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# MEASURE AND CONTINUOUS VECTOR FIELD AT A BOUNDARY I: PROPAGATION EQUATIONS AND WAVE OBSERVABILITY

NICOLAS BURQ, BELHASSEN DEHMAN, AND JÉRÔME LE ROUSSEAU

ABSTRACT. The celebrated geometric control condition of Bardos, Lebeau, and Rauch is necessary and sufficient for wave observability [1, 7] and exact controllability. It requires that any point in phase-space be transported by the generalized geodesic flow to the region of observation in some finite time. The initial smoothness ( $\mathcal{C}^\infty$ ) required on the coefficients of the metric to prove that exact control and geometric control are essentially equivalent was subsequently relaxed to  $\mathcal{C}^2$ -metrics/coefficients and  $\mathcal{C}^3$ -domains [2], which is close to the optimal smoothness required to preserve a generalized geodesic flow. In this article, we investigate a natural generalization of the geometric control condition that makes sense for  $\mathcal{C}^1$ -metrics and we prove that wave observability holds under this condition. Moreover, we establish that the observability property is stable under rougher (Lipschitz) perturbation of the metric. We also provide a geometric necessary condition for wave observability to hold. Transport equations that describe the propagation of semi-classical measures are at the heart of the arguments. They are natural extensions to geometries with boundaries of usual transport equations. This article is mainly dedicated to the proof of such propagation equations in this very rough context.

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## 1. INTRODUCTION

An observability inequality for the wave equation is an estimate of the energy of a solution by a “recording” of this solution in a restricted domain for some time  $T > 0$ . This restricted domain can be in the interior of the region  $\Omega$  where waves propagate or in a part of its boundary. One interest in such an inequality lies in its consequences in terms of exact controllability and stabilization. The observability property has been intensively studied during the last decades. Until the end of the 80’s, most of the results were proven under a (global) geometrical assumption called  $\Gamma$ -condition and introduced by J.-L. Lions [26], essentially based on a multiplier method. Later, following Rauch and Taylor [28], Bardos, Lebeau, and Rauch proved observability inequalities from part of the boundary in their seminal article [1], and as a consequence, boundary stabilization, under a microlocal condition, that is, a property in the cotangent bundle  $T^*(\mathbb{R} \times \Omega)$ , the so-called geometric control condition (GCC in short), exhibiting a connection between the set on which observation is performed and the generalized geodesics of the wave operator. In addition, taking into account the work of [7], it is now classical that observability (with stability with respect to the observation set) is equivalent to the GCC. In terms of geodesics, the GCC reads as follows:

*for any point  $x$  and any tangent vector  $v$ , the generalized geodesic initiated at  $(x, v)$  enters the observation region in some time  $T > 0$ .*

Generalized geodesics follow the laws of geometrical optics at boundary points: reflection if the boundary is hit transversally and possible gliding if hit tangentially.

The proofs of the results in [1, 7] are based on microlocal tools, namely, the propagation of wavefront sets or that of microlocal defect measures. Let us notice here that despite their high efficiency and robustness, these methods present the great disadvantage of requiring a lot of regularity for the domain and for the metric/coefficients. Starting from the original result developed in the framework of the Melrose-Sjöstrand  $\mathcal{C}^\infty$ -singularity propagation results, thus

requiring  $\mathcal{C}^\infty$ -smoothness, the theory has been subsequently developed in the framework of microlocal defect measures allowing one to relax the assumptions down to a  $\mathcal{C}^2$ -metric [2], which barely misses the natural smoothness ( $W^{2,\infty}$ ) required to define a geodesic flow (away from any boundary). An important remark is that below this smoothness threshold, for instance for  $\mathcal{C}^1$ -metrics that we will consider, generalized geodesics may still exist as integral curves of the  $\mathcal{C}^0$ -Hamiltonian vector field in the interior of the domain but *uniqueness is lost* in general. A natural question lies in the understanding of the relationship between those nonunique integral curves and the observability property for such a rough metric.

Many attempts were made in the last years (see, for instance, the works [14] in dimension 1, and [12]). In the present article, we reach the lowest possible regularity level for the mere existence of geodesics (with a gain of a full derivative with respect to all previous geometric results) and lowest possible regularity level for observability to hold as exhibited by the counter-example in [9, 10].

We prove the following result: for a  $\mathcal{C}^1$ -metric, observability holds, and consequently exact controllability, if a generalized geometric control condition is satisfied, that is,

*all generalized geodesics enter the observation region in some time  $T > 0$ .*

In particular, if considering a point  $x$  and some tangent vector  $v$ , *all* generalized geodesics initiated at this point with  $v$  as initial direction fulfill this property. When uniqueness of generalized geodesics holds the above condition coincides with the usual GCC. We thus keep the ‘‘GCC’’ denomination.

Moreover, our proof allows us to go beyond the  $\mathcal{C}^1$ -threshold in a perturbative regime and consider cases where the notion of geodesics is lost. We prove that if a reference  $\mathcal{C}^1$ -metric  $g$  satisfies the GCC for some time  $T > 0$ , for any other metric  $\tilde{g}$  close enough to  $g$  in the Lipschitz topology the observability property holds also, moreover in the same time  $T$ . We insist once more on the fact that Lipschitz metrics are too rough to even define geodesics since the associated Hamiltonian vector field is only  $L^\infty$  and hence integral curves do not make sense in general.

The GCC stated above stands as a sufficient condition for observability to hold. We also provide a necessary condition that is natural in the sense that it coincides with the usual necessary condition if uniqueness of the flow holds [7].

Our proof of the observability property relies on two key results on semi-classical measures that appear naturally if the energy of waves concentrates asymptotically:

- The derivation of a transport equation that describes the propagation of such semi-classical measures; this is one of the main contribution of the present article; see Theorem 6.1. Its proof relies on commutator analysis and finely tuned properties of semi-classical operators to address the low regularity level of metric/coefficients and the boundary/manifold.
- The description of the support of measures that are solutions to the above propagation equation in terms of generalized bicharacteristics (whose projection on the base manifold are geodesics) even if uniqueness of such curves fails to hold; this is the main contribution of the companion article [5] stated in Theorem 2.14 here.

In section 1.5 below, we describe how these two results are used in the structure of the proof of observability inequalities.

An important difficulty in the present article is the presence of a boundary. In the case of a compact manifold without boundary, we refer to our much less technically demanding article [4] where both parts of this program are achieved in that simpler setting.

Our proof of a necessary condition for wave observability to hold is quite similar and based on:

- the derivation of a transport equation for semi-classical measures across an isochrone  $\{t = \text{Cst}\}$ ; see Theorem 10.7;
- the use of this measure equation to ensure that a maximal generalized bicharacteristic lies entirely in the support of the measure.

**1.1. Metrics, elliptic operators and wave equations.** Consider a compact connected Riemannian manifold  $\mathcal{M}$  of dimension  $d$  with boundary, endowed with a metric  $g = (g_{ij})$ . At first  $\mathcal{M}$  and its boundary are assumed  $W^{2,\infty}$  and the metric is assumed Lipschitz. Denote by  $\mu_g$  the canonical positive Riemannian density on  $\mathcal{M}$ , that is, the density measure associated with the density function  $(\det g)^{1/2}$ . We also consider a positive Lipschitz function  $\kappa$  and we define the density  $\kappa\mu_g$ .

The  $L^2$ -inner product and norm are considered with respect to this density  $\kappa\mu_g$ , that is,

$$(1.1) \quad (u, v)_{L^2(\mathcal{M})} = \int_{\mathcal{M}} u\bar{v} \kappa\mu_g, \quad \|u\|_{L^2(\mathcal{M})}^2 = \int_{\mathcal{M}} |u|^2 \kappa\mu_g.$$

We denote by  $L^2V(\mathcal{M})$  the space of  $L^2$ -vector fields on  $\mathcal{M}$ , equipped with the norm

$$\|v\|_{L^2V(\mathcal{M})}^2 = \int_{\mathcal{M}} g(v, \bar{v}) \kappa\mu_g, \quad v \in L^2V(\mathcal{M}).$$

We write

$$(\cdot, \cdot)_{L^2(\mathcal{M}, \kappa \mu_g)}, \quad \|\cdot\|_{L^2(\mathcal{M}, \kappa \mu_g)}, \quad \text{and} \quad \|\cdot\|_{L^2 V(\mathcal{M}, \kappa \mu_g)}^2,$$

if needed for clarity in particular if different metrics and functions  $\kappa$  are considered simultaneously.

Recall that the Riemannian gradient and divergence are given by

$$g(\nabla_g f, v) = v(f) \quad \text{and} \quad \int_{\mathcal{M}} f \operatorname{div}_g v \mu_g = - \int_{\mathcal{M}} v(f) \mu_g,$$

for  $f$  a function and  $v$  a vector field with supports away from the boundary, yielding in local coordinates

$$(\nabla_g f)^i = \sum_{1 \leq j \leq d} g^{ij} \partial_{x_j} f, \quad \operatorname{div}_g v = (\det g)^{-1/2} \sum_{1 \leq i \leq d} \partial_{x_i} ((\det g)^{1/2} v^i),$$

with  $(g_x^{ij}) = (g_{x,ij})^{-1}$ . With the Poincaré inequality a norm on  $H_0^1(\mathcal{M})$  is

$$\|u\|_{H_0^1(\mathcal{M})} = \|\nabla_g u\|_{L^2 V(\mathcal{M})}.$$

We introduce the elliptic operator  $A = A_{\kappa, g} = \kappa^{-1} \operatorname{div}_g(\kappa \nabla_g)$ , that is, in local coordinates

$$(1.2) \quad Af = \kappa^{-1} (\det g)^{-1/2} \sum_{1 \leq i, j \leq d} \partial_{x_i} (\kappa (\det g)^{1/2} g^{ij}(x) \partial_{x_j} f).$$

The operator  $A$  is unbounded on  $L^2(\mathcal{M})$ . With the domain  $D(A) = H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$  one finds that  $A$  is selfadjoint, with respect to the  $L^2$ -inner product given in (1.1), and negative.

With the elliptic operator  $A = A_{\kappa, g}$  one also defines the wave operator

$$P = P_{\kappa, g} = \partial_t^2 - A_{\kappa, g}.$$

Consider the wave equation

$$(1.3) \quad \begin{cases} P_{\kappa, g} u = f & \text{in } \mathbb{R} \times \mathcal{M}, \\ u = 0 & \text{in } \mathbb{R} \times \partial \mathcal{M}, \\ u|_{t=0} = \underline{u}^0, \quad \partial_t u|_{t=0} = \underline{u}^1 & \text{in } \mathcal{M}. \end{cases}$$

Solutions are given by the following result.

**Proposition 1.1.** *Consider  $\kappa$  and  $g$  both Lipschitz. Let  $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  and  $f \in L_{\text{loc}}^1(\mathbb{R}, L^2(\mathcal{M}))$ . There exists a unique*

$$u \in \mathcal{C}^0(\mathbb{R}; H_0^1(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathcal{M})),$$

that is, a weak solution of (1.3), meaning  $u|_{t=0} = \underline{u}^0$  and  $\partial_t u|_{t=0} = \underline{u}^1$  and  $P_{\kappa, g} u = f$  holds in  $\mathcal{D}'(\mathbb{R} \times \mathcal{M})$ . The map

$$(1.4) \quad H_0^1(\mathcal{M}) \oplus L^2(\mathcal{M}) \oplus L_{\text{loc}}^1(\mathbb{R}, L^2(\mathcal{M})) \rightarrow \mathcal{C}^0(\mathbb{R}; H_0^1(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathcal{M})) \\ (\underline{u}^0, \underline{u}^1, f) \mapsto u,$$

is continuous.

One denotes by

$$\mathcal{E}_{\kappa, g}(u)(t) = \frac{1}{2} (\|\nabla_g u(t)\|_{L^2 V(\mathcal{M})}^2 + \|\partial_t u(t)\|_{L^2(\mathcal{M})}^2)$$

the energy of  $u$  at time  $t$ . For any  $T > 0$  there exists  $C_T > 0$  such that

$$\sup_{|t| \leq T} \mathcal{E}_{\kappa, g}(u)(t)^{1/2} \leq C_T (\mathcal{E}_{\kappa, g}(u)(0)^{1/2} + \|f\|_{L^1(-T, T; L^2(\mathcal{M}))}).$$

If  $f = 0$ , then equation (1.3) is homogeneous. This is the case we will most often consider for the issue of observability. Then, for the weak solution  $u$ , the energy is independent of time  $t$  that is,

$$\mathcal{E}_{\kappa, g}(u)(t) = \mathcal{E}_{\kappa, g}(u)(0) = \frac{1}{2} (\|\nabla_g \underline{u}^0\|_{L^2 V(\mathcal{M})}^2 + \|\underline{u}^1\|_{L^2(\mathcal{M})}^2).$$

In such case, we simply write  $\mathcal{E}_{\kappa, g}(u)$ .

**1.2. Regularity levels for manifolds, metrics and coefficients.** Two classes of regularity levels will be of importance in what follows. A first one for which microlocal methods apply (the spaces  $\mathcal{X}^1$  and  $\mathcal{X}^2$  below) and a second one for which basic results (uniqueness, traces, etc) remain true (the space  $\mathcal{Y}^1$  below). More precisely denote by  $X^k$  (resp.  $Y^k$ ) the sets of manifolds  $\mathcal{M}$  as above of class  $\mathcal{C}^k$  (resp.  $W^{k, \infty}$ ). This regularity of the manifold includes that of its boundary. Set

$$\mathcal{X}^k = \{(\mathcal{M}, \kappa, g); \mathcal{M} \in X^{1+k}, \kappa \in \mathcal{C}^k(\mathcal{M}) \text{ and } g \text{ is a } \mathcal{C}^k\text{-metric on } \mathcal{M}\}, \\ \mathcal{Y}^k = \{(\mathcal{M}, \kappa, g); \mathcal{M} \in Y^{1+k}, \kappa \in W^{k, \infty}(\mathcal{M}) \text{ and } g \text{ is a } W^{k, \infty}\text{-metric on } \mathcal{M}\},$$



$k \in \mathbb{N}$ . The levels of regularity we use in what follows are  $\mathcal{X}^1$  and  $\mathcal{X}^2$  on the one hand, and  $\mathcal{Y}^1$  on the other hand.

Observe that having the manifold  $\mathcal{M}$  with one order of regularity higher than that for  $g$  is consistent with the transformation rules of a 2-covariant tensor on  $\mathcal{M}$ . Next, the regularity of the function  $\kappa$  is set equal to that of  $g$  because of the definition of the elliptic operator  $A_{\kappa,g}$  in (1.2).

With  $\mathcal{M}$  of class  $\mathcal{C}^{1+k}$  (resp.  $W^{1+k,\infty}$ ) the same holds for  $\partial\mathcal{M}$ . Once an atlas is given on  $\mathcal{M}$  as in Section 2.1, this is quite clear.

**Remark 1.2.** *Note that  $\mathcal{Y}^1$  exhibits a ‘tiny’ loss of regularity if compared to  $\mathcal{X}^1$ . Yet, this loss is more like an abyss as far as the geometry underlying wave propagation is concerned. In fact, if considering a  $W^{1,\infty}$ -metric  $g$  the Hamiltonian vector field that defines the bicharacteristics at higher levels of regularity is only  $L^\infty$  here. Hence, the existence of bicharacteristics is not guaranteed. As a result a  $W^{1,\infty}$ -metric is too rough to state the usual GCC and consequently also to implement standard microlocal tools.*

Based on the previous remark we will exploit the geometry of wave propagation available for some  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$  yet consider some  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  sufficiently close to  $(\mathcal{M}, \kappa, g)$ . Such closedness will be understood as follows.

**Definition 1.3.** *Consider on the one hand  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$  and  $\omega$  an open subset of  $\mathcal{M}$  (resp.  $\Gamma$  an open subset of  $\partial\mathcal{M}$ ) and, on the other hand,  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  and  $\tilde{\omega}$  an open subset of  $\tilde{\mathcal{M}}$  (resp.  $\tilde{\Gamma}$  an open subset of  $\partial\tilde{\mathcal{M}}$ ). Let  $\varepsilon > 0$ . One says that  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\omega})$  (resp.  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\Gamma})$ ) is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \omega)$  (resp.  $(\mathcal{M}, \kappa, g, \Gamma)$ ) in the  $\mathcal{Y}^1$ -topology if the following holds*

- (1) *There exists a  $W^{2,\infty}$ -diffeomorphism  $\psi : \mathcal{M} \rightarrow \tilde{\mathcal{M}}$  such that  $\psi(\omega) = \tilde{\omega}$  (resp.  $\psi(\Gamma) = \tilde{\Gamma}$ ).*
- (2) *One has  $\|\psi^*\tilde{\kappa} - \kappa\|_{W^{1,\infty}(\mathcal{M})} + \|\psi^*\tilde{g} - g\|_{W^{1,\infty}\mathcal{T}_2^0(\mathcal{M})} \leq \varepsilon$ , where  $\|\cdot\|_{W^{1,\infty}\mathcal{T}_2^0(\mathcal{M})}$  denotes the  $W^{1,\infty}$ -norm for 2-covariant tensors on  $\mathcal{M}$ .*

**1.3. Interior and boundary observability.** Consider the following homogeneous version of the wave equation:

$$(1.5) \quad \begin{cases} P_{\kappa,g} u = 0 & \text{in } \mathbb{R} \times \mathcal{M}, \\ u = 0 & \text{in } \mathbb{R} \times \partial\mathcal{M}, \\ u|_{t=0} = \underline{u}^0, \quad \partial_t u|_{t=0} = \underline{u}^1 & \text{in } \mathcal{M}. \end{cases}$$

Let  $\omega$  be a nonempty open subset of  $\mathcal{M}$  and  $T > 0$ . Observability of the wave equation from  $\omega$  in time  $T$  is the following notion.

**Definition 1.4** (interior observability). *Let  $\omega$  be a nonempty open subset of  $\mathcal{M}$ . One says that the homogeneous wave equation is observable from  $\omega$  in time  $T > 0$  if there exists  $C_{\text{obs}} > 0$  such that for any  $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  one has*

$$(1.6) \quad \mathcal{E}_{\kappa,g}(u) \leq C_{\text{obs}} \|\mathbf{1}_{]0,T[ \times \omega} \partial_t u\|_{L^2(\mathbb{R} \times \mathcal{M})}^2,$$

for the weak solution  $u$  to (1.5).

Let  $\Gamma$  be a nonempty open subset of  $\partial\mathcal{M}$ . Observability of the wave equation from  $\Gamma$  in time  $T$  is the following notion.

**Definition 1.5** (boundary observability). *Let  $\Gamma$  be a nonempty open subset of  $\partial\mathcal{M}$ . One says that the homogeneous wave equation is observable from  $\Gamma$  in time  $T > 0$  if there exists  $C_{\text{obs}} > 0$  such that for any  $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  one has*

$$(1.7) \quad \mathcal{E}_{\kappa,g}(u) \leq C_{\text{obs}} \|\mathbf{1}_{]0,T[ \times \Gamma} \partial_n u|_{\mathbb{R} \times \partial\mathcal{M}}\|_{L^2(\mathbb{R} \times \partial\mathcal{M})}^2,$$

for the weak solution  $u$  to (1.5).

**1.4. Main results and open questions.** The following observability results were proven in [2].

**Theorem 1.6** (Burq, 97). *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^2$ .*

**Interior observability.** *Let  $\omega$  be an open subset of  $\mathcal{M}$  that satisfies the interior geometric control condition (see Definition 2.10 below for a precise description) associated with the infimum time  $T_{GCC}(\omega)$ . Let  $T > T_{GCC}(\omega)$ . Then, the wave equation is observable from  $\omega$  in time  $T$ .*

**Boundary observability.** *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$  such that  $\Gamma$  satisfies the boundary geometric control condition (see Definition 2.13 below for a precise description) associated with the infimum time  $T_{GCC}(\Gamma)$ . Let  $T > T_{GCC}(\Gamma)$ . Then, the wave equation is observable from  $\Gamma$  in time  $T$ .*

The proof of these results essentially relies on pseudo-differential calculus and microlocal tools, namely the microlocal defect measures and their localization and propagation properties.

One of our main contributions in the present article is to improve upon the regularity assumptions on the metric  $g$ , the function  $\kappa$  and the manifold  $\mathcal{M}$ .

**Theorem 1.7.** *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$ . The two conclusions of Theorem 1.6 hold.*

In fact, we prove a stronger result, namely that these observability results are stable by small perturbations in  $\mathcal{Y}^1$ , that is, perturbations that are slightly less smooth.

**Theorem 1.8.** *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$ .*

**Interior observability.** *Let  $\omega$  be an open subset of  $\mathcal{M}$  that satisfies the interior geometric control condition associated with the infimum time  $T_{GCC}(\omega)$ . Let  $T > T_{GCC}(\omega)$ . Then, there exists  $\varepsilon > 0$  such that if  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\omega})$  is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \omega)$  in the  $\mathcal{Y}^1$ -topology in the sense of Definition 1.3 for  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  and  $\tilde{\omega}$  an open subset of  $\tilde{\mathcal{M}}$ , then the wave equation associated with  $P_{\tilde{\kappa}, \tilde{g}}$  on  $\tilde{\mathcal{M}}$  is interior observable from  $\tilde{\omega}$  in time  $T$ .*

**Boundary observability.** *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$  such that  $\Gamma$  satisfies the boundary geometric control condition associated with the infimum time  $T_{GCC}(\Gamma)$ . Let  $T > T_{GCC}(\Gamma)$ . Then, there exists  $\varepsilon > 0$  such that if  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\Gamma})$  is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \Gamma)$  in the  $\mathcal{Y}^1$ -topology in the sense of Definition 1.3 for  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  and  $\tilde{\Gamma}$  an open subset of  $\partial\tilde{\mathcal{M}}$ , then the wave equation associated with  $P_{\tilde{\kappa}, \tilde{g}}$  on  $\tilde{\mathcal{M}}$  is boundary observable from  $\tilde{\Gamma}$  in time  $T$ .*

**Remark 1.9.** *First, as pointed out in Remark 1.2 above, note that standard microlocal tools cannot be used at the  $W^{1, \infty}$  level of regularity of  $\tilde{\kappa}$  and the metric  $\tilde{g}$ .*

*Second, Theorem 1.7 shows that the observability property is stable by small Lipschitz perturbations around rough ( $\mathcal{C}^1$ ) metrics satisfying GCC. We exhibited in [4, Remark 1.13] an example showing that the observation property is not stable by small (even smooth) perturbations of the geometry/metric around geometries satisfying only the observation property. This counter example is actually connected to an example due to G. Lebeau [25]. In particular, this shows that our perturbation argument will have to be performed on the proof of the fact that the geometric control condition implies observability and not on the final property itself. Since Theorem 1.7 is a straightforward consequence of Theorem 1.8 we will hence focus on the proof of Theorem 1.8 in what follows.*

In the two kind of observability-inequality results stated above, the GCC of Definitions 2.10 and 2.13 appear as sufficient conditions. We also formulate the following weak GCC condition

*For all point in the tangent bundle, at least one generalized geodesics initiated at this point enters any region larger than the observation region in some time  $T > 0$ .*

A more precise definition is given in Section 10.1. Observe that this condition reduces to the classical necessary condition in the case of uniqueness of generalized bicharacteristics.

We prove the following result proven in Section 10.

**Theorem 1.10** (Necessary geometric control conditions). *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$ .*

**Interior observability.** *Let  $\omega$  be an open subset of  $\mathcal{M}$  and  $T > 0$  such that interior observability holds from  $\omega$  in time  $T$ . Then the weak geometric control condition holds.*

**Boundary observability.** *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$  and  $T > 0$  such that boundary observability holds from  $\Gamma$  in time  $T$ . Then the weak geometric control condition holds.*

In the framework of the present article, sharpness of GCC as a sufficient condition for observability is an open question. Sharpness of the weak GCC as a necessary condition for observability is also open. In the case where generalized geodesics are not unique, there is quite a gap between the GCC and the weak GCC. Note that this gap closes as soon as uniqueness holds. Note also that a lack of uniqueness of generalized geodesics can be connected to a low regularity of the coefficients, as in the present setting. However, even in the case of smooth coefficients and a smooth manifold, an infinite contact order of a generalized geodesics with the boundary can be a source of nonuniqueness. We refer to the Taylor example [29] (see also [20, Example 24.3.11]). No thorough study of nonuniqueness issues at boundary has been carried out for  $\mathcal{C}^k$  coefficients,  $k \geq 2$ , up to our knowledge.

**1.5. Method to prove observability and outline.** We present here the scheme of the proof of the observability inequalities even though some material needs to be introduced. Yet, this will provide the reader with a road map.

The strategy of the proof of observability follows the following steps:

- First, we reduce the observation estimate for general data to a high-frequency observation estimate for semi-classically localized initial data. This step is quite classical [6].
- Second, we proceed by contradiction and obtain sequences of  $L^2$ -normalized initial data that are spectrally localized and vanishing asymptotically in the observation region.
- Third, associated with these sequences is a semi-classical measure  $\mu$  that characterises in phase space points where mass concentrates asymptotically. The main result of the present article is a propagation equation provided in Theorem 6.1 and fulfilled by the measure  $\mu$ .
- Fourth, we exploit the result of the companion article [5] stated here in Theorem 2.14 and we deduce that the support of  $\mu$  is a union of maximal generalized bicharacteristics. This leads to a contradiction with  $\mu$  vanishing in the observation region.

The remainder of the article is organized as follows.

In Section 2 we introduce the necessary geometrical notions to precisely state the geometric control condition (GCC) in our low regularity framework and to state the main result of the companion article [5]. In Section 3 we perform some classical *a priori* estimates for the normal derivatives of solutions to wave equations and we recall that an observability inequality is equivalent to an exact controllability result for the wave equation. Section 4 is devoted to a semi-classical reduction of observability estimates and the definition of semi-classical measures. In Section 5, we recall and introduce some aspects of semi-classical pseudo-differential operators with minimal regularity properties of the symbols. We also recall the notions of semi-classical measures and some of their properties. In Section 6.2 we write the contradiction argument that leads to the proof of a semi-classical observability inequality. This generates a semi-classical measure associated with a sequence of solutions to the wave equation and a semi-classical measure associated with their normal derivatives on the boundary. The measure-propagation equation that links these two measures is stated in Theorem 6.1 of Section 6.1 allowing one to conclude the proof of the semi-classical observability inequality. Sections 7 to 9 are dedicated to the proof of the measure-propagation equation of Theorem 6.1. Section 7 exposes the commutator argument that is the foundation of the measure-propagation equation. In Section 8 we present a Weierstrass division argument to be applied to the test functions to prove the measure-propagation equation. This leads to symbols with low regularity and low decay in the conormal direction. Further analysis for such symbols and associated operators is provided. Finally, in Section 9 the different symbols obtained in the Weierstrass division are quantized leading to the proof of the measure-propagation equation.

In Section 10 we prove that observability implies a weak GCC (Theorem 1.10). The proof is based on the measure equation that is stated in Theorem 10.7 and proven in Section 11.

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## 2. GEOMETRY

In this section we define the basic notions required to understand the statements of our results. We will work in special *quasi-normal geodesic* coordinates near the boundary. We refer to [5, Section 5] for a more thorough and intrinsic presentation of the geometry.

Here,  $\mathcal{M}$  is a  $\mathcal{C}^2$ -compact connected Riemannian manifold of dimension  $d$  with boundary, endowed with a  $\mathcal{C}^1$ -metric  $g$ . An example would be a bounded open subset  $\Omega$  of  $\mathbb{R}^d$  with a  $\mathcal{C}^2$ -boundary, that is, with the boundary given locally by  $\varphi(x) = 0$  with  $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$  and  $d\varphi \neq 0$ . Then  $\mathcal{M} = \Omega \cup \partial\Omega$  and one can simply consider the Euclidean metric. In the spirit of this simple example, consider an open  $d$ -dimensional manifold<sup>1</sup>  $\tilde{\mathcal{M}}$  such that  $\mathcal{M} \subset \tilde{\mathcal{M}}$  and extend the metric  $g$  to a neighborhood of  $\mathcal{M}$  in a  $\mathcal{C}^1$ -manner.

**2.1. Local coordinates.** Equip a compact neighborhood  $\hat{\mathcal{M}}$  of  $\mathcal{M}$  in  $\tilde{\mathcal{M}}$  with a finite  $\mathcal{C}^2$ -atlas. A local chart is denoted  $(O, \phi)$  with  $O$  an open subset of  $\hat{\mathcal{M}}$  and  $\phi$  a one-to-one map from  $O$  onto an open subset of  $\mathbb{R}^d$ . Charts can be chosen so that

$$\begin{aligned} \phi(O \cap \mathcal{M}) &= \phi(O) \cap \{x_d \geq 0\} \text{ is an open subset of } \overline{\mathbb{R}_+^d}, \\ \phi(O \cap \partial\mathcal{M}) &= \phi(O) \cap \{x_d = 0\}, \text{ and } \phi(O \setminus \mathcal{M}) = \phi(O) \cap \{x_d < 0\}, \end{aligned}$$

if  $O \cap \partial\mathcal{M} \neq \emptyset$ . Denote the local coordinates by  $x = (x', x_d)$  with  $x' \in \mathbb{R}^{d-1}$ . Note that  $\mathcal{M}$  being compact it contains its boundary  $\partial\mathcal{M}$ .

In a local chart, the metric  $g$  is given by  $g_x = g_{ij}(x)dx^i \otimes dx^j$ , where  $g_{ij} \in \mathcal{C}^1(\phi(O))$ . We use the classical notation  $(g^{ij}(x))_{i,j}$  for the inverse of  $(g_{ij}(x))_{i,j}$ . The metric  $g_x = (g_{ij}(x))_{i,j}$  provides an inner product on  $T_x\mathcal{M}$ . The metric  $g_x^* = g^{ij}(x)d\xi_i \otimes d\xi_j$  provides an inner product on  $T_x^*\mathcal{M}$ , denoted  $g_x^*(\xi, \tilde{\xi})$ , for  $\xi, \tilde{\xi} \in T_x^*\mathcal{M}$ . Define the associated norm

$$|\xi|_x = g_x^*(\xi, \xi)^{1/2}.$$

Near a boundary point, local coordinates are chosen according to the following proposition. They simplify the exposition of some geometrical notions and are key in arguments developed in what follows.

**Proposition 2.1** (quasi-normal geodesic coordinates). *Suppose  $m^0 \in \partial\mathcal{M}$ . There exists a  $\mathcal{C}^2$ -local chart  $(O, \phi)$  such that  $m^0 \in O$ ,  $\phi(m) = (x', z)$ , with  $x' \in \mathbb{R}^{d-1}$  and  $z \in \mathbb{R}$ , and*

<sup>1</sup>The manifold  $\tilde{\mathcal{M}}$  can be constructed by embedding  $\mathcal{M}$  in  $\mathbb{R}^{2d}$  thanks to the Whitney theorem [32].

- (1)  $\phi(O \cap \mathcal{M}) = \{z \geq 0\} \cap \phi(O)$ ,  $\phi(O \cap \partial\mathcal{M}) = \{z = 0\} \cap \phi(O)$ , and  $\phi(O \setminus \mathcal{M}) = \{z < 0\} \cap \phi(O)$  ;  
(2) *at the boundary, the representative of the metric has the form*

$$g(x', z = 0) = \sum_{1 \leq i, j \leq d-1} g_{ij}(x', z = 0) dx^i \otimes dx^j + |dz|^2.$$

In other words the matrix of  $g = (g_{ij})$  has the block-diagonal form *at the boundary*

$$(2.1) \quad g(x', z = 0) = \begin{pmatrix} & & & 0 \\ & * & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Naturally, the same form holds for  $g_x^* = (g^{ij}(x))$  at the boundary. One deduces that

$$g_{jd}(x', z) = zh_{jd}(x', z) \quad \text{and} \quad g_{dd}(x', z) = 1 + zh_{dd}(x', z),$$

for some continuous functions  $h_{jd}$ ,  $j = 1, \dots, d$ .

Proposition 2.1 can be found in [8] with a different regularity level. A proof of Proposition 2.1 at the regularity level we consider is written in Appendix B of [5] with a generalization to other levels of regularity.

**Remark 2.2.** *Because of the low regularity of  $g$  and  $\mathcal{M}$  one cannot choose normal geodesic coordinates, that is, local coordinates for which  $g_{jd} = g_{dj} = 0$  for  $j \neq d$  and  $g_{dd} = 1$  near a point  $m^0$  of the boundary. The coordinates that Proposition 2.1 provides only have this property in a neighborhood of  $m^0$  within the boundary  $\partial\mathcal{M}$ .*

One sets  $\mathcal{L} = \mathbb{R} \times \mathcal{M}$  and  $\hat{\mathcal{L}} = \mathbb{R} \times \hat{\mathcal{M}}$ . From a local chart  $(O, \phi)$  in the atlas for  $\hat{\mathcal{M}}$  one defines a map  $\phi_{\mathcal{L}} : (t, m) \mapsto (t, \phi(m))$  from  $\mathcal{O} = \mathbb{R} \times O$  onto  $\mathbb{R} \times \phi(O)$ , yielding a local chart  $(\mathcal{O}, \phi_{\mathcal{L}})$  for  $\hat{\mathcal{L}}$  and thus a finite atlas.

For  $x = \phi(m)$ ,  $m \in O \cap \mathcal{M}$ , denote by  $v = (v', v^d)$  and  $\xi = (\xi', \xi_d)$  the associated coordinates in  $T_m\mathcal{M}$  and  $T_m^*\mathcal{M}$ , with  $v', \xi' \in \mathbb{R}^{d-1}$  and  $v^d, \xi_d \in \mathbb{R}$ . We write  $T_x\mathcal{M}$  and  $T_x^*\mathcal{M}$  by abuse of notation. In what follows, it will be convenient to write  $z$  in place of  $x_d$ , in particular for the local coordinates given by Proposition 2.1. Accordingly we denote the associated cotangent variable  $\xi_d$  by the letter  $\zeta$ , that is,  $\xi = (\xi', \zeta)$ . We however do not change the notation for the associated tangent variable  $v^d$ . With local charts at the boundary given by Proposition 2.1, if  $x \in \partial\mathcal{M}$  and  $v \in T_x\partial\mathcal{M}$  then  $v = (v', 0)$  and we use the bijective map  $(\xi', 0) \mapsto \xi'$  to parameterize  $T_x^*\partial\mathcal{M}$ .

Also classically set

$$T\mathcal{M} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x\mathcal{M}, \quad T^*\mathcal{M} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x^*\mathcal{M}$$

$$(\text{resp. } T\hat{\mathcal{M}} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x\hat{\mathcal{M}}, \quad T^*\hat{\mathcal{M}} = \bigcup_{x \in \mathcal{M}} \{x\} \times T_x^*\hat{\mathcal{M}}).$$

With  $\mathcal{M}$  containing its boundary  $\partial\mathcal{M}$ , one sees that  $T\mathcal{M}$  (resp.  $T^*\mathcal{M}$ ) contains  $\{x\} \times T_x\mathcal{M}$  (resp.  $\{x\} \times T_x^*\mathcal{M}$ ) for  $x \in \partial\mathcal{M}$ . We denote by  $\partial(T^*\mathcal{M})$  the boundary of  $T^*\mathcal{M}$  that is the set of  $(x, \xi)$  with  $x \in \partial\mathcal{M}$ . In the local coordinates,  $\partial(T^*\mathcal{M})$  is given by  $\{z = 0\}$  and  $T^*\mathcal{M}$  by  $\{z \geq 0\}$ .

In the associated local chart on  $\mathcal{L}$ , the representative of  $(t, m) \in \mathcal{L}$  is  $(t, x) = (t, x', z)$ . We use the letter  $\varrho$  to denote an element of  $T^*\mathcal{L}$ , that is,  $\varrho = (t, x; \tau, \xi)$  with  $(t, x) \in \mathcal{L}$ ,  $\tau \in \mathbb{R}$  and  $\xi \in T_x^*\mathcal{M}$ . Classically, we write  $T^*\mathcal{L} \setminus 0$  for the set of points  $\varrho = (t, x; \tau, \xi)$  with  $(\tau, \xi) \neq 0$ . The boundary  $\partial(T^*\mathcal{L})$  is the set of points  $\varrho = (t, x; \tau, \xi)$  such that  $x \in \partial\mathcal{M}$ . Note that  $\partial(T^*\mathcal{L})$  is locally given by  $\{z = 0\}$  and  $T^*\mathcal{L}$  is locally given by  $\{z \geq 0\}$ .

**2.2. Wave operators and bicharacteristics.** On the manifold  $\mathcal{M}$  consider the elliptic operator  $A = A_{\kappa, g} = \kappa^{-1} \operatorname{div}_g(\kappa \nabla_g)$ , that is, in local coordinates

$$Af = \kappa^{-1}(\det g)^{-1/2} \sum_{1 \leq i, j \leq d} \partial_{x_i}(\kappa(\det g)^{1/2} g^{ij}(x) \partial_{x_j} f).$$

Its principal symbol is simply  $a(x, \xi) = -g_x^*(\xi, \xi) = -g_x^{ij} \xi_i \xi_j = -|\xi|_x^2$ . Note that for  $\kappa = 1$ , one has  $A = \Delta_g$ , the Laplace-Beltrami operator associated with  $g$  on  $\mathcal{M}$ . Together with  $A$  consider the wave operator  $P_{\kappa, g} = \partial_t^2 - A_{\kappa, g}$ . Its principal symbol in a local chart is given by

$$p(\varrho) = -\tau^2 + |\xi|_x^2.$$

Note that  $p(\varrho)$  is smooth in the variables  $(\tau, \xi)$  and  $\mathcal{C}^1$  in  $x$ .

For a function  $f$  of the variable  $\varrho$ , the Hamiltonian vector field  $H_f$  is defined by  $H_f(h) = \{f, h\}$ , where  $\{., .\}$  is the Poisson bracket. In local coordinates one has

$$\begin{aligned} H_p(\varrho) &= \partial_\tau p(\varrho) \partial_t + \nabla_\xi p(\varrho) \cdot \nabla_x - \nabla_x p(\varrho) \cdot \nabla_\xi \\ &= -2\tau \partial_t + 2g^{ij}(x) \xi_i \partial_{x_j} - \partial_{x_k} g^{ij}(x) \xi_i \xi_j \partial_{\xi_k}. \end{aligned}$$

Recall the following definition.



**Definition 2.3.** *Suppose  $V$  is an open subset of  $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$  and  $J \subset \mathbb{R}$  is an interval. A  $\mathcal{C}^1$ -map  $\gamma : J \rightarrow V \cap \text{Char } p$  is called a bicharacteristic in  $V$  if*

$$\frac{d}{ds}\gamma(s) = \text{H}_p(\gamma(s)), \quad s \in J.$$

*It is called maximal in  $V$  if it cannot be extended by another bicharacteristic also valued in  $V$ .*

Note that  ${}^t\text{H}_p f(\varrho) = 2\tau\partial_t f(\varrho) - 2\partial_{x_j}(g^{ij}(x)\xi_i f(\varrho)) + \partial_{\xi_k}(\partial_{x_k}g^{ij}(x)\xi_i\xi_j f(\varrho))$  and deduce

$${}^t\text{H}_p = -\text{H}_p.$$

Recall also that

$$(2.2) \quad \text{H}_p f(\gamma(s)) = \frac{d}{ds}f(\gamma(s)), \quad \text{if } \gamma \text{ is a bicharacteristic.}$$

**2.3. A partition of the cotangent bundle at the boundary.** Denote by  $\parallel\partial(T^*\mathcal{L}) \subset \partial(T^*\mathcal{L})$  the bundle of points  $\varrho = (\varrho', 0) = (t, x', z = 0, \tau, \xi', 0) \in T^*\mathcal{L}$  for  $\varrho' = (t, x', z = 0, \tau, \xi') \in T^*\partial\mathcal{L}$ . Identifying  $\varrho'$  and  $(\varrho', 0)$  as presented above thanks to the chosen local coordinates allows one to indentify  $\parallel\partial(T^*\mathcal{L})$  and  $T^*\partial\mathcal{L}$ .

Denote by  $\pi_{\parallel}$  the map from  $\partial(T^*\mathcal{L})$  into  $\parallel\partial(T^*\mathcal{L})$  given by

$$\pi_{\parallel}(t, x', z = 0, \tau, \xi', \zeta) = (t, x', z = 0, \tau, \xi', 0).$$

**Definition 2.4** (elliptic, glancing, and hyperbolic regions). *One partitions  $\parallel\partial(T^*\mathcal{L})$  into three homogeneous regions.*

- (1) *The elliptic region  $\parallel\mathcal{E}_{\partial} = \parallel\partial(T^*\mathcal{L}) \cap \{p > 0\}$ ; if  $\varrho \in \parallel\mathcal{E}_{\partial}$  it is called an elliptic point.*
- (2) *The glancing region  $\parallel\mathcal{G}_{\partial} = \parallel\partial(T^*\mathcal{L}) \cap \{p = 0\}$ ; if  $\varrho \in \parallel\mathcal{G}_{\partial}$  it is called a glancing point.*
- (3) *The hyperbolic region  $\parallel\mathcal{H}_{\partial} = \parallel\partial(T^*\mathcal{L}) \cap \{p < 0\}$ ; if  $\varrho \in \parallel\mathcal{H}_{\partial}$  it is called a hyperbolic point.*

Since  $p(\varrho) = -\tau^2 + \zeta^2 + g_x(\xi', \xi')_x$  by (2.1) if  $\varrho \in \partial(T^*\mathcal{L})$ , one has the following properties:

- (1) If  $\varrho \in \parallel\mathcal{E}_{\partial}$  then  $\pi_{\parallel}^{-1}(\{\varrho\}) \cap \text{Char } p = \emptyset$ .
- (2) If  $\varrho \in \parallel\mathcal{G}_{\partial}$  then  $\pi_{\parallel}^{-1}(\{\varrho\}) \cap \text{Char } p = \{\varrho\}$ .

