n \geq n_0$ and $s_n \in B_n \cap [s^0-\delta_1, s^0+\delta_1]$, that is, s_n associated with a hyperbolic point of γ_n for $s \in [s^0-\delta_1, s^0+\delta_1]$. One has $\mathrm{H}_p\, z(s_n^+) > 0$. As $\gamma_n(s)$ remains in V^0 one concludes that $\mathrm{H}_p\, z\big(\gamma_n(s)\big) > 0$ and $z_n(s) > 0$ for $s \in]s_n, s^0+\delta_1]$. Similarly $\mathrm{H}_p\, z\big(\gamma_n(s)\big) < 0$ and $z_n(s) > 0$ for $s \in [s^0-\delta_1, s_n[$. Hence, γ_n has at most one isolated hyperbolic point in the interval $[s^0-\delta_1, s^0+\delta_1]$. More precisely one has

$$
z_n(s) \geq C_2(s-s_n)^2/2, \qquad s \in [s^0-\delta_1, s^0+\delta_1]. \tag{6.27}
$$

Assume that σ^0 is an accumulation point of the sequence $(s_n)_n$ and s_{n_p} a subsequence that converges to σ^0. For ${}^{\mathsf{G}}\gamma(s) = \big(t(s), x'(s), z(s), \tau(s), \xi(s)\big)$, by (6.27) one obtains in the limit $n_p \to \infty$

$$
z(s) \geq C_2(s-\sigma^0)^2/2, \qquad s \in [s^0-\delta_1, s^0+\delta_1], \tag{6.28}
$$

implying that $\sigma^0 = s^0$. Thus, one concludes that $s_n \to s^0$ as $n \to \infty$.

Consider $0 < \varepsilon < \delta_1/2$. For n sufficiently large one has $s_n < s^0 + \varepsilon$ and one has

$$
\gamma_n(s) = \gamma_n(s^0+2\varepsilon) + \int_{s^0+2\varepsilon}^{s} \mathrm{H}_p\left(\gamma_n(\sigma)\right) d\sigma, \qquad s \in [s^0+2\varepsilon, s^0+\delta_1].
$$

Since $z_n(s) \geq C_2(s^0+2\varepsilon-s_n)^2 \geq C_2\varepsilon^2$, $\gamma_n(s)$ remains away from $\mathcal{H}_\partial$ for $s \in [s^0+2\varepsilon, s^0+\delta_1]$ and in the limit $n \to \infty$ one finds

$$
{}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}\gamma(s^0+2\varepsilon) + \int_{s^0+2\varepsilon}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \qquad s \in [s^0+2\varepsilon, s^0+\delta_1].
$$

Since ${}^{\mathsf{G}}\gamma(s^0) \in \mathcal{G}^{\mathrm{d}}_\partial$ one has ${}^{\mathsf{G}}\gamma(s^{0,-}) = {}^{\mathsf{G}}\gamma(s^{0,+}) = \varrho^1$ as the continuity of ${}^{c}\gamma$ implies the continuity of ${}^{\mathsf{G}}\gamma$ away from points of $\mathcal{H}_\partial$. Thus, letting $\varepsilon \to 0^+$, one finds

$$ {}^{\mathsf{G}}\gamma(s) = \varrho^1 + \int_{s^0}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \qquad s \in [s^0, s^0 + \delta_1], $$

using that ${}^{\mathsf{G}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ by (6.28) for $s > s^0$. Similarly, one has

$$ {}^{\mathsf{G}}\gamma(s) = \varrho^1 + \int_{s^0}^{s} \mathrm{H}_p\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, \qquad s \in [s^0 - \delta_1, s^0]. $$

As in Case (1) this means that ${}^{\mathsf{G}}\gamma$ is a bicharacteristic in a neighborhood of ϱ^1.

(4) Points in $\mathcal{G}^{\mathbf{3}}_{\boldsymbol{\partial}} \cup \mathcal{G}^{\mathbf{g}}_{\boldsymbol{\partial}}$. Suppose $\varrho^1 = {}^{\mathsf{G}}\gamma(s^0) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial$. An example of such a point is naturally $\varrho^0 = {}^{\mathsf{G}}\gamma(0)$ where the constuction of ${}^{\mathsf{G}}\gamma$ is initiated.

Applying $\pi_{\|}$ to (6.18) gives

$$ {}^{\|}\gamma_n(s) = {}^{\|}\gamma_n(s_1^+) + \int_{s_1}^{s} \mathrm{H}_p^{\mathcal{G}}\left(\gamma_n(\sigma)\right) d\sigma. $$

for $[s_1, s]$ within a maximal bicharateristic and this extends to any whole maximal broken bicharacteristic in the construction made above. With (6.20) one has

$$ {}^{\|}\gamma_n(s) = {}^{\|}\gamma_n(s_1) + \int_{s_1}^{s} \mathrm{H}_p^{\mathcal{G}}\left(\gamma_n(\sigma)\right) d\sigma + \int_{s_1}^{s} {}^{\|}e_n(\sigma)\, d\sigma, $$

for any $s, s^1 \in [0, S_{\max}]$, with the errors $|{}^{\|}e_n|$ going to zero *uniformly* as $n \to +\infty$. Note that $\mathrm{H}_p^{\mathcal{G}}\left(\gamma_n(\sigma)\right)$ may only be discontinuous at hyperbolic points that form a discrete set and thus a countable set for each n. Dominated convergence yields

$$ \pi_{\|}\big({}^{\mathsf{G}}\gamma(s)\big) = \pi_{\|}\big({}^{\mathsf{G}}\gamma(s^0)\big) + \int_{s^0}^{s} \mathrm{H}_p^{\mathcal{G}}\left({}^{\mathsf{G}}\gamma(\sigma)\right) d\sigma, $$

for $s \in [0, S_{\max}]$. From the continuity of the gliding vector field $\mathrm{H}_p^{\mathcal{G}}$ and the continuity of ${}^{\mathsf{G}}\gamma(s)$ at $s = s^0$ one has $\mathrm{H}_p^{\mathcal{G}}\left({}^{\mathsf{G}}\gamma(s)\right) = \mathrm{H}_p^{\mathcal{G}}(\varrho^1) + o(1)$ as $s \to s^0$, yielding

$$ \pi_{\|}\big({}^{\mathsf{G}}\gamma(s)\big) = \pi_{\|}\big({}^{\mathsf{G}}\gamma(s^0)\big) + (s - s^0)\, \mathrm{H}_p^{\mathcal{G}}(\varrho^1) + (s - s^0)o(1), $$

implying that $\pi_{\|}\big({}^{\mathsf{G}}\gamma(s)\big)$ is differentiable at $s = s_0$ and

$$ \frac{d}{ds}\pi_{\|}\big({}^{\mathsf{G}}\gamma(s)\big)(s^0) = \mathrm{H}_p^{\mathcal{G}}(\varrho^1). $$

Parts (1), (2), and (3) show that the first two assumption of Proposition 5.31 are fulfilled by $s \mapsto {}^{\mathsf{G}}\gamma(s)$. Hence, Proposition 5.31 applies (one may need to change s into $s^0 - s$ depending if one considers $s > s^0$ or $s < s^0$). Consequently, ${}^{\mathsf{G}}\gamma$ is differentiable at $s = s^0$ and

$$\frac{d}{ds}{}^{\mathsf{G}}\gamma(s^0) = \mathrm{H}_p^{\mathcal{G}}(\varrho^1).$$

This concludes the proof of Lemma 6.15. □

Proof of Lemma 6.16. First, consider the case $M_n^{[s_1,s_2]} = \emptyset$, meaning that all points $\gamma_n(s)$ are in affine parts of the construction for $s \in [s_1, s_2]$. Thus, there exists $r_0 \in [0, S_{\max}[$ and $N > 0$ such that a sequence of N affine parts is initiated at $\gamma(r_0)$. For some $0 \le \ell_1 \le \ell_2 \le N$, one has

$$s_j \in [r_{\ell_j}, r_{\ell_j+1}], \quad j = 1, 2 \ \text{ for } r_\ell = r_0 + \ell\delta_n,$$

If $\ell_1 = \ell_2 = \ell$ one as

$$\gamma_n(s_j) = \gamma_n(r_\ell) + \frac{s_j - r_\ell}{\delta_n}\big(\gamma_n(r_{\ell+1}) - \gamma_n(r_\ell)\big)$$

yielding $\vartheta_n(s_j) = \frac{s_j - r_\ell}{\delta_n}\vartheta_n(r_{\ell+1})$, as $\vartheta_n(r_\ell) = 0$ since $\gamma_n(r_\ell) \in \mathcal{G}_\partial$. One thus obtains

$$\vartheta_n(s_2) - \vartheta_n(s_1) = \frac{s_2 - s_1}{\delta_n}\vartheta_n(r_{\ell+1}).$$

As $|\vartheta_n(r_{\ell+1})| \lesssim \delta_n^{1/2}$ by Proposition 6.9 and $0 < s_2 - s_1 \le \delta_n$ one obtains

$$Z_{[s_1,s_2]} = |\vartheta_n(s_2) - \vartheta_n(s_1)| \lesssim \sqrt{s_2 - s_1}. \tag{6.29}$$

If now $\ell_1 < \ell_2$ one writes

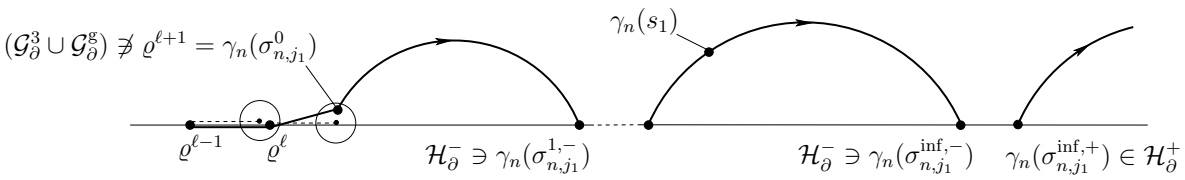
$$\vartheta_n(s_2) - \vartheta_n(s_1) = \vartheta_n(s_2) - \vartheta_n(r_{\ell_2}) + \vartheta_n(r_{\ell_1+1}) - \vartheta_n(s_1),$$

since $\vartheta_n(r_{\ell_2}) = \vartheta_n(r_{\ell_1+1}) = 0$. The argument that led to the previous estimate (6.29) gives