(3) If  $\varrho = (t, x', z = 0, \tau, \xi', 0) \in \parallel \mathcal{H}_\partial$  then  $\pi_\parallel^{-1}(\{\varrho\}) \cap \text{Char } p = \{\varrho^-, \varrho^+\}$ , where

$$(2.3) \quad \varrho^\pm = (t, x', z = 0, \tau, \xi^\pm), \quad \text{where } \xi^\pm = (\xi', \zeta^\pm) \text{ with } \zeta^\pm = \pm \sqrt{-p(\varrho)}.$$

Associated with the previous partition of  $\parallel \partial(T^*\mathcal{L})$  is a partition of  $\text{Char } p \cap \partial(T^*\mathcal{L})$ . Indeed, if  $\varrho \in \text{Char } p \cap \partial(T^*\mathcal{L})$  then  $\pi_\parallel(\varrho) \in \parallel \partial(T^*\mathcal{L})$  and  $p(\pi_\parallel(\varrho)) \leq 0$ . Note that having  $\varrho \in \text{Char } p \cap \partial(T^*\mathcal{L})$  and  $p(\pi_\parallel(\varrho)) = 0$  is equivalent to having  $\varrho \in \parallel \mathcal{G}_\partial$ .

**Definition 2.5** (partition of  $\text{Char } p$  at the boundary). *One partitions  $\text{Char } p \cap \partial(T^*\mathcal{L})$  into two homogeneous regions  $\mathcal{G}_\partial$  and  $\mathcal{H}_\partial$ :*

- (1)  $\mathcal{G}_\partial = \parallel \mathcal{G}_\partial$ ;  $\varrho \in \mathcal{G}_\partial \Leftrightarrow \varrho \in \text{Char } p$  and  $\pi_\parallel(\varrho) = \varrho$ .
- (2)  $\varrho \in \mathcal{H}_\partial$  if  $\varrho \in \text{Char } p$  and  $\pi_\parallel(\varrho) \in \parallel \mathcal{H}_\partial$ . It is also called a hyperbolic point. If  $\varrho = (t, x', z = 0, \tau, \xi', \zeta)$  one says that  $\varrho \in \mathcal{H}_\partial^+$  if  $\zeta > 0$  and  $\varrho \in \mathcal{H}_\partial^-$  if  $\zeta < 0$ .

Thus, if  $\varrho \in \parallel \mathcal{H}_\partial$  then  $\pi_\parallel^{-1}(\{\varrho\}) \cap \text{Char } p = \{\varrho^-, \varrho^+\}$  with  $\varrho^+ \in \mathcal{H}_\partial^+$  and  $\varrho^- \in \mathcal{H}_\partial^-$ , with  $\varrho^\pm$  as given in (2.3).

Introducing the following involution on  $\partial(T^*\mathcal{L})$

$$\Sigma(t, x', z = 0, \tau, \xi', \zeta) = (t, x', z = 0, \tau, \xi', -\zeta),$$

one finds that  $\Sigma(\varrho^-) = \varrho^+$  if  $\varrho \in \parallel \mathcal{H}_\partial$ . Thus,  $\Sigma$  is a one-to-one map from  $\mathcal{H}_\partial^-$  onto  $\mathcal{H}_\partial^+$ .

**2.4. Glancing region, gliding vector field, and generalized bicharacteristics.** One computes

$$\mathbb{H}_p z(\varrho) = \mathbb{H}_p z(x, \xi) = 2g^{dj}(x)\xi_j \quad (\text{recall } x_d = z \text{ and } \xi_d = \zeta).$$

Observe that  $\mathbb{H}_p z$  is a  $\mathcal{C}^1$ -function and that  $\mathbb{H}_p z|_{z=0} = 2\zeta$  in the present local coordinates. Hence, locally one has

$$\parallel \mathcal{G}_\partial = \mathcal{G}_\partial = \{z = \mathbb{H}_p z = p = 0\} \quad \text{and} \quad \mathcal{H}_\partial^\pm = \{z = p = 0, \mathbb{H}_p z \gtrless 0\}.$$

With (2.2) this means that a bicharacteristic going through a point  $\varrho \in \mathcal{H}_\partial$  has a contact of order exactly one with the boundary: it is transverse to  $\partial(T^*\mathcal{L})$ . A bicharacteristic going through a point  $\varrho \in \mathcal{G}_\partial$  has a contact of order greater than or equal to two: it is tangent to  $\partial(T^*\mathcal{L})$ .

One can further compute  $\mathbb{H}_p^2 z$ . It is a continuous and gives the following partition of  $\mathcal{G}_\partial$ .

**Definition 2.6** (partition of  $\mathcal{G}_\partial$ ). *Introduce*

$$\begin{aligned}\mathcal{G}_\partial^d &= \{\varrho \in \mathcal{G}_\partial; H_p^2 z(\varrho) > 0\}, \\ \mathcal{G}_\partial^3 &= \{\varrho \in \mathcal{G}_\partial; H_p^2 z(\varrho) = 0\}, \\ \mathcal{G}_\partial^g &= \{\varrho \in \mathcal{G}_\partial; H_p^2 z(\varrho) < 0\}.\end{aligned}$$

One calls  $\mathcal{G}_\partial^d$  the diffractive set,  $\mathcal{G}_\partial^g$  the gliding set. One calls  $\mathcal{G}_\partial^3$  the glancing set of order three: if  $\varrho^0 \in \mathcal{G}_\partial^3$  a bicharacteristic that goes through  $\varrho^0$  has a contact with the boundary of order greater than or equal to three.

On  $\|\partial(T^*\mathcal{L})$  one defines

$$H_p^{\mathcal{G}}(\varrho) = \left( H_p + \frac{H_p^2 z}{H_z^2 p} H_z \right)(\varrho),$$

referred to as the *gliding vector field*. In the present coordinates one has  $H_z^2 p = 2$ . Define the following vector field on  $T^*\mathcal{L}$

$${}^{\mathcal{G}}X(\varrho) = \begin{cases} H_p(\varrho) & \text{if } \varrho \in T^*\mathcal{L} \setminus \mathcal{G}_\partial^g, \\ H_p^{\mathcal{G}}(\varrho) & \text{if } \varrho \in \mathcal{G}_\partial^g, \end{cases}$$

that is,  ${}^{\mathcal{G}}X = H_p + \mathbf{1}_{\mathcal{G}_\partial^g}(H_p^{\mathcal{G}} - H_p)$ . More explanations on the vector field  $H_p^{\mathcal{G}}$  are given in Section 5 in the companion article [5].

**Definition 2.7** (generalized bicharacteristic). *Let  $J \subset \mathbb{R}$  be an interval,  $B$  a discrete subset of  $J$ , and*

$${}^{\mathcal{G}}\gamma : J \setminus B \rightarrow \text{Char } p \cap T^*\mathcal{L}.$$

One says that  ${}^{\mathcal{G}}\gamma$  is a generalized bicharacteristic if the following properties hold:

- (1) For  $s \in J \setminus B$ ,  ${}^{\mathcal{G}}\gamma(s) \notin \mathcal{H}_\partial$  and the map  ${}^{\mathcal{G}}\gamma$  is differentiable at  $s$  with

$$\frac{d}{ds} {}^{\mathcal{G}}\gamma(s) = {}^{\mathcal{G}}X({}^{\mathcal{G}}\gamma(s)).$$

- (2) If  $S \in B$ , then  ${}^{\mathcal{G}}\gamma(s) \in T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$  for  $s \in J \setminus B$  sufficiently close to  $S$  and moreover

- (a) if  $[S - \varepsilon, S] \subset J$  for some  $\varepsilon > 0$ , then  ${}^{\mathcal{G}}\gamma(S^-) = \lim_{s \rightarrow S^-} {}^{\mathcal{G}}\gamma(s) \in \mathcal{H}_\partial^-$ ;
- (b) if  $[S, S + \varepsilon] \subset J$  for some  $\varepsilon > 0$ , then  ${}^{\mathcal{G}}\gamma(S^+) = \lim_{s \rightarrow S^+} {}^{\mathcal{G}}\gamma(s) \in \mathcal{H}_\partial^+$ ;
- (c) and if  $[S - \varepsilon, S + \varepsilon] \subset J$  for some  $\varepsilon > 0$ , then  ${}^{\mathcal{G}}\gamma(S^+) = \Sigma({}^{\mathcal{G}}\gamma(S^-))$ .

Recall that  $T^*\mathcal{L}$  contains its boundary  $\partial(T^*\mathcal{L})$ ; as a result a generalized bicharacteristic  ${}^G\gamma(s)$  may lie in the boundary for  $s$  in some interval. Details on generalized bicharacteristics can be found in Section 5 of the companion article [5].

When one refers to a (generalized) bicharacteristic one often means the points visited in  $T^*\mathcal{L}$  by  $s \mapsto {}^G\gamma(s)$  as  $s$  varies, that is,

$$\{{}^G\gamma(s); s \in J \setminus B\}.$$

Observe however that this set may not be a closed set if  $B \neq \emptyset$  as its intersection with  $\mathcal{H}_\partial$  is empty. Consequently, we rather use its closure to describe the set of reached points.

**Definition 2.8** (generalized bicharacteristic). *By generalized bicharacteristic one also refers to*

$$\bar{{}^G\gamma} = \overline{\{{}^G\gamma(s); s \in J \setminus B\}} = \{{}^G\gamma(s); s \in J \setminus B\} \cup \bigcup_{s \in B} \{{}^G\gamma(s^-), {}^G\gamma(s^+)\}.$$

The following theorem states that for every point of  $T^*\mathcal{L}$  one can find a maximal generalized bicharacteristic that goes through this point.

**Theorem 2.9.** *Suppose  $J \setminus B \ni s \mapsto {}^G\gamma(s) = (t(s), x(s), \tau(s), \xi(s))$  is a generalized bicharacteristic. If  ${}^G\gamma$  is maximal then  $J = \mathbb{R}$ . Moreover,  $t(\mathbb{R}) = \mathbb{R}$  if  $\tau(s) = \text{Cst} \neq 0$ .*

*If  $\varrho^0 \in \text{Char } p \cap T^*\mathcal{L}$  there exists a maximal generalized bicharacteristic  $s \mapsto {}^G\gamma(s)$  with  $s \in \mathbb{R} \setminus B$  such that  ${}^G\gamma(0) = \varrho^0$  if  $\varrho^0 \notin \mathcal{H}_\partial$  and  ${}^G\gamma(0^\pm) = \varrho^0$  if  $\varrho^0 \in \mathcal{H}_\partial^\pm$ .*

Note that there is no uniqueness of such a maximal generalized bicharacteristic because of the limited smoothness of  ${}^G X$ . This result is classical in the case of smooth coefficients; see [27] or [20, Section 24.3]. Here, in the case of the present limited smoothness it can be proven with the arguments developed in the companion article; see [5, Appendix A] for a proof.

**2.5. Geometric control conditions.** In the present low regularity framework we state the geometric control conditions (GCC) that coincide with the usual definitions found in the literature. First, we state the interior geometric control condition.

**Definition 2.10** (interior geometric control). *Let  $\omega$  be an open subset of  $\mathcal{M}$ . One says that  $\omega$  controls geometrically the manifold  $\mathcal{M}$  if there exists  $T > 0$  such that any generalized bicharacteristic reaches a point above  $]0, T[ \times \omega$ . One says that  $(\omega, T)$  fulfills GCC. In such case, one sets*

$$T_{GCC}(\omega) = \inf\{T > 0; (\omega, T) \text{ fulfills GCC}\}.$$

To state the boundary geometric control we introduce the notion of boundary escape point.

**Definition 2.11** (boundary escape point). (1) *a point  $\rho \in \partial(T^*\mathcal{L})$  is said to be a boundary escape point in the future if locally in time all bicharacteristics initiated at  $\rho$  immediately leave  $T^*\mathcal{L}$  in the future. One denotes by  $\mathcal{B}_{esc}^F$  the set of all such points*

(2) *a point  $\rho \in \partial(T^*\mathcal{L})$  is said to be a boundary escape point in the past if locally in time all bicharacteristics initiated at  $\rho$  immediately leave  $T^*\mathcal{L}$  in the past. One denotes by  $\mathcal{B}_{esc}^P$  the set of all such points*

*Moreover  $\mathcal{B}_{esc} = \mathcal{B}_{esc}^F \cup \mathcal{B}_{esc}^P$  is called the boundary escape set and points in  $\mathcal{B}_{esc}$  are the boundary escape points.*

The reader should note that the definition of escape points relies on bicharacteristics, that is, integral curves of  $H_p$  in  $\text{Char } p \subset T^*\hat{\mathcal{L}}$ , and not on the notion of generalized bicharacteristics. The latter curves do remain in  $T^*\mathcal{L}$ .

**Lemma 2.12.** *The following properties hold.*

- (1)  $\mathcal{H}_{\partial}^- \subset \mathcal{B}_{esc}^F \setminus \mathcal{B}_{esc}^P$  and  $\mathcal{H}_{\partial}^+ \subset \mathcal{B}_{esc}^P \setminus \mathcal{B}_{esc}^F$ .
- (2)  $\mathcal{G}_{\partial}^g \subset \mathcal{B}_{esc}^F \cap \mathcal{B}_{esc}^P$ .
- (3)  $\mathcal{G}_{\partial}^d \cap \mathcal{B}_{esc} = \emptyset$ .
- (4)  $\mathcal{G}_{\partial} \setminus \mathcal{B}_{esc} \subset \mathcal{G}_{\partial}^d \cup \mathcal{G}_{\partial}^3$ .

**Definition 2.13** (boundary geometric control). *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$ . One says that  $\Gamma$  controls geometrically the manifold  $\mathcal{M}$  if there exists  $T > 0$  such that any generalized bicharacteristic encounters a boundary escape point above  $]0, T[ \times \Gamma$ . One says that  $(\Gamma, T)$  fulfills GCC. In such case, one sets*

$$T_{GCC}(\Gamma) = \inf\{T > 0; (\Gamma, T) \text{ fulfills GCC}\}.$$

**2.6. Invariant measure supports.** For a manifold  $\mathcal{M} \in \mathcal{X}^1$  we will consider an extension  $\tilde{\mathcal{M}}$  as in the beginning of Section 2. The following result is proven in the companion article [5].

**Theorem 2.14.** *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$  and let  $\mu$  and  $\nu$  be two nonnegative measure densities on  $T^*\hat{\mathcal{L}}$  and  $T^*\partial\mathcal{L} \simeq {}^{\parallel}\partial(T^*\mathcal{L})$  respectively that fulfill the following properties:*

- (1)  $\text{supp } \mu \subset \text{Char } p \cap T^*\mathcal{L} \setminus 0$ .  
 (2) *One has, in the sense of distributions,*

$$(2.4) \quad \mathbb{H}_p \mu = -{}^t \mathbb{H}_p \mu = \int_{\varrho \in \|\mathcal{H}_\partial \cup \mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathbf{n}_x \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}}} d\nu(\varrho),$$

where  $\varrho^\pm$  and  $\xi^\pm$  are as given in (2.3). Here,  $\mathbf{n}_x$  stands for the unitary inward pointing normal vector in the sense of the metric.

Then, the support of the measure  $\mu$  is a union of maximal generalized bicharacteristics.

With the notation of Definitions 2.7 and 2.8, the result of Theorem 2.14 means that if  $\varrho \in \text{supp } \mu$ , there exists a maximal generalized bicharacteristic  $s \rightarrow {}^G \gamma(s)$ ,  $s \in \mathbb{R} \setminus B$  such that  $\varrho \in {}^G \bar{\gamma} \subset \text{supp } \mu$ .

The identification  $T^*\partial\mathcal{L} \simeq \|\partial(T^*\mathcal{L})$  is explained in Section 2.3.

**Remark 2.15.** *If  $\varrho \in \|\mathcal{G}_\partial$  then  $\varrho^-$  and  $\varrho^+$  coincide with  $\varrho$  and  $\xi^+ = \xi^-$ . The value of the integrand in (2.4) thus requires some explanation in this case. In fact, first consider  $\varrho^0 = (\varrho^{0'}, 0) \in \|\mathcal{H}_\partial$  with  $\varrho^{0'} = (t^0, x^0, z = 0, \tau^0, \xi^{0'})$ . Then  $\varrho^{0,\pm} \neq \varrho^0$  and (2.3) give  $\xi^{0,+} - \xi^{0,-} = 2\zeta dz$ , yielding  $\langle \xi^{0,+} - \xi^{0,-}, \mathbf{n}_{x^0} \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}} = 2\zeta$  since  $\mathbf{n}_x = \partial_z$  in the coordinates we consider here. Considering a  $\mathcal{C}^1$ -test function  $q(\varrho)$  one has*

$$\langle \delta_{\varrho^{0,+}} - \delta_{\varrho^{0,-}}, q \rangle = q(\varrho^{0'}, \zeta) - q(\varrho^{0'}, -\zeta).$$

The integrand is thus

$$\frac{q(\varrho^{0'}, \zeta) - q(\varrho^{0'}, -\zeta)}{2\zeta}.$$

If now a sequence  $(\varrho^{(n)})_n \subset \|\mathcal{H}_\partial$  converges to  $\varrho \in \|\mathcal{G}_\partial$  then

$$\frac{\langle \delta_{\varrho^{(n),+}} - \delta_{\varrho^{(n),-}}, q \rangle}{\langle \xi^{(n),+} - \xi^{(n),-}, \mathbf{n}_x \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}}} \rightarrow \partial_\zeta q(\varrho).$$

The integrand in (2.4) for  $\varrho \in \|\mathcal{G}_\partial$  is thus to be understood as the derivative with respect to the variable  $\zeta$  at  $\zeta = 0$ . Note that this interpretation is very coordinate dependent. We give a more geometrical interpretation using more intrinsic coordinates in the companion article [5, Section 5.7].

This result was proven in [2, Théorème 3] in the case  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^2$ . The proof of the result of Theorem 2.14 in [5] is more intricate due to the lower regularity of the metric  $g$  and the function  $\kappa$ .

### 3. A PRIORI ESTIMATES AND EXACT CONTROLLABILITY

In this section we consider  $(\mathcal{M}, \kappa, g) \in \mathcal{Y}^1$ , that is,  $\mathcal{M}$  is  $W^{2,\infty}$  and both  $\kappa$  and the metric  $g$  are Lipschitz.

First we recall a classical *a priori* estimation for the normal derivative of a solution to the wave equation. Second, we recall the equivalence between observability and exact controllability.

**3.1. Normal derivative estimation.** Denote by  $\mathbf{n}$  the unitary normal inward pointing vector field to  $\partial\mathcal{M}$  in the sense of the metric. It has the regularity of the metric, that is Lipschitz here. For a function  $w$  and  $x \in \partial\mathcal{M}$  then  $\partial_{\mathbf{n}}w(x) = \mathbf{n}(w)(x) = dw(x)(\mathbf{n}_x)$ . In the quasi-normal geodesic coordinates of Proposition 2.1 that can also be obtained in the  $\mathcal{Y}^1$ -regularity setting [5, Appendix B] one has  $n = \partial_{x_d}$  and thus  $\partial_{\mathbf{n}}w = \partial_d w$ .

**Proposition 3.1.** *Assume that  $(\mathcal{M}, \kappa, g) \in \mathcal{Y}^1$ . For any  $T > 0$  there exists  $C > 0$  depending only on  $T$ ,  $\mathcal{M}$ ,  $\|\kappa\|_{W^{1,\infty}(\mathcal{M})}$ ,  $\|g\|_{W^{1,\infty}\mathcal{T}_2^0(\mathcal{M})}$  such that for any  $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$  and  $f \in L_{\text{loc}}^2(\mathcal{L})$ , if  $u$  is the solution to the wave equation (1.3) then*

$$\begin{aligned} & \|\partial_{\mathbf{n}}u\|_{L^2([0,T] \times \partial\mathcal{M})}^2 \\ & \leq C \left( \int_{-1}^{T+1} \mathcal{E}_{\kappa,g}(u)(t) dt + \|\nabla_g u\|_{L^2(-1,T+1;L^2V(\mathcal{M}))} \|f\|_{L^2([-1,T+1] \times \mathcal{M})} \right). \end{aligned}$$

Below we will use the Neumann trace as an observation operator for the wave equation. In this context, with  $f = 0$ , Proposition 3.1 provides a so-called admissibility result; see for instance [31].

Note that a more usual and natural form of the estimation is simply

$$\|\partial_{\mathbf{n}}u\|_{L^2([0,T] \times \partial\mathcal{M})}^2 \lesssim \int_{-1}^{T+1} \mathcal{E}_{\kappa,g}(u)(t) dt + \|f\|_{L^2([-1,T+1] \times \mathcal{M})}^2.$$

This form is however not sufficient in one argument we use in what follows; we refer to the use of Proposition 3.1 made below (7.3).

Note that since  $u$  vanishes on  $\partial\mathcal{M}$  one has  $\|\nabla_g u|_{\partial\mathcal{M}}\| = |\partial_{\mathbf{n}}u|$ . The result of the previous proposition thus can be transferred to  $\|\nabla_g u|_{\mathbb{R} \times \partial\mathcal{M}}\|_{L^2([0,T] \times \partial\mathcal{M})}$ .

The proof of Proposition 3.1 follows from an examination of the standard proof and a careful handling of the low regularity metric. We will also need to approach the weak solution to the wave equation 1.3 by a sequence of strong solutions.

**Proposition 3.2.** *Suppose that  $(\underline{u}^0, \underline{u}^1) \in (H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \times H_0^1(\mathcal{M})$  and that  $f \in L_{\text{loc}}^1(\mathbb{R}; \mathcal{M})$ . There exists a unique*

$$u \in \mathcal{C}^0(\mathbb{R}; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; H_0^1(\mathcal{M})) \cap \mathcal{C}^2(\mathbb{R}; L^2(\mathcal{M}))$$

that is a strong solution of (1.3) meaning that  $(u, \partial_t u)|_{t=0} = (\underline{u}^0, \underline{u}^1)$  and  $P_{\kappa, g} u = f$  holds in  $L_{\text{loc}}^1(\mathbb{R}; L^2(\mathcal{M}))$ .

Note that a strong solution is also a weak solution. Then, if  $(\underline{u}^0, \underline{u}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ ,  $f \in L_{\text{loc}}^1(\mathbb{R}; L^2(\mathcal{M}))$  and  $u$  is the weak solution to (1.3) given by Proposition 1.1 and if  $(u_n^0, u_n^1)_n \subset (H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \times H_0^1(\mathcal{M})$ ,  $(f_n)_n \subset L_{\text{loc}}^1(\mathbb{R}; H_0^1(\mathcal{M}))$ , with  $(u_n)_n$  the sequence of associated strong solutions, are such that  $(u_n^0, u_n^1) \rightarrow (\underline{u}^0, \underline{u}^1)$  in  $H_0^1(\mathcal{M}) \oplus L^2(\mathcal{M})$ , and  $f_n \rightarrow f$  in  $L_{\text{loc}}^1(\mathbb{R}; L^2(\mathcal{M}))$  then  $u_n \rightarrow u$  in  $\mathcal{C}^0(\mathbb{R}; H_0^1(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathcal{M}))$  from the continuity of the map (1.4).

**Proof of Proposition 3.1.** First we consider the case of a strong solution

$$u \in \mathcal{C}^0(\mathbb{R}; H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; H_0^1(\mathcal{M})) \cap \mathcal{C}^2(\mathbb{R}; L^2(\mathcal{M})),$$

with  $f \in L_{\text{loc}}^2(\mathcal{L})$ .

Consider a *Lipschitz* vector field  $X$  that coincides with  $\mathbf{n}$  on the boundary. We view  $X$  as a first-order differential operator. For  $\chi \in \mathcal{C}_c^\infty(-1, T+1)$ , non-negative and equal to 1 on  $]0, T[$ , one finds that  $[P, \chi(t)X]u \in \mathcal{C}^0(\mathbb{R}; L^2(\mathcal{M}))$ . Set  $I = ([P, \chi(t)X]u, u)_{L^2(\mathcal{L})}$ . With the Green formula, that is, two integrations by parts, one finds

$$\begin{aligned} (3.1) \quad \|\partial_{\mathbf{n}} u\|_{L^2(]0, T[ \times \partial \mathcal{M})}^2 &\leq (\chi(t)X u|_{\partial \mathcal{M}}, \partial_{\mathbf{n}} u)_{L^2(\partial \mathcal{L})} \\ &\leq I - (\chi(t)X u, f)_{L^2(\mathcal{L})} + (f, \chi(t)X^* u)_{L^2(\mathcal{L})} \\ &\lesssim I + \|\nabla_g u\|_{L^2(-1, T+1; L^2 V(\mathcal{M}))} \|f\|_{L^2(]-1, T+1[ \times \mathcal{M})}. \end{aligned}$$

Writing  $[P, \chi(t)X] = [\partial_t^2, \chi(t)]X - \chi(t)[A, X]$ . One has  $I = J - K$  with

$$\begin{aligned} J &= \int_{\mathbb{R}} \langle [\partial_t^2, \chi(t)]X u, \bar{u} \rangle_{H^{-1}(\mathcal{M}), H_0^1(\mathcal{M})} dt, \\ K &= \int_{\mathbb{R}} \langle \chi(t)[A, X]u, \bar{u} \rangle_{H^{-1}(\mathcal{M}), H_0^1(\mathcal{M})} dt. \end{aligned}$$

Since  $[\partial_t^2, \chi]$  is a first-order operator and compactly supported (in time) we can integrate by parts in the time variable and obtain the bound, using the Poincaré inequality,

$$(3.2) \quad |J| \lesssim \int_{-1}^{T+1} \left( \|\partial_t u\|_{L^2(\mathcal{M})} + \|u\|_{L^2(\mathcal{M})} \right) \|\nabla_g u\|_{L^2 V(\mathcal{M})} dt \lesssim \int_{-1}^{T+1} \mathcal{E}_{\kappa, g}(u)(t) dt.$$



To estimate  $|K|$  we use a partition of unity subordinated to an atlas on  $\mathcal{M}$  and we consider the commutator  $[A, X]$  in local coordinates. Recall that  $A$  takes the form

$$A = \tilde{\kappa}^{-1} \partial_{x_i} \circ \tilde{\kappa} g^{ij} \partial_{x_j}$$

with  $\tilde{\kappa} = \kappa(\det g)^{1/2}$  where we use Einstein's summation convention. One thus finds

$$(3.3) \quad [A, X] = \tilde{\kappa}^{-1} \partial_{x_i} \circ \tilde{\kappa} g^{ij} [\partial_{x_j}, X] + \tilde{\kappa}^{-1} \partial_{x_i} \circ [\tilde{\kappa} g^{ij}, X] \circ \partial_{x_j} \\ + \tilde{\kappa}^{-1} \circ [\partial_{x_i}, X] \circ \tilde{\kappa} g^{ij} \partial_{x_j} + [\tilde{\kappa}^{-1}, X] \circ \partial_{x_i} \circ \tilde{\kappa} g^{ij} \partial_{x_j}.$$

Write  $K = K_1 + \dots + K_4$  in association with the four terms in (3.3). Since  $X$  has Lipschitz coefficients then  $[\partial_{x_k}, X]$  is a vector field with bounded coefficients and  $[\tilde{\kappa} g^{ij}, X]$  is a bounded function in the local coordinates. Thus, with an integration by parts in space the contribution  $|K_1|$ ,  $|K_2|$ , and  $|K_3|$  can be estimated by

$$(3.4) \quad |K_1| + |K_2| + |K_3| \lesssim \int_{\mathbb{R}} \chi(t) \|\nabla_g u\|_{L^2 V(\mathcal{M})}^2 dt.$$

For the term  $K_4$  since  $[\tilde{\kappa}^{-1}, X]$  is only bounded, an integration by parts in space is not possible. Instead, exploiting that  $u$  is a solution to the homogenous wave equation one writes

$$[\tilde{\kappa}^{-1}, X] \circ \partial_{x_i} \circ \tilde{\kappa} g^{ij} \partial_{x_j} u = [\tilde{\kappa}^{-1}, X] \tilde{\kappa} A u = [\tilde{\kappa}^{-1}, X] \tilde{\kappa} \partial_t^2 u.$$

This now allows one to perform an integration by parts with respect to the time variable yielding

$$(3.5) \quad |K_4| \lesssim \int_{-1}^{T+1} \|\partial_t u\|_{L^2(\mathcal{M})}^2 dt + \int_{-1}^{T+1} \|u\|_{L^2(\mathcal{M})} \|\partial_t u\|_{L^2(\mathcal{M})} dt \\ + \|u\|_{L^2([-1, T+1] \times \mathcal{M})} \|f\|_{L^2([-1, T+1] \times \mathcal{M})} \\ \lesssim \int_{-1}^{T+1} \mathcal{E}_{\kappa, g}(u)(t) dt + \|\nabla_g u\|_{L^2(-1, T+1; L^2 V(\mathcal{M}))} \|f\|_{L^2([-1, T+1] \times \mathcal{M})}.$$

Combining (3.1), (3.2), (3.4) and (3.5) gives

$$(3.6) \quad \|\partial_n u\|_{L^2([0, T] \times \partial \mathcal{M})}^2 \lesssim \int_{-1}^{T+1} \mathcal{E}_{\kappa, g}(u)(t) dt \\ + \|\nabla_g u\|_{L^2(-1, T+1; L^2 V(\mathcal{M}))} \|f\|_{L^2([-1, T+1] \times \mathcal{M})}.$$

If  $u$  is now a weak solution, if one approaches  $u$  by a sequence of strong solutions as described below Proposition 3.2 one finds that the normal trace  $\partial_n u|_{\partial \mathcal{M}}$  makes sense in  $L^2([0, T] \times \partial \mathcal{M})$  and (3.6) remains true for the weak solution.  $\square$

**3.2. Exact controllability notions.** We make the classical connexion between the observability properties of the *homogeneous* wave equation given in (1.5) and the exact controllability of the wave equations. We will consider two different wave equations here, one with an interior source term and a homogeneous Dirichlet boundary condition and one with a boundary source term through the Dirichlet boundary condition. In each case we describe what is meant by exact controllability.

**3.2.1. Exact interior controllability.** Suppose  $\omega$  is an open subset of  $\mathcal{M}$ . The notion of exact interior controllability for the wave equation on  $\mathcal{M}$  from  $\omega$  in time  $T$  is stated as follows.

**Definition 3.3** (exact interior controllability in  $H_0^1(\mathcal{M}) \oplus L^2(\mathcal{M})$ ). *One says that the wave equation is exactly controllable from  $\omega$  in time  $T > 0$  if for any  $(\underline{y}^0, \underline{y}^1) \in H_0^1(\mathcal{M}) \times L^2(\mathcal{M})$ , there exists  $f \in L^2(]0, T[ \times \mathcal{M})$  such that the weak solution  $y$  to*

$$P_{\kappa,g}y = \mathbf{1}_{]0,T[ \times \omega} f, \quad y|_{\mathbb{R} \times \partial\mathcal{M}} = 0, \quad (y, \partial_t y)|_{t=0} = (\underline{y}^0, \underline{y}^1),$$

as given by Proposition 1.1 satisfies  $(y, \partial_t y)|_{t=T} = (0, 0)$ . The function  $f$  is called the control function or simply the control.

**3.2.2. Exact boundary controllability.** Consider the nonhomogeneous wave equation with source term given by a Dirichlet boundary condition.

$$(3.7) \quad \begin{cases} P_{\kappa,g}y = 0 & \text{in } \mathbb{R} \times \mathcal{M}, \\ y = f_\partial & \text{on } \mathbb{R} \times \partial\mathcal{M}, \\ y|_{t=0} = \underline{y}^0, \partial_t y|_{t=0} = \underline{y}^1 & \text{in } \mathcal{M}, \end{cases}$$

Standard results show that it is well-posed.

**Proposition 3.4.** *Consider  $\kappa$  and  $g$  both Lipschitz. Let  $(\underline{y}^0, \underline{y}^1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$  and  $f_\partial \in L_{\text{loc}}^2(\mathbb{R} \times \partial\mathcal{M})$ . There exists a unique*

$$y \in \mathcal{C}^0(\mathbb{R}; L^2(\mathcal{M})) \cap \mathcal{C}^1(\mathbb{R}; H^{-1}(\mathcal{M})).$$

that is a weak solution of (3.7).

Let  $\Gamma$  be a nonempty open subset of  $\partial\mathcal{M}$  and  $T > 0$ . The notion of exact boundary controllability for the wave equation from  $\Gamma$  in time  $T$  is stated as follows.

**Definition 3.5** (exact boundary controllability in  $L^2(\mathcal{M}) \oplus H^{-1}(\mathcal{M})$ ). *One says that the wave equation is exactly controllable from  $\Gamma$  in time  $T > 0$  if for any  $(\underline{y}^0, \underline{y}^1) \in L^2(\mathcal{M}) \times H^{-1}(\mathcal{M})$ , there exists  $f_\partial \in L^2(]0, T[ \times \partial\mathcal{M})$  such that the weak solution  $y$  to*

$$P_{\kappa, g} y = 0, \quad y|_{\mathbb{R} \times \partial\mathcal{M}} = \mathbf{1}_{]0, T[ \times \Gamma} f_\partial, \quad (y, \partial_t y)|_{t=0} = (\underline{y}^0, \underline{y}^1),$$

as given by Proposition 3.4 satisfies  $(y, \partial_t y)|_{t=T} = (0, 0)$ . The function  $f_\partial$  is called the control function or simply the control.

### 3.3. Exact controllability equivalent to observability and corollaries.

The following proposition is standard and states that in the two cases we consider exact controllability is equivalent to an observability inequality.

- Proposition 3.6.** (1) *Let  $\omega$  be an open subset of  $\mathcal{M}$  and  $T > 0$ . The wave equation is exactly controllable from  $\omega$  in time  $T$  if and only if the homogeneous wave equation is observable from  $\omega$  in time  $T$ .*
- (2) *Let  $\Gamma$  be a nonempty open subset of  $\partial\mathcal{M}$  and  $T > 0$ . The wave equation is exactly controllable from  $\Gamma$  in time  $T$  if and only if the homogeneous wave equation is observable from  $\Gamma$  in time  $T$ .*

With the previous proposition and Theorem 1.8 one deduces the following corollary.

**Corollary 3.7** (Exact controllability result). *Let  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$ .*

**Interior exact controllability.** *Let  $\omega$  be an open subset of  $\mathcal{M}$  that satisfies the interior geometric control condition associated with the infimum time  $T_{GCC}(\omega)$ . Let  $T > T_{GCC}(\omega)$ . Then, there exists  $\varepsilon > 0$  such that if  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\omega})$  is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \omega)$  in the  $\mathcal{Y}^1$ -topology in the sense of Definition 1.3 for  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  and  $\tilde{\omega}$  an open subset of  $\tilde{\mathcal{M}}$ , then the wave equation associated with  $P_{\tilde{\kappa}, \tilde{g}}$  on  $\tilde{\mathcal{M}}$  is exactly controllable from  $\tilde{\omega}$  in time  $T$ .*

**Boundary exact controllability.** *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$  such that  $\Gamma$  satisfies the boundary geometric control condition associated with the infimum time  $T_{GCC}(\Gamma)$ . Let  $T > T_{GCC}(\Gamma)$ . Then, there exists  $\varepsilon > 0$  such that if  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\Gamma})$  is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \Gamma)$  in the  $\mathcal{Y}^1$ -topology in the sense of Definition 1.3 for  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}) \in \mathcal{Y}^1$  and  $\tilde{\Gamma}$  an open subset of  $\partial\tilde{\mathcal{M}}$ , then the wave equation associated with  $P_{\tilde{\kappa}, \tilde{g}}$  on  $\tilde{\mathcal{M}}$  is exactly controllable from  $\tilde{\Gamma}$  in time  $T$ .*

## 4. SEMI-CLASSICAL REDUCTION

In [4], on a compact manifold without boundary, we proved an interior observability estimate for the Klein-Gordon equation by means of microlocal defect

measures. The more general case we consider here, in the presence of a boundary is technically more involved and requires a semi-classical approach. We recall in this section how observability estimates as in Definitions 1.4 and 1.5 can be obtained from counterpart semi-classical observability estimates.

**4.1. Dyadic decomposition.** Consider  $(\kappa, g) \in \mathcal{X}^1(\mathcal{M})$  and the associated operator  $A = A_{\kappa, g}$  with Dirichlet boundary conditions. Denote by  $\lambda_\nu$  the nondecreasing sequence of positive eigenvalues of  $-A$  that goes to  $+\infty$  and consider  $(e_\nu)_\nu$  a Hilbert basis of  $L^2(\mathcal{M}) = L^2(\mathcal{M}, \kappa\mu_g)$  of associated *real* eigenfunctions.

Let  $0 < \alpha < 1$ ,  $\varrho \in ]1, 1/\alpha[$  and set  $h_k = \varrho^{-|k|}$  and

$$J_k = \{\nu \in \mathbb{N}; \alpha \leq h_k \sqrt{\lambda_\nu} < \alpha^{-1}\} = \{\nu \in \mathbb{N}; \alpha \varrho^{|k|} \leq \sqrt{\lambda_\nu} < \varrho^{|k|}/\alpha\},$$

for  $k \in \mathbb{Z}^*$ . Introduce

$$E_k = \text{span}\{e_\nu; \nu \in J_k\},$$

equipped with the  $L^2$ -norm  $\|u\|_{L^2(\mathcal{M})}^2 = \|u\|_{L^2(\mathcal{M}, \kappa\mu_g)}^2 = \sum_{\nu \in J_k} |u_\nu|^2$  for  $u = \sum_{\nu \in J_k} u_\nu e_\nu \in E_k$ . Observe that if  $u \in E_k$  then  $A^n u \in E_k$ . Hence,  $E_k$  is a subspace of all the iterated domains of  $A$ .

At this stage it is important to note that  $J_{-k} = J_k$  implying  $E_{-k} = E_k$ . However, we will identify  $u \in E_k$  with the following solution of the wave equation

$$u = \sum_{\nu \in J_k} e^{\text{sgn}(k)it\sqrt{\lambda_\nu}} u_\nu e_\nu.$$

The sign of  $k$  here becomes important. Yet, note that  $u \in E_k$  if and only if  $\bar{u} \in E_{-k}$ , through this identification since the eigenfunctions  $e_\nu$  are chosen real.

Following up, we identify  $\partial_t^\ell u$  with  $u = \sum_{\nu \in J_k} (i \text{sgn}(k))^\ell \lambda_\nu^{\ell/2} u_\nu e_\nu \in E_k$ , its value at  $t = 0$ . Similarly, one identifies  $A^s u$  with  $\sum_{\nu \in J_k} \lambda_\nu^s u_\nu e_\nu \in E_k$ .

**Lemma 4.1.** *For  $u \in E_k$ , the norms*

$$\|h_k \nabla_g u\|_{L^2 V(\mathcal{M})} \quad \text{and} \quad \|h_k \partial_t u\|_{L^2(\mathcal{M})}$$

*are both equivalent to  $\|u\|_{L^2(\mathcal{M})}$ , uniformly with respect to  $k \in \mathbb{Z}$ . More generally, for  $\ell \in \mathbb{N}$  and  $s \in \mathbb{R}$ , the norm  $h_k^{\ell+2s} \|\partial_t^\ell A^s u\|_{L^2(\mathcal{M})}$  is equivalent to  $\|u\|_{L^2(\mathcal{M})}$  for  $u \in E_k$ , uniformly with respect to  $k \in \mathbb{Z}^*$ .*

**Proof.** For the first result one writes

$$\|h_k \nabla_g u\|_{L^2 V(\mathcal{M})}^2 = (h_k^2 A u, u)_{L^2(\mathcal{M})} = \|h_k \partial_t u\|_{L^2(\mathcal{M})}^2.$$

Then one concludes using  $h_k \lambda_\nu^{1/2} \approx 1$  for  $\nu \in J_k$ . □

As a consequence the  $L^2$ -norm and the square root of the semi-classical energy  $\mathcal{E}^h(u)$

$$(4.1) \quad \mathcal{E}^h(u) = \frac{1}{2} (\|h_k \nabla_g u\|_{L^2 V(\mathcal{M})}^2 + \|h_k \partial_t u\|_{L^2(\mathcal{M})}^2) = h_k^2 \mathcal{E}(u),$$

are equivalent on  $E_k$ . Note that for  $u \in E_k$  both terms in the semi-classical energy coincide; this is not the case in general for a solution of the wave equation.

We introduce the following sets of sequences of functions

$$\begin{aligned} B &= \{(u^k)_{k \in \mathbb{Z}^*}; u^k \in E_k \text{ and } \|u^k\|_{L^2(\mathcal{M})} \leq 1\}, \\ B^\pm &= \{(u^k)_{k \in \pm \mathbb{N}^*}; u^k \in E_k \text{ and } \|u^k\|_{L^2(\mathcal{M})} \leq 1\}. \end{aligned}$$

**4.2. Semi-classical observation.** The result of Proposition 4.3 below for boundary observation is proven in [2, Section 4] following a strategy of G. Lebeau [24]. The result of Proposition 4.2 for interior observation can be proven similarly. In [2] domains and metrics are smoother, yet lowering the regularity does not affect the proof that is only based on semi-classical analysis arguments with respect to the time variable. In fact, the proofs of both propositions can be carried out within an abstract framework that can be found in [6].