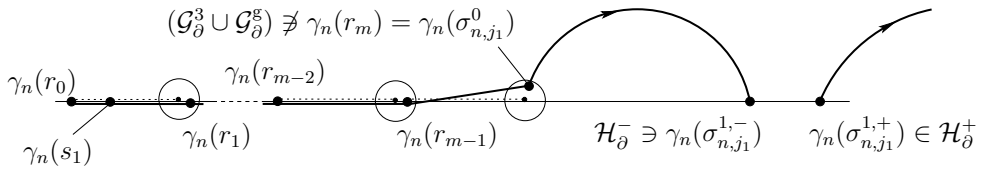
$$|\vartheta_n(s_2) - \vartheta_n(r_{\ell_2})| + |\vartheta_n(r_{\ell_1+1}) - \vartheta_n(s_1)| \lesssim \sqrt{s_2 - r_{\ell_2}} + \sqrt{r_{\ell_1+1} - s_1},$$

yielding also in this second case

$$Z_{[s_1,s_2]} = |\vartheta_n(s_2) - \vartheta_n(s_1)| \lesssim \sqrt{s_2 - s_1}.$$



(a) $\gamma_n(s_1)$ in the j_1th broken bicharacteristic.



(b) $\gamma_n(s_1)$ in some affine part of γ_n upstream from the j_1th broken bicharacteristic is initiated. Here $\sigma^{\mathrm{inf}}_{n,j_1} = \sigma^0_{n,j_1}$.

FIGURE 9. Two possible locations of $\gamma_n(s_1^+)$, upstream from or within the j_1th broken bicharacteristic.

Second, consider the case $M_n^{[s_1,s_2]} \neq \emptyset$. Start with the case of $Z_{[s_1,s_2]}$ given by (6.24), that is, $\sigma^{\mathrm{sup}}_{n,j_N} \leq s_2 < \sigma^{\infty}_{n,j_N}$.

By (6.18) and (6.19), since the following terms only concern parts of bicharacteristics, one obtains

$$
\begin{aligned}
(6.30)\qquad & \sum_{\ell^{\mathrm{inf}}_{n,j_1} \leq \ell \leq L_{n,j_1}} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_1}) - \vartheta_n(\sigma^{\ell,+}_{n,j_1})\right| + \sum_{0 \leq \ell \leq \ell^{\mathrm{sup}}_{n,j_N}-1} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_N}) - \vartheta_n(\sigma^{\ell,+}_{n,j_N})\right| \\
& \lesssim \sum_{\ell^{\mathrm{inf}}_{n,j_1} \leq \ell \leq L_{n,j_1}} \left(\sigma^{\ell+1}_{n,j_1} - \sigma^{\ell}_{n,j_1}\right) + \sum_{0 \leq \ell \leq \ell^{\mathrm{sup}}_{n,j_N}-1} \left(\sigma^{\ell+1}_{n,j_N} - \sigma^{\ell}_{n,j_N}\right) \\
& \lesssim \sigma^{\infty}_{n,j_1} - \sigma^{\mathrm{inf}}_{n,j_1} + \sigma^{\mathrm{sup}}_{n,j_N} - \sigma^{0}_{n,j_N}.
\end{aligned}
$$

Now, consider the interval $]s_1, \sigma^{\mathrm{inf}}_{n,j_1}[$ and estimate the term $\left|\vartheta_n(\sigma^{\mathrm{inf},-}_{n,j_1}) - \vartheta_n(s_1^+)\right|$. In fact $\gamma_n(s_1^+)$ is either (1) located on the j_1th broken bicharacteristic, or (2) located on some affine part of γ_N that stands upstream from the initiation of j_1th broken bicharacteristic. The two situations are illustrated in Figure 9. In the first configuration, arguing as above one finds

$$
(6.31)\qquad \left|\vartheta_n(\sigma^{\mathrm{inf},-}_{n,j_1}) - \vartheta_n(s_1^+)\right| \lesssim \sigma^{\mathrm{inf}}_{n,j_1} - s_1.
$$

Let us now perform the estimation in the second configuration; then $\sigma^{\inf}_{n,j_1} = \sigma^0_{n,j_1}$. Suppose the piecewise affine part of γ_n upstream from the j_1th broken bicharacteristic is composed of at least m pieces. Assume that

$$r_0 \leq s_1 \leq r_1 = r_0 + \delta_n < \cdots < r_m = r_0 + m\delta_n = \sigma^0_{n,j_1},$$

with $\gamma_n(r_0), \ldots, \gamma_n(r_{m-1}) \in \operatorname{supp}\mu \cap (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$ and $\gamma_n(r_m) \in \operatorname{supp}\mu \setminus (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)$, and γ_n affine between these points. The point $\gamma_n(s_1)$ is here located on the affine part joinging $\gamma_n(r_0)$ and $\gamma_n(r_1)$. It is not excluded here that there could be additional affine parts upstream from the point $\gamma_n(r_0)$.

The j_1th broken bicharacteristic is initiated at $s = r_m = \sigma^0_{n,j_1} = \sigma^{\inf,-}_{n,j_1}$; See Figure 9(b). Estimates obtained above in the case $M_n^{[s_1,s_2]} = \emptyset$ give

$$\left|\vartheta_n(\sigma^{\inf,-}_{n,j_1}) - \vartheta_n(s_1^+)\right| \lesssim \sqrt{\sigma^{\inf}_{n,j_1} - s_1} \lesssim \sqrt{s_2 - s_1} \tag{6.32}$$

To take into account both configurations, the first one given by (6.31) and the second one given by (6.32) one writes

$$\left|\vartheta_n(\sigma^{\inf,-}_{n,j_1}) - \vartheta_n(s_1^+)\right| \lesssim \sigma^{\inf}_{n,j_1} - s_1 + \sqrt{s_2 - s_1}. \tag{6.33}$$

Similarly, one finds

$$\left|\vartheta_n(\sigma^{0,-}_{n,j_N})\right| = \left|\vartheta_n(\sigma^{0,-}_{n,j_N}) - \vartheta_n(\sigma^{\infty,+}_{n,j_{N-1}})\right| \lesssim \sqrt{\sigma^0_{n,j_N} - \sigma^\infty_{n,j_{N-1}}} \lesssim \sqrt{s_2 - s_1}. \tag{6.34}$$

Finally, since $\sigma^{\sup}_{n,j_N} \leq s_2 < \sigma^\infty_{n,j_N}$ one has as in (6.30)

$$\left|\vartheta_n(s_2^-) - \vartheta_n(\sigma^{\sup,+}_{n,j_N})\right| \lesssim s_2 - \sigma^{\sup}_{n,j_N}, \tag{6.35}$$

since this term only concerns parts of bicharacteristics.

Summing estimates (6.30), (6.33), (6.34), and (6.35) one finds the estimate

$$Z_{[s_1,s_2]} \lesssim s_2 - s_1 + \sqrt{s_2 - s_1} + \sigma^\infty_{n,j_1} - \sigma^0_{n,j_N} \lesssim s_2 - s_1 + \sqrt{s_2 - s_1}.$$

for $Z_{[s_1,s_2]}$ as given by (6.24).

Treat now the case of $Z_{[s_1,s_2]}$ given by (6.25), that is, if $\sigma^\infty_{n,j_N} = s_2$. As above one has

$$\sum_{\ell^{\inf}_{n,j_1} \leq \ell \leq L_{n,j_1}} \left|\vartheta_n(\sigma^{\ell+1,-}_{n,j_1}) - \vartheta_n(\sigma^{\ell,+}_{n,j_1})\right| \lesssim \sigma^\infty_{n,j_1} - \sigma^{\inf}_{n,j_1}. \tag{6.36}$$

Estimation (6.33) holds and yield

$$Z_{[s_1,s_2]} \lesssim \sigma^{\infty}_{n,j_1} - s_1 + \sqrt{s_2 - s_1} \lesssim s_2 - s_1 + \sqrt{s_2 - s_1},$$

for $Z_{[s_1,s_2]}$ as given by (6.25).

Finally, Treat the case of $Z_{[s_1,s_2]}$ given by (6.26), that is, if $\sigma^{\infty}_{n,j_N} < s_2$. The $\gamma(s_2)$ lies in some affine part. Similarly to (6.32) one has

$$\left|\vartheta_n(s_2^-)\right| \lesssim \sqrt{s_2 - s_1}.$$

Estimations (6.36) and (6.33) hold and yield

$$Z_{[s_1,s_2]} \lesssim \sigma^{\infty}_{n,j_1} - s_1 + \sqrt{s_2 - s_1} \lesssim s_2 - s_1 + \sqrt{s_2 - s_1},$$

for $Z_{[s_1,s_2]}$ as given by (6.26). This concludes the proof of Lemma 6.16. □

7. Mass property of the boundary measure

Here, we prove Proposition 3.5. Locally, we first use the quasi-normal geodesic coordinates of Proposition 2.1. With Assumption 3.1 one has $|\tau| \geq C_0 > 0$ in $\operatorname{supp}\mu$. Consider $\varrho = (t, x', z = 0, \tau, \xi', \zeta = 0) \in {}^{\|}\mathcal{H}_{\partial}$ with $|\tau| < C_0$. Then, $\varrho^{\pm} \notin \operatorname{supp}\mu$. In a neighborhood of $\varrho^{\pm}$ Equation 3.1 reads

$$\int_{\varrho \in {}^{\|}\mathcal{H}_{\partial}} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x \rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}}}\, d\nu(\varrho) = 0,$$

implying that $\varrho \notin \operatorname{supp}\nu$. Thus $\operatorname{supp}\nu \cap {}^{\|}\mathcal{H}_{\partial} \cap \{|\tau| < C_0\} = \emptyset$. Consider now $\varrho \in {}^{\|}\mathcal{G}_{\partial} = \mathcal{G}_{\partial}$ with $|\tau| < C_0$. With Remark 3.3 one finds $\int_{{}^{\|}\mathcal{G}_{\partial}} \partial_{\zeta} a\, d\nu = 0$, for any $a \in \mathscr{C}^1_c(T^*\hat{\mathcal{L}})$ supported near ϱ. Yet, any $\mathscr{C}^0$-function on ${}^{\|}\partial(T^*\mathcal{L}) \simeq T^*\partial\mathcal{L}$ can take the form $\partial_{\zeta} a_{|_{\zeta=0}^{z=0}}$ implying that $\operatorname{supp}\nu \cap {}^{\|}\mathcal{G}_{\partial} \cap \{|\tau| < C_0\} = \emptyset$. This gives the first result.

The glancing set $\mathcal{G}_{\partial}$ is given by $\{z = \mathrm{H}_p\, z = p = 0\} = \{z = \zeta = p = 0\}$. The set $\mathcal{G}^{\mathrm{d}}_{\partial} \cup \mathcal{G}^3_{\partial}$ is given by $\{z = \mathrm{H}_p\, z = p = 0 \text{ and } \mathrm{H}^2_p\, z \geq 0\}$. Define $r(x, \tau, \xi') = \tau^2 - \sum_{1 \leq i,j \leq d-1} g^{ij}(x)\xi_i\xi_j$. One has

$$p(x, \tau, \xi', \zeta) = (1 + z h_{dd}(x))\zeta^2 + z \sum_{1 \leq j \leq d-1} h_{jd}(x)\xi_j\zeta - r(x, \tau, \xi').$$

Note that $\mathcal{G}_{\partial}$ is also given by $\{z = r = p = 0\}$.

As $\partial_\tau r = 2\tau \neq 0$ in $\operatorname{supp}\nu$ by the first part of the proposition, one finds that $r(x,\tau,\xi')$ can be used as a coordinate on $\{z = \mathrm{H}_p\, z = 0\}$ near $\operatorname{supp}\nu$. Denote by $\sigma \in \mathbb{R}^{2d-1}$ coordinates such that $(z, \mathrm{H}_p\, z, r, \sigma)$ are local coordinates of $T^*\hat{\mathcal{L}}$.

Consider $\psi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\psi(s) = 1$ for $s \geq 0$ and $\psi(s) = 0$ for $s \leq -1/2$. Consider also $b \in \mathscr{C}^\infty_c(\mathbb{R}^{2d-1})$ such that $b(z,r,\sigma)$ is independent of z and r in a neighborhood of $\{z = r = 0\}$. In $\operatorname{supp} b$, $|\tau|$ is bounded. Hence, $|\zeta| \leq C_1$ in $\operatorname{supp} b \cap \operatorname{Char} p$ for some $C_1 > 0$. Pick $\varphi \in \mathscr{C}^\infty_c(\mathbb{R})$ such that $\varphi(s) = 1$ for $|s| \leq 2C_1 + 1$. For $\varepsilon > 0$ and $\alpha > 0$, set

$$a_{\varepsilon,\alpha}(\varrho) = (\mathrm{H}_p\, z)\varphi(\mathrm{H}_p\, z)\psi\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho)\, b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma),$$

where $\phi_\alpha(\varrho) = \alpha^{-2d-2}\phi(\varrho/\alpha)$ with $\int \phi = 1$, that is, an approximation to the identity. One has

$$\begin{aligned}
\mathrm{H}_p\, a_{\varepsilon,\alpha}(\varrho) = (\mathrm{H}_p^2\, z)\big(&\varphi(\mathrm{H}_p\, z)\\
&+ (\mathrm{H}_p\, z)\varphi'(\mathrm{H}_p\, z)\big)\psi\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho) b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma)\\
&+ \varepsilon^{-1/2}(\mathrm{H}_p\, z)^2\varphi(\mathrm{H}_p\, z)\psi\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho)\partial_1 b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma)\\
&+ \varepsilon^{-1}(\mathrm{H}_p\, r)(\mathrm{H}_p\, z)\varphi(\mathrm{H}_p\, z)\psi\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho)\partial_2 b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma)\\
&+ (\mathrm{H}_p\, z)\varphi(\mathrm{H}_p\, z)\psi\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho) d_3 b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma)(\mathrm{H}_p\, \sigma)\\
&+ \varepsilon^{-1/4}\big((\mathrm{H}_p\, \phi_\alpha) * \mathrm{H}_p^2\, z\big)(\mathrm{H}_p\, z)\varphi(\mathrm{H}_p\, z)\\
&\quad \times \psi'\big(\varepsilon^{-1/4}\phi_\alpha * \mathrm{H}_p^2\, z\big)(\varrho) b(\varepsilon^{-1/2}z, \varepsilon^{-1}r, \sigma).
\end{aligned}$$

One has $|z| \lesssim \varepsilon^{1/2}$, $|r| \lesssim \varepsilon$ in $\operatorname{supp} a$. Since $\operatorname{supp}\mu \subset \operatorname{Char} p$, solving $p = 0$ for ζ one finds that $|\zeta| \lesssim \varepsilon^{1/2}$. The estimate $|\,\mathrm{H}_p\, z| = |\partial_\zeta p| \lesssim \varepsilon^{1/2}$ follows. As $\mathrm{H}_p\, r = \mathrm{H}_p(p + r)$ one finds $|\,\mathrm{H}_p\, r| \lesssim \varepsilon^{1/2}$. These estimates and the dominated convergence theorem give

$$\langle {}^t\mathrm{H}_p\, \mu, a_{\varepsilon,\alpha}\rangle \underset{\varepsilon\to 0}{\longrightarrow} \langle \mu, \mathrm{H}_p^2\, z(\varrho) b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{A_\alpha}\rangle,$$

using that $\operatorname{supp}\mu \subset \operatorname{Char} p$, with $A_\alpha = \{\phi_\alpha * \mathrm{H}_p^2\, z(\varrho) \geq 0\}$. One also obtains

$$\langle f\mu, a_{\varepsilon,\alpha}\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

If $\varrho \in {}^{\|}\mathcal{H}_\partial$ one has $\mathrm{H}_p\, z(\varrho^+) = 2\zeta^+ = -\,\mathrm{H}_p\, z(\varrho^-) = -2\zeta^-$ and $\varphi(2\zeta^+) = \varphi(2\zeta^-) = 1$. One obtains

$$\Big\langle \int_{\varrho\in{}^{\|}\mathcal{H}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\, d\nu(\varrho), a_{\varepsilon,\alpha}\Big\rangle = \int_{\varrho\in{}^{\|}\mathcal{H}_\partial} \frac{a_{\varepsilon,\alpha}(\varrho^+) - a_{\varepsilon,\alpha}(\varrho-)}{2\zeta^+}\, d\nu(\varrho)$$
$$= \int_{\varrho\in{}^{\|}\mathcal{H}_\partial} b(0,\varepsilon^{-1}r,\sigma)\big(\psi\big(\varepsilon^{-1/2}\phi_\alpha * \mathrm{H}_p^2\, z(\varrho)\big)(\varrho^+) + \psi\big(\varepsilon^{-1/2}\phi_\alpha * \mathrm{H}_p^2\, z(\varrho)\big)(\varrho^-)\big)\, d\nu(\varrho).$$

Since $r > 0$ in ${}^{\|}\mathcal{H}_\partial$, with the support property of b, by dominated convergence one finds

$$\Big\langle \int_{\varrho\in{}^{\|}\mathcal{H}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\, d\nu(\varrho), a_{\varepsilon,\alpha}\Big\rangle \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Next, as ${}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial$ is also given by $\{z = r = 0\}$ in ${}^{\|}T^*\mathcal{L}$ one has

$$\Big\langle \int_{\varrho\in{}^{\|}\mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\, d\nu(\varrho), a_{\varepsilon,\alpha}\Big\rangle = \langle \nu, 2b(0,0,\sigma)\psi\big(\varepsilon^{-1/2}\phi_\alpha * \mathrm{H}_p^2\, z\big)\rangle.$$

By dominated convergence one finds

$$\Big\langle \int_{\varrho\in{}^{\|}\mathcal{H}_\partial\cup{}^{\|}\mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\, d\nu(\varrho), a_{\varepsilon,\alpha}\Big\rangle \underset{\varepsilon\to 0}{\longrightarrow} 2\langle \nu, b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{A_\alpha}\rangle.$$

The measure equation of Assumption 3.2 gives

$$\langle \mu, \mathrm{H}_p^2\, z(\varrho) b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{A_\alpha}\rangle = -2\langle \nu, b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{A_\alpha}\rangle.$$

Letting $\alpha \to 0$ gives by dominated convergence

$$\langle \mu, \mathrm{H}_p^2\, z(\varrho) b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{\mathrm{H}_p^2 z\geq 0}\rangle = -2\langle \nu, b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{\mathrm{H}_p^2 z\geq 0}\rangle.$$