**Proposition 4.2** (interior semi-classical observation implies classical observation). *Let  $\omega$  be a nonempty open subset of  $\mathcal{M}$ . Assume that there exists  $C > 0$ ,  $k_0 > 0$ , and  $\delta > 0$ , such that for any  $U = (u^k)_{k \in \mathbb{N}} \in B^+$  and any  $k \geq k_0$  one has*

$$(4.2) \quad \|u^k|_{t=0}\|_{L^2(\mathcal{M})} \leq C \|\mathbf{1}_{I \times \omega} h_k \partial_t u^k\|_{L^2(\mathbb{R} \times \mathcal{M})}, \quad \text{with } I = ]\delta, T - \delta[.$$

*Then, the homogeneous wave equation is observable from  $\omega$  in time  $T > 0$  in the sense of Definition 1.4.*

**Proposition 4.3** (boundary semi-classical observation implies classical observation). *Let  $\Gamma$  be a nonempty open subset of  $\partial\mathcal{M}$ . Assume that there exists  $C > 0$ ,  $k_0 > 0$ , and  $\delta > 0$  such that for any  $U = (u^k)_{k \in \mathbb{N}} \in B^+$  and any  $k \geq k_0$ , one has*

$$(4.3) \quad \|u^k|_{t=0}\|_{L^2(\mathcal{M})}^2 \leq C \|\mathbf{1}_{I \times \Gamma} h_k \partial_n u^k\|_{L^2(\mathbb{R} \times \partial\mathcal{M})}, \quad \text{with } I = ]\delta, T - \delta[.$$

*Then, the homogeneous wave equation is observable from  $\Gamma$  in time  $T > 0$  (in the sense of Definition 1.5).*

Propositions 4.2 and 4.3 state that if an observability inequality as in Definition 1.4 or Definition 1.5 holds in  $E_k$  uniformly for large  $|k|$ , then it also holds

for any initial data with possibly a small loss (here  $\delta$  on each side) in the time interval required for observation.

The proof presented in [6] is based on several properties of the observation operator  $\mathbf{L}$ , here  $\mathbf{L} = \mathbf{1}_{I \times \omega} h_k \partial_t$  in the first case and  $\mathbf{L} = \mathbf{1}_{I \times \Gamma} h_k \partial_n$  in the second case:

- (1) a unique continuation property,  $\mathbf{L} u = 0$  implying  $u = 0$  for eigenfunctions of the operator  $A_{\kappa, g}$ ; this condition holds in the both cases we consider; see for instance [19, Theorem 2.4] and [22, Theorems 5.11 and 5.13].
- (2) an optional admissibility condition, here given by Proposition 3.1 in the case  $\mathbf{L} = \mathbf{1}_{I \times \Gamma} h_k \partial_n$ . In the first case,  $\mathbf{L} = \mathbf{1}_{I \times \omega} h_k \partial_t$  the admissibility condition is trivial.

**Remark 4.4.** In (4.1), we pointed out that the  $L^2$ -norm  $\|\cdot\|_{L^2(\mathcal{M})}$  is equivalent to the square root of the semi-classical energy  $\mathcal{E}^h(u)$ , uniformly in  $k$ . Here,  $\mathcal{E}^h(u)$  is constant w.r.t. time  $t$ , since  $u_k$  is solution of the homogeneous wave equation. Consequently, one can also replace the l.h.s. in (4.3) and (4.2) by

$$\|u^k\|_{L^\infty(\mathbb{R}; L^2(\mathcal{M}))}^2 \quad \text{or} \quad \|u^k\|_{L^2(J \times \mathcal{M})}^2,$$

for any finite interval  $J \subset \mathbb{R}$ .

## 5. SEMI-CLASSICAL OPERATORS AND MEASURES

**5.1. The Schur lemma.** Here, we recall a result that is important in our analysis of some semi-classical operators on  $\mathbb{R}^d$  in what follows.

**Lemma 5.1** (Schur's Lemma). *Let  $K(x, y)$  be a measurable function on  $\mathbb{R}^d \times \mathbb{R}^d$  such that  $K(x, \cdot)$  and  $K(\cdot, y)$  are  $L^1$ -functions for almost all  $x$  and  $y$  in  $\mathbb{R}^d$  respectively, with moreover*

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \|K(x, \cdot)\|_{L^1(\mathbb{R}^d)} \leq A \quad \text{and} \quad \operatorname{ess\,sup}_{y \in \mathbb{R}^d} \|K(\cdot, y)\|_{L^1(\mathbb{R}^d)} \leq B,$$

for some  $A \geq 0$  and  $B \geq 0$ . Then, the operator  $\mathcal{K}$  with Schwartz kernel  $K(\cdot, \cdot)$  extends as a continuous operator on  $L^2(\mathbb{R}^d)$  with  $\|\mathcal{K}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq (AB)^{1/2}$ .

Assume that the kernel of the operator  $\mathcal{K}$  is of the form

$$K(x, y) = h^{-d} k\left(x, \frac{x-y}{h}\right),$$

for some measurable function  $k$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$ . Changes of variables give

$$\begin{aligned}\|K(x, \cdot)\|_{L^1(\mathbb{R}^d)} &= \|k(x, \cdot)\|_{L^1(\mathbb{R}^d)}, \\ \|K(\cdot, y)\|_{L^1(\mathbb{R}^d)} &= \|k(y + h \cdot, \cdot)\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

The Schur Lemma can be translated accordingly.

**Lemma 5.2.** *Let the operator  $\mathcal{K}$  have Schwartz kernel  $K(x, y) = h^{-d} k(x, \frac{x-y}{h})$  with the function  $k$  satisfying*

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \|k(x, \cdot)\|_{L^1(\mathbb{R}^d)} \leq A \quad \text{and} \quad \operatorname{ess\,sup}_{y \in \mathbb{R}^d} \|k(y + h \cdot, \cdot)\|_{L^1(\mathbb{R}^d)} \leq B,$$

for some  $A \geq 0$  and  $B \geq 0$ . Then, the operator  $\mathcal{K}$  with Schwartz kernel  $K(\cdot, \cdot)$  extends as a continuous operator on  $L^2(\mathbb{R}^d)$  with  $\|\mathcal{K}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq (AB)^{1/2}$ .

**Corollary 5.3.** *Let the operator  $\mathcal{K}$  have Schwartz kernel  $K(x, y) = h^{-d} k(x, \frac{x-y}{h})$  with the function  $k$  satisfying*

$$|k(x, v)| \leq L_0 \langle v \rangle^{-d-\delta}, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d,$$

for some  $\delta > 0$  and  $L_0 > 0$ . Then,  $\mathcal{K}$  extends as a continuous operator on  $L^2(\mathbb{R}^d)$  with  $\|\mathcal{K}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_{d,\delta} L_0$  for some  $C_{d,\delta} > 0$ .

**5.2. Semi-classical operators on  $\mathbb{R}^d$ .** We recall and develop here some aspects of semi-classical pseudo-differential operators associated with symbols with fairly low regularity.

Let  $h_0 > 0$ . In the semi-classical setting we denote by  $h \in (0, h_0]$  a small parameter.

**Definition 5.4** (symbols). *Let  $m, n \in \mathbb{N} \cup \{+\infty\}$ , with  $n \geq d + 1$ , and  $N \in \mathbb{R}^+$ . Denote by  $\Sigma^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  the space of all functions  $a(x, \xi)$ ,  $x \in \mathbb{R}^d$ ,  $\xi \in \mathbb{R}^d$ , such that  $\partial_x^\alpha \partial_\xi^\beta a \in L^1_{\text{loc}}(\mathbb{R}^{2d})$  for  $\alpha, \beta \in \mathbb{N}^d$  with  $|\alpha| \leq m$ ,  $|\beta| \leq n$ , and*

$$(5.1) \quad M_{m,n}^{-N}(a) := \max_{\substack{|\alpha| \leq m \\ |\beta| \leq n}} \operatorname{ess\,sup}_{(x,\xi)} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \langle \xi \rangle^N < \infty.$$

*In addition, one sets  $\Sigma_0^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$ ,  $n \geq d + 1$ , as the set of all symbols  $a \in \Sigma^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  with moreover  $\partial_x^\alpha \partial_\xi^\beta a \in \mathcal{C}_0(\mathbb{R}^{2d})$  for  $\alpha, \beta \in \mathbb{N}^d$  with  $|\alpha| \leq m$  and  $|\beta| \leq n - 1 - d$ .*

Recall that  $\mathcal{C}_0(\mathbb{R}^{2d})$  is the space of continuous functions on  $\mathbb{R}^{2d}$  that converge to 0 at infinity. Both spaces  $\Sigma^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  and  $\Sigma_0^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  are

complete if equipped with the norm  $M_{m,n}^{-N}(\cdot)$ . The space  $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$  is dense in  $\Sigma_0^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  for  $N > 0$ .

At first, we will be interested in the case  $N = d + 1$ . Since

$$\Sigma^{m',n'}(\langle \xi \rangle^{-N'}; \mathbb{R}^{2d}) \subset \Sigma^{m,n}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$$

if  $m' \geq m$ ,  $n' \geq n$ , and  $N' \geq N$ , set

$$\Sigma(\mathbb{R}^{2d}) = \Sigma^{0,d+1}(\langle \xi \rangle^{-(d+1)}; \mathbb{R}^{2d}).$$

Set also

$$\Sigma_0(\mathbb{R}^{2d}) = \Sigma_0^{0,d+1}(\langle \xi \rangle^{-(d+1)}; \mathbb{R}^{2d}).$$

Faster decay with respect to  $\xi$  will be considered, starting in Section 7.2. For simplicity, we will use

$$(5.2) \quad \Sigma_0^{\infty,\infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d}) = \bigcap_{N \geq 0} \Sigma_0^{\infty,\infty}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$$

in those later sections.

**Definition 5.5** (semi-classical operators). *For  $u \in \mathcal{S}(\mathbb{R}^d)$  and  $a \in \Sigma(\mathbb{R}^{2d})$  one sets*

$$(5.3) \quad \text{Op}^h(a)u(x) = a(x, hD_x)u(x) = (2\pi)^{-d} \int e^{ix \cdot \xi} a(x, h\xi) \hat{u}(\xi) d\xi.$$

The Schwartz kernel of  $\text{Op}^h(a)$  is given by

$$\begin{aligned} K_a(x, y) &= (2\pi)^{-d} \int e^{i(x-y) \cdot \xi} a(x, h\xi) d\xi = (2\pi h)^{-d} \int e^{i\frac{x-y}{h} \cdot \xi} a(x, \xi) d\xi \\ &= h^{-d} k_a\left(x, \frac{x-y}{h}\right), \end{aligned}$$

with

$$(5.4) \quad k_a(x, v) = (2\pi)^{-d} \int e^{iv \cdot \xi} a(x, \xi) d\xi.$$

Note that (5.4) is well defined in the sense of classical integrals by the decay property in the variable  $\xi$  of the symbol  $a$ . Observe that  $L \exp(iv \cdot \xi) = \exp(iv \cdot \xi)$  with  $L = (1 - iv \cdot \nabla_\xi) / \langle v \rangle^2$  leading to, with integrations by parts,

$$k_a(x, v) = (2\pi)^{-d} \int e^{iv \cdot \xi} ({}^tL)^N a(x, \xi) d\xi,$$



for  $N \leq d + 1$ , with  ${}^tL = (1 + iv \cdot \nabla_\xi) / \langle v \rangle^2$ . One then obtains

$$(5.5) \quad |k_a(x, v)| \lesssim M_{0,d+1}^{-(d+1)}(a) \langle v \rangle^{-(d+1)}, \quad v \in \mathbb{R}^d, \quad x \in \mathbb{R}^d \text{ a.e..}$$

With Corollary 5.3 one deduces the boundedness of  $\text{Op}^h(a)$  on  $L^2(\mathbb{R}^d)$  with  $a$  as above.

**Lemma 5.6.** *Let  $a \in \Sigma(\mathbb{R}^{2d})$ . Then  $\text{Op}^h(a)$  extends as a uniformly bounded operator on  $L^2(\mathbb{R}^d)$  and*

$$\|\text{Op}^h(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_d M_{0,d+1}^{-(d+1)}(a).$$

The following remark will be used in what follows.

**Remark 5.7.** *If  $a \in \Sigma^{0,d+2}(\langle \xi \rangle^{-(d+1)}; \mathbb{R}^{2d})$ , note that one has*

$$(5.6) \quad k_{\partial_{\xi_j} a}(x, v) = -iv_j k_a(x, v), \quad j = 1, \dots, d.$$

*In fact, with an integration by parts one has*

$$\begin{aligned} k_{\partial_{\xi_j} a}(x, v) &= (2\pi)^{-d} \int e^{iv \cdot \xi} \partial_{\xi_j} a(x, \xi) d\xi \\ &= -(2\pi)^{-d} \int \partial_{\xi_j} (e^{iv \cdot \xi}) a(x, \xi) d\xi = -iv_j k_a(x, v). \end{aligned}$$

**Lemma 5.8.** *Let  $a \in \Sigma_0(\mathbb{R}^{2d})$ . Then,  $k_a(x, v) \rightarrow 0$  as  $|x| \rightarrow \infty$  uniformly with respect to  $v \in \mathbb{R}^d$ .*

**Proof.** One writes  $|k_a(x, v)| \leq g(x) = (2\pi)^{-d} \int |a(x, \xi)| d\xi$ . Since  $|a(x, \xi)| \lesssim \langle \xi \rangle^{-d-1}$ , one finds that  $g(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  by the dominated-convergence theorem.  $\square$

The following lemma will be of great use in what follows.

**Lemma 5.9.** *Let  $a \in \Sigma_0(\mathbb{R}^{2d})$ . Let  $\rho_h(x, v) \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R}^d)$  be such that*

- (1)  $\|\rho_h\|_{L^\infty} \leq C_0$  uniformly in  $h$ ,
- (2)  $\rho_h(x, v) \rightarrow 0$  as  $h \rightarrow 0$  uniformly for  $(x, v)$  in any compact set.

*Then, one has*

$$\text{ess sup}_{x \in \mathbb{R}^d} \|\rho_h k_a(x, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0 \quad \text{and} \quad \text{ess sup}_{x \in \mathbb{R}^d} \|\rho_h k_a(x + h, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0,$$

*as  $h \rightarrow 0$ , with  $k_a(x, v)$  as in (5.4).*

**Proof.** Set  $m_h = \rho_h k_a$ . For  $R > 0$ , by (5.5) one writes

$$\int_{|v|>R} |m_h(x, v)| dv \lesssim C_0 \int_{|v|>R} \langle v \rangle^{-(d+1)} dv,$$

for almost all  $x \in \mathbb{R}^d$ . Let  $\varepsilon > 0$ . For  $R > 0$  chosen sufficiently large one has

$$(5.7) \quad \int_{\mathbb{R}^d} |m_h(x, v)| dv \leq \varepsilon + \int_{|v|\leq R} |m_h(x, v)| dv, \quad x \in \mathbb{R}^d \text{ a.e..}$$

Next, one writes

$$\int_{|v|\leq R} |m_h(x, v)| dv \leq C_0 \int_{|v|\leq R} |k_a(x, v)| dv.$$

Thus, by Lemma 5.8, for some  $R' > 0$ , one has  $\int_{|v|\leq R} |m_h(x, v)| dv \leq \varepsilon/C_0$  for  $|x| \geq R'$ . One thus has

$$\text{ess sup}_{|x|\geq R'} \int_{\mathbb{R}^d} |m_h(x, v)| dv \leq 2\varepsilon.$$

Consider now the case  $|x| \leq R'$ . By hypothesis  $|\rho_h(x, v)| \rightarrow 0$  as  $h \rightarrow 0$  uniformly with respect to  $x$  and  $v$  if  $|x| \leq R'$  and  $|v| \leq R$ . With (5.5) one finds

$$\int_{|v|\leq R} |m_h(x, v)| dv \lesssim \int_{|v|\leq R} |\rho_h(x, v)| dv,$$

for almost all  $x$  such that  $|x| \leq R'$ . One thus finds that  $\int_{|v|\leq R} |m_h(x, v)| dv \leq \varepsilon$  for such  $x$  and for  $h > 0$  chosen sufficiently small. With (5.7) one thus concludes that

$$\text{ess sup}_{|x|\leq R'} \int_{\mathbb{R}^d} |m_h(x, v)| dv \leq 2\varepsilon,$$

if  $h > 0$  is chosen sufficiently small and thus  $\text{ess sup}_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |m_h(x, v)| dv \leq 2\varepsilon$ .

One obtains *mutatis mutandis* that  $\text{ess sup}_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |m_h(x + hv, v)| dv \leq 2\varepsilon$ , for  $h$  chosen sufficiently small.  $\square$

**Proposition 5.10.** *Let  $a \in \Sigma_0(\mathbb{R}^{2d})$ .*

(1) *Consider  $\theta \in \mathcal{C}^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ . one has*

$$(5.8) \quad \lim_{h \rightarrow 0} \|[a(x, hD_x), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = 0.$$

(2) More generally, if  $(\theta_k)_{k \in \mathbb{N}} \subset L^\infty(\mathbb{R}^d)$ ,  $\theta \in \mathcal{C}^0(\mathbb{R}^{d+1})$  is such that  $\|\theta_k - \theta\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow +\infty$ , then

$$(5.9) \quad \|[a(x, hD_x), \theta_k]\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = o(1)_{h \rightarrow 0 \text{ and } k \rightarrow \infty}.$$

(3) Assume in addition that  $a \in \Sigma_0^{0,2+d}(\langle \xi \rangle^{-(1+d)}; \mathbb{R}^{2d})$  and consider  $\theta \in W^{1,\infty}(\mathbb{R}^d)$  then one has

$$\left\| [\text{Op}^h(a), \theta] \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = O(h).$$

(4) For  $a \in \Sigma_0^{0,2+d}(\langle \xi \rangle^{-(1+d)}; \mathbb{R}^{2d})$ , if moreover  $\theta \in \mathcal{C}^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$  then one has the following properties

$$(5.10) \quad \left\| [\text{Op}^h(a), \theta] + ih \sum_{j=1}^d \frac{\partial \theta}{\partial x_j} \text{Op}^h\left(\frac{\partial a}{\partial \xi_j}\right) \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = o(h),$$

and

$$(5.11) \quad \left\| [\text{Op}^h(a), \theta] + ih \sum_{j=1}^d \text{Op}^h\left(\frac{\partial a}{\partial \xi_j}\right) \frac{\partial \theta}{\partial x_j} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = o(h).$$

(5) More generally, if  $(\theta_k)_{k \in \mathbb{N}} \subset \text{Lip}(\mathbb{R}^d)$  is such that  $\|\theta_k - \theta\|_{\text{Lip}} \rightarrow 0$  as  $k \rightarrow +\infty$ , then

$$(5.12) \quad \left\| [\text{Op}^h(a), \theta_k] + ih \sum_{j=1}^d \frac{\partial \theta_k}{\partial x_j} \text{Op}^h\left(\frac{\partial a}{\partial \xi_j}\right) \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = ho(1)_{h \rightarrow 0 \text{ and } k \rightarrow \infty},$$

and

$$\left\| [\text{Op}^h(a), \theta_k] + ih \sum_{j=1}^d \text{Op}^h\left(\frac{\partial a}{\partial \xi_j}\right) \frac{\partial \theta_k}{\partial x_j} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = ho(1)_{h \rightarrow 0 \text{ and } k \rightarrow \infty}.$$

(6) Finally, assume that  $a \in \Sigma_0^{0,N+1+d}(\langle \xi \rangle^{-(d+1)}; \mathbb{R}^{2d})$ . Let  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  be such that  $\phi a = 0$ . One has

$$(5.13) \quad \|\text{Op}^h(a) \circ \phi\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = o(h^N).$$

**Proof.** The kernel of the operator  $[a(x, hD_x), \theta]$  is given by

$$(5.14) \quad K(x, y) = K_a(x, y)(\theta(y) - \theta(x)) = h^{-d} m_h(x, \frac{x-y}{h}),$$

with  $m_h(x, v) = k_a(x, v)(\theta(x-hv) - \theta(x))$ . Since  $\theta$  is continuous it is uniformly continuous on any compact set. Thus, one finds  $|\theta(x-hv) - \theta(x)| \rightarrow 0$  as  $h \rightarrow 0$  uniformly with respect to  $x$  and  $v$  if  $|x| \leq R'$  and  $|v| \leq R$ . With Lemma 5.9 one obtains that

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \|m_h(x, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0 \quad \text{and} \quad \operatorname{ess\,sup}_{y \in \mathbb{R}^d} \|m_h(x+h\cdot, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0,$$

as  $h \rightarrow 0$ . With Lemma 5.2 one concludes that the limit in (5.8) holds.

To obtain (5.9) one writes

$$[a(x, hD_x), \theta_k] = a(x, hD_x)(\theta_k - \theta) - (\theta_k - \theta)a(x, hD_x) + [a(x, hD_x), \theta].$$

Assume now that  $a$  and  $\theta$  fulfill the requirements of point (3). The kernel of the operator  $\operatorname{Op}^h(a), \theta]$  is given by (5.14). With the first-order Taylor formula one writes

$$\theta(x-hv) - \theta(x) = -h \int_0^1 d_x \theta(x-shv)(v) ds = -h \sum_j v_j \Theta_j(x, hv),$$

with  $\Theta_j(x, hv) = \int_0^1 \partial_j \theta(x-shv) ds$ . With the additional regularity of  $a(x, \xi)$  and Remark 5.7 one finds

$$m_h(x, v) = -ih \sum_j k_{\partial_{\xi_j} a}(x, v) \Theta_j(x, hv),$$

yielding

$$|m_h(x, v)| \lesssim h \sum_j M_{0, d+1}^{-(d+1)}(\partial_{\xi_j} a) \langle v \rangle^{-(d+1)} \lesssim h M_{0, d+2}^{-(d+1)}(a) \langle v \rangle^{-(d+1)},$$

as  $|\Theta_j(x, hv)| \lesssim 1$  uniformly in  $x, v$  and  $h$ . With Corollary 5.3 one deduces the result of point (3).

Assume now that  $a$  and  $\theta$  fulfill the requirements of point (4) of the proposition and denote by  $\mathbf{K}(x, y)$  the kernel of  $[\operatorname{Op}^h(a), \theta] + ih \sum_{j=1}^d \partial_{x_j} \theta \operatorname{Op}^h(\partial_{\xi_j} a)$ . One has  $\mathbf{K}(x, y) = h^{-d} r_h(x, (x-y)/h)$  with

$$r_h(x, v) = k_a(x, v)(\theta(x-hv) - \theta(x) + hd_x \theta(x)(v)),$$

using Remark 5.7. The first-order Taylor formula gives

$$\theta(x - hv) - \theta(x) + hd_x\theta(x)(v) = h \int_0^1 (d_x\theta(x) - d_x\theta(x - shv))(v)ds.$$

Setting  $A_h^j(x, v) = \int_0^1 (\partial_j\theta(x) - \partial_j\theta(x - shv))ds$ , one finds

$$r_h(x, v) = ih \sum_j A_h^j(x, v)k_{\partial_{\xi_j}a}(x, v),$$

using again Remark 5.7. Since  $\theta \in \mathcal{C}^1(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$  one finds  $A_h^j \in \mathcal{C}^0(\mathbb{R}^{2d})$  and  $\|A_h^j\|_{L^\infty(\mathbb{R}^{2d})} \leq C_0$  for some  $C_0 > 0$  uniformly with respect to  $h$ . Moreover, if  $L$  is a compact of  $\mathbb{R}^{2d}$ , and  $0 \leq h \leq h_0$ , if  $x, v \in L$  then  $x - shv$  remains in a compact set of  $\mathbb{R}$  where  $\partial_j\theta$  is uniformly continuous. One concludes that  $A_h^j(x, v) \rightarrow 0$  as  $h \rightarrow 0$  uniformly with respect to  $(x, v) \in L$ . Since  $\partial_{\xi_j}a \in \Sigma_0(\mathbb{R}^{2d})$ , with Lemma 5.9 one concludes that

$$\operatorname{ess\,sup}_{x \in \mathbb{R}^d} \|r_h(x, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0 \quad \text{and} \quad \operatorname{ess\,sup}_{y \in \mathbb{R}^d} \|r_h(x + h\cdot, \cdot)\|_{L^1(\mathbb{R}^d)} \rightarrow 0,$$

as  $h \rightarrow 0$ . With Lemma 5.2 one concludes that the limit in (5.10) holds.

Following the same strategy, one finds that the kernel of the operator

$$[\operatorname{Op}^h(a), \theta] - i \sum_j \operatorname{Op}^h(\partial_{\xi_j}a) \partial_j\theta(x)$$

is given by  $h^{-d}\tilde{r}_h(x, (x - y)/h)$ , with

$$\tilde{r}_h(x, v) = k_a(x, v)(\theta(x - hv) - \theta(x) + hd\theta(x - hv)(v))$$

and applying the Taylor formula as above and Lemma 5.9 one obtains the limit in (5.11) by Lemma 5.2.

To prove (5.12), as above one writes

$$[a(x, hD_x), \theta_k] = [a(x, hD_x), \theta_k - \theta] + [a(x, hD_x), \theta].$$

and by point (2) of the proposition one observe that it suffices to prove

$$(5.15) \quad \|[a(x, hD_x), \theta_k - \theta]\|_{\mathcal{L}(L^2)} = ho(1),$$

as  $k \rightarrow +\infty$ . Set  $\alpha_k = \theta_k - \theta$ . The kernel of  $[a(x, hD_x), \alpha_k]$  is given by  $L(x, y) = h^{-d}q_h(x, (x-y)/h)$  with

$$\begin{aligned} q_h(x, v) &= k_a(x, v)(\alpha_k(x - hv) - \alpha_k(x)) = -hk_a(x, v) \int_0^1 d_x \alpha_k(x - shv)(v) ds \\ &= -h \sum_{j=1}^d Q_{h,k}^j k_{\partial_{\xi_j} a}(x, v), \end{aligned}$$

where  $Q_{h,k}^j(x, v) = \int_0^1 \partial_j \alpha_k(x - thv) dt$ . Arguing as above one obtains (5.15).

Finally, we consider the last statement of the proposition, with  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  such that  $a\phi = 0$ . The kernel of  $\text{Op}^h(a)\phi$  reads  $h^{-d}s_h(x, (x-y)/h)$  with  $s_h(x, v) = k_a(x, v)\phi(x - hv)$ . With the Taylor formula one writes

$$\begin{aligned} \phi(x - hv) &= \sum_{j \leq N-1} \frac{(-h)^j}{j!} d^j \phi(x)(v, \dots, v) \\ &\quad + \frac{(-h)^N}{(N-1)!} \int_0^1 d^N \phi(x - shv)(v, \dots, v)(1-s)^{N-1} ds, \end{aligned}$$

which we write

$$\phi(x - hv) = \sum_{j \leq N} \frac{(-h)^j}{j!} d^j \phi(x)(v, \dots, v) + R_h^N(x, v),$$

with

$$\begin{aligned} R_h^N(x, v) &= \frac{(-h)^N}{(N-1)!} \int_0^1 d^N \phi(x - shv)(v, \dots, v)(1-s)^{N-1} ds \\ &\quad - \frac{(-h)^N}{N!} d^N \phi(x)(v, \dots, v) \\ &= \frac{(-h)^N}{(N-1)!} \int_0^1 (d^N \phi(x - shv) - d^N \phi(x))(v, \dots, v)(1-s)^{N-1} ds. \end{aligned}$$

Since  $\phi a = 0$  the same holds for  $\partial_x^\beta \phi a$  for any  $\beta$  and  $s_h(x, v) = k_a(x, v)R_h^N(x, v)$ . One has

$$R_h^N(x, v) = h^N \sum_{|\beta|=N} v^\beta \psi_{h,\beta}^N(x, v),$$

with  $\|\psi_{h,\beta}^N\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \leq C_0$  for some  $C_0 > 0$  uniformly with respect to  $h$  and  $\psi_{h,\beta}^N(x, v) \rightarrow 0$  as  $h \rightarrow 0$  uniformly with respect to  $(x, v)$  in a compact set. Iterating (5.6), one finds

$$k_a(x, v)R_h^N(x, v) = h^N \sum_{|\beta|=N} i^{|\beta|} k_{\partial_\xi^\beta a}(x, v) \psi_{h,\beta}^N(x, v),$$

and thus Lemma 5.9 and Lemma 5.2. imply that (5.13) holds.  $\square$

Let  $K$  be a compact set of  $\mathbb{R}^d$  and  $a \in \Sigma_0(\mathbb{R}^{2d})$  such that  $\text{supp } a \subset K \times \mathbb{R}^d$ . For these particular symbols, if  $\phi \in \mathcal{C}_c^0(\mathbb{R}^d)$  is equal to 1 on the  $x$ -projection of  $\text{supp } a$  then

$$\|\text{Op}^h(a)(1 - \phi)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} = o(1), \quad h \rightarrow 0,$$

by Proposition 5.10. In fact, we will be inclined to define semi-classical operators up to operators in  $\mathcal{L}(L^2(\mathbb{R}^d))$  whose norm is  $o(1)$  as  $h \rightarrow 0$ . Then we denote  $[\text{Op}^h](a)$  the class of operators defined by  $\text{Op}^h(a)(\phi u)$  where  $\phi$  is as above. This is further explained by our intention to use semi-classical operators on manifolds, here  $\mathcal{M}$  or  $\mathcal{L}$ , that we now present.

**5.3. Tangential symbols and operators.** In what follows we also use tangential operator. They are associated with symbols of the form  $a(y, \eta')$  with  $y \in \mathbb{R}^d$  and  $\eta' \in \mathbb{R}^{d-1}$ .

**Definition 5.11** (tangential symbols). *Let  $m, n \in \mathbb{N} \cup \{+\infty\}$ , with  $n \geq d$ , and  $N \in \mathbb{R}^+$ . Denote by  $\Sigma_{\mathbb{T}}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$  the space of all functions  $a(y, \eta)$ ,  $y \in \mathbb{R}^d$ ,  $\eta' \in \mathbb{R}^{d-1}$ , such that  $\partial_y^\alpha \partial_{\eta'}^\beta a \in L_{\text{loc}}^1(\mathbb{R}^{2d-1})$  for  $\alpha \in \mathbb{N}^d, \beta \in \mathbb{N}^{d-1}$  with  $|\alpha| \leq m, |\beta| \leq n$ , and for some  $C_{\alpha,\beta} > 0$ ,*

$$|\partial_y^\alpha \partial_{\eta'}^\beta a(y, \eta')| \leq C_{\alpha,\beta} \langle \eta' \rangle^{-N}, \quad y \in \mathbb{R}^d, \eta' \in \mathbb{R}^{d-1}.$$

*In addition, one sets  $\Sigma_{\mathbb{T},0}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$ ,  $n \geq d$ , as the set of all symbols  $a \in \Sigma_{\mathbb{T}}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$  with moreover  $\partial_y^\alpha \partial_{\eta'}^\beta a \in \mathcal{C}_0(\mathbb{R}^{2d-1})$  for  $\alpha \in \mathbb{N}^d, \beta \in \mathbb{N}^{d-1}$  with  $|\alpha| \leq m$  and  $|\beta| \leq n - d$ .*

Equipped with the norm

$$M_{\mathbb{T},m,n}^{-N}(a) = \max_{\substack{|\alpha| \leq m \\ |\beta| \leq n}} \text{ess sup}_{(y,\eta') \in \mathbb{R}^d \times \mathbb{R}^{d-1}} |\partial_y^\alpha \partial_{\eta'}^\beta a(y, \eta')| \langle \eta' \rangle^N,$$

both spaces  $\Sigma_{\mathbb{T}}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$  and  $\Sigma_{\mathbb{T},0}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$  are complete.

Note that  $\Sigma_{\mathbb{T}}^{m',n'}(\langle \eta' \rangle^{-N'}; \mathbb{R}^d \times \mathbb{R}^{d-1}) \subset \Sigma_{\mathbb{T}}^{m,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^d \times \mathbb{R}^{d-1})$  if  $m' \geq m$ ,  $n' \geq n$ , and  $N' \geq N$ . The case  $N = d$  is of interest similarly to symbols defined in Section 5.2. Set

$$\begin{aligned}\Sigma_{\mathbb{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1}) &= \Sigma_{\mathbb{T}}^{0,d}(\langle \eta' \rangle^{-d}; \mathbb{R}^d \times \mathbb{R}^{d-1}), \\ \Sigma_{\mathbb{T},0}(\mathbb{R}^d \times \mathbb{R}^{d-1}) &= \Sigma_{\mathbb{T},0}^{0,d}(\langle \eta' \rangle^{-d}; \mathbb{R}^d \times \mathbb{R}^{d-1}),\end{aligned}$$

and

$$N_n(a) = M_{\mathbb{T},0,n}^{-d}(a) = \max_{|\beta| \leq n} \operatorname{ess\,sup}_{(y,\eta')} |\partial_{\eta'}^{\beta} a(y, \eta')| \langle \eta' \rangle^d.$$

With  $y = (y', z)$ , observe that  $N_n(a)$  corresponds to  $M_{0,n}^{-d}(a)$  in (5.1) with  $z$  acting as a parameter.

For  $a \in \Sigma_{\mathbb{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1})$ , the associated operator is defined by

$$\operatorname{Op}^h(a)u(y) = a(y, hD'_y)u(z, y') = (2\pi)^{1-d} \int_{\mathbb{R}^{d-1}} e^{iy' \cdot \eta'} a(z, y', h\eta') \hat{u}(z, \eta') d\eta',$$

where the Fourier transformation acts in the  $y'$  variables. In fact, the action of  $\operatorname{Op}^h(a)$  is through the Schwartz kernel

$$K(y, \tilde{y}) = K_a(y', \tilde{y}'; z) \otimes \delta_{z-\tilde{z}},$$

with the tangential kernel

$$(5.16) \quad K_a(y', \tilde{y}'; z) = (2\pi)^{1-d} \int_{\mathbb{R}^{d-1}} e^{i(y'-\tilde{y}') \cdot \eta'} a(y', z, h\eta') d\eta'.$$

Then, one has

$$(5.17) \quad \operatorname{Op}^h(a)u(y', z) = \int_{\mathbb{R}^{d-1}} K_a(y', \tilde{y}'; z) u(\tilde{y}', z) d\tilde{y}'.$$

If  $a \in \Sigma_{\mathbb{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1})$  one finds  $K_a(y', \tilde{y}'; z) = h^{1-d} k_a(y, \frac{y'-\tilde{y}';z}{h})$  and

$$|k_a(y', v; z)| \leq CN_d(a) \langle v \rangle^{-d}, \quad v \in \mathbb{R}^{d-1}, \quad z \in \mathbb{R}, \quad y' \in \mathbb{R}^{d-1} \text{ a.e.},$$

as in (5.5). With Corollary 5.3 one has

$$\|\operatorname{Op}^h(a)u(\cdot, z)\|_{L^2(\mathbb{R}^{d-1})} \leq C_d N_d(a) \|u(\cdot, z)\|_{L^2(\mathbb{R}^{d-1})}, \quad z \in \mathbb{R} \text{ a.e.},$$



for some  $C_d > 0$  uniform with respect to  $z$ , yielding

$$\begin{aligned} \|\text{Op}^h(a)u\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}} \|\text{Op}^h(a)u(\cdot, z)\|_{L^2(\mathbb{R}^{d-1})}^2 dz \lesssim N_d(a)^2 \int_{\mathbb{R}} \|u(\cdot, z)\|_{L^2(\mathbb{R}^{d-1})}^2 dz \\ &\lesssim N_d(a)^2 \|u\|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

that is, the following continuity result.

**Lemma 5.12.** *Let  $a(y, \eta') \in \Sigma_{\top}(\mathbb{R}^d \times \mathbb{R}^{d-1})$ . Then  $\text{Op}^h(a)$  extends as a uniformly bounded operator on  $L^2(\mathbb{R}^d)$  and*

$$\|\text{Op}^h(a)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_d N_d(a).$$

In what follows, we also use symbols of the form  $a(y, \eta) = b(y, \eta')f(\zeta)$  with  $b(y, \eta') \in \Sigma_{\top}(\mathbb{R}^d \times \mathbb{R}^{d-1})$  and  $f(\zeta)$  a Fourier multiplier; see for instance Proposition 8.2 In fact, one has

$$(5.18) \quad \text{Op}^h(b(y, \eta')f(\zeta)) = \text{Op}^h(b)f(hD_z),$$

and thus one can write

$$(5.19) \quad \|\text{Op}^h(b(y, \eta')f(\zeta))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \|\text{Op}^h(b)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \|f(hD_z)\|_{\mathcal{L}(L^2(\mathbb{R}^d))},$$

that we will use several times in what follows. Since

$$f(hD_z) = \mathcal{F}_{\zeta \rightarrow z}^{-1} f(h\zeta) \mathcal{F}_{z \rightarrow \zeta}$$

if  $f$  is bounded one finds

$$(5.20) \quad \|f(hD_z)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq \|f\|_{L^\infty},$$

since the Fourier transformation  $\mathcal{F}_{z \rightarrow \zeta}$  is a an isometry on  $L_z^2(\mathbb{R}; L_{y'}^2(\mathbb{R}^{d-1}))$ .

Similarly  $f(hD_z)$  has kernel on  $\mathbb{R}^d$  given by

$$\delta_{y'-\tilde{y}'} \otimes K_f(y; z, \tilde{z}),$$

with the part only acting in the  $z$  variable given by

$$(5.21) \quad K_f(z, \tilde{z}) = h^{-1} k_f((z - \tilde{z})/h),$$

with

$$(5.22) \quad k_f(v) = (2\pi)^{-1} \int_{\mathbb{R}} e^{iv \cdot \zeta} f(\zeta) d\zeta = \check{f}(v).$$

**5.4. Semi-classical operators on a manifold.** Let  $\mathcal{N}$  be a  $\mathcal{C}^1$ -manifold of dimension  $d$  equipped with a density measure  $\rho$  that allows one to define  $L^2(\mathcal{N})$ . We denote by  $\mathcal{P}(\mathcal{N})$  the algebra of bounded operators  $B_h$  on  $L^2(\mathcal{N})$ , depending on  $h \in (0, h_0]$  as a parameter, and by  $\mathcal{R}(\mathcal{N})$  the ideal of  $\mathcal{P}(\mathcal{N})$  of the operators  $B_h$  such that  $\|B_h\|_{\mathcal{L}(L^2)} = o(1)$ . Set  $\mathcal{Q}(\mathcal{N}) = \mathcal{P}(\mathcal{N})/\mathcal{R}(\mathcal{N})$ .

The following lemma is key towards the notion of semi-classical operators on a manifold.

**Lemma 5.13** ([16, Lemme 1.10]). *Consider  $\psi : V \rightarrow U$  a  $\mathcal{C}^1$ -diffeomorphism between two open subsets of  $\mathbb{R}^d$ . Let  $a \in \mathcal{C}_c^0(U \times \mathbb{R}^d)$  be such that  $\partial_\xi^\beta a \in \mathcal{C}_c^0(U \times \mathbb{R}^d)$  for  $|\beta| \leq d + 1$ . Set  $b(y, \eta) = a(\psi(y), {}^t d\psi_y^{-1}(\eta)) \in \mathcal{C}_c^0(V \times \mathbb{R}^d)$ . Then, for any compact set  $K \subset U$  one has*

$$\|a(x, hD_x)u \circ \psi - b(y, hD_y)(u \circ \psi)\|_{L^2(V)} = o(1)\|u\|_{L^2(U)}, \quad h \rightarrow 0,$$

uniformly with respect to  $u \in L^2(U)$  with support in  $K$ .

**Definition 5.14.** Let  $\mathcal{N}$  be a  $\mathcal{C}^1$ -manifold of dimension  $d$ . Denote by  $\Sigma_c(T^*\mathcal{N})$  the space of functions  $a \in \mathcal{C}_c^0(T^*\mathcal{N})$  such that for  $|\beta| \leq d + 1$ , one has  $\partial_\xi^\beta a \in \mathcal{C}_c^0(T^*\mathcal{N})$ .

For  $a \in \Sigma_c(T^*\mathcal{N})$  and a chart  $\mathcal{C} = (\mathcal{O}, \phi)$  we denote by  $a^{\mathcal{C}}$  the local representative of  $a$  in this chart. Consider two local charts  $\mathcal{C}_1 = (\mathcal{O}_1, \phi_1)$  and  $\mathcal{C}_2 = (\mathcal{O}_2, \phi_2)$  with  $W = \mathcal{O}_1 \cap \mathcal{O}_2 \neq \emptyset$  and  $a \in \Sigma_c(T^*\mathcal{N})$  supported in  $W$ . Then, the representatives  $a^{\mathcal{C}_1}$  and  $a^{\mathcal{C}_2}$  fulfill the assumption of Lemma 5.13 with  $U = \phi_1(W)$ ,  $V = \phi_2(W)$  and  $\psi = \phi_1 \circ \phi_2^{-1}$ .

Consider a chart  $\mathcal{C} = (\mathcal{O}, \phi)$  as above,  $a \in \Sigma_c(T^*\mathcal{N})$  and  $\theta, \chi \in \mathcal{C}_c(\mathcal{O})$ ,  $\theta \equiv 1$  in a neighborhood of  $\text{supp } \chi$ . For  $u \in L^2(\mathcal{N})$  one may compute

$$\phi^* \circ (\chi a)^{\mathcal{C}}(x, hD_x) \circ (\phi^{-1})^*(\theta u),$$

yielding an  $L^2$ -function on  $\mathcal{N}$ .