One has $\mathbf{1}_{\mathcal{G}_\partial}\mathbf{1}_{\mathrm{H}_p^2 z\geq 0} = \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}\cup\mathcal{G}_\partial^3}$. If $b \geq 0$ one finds that both sides have opposite signs since μ and ν are both nonnegative measures. Hence, both side vanish if $b \geq 0$. Thus, on the one hand, $\langle \nu, b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}\cup\mathcal{G}_\partial^3}\rangle = 0$ for any $b \geq 0$, yielding $\langle \nu, \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}\cup\mathcal{G}_\partial^3}\rangle = 0$. On the other hand, $\langle \mu, \mathrm{H}_p^2\, z(\varrho) b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}\cup\mathcal{G}_\partial^3}\rangle = 0$ for any $b \geq 0$. Since $\mathrm{H}_p^2 z$ vanishes in $\mathcal{G}_\partial^3$ one obtains $\langle \mu, \mathrm{H}_p^2\, z(\varrho) b(0,0,\sigma)\mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}}\rangle = 0$ for any $b \geq 0$. One concludes that $\langle \mu, \mathbf{1}_{\mathcal{G}_\partial^{\mathrm{d}}}\rangle = 0$ as $\mathrm{H}_p^2\, z > 0$ on $\mathcal{G}_\partial^{\mathrm{d}}$. □

Appendix A. Mesure associated with a single generalized bicharacteristic

Consider a generalized bicharacteristic ${}^{\mathsf{G}}\gamma : \mathbb{R} \setminus B \to T^*\mathcal{L}$ according to Definition 2.8 or equivalently Definition 5.32. One wishes to introduce the measure $\delta_{\mathsf{G}\bar{\gamma}}$ as the linear measure supported on ${}^{\mathsf{G}}\bar{\gamma}$ (see Definition 2.9), that is,

$$\langle \delta_{\mathsf{G}\bar{\gamma}}, a \rangle = \int_{\mathbb{R}\setminus B} a\big({}^{\mathsf{G}}\gamma(s)\big)\, ds, \qquad a \in \mathscr{C}^0_c(T^*\hat{\mathcal{L}}). \tag{A.1}$$

First, observe that the curve ${}^{\mathsf{G}}\gamma$ is noncontinuous at hyperbolic points and therefore not rectifiable. However, since B is a discrete set, upon defining the value of ${}^{\mathsf{G}}\gamma(s) = {}^{\mathsf{G}}\gamma(s^-)$ for $s \in B$, one finds that ${}^{\mathsf{G}}\gamma$ is left continuous and therefore measurable. Second, observe that $I_K = \{s \in \mathbb{R}\setminus B;\ {}^{\mathsf{G}}\gamma(s) \in K\}$ is bounded for K a compact set of $T^*\hat{\mathcal{L}}$, using that $\frac{dt}{ds}(s) = -2\tau \neq 0$ and $\frac{d\tau}{ds}(s) = 0$. Consequently, given $a \in \mathscr{C}^0_c(T^*\hat{\mathcal{L}})$ with $\operatorname{supp} a \subset K$ one sees that the integral in (A.1) is sensible and

$$|\langle \delta_{\mathsf{G}\bar{\gamma}}, a \rangle| \leq \|a\|_{L^\infty}\, |I_K|,$$

meaning that $\delta_{\mathsf{G}\bar{\gamma}}$ is a Radon measure.

The following theorem states that the measure $\delta_{\mathsf{G}\bar{\gamma}}$ fulfills a transport equation of the form considered in the present article.

Theorem A.1. *Consider a nontrivial generalized bicharacteristic ${}^{\mathsf{G}}\gamma : \mathbb{R} \setminus B \to T^*\mathcal{L}$. Then (A.1) yields a measure $\mu = \delta_{\mathsf{G}\bar{\gamma}}$ that fulfills the transport equation of (3.1) for some nonnegative measure ν on ${}^{\parallel}\partial(T^*\mathcal{L})$.*

Proof. To lighten notation write γ in place of ${}^{\mathsf{G}}\gamma$. Having γ not trivial means that the cotangent component of $\gamma(s)$ is not zero, and thus neither H_p nor $\mathrm{H}^{\mathcal{G}}_p$ vanish at any point of the bicharacteristic.

Partition $\mathbb{R} \setminus B$ into

$$L = \{s \in \mathbb{R} \setminus B;\ \gamma(s) \in T^*\mathcal{L} \setminus (\mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial)\},$$
$$G = \{s \in \mathbb{R} \setminus B;\ \gamma(s) \in \mathcal{G}^3_\partial \cup \mathcal{G}^{\mathrm{g}}_\partial\}.$$

As equation (3.1) is of geometrical nature, it suffices to check that it holds in some local chart. The argument is simple away from the boundary. We thus choose a local chart $(\mathcal{O}, \phi)$ at the boundary of $\partial\mathcal{L}$, see Section 2.1, and we use the quasi-geodesic coordinates of Proposition 2.1 to simplify some computations.

Consider $a \in \mathscr{C}_c^1\big(T^*(\mathcal{O})\big)$. One writes

$$\langle {}^t\mathrm{H}_p\,\mu, a\rangle = \langle \mu, \mathrm{H}_p\,a\rangle = \int_{\mathbb{R}\setminus B} (\mathrm{H}_p\,a)\big(\gamma(s)\big)\,ds.$$

If $s \in L$ then

$$(\mathrm{H}_p\,a)\big(\gamma(s)\big) = \frac{d}{ds} a\big(\gamma(s)\big). \tag{A.2}$$

If $s \in G$ then

$$\mathrm{H}_p\left(\gamma(s)\right) = \mathrm{H}_p^{\mathcal{G}}\left(\gamma(s)\right) - \frac{\mathrm{H}_p^2\, z}{\mathrm{H}_z^2\, p}\big(\gamma(s)\big)\,\mathrm{H}_z = \mathrm{H}_p^{\mathcal{G}}\left(\gamma(s)\right) + \frac{1}{2}\,\mathrm{H}_p^2\, z\big(\gamma(s)\big)\partial_\zeta,$$

by (2.8) (see also Lemma 5.18) and using that $\mathrm{H}_z^2\, p = 2$ at $z = 0$ in the chosen quasi-geodesic coordinates. Thus one finds

$$(\mathrm{H}_p\,a)\big(\gamma(s)\big) = \frac{d}{ds} a\big(\gamma(s)\big) + \frac{1}{2}\big(\,\mathrm{H}_p^2\, z\,\partial_\zeta a\big)\big(\gamma(s)\big). \tag{A.3}$$

With (A.2) and (A.3) one obtains

$$\langle {}^t\mathrm{H}_p\,\mu, a\rangle = \frac{1}{2} A_1 + A_2, \tag{A.4}$$

with

$$A_1 = \int_G (\mathrm{H}_p^2\, z)\partial_\zeta a\big(\gamma(s)\big) ds \quad \text{and} \quad A_2 = \int_{\mathbb{R}\setminus B} \frac{d}{ds} a\big(\gamma(s)\big) ds.$$

First, consider the term A_1 in (A.4). With (3.2), for $\varrho \in \mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^{\mathrm{g}}$ such as $\gamma(s)$ for $s \in G$ one can write

$$\partial_\zeta a(\varrho) = \lim \frac{\langle \delta_{\varrho^{(n)+}} - \delta_{\varrho^{(n)-}}, a\rangle}{\langle \xi^{(n)+} - \xi^{(n)-}, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}},$$

for a sequence $(\varrho^{(n)})_n \subset {}^{\|}\mathcal{H}_\partial$ that converges to ϱ. With the notation understanding used for the measure propagation equation (3.1) for the part of the integration performed on ${}^{\|}\mathcal{G}_\partial = \mathcal{G}_\partial$, one thus finds

$$A_1 = -\int_{{}^{\|}\mathcal{H}_\partial \cup {}^{\|}\mathcal{G}_\partial} \frac{\langle \delta_{\varrho^+} - \delta_{\varrho^-}, a\rangle}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}} d\nu_G, \tag{A.5}$$

with $d\nu_G = -\,\mathrm{H}_p^2\, z\, \mathbf{1}_{\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^g}\, \delta_{\mathfrak{G}_{\bar\gamma}}$, measure on ${}^{\|}\partial(T^*\mathcal{L})$. It is nonnegative as one has $-(\mathrm{H}_p^2\, z)\mathbf{1}_{\mathcal{G}_\partial^3 \cup \mathcal{G}_\partial^g} \geq 0$.

Second, before considering the term A_2 in (A.4), with ${}^{\|}\gamma(s) = \pi_{\|}\big(\gamma(s)\big)$ set

$$ {}^{\|}\mathcal{H}_\partial^\gamma = \{{}^{\|}\gamma(s);\ s \in B\} \subset {}^{\|}\mathcal{H}_\partial. $$

Like B, this is a discrete set. Write $\gamma(s)$ in the $\mathscr{C}^1$-variables $({}^{\|}\varrho, \vartheta)$, that is, $\gamma(s) = \big({}^{\|}\gamma(s), \vartheta(s)\big)$. Recall that $\vartheta = \mathrm{H}_p\, z$ and $\vartheta_{|z=0} = 2\zeta$ in the used quasi-geodesic coordinates. Set $[\vartheta]_s = \vartheta(s^+) - \vartheta(s^-)$ for $s \in B$. Note that $[\vartheta]_s = 2\vartheta(s^+) > 0$ since $\vartheta(s^+) > 0$ and $\vartheta(s^-) = -\vartheta(s^+)$. One needs the following lemma whose proof is given below.

Lemma A.2.

(1) *Suppose I is a bounded interval. The series $\sum_{s\in B\cap I} \vartheta(s^+)$ is absolutely convergent.*

(2) *The series $\nu_H = \sum_{\varrho\in {}^{\|}\mathcal{H}_\partial^\gamma} \langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M}, T_x\mathcal{M}}\ \delta_\varrho$ yields a nonnegative Radon measure on ${}^{\|}\partial(T^*\mathcal{L})$.*

Third, consider the term A_2 in (A.4). Recall from the main text that ${}^{\|}\gamma(s)$ is defined in $\mathbb{R}\setminus B$ and yet can be continuously extended to $\mathbb{R}$ since ${}^{\|}\gamma(s^-) = {}^{\|}\gamma(s^+) \in {}^{\|}\mathcal{H}_\partial$ if $s \in B$. Moreover this extended function is Lipschitz on $\mathbb{R}$.

Express the $\mathscr{C}_c^1$-function $a(\varrho)$ in terms of the variables $({}^{\|}\varrho, \vartheta)$. The term A_2 in (A.4) reads

$$ A_2 = \int_{\mathbb{R}\setminus B} \frac{d}{ds} b(s) ds \quad \text{with} \quad b(s) = a\big({}^{\|}\gamma(s), \vartheta(s)\big) - a({}^{\|}\gamma(s), 0), $$

using that $\int_{\mathbb{R}\setminus B} \frac{d}{ds} a\big({}^{\|}\gamma(s), 0\big) ds = \int_{\mathbb{R}} \frac{d}{ds} a\big({}^{\|}\gamma(s), 0\big) ds = 0$ since $a\big({}^{\|}\gamma(s), 0\big)$ is Lipschitz thus absolutely continuous.

Note that $s \in \operatorname{supp} b$ implies $\gamma(s) \in \operatorname{supp} a$ or $({}^{\|}\gamma(s), 0) \in \operatorname{supp} a$. Hence, there exists an open bounded interval J such that $\operatorname{supp} b \subset J$ yielding

$$ A_2 = \int_{J\setminus B} \frac{d}{ds} b(s) ds. $$

One has

$$ \begin{aligned} \frac{d}{ds} b\big({}^{\|}\gamma(s), \vartheta(s)\big) = {} & d_{{}^{\|}\varrho} a\big({}^{\|}\gamma(s), \vartheta(s)\big)\big({}^{\|}\gamma'(s)\big) - d_{{}^{\|}\varrho} a\big({}^{\|}\gamma(s), 0\big)\big({}^{\|}\gamma'(s)\big) \\ & + \vartheta'(s) \partial_\zeta a\big({}^{\|}\gamma(s), \vartheta(s)\big) \text{ a.e.} \end{aligned} $$

As $(\overline{B}\setminus B)\subset G$ by Lemma 5.38, if $s\in\overline{B}\setminus B$ one has $\vartheta(s)=\mathrm{H}_p\,z\big(\gamma(s)\big)=0$ and $\vartheta'(s)=\mathrm{H}_p^{\mathcal{G}}\,\mathrm{H}_p\,z\big(\gamma(s)\big)=0$ yielding $\frac{d}{ds}b\big({}^{\|}\gamma(s),\vartheta(s)\big)=0$. One concludes that

$$A_2=\int_{J\setminus\overline{B}}\frac{d}{ds}b(s)ds.$$

If $s\in J\setminus\overline{B}$, there exists $\varepsilon>0$ such that $[s-\varepsilon,s+\varepsilon]\subset J$ and $[s-\varepsilon,s+\varepsilon]\cap\overline{B}=\emptyset$. Denote by $I_s\subset J$ the largest interval such that $s\in I_s$ and $I_s\cap\overline{B}=\emptyset$. For $n\in\mathbb{N}^*$ set

$$R_n=\{s\in J\setminus\overline{B};\ |I_s|\geq 1/n\}.$$

Observe that R_n is a finite union of disjoint intervals $I_{n,1},\dots,I_{n,k(n)}$ all subsets of J. Note that $R_n\subset R_{n+1}$, and $J\setminus\overline{B}=\cup_n R_n$, thus writing $J\setminus\overline{B}$ as an at-most-countable union of disjoint intervals. Note that

$$B\subset E=\{\inf I_{n,j},\sup I_{n,j};\ n\in\mathbb{N},j=1,\dots,k(n)\}.$$

and

$$E\setminus B\subset(\overline{B}\setminus B)\cup\{\inf J,\sup J\}.$$

One obtains

$$A_2=\lim_{n\to+\infty}\int_{R_n}\frac{d}{ds}b(s)ds=\lim_{n\to+\infty}\sum_{j=1}^{k(n)}\int_{I_{n,j}}\frac{d}{ds}b(s)ds. \tag{A.6}$$

One computes

$$\int_{I_{n,j}}\frac{d}{ds}b(s)ds=b(\sup I_{n,j}^-)-b(\inf I_{n,j}^+). \tag{A.7}$$

Note that if $s\in(\overline{B}\setminus B)\cup\{\inf J,\sup J\}$ then $b(s)=0$. Hence, non vanishing terms on the rhs of (A.7) correspond only to the cases $s=\sup I_{n,j}$ lying in $B\cap J$ and $s=\inf I_{n,j}$ lying in $B\cap J$. If $s\in B\cap J$ the term $b(s^-)$ appears exactly once on the r.h.s. of (A.6) as in (A.7). The same holds for the term $-b(s^+)$. One has

$$|b(s^\pm)|=|a\big({}^{\|}\gamma(s),\vartheta(s^\pm)\big)-a({}^{\|}\gamma(s),0)|\leq C|\vartheta(s^\pm)|.$$

By Lemma A.2 the series $\sum_{s\in B\cap J}|\vartheta(s^+)|$ is absolutely convergent using that J is a bounded interval. As $|\vartheta(s^-)|=|\vartheta(s^+)|$ the same holds for $\sum_{s\in B\cap J}|\vartheta(s^-)|$. Hence,

the series $\sum_{s\in B\cap J} b(s^-)$ and $\sum_{s\in B\cap J} b(s^+)$ are absolutely convergent. Summation order is thus not of importance and with (A.6) one obtains

$$A_2 = \sum_{s\in B\cap J} \big(b(s^-) - b(s^+)\big) = -\sum_{s\in B\cap J} [a\circ\gamma]_s \tag{A.8}$$
$$= -\int_{\|\mathcal{H}_\partial\cup\|\mathcal{G}_\partial} \frac{\langle \delta_{\varrho^+} - \delta_{\varrho^-}, a\rangle}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}}}\, d\nu_H(\varrho).$$

with ν_H as given in Lemma A.2. Combining (A.5) and (A.8) yields the result with $\nu = \frac{1}{2}\nu_G + \nu_H$. □

Proof of Lemma A.2. The first result is trivial if $\#I\cap B$ is finite. Assume it is infinite. At a point $s\notin B$ one has $\vartheta'(s) = \mathrm{H}_p^2\, z\big(\gamma(s)\big)$ or $\vartheta'(s) = (\mathrm{H}_p^{\mathcal{G}}\,\mathrm{H}_p\, z)\big(\gamma(s)\big) = 0$. It implies that $\vartheta'(s)$ is bounded on $I\setminus B$.

Pick $s\in I\cap B$ with $s < \sup(I\cap\overline{B})$. There exists $\tilde{s}\in I\cap\overline{B}$ with $s<\tilde{s}$ and $]s,\tilde{s}[\cap\overline{B} = \emptyset$. One has $\vartheta(\tilde{s})\le 0$ (<0 if $\tilde{s}\in B$ and $=0$ if $\tilde{s}\in\overline{B}$). Thus one has

$$\big|\vartheta(s^+)\big| \le \sup_{I\setminus B} |\vartheta'|\,|\tilde{s}-s|.$$

If $s\in I\cap B$ and $s = \sup(I\cap\overline{B})$ and $s > \inf(I\cap\overline{B})$ on picks $\tilde{s}\in I\cap\overline{B}$ with $\tilde{s}<s$ and argues similarly. This implies $\sum_{s\in I\cap B}\big|\vartheta(s^+)\big| \le |I|\sup_{I\setminus B}|\vartheta'|$.

If $\varrho\in{}^{\|}\mathcal{H}_\partial^\gamma$, as $\vartheta_{|z=0} = 2\zeta$ in the present coordinates, by (2.6) one has

$$\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T_x^*\mathcal{M},T_x\mathcal{M}} = [\mathrm{H}_p\, z]_\varrho, \quad \text{with} \quad [\mathrm{H}_p\, z]_\varrho = \mathrm{H}_p\, z(\varrho^+) - \mathrm{H}_p\, z(\varrho^-).$$

The series reads $\nu_H = \frac{1}{2}\sum_{\varrho\in{}^{\|}\mathcal{H}_\partial^\gamma}[\mathrm{H}_p\, z]_\varrho\,\delta_\varrho$. As $[\mathrm{H}_p\, z]_\varrho > 0$, this is a positive measure. Continuity remains to be proven. Consider a smooth test function a with compact support and set $b(s) = a\big({}^{\|}\gamma(s)\big)$ and $\vartheta(s) = \mathrm{H}_p\, z\big(\gamma(s)\big)$ as above. One has $\langle\nu_H, a\rangle = \langle\tilde{\nu}_H, b\rangle$ where

$$\tilde{\nu}_H = \frac{1}{2}\sum_{s\in B}[\vartheta]_s\,\delta_s \quad \text{with} \quad [\vartheta]_s = \vartheta(s^+) - \vartheta(s^-) > 0.$$

Continuity of ν_H is equivalent to that of $\tilde{\nu}_H$. This follows from the first part since the series $\sum_{s\in B\cap I}\vartheta(s^+)$ and $\sum_{s\in B\cap I}\vartheta(s^-)$ converge; recall that $\vartheta(s^-) = -\vartheta(s^+)$. □

Appendix B. Existence and continuity properties of generalized bicharacteristics

B.1. **Existence of generalized bicharacteristics.** Here, we show how our main result, Theorem 3.4, can be used to prove Theorem 2.10, that is, the existence of generalized bicharacteristics without having to carry out a very similar and rather long proof.