Consider now a locally finite  $\mathcal{C}^1$ -partition of unity  $(\chi_i)_{i \in \mathcal{I}}$  subordinated to a given atlas  $\mathcal{A} = (\mathcal{C}_i)_{i \in \mathcal{I}}$ ,  $\mathcal{C}_i = (\mathcal{O}_i, \phi_i)$  and a family of localisation functions  $(\theta_i)_{i \in \mathcal{I}}$  with  $\text{supp } \theta_i \subset \mathcal{O}_i$  and  $\theta_i \equiv 1$  on  $\text{supp } \chi_i$ . We form

$$Au = \sum_{i \in \mathcal{I}} \phi_i^* \circ (\chi_i a)^{\mathcal{C}_i}(x, hD_x) \circ (\phi_i^{-1})^*(\theta_i u).$$

From Lemma 5.13, the class of the operator  $A$  defined above in  $\mathcal{Q}(\mathcal{N})$  is independent of the choice of the atlas  $\mathcal{A}$ , the partition of unity  $(\chi_i)_{i \in \mathcal{I}}$ , and the localisation functions  $(\theta_i)_{i \in \mathcal{I}}$ . We denote this class by  $[\text{Op}^h](a)$ .

Let  $\varphi \in \mathcal{C}^0(\mathcal{N})$ . Let  $B_h$  and  $\tilde{B}_h$  be two representatives of a class in  $\mathcal{Q}(\mathcal{N})$ , that is,  $[B_h] = [\tilde{B}_h]$ . Observe that  $[\varphi B_h] = [\varphi \tilde{B}_h]$ , thus defining a multiplication by the function  $\varphi$  on  $\mathcal{Q}(\mathcal{N})$ , which one writes  $[\varphi B_h] = \varphi[B_h]$ . If  $a \in \Sigma_c(T^*\mathcal{N})$  one has

$$[\text{Op}^h](\varphi a) = \varphi[\text{Op}^h](a).$$

**5.5. Semi-classical measures.** This section is borrowed from [16] and [2]; it recalls the basic properties of semi-classical measures.

In what follows, we call a sequence of scales  $H = (h_k)_k$  a sequence of positive real numbers that converges to 0. If such a sequence of scales is used we will write  $\text{Op}^h$  in place of  $\text{Op}^{h_k}$  for concision if no confusion can arise.

**Definition 5.15** (semi-classical measure). *Let  $H = (h_k)_k$  be a sequence of scales and  $(u_k)_k$  be a bounded sequence of  $L^2(\mathbb{R}^d)$ . Let  $\mu$  be a nonnegative Radon measure on  $\mathbb{R}^{2d}$ . One says that  $(u_k)_k$  admits  $\mu$  as its semi-classical measure (s.c.m.) at scale  $H = (h_k)_k$  if one has*

$$(5.23) \quad \lim_{k \rightarrow +\infty} (\text{Op}^h(a)u_k, u_k)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} a(x, \xi) d\mu(x, \xi) = \langle \mu, a \rangle,$$

for any  $a \in \Sigma_0(\mathbb{R}^{2d})$ .

**Definition 5.16** (mass leakage at infinity). *One says that no mass leaks at infinity at scale  $H$  if one has*

$$\lim_{R \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \left( \int_{|x| \geq R} |u_k(x)|^2 dx + \int_{h_k|\xi| \geq R} |\hat{u}_k(\xi)|^2 d\xi \right) = 0.$$

*One says that there is some mass leakage at scale  $H$  at infinity otherwise.*

**Lemma 5.17.** *Suppose that  $(h_k^s |D_x|^s u_k)_k$  is  $L^2$ -bounded for some  $s > 0$ . Then*

$$\lim_{R \rightarrow +\infty} \limsup_{k \rightarrow +\infty} \int_{h_k|\xi| \geq R} |\hat{u}_k(\xi)|^2 d\xi = 0.$$

**Proof.** Write  $\int_{h_k|\xi| \geq R} |\hat{u}_k(\xi)|^2 d\xi \leq R^{-2s} \int_{\mathbb{R}^2} h_k^{2s} |\xi|^{2s} |\hat{u}_k(\xi)|^2 d\xi \lesssim R^{-2s}$ .  $\square$

The following proposition states that up to a subsequence extraction, every bounded sequence in  $L^2(\mathbb{R}^d)$  admits a s.c.m. at some given scale. It moreover provides a criterium for mass conservation in the limiting process.

**Proposition 5.18** ([16, Propositions 1.4 and 1.6]). *For any sequence of scales  $H = (h_k)_k$ , and any bounded sequence  $(u_k)_k \subset L^2(\mathbb{R}^d)$ , there exist a subsequence  $(k_n)_{n \in \mathbb{N}}$  and a nonnegative measure  $\mu$  on  $\mathbb{R}^{2d}$  such that the following properties hold:*

- (1)  $\mu$  is the s.c.m. for the sequence  $(u_{k_n})_n$  at scale  $(h_{k_n})_n$   
 (2) If no mass leaks at infinity at scale  $H$  in the sense of Definition 5.16, then

$$(5.24) \quad \lim_{n \rightarrow +\infty} \|u_{k_n}\|_{L^2(\mathbb{R}^d)}^2 = \mu(\mathbb{R}^{2d}).$$

meaning mass is preserved in the limiting process.

**Lemma 5.19.** *Assume that  $\mu$  is the s.c.m. for the sequence  $(u_k)_k$  at scale  $(h_k)_k$ . Let  $(a_k)_k \subset \Sigma_0(\mathbb{R}^{2d})$  be converging in  $\Sigma_0(\mathbb{R}^{2d})$  to some  $a$ , and  $(b_k)_k, (b'_k)_k \subset L^\infty(\mathbb{R}^d)$  that converges uniformly to some  $b, b' \in \mathcal{C}^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  respectively. Then*

$$\lim_{k \rightarrow +\infty} (b'_k \text{Op}^h(a_k) b_k u_k, u_k)_{L^2(\mathbb{R}^d)} = \langle \mu, b b' a \rangle.$$

**Proof.** One writes

$$(5.25) \quad \begin{aligned} b'_k \text{Op}^h(a_k) b_k &= (b'_k - b') \text{Op}^h(a_k) b_k + b' \text{Op}^h(a_k - a) b_k + b' \text{Op}^h(a) (b_k - b) \\ &\quad + b' \text{Op}^h(a) b. \end{aligned}$$

Convergence in  $\Sigma_0(\mathbb{R}^{2d})$  shows that the operator norms  $\|\text{Op}^h(a) - \text{Op}^h(a_k)\|_{\mathcal{L}(L^2)}$  converge to 0 uniformly with respect to  $h > 0$  by Lemma 5.6 and  $\text{Op}^h(a_k)$  is uniformly bounded in  $\mathcal{L}(L^2)$ . With the convergences of  $(b_k)_k, (b'_k)_k$  one sees that the first three terms in (5.25) contribute with a vanishing limit because of the  $L^2$ -boundedness of  $(u_k)_k$ . It thus suffices to study the limit of  $(b' \text{Op}^h(a) b u_k, u_k)_{L^2(\mathbb{R}^d)}$ . One writes

$$(b' \text{Op}^h(a) b u_k, u_k)_{L^2(\mathbb{R}^d)} = (b' [\text{Op}^h(a), b] u_k, u_k)_{L^2(\mathbb{R}^d)} + (b' b \text{Op}^h(a) u_k, u_k)_{L^2(\mathbb{R}^d)}.$$

Since  $b' b a \in \Sigma_0(\mathbb{R}^{2d})$  and  $b' b \text{Op}^h(a) = \text{Op}^h(b' b a)$  the result follows from (5.8), the  $L^2$ -boundedness of  $(u_k)_k$ , and (5.23).  $\square$

A consequence is the following result.

**Corollary 5.20.** *Assume that  $\mu$  is the s.c.m. for a sequence  $(u_k)_k$  at scale  $(h_k)_k$  and let  $\theta \in \mathcal{C}^0(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ . Then  $|\theta|^2 \mu$  is the s.c.m. for the sequence  $(\theta u_k)_k$  at scale  $(h_k)_k$ .*

The convergence in (5.23) can be extended to more general symbols and semi-classical operators.

**Proposition 5.21.** *Let  $(u_k)_k$  be bounded in  $L^2(\mathbb{R}^d)$ . Suppose  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$  is such that  $(\varphi(h_k D)u_k)_k$  is bounded in  $L^2(\mathbb{R}^d)$ . Suppose  $\mu$  is the s.c.m. for the sequence  $(u_k)_k$  at scale  $(h_k)_k$  and there is no mass leakage at infinity at scale  $H = (h_k)_k$  in the sense of Definition 5.16 for the sequences  $(u_k)_k$  and  $(\varphi(h_k D)u_k)_k$ . Suppose  $a(x, \xi)$  (or  $a(x, \xi')$ , that is, a tangential symbol), continuous in  $x$  and  $(d+1)$ -times differentiable in  $\xi$ , is such that  $\text{Op}^h(a)$  is bounded on  $L^2(\mathbb{R}^d)$ . Then, one has*

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(a)\varphi(h_k D)u_k, u_k)_{L^2(\mathbb{R}^d)} = \langle \mu, a(x, \xi)\varphi(\xi) \rangle.$$

A Typical example is  $a \in \Sigma_{\mathbb{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1})$  by Lemma 5.12. Other examples are  $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$  or  $a \in S^0(\mathbb{R}^d \times \mathbb{R}^{d-1})$ , with  $S^0$  denoting the usual class of symbols of order 0; see [20, Definition 18.1.1]. The result also applies to any  $a \in S(1, g)$  for any slowly-varying temperate metric  $g$  in the sense of the Weyl-Hörmander calculus [20, Section 18.4-18.5]; such generality is not needed here.

**Remark 5.22.** *An inspection of the proof shows that a sharper assumption is*

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \left( \int_{|x| \geq R} |u_{k_n}(x)|^2 dx + \int_{h_{k_n}|\xi| \geq R} |\varphi(\widehat{h_k D})u_{k_n}(\xi)|^2 d\xi \right) = 0.$$

Note also that Lemma 5.19 also holds in the tangential case, for instance for  $a \in \Sigma_{\mathbb{T}}(\mathbb{R}^d \times \mathbb{R}^{d-1})$ .

**Proof.** The proof is the same in both cases and is along the line of Proposition 1.6-(iii) in [16], yet simpler. With the no mass-leakage assumption and since  $\text{Op}^h(a)$  is bounded on  $L^2(\mathbb{R}^d)$  one finds

$$(5.26) \quad \lim_{R \rightarrow +\infty} \limsup_{k \rightarrow +\infty} |(\text{Op}^h(a)\varphi(h_k D)u_k, u_k)_{L^2(\mathbb{R}^d)} - (\text{Op}^h(a_R)u_k, u_k)_{L^2(\mathbb{R}^d)}| = 0,$$

with  $a_R(x, \xi) = \chi(x/R)\chi(\xi/R)a(x, \xi)\varphi(\xi)$ . Since  $a_R \in \Sigma_0(\mathbb{R}^{2d})$  with (5.23) one has

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(a_R)u_k, u_k)_{L^2(\mathbb{R}^d)} = \langle \mu, a_R \rangle.$$

With (5.26) one concludes by means of the dominated-convergence theorem, since  $\mu$  has finite mass by (5.24).  $\square$

We now extend the notion of semi-classical measures to the case of manifolds. As above  $\mathcal{N}$  is a  $\mathcal{C}^1$ -manifold of dimension  $d$  equipped with a density measure  $\rho$  that allows one to define  $L^2(\mathcal{N})$ . For some basic details on density measures on manifold we refer for instance to [23, Section 16.2].

Set  $\lambda = \ell^\infty/c_0$  as the space of bounded sequences modulo the space of sequences converging to 0. Let  $U = (u_k)_k$  be a bounded sequence in  $L^2(\mathcal{N})$  and  $H = (h_k)_k$  be a sequence of scales. For  $a \in \Sigma_c(T^*\mathcal{N})$ , denote by

$$[[[\text{Op}^h](a)u_k, u_k]_{L^2(\mathcal{N}, \rho)}]_\lambda$$

the class in  $\lambda$  of the sequence  $([\text{Op}^h](a)u_k, u_k)_{L^2(\mathcal{N}, \rho)}$ .

If now  $U = (u_k)_k$  is bounded in  $L^2_{\text{loc}}(\mathcal{N})$  it is sensible to compute

$$([\text{Op}^{h_n}](a)\psi u_{k_n}, u_{k_n})_{L^2(\mathcal{N}, \rho)}$$

for  $a \in \Sigma_c(T^*\mathcal{N})$  and  $\psi \in \mathcal{C}_c^\infty(\mathcal{N})$  with  $\psi = 1$  on  $\text{supp } a$ .

**Definition 5.23.** Let  $U = (u_k)_k$  be a bounded sequence in  $L^2_{\text{loc}}(\mathcal{N})$  and  $H = (h_k)_k$  a sequence of scales. Denote by  $\mathcal{M}^+(U)$  the set of measures  $\mu$  on  $T^*\mathcal{N}$  such that there exists a subsequence  $k_n$  such that

$$\lim_{n \rightarrow +\infty} [[[\text{Op}^{h_n}](a)\psi u_{k_n}, u_{k_n})_{L^2(\mathcal{N}, \rho)}]_\lambda = \langle \mu, a \rangle,$$

for any  $a \in \Sigma_c(T^*\mathcal{N})$  and  $\psi \in \mathcal{C}_c^\infty(\mathcal{N})$  with  $\psi = 1$  on  $\text{supp } a$ .

What follows explains that this definition is sensible in the sense that it is independent of the choice made for the function  $\psi$ . In particular, this coincides with the definition of a s.c.m. in the case of a  $L^2$ -bounded sequence.

If  $\mu$  is the s.c.m. associated with  $U = (u_k)_k$ , then in any local chart  $\mathcal{C} = (\mathcal{O}, \phi)$ , denote by  $\mu^{\mathcal{C}}$  the local representative of  $\mu$ , that is,  $(\phi^{-1})^*\mu$ . Denote also by  $u_k^{\mathcal{C}}$  the local representative of  $u_k$ , that is,  $u_k^{\mathcal{C}} = (\phi^{-1})^*u_k = u_k \circ \phi^{-1}$ . Then, if  $K \subset \phi(\mathcal{O})$  is compact,  $a \in \Sigma_0(\mathbb{R}^{2d})$  with  $\text{supp } a \subset K \times \mathbb{R}^d$ , and  $\psi \in \mathcal{C}_c^\infty(\phi(\mathcal{O}))$  equal to 1 in a neighborhood of the  $x$ -projection of  $\text{supp } a$  one has

$$(5.27) \quad \lim_{k \rightarrow +\infty} (\text{Op}^h(a)\psi u_k^{\mathcal{C}}, u_k^{\mathcal{C}})_{L^2(\mathbb{R}^d, \rho)} = \langle \mu^{\mathcal{C}}, a \rangle.$$

In what follows  $\rho$  will be given by  $\kappa\mu_g$ , that is, in local coordinates  $\rho^{\mathcal{C}} = \kappa^{\mathcal{C}} \det(g^{\mathcal{C}})^{1/2} dx$ . One then has

$$\mu^{\mathcal{C}} = \kappa^{\mathcal{C}} \det(g^{\mathcal{C}})^{1/2} m,$$

if  $m$  is the s.c.m. associated with  $(u_k^{\mathcal{C}})_k$  yet using the  $L^2$ -inner product given by the Lebesgue measure as in Definition 5.15, that is,

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(a)\psi u_k^{\mathcal{C}}, u_k^{\mathcal{C}})_{L^2(\mathbb{R}^d, dx)} = \langle m, a \rangle.$$

Some of the properties of s.c.m. on  $\mathbb{R}^d$  can then be extended to the case on a manifold. For instance one has the following result.

**Lemma 5.24.** *Assume that  $\mu$  is the s.c.m. of a sequence  $U = (u_k)_k$  at scale  $(h_k)_k$  on  $\mathcal{N}$ . Let  $(a_k)_k \subset \Sigma_c(T^*\mathcal{N})$  be converging in  $\Sigma_c(T^*\mathcal{N})$  to some  $a$ , and  $(b_k)_k, (b'_k)_k \subset L^\infty(\mathcal{N})$  that converges uniformly to some  $b, b' \in \mathcal{C}^0(\mathcal{N}) \cap L^\infty(\mathcal{N})$  respectively. Then*

$$\lim_{k \rightarrow +\infty} [(b'_k[\text{Op}^h](a_k)b_k u_k, u_k)_{L^2(\mathcal{N})}]_\lambda = \langle \mu, bb'a \rangle.$$

The local chart version is

$$(5.28) \quad \lim_{k \rightarrow +\infty} (b'_k \text{Op}^h(a_k)b_k \psi u_k^{\mathcal{C}}, u_k^{\mathcal{C}})_{L^2(\mathbb{R}^d, \rho)} = \langle \mu^{\mathcal{C}}, bb'a \rangle,$$

for  $a$  and  $\psi$  as given for (5.27) and  $a_k, b_k, b'_k$  also defined locally accordingly.

The following result that yields the existence of s.c.m..

**Proposition 5.25.** *Suppose  $H = (h_k)_k$  is a sequence of scales and  $U = (u_k)_k$  a sequence of functions on  $\mathcal{N}$ .*

- (1) *If  $U = (u_k)_k$  is bounded in  $L^2(\mathcal{N})$  the set  $\mathcal{M}^+(U)$  is nonempty.*
- (2) *Suppose  $\mathcal{N}$  is countable at infinity. If  $U = (u_k)_k$  is bounded in  $L^2_{\text{loc}}(\mathcal{N})$  the set  $\mathcal{M}^+(U)$  is nonempty.*

**Proof.** The result of the first part, that is, if  $U = (u_k)_k$  is bounded in  $L^2(\mathcal{N})$ , holds by [16, Section 1].

For the second part, as  $\mathcal{N}$  is countable at infinity, there exists a sequence of open sets  $(\mathcal{O}_n)_n$  with  $\mathcal{O}_n \Subset \mathcal{O}_{n+1} \Subset \mathcal{N}$  and  $\cup_n \mathcal{O}_n = \mathcal{N}$ . The sequence  $(u_k)_k$  is  $L^2$ -bounded on  $\mathcal{O}_1$ . Suppose  $a \in \Sigma_c(T^*\mathcal{N})$  supported in  $\mathcal{O}_1$  and  $\psi \in \mathcal{C}_c^\infty(\mathcal{N})$  with  $\psi = 1$  on  $\mathcal{O}_1$ . One has

$$[[[\text{Op}^{h_n}](a)\psi u_{k_n}, u_{k_n}]_{L^2(\mathcal{N}, \rho)}]_\lambda = [[([\text{Op}^{h_n}](a)\psi u_{k_n}, \psi u_{k_n})_{L^2(\mathcal{N}, \rho)}]_\lambda].$$

The sequence  $(\psi u_k)_k$  is  $L^2$ -bounded. By the first part, there exists an inscreasing function  $\varphi_1 : \mathbb{N} \rightarrow \mathbb{N}$  and a measure  $\mu_1$  on  $T^*(\mathcal{O}_1)$  that is the s.c.m. for the subsequence  $(\psi u_{\varphi_1(k)})_k = (u_{\varphi_1(k)})_k$  on  $\mathcal{O}_1$ . With the same reasoning there exists an inscreasing function  $\psi_2 : \mathbb{N} \rightarrow \mathbb{N}$  and measure  $\mu_2$  on  $T^*(\mathcal{O}_2)$  that is the s.c.m. for the subsequence  $(u_{\varphi_2(k)})_k$  on  $\mathcal{O}_2$ , with  $\varphi_2 = \psi_2 \circ \varphi_1$ . One has  $\mu_2 = \mu_1$  on  $T^*(\mathcal{O}_1)$ . One proceeds by induction yielding two sequences of inscreasing functions  $\varphi_n : \mathbb{N} \rightarrow \mathbb{N}$  and  $\psi_n : \mathbb{N} \rightarrow \mathbb{N}$ , with  $\varphi_{n+1} = \psi_{n+1} \circ \varphi_n$ , and a sequence of measures  $\mu_n$  on  $T^*(\mathcal{O}_n)$ , with  $\mu_n$  the s.c.m. of  $(u_{\varphi_n(k)})_k$  on  $\mathcal{O}_n$ . Moreover, for  $\ell \in \mathbb{N}$ , one has  $\mu_{n+\ell} = \mu_n$  on  $T^*(\mathcal{O}_n)$ .

There exists a unique measure  $\mu$  on  $T^*(\mathcal{N})$  such that  $\mu = \mu_n$  on  $T^*(\mathcal{O}_n)$ . A diagonal extraction yields the subsequence  $(u_{\varphi_k(k)})_k$  implying that  $\mu_n$  is its s.c.m. on  $\mathcal{O}_n$  for any  $n \in \mathbb{N}$ . Hence,  $\mu$  is its s.c.m. on  $\mathcal{N}$ .  $\square$

The notion of s.c.m. can be extended to vector valued sequences. If  $N \in \mathbb{N}^*$ , denote by  $\mathcal{M}(T^*\mathcal{N}; \mathbb{M}_N(\mathbb{C}))$  the space of  $N \times N$ -matrix valued Radon measures on  $T^*\mathcal{N}$ , and by  $\mathcal{M}^+(T^*\mathcal{N}; \mathbb{M}_N(\mathbb{C}))$  the subspace formed by nonnegative Hermitian such measures.

**Definition 5.26** (Hermitian measures). *Suppose  $N \in \mathbb{N}^*$  and  $U = (u_k)_k$  is a bounded sequence in  $L^2_{\text{loc}}(\mathcal{N}; \mathbb{C}^N)$  and  $H = (h_k)_k$  a sequence of scales. Denote by  $\mathcal{M}^+(U)$  the set of measures  $\mu \in \mathcal{M}^+(T^*\mathcal{N}; \mathbb{M}_N(\mathbb{C}))$  such that there exists a subsequence  $k_n$  such that*

$$\lim_{n \rightarrow +\infty} [([\text{Op}^{h_n}](a)\psi u_{k_n}, u_{k_n})_{L^2(\mathcal{N}, \rho)}]_\lambda = \langle \text{tr}(a\mu), 1 \rangle = \int_{T^*\mathcal{N}} \text{tr}(a(x, \xi)) d\mu(x, \xi),$$

for any  $N \times N$  matrix  $a$  with entries in  $\Sigma_c(T^*\mathcal{N})$ , and  $[\text{Op}^{h_n}](a)$  the associated class of matrix valued operators, and  $\psi \in \mathcal{C}_c^\infty(\mathcal{N})$  with  $\psi = 1$  on  $\text{supp } a$ .

We refer the reader to [15, 3]. Each element of the matrix valued measure can also be understood as follows:

$$\lim_{n \rightarrow +\infty} [([\text{Op}^{h_n}](a)\psi u_{i,k_n}, u_{j,k_n})_{L^2(\mathcal{N}, \rho)}]_\lambda = \langle \mu_{ij}, a \rangle, \quad a \in \Sigma_c(T^*\mathcal{N}).$$

Each diagonal term is nonnegative. One finds that

$$\mu_{ij} \leq \mu_{ii}^{1/2} \mu_{jj}^{1/2},$$

in the sense that  $|\langle \mu_{ij}, ab \rangle|^2 \leq \langle \mu_{ii}, |a|^2 \rangle \langle \mu_{jj}, |b|^2 \rangle$ .

The counterpart to Proposition 5.25 is the following result.

**Proposition 5.27.** *Suppose  $N \in \mathbb{N}^*$  and  $H = (h_k)_k$  is a sequence of scales and  $U = (u_k)_k$  a sequence of function on  $\mathcal{N}$  valued in  $\mathbb{C}^N$ .*

- (1) *If  $U = (u_k)_k$  is bounded in  $L^2(\mathcal{N}; \mathbb{C}^N)$  the set  $\mathcal{M}^+(U)$  is nonempty.*
- (2) *Suppose  $\mathcal{N}$  is countable at infinity. If  $U = (u_k)_k$  is bounded in  $L^2_{\text{loc}}(\mathcal{N}; \mathbb{C}^N)$  the set  $\mathcal{M}^+(U)$  is nonempty.*

## 6. THE MEASURE PROPAGATION EQUATION AND PROOF OF OBSERVABILITY

We first state a result that is at the heart of the proof of Theorems 1.7 and 1.8. It expresses how a s.c.m.  $\mu$  associated with solutions to wave equations varies in the direction of the hamiltonian vector field  $H_p$ , in particular at the boundary  $\partial\mathcal{L} = \mathbb{R} \times \partial\mathcal{M}$  where this variation is connected to a s.c.m.  $\nu$  associated with the Neumann trace.



**6.1. The measure propagation equation.** Suppose  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$ ,  $H = (h_k)$  is a scale, and  $(\kappa_k, g_k)_k$  such that  $(\mathcal{M}, \kappa_k, g_k)$  converges to  $(\mathcal{M}, \kappa, g)$  in the  $\mathcal{Y}^1$  topology.

Suppose  $(u_k)_k$  are weak-solutions to

$$\partial_t^2 u_k - A_{\kappa_k, g_k} u_k = f_k,$$

with homogeneous Dirichlet boundary condition, as given in Proposition 1.1. Extend  $u_k$  and  $f_k$  by zero to  $\hat{\mathcal{L}}$ .

Suppose  $(u_k)_k$  is bounded in  $L_{\text{loc}}^2(\hat{\mathcal{L}})$ ,  $(h_k \partial_{\mathbf{n}_k} u_k|_{\partial\mathcal{L}})_k$  is bounded in  $L_{\text{loc}}^2(\partial\mathcal{L})$ , and  $(h_k f_k)_k$  is bounded in  $L_{\text{loc}}^2(\hat{\mathcal{L}})$ . With Proposition 5.27, a Hermitian  $2 \times 2$  s.c.m.  $M$  on  $T^*(\hat{\mathcal{L}})$  is associated with a subsequence at scale  $H$  of  $(u_k, h_k f_k)$ . Write

$$M = \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix}.$$

Set  $\mu = M_{0,0}$ . Similarly, with Proposition 5.25, there exists a measure  $\nu$  on  $T^*\partial\mathcal{L}$  such that the s.c.m. measure associated with (a subsequence of)  $h_k \psi(t) \partial_{\mathbf{n}_k} u_k|_{\partial\mathcal{L}}$  is  $|\psi(t)|^2 \nu$  at scale  $H$ .

**Theorem 6.1.** *Suppose that*

$$(6.1) \quad \text{supp } \mu \subset \text{Char } p \cap T^*\mathcal{L} \setminus 0 \quad \text{and} \quad \text{supp } \nu \subset T^*\partial\mathcal{L} \setminus 0.$$

*Then, the two measures  $\mu$  and  $\nu$  fulfill, in sense of density distributions,*

$$(6.2) \quad \mathbb{H}_p \mu = -{}^t \mathbb{H}_p \mu = 2 \text{Im } M_{0,1} + \int_{\varrho \in \|\mathcal{H}_\partial \cup \|\mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathbf{n}_x \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}}} d\nu(\varrho).$$

*The hyperbolic set  $\|\mathcal{H}_\partial$  and the glancing set  $\|\mathcal{G}_\partial$  are introduced in Definition 2.4 and  $\varrho^\pm$  and  $\xi^\pm$  are as given in (2.3). The vector field  $\mathbf{n}_x$  is the unitary inward pointing normal vector in the sense of the metric  $g$ .*

Here,  $p = p_{\kappa, g}$  and thus  $\mathbb{H}_p = \mathbb{H}_{p_{\kappa, g}}$ , the sets  $\|\mathcal{H}_\partial$  and  $\|\mathcal{G}_\partial$  are constructed with respect to the metric  $g$  as in Section 2.3. Recall that we identify  $T^*\partial\mathcal{L}$  and  $\|T^*\mathcal{L}$ . Hence, the measure  $\nu$  defined on  $T^*\partial\mathcal{L}$  is also a measure on  $\|T^*\mathcal{L}$ . The integral performed on  $\|\mathcal{H}_\partial \cup \|\mathcal{G}_\partial$  thus makes sense. Also, the meaning of the right hand side is explained in Remark 2.15.

Sections 7 to 9 are dedicated to the proof of Theorem 6.1. The result of Theorem 6.1 is key in the proof the main observability result as presented in the next section. There, one only considers the case  $f_k = 0$  implying  $M_{0,1} = 0$ . The addition of the source term  $f_k$  does not provide any complication for the proof Theorem 6.1, hence this slight generalization that can be of use elsewhere, in particular for the study of stabilization issues.

**6.2. Proof of the observability results.** Here, we provide the proof of Theorem 1.8 based on the measure equation of Theorem 6.1. Suppose  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$  and  $\omega$  is an open subset of  $\mathcal{M}$  (resp.  $\Gamma$  an open subset of  $\partial\mathcal{M}$ ) such that the interior geometric control condition of Definition 2.10 (resp. the boundary geometric control condition of Definition 2.13) is fulfilled and we consider some time  $T > T_{GCC}(\omega)$  (resp.  $T > T_{GCC}(\Gamma)$ ). We also consider  $\delta > 0$  such that  $T - 2\delta > T_{GCC}(\omega)$  (resp.  $T - 2\delta > T_{GCC}(\Gamma)$ ).

According to Propositions 4.2 (resp. Proposition 4.3), to achieve the observability inequalities of Theorem 1.8 for the time interval  $]0, T[$  it suffices to prove the semi-classical observability inequality (4.2) (resp. (4.3)) for the time interval  $I = ]\delta, T - \delta[$  for any  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\omega})$  (resp.  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\Gamma})$ ) that is  $\varepsilon$ -close to  $(\mathcal{M}, \kappa, g, \omega)$  (resp.  $(\mathcal{M}, \kappa, g, \Gamma)$ ) in the  $\mathcal{Y}^1$ -topology in the sense of Definition 1.3 and  $\varepsilon > 0$  chosen sufficiently small. We perform a contradiction argument based on propagation properties of semi-classical defect measures.

Below we consider a sequence  $(\mathcal{M}_n, \kappa_n, g_n)$  that converges to  $(\mathcal{M}, \kappa, g)$  in the  $\mathcal{Y}^1$  topology. For  $k \in \mathbb{Z}^*$ , one denotes by  $E_{n,k}$ , the space of functions defined in Section 4.1, here built on the elliptic operator  $A_{\kappa_n, g_n}$  on  $\mathcal{M}_n$ .

**6.2.1. Initiation of the contradiction argument.** In the case of an interior observation, we assume that (4.2) does not hold for some  $(\tilde{\mathcal{M}}, \tilde{\kappa}, \tilde{g}, \tilde{\omega})$  arbitrary close to  $(\mathcal{M}, \kappa, g, \omega)$  in the sense recalled above. Thus, there exists a sequence  $(\mathcal{M}_n, \kappa_n, g_n, \omega_n)_n$  that converges to  $(\mathcal{M}, \kappa, g, \omega)$  in the  $\mathcal{Y}^1$  topology, and for each  $n \in \mathbb{N}$  and each  $k \in \mathbb{N}^*$  there exists  $\ell(n, k) \in \mathbb{N}$ , with  $\ell(n, k) \geq k$  and  $u_n^{\ell(n,k)} \in E_{n, \ell(n,k)}$ , such that

$$(6.3) \quad 1 = \|u_n^{\ell(n,k)}|_{t=0}\|_{L^2(\mathcal{M}_n, \kappa_n \mu_{g_n})} \geq k \|\mathbf{1}_{I \times \omega_n} h_{\ell(n,k)} \partial_t u_n^{\ell(n,k)}\|_{L^2(\mathbb{R} \times \mathcal{M}_n, \kappa_n \mu_{g_n} dt)},$$

with  $I = ]\delta, T - \delta[$ . Note that we have normalized the l.h.s. of (6.3) to be equal to 1. The notation  $u_n^{\ell(n,k)}$  may seem very cumbersome at this stage; it will be greatly simplified by a diagonal extraction in what follows shortly.

Similarly, in the case of a boundary observation we assume that there exists a sequence  $(\mathcal{M}_n, \kappa_n, g_n, \Gamma_n)$  that converges to  $(\mathcal{M}, \kappa, g, \Gamma)$  in the  $\mathcal{Y}^1$  topology, and for each  $n \in \mathbb{N}$  a sequence  $(u_n^{\ell(n,k)})_{k \in \mathbb{N}}$ , with  $u_n^{\ell(n,k)} \in E_{n, \ell(n,k)}$  and  $\ell(n, k) \geq k$ , such that

$$(6.4) \quad 1 = \|u_n^{\ell(n,k)}|_{t=0}\|_{L^2(\mathcal{M}_n, \kappa_n \mu_{g_n})} \geq k \|\mathbf{1}_{I \times \Gamma_n} h_{\ell(n,k)} \partial_{\mathbf{n}_n} u_n^{\ell(n,k)}\|_{L^2(\mathbb{R} \times \partial\mathcal{M}_n, \kappa_n \mu_{g_n} dt)},$$

where  $\mathbf{n}_n$  is the normal to the boundary  $\partial\mathcal{M}_n$  in the sense of the metric  $g_n$ .

We now proceed with a diagonal extraction along with a zero-extension of the solutions outside  $\mathcal{L}$ . Set  $u_k = u_k^{\ell(k,k)} \mathbf{1}_{\mathcal{L}}$ , that is, the extension by 0 of the

function  $u_k^{\ell(k,k)}$  to  $\hat{\mathcal{L}} = \mathbb{R} \times \hat{\mathcal{M}}$  (see Section 2) and  $v_k = h_k \partial_{n_k} u_k|_{\partial \mathcal{L}}$  its normal partial derivative (in the sense of  $g_k$ ). In what follows we will denote  $h_{\ell(k,k)}$  and  $J_{\ell(k,k)}$  by  $h_k$  and  $J_k$  for simplicity. Yet, there will be no possible confusion.

First, with the  $W^{2,\infty}$ -diffeomorphism of Definition 1.3 the analysis can be pulled back from  $\mathcal{M}_k$  to  $\mathcal{M}$  for each  $k \in \mathbb{N}$ . By abuse of notation we use the same letters for the pullbacked functions and metric. Hence, without loss of generality we may assume that  $\mathcal{M}_k = \mathcal{M}$ .

Second, observe that since  $\|u_k|_{t=0}\|_{L^2(\mathcal{M}, \kappa_k \mu_{g_k})} = 1$  one has

$$\|u_k|_{t=0}\|_{L^2(\mathcal{M})} = 1 + o(1)_{k \rightarrow \infty}.$$

If no precision is given, the  $L^2$ -norm on  $\mathcal{M}$  is given by the density measure  $\kappa \mu_g$  in what follows.

From Lemma 4.1 and Remark 4.4 one obtains that

$$(6.5) \quad \begin{aligned} 1 &\approx \|u_k(t, \cdot)\|_{L^2(\mathcal{M}, \kappa_k \mu_{g_k})} \approx \|h_k \partial_t u_k(t, \cdot)\|_{L^2(\mathcal{M})} \approx \|h_k \nabla_{g_k} u_k(t, \cdot)\|_{L^2(\mathcal{M})} \\ &\approx \|h_k^2 A_{\kappa_k, g_k} u_k(t, \cdot)\|_{L^2(\mathcal{M})}, \end{aligned}$$

for any  $t \in \mathbb{R}$  and  $k$  large. From ellipticity up to the boundary one deduces [18, Theorem 8.12]<sup>2</sup>

$$(6.6) \quad \|h_k^2 u_k(t, \cdot)\|_{H^2(\mathcal{M})} \approx 1,$$

for any  $t \in \mathbb{R}$ .

6.2.2. *Measures for the wave equations.* From Proposition 3.1 in the case  $f = 0$ , (6.5), and (6.3) and (6.4) (and the fact that  $\kappa_n$  converges to  $\kappa$  and  $g_n$  to  $g$  in the sense given in Definition 1.3) one obtains the following proposition.

**Proposition 6.2.** *The sequences  $u_k \in L^\infty(\mathbb{R}; L^2(\hat{\mathcal{M}}))$  and  $v_k \in L^2_{loc}(\partial \mathcal{L})$  satisfy*

(1) *For any bounded interval  $J \subset \mathbb{R}$  there exists  $C > 0$  such that*

$$\|u_k\|_{L^2(J \times \hat{\mathcal{M}})} + \|v_k\|_{L^2(J \times \partial \mathcal{M})} \leq C.$$

(2) *With  $I = ]\delta, T - \delta[$ , one has*

- $\lim_{k \rightarrow +\infty} \|u_k\|_{L^2(I \times \omega)} = 0$ , if the case (6.3) holds, that is, for interior observability,
- $\lim_{k \rightarrow +\infty} \|v_k\|_{L^2(I \times \Gamma)} = 0$ , if the case (6.4) holds, that is, for boundary observability.

<sup>2</sup>Notice that in [18] the boundary is assumed  $\mathcal{C}^2$ . Still,  $W^{1,\infty}$ -regularity suffices to reach the conclusion since it is enough to make the boundary straight in local coordinates and apply the argumentation therein.

Recall that we consider on  $\partial\mathcal{M}$  the density measure  $\kappa\mu_{g_\partial}$ . From Propositions 6.2 and 5.25 we deduce the following result.

**Proposition 6.3.** (1) *There exists a semi-classical measure  $\mu$  on  $T^*\hat{\mathcal{L}}$  associated with a subsequence of  $(u_k)_k$ .*  
 (2) *There exists a semi-classical measure  $\nu$  on  $T^*\partial\mathcal{L}$  associated with a subsequence of  $(v_k)_k$ .*

By abuse of notation we will use the notation  $(u_k)_k$  and  $(v_k)_k$  for both subsequences. Then one has

$$\begin{aligned}\langle \mu, a \rangle &= \lim_{k \rightarrow +\infty} [([\text{Op}^h](a)u_k, u_k)_{L^2(\hat{\mathcal{L}})}]_\lambda, & a \in \Sigma_c(T^*\hat{\mathcal{L}}), \\ \langle \nu, b \rangle &= \lim_{k \rightarrow +\infty} [([\text{Op}^h](b)v_k, v_k)_{L^2(\partial\mathcal{L})}]_\lambda, & b \in \Sigma_c(T^*\partial\mathcal{L}),\end{aligned}$$

where both limits are understood in the sense given in Definition 5.23. Recall that the spaces of symbols  $\Sigma_c(T^*\hat{\mathcal{L}})$  and  $\Sigma_c(T^*\partial\mathcal{L})$  are introduced in Definition 5.14.

**Proposition 6.4.** *The three following properties hold.*

- (1) *If  $J \subset \mathbb{R}$  is a bounded nonempty open interval, one has  $\mu(T^*(J \times \hat{\mathcal{M}})) > 0$ .*
- (2) *One has*

$$(6.7) \quad \text{supp } \mu \subset \text{Char } p \cap T^*\mathcal{L} \cap \{\alpha \leq \tau \leq \alpha^{-1}\},$$

$$(6.8) \quad \text{supp } \nu \subset T^*\partial\mathcal{L} \cap \{\alpha \leq \tau \leq \alpha^{-1}\}.$$

- (3) *With  $I = ]\delta, T - \delta[$  as above one has*
  - *the measure  $\mu$  vanishes on  $T^*(I \times \omega)$ , in the case of an interior observation,*
  - *the measure  $\nu$  vanishes on  $T^*(I \times \Gamma)$ , in the case of a boundary observation.*

**Proof.** Consider a finite  $\mathcal{C}^2$ -partition of unity  $(\chi_i)_{i \in \mathcal{I}}$  subordinated to a given atlas of  $\mathcal{M}$ ; see Section 2. Let  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$  be nonvanishing. From (6.5) one has

$$\|\varphi(t)\chi_i u_k\|_{L^2(\hat{\mathcal{L}})} \gtrsim 1, \quad \text{for some } i \in \mathcal{I}.$$

The semi-classical measure associated with  $(\varphi(t)\chi_i u_k)_k$  is  $|\varphi(t)\chi_i|^2 \mu$  by Lemma 5.24 for the  $L^2$ -inner product associated with the density measure  $\kappa\mu_g dt$ . In a local chart  $\mathcal{C} = (\mathcal{O}, \phi_{\mathcal{L}})$  of  $\hat{\mathcal{L}}$ , where  $\mathcal{O} = \mathbb{R} \times O$  and  $\phi_{\mathcal{L}}(t, x) = (t, \phi(x))$  (see Section 2.1), with  $\text{supp } \chi_i \subset O$ , from (6.5) and Lemma 5.17 one has

$$\|\mathbf{1}_{h_k|(\tau, \xi)| \geq R} \widehat{\varphi(t)\chi_i u_k}\|_{L^2(\mathbb{R}^{d+1})} \lesssim R^{-1},$$

There is thus no mass leakage at infinity at scale  $H$  in the sense of Definition 5.16.

Denote by  $\mu^{\mathcal{C}}$  the local representative of  $\mu$ . Recall that  $\tilde{\kappa} = \kappa(\det g)^{1/2}$ . Using that the s.c.m. of the local representative of  $\varphi(t)\chi_i u_k$  with a  $L^2$ -inner product associated with the Lebesgue measure  $dx$  is  $m = \tilde{\kappa}^{-1}|\varphi(t)\chi_i|^2\mu^{\mathcal{C}}$ , with Proposition 5.18 one finds

$$m(\mathbb{R}^{2d+2}) = \lim_{n \rightarrow +\infty} \|\varphi(t)\chi_i u_k\|_{L^2(\mathbb{R}^d)}^2 \gtrsim \lim_{n \rightarrow +\infty} \|\varphi(t)\chi_i u_k\|_{L^2(\hat{\mathcal{L}})}^2 \gtrsim 1,$$

hence the first result.