As in the main text, the manifold $T^*\mathcal{L}$ is slightly extended beyond its boundary and the metric g and the wave symbol p are extended in a $\mathscr{C}^1$ manner. With the Liouville measure $\omega^{d+1}/(d+1)!$ one identifies a function with a measure on $T^*\hat{\mathcal{L}}$, for instance $\mathbf{1}_{T^*\mathcal{L}}$.

Suppose $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in T^*\mathcal{L} \cap \operatorname{Char} p$. If $\tau^0 = 0$ then $(\tau^0, \xi^0) = 0$ and ${}^{\mathsf{G}}\gamma(s) = \varrho^0$, $s \in \mathbb{R}$, is a maximal generalized bicharacteristic that goes through ϱ^0. Assume now that $\tau^0 \neq 0$.

Note that, away from the zero section, $\operatorname{Char} p$ is a $\mathscr{C}^1$-submanifold of codimension one of $T^*\hat{\mathcal{L}}$, and consider the uniform measure density μ_p on $\operatorname{Char} p \cap T^*\mathcal{L} \cap \{\tau = \tau^0\}$, that is, $\mu_p = \mathbf{1}_{T^*\mathcal{L}}\, \delta_{p=0}\delta_{\tau=\tau^0}$. Note that this product of measures makes sense if one considers the product of distributions with a wavefront set criterium; see for instance Section 9.2 in [16].

One has $\mathrm{H}_p\, p = 0$, meaning that the Hamiltonian vector field H_p is tangent to $\operatorname{Char} p$, and $\mathrm{H}_p\, \tau = 0$, and thus $\mathrm{H}_p\, \delta_{p=0}\delta_{\tau=\tau^0} = 0$ impling that $\mathrm{H}_p\, \mu_p = 0$ away from $\partial(T^*\mathcal{L})$.

With the same arguments used in Proposition 5.13 one finds that $\operatorname{Char} p \cap \partial(T^*\mathcal{L})$ is a submanifold of codimension two, locally given by $\{p = 0, z = 0\}$. One has $\operatorname{Char} p \cap \partial(T^*\mathcal{L}) = \mathcal{H}_\partial \cup \mathcal{G}_\partial$ as expressed below Definition 5.8.

Near the boundary, in a local chart as in (2.1), use the quasi-normal geodesic coordinates of Proposition 2.1. One has $\mu_p = \mathbf{1}_{z\geq 0}\, \delta_{p=0}\delta_{\tau=\tau^0}$. The Leibnitz rule applies to this distribution product and one finds

$$\mathrm{H}_p\, \mu_p = (\mathrm{H}_p\, \mathbf{1}_{z\geq 0})\, \delta_{p=0}\delta_{\tau=\tau^0} = (\mathrm{H}_p\, z)\delta_{z=0}\delta_{p=0}\delta_{\tau=\tau^0}.$$

As above the product $\delta_{z=0}\delta_{p=0}\delta_{\tau=\tau^0}$ makes sense under the wavefront set criterium and $\mathrm{H}_p\, \mu_p$ is a measure. The measure $\delta_{z=0}\delta_{p=0}$ is the uniform positive measure ℓ for $\operatorname{Char} p \cap \{z = 0\}$ inherited from the Liouville measure. Note that $\mathbf{1}_{\mathcal{G}_\partial}\, \mathrm{H}_p\, \mu_p = 0$, since $\mathrm{H}_p\, z = 0$ on $\mathcal{G}_\partial$. Hence one can write

$$\mathrm{H}_p\, \mu_p = \mathbf{1}_{\mathcal{H}_\partial}\, \mathrm{H}_p\, \mu_p,$$

and

$$\langle \mathrm{H}_p\, \mu_p, a\rangle = \int_{\mathcal{H}_\partial} (\mathrm{H}_p\, z) a_{|\substack{z=0\\p=0}}(\varrho)\, d\ell_{\tau^0}(\varrho),$$

with $\ell_{\tau^0} = \delta_{\tau=\tau^0}\,\ell$. Since $(z,\zeta) \mapsto (-z,-\zeta)$ leaves ω^{d+1} and thus ℓ invariant and moreover exchanges $\mathcal{H}^+_\partial$ and $\mathcal{H}^-_\partial$ (as $\mathrm{H}_p\, z = 2\zeta$ at $z = 0$ in the used quasi-geodesic coordinates), one finds

$$\langle \mathrm{H}_p\,\mu_p, a\rangle = \int_{\mathcal{H}^+_\partial} \Big((\mathrm{H}_p\, z) a_{|\substack{z=0\\p=0}}(\varrho) + (\mathrm{H}_p\, z) a_{|\substack{z=0\\p=0}}\big(\Sigma(\varrho)\big)\Big) d\ell_{\tau^0}(\varrho),$$

with Σ defined in (2.7). Denoting by ${}^{\|}\ell_{\tau^0}$ the pullback of the measure ℓ_{τ^0} by the diffeomorphism

$$\begin{aligned} {}^{\|}\mathcal{H}_\partial &\to \mathcal{H}^+_\partial \\ \varrho &\mapsto \varrho^+, \end{aligned}$$

one obtains

$$\langle \mathrm{H}_p\,\mu_p, a\rangle = \int_{{}^{\|}\mathcal{H}_\partial} \Big((\mathrm{H}_p\, z) a_{|\substack{z=0\\p=0}}(\varrho^+) + (\mathrm{H}_p\, z) a_{|\substack{z=0\\p=0}}(\varrho^-)\Big) d\,{}^{\|}\ell_{\tau^0}(\varrho),$$

where $\varrho^+ \in \mathcal{H}^+_\partial$ and $\varrho^- = \Sigma(\varrho^+) \in \mathcal{H}^-_\partial$ with $\pi_{\|}(\varrho^+) = \pi_{\|}(\varrho^-) = \varrho$ if $\varrho \in {}^{\|}\mathcal{H}_\partial$ as in Section 2.3. One has $0 < \mathrm{H}_p\, z(\varrho^+) = -\,\mathrm{H}_p\, z(\varrho^-)$ yielding

$$\langle \mathrm{H}_p\,\mu_p, a\rangle = \int_{{}^{\|}\mathcal{H}_\partial} (\mathrm{H}_p\, z)(\varrho^+)\,\big\langle \delta_{\varrho^+} - \delta_{\varrho^-}, a\big\rangle\, d\,{}^{\|}\ell_{\tau^0}(\varrho).$$

As computed in (5.38) one has $\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}} = 2\alpha(x)\,\mathrm{H}_p\, z(\varrho^+) = \mathrm{H}_p\, z(\varrho^+)$ since $\alpha = 1/2$ at $z = 0$ in the present quasi-geodesic coordinates. One can thus write

$$\mathrm{H}_p\, z(\varrho^+) = \frac{\mathrm{H}_p\, z(\varrho^+)^2}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}}},$$

implying

$$\langle \mathrm{H}_p\,\mu_p, a\rangle = \int_{{}^{\|}\mathcal{H}_\partial} \frac{\langle \delta_{\varrho^+} - \delta_{\varrho^-}, a\rangle}{\langle \xi^+ - \xi^-, \mathsf{n}_x\rangle_{T^*_x\mathcal{M}, T_x\mathcal{M}}} d\nu(\varrho),$$

with ν a nonnegative measure on ${}^{\|}\mathcal{H}_\partial$ given by $d\nu(\varrho) = \mathrm{H}_p\, z(\varrho^+)^2 d\,{}^{\|}\ell_{\tau^0}(\varrho)$. This is precisely the form of the equation one has in Assumption 3.2 for the semiclassical measure. The other condition on this measure stated in Assumption 3.1 is obvious here: the measure μ_p is supported in $\mathrm{Char}\, p \cap T^*\mathcal{L} \setminus 0$. Consequently, all the constructions made in Section 6 can be carried out with μ replaced by μ_p. Theorem 3.4 then implies Theorem 2.10.

Remark B.1. Note that one can replace the use of $\delta_{\tau=\tau^0}$ with $\mathbf{1}_{\tau^1<|\tau|<\tau^2}$ for some $0 < \tau^1 < |\tau^0| < \tau^2 < \infty$. The argument remains the same.

B.2. **Continuity properties of generalized bicharacteristics.** Here, we prove Proposition 2.11. One proceeds by contradiction. Then, there exists $\varrho^0 \in \operatorname{Char} p \cap T^*\mathcal{L}$ and $T > 0$ and $\varepsilon^0 > 0$, and for all $n \in \mathbb{N}^*$ there exists ϱ_n, ϱ_n^1 such that $\operatorname{dist}(\varrho_n^1, \varrho^0) \leq 1/n$, $\varrho_n \in \Gamma^T(\varrho_n^1)$, and $\operatorname{dist}\big(\varrho_n, \Gamma^T(\varrho^0)\big) \geq \varepsilon^0$.

Write $\varrho_n^1 = (t_n^1, x_n^1, \tau_n^1, \xi_n^1)$ and $\varrho_n = (t_n, x_n, \tau_n, \xi_n)$. One has $\varrho_n = {}^{\mathsf{G}}\hat{\gamma}_n(s_n)$ for some $s_n \in \mathbb{R}$ with ${}^{\mathsf{G}}\hat{\gamma}_n$ a generalized bicharacteristic such that ${}^{\mathsf{G}}\hat{\gamma}_n(0) = \varrho_n^1$. Possibly $\varrho_n = {}^{\mathsf{G}}\hat{\gamma}_n(s_n^{\pm})$, if $\varrho_n \in \mathcal{H}_\partial$. For all generalized bicharacteristics, τ is constant; thus, $\tau_n = \tau_n^1$.

If $\varrho^0 = (t^0, x^0, \tau^0, \xi^0)$, the assumption states $(\tau^0, \xi^0) \neq 0$. Since $\varrho^0 \in \operatorname{Char} p$ one has $\tau^0 \neq 0$. One has $|\tau^0|/2 \leq |\tau_n^1| \leq 2|\tau^0|$ for n sufficiently large. Set $S = 2T/|\tau^0|$. Denote ${}^{\mathsf{G}}\hat{\gamma}_n(s) = (\hat{t}_n(s), \hat{x}_n(s), \tau_n^1, \hat{\xi}_n(s))$. For $|s| > S/2$ one has $|\hat{t}_n(s) - t_n^1| > |\tau_n^1| S > T$. This gives $s_n \in [-S/2, S/2]$ for the value of s_n introduced above.

Set

$$
{}^{c,\mathsf{G}}\hat{\gamma}_n(s) = \begin{cases} {}^c\phi\big({}^{\mathsf{G}}\hat{\gamma}_n(s)\big) = {}^{\mathsf{G}}\hat{\gamma}_n(s) & \text{if } s \notin B_n \\ {}^c\phi\big(\lim_{s' \to s^-} {}^{\mathsf{G}}\hat{\gamma}_n(s)\big) = {}^c\phi\big(\lim_{s' \to s^+} {}^{\mathsf{G}}\hat{\gamma}_n(s)\big) & \text{if } s \in B_n. \end{cases}
$$

The map ${}^c\phi$ is defined in Section 5.5. The curve ${}^{c,\mathsf{G}}\hat{\gamma}_n(s)$ is continuous with values in the compressed cotangent bundle ${}^cT^*\mathcal{L}$.

The sequence of functions ${}^{c,\mathsf{G}}\hat{\gamma}_{n|[-S,S]}$ is equicontinuous. By the Arzelà-Ascoli theorem one can extract a subsequence $(s \mapsto {}^{c,\mathsf{G}}\hat{\gamma}_{n_p})_{p \in \mathbb{N}}$ that converges uniformly to a curve ${}^{c,\mathsf{G}}\hat{\gamma}(s)$ in ${}^cT^*\mathcal{L}$ for $s \in [-S, S]$ and s_{n_p} converges to some $S' \in [-S/2, S/2]$. Write n in place of n_p for simplicity. In particular, one has ${}^{c,\mathsf{G}}\hat{\gamma}(0) = {}^c\phi(\varrho^0)$. Set

$$
B = \{s \in [-S, S];\ {}^{c,\mathsf{G}}\hat{\gamma}(s) \in {}^c\phi(\mathcal{H}_\partial)\}.
$$

For $s \in [-S, S] \setminus B$ one defines ${}^{\mathsf{G}}\hat{\gamma}(s) = {}^c\phi^{-1}({}^{c,\mathsf{G}}\hat{\gamma}(s))$. One has ${}^{\mathsf{G}}\hat{\gamma}(0) = \varrho^0$, with the understanding that ${}^{\mathsf{G}}\hat{\gamma}(0^{\pm}) = \varrho^0$ if $\varrho^0 \in \mathcal{H}_\partial^{\pm}$. With the same sequence of arguments as in Section 6.5.1 one obtains that ${}^{\mathsf{G}}\hat{\gamma}(s)$ is a generalized bicharacteristic for $s \in [-S, S]$. This generalized bicharacteristic can be extended to a maximal generalized bicharacteristic by Theorem 2.10.

Assume first $S' \notin B$. One has ${}^{c,\mathsf{G}}\hat{\gamma}_n(s_n) \to {}^{c,\mathsf{G}}\hat{\gamma}(S')$ with the equicontinuity property and the uniform convergence. Then, $\varrho_n = {}^{\mathsf{G}}\hat{\gamma}_n(s_n) \to {}^{\mathsf{G}}\hat{\gamma}(S')$. One has $t_n - t_n^1 = -2\tau_n^1 s_n$. With $|t_n - t_n^1| \leq T$ this gives $|\tau_n^1 s_n| \leq T/2$. In the limit, this gives $|\tau^0 S'| \leq T/2$, implying $|\hat{t}(S') - t^0| \leq T$. Hence, ϱ_n converges to a point of $\Gamma^T(\varrho^0)$. A contradiction.

Assume second that $S' \in B$. One has ${}^{c,\mathsf{G}}\hat{\gamma}_n(s_n) \to {}^{c,\mathsf{G}}\hat{\gamma}(S')$. This means that one can find a subsequence of ϱ^n that converges either to ${}^{\mathsf{G}}\hat{\gamma}(S'^+)$ or to ${}^{\mathsf{G}}\hat{\gamma}(S'^-)$. Then, one reaches the same contradiction. □

Appendix C. Quasi-normal geodesic coordinates

This section is devoted to the proof of Proposition 2.1. In fact, we prove the following more general result that is adatpted to various levels of regularity.

Proposition C.1 (quasi-normal coordinates)**.** *Suppose $k \geq 0$ and $\mathcal{M}$ is a d-dimensional manifold of class $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$) equipped with a metric of class $\mathscr{C}^k$ (resp. $W^{k,\infty}$). Suppose $\mathcal{N}$ is a submanifold of class $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$) and codimension one and $m^0 \in \mathcal{N}$. There exists a local chart (O, ϕ) with regularity consistent with that of $\mathcal{M}$, that is, $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$), such that $m^0 \in O$, $\phi(m) = (x', z)$ with $x' \in \mathbb{R}^{d-1}$ and $z \in \mathbb{R}$ and*

1. *the vector field ∂_z is transverse to $\mathcal{N}$;*
2. *$\phi(O \cap \mathcal{N}) = \{z = 0\} \cap \phi(O)$;*
3. *on $\mathcal{N}$ the representative of the metric reads*

$$g(x', 0) = \sum_{1 \leq i,j \leq d-1} g_{ij}(0, x') dx^i \otimes dx^j + |dz|^2.$$

If $k \geq 1$ one has

$$g_{jd}(x', z) = z h_{jd}(x', z) \quad \text{and} \quad g_{dd}(x', z) = 1 + z h_{dd}(x', z),$$

for some h_{jd}, $j = 1, \dots, d$, of class $\mathscr{C}^{k-1}$ (resp. $W^{k-1,\infty}$).

Proof. Consider a local chart (O^0, ϕ^0) such that $m^0 \in O^0$, with coordinates (x', x_d) with $\phi^0(O^0 \cap \mathcal{N})$ given by $\{x_d = 0\}$. Without loss of generality we may assume $x^0 = \phi^0(m^0) = 0$. Denote by g^0 the representative of the metric in this local chart. In these coordinates, at $(x', 0)$ the one form $\nu_{x'} = \big(g^{0,dd}(x', 0)\big)^{-1/2} dx^d$ is unitary with respect to $g_x^* = \big(g^{ij}(x)\big) = \big(g_{ij}(x)\big)^{-1}$ and $n_{x'} = \nu_{x'}^\sharp$ is a unitary vector field with respect to $g_x = \big(g_{ij}(x)\big)$ and orthogonal to $T_{(x',0)}\mathcal{N}$ in the sense of the metric g_x, thus transverse to $\mathcal{N}$. Its coodinates are given by

$$n^j_{x'} = (g^{0,dd})^{-1/2} g^{0,jd}(x', 0).$$

Note that the vector field n defined along $\mathcal{N}$ has the regularity of the metric, that is, $\mathscr{C}^k$ (resp. $W^{k,\infty}$).

Suppose $R > 0$ and $E > 0$ are such that $B'(0, 2R) \times [-E, E] \subset \phi^0(O^0)$, where $B'(0, a)$ denotes the Euclidean ball of radius a centered at $x^0 = 0$ in the x'

variables. Suppose $\chi \in \mathscr{C}_c^\infty(\mathbb{R}^{d-1})$ with $\chi = 1$ in a neighborhood of $B'(0,R)$ and $\operatorname{supp}\chi \subset B'(0,2R)$. Introduce the vector field

$$m(x',z) = e^{1-\langle zD_{x'}\rangle}(\chi n)_{x'},$$

with z acting as a parameter at this stage and where $e^{1-\langle zD_{x'}\rangle}$ stands as the operator associated with the Fourier multiplier $e^{1-\langle z\xi'\rangle}$ with the usual notation $\langle u\rangle = \sqrt{1+|u|^2}$ for $u \in \mathbb{R}^{d-1}$.

Lemma C.2. *The function $(x',z) \mapsto m(x',z)$ is of class $\mathscr{C}^k$ (resp. $W^{k,\infty}$). Moreover the function $(x',z) \mapsto zm(x',z)$ is of class $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$).*

A proof of this lemma is given below.

Consider now the $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$) function

$$\phi(x',z) = (x',0) + zm(x',z).$$

Observe that $\partial_{x_j}\phi(x',0) = e_j$, $j = 1,\dots,d-1$, the Euclidean unit vector in the j-direction and $\partial_z\phi(x',0) = m(x',0) = \chi n_{x'} = n_{x'}$ if $x' \in B'(0,R)$. The Jacobian matrix of ϕ is thus full rank in a neighborhood of x^0. Thus, ϕ is a local $\mathscr{C}^{1+k}$ (resp. $W^{1+k,\infty}$) diffeomorphism, and (x',z) provides local coordinates for a neighborhood of x^0 in $\mathbb{R}^d$ of the form $(x',z) \in B'(0,r)\times]-e,e[$ for some $0 < r < R$ and $0 < e < E$ chosen sufficiently small, and thus coordinates for an open neighborhood of m^0 in $\mathcal{M}$. These coordinates have the announced regularity.

We claim that ∂_z is the representative of $n_{x'}$ at $x = (x', z = 0)$ in the (x',z) coordinates. In fact, consider a function on $\mathcal{M}$ with f as its representative in the original coordinates (x',x_d) and $\tilde f$ its representative in the (x',z) coordinates. Then $\tilde f = f \circ \phi$ and one has

$$\partial_z\tilde f(x',z) = df\big(\phi(x',z)\big)\big(\partial_z\phi(x',z)\big).$$

It was seen above that $\partial_z\phi(x',0) = n_{x'}$ for $x' \in B'(0,r)$ implying

$$\partial_z\tilde f(x',0) = df(x',0)(n_{x'}) = n_{x'}(f),$$

which is precisely our claim.