We place ourselves in a local chart  $(\mathcal{O}, \phi_{\mathcal{L}})$  of  $\hat{\mathcal{L}}$ . Here,  $\partial\mathcal{L}$  is given by  $\{z=0\}$ . Let  $b \in \mathcal{C}_c^\infty(\mathbb{R}^{2d+2})$  with  $\text{supp } b \subset \phi_{\mathcal{L}}(\mathcal{O}) \times \mathbb{R}^{d+1}$  and  $\psi, \tilde{\psi} \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  with  $\psi$  equal to 1 in a neighborhood of the  $(t, x)$ -projection of  $\text{supp } b$  and  $\tilde{\psi}$  equal to 1 in a neighborhood of  $\text{supp } \psi$ . Here,  $\text{Op}^h(b) = b(t, x, h_k D_t, h_k D_x)$ . One has, for any  $s \geq 0$ ,

$$(6.9) \quad \|\text{Op}^h(b)\|_{\mathcal{L}(H^{-s}(\mathbb{R}^{d+1}), L^2(\mathbb{R}^{d+1}))} \leq C_s h^{-s}.$$

In fact, one uses that  $\text{Op}^h(b) \text{Op}^h(\langle \xi \rangle^s) = \text{Op}^h(b \langle \xi \rangle^s)$  is bounded on  $L^2$  and  $h^s \langle \xi \rangle^s \leq \langle h\xi \rangle^s$  yielding

$$\begin{aligned} h^s \|\text{Op}^h(b)u\|_{L^2(\mathbb{R}^{d+1})} &\lesssim h^s \|\text{Op}^h(\langle \xi \rangle^{-s})u\|_{L^2(\mathbb{R}^{d+1})} = h^s \|\langle h\xi \rangle^{-s} \hat{u}\|_{L^2(\mathbb{R}^{d+1})} \\ &\lesssim \|\langle \xi \rangle^{-s} \hat{u}\|_{L^2(\mathbb{R}^{d+1})} = \|u\|_{H^{-s}(\mathbb{R}^{d+1})}. \end{aligned}$$

One has

$$(6.10) \quad h_k^2 P_{\kappa_k, g_k} u_k = -h_k v_k \otimes \delta_{z=0}.$$

Note that  $v_k \otimes \delta_{z=0}$  is bounded in  $H^{-s}(\mathbb{R}_z; L_{\text{loc}}^2(\mathbb{R}_{y'}^d)) \subset H_{\text{loc}}^{-s}(\mathbb{R}^{1+d})$  if  $s > 1/2$ . Thus, with (6.9) one finds  $\|\text{Op}^h(b)\psi h_k v_k \otimes \delta_{z=0}\|_{L^2} \leq h_k^{1-s}$ . With  $1/2 < s < 1$ , with (6.10) and since  $(u_k)_k$  is bounded in  $L^2$ , one concludes that

$$\begin{aligned} &\lim_{k \rightarrow +\infty} h_k^2 (\text{Op}^h(b)\psi P_{\kappa_k, g_k} u_k, u_k)_{L^2(\mathbb{R}^{d+1}, \kappa \mu_g dt)} \\ &= - \lim_{k \rightarrow +\infty} (\text{Op}^h(b)\psi h_k v_k \otimes \delta_{z=0}, u_k)_{L^2(\mathbb{R}^{d+1}, \kappa \mu_g dt)} = 0. \end{aligned}$$

Now, one has

$$\begin{aligned} \text{Op}^h(b)\psi h_k^2 \partial_t^2 &= -\text{Op}^h(\tau^2 b)\psi + h_k^2 \text{Op}^h(b)[\psi, \partial_t^2] \\ &= -\text{Op}^h(\tau^2 b)\psi + h_k^2 \mathcal{O}(1)_{\mathcal{L}(H^1, L^2)}. \end{aligned}$$

With the form of  $A_{\kappa_k, g_k}$  in local coordinates one writes, with  $\tilde{\kappa}_k = \kappa_k(\det g_k)^{1/2}$ ,

$$\begin{aligned}
 & \text{Op}^h(b)\psi h_k^2 A_{\kappa_k, g_k} \\
 &= h_k^2 \text{Op}^h(b) \sum_{1 \leq i, j \leq d} \left( \partial_{x_i} \psi g_k^{ij}(x) \partial_{x_j} + [\psi \tilde{\kappa}_k^{-1}, \partial_{x_i}] \tilde{\kappa}_k g_k^{ij}(x) \partial_{x_j} \right) \\
 &= \sum_{1 \leq i, j \leq d} i \text{Op}^h(b \xi_i) \psi g_k^{ij}(x) h_k \partial_{x_j} \tilde{\psi} + h_k^2 O(1)_{\mathcal{L}(H^1, L^2)} \\
 &= \sum_{1 \leq i, j \leq d} \left( -\psi g_k^{ij}(x) \text{Op}^h(b \xi_i \xi_j) + i[\text{Op}^h(b \xi_i), \psi g_k^{ij}(x)] h_k \partial_{x_j} \right) \tilde{\psi} \\
 &\quad + h_k^2 O(1)_{\mathcal{L}(H^1, L^2)} \\
 &= - \sum_{1 \leq i, j \leq d} \psi g_k^{ij}(x) \text{Op}^h(b \xi_i \xi_j) \tilde{\psi} + h_k^2 O(1)_{\mathcal{L}(H^1, L^2)},
 \end{aligned}$$

where in the last line we used Proposition 5.10. From Lemma 5.24 and (5.28) one finds

$$0 = \lim_{k \rightarrow +\infty} h_k^2 \left( \text{Op}^h(b) \psi P_{\kappa_k, g_k} u_k, u_k \right)_{L^2(\hat{\mathcal{L}}, \kappa \mu_g dt)} = \langle \mu^c, b p_{\kappa, g} \rangle.$$

Since  $b$  is arbitrary in  $\mathcal{C}_c^\infty(\mathbb{R}^{2d+2})$  this implies that  $\text{supp } \mu^c \subset \text{Char}(p_{\kappa, g})$ , which is the first inclusion in (6.7).

To prove the second property, that is  $\tau \in [\alpha, \alpha^{-1}]$ , consider  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$  such that  $\varphi \equiv 0$  in a neighborhood of  $[\alpha, \alpha^{-1}]$ , say  $[(1-\varepsilon)\alpha, (1+\varepsilon)\alpha^{-1}]$  for some  $\varepsilon > 0$  to be kept fixed, and now  $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ . We write

$$\varphi(h_k D_t) \psi(t) u_k = \sum_{\nu \in J_k} \varphi(h_k D_t) (\psi(t) e^{it\sqrt{\lambda_\nu}}) u_\nu e_\nu(x).$$

The Fourier transform of  $\varphi(h_k D_t) (\psi(t) e^{it\sqrt{\lambda_\nu}})$  is  $\varphi(h_k \tau) \hat{\psi}(\tau - \sqrt{\lambda_\nu})$ . As  $h_k \sqrt{\lambda_\nu} \in [\alpha, \alpha^{-1}]$  if  $\nu \in J_k$  and  $h_k \tau \notin [(1-\varepsilon)\alpha, (1+\varepsilon)\alpha^{-1}]$  if in the support of  $\varphi$ , then

$$|\tau - \sqrt{\lambda_\nu}| \gtrsim |\tau| + h_k^{-1},$$

in the support of the above Fourier transform. With the rapid decay of  $\hat{\psi}$  one finds, for any  $N \in \mathbb{N}$ ,

$$|\varphi(h_k \tau) \hat{\psi}(\tau - \sqrt{\lambda_\nu})| \leq C_N (|\tau|^{-1} + h_k)^N,$$

leading to

$$(6.11) \quad \|\varphi(h_k D_t) (\psi(t) e^{it\sqrt{\lambda_\nu}})\|_{H^\ell(\mathbb{R})} \leq C'_N h_k^N$$

for any  $\ell \geq 0$ . With  $\ell = 0$ , one deduces

$$\begin{aligned} & \|\varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)}^2 \\ & \approx \|\varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathcal{L}, \kappa_k \mu_{g_k} dt)}^2 = \sum_{\nu \in J_k} |u_\nu|^2 \|\varphi(h_k D_t)(\psi(t)e^{it\sqrt{\lambda_\nu}})\|_{L^2(\mathbb{R})}^2 \\ & \leq C_N h_k^N \sum_{\nu \in J_k} |u_\nu|^2 = C_N h_k^N \|u_k|_{t=0}\|_{L^2(\mathcal{M}, \kappa_k \mu_{g_k})}^2 \lesssim C_N h_k^N, \end{aligned}$$

for any  $N \in \mathbb{N}$ , using (6.3) or (6.4), implying  $\langle \mu, |\varphi(\tau)\psi(t)|^2 \rangle = 0$ , which gives the last inclusion in (6.7) since  $\mu$  is nonnegative.

We now prove the inclusion in (6.8). One has

$$\begin{aligned} \|\varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathbb{R}; H^2(\mathcal{M}, \kappa_k \mu_{g_k}))} & \lesssim \|\varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)} \\ & \quad + \|\mathbf{H}_g \varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)}, \end{aligned}$$

where  $\mathbf{H}_g$  denotes the Hessian operator. Using the elliptic regularity and that  $u_k$  is solution to the wave equation one obtains

$$\begin{aligned} \|\mathbf{H}_g \varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)} & \lesssim \|\varphi(h_k D_t)\psi(t)A_{\kappa_k, g_k} u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)} \\ & \lesssim \|\varphi(h_k D_t)\psi(t)\partial_t^2 u_k\|_{L^2(\mathcal{L}, \kappa \mu_g dt)} \\ & \lesssim \|\varphi(h_k D_t)\psi(t)\partial_t^2 u_k\|_{L^2(\mathcal{L}, \kappa_k \mu_{g_k} dt)}. \end{aligned}$$

Then, one writes

$$\begin{aligned} \|\varphi(h_k D_t)\psi(t)\partial_t^2 u_k\|_{L^2(\mathcal{L}, \kappa_k \mu_{g_k} dt)}^2 & = \sum_{\nu \in J_k} |u_\nu|^2 \lambda_\nu^2 \|\varphi(h_k D_t)(\psi(t)e^{it\sqrt{\lambda_\nu}})\|_{L^2(\mathbb{R})}^2 \\ & \lesssim C_N h_k^N \|u_k|_{t=0}\|_{L^2(\mathcal{M}, \kappa_k \mu_{g_k})}^2 \lesssim C_N h_k^N, \end{aligned}$$

for any  $N \in \mathbb{N}$ , using (6.11) and that  $h_k^2 \lambda_\nu \lesssim 1$  for  $\nu \in J_k$ . Hence, one has

$$\|\varphi(h_k D_t)\psi(t)u_k\|_{L^2(\mathbb{R}; H^2(\mathcal{M}, \kappa_k \mu_{g_k}))}^2 \lesssim C_N h_k^N,$$

implying, by the trace formula,

$$\|\varphi(h_k D_t)\psi(t)h_k \partial_{n_k} u_k|_{\partial \mathcal{L}}\|_{L^2(\partial \mathcal{L}, \kappa_k \mu_{g_k \partial} dt)}^2 \lesssim C_N h_k^N.$$

This yields  $\langle \nu, |\varphi(\tau)\psi(t)|^2 \rangle = 0$ , which gives (6.8) since  $\nu$  is nonnegative.  $\square$

6.2.3. *Final step of the proof of observability.* With Theorem 6.1 at hand we can complete the proof of the observability results of Theorem 1.8. The two measures  $\mu$  and  $\nu$  given by Proposition 6.3, with scale  $h_k = \varrho^{-|k|}$  of Section 4.1, fulfill the assumptions of Theorem 6.1 by Proposition 6.4. Hence, one has

$$\mathbb{H}_p \mu = \int_{\varrho \in \|\mathcal{H}_\partial \cup \mathcal{G}_\partial} \frac{\delta_{\varrho^+} - \delta_{\varrho^-}}{\langle \xi^+ - \xi^-, \mathbf{n}_x \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}}} d\nu(\varrho),$$

since the considered wave equations are homogeneous here. Theorem 2.14 recalled from the companion article [5] implies that the support of the measure  $\mu$  is a union of maximal generalized bicharacteristics.

Recall that  $I = ]\delta, T - \delta[$ . Let  $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \text{supp } \mu$ , with  $t^0 \in I$ . According the first point of Proposition 6.4 such a point exists. Then, there exists a generalized bicharacteristic  ${}^G\gamma$  with  ${}^G\gamma(0) = \varrho^0$  such that  ${}^G\bar{\gamma} \subset \text{supp } \mu$ .

#### Case of an interior observation.

With the interior geometric control condition fulfilled by  $(\omega, T - 2\delta)$  (Definition 2.10) the bicharacteristic  ${}^G\gamma$  reaches a point above  $I \times \omega$ . Yet, from the last point of Proposition 6.4, the measure  $\mu$  vanishes above  $I \times \omega$ , which gives a contradiction and concludes the proof in this case.

#### Case of a boundary observation.

With the boundary geometric control condition fulfilled by  $(\Gamma, T - 2\delta)$  (Definition 2.13), there exists  $s \in \mathbb{R}$  such that  $t(s) \in I$  and

- (1) either  $\varrho^1 = \lim_{s \rightarrow s^-} {}^G\gamma(s) \in \mathcal{B}_{esc}^F$ ;
- (2) or  $\varrho^1 = \lim_{s \rightarrow s^+} {}^G\gamma(s) \in \mathcal{B}_{esc}^P$ .

The sets  $\mathcal{B}_{esc}^F$  and  $\mathcal{B}_{esc}^P$  are introduced in Definition 2.11. With the measure  $\nu$  vanishing above  $I \times \Gamma$  by the last point of Proposition 6.4, the measure propagation equation is locally  ${}^t\mathbb{H}_p \mu = 0$ . Thus locally, the support of  $\mu$  is a union of *bicharacteristics*. In both possibilities, all such bicharacteristics exit  $T^*\mathcal{L}$  reaching a region where  $\mu$  vanishes, which gives a contradiction and concludes the proof in this case.

## 7. PROOF OF THE PROPAGATION EQUATION I

7.1. **Preliminary remarks and observations.** Recall that  $u_k$  is the zero-extension to  $\hat{\mathcal{L}}$  of a weak solution to the wave equation in  $\mathcal{L}$ . With the homogeneous Dirichlet condition this extension is  $H^1$ .

Consider  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  with  $0 \notin \text{supp } \chi$ . Since the coefficient of the wave operator are independent of time  $t$  then  $\tilde{u}_k = \chi(hD_t)u_k$  also solves the wave equation in  $\mathcal{L}$  with  $\tilde{f}_k = \chi(hD_t)f_k$  as source term and the Hermitian s.c.m. of



$(\tilde{u}_k, h_k \tilde{f}_k)$  is  $|\chi(\tau)|^2 M$ . Similarly, the s.c.m. of  $(h_k \partial_{\mathfrak{n}_k} \chi(hD_t) u_k|_{\partial \mathcal{L}})_k$  is  $|\chi(\tau)|^2 \nu$ . If we prove that (6.2) holds for  $M$  and  $\nu$  replaced by  $|\chi(\tau)|^2 M$  and  $|\chi(\tau)|^2 \nu$  then, using (6.1), one finds that (6.2) holds also for  $M$  and  $\nu$  by the dominated-convergence theorem. Without any loss of generality we may thus replace  $u_k$  by  $\tilde{u}_k$  and  $f_k$  by  $\tilde{f}_k$ . Then, there exists  $0 < C_{\mu,0} < 1 < C_{\mu,1} < \infty$  such that

$$(7.1) \quad \begin{aligned} \text{supp } \mu &\subset \text{Char } p \cap T^* \mathcal{L} \cap \{C_{\mu,0} \leq |\tau| \leq C_{\mu,1}\}, \\ \text{supp } M_{0,1} &\subset T^* \mathcal{L} \cap \{C_{\mu,0} \leq |\tau| \leq C_{\mu,1}\}, \end{aligned}$$

and

$$(7.2) \quad \text{supp } \nu \subset T^* \partial \mathcal{L} \cap \{C_{\mu,0} \leq |\tau| \leq C_{\mu,1}\}.$$

If no precision is given, the  $L^2$ -norm on  $\mathcal{M}$  is given by the density measure  $\kappa \mu_g$  in what follows.

Suppose  $I$  is a time interval. With the  $\tau$ -microlocalisation performed above, one has

$$(7.3) \quad \|u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 \partial_t^2 u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 A_{\kappa_k, g_k} u_k + h_k^2 f_k\|_{L^2(I \times \mathcal{M})}.$$

using the wave equation. Assume that a subsequence of  $u_k$  converges to 0 in  $L^2(I \times \mathcal{M})$ . This gives  $\mu = 0$  on  $T^*(I \times \mathcal{M})$ . With (7.3), one finds that  $\|h_k^2 \partial_t^2 u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0$  and  $\|h_k^2 A_{\kappa_k, g_k} u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0$  also, using that  $h_k f_k$  is  $L^2_{\text{loc}}$ -bounded. Ellipticity up to the boundary gives  $\|h_k^2 u_k\|_{H^2(I \times \mathcal{M})} \rightarrow 0$  and interpolation gives

$$\|h_k \partial_t u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0 \quad \text{and} \quad \|h_k \nabla_{g_k} u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0.$$

With Proposition 3.1 one concludes that  $\|h_k \partial_{\mathfrak{n}} u_k\|_{L^2(I \times \partial \mathcal{M})} \rightarrow 0$ . Hence, all terms in the measure equation vanish, in this case. One may thus assume that  $\|u_k\|_{L^2(I \times \mathcal{M})} \gtrsim 1$ , for any interval  $I$ .

With the arguments given just above, one has

$$(7.4) \quad \begin{aligned} 1 &\approx \|u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 \partial_t^2 u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 A_{\kappa_k, g_k} u_k\|_{L^2(I \times \mathcal{M})} \\ &\approx \|h_k^2 u_k\|_{H^2(I \times \mathcal{M})} \approx \|h_k \partial_t u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k \nabla_{g_k} u_k\|_{L^2(I \times \mathcal{M})}, \end{aligned}$$

and more generally one has

$$(7.5) \quad h_k^{\ell+2s} \|\partial_t^\ell A_{\kappa_k, g_k}^s u_k\|_{L^2(\mathcal{M})} \approx 1,$$

using that  $\|h_k^\ell \partial_t^\ell f_k\|_{L^2(I \times \mathcal{M})} \approx \|f_k\|_{L^2(I \times \mathcal{M})}$ .

Note that equation (6.2) is local. Consequently, the proof can be carried out in local charts. It greatly simplifies away from the boundary: there (6.2) is  $\mathbb{H}_p \mu = 2 \operatorname{Im} M_{0,1}$ , which follows from Proposition 7.2 below. We will thus only consider the case of a local chart  $\mathcal{C} = (\mathcal{O}, \phi_{\mathcal{L}})$  near a boundary point, where the boundary is given by  $\{z = x_d = 0\}$  and  $\mathcal{M} = \{z > 0\}$ ; see Section 2.1. By abuse of notation we will use the notation  $A$ ,  $\kappa$ ,  $g$ ,  $u_k$  and  $\mu$  for their representative in the local chart.

For concision we will write  $y = (t, x)$ ,  $y' = (t, x')$ ,  $\eta = (\tau, \xi)$ , and  $\eta' = (\tau, \xi')$ .

A consequence of (7.4)–(7.5) that we will use is as follows in  $\mathcal{C}$ , for  $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1})$ ,

$$(7.6) \quad \sum_{1 \leq i, j \leq d-1} \|\psi h_k^2 D_i D_j \underline{u}^k\|_{L^2(\mathbb{R}^{d+1})} + \sum_{1 \leq j \leq d-1} \|\psi h_k^2 D_j D_d \underline{u}^k\|_{L^2(\mathbb{R}^{d+1})} + \|\psi h_k^2 D_d^2 \underline{u}^k\|_{L^2(\mathbb{R}_+^{d+1})} \lesssim 1.$$

Note that the last term  $h_k^2 D_d^2 \underline{u}^k$  does not lie in  $L^2(\mathbb{R}^{d+1})$  in general as  $D_d u_k$  is not continuous across  $z = 0$ . This explains the computation of its norm only on  $\mathbb{R}_+^{d+1}$ .

As mentioned in Section 3.1, in the quasi-normal geodesic coordinates of Proposition 2.1 one has  $\partial_n = \partial_d$ , if  $\mathbf{n}$  is the unitary normal inward pointing vector field to  $\partial\mathcal{M}$ . Here, we will not use a different coordinate system if the metric  $g_k$  varies. In the chosen local chart  $\mathcal{C}$ , quasi-normal geodesic coordinates adapted to the “reference” metric  $g$  will be kept fixed. In such coordinates one has  $\mathbf{n}_k = \sum_j g_k^{dj} \partial_j$ . For a function like  $u_k$  that vanishes at  $z = 0$  one has  $\partial_{\mathbf{n}_k} u_k|_{\partial\mathcal{L}} = g_k^{dd} \partial_d u_k|_{z=0}$ . Hence

$$(7.7) \quad v_k = h_k g_k^{dd} \partial_z u_k|_{z=0^+},$$

in local coordinates. Note that  $g_k^{dd}|_{z=0} = 1 + o(1)$  as  $k \rightarrow \infty$ , since  $g^{dd}|_{z=0} = 1$  in the chosen quasi-normal geodesic coordinates; see Section 2.

From the “jump formula” the sequences  $u_k$  and  $v_k$  satisfy

$$(7.8) \quad h_k^2 (\partial_t^2 - A_{\kappa_k, g_k}) u_k = h_k^2 f_k - h_k v_k \otimes \delta_{\partial\mathcal{L}}.$$

The proof follows the lines of the proof of [16, Theorem 2.3] (or also [2, Théorème 4]). The main differences are as follows.

- The sequence  $(u_k)_k$  is solution to a *family of* wave equations associated with Lipschitz metrics.

- On the l.h.s. of (7.8), the wave operator is  $k$  dependent. In fact, in the application we make in Section 6.2, the sequence  $(u_k)_k$  is not spectrally localized with respect to a fixed operator, but rather with respect to the family of operators  $A_k = A_{\kappa_k, g_k}$ ,

$$u_k = 1_{-h_k^2 A_k \in [\alpha^2, \alpha^{-2}]} u_k.$$

- With respect to [2, Théorème 4] the result here assumes less smoothness ( $W^{1, \infty}$  as opposed to  $\mathcal{C}^2$ ), while the difference with respect to [16, Theorem 2.3] is more subtle: in [16], the authors study only rough ( $W^{2, \infty}$ ) domains of  $\mathbb{R}^d$  with the usual flat metric, which ensures the existence of local coordinate systems that are *regular* with respect to the variable transverse to the boundary, which simplifies greatly the analysis. Note that neither the result of [16] nor its proof are preserved by change of coordinates. To the opposite the result and proofs in the present article are coordinate invariant and thus geometrical.

The lack of regularity and the geometrical framework we consider, if compared with [2, 16], generate technical difficulties. In Sections 8–9, we will use a particular decomposition of symbols based on the Weierstrass preparation theorem. This allows one to express symbols as a first-order polynomial in  $\zeta$ , the dual variable of  $z = x_d$ , with coefficients that are tangential symbols, and a remainder term. An issue is then the handling of the different terms that lack decay in  $\zeta$ , even if the initial symbol is smooth with fast decay. This is a main reason for the introduction of *ad hoc* symbol spaces, taking into account both the low regularity in the  $x$  variable (that originates from the regularity of the metric we consider) and this low decay in the  $\zeta$  variable. This makes some of the statements quite technical even though we made an effort to minimize this aspect.

**7.2. Commutator analysis.** To establish the propagation equation of Theorem 6.1 we carefully compute a commutator. In fact, assume for a second that  $p$  and  $b$  are smooth symbols  $-ih_k H_p b = -ih_k \{p, b\}$  is the principal symbol of the commutator  $[\text{Op}^h(p), \text{Op}^h(b)]$ . Hence, to find the value of  $\langle \mu, H_p b \rangle$  it is very natural to analyse the limit of

$$(h_k [P_{\kappa_k, g_k}, \text{Op}^h(b)] u_k, u_k)_{L^2}.$$

Technicalities arise because of the limited smoothness of the coefficients of  $P_{\kappa_k, g_k}$  that prevent one from using standard semi-classical calculus results. However, there is no restriction on the smoothness and decay properties of the test symbol  $b$ . In fact, in the course of the proof of the Proposition 7.2, differentiations of the

symbol  $b$  with respect to  $x$  are needed as well as some decay in  $\xi$ . For simplicity, symbols in  $\Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d})$  are thus considered; see Definition 5.4 and (5.2). A classical result is the following one.

**Lemma 7.1.** *Let  $b \in \Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d})$  and  $s, s' \in \mathbb{R}$ . There exists  $C_{s, s'} > 0$  such that*

$$\| \text{Op}^h(\langle \xi \rangle^{s'}) \text{Op}^h(b)u \|_{L^2(\mathbb{R}^d)} \leq C_{s, s'} \| \text{Op}^h(\langle \xi \rangle^s)u \|_{L^2(\mathbb{R}^d)}, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

This means that  $\text{Op}^h(b)$  sends any *semi-classical* Sobolev space in the intersection of all semi-classical Sobolev spaces. This is due to both the smoothness in  $x$  and  $\xi$  and the fast decay in  $\xi$  of  $b$ .

Here, and in what follows one writes  $H_z^\delta L_{y'}^2$  in place of  $H^\delta(\mathbb{R}_z; L^2(\mathbb{R}_{y'}^d))$  in norm indices or duality bracket indices for the sake of concision for  $\delta \in [-1, 1]$ . For a density measure  $\rho$  on  $\mathbb{R}^N$  we will denote by

$$(\cdot, \cdot)_{H_z^{-\delta} L_{y'}^2, H_z^\delta L_{y'}^2}^\rho,$$

the complex duality bracket understood with  $L^2(\mathbb{R}^N, \rho)$  as a pivot space.

**Proposition 7.2.** *Suppose  $b \in \Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d+2})$  with  $\text{supp } b \subset K \times \mathbb{R}^{d+1}$ , for  $K$  a compact of  $\phi_{\mathcal{L}}(\mathcal{O})$ , and suppose  $\psi \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  be equal to 1 in a neighborhood of the  $y$ -projection of  $\text{supp } b$ . Set*

$$(7.9) \quad \begin{aligned} L_k(b, \psi) &= i \left( \text{Op}^h(b) \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \psi \text{Op}^h(b)^* u_k \right)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

One has  $\lim_{k \rightarrow +\infty} L_k(b, \psi) = -\langle \mu, H_{p_{\kappa, g}} b \rangle - 2\langle \text{Im } M_{0,1}, b \rangle$ .

Recall that  $v_k \otimes \delta_{z=0}$  lies in  $H_{\text{loc}}^{-s}(\mathbb{R}_z; L^2(\mathbb{R}_{y'}^d))$  for any  $s > 1/2$ , hence one finds the duality brackets appearing in the definition of  $L_k$ . Here,  $\text{Op}^h(b)^*$  stands for the adjoint in the sense of the  $L^2(\mathbb{R}^{d+1}, \kappa_k \mu_{g_k} dt)$ -inner product.

**Proof.** Using  $L^2(\mathbb{R}^{d+1}, \kappa_k \mu_{g_k} dt)$  as a pivot space and the symmetry of  $P_{\kappa_k, g_k}$  for the associated inner product and using (7.8) one has

$$\begin{aligned}
& ([P_{\kappa_k, g_k}, \text{Op}^h(b)\psi]u_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&= (\text{Op}^h(b)\psi u_k, P_{\kappa_k, g_k} u_k)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} - (\text{Op}^h(b)\psi P_{\kappa_k, g_k} u_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&= (\text{Op}^h(b)\psi u_k, f_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} - (\text{Op}^h(b)\psi f_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&\quad - h_k^{-1} (\text{Op}^h(b)\psi u_k, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\
&\quad + h_k^{-1} (\text{Op}^h(b)\psi v_k \otimes \delta_{z=0}, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&= (\text{Op}^h(b)\psi u_k, f_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} - (\text{Op}^h(b)\psi f_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&\quad + i h_k^{-1} L_k(b, \psi).
\end{aligned}$$

Since one has

$$\begin{aligned}
& \lim_{k \rightarrow +\infty} \frac{h_k}{i} ((\text{Op}^h(b)\psi u_k, f_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} - (\text{Op}^h(b)\psi f_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)}) \\
&= \frac{1}{i} (\langle M_{0,1}, b \rangle - \langle M_{1,0}, b \rangle) = 2 \langle \text{Im } M_{0,1}, b \rangle,
\end{aligned}$$

the result follows if one proves

$$\lim_{k \rightarrow +\infty} I_k = -\langle \mu, \mathbf{H}_{p_{\kappa, g}} b \rangle,$$

with

$$\begin{aligned}
I_k &= \frac{h_k}{i} ([P_{\kappa_k, g_k}, \text{Op}^h(b)\psi]u_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa_k \mu_{g_k} dt)} \\
&= \frac{1}{i h_k} (\tilde{\kappa}^{-1} \tilde{\kappa}_k [h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)\psi]u_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa \mu_g dt)}.
\end{aligned}$$

Recall that  $\tilde{\kappa} = \kappa \det(g)^{1/2}$  and  $\tilde{\kappa}_k = \kappa_k \det(g_k)^{1/2}$ . First one writes

$$[h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)\psi] = [h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)]\psi + \text{Op}^h(b)[h_k^2 P_{\kappa_k, g_k}, \psi].$$

Since  $[P_{\kappa_k, g_k}, \psi]$  is a differential operator of order one with Lipschitz coefficients one finds

$$[h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)\psi] = [h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)]\psi + h_k^2 \text{Op}^h(b)O(1)_{\mathcal{L}(H^1, L^2)},$$

and one obtains

$$(7.10) \quad \lim_{k \rightarrow +\infty} I_k = \lim_{k \rightarrow +\infty} \frac{1}{ih_k} (\tilde{\kappa}^{-1} \tilde{\kappa}_k [h_k^2 P_{\kappa_k, g_k}, \text{Op}^h(b)] \psi u_k, u_k)_{L^2(\mathbb{R}^{d+1}; \kappa \mu_g dt)}.$$

According to symbolic calculus one has

$$(7.11) \quad [h_k^2 \partial_t^2, \text{Op}^h(b)] = ih_k \text{Op}^h(2\tau \partial_t b) + o(h_k)_{\mathcal{L}(L^2)}.$$

The contribution of (7.11) to the limit in (7.10) is then

$$(7.12) \quad \lim_{k \rightarrow +\infty} \frac{1}{ih_k} (\tilde{\kappa}^{-1} \tilde{\kappa}_k [h_k^2 \partial_t^2, \text{Op}^h(b)] \psi u_k, u_k)_{L^2(\mathbb{R}^{d+1}, \kappa \mu_g dt)} = \langle \mu, 2\tau \partial_t b \rangle = \langle \mu, \{\tau^2, b\} \rangle,$$

by Lemma 5.24 and (5.28).

Next, with repeated indices convention, one writes

$$A_{\kappa_k, g_k} = \rho_k^j \partial_j + g_k^{ij} \partial_i \partial_j \quad \text{with} \quad \rho_k^j = \tilde{\kappa}_k^{-1} [\partial_i, \tilde{\kappa}_k g_k^{ij}],$$

with  $(\rho_k^j)_k \subset L^\infty$  that converges to some  $\rho^j \in \mathcal{C}^0 \cap L^\infty$ . One computes

$$[h_k^2 A_{\kappa_k, g_k}, \text{Op}^h(b)] = A_1 + A_2 + A_3 + A_4,$$

with

$$\begin{aligned} A_1 &= h_k^2 \rho_k^j [\partial_j, \text{Op}^h(b)], & A_2 &= h_k^2 [\rho_k^j, \text{Op}^h(b)] \partial_j, \\ A_3 &= h_k^2 g_k^{ij} [\partial_i \partial_j, \text{Op}^h(b)], & A_4 &= h_k^2 [g_k^{ij}, \text{Op}^h(b)] \partial_i \partial_j. \end{aligned}$$

One writes

$$A_1 = h_k^2 \rho_k^j \text{Op}^h(\partial_{x_j} b) = O(h_k^2)_{\mathcal{L}(L^2)},$$

with Lemma 7.1 since  $\partial_{x_j} b \in \Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d+2})$  (one can also argue that  $\partial_{x_j} b \in \Sigma(\mathbb{R}^{2d+2})$  and use Lemma 5.6). With (5.9) in Proposition 5.10 one finds

$$A_2 = h_k^2 \sum_j o(1)_{\mathcal{L}(L^2)} \partial_j = o(h_k^2)_{\mathcal{L}(H^1, L^2)}.$$

Thus, with (7.4) one finds that the contributions of  $A_1$  and  $A_2$  to the limit in (7.10) are 0.

Next, one writes

$$\begin{aligned} A_3 &= h_k^2 g_k^{ij} [\partial_i \partial_j, \text{Op}^h(b)] = h_k i g_k^{ij} \text{Op}^h(\xi_i \partial_{x_j} b + \xi_j \partial_{x_i} b) + h_k^2 g_k^{ij} \text{Op}^h(\partial_{x_i} \partial_{x_j} b) \\ &= h_k i g_k^{ij} \text{Op}^h(\{\xi_i \xi_j, b\}) + h_k^2 O(1)_{\mathcal{L}(L^2)}, \end{aligned}$$

as  $\partial_{x_i}\partial_{x_j}b \in \Sigma_0^{\infty,\infty}(\langle\xi\rangle^{-\infty}; \mathbb{R}^{2d+2})$ . Since  $\{\xi_i\xi_j, b\} \in \Sigma_0(\mathbb{R}^{2d+2})$ , with (7.4) one finds that the contributions of  $A_3$  to the limit in (7.10) is

$$(7.13) \quad - \lim_{k \rightarrow +\infty} \frac{1}{ih_k} (\tilde{\kappa}^{-1}\tilde{\kappa}_k A_3 \psi u_k, u_k)_{L^2(\mathbb{R}^{d+1}, \kappa \mu_g dt)} = -\langle \mu, g^{ij} \{\xi_i \xi_j, b\} \rangle,$$

by Lemma 5.24 and (5.28). Finally, with (5.12) in Proposition 5.10 one writes

$$\begin{aligned} A_4 &= ih_k^3 \sum_{\ell} (\partial_{x_\ell} g_k^{ij}) \text{Op}^h(\partial_{\xi_\ell} b) \partial_i \partial_j + o(h_k^3)_{\mathcal{L}(H^2, L^2)} \\ &= -ih_k \sum_{\ell} (\partial_{x_\ell} g_k^{ij}) \text{Op}^h(\xi_i \xi_j \partial_{\xi_\ell} b) + o(h_k^3)_{\mathcal{L}(H^2, L^2)}, \end{aligned}$$

implying with (7.4) that the contributions of  $A_4$  to the limit in (7.10) is

$$(7.14) \quad - \lim_{k \rightarrow +\infty} \frac{1}{ih_k} (\tilde{\kappa}^{-1}\tilde{\kappa}_k A_4 \psi u_k, u_k)_{L^2(\mathbb{R}^{d+1}, \kappa \mu_g dt)} = \sum_{\ell} \langle \mu, \xi_i \xi_j \partial_{x_\ell} g^{ij} \partial_{\xi_\ell} b \rangle \\ = -\langle \mu, \xi_i \xi_j \{g^{ij}, b\} \rangle.$$

Gathering (7.12), (7.13) and (7.14) and writing

$$\{\tau^2, b\} - g^{ij} \{\xi_i \xi_j, b\} - \xi_i \xi_j \{g^{ij}, b\} = \{\tau^2 - g^{ij} \xi_i \xi_j, b\} = -\{p_{\kappa, g}, b\} = -\mathbb{H}_{p_{\kappa, g}} b,$$

one obtains the result of the proposition.  $\square$

**7.3. Time microlocalization.** Above in Proposition 7.2, we defined  $L_k(b, \psi)$  for symbols  $b$  in  $\Sigma_0^{\infty,\infty}(\langle\xi\rangle^{-\infty}; \mathbb{R}^{2d})$  and we are interested in the limit of  $L_k(b, \psi)$  as  $k \rightarrow +\infty$ . With the support properties of the measure  $\mu$  given in (7.1) one obtains the following lemma.

**Lemma 7.3.** *Suppose  $\chi \in \mathcal{C}_c^\infty(C_{\mu,0}^2, C_{\mu,1}^2)$  be equal to 1 on a neighborhood of  $[C_{\mu,0}, C_{\mu,1}]$ . Let  $b \in \Sigma_0^{\infty,\infty}(\langle\xi\rangle^{-\infty}; \mathbb{R}^{2d})$  and  $\psi$  be as in Proposition 7.2. Then, one has*

$$\lim_{k \rightarrow +\infty} L_k((1 - \chi)(\tau) b, \psi) = 0.$$

## 8. MORE ON SEMI-CLASSICAL SYMBOLS AND OPERATORS

**8.1. Preparation theorem: Euclidean symbol division.** For technical reasons, it is convenient to consider symbols with finer properties here and in what follows.

**Definition 8.1.** *One says that  $a \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  if  $a \in \Sigma_0^{\infty,\infty}(\langle\xi\rangle^{-\infty}; \mathbb{R}^{(2d+2)})$  and satisfies moreover the following properties*

- $a(y, \eta)$  is compactly supported in the  $y$  variable;
- $a(y, \eta)$  has a compactly supported Fourier transform in the  $\eta$  variable and, consequently, is holomorphic with respect to the  $\eta$  variable.

This choice is possible and relevant observing that  $\Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  is dense in  $\Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d+2})$ , the symbol classes we consider in Proposition 7.2. Recall that  $y = (t, x)$  and  $\eta = (\tau, \xi)$ ,  $x = (z, y')$  (the boundary is given by  $\{z = 0\}$ ). See the beginning of Section 7.

Below we will need the following quantification of the decay of a symbol  $a \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  with the  $\eta$  variable allowed to slightly depart from the real axis: for any  $R > 0$ ,  $\alpha, \beta \in \mathbb{N}^{d+1}$ , and  $N \in \mathbb{N}$  there exists  $C_{\alpha, \beta, N, R} > 0$  such that

$$(8.1) \quad |\partial_y^\alpha \partial_\eta^\beta a(y, \eta)| \leq C_{\alpha, \beta, N, R} \langle \eta \rangle^{-N}, \quad y \in \mathbb{R}^{d+1}, \eta \in \mathbb{C}^{d+1} \text{ with } |\operatorname{Im} \eta| \leq R.$$

This is given by the Paley-Wiener theorem; see for instance [21, Theorem 7.3.1].

The following proposition gives a decomposition of a symbol  $b \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$ . For our purpose, that is, using such symbols in the identity given by Proposition 7.2, with Lemma 7.3 it suffices to work with a time-frequency truncated symbol. Recall that  $\eta = (\eta', \zeta)$  with  $\eta' = (\tau, \xi')$  and  $\zeta$  is the dual variable to  $z = x_d$ .

**Proposition 8.2** (Euclidean symbol division). *Let  $\chi \in \mathcal{C}_c^\infty(C_{\mu, 0}^2, C_{\mu, 1}^2)$  be equal to 1 on a neighborhood of  $[C_{\mu, 0}, C_{\mu, 1}]$  and  $b(y, \eta) \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$ . For  $k \in \mathbb{N}$ , there exist  $b_{0,k}(y, \eta')$ ,  $b_{1,k}(y, \eta')$  and  $q_k(y, \eta)$  such that*

$$(8.2) \quad \chi(\tau) b(y, \eta', \zeta) = b_{0,k}(y, \eta') + b_{1,k}(y, \eta') \zeta + q_k(y, \eta', \zeta) p_k(y, \eta', \zeta),$$

with the following symbol properties

$$(8.3) \quad |\partial_y^\alpha \partial_{\eta'}^\beta b_{j,k}(y, \eta')| \leq C_{N, \beta} \langle \eta' \rangle^{-N},$$

$$\text{for } N \in \mathbb{N}, \alpha \in \mathbb{N}^{d+1}, |\alpha| \leq 1, \beta \in \mathbb{N}^d, j = 0, 1, y \in \mathbb{R}^{d+1}, \eta' \in \mathbb{R}^d,$$

and

$$(8.4) \quad |\partial_y^\alpha \partial_{\eta'}^\beta \partial_\zeta^\delta q_k(y, \eta', \zeta)| \leq C_{N, \beta, \delta} \langle \eta' \rangle^{-N} \langle \zeta \rangle^{-1-\delta},$$

$$\text{for } N \in \mathbb{N}, \alpha \in \mathbb{N}^{d+1}, |\alpha| \leq 1, \beta \in \mathbb{N}^d, \delta \in \mathbb{N}, y \in \mathbb{R}^{d+1}, (\eta', \zeta) \in \mathbb{R}^{d+1},$$

uniformly with respect to  $k \in \mathbb{N}$ . Moreover,  $q_k$  admits a polyhomogeneous development in the  $\zeta$  variable: there exist  $q_k^j(y, \eta')$ ,  $j \in \mathbb{N}^*$ , such that

$$(8.5) \quad |\partial_y^\alpha \partial_{\eta'}^\beta q_k^j(y, \eta')| \leq C_{N, \beta} \langle \eta' \rangle^{-N},$$

$$\text{for } N \in \mathbb{N}, \alpha \in \mathbb{N}^{d+1}, |\alpha| \leq 1, \beta \in \mathbb{N}^d, y \in \mathbb{R}^{d+1}, \eta' \in \mathbb{R}^d,$$



uniformly with respect to  $k \in \mathbb{N}$ , and  $q_k \sim \sum_{j \geq 1} q_k^j \zeta^{-j}$  in the following sense: for  $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$  equal to 1 near 0 one has

(8.6)

$$\left| \partial_y^\alpha \partial_{\eta'}^\beta \partial_\zeta^\delta \left( q_k(y, \eta', \zeta) - (1 - \phi(\zeta)) \sum_{j=1}^M q_k^j(y, \eta') \zeta^{-j} \right) \right| \leq C_{N,M,\beta,\delta} \langle \eta' \rangle^{-N} \langle \zeta \rangle^{-M-1-\delta},$$

for  $M, N \in \mathbb{N}$ ,  $\alpha \in \mathbb{N}^{d+1}$ ,  $|\alpha| \leq 1$ ,  $\beta \in \mathbb{N}^d$ ,  $\delta \in \mathbb{N}$ ,  $y \in \mathbb{R}^{d+1}$ ,  $(\eta', \zeta) \in \mathbb{R}^{d+1}$ .

The decomposition of symbols given in Proposition 8.2 makes tangential symbols appear; they are introduced in Section 5.3. Observe that  $q_k$  has limited decay in  $\zeta$ . Yet, the polyhomogeneous development will be used in what follows.

**Proof.** Recall that  $p_k(y, \eta) = -\tau^2 + \sum_{i,j} g_k^{i,j}(x) \xi_i \xi_j$  is the principal symbol of  $P_{\kappa_k, g_k}$ . For  $\tau \in \text{supp } \chi$ , and  $\varrho' = (y, \eta')$ , with  $\eta' = (\tau, \xi')$ , having  $p_k(\varrho', \zeta) = 0$  reads

$$g_k^{ij} \xi_i \xi_j = \tau^2 \in (C_{\mu,0}^4, C_{\mu,1}^4), \quad \xi = (\xi', \zeta),$$

meaning that  $|\xi'| + |\zeta| \lesssim C_{\mu,1}^2$  if  $y = (t, x)$  remains in a bounded domain. Hence for  $\text{supp } b \subset K \times \mathbb{R}^{d+1}$  with  $K$  compact of  $\mathbb{R}^{d+1}$  one sees that there exist a bounded domain  $L'$  of  $\mathbb{R}^{d-1}$  and  $R > 0$

(8.7)

$$\begin{aligned} y = (t, x', z) \in K, \quad \eta' = (\tau, \xi') \in \mathbb{R}^d, \quad \zeta \in \mathbb{C}, \quad \tau \in \text{supp } \chi \quad \text{and} \quad p_k(y, \eta', \zeta) = 0 \\ \Rightarrow \quad \xi' \in L' \quad \text{and} \quad |\zeta| < R. \end{aligned}$$

For  $r \geq R$  we will consider the rectangular curve in the complex plane postively oriented and made with the following pieces

$$\begin{aligned} L_{r,R} = \{z \in \mathbb{C}; -r \leq \text{Re } z \leq r \text{ and } \text{Im } z = \pm R\} \\ \cup \{z \in \mathbb{C}; \text{Re } z = \pm r \text{ and } -R \leq \text{Im } z \leq R\}, \end{aligned}$$

that encloses the open ball centred at 0 with radius  $R$ . The important aspect of this contour is that the distance from the real axis is bounded by  $R$  from above allowing one to use the estimation (8.1)

Consider  $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}^{d-1})$  that is equal to 1 in a neighborhood of  $L'$ . We decompose symbols in  $\Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  according to Weierstrass preparation Theorem [21, Section 7.5]. Let  $b(y, \eta', \zeta) \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$ . One may write for  $|\zeta| < r$ , with  $r \geq R$ ,

$$\chi(\tau) \tilde{\chi}(\xi') b(y, \eta', \zeta) = \frac{\chi(\tau) \tilde{\chi}(\xi')}{2i\pi} \int_{L_{r,R}} b(y, \eta', \tilde{\zeta}) \frac{d\tilde{\zeta}}{\tilde{\zeta} - \zeta}.$$

Following the proof of [21, Theorem 7.5.2], using (8.7), one further writes

$$\chi(\tau)\tilde{\chi}(\xi')b(y, \eta', \zeta) = \tilde{b}_k(y, \eta', \zeta) + \tilde{r}_k(y, \eta) p_k(y, \eta),$$

with

$$(8.8) \quad \tilde{b}_k(y, \eta', \zeta) = \frac{\chi(\tau)\tilde{\chi}(\xi')}{2i\pi} \int_{L_{r,R}} \frac{b(y, \eta', \tilde{\zeta})}{p_k(y, \eta', \tilde{\zeta})} \frac{p_k(y, \eta', \tilde{\zeta}) - p_k(y, \eta', \zeta)}{\tilde{\zeta} - \zeta} d\tilde{\zeta}$$

and

$$\tilde{r}_k(y, \eta', \zeta) = \frac{\chi(\tau)\tilde{\chi}(\xi')}{2i\pi} \int_{L_{r,R}} \frac{b(y, \eta', \tilde{\zeta})}{p_k(y, \eta', \tilde{\zeta})} \frac{d\tilde{\zeta}}{\tilde{\zeta} - \zeta}.$$

Observing that  $(p_k(y, \eta', \tilde{\zeta}) - p_k(y, \eta', \zeta))/(\tilde{\zeta} - \zeta)$  is a first-order polynomial in  $\zeta$  one finds that  $\tilde{b}_k$  has the form

$$(8.9) \quad \tilde{b}_k(y, \eta', \zeta) = b_{0,k}(y, \eta') + b_{1,k}(y, \eta')\zeta.$$

It is important to notice that the values of  $\tilde{r}_k$ ,  $b_{0,k}$ , and  $b_{1,k}$  are independent of the value of  $r$ , provided that  $r > |\zeta|$ . From (8.1) and the explicit formula (8.8) one deduces that (8.3) holds uniformly with respect to  $k \in \mathbb{N}$ .

Setting

$$(8.10) \quad q_k(y, \eta', \zeta) = \tilde{r}_k(y, \eta', \zeta) + (1 - \tilde{\chi})(\xi')\chi(\tau) \frac{b(y, \eta', \zeta)}{p_k(y, \eta', \zeta)},$$

where the second term is properly defined by (8.7), one has

$$(8.11) \quad \chi(\tau)b(y, \eta', \zeta) = b_{0,k}(y, \eta') + b_{1,k}(y, \eta')\zeta + q_k(y, \eta', \zeta) p_k(y, \eta', \zeta).$$

Using that  $q_k$  is smooth in the  $\eta', \zeta$  variables and that  $p_k(y, \eta', \zeta)$  is invertible for  $|(\xi', \zeta)|$  large and  $\tau \in \text{supp } \chi$ , with (8.3) and (8.11) by induction one finds that (8.4) holds uniformly with respect to  $k \in \mathbb{N}$ .

We now consider the polyhomogeneous development of  $q_k$  in the  $\zeta$  variable. Observe that the second term on the r.h.s. of (8.10) can be estimated by the remainder in (8.6). Hence, it suffices to consider the term  $\tilde{r}_k$ . In the support of this term one has  $|\eta'| = |(\xi', \tau)|$  bounded. Observe that it suffices to have the polyhomogeneous development for  $|\zeta|$  large. With (8.7), if  $|\zeta| \geq R$  one has  $p_k(y, \eta) \neq 0$  and one can write

$$(8.12) \quad \tilde{r}_k = \frac{\chi(\tau)\tilde{\chi}(\xi')b(y, \eta', \zeta) - b_{0,k}(y, \eta') - b_{1,k}(y, \eta')\zeta}{p_k(y, \eta)},$$

and  $p_k(y, \eta)$  takes the form

$$p_k(y, \eta) = (\zeta - \rho(y, \eta'))(\zeta - \rho'(y, \eta')),$$

with the two roots having the same regularity as the coefficients in the  $x$  variable and homogeneous of degree one in  $\eta'$ , a classical result based on the Rouché theorem; see for instance [23, Section 6.A]. Observe that the first term on the r.h.s. of (8.12) can be estimated by the remainder in (8.6). For the other terms one writes

$$\frac{b_{0,k}(y, \eta') + b_{1,k}(y, \eta')\zeta}{p_k(y, \eta)} = \frac{b_{0,k}(y, \eta')/\zeta^2 + b_{1,k}(y, \eta')/\zeta}{(1 - \rho(y, \eta')/\zeta)(1 - \rho'(y, \eta')/\zeta)}.$$

Since here  $|\eta'|$  is bounded and  $y$  remains in a compact domain, for  $|\zeta|$  sufficiently large one obtains the sought polyhomogeneous development with a truncated Neumann series.  $\square$

**Remark 8.3.** *Recall that the symbol  $p_k(y, \eta)$  is in fact smooth in  $t$  since independent of  $t$ . Hence, estimates (8.2)-(8.6) remain valid with an arbitrary number of derivatives in  $t$ . This is however not needed in what follows.*

**8.2. Low regularity/low conormal decay symbolic calculus.** The limited smoothness with respect to the  $x$  variable of the symbols obtained in Proposition 8.2 and their limited decay in the  $\zeta$  variable force us to investigate the symbolic calculus properties of operators with low regularity ( $W^{1,\infty}$ ) that we will need in what follows.

In (8.4) and (8.5) we have found symbol estimates with distinct decays in the variables  $\eta'$  and  $\zeta$ . For a symbol  $a(y, \eta)$  we thus set

$$(8.13) \quad \tilde{N}_\ell(a) = \max_{|\alpha| \leq \ell} \operatorname{ess\,sup}_{(y, \eta)} |\partial_\eta^\alpha a_2(y, \eta)| \langle \zeta \rangle^2 \langle \eta' \rangle^{d+1}.$$

Observe the difference with  $M_{0,\ell}^{-(d+1)}(a)$  in (5.1).

**Lemma 8.4.** *Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  be equal to 1 in a neighborhood of 0. Consider a symbol  $a(y, \eta)$  that is compactly supported in the  $y$  variable and of the form*

$$a(y, \eta) = a_0(y, \eta') + a_1(y, \eta') \frac{(1 - \chi(\zeta))}{\zeta} + a_2(y, \eta),$$

with  $a_j \in \Sigma_{\mathbb{T}}(\mathbb{R}^{d+1} \times \mathbb{R}^d)$ ,  $j = 0, 1$ , and  $\tilde{N}_{d+2}(a_2) < +\infty$ . Then the operator  $a(y, hD_y)$  is bounded on  $L^2(\mathbb{R}^{d+1})$  and

$$\|a(y, hD_y)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \leq C(N_{d+1}(a_0) + N_{d+1}(a_1) + \tilde{N}_{d+2}(a_2)).$$

**Proof.** The estimate of the contribution associated with  $a_0$  is given by Lemma 5.12. The contribution associated with  $a_1$  is given by (5.19), Lemma 5.12 and 5.20.

The contribution of the symbol  $a_2$  is dealt with by using the same method as in the proof of Lemma 5.6. In fact, the kernel of  $a_2(y, hD_y)$  is given by  $K(y, \tilde{y}) = h^{-d-1}k(y, (y - \tilde{y})/h)$  with

$$k(y, v) = (2\pi)^{-d-1} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} a_2(y, \eta) d\eta = (2\pi)^{-d-1} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} ({}^tL)^{d+2} a_2(y, \eta) d\eta,$$

with  $L = (1 - iv \cdot \nabla_\eta) / \langle v \rangle^2$  and  ${}^tL = (1 + iv \cdot \nabla_\eta) / \langle v \rangle^2$  since  $L \exp(iv \cdot \eta) = \exp(iv \cdot \eta)$ . Using (8.13) and that  $\langle \zeta \rangle^{-2} \langle \eta' \rangle^{-(d+1)}$  is integrable one finds

$$|k(y, v)| \lesssim \tilde{N}_{d+2}(a_2) \langle v \rangle^{-(d+2)} \int_{\mathbb{R}} \langle \eta' \rangle^{-d-1} \langle \zeta \rangle^{-2} d\eta' d\zeta \lesssim \tilde{N}_{d+2}(a_2) \langle v \rangle^{-(d+2)}.$$

One concludes with Corollary 5.3.  $\square$

An inspection of the part of the proof of Lemma 8.4 dedicated to the term  $a_1(y, \eta')$  shows that multiplying  $a_0(y, \eta')$  and  $a_1(y, \eta')$  by a uniformly bounded function of  $\zeta$  leaves the result unchanged.

**Lemma 8.4'.** *Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  be equal to 1 in a neighborhood of 0. Let  $m(\zeta, h)$  be a bounded function uniformly with respect to  $h > 0$ . Consider a symbol  $a(y, \eta)$  that is compactly supported in the  $y$  variable and of the form*

$$a(y, \eta, h) = a_0(y, \eta') m(\zeta, h) + a_1(y, \eta') \frac{(1 - \chi(\zeta)) m(\zeta, h)}{\zeta}$$

with  $a_j \in \Sigma_{\mathbb{T}}(\mathbb{R}^{d+1} \times \mathbb{R}^d)$ ,  $j = 0, 1$ . Then, the operator  $a(y, hD_y, hx)$  is bounded on  $L^2(\mathbb{R}^{d+1})$  and

$$\|a(y, hD_y, h)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \leq C(N_{d+1}(a_0) + N_{d+1}(a_1)).$$

**Lemma 8.5.** *Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  be equal to 1 in a neighborhood of 0. Consider a symbol  $a(y, \eta)$  that is compactly supported in the  $y$  variable and of the form*

$$a(y, \eta) = a_0(y, \eta') + a_1(y, \eta') \frac{(1 - \chi(\zeta))}{\zeta} + a_2(y, \eta).$$

- (1) *Assume that  $N_{d+2}(a_j) < \infty$ ,  $j = 0, 1$ , that is,  $a_j \in \Sigma_{\mathbb{T}}^{0, d+2}(\langle \eta' \rangle^{-d-1}; \mathbb{R}^{d+1} \times \mathbb{R}^d)$ , and  $\tilde{N}_{d+3}(a_2) < \infty$ . Then, for  $\theta \in W^{1, \infty}(\mathbb{R}_y^{d+1})$ , one has*

(8.14)

$$\|[a(y, hD_y), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \leq Ch(N_{d+2}(a_0) + N_{d+2}(a_1) + \tilde{N}_{d+3}(a_2)) \|\theta\|_{W^{1, \infty}}.$$

- (2) Assume that  $N_{d+2}(\nabla_{y'}a_0) < \infty$ ,  $N_{d+2}(\nabla_y a_1) < \infty$ , and  $\tilde{N}_{d+3}(\nabla_y a_2) < \infty$ . Then,

$$(8.15) \quad \begin{aligned} \|\bar{a}(y, hD_y)^* - a(y, hD_y)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \\ \leq Ch(N_{d+2}(\nabla_{y'}a_0) + N_{d+2}(\nabla_y a_1) + \tilde{N}_{d+3}(\nabla_y a_2)). \end{aligned}$$

The adjoint is understood with respect to the inner product  $L^2(\mathbb{R}^{d+1}, dxdt)$ .

**Proof.** First, we consider the contribution of  $a_2$  to the commutator. The kernel of the commutator is then given by  $K_2(y, \tilde{y}) = h^{-(d+1)}k_2(y, (y - \tilde{y})/h)$

$$\begin{aligned} k_2(y, v) &= (2\pi)^{-(d+1)} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} (\theta(y - hv) - \theta(y)) a_2(y, \eta) d\eta \\ &= (2\pi)^{-(d+1)} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} (\theta(y - hv) - \theta(y)) ({}^tL)^{d+3} a_2(y, \eta) d\eta, \end{aligned}$$

with  $L = (1 - iv \cdot \nabla_\eta) / \langle v \rangle^2$  and  ${}^tL = (1 + iv \cdot \nabla_\eta) / \langle v \rangle^2$  since  $L \exp(iv \cdot \eta) = \exp(iv \cdot \eta)$ . Using that  $\ell(y, hv) = (\theta(y - hv) - \theta(y)) / \|hv\|$  is bounded one can write

$$k_2(y, v) = h(2\pi)^{-(d+1)} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} \ell(y, hv) \|v\| ({}^tL)^{d+3} a_2(y, \eta) d\eta.$$

With the form of  ${}^tL$  and (8.13) one obtains

$$\begin{aligned} |k_2(y, v)| &\lesssim h \tilde{N}_{d+3}(a_2) \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-(d+2)} \int_{\mathbb{R}^d} \langle \eta' \rangle^{-d-1} d\eta' \int_{\mathbb{R}} \langle \zeta \rangle^{-2} d\zeta \\ &\lesssim h \tilde{N}_{d+3}(a_2) \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-(d+2)}. \end{aligned}$$

One concludes with Corollary 5.3 as for the proof of Lemma 5.12.

Second, we consider the contribution of  $a_0$  to the commutator. Since  $[\text{Op}^h(a_0), \theta]$  is tangential one can consider its action in the  $y'$  variable only. As in (5.16)–(5.17) one writes

$$\text{Op}^h(a_0)u(y', z) = \int_{\mathbb{R}^d} K_{a_0}(y', \tilde{y}'; z) u(\tilde{y}', z) d\tilde{y}',$$

with

$$K_{a_0}(y', \tilde{y}'; z) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(y' - \tilde{y}') \cdot \eta'} a_0(y', z, h\eta') d\eta',$$

with  $z$  as a parameter. The associated tangential kernel for the commutator is

$$K_0(y', \tilde{y}'; z) = K_{a_0}(y', \tilde{y}'; z)(\theta(\tilde{y}', z) - \theta(y', z)) = h^{-d}k_0(y', (y' - \tilde{y}')/h; z),$$

with

$$k_0(y', v; z) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{iv \cdot \eta'} (\theta(y' - hv, z) - \theta(y', z)) a_0(y, \eta') d\eta'.$$

Note that here  $v \in \mathbb{R}^d$ . With the same argument as above one finds

$$k_0(y', v; z) = h(2\pi)^{-d} \int_{\mathbb{R}^d} e^{iv \cdot \eta'} \ell'(y', hv) \|v\| ({}^tL_0)^{d+2} a_0(y, \eta') d\eta'.$$

with  $\ell'(y', hv, z) = (\theta(y' - hv, z) - \theta(y', z)) / \|hv\|$  and  ${}^tL_0 = (1 + iv \cdot \nabla_{\eta'}) / \langle v \rangle^2$  yielding

$$\begin{aligned} |k_0(y', v; z)| &\lesssim hN_{d+2}(a_0) \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-(d+1)} \int_{\mathbb{R}^d} \langle \eta' \rangle^{-d-1} d\eta' \\ &\lesssim hN_{d+2}(a_0) \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-(d+1)}. \end{aligned}$$

One concludes with Corollary 5.3.

Third, we consider the contribution of  $a_1$  to the commutator. Set  $f(\zeta) = (1 - \chi(\zeta)) / \zeta$ . As observed in (5.18) one has  $\text{Op}^h(a_1 f(\zeta)) = \text{Op}^h(a_1) f(hD_z)$  allowing one to write

$$[\text{Op}^h(a_1 f(\zeta)), \theta] = \text{Op}^h(a_1) [f(hD_z), \theta] + [\text{Op}^h(a_1), \theta] f(hD_z).$$

By (5.20) one has  $\|f(hD_z)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim 1$  and  $[\text{Op}^h(a_1), \theta] \lesssim hN_{d+2}(a_1) \|\theta\|_{W^{1,\infty}}$  similarly to the treatment of the term associated with  $a_0$ , yielding

$$\|[\text{Op}^h(a_1), \theta] f(hD_z)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim hN_{d+2}(a_1) \|\theta\|_{W^{1,\infty}}.$$

With (5.21)–(5.22) the commutator  $[f(hD_z), \theta]$  has tangential kernel acting only in the  $z$  variable

$$K(y; z, \tilde{z}) = h^{-1}k(y; z, (z - \tilde{z})/h)$$

with

$$k(y; z, v) = (2\pi)^{-1} \int_{\mathbb{R}} e^{iv\zeta} (\theta(y', z - hv) - \theta(y', z)) f(\zeta) d\zeta,$$

where  $v \in \mathbb{R}$  here. One writes

$$k(y; z, v) = h(2\pi)^{-1} \int_{\mathbb{R}} v e^{iv\zeta} \ell_d(y', z, hv) f(\zeta) d\zeta,$$

with  $\ell_d(y', z, hv) = (\theta(y', z - hv) - \theta(y', z)) / (hv)$  with  $|\ell_d(y', z, hv)| \leq \|\theta\|_{W^{1,\infty}}$ . Since  $v e^{iv\zeta} = -i \partial_\zeta e^{iv\zeta}$ , with an integration by parts, one finds

$$k(y; z, v) = ih(2\pi)^{-1} \int_{\mathbb{R}} e^{iv\zeta} \ell_d(y', z, hv) \partial_\zeta f(\zeta) d\zeta.$$

Moreover with  ${}^tL_f = (1 + iv \partial_\zeta) / \langle v \rangle^2$  one writes

$$k(y; z, v) = ih(2\pi)^{-1} \int_{\mathbb{R}} e^{iv\zeta} \ell_d(y', z, hv) ({}^tL_f)^2 \partial_\zeta f(\zeta) d\zeta.$$

Since  $|({}^tL_f)^2 \partial_\zeta f(\zeta)| \lesssim \langle v \rangle^{-2} \langle \zeta \rangle^{-2}$  one finds  $|k(y; z, v)| \lesssim h \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-2}$ , implying

$$\|[f(hD_z), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h \|\theta\|_{W^{1,\infty}}.$$

With Lemma 5.12 one obtains

$$\|\text{Op}^h(a_1)[f(hD_z), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h N_{d+1}(a_1) \|\theta\|_{W^{1,\infty}} \lesssim h N_{d+2}(a_1) \|\theta\|_{W^{1,\infty}}.$$

This concludes the proof of the estimation of the commutator norm.

We now turn to the proof of the estimate for the adjoint. We will observe that the proof is in fact along the same lines as that for the commutator. We start with the contribution of  $a_2(y, \eta)$ . The kernel of the operator  $\text{Op}^h(\bar{a}_2)^* - \text{Op}^h(a_2)$  is given by  $K(y, \tilde{y}) = h^{-d-1} k(y, (y - \tilde{y})/h)$  with

$$k(y, v) = (2\pi)^{-d-1} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} (a_2(y - hv, \eta) - a_2(y, \eta)) d\eta,$$

with  $v \in \mathbb{R}^{d+1}$ , and one writes

$$k(y, v) = (2\pi)^{-d-1} \int_{\mathbb{R}^{d+1}} e^{iv \cdot \eta} ({}^tL)^{d+3} (a_2(y - hv, \eta) - a_2(y, \eta)) d\eta,$$

with  ${}^tL = (1 + iv \cdot \nabla_\eta) / \langle v \rangle^2$ . Since

$$\begin{aligned} |({}^tL)^{d+3} (a_2(y - hv, \eta) - a_2(y, \eta))| &\leq h \|v\| \|\nabla_y ({}^tL)^{d+3} a_2(y, \eta)\|_{L^\infty} \\ &\lesssim h \langle v \rangle^{-d-2} \tilde{N}_{d+3}(\nabla_y a_2) \langle \eta' \rangle^{-d-1} \langle \zeta \rangle^{-2}. \end{aligned}$$

The sought estimate follows.

For the contribution of the symbol  $a_0(y, \eta)$ , the *tangential* kernel of the operator  $\text{Op}^h(\bar{a}_0)^* - \text{Op}^h(a_0)$  is given by  $K(y', \tilde{y}') = h^{-d}k(y', (y' - \tilde{y}')/h)$  with

$$k(y', v) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{iv \cdot \eta'} (a_0(y' - hv, \eta') - a_0(y', \eta')) d\eta',$$

with here  $v \in \mathbb{R}^d$ , and one writes

$$k(y', v) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{iv \cdot \eta'} ({}^tL)^{d+2} (a_0(y' - hv, \eta') - a_0(y', \eta')) d\eta',$$

with  ${}^tL = (1 + iv \cdot \nabla_{\eta'}) / \langle v \rangle^2$  and one finds similarly

$$|({}^tL)^{d+2} (a_0(y - hv, \eta) - a_0(y, \eta))| \lesssim h \langle v \rangle^{-d-1} N_{d+2}(\nabla_{y'} a_0) \langle \eta' \rangle^{-d-1}.$$

The sought estimate follows.

For the contribution of  $a_1(y, \eta)$  one writes

$$\begin{aligned} & \text{Op}^h(\bar{a}_1 \bar{f}(\zeta))^* - \text{Op}^h(a_1 f(\zeta)) \\ &= f(hD_z) \bar{a}_1(y, hD_y)^* - a_1(y, hD_y) f(hD_z) \\ &= f(hD_z) (\bar{a}_1(y, hD_y)^* - a_1(y, hD_y)) + [f(hD_z), a_1(y, hD_y)], \end{aligned}$$

using that  $\bar{f}(hD_z)^* = f(hD_z)$ . With (5.20) and applying the argument made for the term associated with  $a_0$  one finds

$$\|f(hD_z) (\bar{a}_1(y, hD_y)^* - a_1(y, hD_y))\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h N_{d+2}(\nabla_{y'} a_1).$$

We now consider the commutator  $[f(hD_z), a_1(y, hD_y)]$ . The kernel of  $f(hD_z)$  is given by

$$K_1(y, \tilde{y}) = \delta_{y' - \tilde{y}'} \otimes K_d(z, \tilde{z}),$$

with

$$K_d(z, \tilde{z}) = (2\pi)^{-1} \int_{\mathbb{R}} e^{i(z - \tilde{z})\zeta} f(h\zeta) d\zeta.$$

The kernel of  $a_1(y, hD_y)$  is given by

$$K_2(y, \tilde{y}) = K'(y', \tilde{y}'; z) \otimes \delta_{z - \tilde{z}},$$



with

$$K'(y', \tilde{y}'; z) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(y' - \tilde{y}') \cdot \eta'} a_1(y', z, h\eta') d\eta'.$$

This leads to the following kernel for the commutator  $[f(hD_z), a_1(y, hD_y)]$

$$K(y, \tilde{y}) = \int_{\mathbb{R}^{d+1}} (K_1(y, \hat{y})K_2(\hat{y}, \tilde{y}) - K_2(y, \hat{y})K_1(\hat{y}, \tilde{y})) d\hat{y},$$

where the integration is understood in the sense of distribution action. Products here make sense; see [21, Theorem 8.2.14]. This gives

$$\begin{aligned} K(y, \tilde{y}) &= (2\pi)^{-d-1} \int_{\mathbb{R}} e^{i(z - \tilde{z})\zeta} f(h\zeta) d\zeta \\ &\quad \times \int_{\mathbb{R}^d} e^{i(y' - \tilde{y}') \cdot \eta'} (a_1(y', \tilde{z}, h\eta') - a_1(y', z, h\eta')) d\eta', \end{aligned}$$

that we write  $K(y, \tilde{y}) = h^{-d-1}k(y, (y - \tilde{y})/h)$  with

$$k(y, v) = (2\pi)^{-d-1} \int_{\mathbb{R}} e^{iw\zeta} f(\zeta) d\zeta \int_{\mathbb{R}^d} e^{iv' \cdot \eta'} (a_1(y', z - hw, \eta') - a_1(y', z, \eta')) d\eta',$$

for  $v = (v', w)$  with  $v' \in \mathbb{R}^d$  and  $w \in \mathbb{R}$ . Considering the bounded function

$$\ell(y, z, hw, \eta') = (a_1(y', z - hw, \eta') - a_1(y', z, \eta')) / (hw),$$

one writes

$$k(y, v) = h(2\pi)^{-d-1} \int_{\mathbb{R}} e^{iw\zeta} w f(\zeta) d\zeta \int_{\mathbb{R}^d} e^{iv' \cdot \eta'} \ell(y, z, hw, \eta') d\eta'.$$

Set  $L_\zeta = (1 - iw\partial_\zeta) / \langle w \rangle^2$  and  $L_{\eta'} = (1 - iv' \cdot \partial_{\eta'}) / \langle v' \rangle^2$ . With  $w e^{iw\zeta} = -i\partial_\zeta e^{iw\zeta}$ ,  $L_\zeta e^{iw\zeta} = e^{iw\zeta}$ , and  $L_{\eta'} e^{iv' \cdot \eta'} = e^{iv' \cdot \eta'}$  one writes

$$k(y, v) = ih(2\pi)^{-d-1} \int_{\mathbb{R}} e^{iw\zeta} ({}^tL_\zeta)^2 \partial_\zeta f(\zeta) d\zeta \int_{\mathbb{R}^d} e^{iv' \cdot \eta'} ({}^tL_{\eta'})^{d+1} \ell(y, z, hw, \eta') d\eta',$$

and one finds

$$|k(y, v)| \lesssim h \langle w \rangle^{-2} \langle v' \rangle^{-d-1} N_{d+1}(\partial_z a_1) \int_{\mathbb{R}} \langle \zeta \rangle^{-2} d\zeta \int_{\mathbb{R}^d} \langle \eta' \rangle^{-d-1} d\eta',$$

since  $({}^tL_{\eta'})^{d+1} \ell(y, z, hw, \eta') \lesssim N_{d+1}(\partial_z a_1) \langle \eta' \rangle^{-d-1}$ . This leads to the conclusion of the proof.  $\square$

Multiplying  $a_0(y, \eta')$  and  $a_1(y, \eta')$  by a sufficiently rapidly decaying function of  $\zeta$  leaves the result *nearly* unchanged.

**Lemma 8.5'.** *Let  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  be equal to 1 in a neighborhood of 0. Let  $m(\zeta, h)$  be a bounded function of  $\zeta$  and  $h > 0$  with moreover*

$$\partial_\zeta^j m(\zeta, h) \in L_\zeta^1, \quad 1 \leq j \leq 3,$$

*uniformly with respect to  $h > 0$ . Consider a symbol  $a(y, \eta)$  that is compactly supported in the  $y$  variable and of the form*

$$a(y, \eta) = a_0(y, \eta')m(\zeta, h) + a_1(y, \eta') \frac{(1 - \chi(\zeta))m(\zeta, h)}{\zeta}$$

(1) *Assume that  $N_{d+2}(a_j) < \infty$ ,  $j = 0, 1$ , that is,  $a_j \in \Sigma_{\mathbb{T}}^{0, d+2}(\langle \eta' \rangle^{-d-1}; \mathbb{R}^{d+1} \times \mathbb{R}^d)$ . Then, for  $\theta \in W^{1, \infty}(\mathbb{R}_y^{d+1})$ , one has*

$$\begin{aligned} \|[a(y, hD_y), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \\ \leq Ch(N_{d+2}(a_0) + N_{d+2}(a_1)) \|\theta\|_{W^{1, \infty}} \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1}. \end{aligned}$$

(2) *Assume that  $N_{d+2}(\nabla_y a_0)$ ,  $N_{d+2}(\nabla_y a_1)$  are finite. Then one has*

$$\begin{aligned} \|\bar{a}(y, hD_y)^* - a(y, hD_y)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \\ \leq Ch(N_{d+2}(\nabla_y a_0) + N_{d+2}(\nabla_y a_1)) \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1}. \end{aligned}$$

*The adjoint is understood with respect to the inner product  $L^2(\mathbb{R}^{d+1}, dxdt)$ .*

Note that  $N_{d+2}(\nabla_{y'} a_0)$  is replaced by  $N_{d+2}(\nabla_y a_0)$  in the second estimate if computed with Lemma 8.5.

**Proof.** Considering the properties of  $m(\zeta, h)$  in both results, only the contribution associated with the tangential symbol  $a_0(y, \eta')$  needs to be analyzed. For the commutator, as for the treatment of the term  $a_1$  in the proof of Lemma 8.5 one estimates the operator norm of  $[m(hD_z, h), \theta]$ . Its tangential kernel is

$$K(y; z, \tilde{z}) = h^{-1}k(y; z, (z - \tilde{z})/h)$$

with

$$k(y; z, v) = (2\pi)^{-1} \int_{\mathbb{R}} e^{iv\zeta} (\theta(y', z - hv) - \theta(y', z)) m(\zeta, h) d\zeta, \quad v \in \mathbb{R}.$$

Following the proof of Lemma 8.5 one obtains

$$k(y; z, v) = ih(2\pi)^{-1} \int_{\mathbb{R}} e^{iv\zeta} \ell_d(y', z, hv) ({}^tL_f)^2 \partial_\zeta m(\zeta, h) d\zeta.$$

where  $\ell_d(y', z, hv) = (\theta(y', z - hv) - \theta(y', z)) / (hv)$  and  ${}^tL_f = (1 + iv\partial_\zeta) / \langle v \rangle^2$ . With the properties of the function  $m(\zeta, h)$  one obtains

$$|k(y; z, v)| \lesssim h \|\theta\|_{W^{1,\infty}} \langle v \rangle^{-2} \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1} \lesssim h \|\theta\|_{W^{1,\infty}} \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1} \langle v \rangle^{-2},$$

implying

$$\|[f(hD_z), \theta]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h \|\theta\|_{W^{1,\infty}} \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1}.$$

Considering the argument for the adjoint given in the proof of Lemma 8.5 one needs to estimate the operator norm of  $[m(hD_z, h), a_0(y, hD_y)]$ . Its kernel reads  $K(y, \tilde{y}) = h^{-d-1} k(y, (y - \tilde{y})/h)$  with

$$k(y, v) = (2\pi)^{-d-1} \int_{\mathbb{R}} e^{iw\zeta} m(\zeta, h) d\zeta \int_{\mathbb{R}^d} e^{iv' \cdot \eta'} (a_0(y', z - hw, \eta') - a_0(y', z, \eta')) d\eta',$$

for  $v = (v', w)$  with  $v' \in \mathbb{R}^d$  and  $w \in \mathbb{R}$ . One obtains

$$k(y, v) = ih(2\pi)^{-d-1} \int_{\mathbb{R}} e^{iw\zeta} ({}^tL_\zeta)^2 \partial_\zeta m(\zeta, h) d\zeta \int_{\mathbb{R}^d} e^{iv' \cdot \eta'} ({}^tL_{\eta'})^{d+1} \ell(y, z, hw, \eta') d\eta',$$

with  $\ell(y, z, hw, \eta') = (a_0(y', z - hw, \eta') - a_0(y', z, \eta')) / (hw)$ . This leads to

$$|k(y, v)| \lesssim h \langle w \rangle^{-2} \langle v' \rangle^{-d-1} N_{d+1}(\partial_z a_0) \sup_{1 \leq j \leq 3} \|d_\zeta^j m\|_{L^1} \int_{\mathbb{R}^d} \langle \eta' \rangle^{-d-1} d\eta',$$

and the conclusion of the proof.  $\square$

**Remark 8.6.** A particular choice of Fourier multiplier  $m(\zeta, h)$  appearing in Lemmata 8.4' and 8.5' is  $m(\zeta, h) = \varphi(h^\beta \zeta)$  for  $\varphi \in \mathcal{S}(\mathbb{R}; \mathbb{R})$  and some  $\beta > 0$ . Indeed, the following properties hold:

- (1) One has  $m(\zeta, h) \lesssim 1$  uniformly with respect to  $h > 0$  as required by Lemmata 8.4' and 8.5'.
- (2) One has obviously

$$\sup_h \sup_{1 \leq j \leq 3} \|\partial_\zeta^j m\|_{L^1} < +\infty.$$

In what follows, we will use  $m(\zeta, h) = \varphi(h^3 \zeta)$  with  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ .

### 8.3. A class of error terms.

**Definition 8.7.** Let  $\mathcal{R}$  be the class of sequences of operators  $(R_k)_k$  bounded on  $L^2(\mathbb{R}^{d+1})$  by  $Ch_k^\delta$ ,  $\delta \geq 0$ , and from  $L^2(\mathbb{R}^{d+1})$  to  $H^1(\mathbb{R}_z; L^2(\mathbb{R}_{y'}^d))$  by  $Ch_k^\rho$ ,  $\rho \geq 0$ , with moreover  $\delta + \rho > 0$ . Denote by  $\mathcal{R}_0$  the class obtained in the case  $(\delta, \rho) = (1, 0)$ .

**Lemma 8.8.** Let  $(f_k)_k$  be a bounded sequence of  $L^2(\mathbb{R}^{d+1})$  and  $(g_k)_k$  be a bounded sequence of  $L^2_{y'}(\mathbb{R}^d)$ . Then, if  $(R_k)_k \in \mathcal{R}$  one has

$$\lim_{k \rightarrow +\infty} \left| (g_k \otimes \delta_{z=0}, R_k f_k)_{H_z^{-1}L^2_{y'}, H_z^1 L^2_{y'}} \right| = 0.$$

**Proof.** As the sequence  $g_k \otimes \delta_{z=0}$  is bounded in  $H^{-\sigma}(\mathbb{R}_z; L^2(\mathbb{R}_{y'}^d))$ , for any  $\sigma > 1/2$ , the result follows from the bound

$$\|R_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}), H_z^\sigma L^2_{y'})} \leq Ch_k^{\frac{\delta+\rho}{2} + (\sigma - \frac{1}{2})(\rho - \delta)},$$

obtained by interpolation and choosing  $\sigma > \frac{1}{2}$  sufficiently close to  $\frac{1}{2}$ .  $\square$

**Corollary 8.9.** Let  $b \in \Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d+2})$  with  $\text{supp } b \subset K \times \mathbb{R}^{d+1}$ , for  $K$  a compact of  $\phi_{\mathcal{L}}(\mathcal{O})$ , and let  $\psi \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  be equal to 1 in a neighborhood of the  $y$ -projection of  $\text{supp } b$ . Let  $L_k(b, \psi)$  be as defined in (7.9). One has  $L_k(b, \psi) = L'_k(b, \psi) + o(1)_{k \rightarrow +\infty}$  with

$$(8.16) \quad \begin{aligned} L'_k(b, \psi) &= i \left( \text{Op}^h(b) \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L^2_{y'}, H_z^{-1} L^2_{y'}}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \text{Op}^h(\bar{b}) \psi u_k \right)_{H_z^{-1} L^2_{y'}, H_z^1 L^2_{y'}}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

**Proof.** One has to prove that

$$\begin{aligned} I_k &= (v_k \otimes \delta_{z=0}, \psi \text{Op}^h(b)^\star u_k)_{H_z^{-1} L^2_{y'}, H_z^1 L^2_{y'}}^{\kappa_k \mu_{g_k} dt} \\ &= (v_k \otimes \delta_{z=0}, \text{Op}^h(\bar{b}) \psi u_k)_{H_z^{-1} L^2_{y'}, H_z^1 L^2_{y'}}^{\kappa_k \mu_{g_k} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

Let  $\tilde{\psi} \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  be equal to 1 in a neighborhood of  $\text{supp } \psi$ . One has

$$I_k = (\tilde{\psi} v_k \otimes \delta_{z=0}, \psi \text{Op}^h(b)^\star \tilde{\psi} u_k)_{H_z^{-1} L^2_{y'}, H_z^1 L^2_{y'}}^{\kappa_k \mu_{g_k} dt}.$$

The adjoint operator with the  $\star$ -notation is here understood in the sense of the inner product  $L^2(\mathbb{R}^{d+1}, \kappa_k \mu_{g_k} dt)$ ; see Proposition 7.2. Thus,  $\text{Op}^h(b)^\star =$

$\tilde{\kappa}_k^{-1} \text{Op}^h(b)^* \tilde{\kappa}_k$  where the adjoint with the usual  $*$ -notation is understood for the inner product  $L^2(\mathbb{R}^{d+1}, dxdt)$ .