Note that $\phi(x',0) = (x',0)$ meaning that $\mathcal{N}$ is locally given by $\{z=0\}$ and one sees that ∂_z is transverse to $\mathcal{N}$ and $\{z>0\}$ coincides locally with $\{x_d>0\}$.

We prove that the metric has the announced structure at a point of $\mathcal{N}$ in the coordinates (x',z). By abuse of notation, still denote the representative of

the metric by g. One has

$$\begin{aligned} g_{ij}(x',0) &= g_{(x',0)}(\partial_{x_i},\partial_{x_j}), \quad 1\le i,j\le d-1,\\ g_{dj}(x',0) &= g_{(x',0)}(\partial_z,\partial_{x_j}), \quad 1\le j\le d-1, \quad \text{and}\\ g_{dd}(x',0) &= g_{(x',0)}(\partial_z,\partial_z). \end{aligned}$$

Since ∂_z is the representative of $n_{x'}$ at $x=(x',z=0)$ in the (x',z) coordinates it follows that

$$g_{dj}(x',0)=0, \quad 1\le j\le d-1, \quad \text{and} \quad g_{dd}(x',0)=1,$$

since $n_{x'}$ is orthogonal to $T_{(x',0)}\mathcal{N}$ and unitary in the sense of the metric g. This concludes the proof of Proposition 2.1. □

Proof of Lemma C.2. Preliminay observation. Suppose $p:\mathbb{R}^{d-1}\to\mathbb{R}$ and $q:\mathbb{R}\to\mathbb{R}$ are both polynomial. Observe that for $z>0$ the operator

$$p(zD_{x'})q(\langle zD_{x'}\rangle)\exp(1-\langle zD_{x'}\rangle)$$

acts as a convolution with the function $k_z(x')=z^{1-d}\ell(x'/z)$ where

$$\ell(x')=\frac{1}{(2\pi)^{d-1}}\int e^{ix'\cdot\xi'}p(\xi')q(\langle\xi'\rangle)e^{1-\langle\xi'\rangle}d\xi'.$$

Note that $\ell\in\mathscr{S}(\mathbb{R}^{d-1})\subset L^1(\mathbb{R}^{d-1})$ as the inverse Fourier transform of a Schwartz function, yielding $k_z\in L^1(\mathbb{R}^{d-1})$ uniformly in $z>0$ and moreover $k_z\to C\delta$ in the sense of measures with $C=\int\ell$. Here, the Dirac measure acts in the x' variable. If one considers a $\mathscr{C}^0$-function $h(x')$ one thus obtains that $p(zD_{x'})q(\langle zD_{x'}\rangle)\exp(1-\langle zD_{x'}\rangle)h$ is a $\mathscr{C}^0$-function in both variables x' and z.

Recall the Faà di Bruno formula for the repeated differentiation of the composition of two functions of one variable:

$$\frac{d^n}{dz^n}(f\circ h)(z)=\sum\frac{n!}{r_1!\cdots r_n!}f^{(r_1+\cdots+r_n)}\big(h(z)\big)\prod_{j=1}^{n}\Big(\frac{h^{(j)}(z)}{j!}\Big)^{r_j}, \tag{C.1}$$

where the sum is carried out for $r_1+2r_2+\cdots+nr_n=n$. Applying (C.1) with $f(z)=\exp(z)$ and $h(z)=1-\langle z\xi'\rangle$ one finds

$$\frac{d^n}{dz^n}e^{1-\langle z\xi'\rangle}=\sum\frac{n!}{r_1!\cdots r_n!}e^{1-\langle z\xi'\rangle}\prod_{j=1}^{n}\Big(-\frac{1}{j!}\frac{d^j}{dz^j}\langle z\xi'\rangle\Big)^{r_j}, \tag{C.2}$$

and applying (C.1) with $f(z) = \sqrt{z}$ and $h(z) = 1 + |\xi'|^2 z^2$, since $h'(z) = 2|\xi'|^2 z$, $h''(z) = 2|\xi'|^2$ and $h^{(3)} = 0$ one finds for some $\alpha_{m_1,m_2} > 0$

$$\frac{d^j}{dz^j}\langle z\xi'\rangle = \sum_{m_1+2m_2=j} \alpha_{m_1,m_2}\, z^{m_1}|\xi'|^{2(m_1+m_2)}\langle z\xi'\rangle^{1-2(m_1+m_2)}. \tag{C.3}$$

Expanding $|\xi'|^{2(m_1+m_2)} = (\xi_1^2 + \cdots + \xi_{d-1}^2)^{m_1+m_2}$ one writes $z^{m_1}|\xi'|^{2(m_1+m_2)}$ as a linear combination of terms of the form

$$(z\xi')^{\beta_1^j}(\xi')^{\beta_2^j}, \quad \text{with } |\beta_1^j| = m_1 \text{ and } |\beta_2^j| = m_1 + 2m_2 = j.$$

Combining (C.2) and (C.3) one obtains $\frac{d^n}{dz^n}\exp(1-\langle z\xi'\rangle)$ as a linear combination of terms of the form

$$(\xi')^\beta p(z\xi')q(\langle z\xi'\rangle)\exp(1-\langle z\xi'\rangle),$$

with $p : \mathbb{R}^{d-1} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ both polynomial, and with $|\beta| = n$ using that $r_1 + 2r_2 + \cdots + nr_n = n$. One thus obtains $\partial_{x'}^\gamma \partial_z^n m$ as a linear combination of terms of the form

$$p(zD_{x'})q(\langle zD_{x'}\rangle)e^{1-\langle zD_{x'}\rangle}(\partial_{x'})^{\beta+\gamma}(\chi n_{x'}),$$

with $|\beta| = n$. Since $\chi n_{x'}$ is $\mathscr{C}^k$ (resp. $W^{k,\infty}$) with respect to x', with the preliminary observation made above, one concludes that $m(x', z)$ is $\mathscr{C}^k$ (resp. $W^{k,\infty}$) with respect to (x', z).

Observe now that for $n \geq 1$

$$\partial_{x'}^\gamma \partial_z^n \big(zm\big) = n\partial_{x'}^\gamma \partial_z^{n-1} m + z\partial_{x'}^\gamma \partial_z^n m$$

One obtains $\partial_{x'}^\gamma \partial_z^n \big(zm\big)$ as a linear combination of terms of the form

$$p(zD_{x'})q(\langle zD_{x'}\rangle)e^{1-\langle zD_{x'}\rangle}(\partial_{x'})^{\tilde\beta+\gamma}(\chi n_{x'}),$$

with $\tilde p : \mathbb{R}^{d-1} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ both polynomial and with $|\tilde\beta| = n-1$. Arguing as above one concludes that $zm(x', z)$ is $\mathscr{C}^{k+1}$ (resp. $W^{k+1,\infty}$) with respect to (x', z). □

References

[1] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[2] L. Ambrosio. Transport equation and Cauchy problem for non-smooth vector fields. In *Calculus of variations and nonlinear partial differential equations. Lectures given at the C.I.M.E. summer school, Cetraro, Italy, June 27–July 2, 2005.*, pages 1–42. Berlin: Springer, 2008.

[3] L. Ambrosio and G. Crippa. Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields. In *Transport equations and multi-D hyperbolic conservation laws*, pages 3–57. Berlin: Springer, 2008.
[4] L. Ambrosio and G. Crippa. Continuity equations and ODE flows with non-smooth velocity. *Proc. R. Soc. Edinb., Sect. A, Math.*, 144(6):1191–1244, 2014.
[5] C. Bardos, G. Lebeau, and J. Rauch. Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. *SIAM J. Control Optim.*, 30:1024–1065, 1992.
[6] N. Burq, B. Dehman, and J. Le Rousseau. Measure propagation along $\mathscr{C}^0$-vector field and wave controllability on a rough compact manifold. *Anal. & PDE, accepted for publication*, page 34 pages, 2021.
[7] N. Burq, B. Dehman, and J. Le Rousseau. Measure and continuous vector field at a boundary I: propagation equation and wave observability. *Submitted, 119 pages*, 2024.
[8] N. Burq, B. Dehman, and J. Le Rousseau. Measure and continuous vector field at a boundary III: Invariance of a wave measure along a flow. *In preparation*, 2024.
[9] N. Burq and P. Gérard. Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(7):749–752, 1997.
[10] N. Burq and I. Moyano. Propagation of smallness and control for heat equations. *J. Eur. Math. Soc., to appear*, 2021. Doi: 10.4171/JEMS/1213.
[11] J.-P. Demailly. *Analyse numérique et équations différentielles*. Grenoble: Presses Univ. de Grenoble; Les Ulis: EDP Sciences, 1996.
[12] R. J. DiPerna and P. L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.
[13] L. C. Evans and W. Gangbo. *Differential equations methods for the Monge-Kantorovich mass transfer problem*, volume 653 of *Mem. Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 1999.
[14] P. Gérard and É. Leichtnam. Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.*, 71(2):559–607, 1993.
[15] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume III. Springer-Verlag, 1985. Second printing 1994.
[16] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume I. Springer-Verlag, second edition, 1990.
[17] G. Lebeau. Équation des ondes amorties. In *Algebraic and geometric methods in mathematical physics (Kaciveli, 1993)*, volume 19 of *Math. Phys. Stud.*, pages 73–109. Kluwer Acad. Publ., Dordrecht, 1996.
[18] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31:593–617, 1978.
[19] H. Whitney. Differentiable manifolds. *Annals of Mathematics*, 37(3):645—-680, 1936.

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