Since  $(\tilde{\psi}u_k)_k$  and  $(\tilde{\psi}v_k)_k$  are bounded in  $L^2(\mathbb{R}^{d+1})$  and in  $L^2_{y'}(\mathbb{R}^d)$  respectively, it suffices to prove that  $\psi \tilde{\kappa}_k^{-1} \text{Op}^h(b)^* \tilde{\kappa}_k - \text{Op}^h(\bar{b})\psi \in \mathcal{R}_0$  by Lemma 8.8. One has

$$\psi \tilde{\kappa}_k^{-1} \text{Op}^h(b)^* \tilde{\kappa}_k - \text{Op}^h(\bar{b})\psi = \psi \tilde{\kappa}_k^{-1} (\text{Op}^h(\tilde{\kappa}_k b)^* - \text{Op}^h(\tilde{\kappa}_k \bar{b})) + [\psi, \text{Op}^h(\bar{b})].$$

From (8.14) and (8.15) one deduces that

$$\|\psi \tilde{\kappa}_k^{-1} \text{Op}^h(b)^* \tilde{\kappa}_k - \text{Op}^h(\bar{b})\psi\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k.$$

To estimate the operator norm from  $L^2(\mathbb{R}^{d+1})$  to  $H^1(\mathbb{R}_z; L^2(\mathbb{R}^d_{y'}))$  we compose with  $D_z = h_k^{-1} \text{Op}^h(\zeta)$  and get

$$\begin{aligned} D_z(\psi \tilde{\kappa}_k^{-1} \text{Op}^h(\bar{b})^* \tilde{\kappa}_k - \text{Op}^h(b)\psi) \\ &= [D_z, \psi \tilde{\kappa}_k^{-1}] \text{Op}^h(b)^* \tilde{\kappa}_k + \psi \tilde{\kappa}_k^{-1} D_z \text{Op}^h(b)^* \tilde{\kappa}_k - [D_z, \text{Op}^h(\bar{b})]\psi - \text{Op}^h(\bar{b})D_z\psi \\ &= D_z(\psi \tilde{\kappa}_k^{-1}) \text{Op}^h(b)^* \tilde{\kappa}_k + h_k^{-1} \psi \tilde{\kappa}_k^{-1} \text{Op}^h(b\zeta)^* \tilde{\kappa}_k - \text{Op}^h(D_z \bar{b})\psi - h_k^{-1} \text{Op}^h(\zeta \bar{b})\psi. \end{aligned}$$

One has

$$\|D_z(\psi \tilde{\kappa}_k^{-1}) \text{Op}^h(b)^* \tilde{\kappa}_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} + \|\text{Op}^h(D_z \bar{b})\psi\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim 1.$$

It thus remains to prove that

$$\|\psi \tilde{\kappa}_k^{-1} \text{Op}^h(b\zeta)^* \tilde{\kappa}_k - \text{Op}^h(\zeta \bar{b})\psi\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k.$$

The argument is the same as for estimate (8.9) with  $b$  replaced by  $\zeta b$ . We conclude using Lemma 8.8 with  $\delta = 1$ ,  $\rho = 0$ .  $\square$

**Corollary 8.10.** *Let  $b \in \Sigma_0^{\infty, \infty}(\langle \xi \rangle^{-\infty}; \mathbb{R}^{2d+2})$  with  $\text{supp } b \subset K \times \mathbb{R}^{d+1}$ , for  $K$  a compact of  $\phi_{\mathcal{L}}(\mathcal{O})$ , and let  $\psi \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  be equal to 1 in a neighborhood of the  $y$ -projection of  $\text{supp } b$ . Let also  $\varphi \in \mathcal{C}_c^\infty([-2, 2]; \mathbb{R})$  and equal to 1 on  $(-1, 1)$ . Let  $L_k(b, \psi)$  be as defined in (7.9). One has  $L_k(b, \psi) = L_k''(b, \psi) + o(1)_{k \rightarrow +\infty}$  with*

$$\begin{aligned} L_k''(b, \psi) &= i(\text{Op}^h(b)\varphi(h_k^3 D_z)\psi u_k, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, \text{Op}^h(\bar{b})\varphi(h_k^3 D_z)\psi u_k)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

**Proof.** Arguing as for Corollary 8.9, starting from the form of  $L'_k(b, \psi)$  given in (8.16) it suffices to prove that

$$\text{Op}^h(b)(1 - \varphi(h_k^3 D_z)) = \text{Op}^h(\gamma_{h_k}) \in \mathcal{R}_0, \quad \gamma_{h_k}(y, \eta) = b(y, \eta)(1 - \varphi(h_k^2 \zeta))$$

In the support of  $1 - \varphi(h_k^2 \zeta)$  one has  $h_k^2 |\zeta| \gtrsim 1$ , which combined with the fast decay of  $b$  in  $\eta$  yields

$$|\partial_y^\alpha \partial_\eta^\beta \gamma_{h_k}(y, \eta)| \lesssim h_k^N \langle \eta \rangle^{-N},$$

for any  $N$ . The result follows from Lemma 7.1.  $\square$

## 9. PROOF OF THE PROPAGATION EQUATION II: SYMBOL QUANTIZATION

From the support property of the semi-classical measure  $\mu$  given in (7.1) if considering the action of  $\mu$  on a symbol in  $\Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  it suffices to work with a time-frequency truncated version. That is, for  $\chi \in \mathcal{C}_c^\infty(C_{\mu,0}^2, C_{\mu,1}^2)$  equal to 1 on a neighborhood of  $[C_{\mu,0}, C_{\mu,1}]$  and  $b(y, \eta) \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  one has  $\langle \mu, (1 - \chi)b \rangle = 0$ , meaning that

$$\lim_{k \rightarrow +\infty} L_k''((1 - \chi(\tau))b, \psi) = \lim_{k \rightarrow +\infty} L_k((1 - \chi(\tau))b, \psi) = 0.$$

With Proposition 7.2, we will thus only consider the action of  $\mu$  on a symbol of the form  $\chi(\tau)b(y, \eta)$  through the limit of  $L_k(\chi(\tau)b, \psi)$  and we will now quantize the Euclidean division of Proposition 8.2. Even though the symbol  $b$  on the l.h.s. of (8.2) exhibits rapid decay in the variable  $\zeta$ , it is not the case for the symbols  $b_{0,k}, b_{1,k}$ , and  $q_k$  on the r.h.s. of (8.2). Following [16], adding a cutoff in the  $\zeta$  variable in the form of  $\varphi(h_k^3 D_z)$ , made possible by Corollary 8.10, acts as a remedy.

Since  $L_k(\cdot, \psi)$  and  $L_k''(\cdot, \psi)$  have the same limit as  $k \rightarrow \infty$  by Corollary 8.10, in what follows, we will study sequentially the limits of  $L_k''(a, \psi)$  as  $k \rightarrow +\infty$  with  $a(y, \eta) = q_k p_k(y, \eta)$ ,  $a(y, \eta') = b_{0,k}(y, \eta')$ , and  $a(y, \eta) = b_{1,k}(y, \eta')\zeta$ .

**9.1. Contribution of  $q_k p_k$ .** We prove that the symbol  $q_k p_k(y, \eta)$  yields a vanishing contribution to the limit of  $L_k''(\chi(\tau)b, \psi)$ .

**Proposition 9.1.** *One has  $L_k''(q_k p_k, \psi) = o(1)_{k \rightarrow +\infty}$ .*

Proving this result requires some preliminary results.

Set  $\varphi_k = \varphi(h_k^3 D_z)$ . Naturally,  $\varphi_k$  is uniformly bounded on  $L^2(\mathbb{R})$  as a uniformly bounded Fourier multiplier. One can view  $\varphi_k$  in various manners: one has

$$\varphi_k = \text{Op}^{h_k}(h_k^2 \zeta) = \text{Op}^{h_k^3}(\zeta).$$

With the second formula, by simply replacing  $h$  by  $h_k^3$  in the analysis of Section 5, with point (3) of Proposition 5.10 one has the following result.

**Lemma 9.2.** *Let  $\theta \in W^{1,\infty}(\mathbb{R})$ . Then,  $\|[\theta, \varphi_k]\|_{\mathcal{L}(L^2(\mathbb{R}))} \leq Ch_k^3$ .*

Set

$$\begin{aligned} Q_k &= \text{Op}^h(q_k), \quad \bar{Q}_k = \text{Op}^h(\bar{q}_k), \quad P_k = h_k^2 P_{\kappa_k, g_k}, \\ G_k &= \text{Op}^h(q_k p_k), \quad \text{and} \quad \bar{G}_k = \text{Op}^h(\bar{q}_k p_k). \end{aligned}$$

Note that

$$P_k = \text{Op}^h(p_k) + h_k^2 P_k^1,$$

where  $P_k^1$  is a differential operator of order one with bounded coefficients.

Because of the form of  $p_k$  and  $P_k$ , one writes  $P_k = P_k^d + P_k^\top$  with

$$\begin{aligned} P_k^d &= \tilde{\kappa}_k^{-1} h_k D_z \tilde{\kappa}_k g_k^{dd}(x) h_k D_z, \\ P_k^\top &= h_k^2 \partial_t^2 + \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} \tilde{\kappa}_k^{-1} h_k D_i \tilde{\kappa}_k g_k^{ij}(x) h_k D_j, \end{aligned}$$

and  $G_k = G_k^d + G_k^\top$  with

$$\begin{aligned} G_k^d &= g_k^{dd}(x) \text{Op}^h(q_k) h_k^2 D_z^2 \\ G_k^\top &= \text{Op}^h(q_k) h_k^2 \partial_t^2 + \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} g_k^{ij}(x) \text{Op}^h(q_k) h_k^2 D_i D_j, \end{aligned}$$

and  $\bar{G}_k = \bar{G}_k^d + \bar{G}_k^\top$  with

$$\begin{aligned} \bar{G}_k^d &= g_k^{dd}(x) \text{Op}^h(\bar{q}_k) h_k^2 D_z^2 \\ \bar{G}_k^\top &= \text{Op}^h(\bar{q}_k) h_k^2 \partial_t^2 + \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} g_k^{ij}(x) \text{Op}^h(\bar{q}_k) h_k^2 D_i D_j. \end{aligned}$$

With this notation one has

$$L_k''(q_k p_k, \psi) = L_k^d(q_k p_k, \psi) + L_k^\top(q_k p_k, \psi),$$

with

$$\begin{aligned} L_k^d(q_k p_k, \psi) &= i \left( G_k^d \varphi_k \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \bar{G}_k^d \varphi_k \psi u_k \right)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

and

$$\begin{aligned} L_k^\top(q_k p_k, \psi) &= i \left( G_k^\top \varphi_k \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \bar{G}_k^\top \varphi_k \psi u_k \right)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

**Lemma 9.3.** (1) One has  $h_k Q_k \in \mathcal{R}_0$ .

(2) Let  $(f_k)_k$  be a bounded sequence in  $W^{1,\infty}(\mathbb{R}^{d+1})$ . One has  $[Q_k, f_k] \in \mathcal{R}_0$ .

(3) The operator  $G_k^\top - Q_k P_k^\top$  is a finite sum of operators that lie in

$$(9.1) \quad \sum_{|\alpha'|=1} \mathcal{R}_0 h_k^2 D_{y'}^{\alpha'} D_z + \sum_{|\alpha|=2} \mathcal{R}_0 h_k^2 D_{y'}^\alpha + \sum_{|\beta|=1} \mathcal{R}_0 h_k D_y^\beta,$$

with  $\mathcal{R}_0$  as given in Definition 8.7. The same holds for  $\bar{G}_k^\top - \bar{Q}_k P_k^\top$

(4) The operators  $Q_k[P_k^\top, \varphi_k]$  and  $\bar{Q}_k[P_k^\top, \varphi_k]$  are also finite sums of operators that lie in the space given by (9.1).

A proof is given below.

Observe that  $\mathcal{R} \varphi_k \subset \mathcal{R}$  as  $\varphi_k$  is uniformly bounded on  $L^2(\mathbb{R}^{d+1})$ . Then, with Lemma 8.8, exploiting (7.4)–(7.5) and the local estimate (7.6), with the third item in Lemma 9.3 one obtains

$$\begin{aligned} L_k^\top(q_k p_k, \psi) &= i \left( Q_k P_k^\top \varphi_k \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \bar{Q}_k P_k^\top \varphi_k \psi u_k \right)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

With the fourth item in Lemma 9.3, with the same argumentation one finds

$$\begin{aligned} L_k^\top(q_k p_k, \psi) &= i \left( Q_k \varphi_k P_k^\top \psi u_k, v_k \otimes \delta_{z=0} \right)_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i \left( v_k \otimes \delta_{z=0}, \bar{Q}_k \varphi_k P_k^\top \psi u_k \right)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

**Lemma 9.4.** Let  $M_k(y, D_y)$  be a differential operator with coefficients that are uniformly bounded with respect to  $k$ . Let  $N \in \mathbb{N}$ . For some  $C_N > 0$  one has

$$\|Q_k \varphi_k [M(y, D_y), \psi] u_k\|_{H_z^1 L_{y'}^2} \leq C h_k^N.$$

The same hold for  $\bar{Q}_k$  in place of  $Q_k$ .



A proof is given below. Applying Lemma 9.4 gives

$$\begin{aligned} L_k^\top(q_k p_k, \psi) &= i(Q_k \varphi_k \psi P_k^\top u_k, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, \bar{Q}_k \varphi_k \psi P_k^\top u_k)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

Our goal is now to handle the terms associated with the operators  $G_k^d$  and  $\bar{G}_k^d$  in  $L_k^d(q_k p_k, \psi)$ . With the forms of  $G_k^d$  and  $\bar{G}_k^d$  and Lemma 9.4, using that  $g_k^{dd}$  is uniformly Lipschitz as  $k \rightarrow +\infty$ , one has

$$\begin{aligned} L_k^d(q_k p_k, \psi) &= i(g_k^{dd} Q_k \varphi_k \psi h_k^2 D_z^2 u_k, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, g_k^{dd} \bar{Q}_k \varphi_k \psi h_k^2 D_z^2 u_k)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

With the "jump formula" one has

$$\begin{aligned} h_k^2 D_z^2 u_k &= -h_k^2 \partial_z u_{kz=0^+} \otimes \delta_{z=0} + h_k^2 \underline{D_z^2} u_k \\ &= -h_k (g_k^{dd})^{-1} v_k \otimes \delta_{z=0} + h_k^2 \underline{D_z^2} u_k, \end{aligned}$$

recalling (7.7), where  $\underline{f}$  denotes the zero-extension of  $f|_{z>0}$ . One writes

$$L_k^d(q_k p_k, \psi) = L_k^\delta(q_k p_k, \psi) + L_k^{\{z>0\}}(q_k p_k, \psi) + o(1)_{k \rightarrow +\infty},$$

with

$$(9.2) \quad \begin{aligned} L_k^\delta(q_k p_k, \psi) &= -i h_k (g_k^{dd} Q_k \varphi_k \psi (g_k^{dd})^{-1} v_k \otimes \delta_{z=0}, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad + i h_k (v_k \otimes \delta_{z=0}, g_k^{dd} \bar{Q}_k \varphi_k \psi (g_k^{dd})^{-1} v_k \otimes \delta_{z=0})_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}, \end{aligned}$$

and

$$\begin{aligned} L_k^{\{z>0\}}(q_k p_k, \psi) &= i(g_k^{dd} Q_k \varphi_k \psi h_k^2 \underline{D_z^2} u_k, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, g_k^{dd} \bar{Q}_k \varphi_k \psi h_k^2 \underline{D_z^2} u_k)_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

With Lemmata 9.2 and 9.3 one has

$$g_k^{dd} Q_k \varphi_k \psi = Q_k \varphi_k \psi g_k^{dd} \pmod{\mathcal{R}_0} \quad \text{and} \quad g_k^{dd} \bar{Q}_k \varphi_k \psi = \bar{Q}_k \varphi_k \psi g_k^{dd} \pmod{\mathcal{R}_0}.$$

With Lemma 8.8, as  $h_k^2 \underline{D_z^2} u_k$  is bounded in  $L^2$  by (7.6), one obtains

$$\begin{aligned} L_k^{\{z>0\}}(q_k p_k, \psi) &= i(Q_k \varphi_k \psi \underline{g_k^{dd} h_k^2 D_z^2 u_k}, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, \bar{Q}_k \varphi_k \psi \underline{g_k^{dd} h_k^2 D_z^2 u_k})_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt}. \end{aligned}$$

One writes

$$g_k^{dd} h_k^2 D_z^2 = \kappa_k^{-1} h_k D_z g_k^{dd} \kappa_k h_k D_z + h_k \kappa_k^{-1} (D_z (g_k^{dd} \kappa_k)) h_k D_z.$$

Since  $h_k Q_k \in \mathcal{R}_0$  by Lemma 9.3 one obtains

$$\begin{aligned} L_k^{\{z>0\}}(q_k p_k, \psi) &= i(Q_k \varphi_k \psi \underline{\kappa_k^{-1} h_k D_z g_k^{dd} \kappa_k h_k D_z u_k}, v_k \otimes \delta_{z=0})_{H_z^1 L_{y'}^2, H_z^{-1} L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad - i(v_k \otimes \delta_{z=0}, \bar{Q}_k \varphi_k \psi \underline{\kappa_k^{-1} h_k D_z g_k^{dd} \kappa_k h_k D_z u_k})_{H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2}^{\kappa_k \mu_{g_k} dt} \\ &\quad + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

One thus obtains

$$L_k^{\{z>0\}}(q_k p_k, \psi) + L_k^\Gamma(q_k p_k, \psi) = o(1)_{k \rightarrow +\infty}.$$

since  $P_k = \kappa_k^{-1} h_k D_z g_k^{dd} \kappa_k h_k D_z + P_k^\Gamma$  and thus

$$L_k''(q_k p_k, \psi) = L_k^\delta(q_k p_k, \psi) + o(1)_{k \rightarrow +\infty},$$

with  $L_k^\delta(q_k p_k, \psi)$  given in (9.2). One then writes

$$L_k^\delta(q_k p_k, \psi) = i h_k \langle N_k v_k \otimes \delta_{z=0}, v_k \otimes \delta_{z=0} \rangle_{H_z^\alpha L_{y'}^2, H_z^{-\alpha} L_{y'}^2}^{\kappa_k \mu_{g_k} dt},$$

for any  $\alpha > 1/2$ , with

$$N_k = (g_k^{dd})^{-1} \psi \varphi_k^* \bar{Q}_k^* g_k^{dd} - g_k^{dd} Q_k \varphi_k \psi (g_k^{dd})^{-1},$$

where the adjoints with the  $\star$ -notation are understood in the sense of the inner product  $L^2(\mathbb{R}^{d+1}, \kappa_k \mu_{g_k} dt)$ , that is,  $\varphi_k^* \bar{Q}_k^* = \tilde{\kappa}_k^{-1} \varphi_k \bar{Q}_k^* \tilde{\kappa}_k$  where the adjoint with the  $*$ -notation is understood for the inner product  $L^2(\mathbb{R}^{d+1}, dx dt)$ . One thus has

$$\begin{aligned} N_k &= (g_k^{dd})^{-1} \psi \tilde{\kappa}_k^{-1} \varphi_k \bar{Q}_k^* \tilde{\kappa}_k g_k^{dd} - g_k^{dd} Q_k \varphi_k \psi (g_k^{dd})^{-1} \\ &= g_k^{dd} \left( \psi (\tilde{\kappa}_k (g_k^{dd})^2)^{-1} \varphi_k \bar{Q}_k^* \tilde{\kappa}_k (g_k^{dd})^2 - Q_k \varphi_k \psi \right) (g_k^{dd})^{-1}, \end{aligned}$$

**Lemma 9.5.** *Let  $(f_k)_k$  be a sequence of functions such that*

$$\|f_k\|_{W^{1,\infty}} + \|1/f_k\|_{W^{1,\infty}} \leq C,$$

*uniformly with respect to  $k$ . Let  $\varepsilon > 0$ . For  $\alpha > \frac{1}{2}$  chosen sufficiently close to  $\frac{1}{2}$ . Then,*

$$\|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(H_z^{-\alpha} L_{y'}^2, H_z^\alpha L_{y'}^2)} = o(h_k^{-\varepsilon})_{k \rightarrow +\infty}.$$

A proof is given below.

As  $\tilde{\kappa}_k$  and  $g_k^{dd}$  and their inverses are Lipschitz uniformly with respect to  $k$ , with Lemma 9.5 one finds that  $L_k^\delta(q_k p_k, \psi) = o(h_k^{1-\varepsilon})_{k \rightarrow +\infty}$  for any  $0 < \varepsilon < 1$ , which concludes the proof of Proposition 9.1.  $\square$

**Proof of Lemma 9.3.** Let  $f_k$  be as in the statement. Both operators  $[Q_k, f_k]$  and  $h_k Q_k$  are bounded in  $\mathcal{L}(L^2(\mathbb{R}^{d+1}))$  by  $Ch_k$  by Lemma 8.5 for the first one and Lemma 8.4 for the second one, recalling the properties of  $q_k$  given in (8.4)–(8.5).

To estimate their operator norm from  $L^2(\mathbb{R}^{d+1})$  to  $H^1(\mathbb{R}_z; L^2(\mathbb{R}_{y'}^d))$  we compose with  $D_z = h_k^{-1} \text{Op}^h(\zeta)$ . On the one hand one gets

$$D_z[Q_k, f_k] = \text{Op}^h(D_z q_k) f_k - \text{Op}^h(D_z(f_k q_k)) + h_k^{-1}[\text{Op}^h(\zeta q_k), f_k].$$

The first two operators are bounded in  $\mathcal{L}(L^2(\mathbb{R}^{d+1}))$  uniformly with respect to  $k$  by Lemma 8.4. For the third operator, using that  $\zeta q_k$  is of the form given in Lemma 8.5 by the polyhomogeneous expansion of  $q_k$  given in (8.5), one also finds a  $k$ -uniform bound in  $\mathcal{L}(L^2(\mathbb{R}^{d+1}))$ . On the other hand, one has

$$D_z h_k Q_k = h_k \text{Op}^h(D_z q_k) + \text{Op}^h(\zeta q_k),$$

that also has a  $k$ -uniform bound in  $\mathcal{L}(L^2(\mathbb{R}^{d+1}))$  by Lemma 8.4. The first two points of the lemma are proven.

For the third point, we provide the proof for  $G'_k$  and  $Q_k$ . The proof of  $\bar{G}'_k$  and  $\bar{Q}_k$  is identical. One sees that it suffices to prove that  $\text{Op}^h(q_k a'_k) - Q_k A'_k$  is a finite sum of operators that lie in the space given by (9.1), with

$$a'_k = \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} g_k^{ij} \xi_i \xi_j \quad \text{and} \quad A'_k = \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} \tilde{\kappa}_k^{-1} h_k D_i \tilde{\kappa}_k g_k^{ij}(x) h_k D_j.$$

One writes

$$A'_k = \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} (g_k^{ij} h_k^2 D_i D_j + h_k \tilde{\kappa}_k^{-1} (D_i (g_k^{ij} \tilde{\kappa}_k)) h_k D_j),$$

and

$$\begin{aligned} Q_k g_k^{ij} h_k^2 D_i D_j &= g_k^{ij} Q_k h_k^2 D_i D_j + [Q_k, g_k^{ij}] h_k^2 D_i D_j \\ &= \text{Op}^h(q_k g_k^{ij} \xi_i \xi_j) + [Q_k, g_k^{ij}] h_k^2 D_i D_j, \end{aligned}$$

yielding

$$Q_k A'_k = \text{Op}^h(q_k a'_k) + \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} [Q_k, g_k^{ij}] h_k^2 D_i D_j + h_k Q_k \tilde{\kappa}_k^{-1} (D_i (g_k^{ij} \tilde{\kappa}_k)) h_k D_j.$$

The result thus amounts to having  $[Q_k, g_k^{ij}] \in \mathcal{R}_0$  and  $h_k Q_k \in \mathcal{R}_0$ , which holds by the first two points of the lemma proven above. This concludes the proof of the third point of Lemma 9.3.

We now turn to the proof of the fourth point. Since  $[\partial_t^2, \varphi_k] = 0$  it suffices to consider  $Q_k [A'_k, \varphi_k]$ . One writes

$$A'_k = \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} (g_k^{ij} h_k^2 D_i D_j + h_k \tilde{\kappa}_k^{-1} (D_i (g_k^{ij} \tilde{\kappa}_k)) h_k D_j),$$

yielding

$$\begin{aligned} [A'_k, \varphi_k] &= \sum_{\substack{1 \leq i, j \leq d \\ (i, j) \neq (d, d)}} \left( [g_k^{ij}, \varphi_k] h_k^2 D_i D_j + h_k \tilde{\kappa}_k^{-1} (D_i (g_k^{ij} \tilde{\kappa}_k)) \varphi_k h_k D_j \right. \\ &\quad \left. - h_k \varphi_k \tilde{\kappa}_k^{-1} (D_i (g_k^{ij} \tilde{\kappa}_k)) h_k D_j \right). \end{aligned}$$

Since  $Q_k$  is bounded on  $L^2(\mathbb{R}^{d+1})$  and also bounded by  $Ch_k^{-1}$  from  $L^2(\mathbb{R}^{d+1})$  to  $H^1(\mathbb{R}_z; L^2(\mathbb{R}_y^d))$  uniformly in  $h_k$  it suffices to prove that  $[g_k^{ij}, \varphi_k]$  is bounded by  $Ch_k$  on  $L^2(\mathbb{R}^{d+1})$ . This is a consequence of Lemma 9.2.  $\square$

**Proof of Lemma 9.4.** First, we prove

$$(9.3) \quad \|Q_k \varphi_k [M_k(y, D_y), \psi] u_k\|_{L^2(\mathbb{R}^{d+1})} \lesssim h_k^N.$$

Second, we prove

$$(9.4) \quad \|D_z Q_k \varphi_k [M_k(y, D_y), \psi] u_k\|_{L^2(\mathbb{R}^{d+1})} \lesssim h_k^N.$$

Together (9.3) and (9.4) give the sought result.

*Proof of (9.3).* Note that  $[M_k(y, D_y), \psi]u_k$  is bounded in  $L^2(\mathbb{R}^{d+1})$  by  $Ch_k^{1-m}$ , where  $m$  is the order of  $M_k(y, D_y)$ , by (7.5).

With the polyhomogeneous development in the  $\zeta$  variable of  $q_k$  given in (8.5)–(8.6) one writes

$$q_k(y, \eta', \zeta) = \frac{1 - \phi(\zeta)}{\zeta} q_k^1(y, \eta') + q_k^a(y, \eta', \zeta),$$

with

$$(9.5) \quad \left| \partial_y^\alpha \partial_{\eta'}^\beta \partial_\zeta^\delta q_k^a(y, \eta', \zeta) \right| \leq C_{N, \beta, \delta} \langle \eta' \rangle^{-N} \langle \zeta \rangle^{-2-\delta},$$

for  $N \in \mathbb{N}$ ,  $\alpha \in \mathbb{N}^{d+1}$ ,  $|\alpha| \leq 1$ ,  $\beta \in \mathbb{N}^d$ ,  $\delta \in \mathbb{N}$ ,  $y \in \mathbb{R}^{d+1}$ ,  $(\eta', \zeta) \in \mathbb{R}^{d+1}$ .

One writes  $Q_k = \text{Op}^h(q_k^1) \text{Op}^h\left(\frac{1-\phi(\zeta)}{\zeta}\right) + \text{Op}^h(q_k^a)$ .

Recall that we work in the local chart  $(\mathcal{O}, \phi_{\mathcal{L}})$  at the boundary. Since  $\psi = 1$  in a neighborhood of the  $y$ -projection of  $\text{supp } b$ , note that  $\text{supp}([M_k(y, D_y), \psi]u_k)$  does not meet the  $y$ -projection of  $\text{supp } q_k$  since  $\text{supp } q_k \subset \text{supp } b$ . Let  $\tilde{\psi}, \hat{\psi} \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  with  $\hat{\psi}$  equal to 1 in a neighborhood of  $\text{supp } \tilde{\psi}$  and with  $\tilde{\psi}$  equal to 1 in a neighborhood of the  $y$ -projection of  $\text{supp } b$  and moreover  $\psi = 1$  in a neighborhood of  $\text{supp } \hat{\psi}$ . One has  $[M_k(y, D_y), \psi] = (1 - \hat{\psi})[M_k(y, D_y), \psi]$  and

$$(9.6) \quad \|\tilde{\psi} \varphi_k(1 - \hat{\psi})\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} + \|\tilde{\psi} \text{Op}^h(\zeta^{-1}(1 - \phi(\zeta))) \varphi_k(1 - \hat{\psi})\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \\ \leq C_N h_k^N, \quad N \in \mathbb{N},$$

by standard calculus. Since  $\text{Op}^h(q_k^1)$  and  $\text{Op}^h(q_k^a)$  are bounded on  $L^2(\mathbb{R}^{d+1})$ , to obtain (9.3) it thus suffices to study the  $L^2$ -boundedness of the operators  $\text{Op}^h(q_k^1)(1 - \tilde{\psi})$  and  $\text{Op}^h(q_k^a)(1 - \tilde{\psi})$ .

The tangential kernel of  $\text{Op}^h(q_k^1)(1 - \tilde{\psi})$  is given by

$$K(y', \tilde{y}') = (2\pi)^{-d} \int e^{i(y' - \tilde{y}') \cdot \eta'} (1 - \tilde{\psi}(\tilde{y}', z)) q_k^1(y', z, h_k \eta') d\eta'.$$

With the joint support properties of  $q_k^1$  and  $\tilde{\psi}$  one finds that  $\|y' - \tilde{y}'\| \geq C > 0$  in the support of the integrand. Since  $L \exp(i(y' - \tilde{y}') \cdot \eta') = \exp(i(y' - \tilde{y}') \cdot \eta')$  with  $L = -i\|y' - \tilde{y}'\|^{-2}(y' - \tilde{y}') \cdot \partial_{\eta'}$  one can write

$$K(y', \tilde{y}') = (2\pi)^{-d} \int e^{i(y' - \tilde{y}') \cdot \eta'} (1 - \tilde{\psi}(\tilde{y}', z)) ({}^t L)^N q_k^1(y', z, h_k \eta') d\eta'.$$

With the estimation (8.5) for  $q_k^1$  one finds

$$|K(y', \tilde{y}')| \lesssim \frac{h_k^N}{\langle y' - \tilde{y}' \rangle^N},$$

which, by the Schur lemma (Lemma 5.1), gives

$$(9.7) \quad \|\text{Op}^h(q_k^1)(1 - \tilde{\psi})\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k^N.$$

The kernel of  $\text{Op}^h(q_k^a)(1 - \tilde{\psi})$  is given by

$$K(y, \tilde{y}) = (2\pi)^{-d-1} \int e^{i(y-\tilde{y})\cdot\eta} (1 - \tilde{\psi}(\tilde{y})) q_k^a(y, h_k\eta) d\eta.$$

Here,  $\|y - \tilde{y}\| \geq C > 0$  in the support of the integrand, yielding

$$K(y, \tilde{y}) = (2\pi)^{-d-1} \int e^{i(y-\tilde{y})\cdot\eta} (1 - \tilde{\psi}(\tilde{y})) ({}^tL)^N q_k^a(y, h_k\eta) d\eta,$$

with  $L = -i\|y - \tilde{y}\|^{-2}(y - \tilde{y}) \cdot \partial_\eta$ , implying with (9.5)

$$|K(y, \tilde{y})| \lesssim \frac{h_k^N}{\langle y - \tilde{y} \rangle^N},$$

which, by the Schur lemma, gives

$$(9.8) \quad \|\text{Op}^h(q_k^a)(1 - \tilde{\psi})\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k^N.$$

Together (9.7) and (9.8) give estimate (9.3).

*Proof of (9.4).* Here, we write

$$q_k(y, \eta', \zeta) = \frac{1 - \phi(\zeta)}{\zeta} q_k^1(y, \eta') + \frac{1 - \phi(\zeta)}{\zeta^2} q_k^2(y, \eta') + q_k^b(y, \eta', \zeta),$$

with

$$\left| \partial_y^\alpha \partial_{\eta'}^\beta \partial_\zeta^\delta q_k^b(y, \eta', \zeta) \right| \leq C_{N,\beta,\delta} \langle \eta' \rangle^{-N} \langle \zeta \rangle^{-3-\delta},$$

$$\text{for } N \in \mathbb{N}, \alpha \in \mathbb{N}^{d+1}, |\alpha| \leq 1, \beta \in \mathbb{N}^d, \delta \in \mathbb{N}, y \in \mathbb{R}^{d+1}, (\eta', \zeta) \in \mathbb{R}^{d+1}.$$

One writes

$$Q_k = \text{Op}^h(q_k^1) \text{Op}^h\left(\frac{1 - \phi(\zeta)}{\zeta}\right) + \text{Op}^h(q_k^2) \text{Op}^h\left(\frac{1 - \phi(\zeta)}{\zeta^2}\right) + \text{Op}^h(q_k^b).$$

One has

$$\begin{aligned} D_z Q_k &= \text{Op}^h(D_z q_k^1) \text{Op}^h\left(\frac{1-\phi(\zeta)}{\zeta}\right) + \text{Op}^h(D_z q_k^2) \text{Op}^h\left(\frac{1-\phi(\zeta)}{\zeta^2}\right) + \text{Op}^h(D_z q_k^b) \\ &\quad + h_k^{-1} \text{Op}^h(q_k^1) \text{Op}^h(1-\phi(\zeta)) + h_k^{-1} \text{Op}^h(q_k^2) \text{Op}^h\left(\frac{1-\phi(\zeta)}{\zeta}\right) \\ &\quad + h_k^{-1} \text{Op}^h(\zeta q_k^b). \end{aligned}$$

Similarly to (9.6) one has

$$\|\tilde{\psi} \text{Op}^h(\zeta^{-2}(1-\phi(\zeta)))\varphi_k(1-\hat{\psi})\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \leq C_N h_k^N, \quad N \in \mathbb{N}.$$

Observe that  $\zeta q_k^b$  has the same symbol properties as  $q_k^a$ . The symbol properties of  $D_z q_k^1$ ,  $D_z q_k^2$ ,  $D_z q_k^b$  also allow one to carry out the same kernel estimations as above yielding (9.4).  $\square$

**Proof of Lemma 9.5.** We claim that

$$(9.9) \quad \|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(H_z^{-1/2} L_{y'}^2, H_z^{1/2} L_{y'}^2)} = O(1),$$

and

$$(9.10) \quad \|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(H_z^{-1}(L_{y'}^2), (H_z^1(L_{y'}^2)))} = O(h_k^{-4}).$$

Interpolation of the two estimations then gives the result of the lemma.

We now prove the claimed estimates.

*Proof of estimate (9.9).* First, one has

$$\begin{aligned} &Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k \\ &= Q_k \varphi_k \psi f_k^{-1} f_k - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k \\ &= [Q_k \varphi_k, \psi f_k^{-1}] f_k + f_k^{-1} \psi (\text{Op}^h(q_k \varphi(h_k^2 \zeta)) - \text{Op}^h(\bar{q}_k \varphi(h_k^2 \zeta))^*) f_k. \end{aligned}$$

With Lemmata 8.5 and 8.5' one has

$$\begin{aligned} &\|[Q_k \varphi_k, \psi f_k^{-1}]\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \\ &\quad + \|\text{Op}^h(q_k \varphi(h_k^2 \zeta)) - \text{Op}^h(\bar{q}_k \varphi(h_k^2 \zeta))^*\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k, \end{aligned}$$

yielding

$$(9.11) \quad \|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k.$$

Second one writes

$$\begin{aligned} & D_z(Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k) \\ &= \text{Op}^h(D_z q_k) \varphi_k \psi - (D_z(\psi f_k^{-1})) \varphi_k \bar{Q}_k^* f_k \\ & \quad + h_k^{-1}(\text{Op}^h(\zeta q_k) \varphi_k \psi - \psi f_k^{-1} \varphi_k \text{Op}^h(\zeta \bar{q}_k)^* f_k). \end{aligned}$$

With Lemma 8.4 one finds

$$\|\text{Op}^h(D_z q_k) \varphi_k \psi\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} + \|(D_z(\psi f_k^{-1})) \varphi_k \text{Op}^h(\bar{q}_k)^* f_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim 1.$$

Arguing as for (9.11) one finds

$$\|\text{Op}^h(\zeta q_k) \varphi_k \psi - \psi f_k^{-1} \varphi_k \text{Op}^h(\zeta \bar{q}_k)^* f_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim h_k.$$

This gives

$$(9.12) \quad \|D_z(Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k)\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}))} \lesssim 1$$

Together, (9.11) and (9.12) give

$$(9.13) \quad \|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}), H_z^1 L_{y'}^2)} \lesssim 1.$$

By duality this implies

$$(9.14) \quad \|Q_k \varphi_k \psi - \psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(H_z^{-1} L_{y'}^2, L^2(\mathbb{R}^{d+1}))} \lesssim 1.$$

An interpolation of (9.13) and (9.14) yields (9.9).

*Proof of estimate (9.10).* Above we computed  $D_z Q_k = \text{Op}^h(D_z q_k) + h_k^{-1} \text{Op}^h(q_k \zeta)$  yielding

$$\|Q_k\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}), H_z^1 L_{y'}^2)} \lesssim h_k^{-1},$$

and thus

$$(9.15) \quad \|Q_k \varphi_k \psi\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}), H_z^1 L_{y'}^2)} \lesssim h_k^{-1}.$$

One also computes

$$D_z Q_k \varphi_k \psi D_z = D_z Q_k \varphi_k D_z \psi - D_z Q_k \varphi_k (D_z \psi).$$



Since  $\varphi_k D_z = h_k^{-3} \text{Op}^h(\varphi(h_k^2 \zeta) h_k^2 \zeta)$  one finds that  $\varphi_k D_z$  is bounded on  $L^2(\mathbb{R}^{d+1})$  by  $Ch_k^{-3}$  thus yielding

$$(9.16) \quad \|Q_k \varphi_k \psi D_z\|_{\mathcal{L}(L^2(\mathbb{R}^{d+1}), H_z^1 L_{y'}^2)} \lesssim h_k^{-4}.$$

Together (9.15) and (9.16) give

$$(9.17) \quad \|Q_k \varphi_k \psi\|_{\mathcal{L}(H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2)} \lesssim h_k^{-4}.$$

The same holds for  $\bar{Q}_k$  in place of  $Q_k$  and by duality one obtains

$$(9.18) \quad \|\psi f_k^{-1} \varphi_k \bar{Q}_k^* f_k\|_{\mathcal{L}(H_z^{-1} L_{y'}^2, H_z^1 L_{y'}^2)} \lesssim h_k^{-4},$$

and together (9.17) and (9.18) yield (9.10).  $\square$

**Remark 9.6.** *Note that the proof we give of (9.10) is far from optimal. However, this has no consequence on the final result of Lemma 9.5.*

**9.2. Contributions of  $b_{0,k}$  and  $b_{1,k}$ .** First, we prove that the symbol  $b_{0,k}(y, \eta')$  yields a vanishing contribution to the limit of  $L_k''(\chi(\tau)b, \psi)$ . Second, we prove that the symbol  $b_{1,k}(y, \eta')$  yields a contribution to the limit of  $L_k''(\chi(\tau)b, \psi)$  as opposed to the other symbols appearing in the Euclidean division of Proposition 8.2. This contribution implies the action of the semi-classical measure  $\nu$  at the boundary.

The tangential nature of  $\text{Op}^h(b_{0,k})$  and  $\text{Op}^h(b_{1,k})$  allows one to consider traces through the action of the Dirac measure  $\delta_{z=0}$ . A key point of the proof of this section is the understanding of traces after the action of the regularizing operator  $\varphi_k$ .

Consider  $w \in L^2(\mathbb{R}^{d+1})$  such that  $w^+ = w|_{\{z>0\}} \in H^1(\mathbb{R}_z^+; L^2(\mathbb{R}_{y'}^d))$  and  $w^- = w|_{\{z<0\}} \in H^1(\mathbb{R}_z^-; L^2(\mathbb{R}_{y'}^d))$ . On the one hand,  $w^+ \in \mathcal{C}^0([0, +\infty[; L^2(\mathbb{R}_{y'}^d))$  and  $w|_{z=0^+} = w^+|_{z=0^+} = \lim_{z \rightarrow 0^+} w(z)$  makes sense in  $L^2(\mathbb{R}_{y'}^d)$  classically. Similarly  $w|_{z=0^-} = w^-|_{z=0^-} = \lim_{z \rightarrow 0^-} w(z)$  makes sense. On the other hand, the trace of  $(\varphi_k w)|_{z=0}$  can be approximated by the mean of the two traces of  $w$  as in the following lemma.

**Lemma 9.7.** *Let  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$  be real valued and equal to 1 near 0. There exists  $C > 0$  such that*

$$\begin{aligned} \|\varphi(h^3 D_z) w|_{z=0} - \frac{1}{2}(w|_{z=0^-} + w|_{z=0^+})\|_{L^2(\mathbb{R}^d)} \\ \leq Ch^{\frac{3}{2}} (\|\partial_z w\|_{L^2(\mathbb{R}_z^-; L^2(\mathbb{R}_{y'}^d))} + \|\partial_z w\|_{L^2(\mathbb{R}_z^+; L^2(\mathbb{R}_{y'}^d))}), \end{aligned}$$

for  $h > 0$  and  $w \in L^2(\mathbb{R}^{d+1})$  such that  $w^\pm \in H^1(\mathbb{R}_z^\pm; L^2(\mathbb{R}_{y'}^d))$ .

**Proof.** By linearity and symmetry it suffices to consider  $w$  such that  $w_{\{z<0\}} = 0$  and prove<sup>3</sup>

$$(9.19) \quad \left\| \varphi(h^3 D_z)w|_{z=0} - \frac{1}{2}w|_{z=0^+} \right\|_{L^2(\mathbb{R}^d)} \leq Ch^{\frac{3}{2}} \|\partial_z w\|_{L^2(\mathbb{R}_z^+; L^2(\mathbb{R}_y^d))}.$$

Denote by  $\hat{\varphi}$  the inverse Fourier transform of  $\varphi$ . The Parseval formula gives

$$(9.20) \quad \begin{aligned} 2\pi\varphi(h^3 D_z)w|_{z=0} &= \int_{\mathbb{R}} \varphi(h^3 \zeta) \hat{w}(\zeta) d\zeta \\ &= \int_{\mathbb{R}} \hat{\varphi}(z) w(h^3 z) dz = \int_{\mathbb{R}^+} \hat{\varphi}(z) w(h^3 z) dz. \end{aligned}$$

For  $z \geq 0$ , with the Cauchy-Schwarz inequality one finds

$$(9.21) \quad \|w(z) - w|_{z=0^+}\|_{L^2(\mathbb{R}^d)} = \left\| \int_0^z \partial_z w(s) ds \right\|_{L^2(\mathbb{R}^d)} \leq z^{1/2} \|\partial_z w\|_{L^2(\mathbb{R}_z^+; L^2(\mathbb{R}_y^d))}.$$

Using that  $\hat{\varphi}$  is even since  $\varphi$  is real valued one has  $\int_{\mathbb{R}^+} \hat{\varphi} = \pi$  since  $\int_{\mathbb{R}} \hat{\varphi} = 2\pi\varphi(0) = 2\pi$ . With (9.20) and (9.21) one thus obtains

$$\begin{aligned} 2\pi \left\| \varphi(h^3 D_z)w|_{z=0} - \frac{1}{2}w|_{z=0^+} \right\|_{L^2(\mathbb{R}^d)} &= \left\| \int_{\mathbb{R}^+} \hat{\varphi}(z) (w(h^3 z) - w|_{z=0^+}) dz \right\|_{L^2(\mathbb{R}^d)} \\ &\leq h^{3/2} \|\partial_z w\|_{L^2(\mathbb{R}_z^+; L^2(\mathbb{R}_y^d))} \int_{\mathbb{R}^+} |\hat{\varphi}(z)| z^{1/2} dz \lesssim h^{3/2} \|\partial_z w\|_{L^2(\mathbb{R}_z^+; L^2(\mathbb{R}_y^d))}, \end{aligned}$$

which is the sought result (9.19).  $\square$

**Proposition 9.8.** *One has  $L_k''(b_{0,k}, \psi) = o(1)_{k \rightarrow +\infty}$ .*

**Proof.** Using that  $\|\psi u_k\|_{H^1(\mathbb{R}_z^+; L^2(\mathbb{R}_y^d))} = O(h_k^{-1})$  by (7.4) one writes

$$\begin{aligned} L_k''(b_{0,k}, \psi) &= i \left( \text{Op}^h(b_{0,k}|_{z=0}) (\varphi_k \psi u_k)|_{z=0^+}, v_k \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k} \partial} \\ &\quad - i \left( v_k, \text{Op}^h(b_{0,k}|_{z=0}) (\varphi_k \psi u_k)|_{z=0^+} \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k} \partial} \\ &= \frac{i}{2} \left( \text{Op}^h(b_{0,k}|_{z=0}) (\psi u_k)|_{z=0^+}, v_k \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k} \partial} \\ &\quad - \frac{i}{2} \left( v_k, \text{Op}^h(b_{0,k}|_{z=0}) (\psi u_k)|_{z=0^+} \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k} \partial} + O(h_k^{1/2}) \\ &= O(h_k^{1/2}), \end{aligned}$$

using the homogeneous Dirichlet boundary condition, that is,  $u_k|_{z=0^+} = 0$ .  $\square$

<sup>3</sup>In what follows, we will actually use Lemma 9.7 in the case of a function vanishing in  $\{z < 0\}$ .

**Proposition 9.9.** *One has*

$$L_k''(b_{1,k}\zeta, \psi) = (\text{Op}^h(b_{1,k}|_{z=0})((g_k^{dd})^{-1}\psi)|_{z=0}v_k, v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} + o(1)_{k \rightarrow +\infty}$$

**Proof.** With (7.6) one has  $\|h_k D_z \psi u_k\|_{H^1(\mathbb{R}_z^+; (L^2(\mathbb{R}_y^d)))} = O(h_k^{-1})$ , which gives

$$\begin{aligned} L_k''(b_{1,k}\zeta, \psi) &= i(\text{Op}^h(b_{1,k}|_{z=0})(\varphi_k h_k D_z \psi u_k)|_{z=0^+}, v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} \\ &\quad - i(v_k, \text{Op}^h(\bar{b}_{1,k}|_{z=0})(\varphi_k h_k D_z \psi u_k)|_{z=0^+})_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} \\ &= \frac{1}{2}(\text{Op}^h(b_{1,k}|_{z=0})(h_k \partial_z \psi u_k)|_{z=0^+}, v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} \\ &\quad + \frac{1}{2}(v_k, \text{Op}^h(\bar{b}_{1,k}|_{z=0})(h_k \partial_z \psi u_k)|_{z=0^+})_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} + O(h_k^{1/2}), \end{aligned}$$

by Lemma 9.7. With (7.7) one has  $v_k = h_k g_k^{dd} \partial_z u_k|_{z=0^+}$  yielding

$$h_k \partial_z \psi u_k|_{z=0^+} = h_k (\partial_z \psi)|_{z=0} u_k|_{z=0^+} + h_k \psi|_{z=0} (\partial_z u_k)|_{z=0^+} = (g_k^{dd})^{-1} \psi|_{z=0} v_k.$$

One then obtains

$$\begin{aligned} L_k''(b_{1,k}\zeta, \psi) &= \frac{1}{2}(\text{Op}^h(b_{1,k}|_{z=0})((g_k^{dd})^{-1}\psi)|_{z=0}v_k, v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} \\ &\quad + \frac{1}{2}(v_k, \text{Op}^h(\bar{b}_{1,k}|_{z=0})((g_k^{dd})^{-1}\psi)|_{z=0}v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} + O(h_k^{1/2}). \end{aligned}$$

One writes

$$\begin{aligned} &(v_k, \text{Op}^h(\bar{b}_{1,k}|_{z=0})((g_k^{dd})^{-1}\psi)|_{z=0}v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt} \\ &= (((g_k^{dd})^{-1}\psi)|_{z=0} \text{Op}^h(\bar{b}_{1,k}|_{z=0})^* v_k, v_k)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k\mu_{g_k\partial}dt}, \end{aligned}$$

with the adjoint operator with the  $\star$ -notation understood in the sense of the inner product  $L^2(\mathbb{R}^d, (\kappa_k)|_{z=0} \mu_{g_k\partial} dt)$ , that is,

$$\text{Op}^h(\bar{b}_{1,k}|_{z=0})^* = (\tilde{\kappa}_k^{-1})|_{z=0} \text{Op}^h(\bar{b}_{1,k}|_{z=0})^* (\tilde{\kappa}_k)|_{z=0},$$

where the adjoint with the  $\star$ -notation is understood in the sense of the inner product  $L^2(\mathbb{R}^d, dx' dt)$ . With the two points of Lemma 8.5 one finds

$$\begin{aligned} &\| \text{Op}^h(b_{1,k}|_{z=0})((g_k^{dd})^{-1}\psi)|_{z=0} \\ &\quad - (\psi \tilde{\kappa}_k^{-1} (g_k^{dd})^{-1})|_{z=0} \text{Op}^h(\bar{b}_{1,k}|_{z=0})^* (\tilde{\kappa}_k)|_{z=0} \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim h_k, \end{aligned}$$

yielding the result.  $\square$

**9.3. Proof conclusion and further support property of the measure  $\nu$ .** With Proposition 7.2, Corollary 8.10, and Propositions 9.1, 9.8, and 9.9 one now has

$$(9.22) \quad -\langle \mu, H_p b \rangle = 2\langle \text{Im } M_{0,1}, b \rangle \\ + \lim_{k \rightarrow \infty} \left( \text{Op}^h (b_{1,k}|_{z=0}) ((g_k^{dd})^{-1} \psi) \Big|_{z=0} v_k, v_k \Big)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k \partial} dt}.$$

Let  $b \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$ . With Proposition 8.2 one writes

$$\chi(\tau) b(y, \eta', \zeta) = b_{0,k}(y, \eta') + b_{1,k}(y, \eta') \zeta + q_k(y, \eta', \zeta) p_k(y, \eta', \zeta),$$

and

$$(9.23) \quad \chi(\tau) b(y, \eta', \zeta) = b_0(y, \eta') + b_1(y, \eta') \zeta + q(y, \eta', \zeta) p(y, \eta', \zeta).$$

With (8.8) and (8.9) one finds that  $N_{d+1}(b_1 - b_{1,k}) = o(1)_{k \rightarrow \infty}$ . Since also  $\|\tilde{\kappa}_k - \tilde{\kappa}\|_{L^\infty} = o(1)_{k \rightarrow \infty}$  one gets

$$\begin{aligned} & \left( \text{Op}^h (b_{1,k}|_{z=0}) ((g_k^{dd})^{-1} \psi) \Big|_{z=0} v_k, v_k \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa_k \mu_{g_k \partial} dt} \\ &= \left( \text{Op}^h (b_1|_{z=0}) ((g_k^{dd})^{-1} \psi) \Big|_{z=0} v_k, v_k \right)_{L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)}^{\kappa \mu_{g \partial} dt} + o(1)_{k \rightarrow +\infty}. \end{aligned}$$

Since the s.c.m. of  $(v_k)$  is  $\nu$ , with (9.22), Lemma 5.24 and (5.28), one obtains

$$(9.24) \quad -\langle \mu, H_p b \rangle = 2\langle \text{Im } M_{0,1}, b \rangle + \langle \nu, b_1|_{z=0} \rangle,$$

as  $\psi|_{z=0} = 1$  in a neighborhood of  $\text{supp}(b_1|_{z=0})$  and as  $\|(g_k^{dd})|_{z=0}^{-1} - 1\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow +\infty$  since  $g|_{z=0}^{dd} = 1$  in the chosen quasi-normal geodesic coordinates associated with the metric  $g$ ; see Proposition 2.1.

With the results obtained above, the description of  $\text{supp } \nu$  in (7.2) can be refined.

**Proposition 9.10.** *One has  $\text{supp } \nu \subset (\|\mathcal{H}_\partial \cup \|\mathcal{G}_\partial) \cap \{C_{\mu,0} \leq \tau \leq C_{\mu,1}\}$ .*

**Proof.** The inclusion  $\tau \in [C_{\mu,0}, C_{\mu,1}]$  is given in (7.2).

Consider  $a(y, \eta') \in \mathcal{C}_c^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^d)$  supported in a neighborhood of  $\{z = 0\}$  and with  $a(y', z = 0, \eta')$  supported in the elliptic region  $\|\mathcal{E}_\partial = (\|\mathcal{H}_\partial \cup \|\mathcal{G}_\partial)^c$ . One has

$$\pi_{\parallel}(\text{Char } p \cap \{z = 0\}) \cap \text{supp } a|_{z=0} = \emptyset.$$

See Section 2.3. If the support of  $a$  is chosen sufficiently small in the  $z$  variable, then this remains true away from  $\{z = 0\}$ , in the sense that

$$a(y', z, \eta') \neq 0 \quad \Rightarrow \quad p(y', z, \eta', \zeta) \neq 0 \quad \forall \zeta \in \mathbb{R}.$$

Because of the homogeneity of  $p = p_{\kappa, g}$  and  $p_k = p_{\kappa_k, g_k}$ , by compactness, the same property holds for  $p_k$  in place of  $p$  for  $k$  chosen sufficiently large. For such choice one can set

$$q_k(y, \eta) = -\frac{a(y, \eta')\zeta}{p_k(z, \zeta)}.$$

It is Lipschitz in  $y$ , smooth and compactly supported in  $\eta'$  and admits a polyhomogeneous development in the  $\zeta$  variable as in (8.5)–(8.6). In fact, it reads

$$0 = a(y, \eta')\zeta + q_k(y, \eta)p_k(z, \zeta),$$

precisely of the form given by the Euclidean symbol division of Proposition 8.2, with  $b = 0$ ,  $b_{0,k} = 0$ , and  $b_{1,k} = a$ .

With (9.22) and the definition of the semi-classical measure  $\nu$ , one finds

$$\langle \nu, a|_{z=0} \rangle = -\langle \mu, 0 \rangle = 0.$$

This gives the result considering the support property of  $a|_{z=0}$ .  $\square$

Let  $b \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  as above, with  $b_1(y, \eta')$  defined by (9.23). Let  $\varrho' = (y, z = 0, \eta') \in \text{supp } \nu$ . With Proposition 9.10 one has  $\varrho' \in \parallel \mathcal{H}_\partial \cup \parallel \mathcal{G}_\partial$  and  $\tau \in [C_{\mu,0}, C_{\mu,1}]$ . Let  $\zeta^\pm$  be defined as in (2.3) and  $\varrho^\pm = (\varrho', \zeta^\pm) \in \mathcal{H}_\partial^\pm \cup \mathcal{G}_\partial$ . With (9.23) one finds

$$b(\varrho^+) - b(\varrho^-) = b_1(\varrho')(\zeta^+ - \zeta^-) \quad \text{if } \varrho' \in \text{supp } \nu.$$

Hence, in  $\text{supp } \nu$  the function  $(b(\varrho^+) - b(\varrho^-))/(\zeta^+ - \zeta^-)$  is well defined even for points  $\varrho' \in \parallel \mathcal{G}_\partial$ . One has

$$b_1(\varrho') = \frac{b(\varrho^+) - b(\varrho^-)}{\zeta^+ - \zeta^-} = \frac{\langle \delta_{\varrho^+} - \delta_{\varrho^-}, b \rangle}{\langle \xi^+ - \xi^-, \mathbf{n}_x \rangle_{T_x^* \mathcal{M}, T_x \mathcal{M}}}, \quad \varrho' \in \text{supp } \nu,$$

since  $\mathbf{n}_x$  is here the unitary inward pointing normal vector in the sense of the metric  $g$ . Hence, from (9.24) and Proposition 9.10 one concludes the proof of Theorem 6.1.  $\square$

## 10. MEASURE EQUATION AT ISOCHRONES AND NECESSARY GEOMETRIC CONTROL CONDITION

In this section and in Section 11, we suppose  $(\mathcal{M}, \kappa, g) \in \mathcal{X}^1$  chosen fixed, and write  $p = p_{\kappa, g}$ . Thus,  $H_p = H_{p_{\kappa, g}}$ .

**10.1. Necessary geometric control condition.** The geometric conditions we formulate here state that given any point of  $\varrho^0 \in T^*\mathcal{L}$  at least one bicharacteristic that goes through  $\varrho^0$  reaches a point above the observation region.

**Definition 10.1** (weak interior geometric control condition). *Let  $\omega$  be an open subset of  $\mathcal{M}$  and  $T > 0$ . One says that  $(\omega, T)$  fulfills the weak interior geometric control condition if for any  $\varrho^0 \in \text{Char } p \cap T^*\mathcal{L}$  and for any neighborhood  $V$  of  $[0, T] \times \bar{\omega}$ , at least one generalized bicharacteristic that goes through  $\varrho^0$  reaches a point above  $V$ .*

**Definition 10.2** (weak boundary geometric control condition). *Let  $\Gamma$  be an open subset of  $\partial\mathcal{M}$  and  $T > 0$ . One says that  $(\Gamma, T)$  fulfills the weak boundary geometric control condition if for any  $\varrho^0 \in \text{Char } p \cap T^*\mathcal{L}$  and any neighborhood  $V_\partial$  of  $[0, T] \times \bar{\Gamma}$ , at least one generalized bicharacteristic that goes through  $\varrho^0$  encounters a boundary escape point (see Definition 2.11) above  $V_\partial$ .*

The following theorem states the result of Theorem 1.10 in the framework of the precise Definitions 10.1 and 10.2.

**Theorem 10.3.** (1) *Interior observability (Definition 1.4) implies the weak interior geometric control condition.*  
 (2) *Boundary observability (Definition 1.5) implies the weak boundary geometric control condition.*

This theorem is proven in Section 10.5. Its proof uses a measure equation similar to that of Theorem 6.1, yet across isochrones  $\{t = \text{Cst}\}$ , and the construction of concentrating initial conditions.

**Remark 10.4.** • *In the case of uniqueness of generalized bicharacteristics, the weak geometric control condition stated here coincides with the usual necessary condition for observability to hold.*  
 • *If one replaces the rough cut-off  $\mathbf{1}_{[0, T] \times \omega}$  (resp.  $\mathbf{1}_{[0, T] \times \Gamma}$ ) by the smoother version  $\mathbf{1}_{[0, T] \times \Theta}(x)$ , then the (properly modified) geometric control condition is a necessary and sufficient condition in the case of uniqueness of generalized bicharacteristics (see [7]). However, when uniqueness does not hold this is no more the case as there is still the discrepancy between the two conditions (necessary: at least one generalized bicharacteristic reaches the set  $[0, T] \times \{\Theta > 0\}$ ; sufficient: all generalized bicharacteristics reach this set).*

For a given point  $\varrho^0 = (x^0, t^0, \tau^0, \xi^0) \in T^*\mathcal{L}$ , the proof of Theorem 10.3 requires the construction of a sequence of initial data spectrally localized such that the measure of the associated sequence of solutions is supported on generalized bicharacteristics passing through  $\varrho^0$ . This is performed in several steps:

- By an explicit calculation, it is possible to do so if one forgets the spectral localization (See Section 10.3).
- We then apply the spectral dyadic projector. Here, the difficulty comes from the low regularity assumptions on the coefficients (see Section 10.4).
- We prove a transport equation which allows one to transfer the information on the traces of the solutions at  $t = t^0$  to  $\{t > t^0\}$  (see Section 10.5).

**10.2. Measure equation at isochrones.** With  $\underline{t} \in \mathbb{R}$ , we consider the isochrone  $\mathcal{I} = \{t = \underline{t}\}$  in  $\mathcal{L}$ . We naturally identify  $\mathcal{I}$  with  $\mathcal{M}$ , and  $T^*\mathcal{M}$  with  $T^*\mathcal{I}$ . For  $(x, \xi) \in T^*\mathcal{M}$ , identified with  $\varrho = (\underline{t}, x, \tau = 0, \xi)$ , the polynomial  $\tau \mapsto p(\underline{t}, x, \tau, \xi)$  has exactly two roots  $\tau^+(\varrho) > 0$  and  $\tau^-(\varrho) = -\tau^+(\varrho) < 0$ . If compared to Section 2.3 one only faces hyperbolic points in the present situation. Set

$$\varrho^\oplus = (\underline{t}, x, \tau^+(\varrho), \xi), \quad \varrho^\ominus = (\underline{t}, x, \tau^-(\varrho), \xi).$$

Denote by  $a_{\kappa,g}(x, \xi)$  the principal symbol of  $A_{\kappa,g}$ , that is,  $a_{\kappa,g}(x, \xi) = -g_x^{ij}\xi_i\xi_j$  in local coordinates. Suppose  $H = (h_k)$  is a scale. For each  $k$ , suppose  $\underline{u}_k^0 \in H_0^1(\mathcal{M})$ ,  $\underline{u}_k^1 \in L^2(\mathcal{M})$ ,  $f_k \in L_{\text{loc}}^2(\mathcal{L})$ , and  $u_k$  is a weak solution to

$$\begin{cases} P_{\kappa,g} u_k = f_k & \text{in } \mathbb{R} \times \mathcal{M}, \\ u_k = 0 & \text{in } \mathbb{R} \times \partial\mathcal{M}, \\ u_k|_{t=\underline{t}} = \underline{u}_k^0, \quad \partial_t u_k|_{t=\underline{t}} = \underline{u}_k^1 & \text{in } \mathcal{M}. \end{cases}$$

One extends the diffent functions by zero outside  $\mathcal{M}$  and  $\mathcal{L}$ . Suppose the following holds.

**Assumption 10.5.** (1) *The sequences  $(\underline{u}_k^0)_k$  and  $(h_k \underline{u}_k^1)_k$  are both bounded in  $L^2(\hat{\mathcal{M}})$  and  $\underline{U}_k = {}^t(\underline{u}_k^0, h_k \underline{u}_k^1)$  admits at scale  $H$  the Hermitian s.c.m. on  $T^*\hat{\mathcal{M}}$*

$$\nu^0 = \begin{pmatrix} \nu_{0,0}^0 & \nu_{0,1}^0 \\ \nu_{1,0}^0 & \nu_{1,1}^0 \end{pmatrix}$$

*supported away from  $\partial\mathcal{M}$ .*

(2) *The sequences  $(u_k)_k$  and  $(h_k f_k)_k$  are both bounded in  $L_{\text{loc}}^2(\hat{\mathcal{L}})$ , and  ${}^t(u_k, h_k f_k)_k$  admits at scale  $H$  the Hermitian s.c.m. on  $T^*\hat{\mathcal{L}}$*

$$M = \begin{pmatrix} M_{0,0} & M_{0,1} \\ M_{1,0} & M_{1,1} \end{pmatrix}.$$

Set  $\mu = M_{0,0}$ .

- (3) No mass leaks at infinity at scale  $H$  for  $(\psi(t)u_k)_k$  and  $(h_k\psi(t)f_k)_k$ , for any  $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ , and there exists  $C > 0$  such that, for any interval  $I \subset \mathbb{R}$ ,

$$(10.1) \quad \|u_k\|_{L^2(I \times \mathcal{M})} + \|h_k f_k\|_{L^2(I \times \mathcal{M})} \leq C|I|, \quad k \in \mathbb{N}.$$

- (4) One has

$$(10.2) \quad \text{supp } \mu \subset \text{Char } p \cap T^*\mathcal{L} \setminus 0 \quad \text{and} \quad \text{supp } \nu^0 \subset T^*\mathcal{M} \setminus 0.$$

The sequence  $(\mathbf{1}_{t > \underline{t}} u_k, \mathbf{1}_{t > \underline{t}} h_k f_k)_k$  admits at scale  $H$  a Hermitian s.c.m.  $M^+$  on  $T^*\hat{\mathcal{L}}$ , with the following natural connection with  $M$ .

**Lemma 10.6.** *One has  $M^+ = \mathbf{1}_{t > \underline{t}} M$ .*

A proof is given below. One sets  $\mu^+ = M_{0,0}^+ = \mathbf{1}_{t > \underline{t}} \mu$ .

At  $t = \underline{t}$  and away from  $\partial\mathcal{L}$  the measure  $\mu$  is solution to  $H_p \mu = 0$ . The measure equation we establish concerns  $\mu^+$  and involves  $M_{0,1}^+$  and the Hermitian measure  $\nu^0$ .

**Theorem 10.7.** *Suppose  $\Omega$  is an open subset of  $\mathcal{M}$  with  $\bar{\Omega} \cap \partial\mathcal{M} = \emptyset$ . In  $T^*(\mathbb{R} \times \Omega)$  one has*

$$(10.3) \quad H_p \mu^+ = -{}^t H_p \mu^+ = 2 \text{Im } M_{0,1}^+ + \int_{\varrho \in T^*\mathcal{M}} \frac{\delta_{\varrho^\oplus} - \delta_{\varrho^\ominus}}{\tau^+ - \tau^-} d(a_{\kappa,g} \nu_{0,0}^0 - \nu_{1,1}^0)(\varrho) \\ + \int_{\varrho \in T^*\mathcal{M}} (\delta_{\varrho^\oplus} + \delta_{\varrho^\ominus}) d \text{Im } \nu_{0,1}^0(\varrho),$$

in the sense of distributions.

A proof is given in Section 11.

**Remark 10.8.** *The open subset  $\Omega$  is introduced as the measure equation (10.3) is only proven to hold away from the boundary  $\partial\mathcal{L}$ .*

In the simpler context of the wave coefficients with constant coefficients, one can find in [17, Proposition 4.4] a result expressing the measure  $\mu$  by means of measures associated with initial conditions. In the more general context we have here, deriving a formula for  $\mu$  or  $\mu^+$  is not possible. Yet, the result of Theorem 10.7 provides a transport equation solved by  $\mu^+$ .



**Proof of Lemma 10.6.** For simplicity we consider  $\underline{t} = 0$  here without loss of generality.

We prove that  $M_{0,1}^+ = \mathbf{1}_{t>\underline{t}} M_{0,1}$ . The proof is the same for the other matrix entries. Set  $v_k = h_k f_k$ . Suppose that  $\beta \in \mathcal{C}_c^\infty(\mathbb{R})$  with  $0 \leq \beta \leq 1$  and  $\beta(0) = 1$ . Then, for  $\beta_n(t) = \beta(nt)$ , with Proposition 5.21 (adapted to Hermitian measures) and dominated convergence, one obtains

$$\lim_{k \rightarrow +\infty} (\beta_n u_k, v_k)_{L^2(\hat{\mathcal{L}})} = \langle M_{0,1}, \beta_n \rangle \xrightarrow{n \rightarrow \infty} \langle M_{0,1}, \mathbf{1}_{\{t=0\}} \rangle,$$

using that no mass leaks at infinity for both  $u_k$  and  $v_k$  by Assumption 10.5. By (10.1) one has  $|(\beta_n u_k, v_k)_{L^2(\hat{\mathcal{L}})}| \lesssim 1/n$  uniformly in  $k$ . Thus one finds

$$(10.4) \quad \mathbf{1}_{\{t=0\}} M_{0,1} = 0.$$

Suppose  $\chi \in \mathcal{C}^\infty(\mathbb{R})$  is such that  $0 \leq \chi \leq 1$  and  $\chi(t) = 0$  if  $t < 0$  and  $\chi(t) = 1$  if  $t > 1$ . Set  $\chi_n(t) = \chi(nt)$ . Consider  $b \in \Sigma_c(T^*\hat{\mathcal{L}})$ ,  $B^h$  a representative of  $[\text{Op}^h(b)]$ , and  $\psi \in \mathcal{C}_c^\infty(\hat{\mathcal{L}})$  with  $\psi = 1$  on the  $(t, x)$ -projection of  $\text{supp}(b)$ . One writes

$$\begin{aligned} & (B^h \psi \mathbf{1}_{t>0} u_k, \mathbf{1}_{t>0} v_k)_{L^2(\hat{\mathcal{L}})} \\ &= (B^h \psi \mathbf{1}_{t>0} u_k, (\mathbf{1}_{t>0} - \chi_n) v_k)_{L^2(\hat{\mathcal{L}})} + (B^h \psi (\mathbf{1}_{t>0} - \chi_n) u_k, \chi_n v_k)_{L^2(\hat{\mathcal{L}})} \\ &+ (B^h \psi \chi_n u_k, \chi_n v_k)_{L^2(\hat{\mathcal{L}})} \end{aligned}$$

Let  $\varepsilon > 0$ . Since  $B^h$  is bounded on  $L^2(\hat{\mathcal{L}})$ ,  $(u_k)_k$  and  $(v_k)_k$  are bounded in  $L_{\text{loc}}^2(\hat{\mathcal{L}})$ , by (10.1) there exists  $n_0 \in \mathbb{N}$  such that,

$$|(B^h \psi \mathbf{1}_{t>0} u_k, (\mathbf{1}_{t>0} - \chi_n) v_k)_{L^2(\hat{\mathcal{L}})} + (B^h \psi (\mathbf{1}_{t>0} - \chi_n) u_k, \chi_n v_k)_{L^2(\hat{\mathcal{L}})}| \leq \varepsilon,$$

uniformly in  $k$ , for  $n \geq n_0$ . There exists also  $n_1 \geq n_0$  such that

$$|\langle M_{0,1}, b(\mathbf{1}_{t>0} - \chi_n^2) \rangle| \leq \varepsilon$$

for  $n \geq n_1$  by dominated convergence using (10.4). One thus concludes that

$$\begin{aligned} & |(B^h \psi \mathbf{1}_{t>0} u_k, \mathbf{1}_{t>0} v_k)_{L^2(\hat{\mathcal{L}})} - \langle M_{0,1}, \mathbf{1}_{t>0} b \rangle| \\ & \leq 2\varepsilon + |(B^h \psi \chi_n u_k, \chi_n v_k)_{L^2(\hat{\mathcal{L}})} - \langle M_{0,1}, \chi_n^2 b \rangle|, \end{aligned}$$

for  $n \geq n_1$  and  $k \in \mathbb{N}$ . Set  $n = n_1$ . Then, there exists  $k_0 \in \mathbb{N}$  such that

$$|(B^h \psi \mathbf{1}_{t>0} u_k, \mathbf{1}_{t>0} v_k)_{L^2(\hat{\mathcal{L}})} - \langle M_{0,1}, \mathbf{1}_{t>0} b \rangle| \leq 3\varepsilon,$$

for  $k \geq k_0$  implying the result.  $\square$

**Remark 10.9.** Similarly one proves that the off-diagonal entries of the measures of  $f^t(\mathbf{1}_{t>0} u_k, h_k f_k)$  and  ${}^t(u_k, \mathbf{1}_{t>0} h_k f_k)$  are also given by  $M_{0,1}^+ = \mathbf{1}_{t>0} M_{0,1}$  and  $M_{1,0}^+ = \mathbf{1}_{t>0} M_{1,0}$ .

**10.3. Concentration at a point.** Pick  $\psi \in \mathcal{S}(\mathbb{R}^d)$ ,  $x^0 \in \mathbb{R}^d$ , and  $\xi^0 \in \mathbb{R}^d \setminus 0$ . Set

$$w_h(x) = h^{-d/4} e^{ix \cdot \xi^0 / h} \psi(h^{-1/2}(x - x^0)).$$

One has  $\|w_h\|_{L^2} = \|\psi\|_{L^2}$  and  $(w_h)_h$  admits the measure  $\|\psi\|_{L^2}^2 \delta_{(x^0, \xi^0)}$  as its semiclassical measure (at scale  $h$ ); this follows from computations based on oscillatory-integral arguments.

**Lemma 10.10.** *For all  $s \in \mathbb{R}$  one has  $\|(-\Delta)^{s/2} w_h\|_{L^2} \sim h^{-s} |\xi^0|^s \|\psi\|_{L^2}$ .*

**Proof.** Write  $\hat{w}_h(\xi) = h^{d/4} \hat{\psi}(h^{1/2}\xi - h^{1/2}\xi^0)$  assuming  $x^0 = 0$  without any loss of generality. One then computes

$$\| |\xi|^s \hat{w}_h \|_{L^2}^2 = h^{-2s} \int |h^{1/2}\xi + \xi^0|^{2s} |\hat{\psi}(\xi)|^2 d\xi \sim h^{-2s} |\xi^0|^{2s} \|\hat{\psi}\|_{L^2}^2,$$

by the dominated-convergence theorem.  $\square$

If  $b(x, \xi) \in \Sigma^{0,0}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  for some  $N$ , note that  $\text{Op}^h(b)$  as in (5.3) makes sense if acting on  $\mathcal{S}(\mathbb{R}^d)$ .

**Proposition 10.11.** *Suppose  $b(x, \xi) \in \Sigma^{0,d+2}(\langle \xi \rangle^{-N}; \mathbb{R}^{2d})$  for some  $N \in \mathbb{R}$ , and*

$$b_j(x, \xi) = \langle \xi \rangle^{-m_j} \int_0^1 \partial_{\xi_j} b(x, s\xi + \xi^0) ds \in \Sigma(\mathbb{R}^{2d}) = \Sigma^{0,d+1}(\langle \xi \rangle^{d+1}; \mathbb{R}^{2d}),$$

$j = 1, \dots, d$ , for some  $m_j \geq 0$ . One has

$$\text{Op}^h(b)w_h = b(x, \xi^0)w_h + \sum_{1 \leq j \leq d} M_{0,d+1}^{-(d+1)}(b_j) O(h^{1/2}) \text{ in } L^2(\mathbb{R}^d) \quad \text{as } h \rightarrow 0.$$

**Proof.** Set  $\psi_h(x) = \psi(x - h^{-1/2}x^0)$  and

$$v_h = h^{-d/4} \psi(h^{-1/2}(x - x^0)) = h^{-d/4} \psi_h(h^{-1/2}x).$$

Then,  $w_h = b(x, hD + \xi_0)v_h$  and  $\text{Op}^h(b)w_h = e^{ix \cdot \xi^0 / h} b(x, hD + \xi_0)v_h$  by standard computations, yielding with  $q(x, \xi) = b(x, \xi + \xi^0) - b(x, \xi^0)$ ,

$$\begin{aligned} \tilde{w}_h &= (\text{Op}^h(b) - b(x, \xi^0))w_h = e^{ix \cdot \xi^0 / h} (b(x, hD + \xi_0) - b(x, \xi^0))v_h \\ &= \frac{h^{-d/4} e^{ix \cdot \xi^0 / h}}{(2\pi)^d} \int e^{ih^{-1/2}x \cdot \xi} q(x, h^{1/2}\xi) \hat{\psi}_h(\xi) d\xi. \end{aligned}$$

Note that  $\|\tilde{w}_h\|_{L^2} = \|u_h\|_{L^2}$  with  $u_h$  given by

$$u_h(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} q(h^{1/2}x, h^{1/2}\xi) \hat{\psi}_h(\xi) d\xi.$$

Write  $q(x, \xi) = \sum_j \xi_j \int_0^1 \partial_{\xi_j} b(x, s\xi + \xi^0) ds$ . This gives  $u_h(x) = h^{1/2} \sum_j u_{j,h}(x)$  with

$$u_{j,h}(x) = \frac{1}{(2\pi)^d} \int e^{ix \cdot \xi} b_j(h^{1/2}x, h^{1/2}\xi) \langle h^{1/2}\xi \rangle^{m_j} \xi_j \hat{\psi}_h(\xi) d\xi,$$

with  $b_j$  as in the proposition statement. Set  $\psi_{h,j} = \langle h^{1/2}D \rangle^{m_j} D_{x_j} \psi_h$ . Then  $u_{j,h}(x) = b_j(h^{1/2}x, h^{1/2}D) \psi_{h,j}(x)$ , that is, a semi-classical operator acting on  $\psi_{h,j}$ , yet with  $h$  replaced by  $h^{1/2}$ . First, observe that  $\|\psi_{h,j}\|_{L^2}$  is bounded uniformly in  $h$ . Second, with Lemma 5.6 one finds

$$\|u_{j,h}\|_{L^2} \lesssim M_{0,d+1}^{-(d+1)}(b_j(h^{1/2}x, \xi)),$$

which concludes the proof since  $M_{0,d+1}^{-(d+1)}(b_j(h^{1/2}x, \xi)) = M_{0,d+1}^{-(d+1)}(b_j)$ .  $\square$

Set  $b_z(x, \xi) = (z + a_{\kappa,g}(x, \xi))^{-1}$  with  $z \in \mathbb{C} \setminus \mathbb{R}$ . Then,  $b_z \in \Sigma^{0,d+2}(\langle \xi \rangle^{-2}; \mathbb{R}^{2d})$ . Define  $b_j$  as in the statement of Proposition 10.11 with  $b_z$  in place of  $b$  and  $m_j = m = 2d + 3$ .

**Lemma 10.12.** *There exists  $C > 0$  such that  $M_{0,d+1}^{-(d+1)}(b_j) \leq C |\operatorname{Im} z|^{-3-d}$ .*

**Proof.** First, one has  $\partial_{\xi_j} b_z(x, \xi) = -(z + a_{\kappa,g}(x, \xi))^{-2} \partial_{\xi_j} a_{\kappa,g}(x, \xi)$ , implying

$$\langle \xi \rangle^{d+1} |b_j(x, \xi)| \lesssim \langle \xi \rangle^{d+1-m} |\operatorname{Im} z|^{-2} (|\xi^0| + |\xi|) \lesssim |\operatorname{Im} z|^{-2},$$

as  $m = 2d + 3$ . Second, note that  $\partial_{\xi}^{\beta} (\langle \xi \rangle^{-m} \partial_{\xi_j} b_z(x, s\xi + \xi^0))$  is equal to a linear combination of terms

$$\partial_{\xi}^{\beta_1} \langle \xi \rangle^{-m} s^{|\beta_2|} (\partial_{\xi}^{\beta_2} \partial_{\xi_j} b_z)(x, s\xi + \xi^0), \quad \text{with } \beta_1 + \beta_2 = \beta.$$

As  $|(\partial_{\xi}^{\beta_2} \partial_{\xi_j} b_z)(x, s\xi + \xi^0)| \lesssim |\operatorname{Im} z|^{-|\beta_2|-2} (|\xi^0| + |\xi|)^{|\beta_2|+1}$  one obtains

$$\langle \xi \rangle^{d+1} |\partial_{\xi}^{\beta} b_j(h^{1/2}x, \xi)| \lesssim \langle \xi \rangle^{d+2+|\beta|-m} |\operatorname{Im} z|^{-|\beta|-2} \lesssim |\operatorname{Im} z|^{-d-3},$$

for  $|\beta| \leq d + 1$  as  $m = 2d + 3$ .  $\square$

**Corollary 10.13.** *Set  $b_z(x, \xi) = (z + a_{\kappa,g}(x, \xi))^{-1}$  with  $z \in \mathbb{C} \setminus \mathbb{R}$ . One has*

$$\operatorname{Op}^h(b_z)w_h = b_z(x, \xi^0)w_h + h^{1/2} |\operatorname{Im} z|^{-d-3} O(1) \text{ in } L^2(\mathbb{R}^d) \quad \text{as } h \rightarrow 0.$$

**10.4. Dyadic projection.** Consider  $(x^0, \xi^0) \in T^*\mathcal{M}$  with  $x^0 \notin \partial\mathcal{M}$  and  $\mathcal{C} = (O, \phi)$  a local chart with  $x^0 \in O$ . In this local chart, introduce  $(w_h)_h$  as above:

$$(10.5) \quad w_h(x) = h^{-d/4} e^{ix \cdot \xi^0/h} \psi(h^{-1/2}(x - x^0)),$$

with  $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  with  $\psi = 1$  in a neighborhood of 0 here. Consider a scale  $H = (h_k)_k$  and choose  $k$  sufficiently large so that  $\text{supp } v_k \subset O$ , with  $v_k = \phi^* w_k$  for  $w_k = w_{h_k}$ .

As above, denote by  $a_{\kappa,g}(x, \xi)$  the symbol of the operator  $A_{\kappa,g}$ , that is, in local coordinates,  $a_{\kappa,g}(x, \xi) = -g^{ij}(x)\xi_i\xi_j$ . Suppose  $0 < \alpha < 1$ , and  $\chi \in \mathcal{C}_c^\infty([\alpha^2, \alpha^{-2}])$ , with  $\chi = 1$  on  $[\alpha, \alpha^{-1}]$ . Here, we prove that  $\chi(-a_{\kappa,g}(x, \xi^0))v_k$  and  $\chi(-h_k^2 A_{\kappa,g})v_k$  coincide in  $L^2(\mathcal{M})$  up to a  $o(1)$  remainder. One can view  $\chi(-h_k A_{\kappa,g})v_k$  as some ‘‘projection’’ of  $v_k$  onto the dyadic subspace  $E_k$  introduced in Section 4.1.

**Lemma 10.14.** *Suppose  $\theta, \tilde{\theta} \in \mathcal{C}_c^\infty(\phi(O))$ , with  $\tilde{\theta} = 1$  on a neighborhood of  $\text{supp } \theta$ . Set  $\hat{\theta} = \phi^* \theta \in \mathcal{C}_c^2(O)$ . For  $z \in \mathbb{C} \setminus \mathbb{R}$  set  $b_z(x, \xi) = (a_{\kappa,g}(x, \xi) + z)^{-1}$ . One has*

$$(h^2 A_{\kappa,g} + z)^{-1} \hat{\theta} = \phi^* \theta \text{Op}^h(b_z) \tilde{\theta}(\phi^{-1})^* + R_h,$$

with  $\|R_h\|_{\mathcal{L}(L^2)} = O(h) |\text{Im}(z)|^{-1} (1 + |z|^{1/2})$ .

A proof is given below. Note that if  $z \in \mathbb{C} \setminus \mathbb{R}$ , the operator  $\text{Op}^h(b_z)$  is well defined and bounded on  $L^2(\mathbb{R}^d)$  by Lemma 10.16 given below.

With  $\chi$  as above, consider  $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{C})$  an almost analytic extension of  $\chi$ . The Helffer-Sjöstrand formula [13] gives

$$(10.6) \quad \chi(-h^2 A_{\kappa,g}) = \frac{1}{2i\pi} \lim_{\epsilon \rightarrow 0^+} \int_{|\text{Im } z| \geq \epsilon} \bar{\partial} \tilde{\chi}(z) (h^2 A_{\kappa,g} + z)^{-1} dz \wedge d\bar{z}.$$

The function  $\tilde{\chi}$  has the following properties:  $\tilde{\chi}|_{\mathbb{R}} = \chi$ , there exists  $C > 0$  such that  $\text{supp } \tilde{\chi} \subset \text{supp } \chi + i[-C, C]$ , and for any  $n \in \mathbb{N}$  there exists  $C_n > 0$  such that

$$|\bar{\partial} \tilde{\chi}(z)| \leq C_n |\text{Im } z|^n.$$

Choose  $\theta, \tilde{\theta}$  as in Lemma 10.14 and  $\hat{\theta} = \phi^*\theta$ . One obtains

$$\begin{aligned}
\chi(-h^2 A_{\kappa,g})\hat{\theta} &= \frac{1}{2i\pi} \lim_{\epsilon \rightarrow 0^+} \int_{|\operatorname{Im} z| \geq \epsilon} \bar{\partial} \tilde{\chi}(z) (h^2 A_{\kappa,g} + z)^{-1} \hat{\theta} dz \wedge d\bar{z} \\
&= \frac{1}{2i\pi} \phi^* \theta \lim_{\epsilon \rightarrow 0^+} \int_{|\operatorname{Im} z| \geq \epsilon} \bar{\partial} \tilde{\chi}(z) \operatorname{Op}^h(b_z) dz \wedge d\bar{z} \tilde{\theta} (\phi^{-1})^* + O(h)_{\mathcal{L}(L^2)} \\
&= \frac{1}{2i\pi} \phi^* \theta \operatorname{Op}^h \left( \lim_{\epsilon \rightarrow 0^+} \int_{|\operatorname{Im} z| \geq \epsilon} \bar{\partial} \tilde{\chi}(z) b_z dz \wedge d\bar{z} \right) \tilde{\theta} (\phi^{-1})^* + O(h)_{\mathcal{L}(L^2)} \\
&= \phi^* \theta \operatorname{Op}^h (\chi(-a_{\kappa,g})) \tilde{\theta} (\phi^{-1})^* + O(h)_{\mathcal{L}(L^2)},
\end{aligned}$$

meaning that

$$(10.7) \quad \chi(-h^2 A_{\kappa,g}) = [\operatorname{Op}^h](\chi(-a_{\kappa,g})),$$

with the notation introduced in Section 5.4.

Consider now  $\theta$  such that  $\hat{\theta} = 1$  on  $\operatorname{supp} v_k$ , for  $k$  sufficiently large. One has

$$\chi(-h^2 A_{\kappa,g})v_k = \phi^* \theta \operatorname{Op}^h (\chi(-a_{\kappa,g}))w_k + O(h)_{L^2},$$

yielding, with Proposition 10.11,

$$(10.8) \quad \chi(-h^2 A_{\kappa,g})v_k = \chi(-a_{\kappa,g}(x, \xi^0))v_k + O(h^{1/2})_{L^2(\mathcal{M})}.$$

Since  $(v_k)_k$  has  $\|\psi\|_{L^2}^2 \delta_{(x^0, \xi^0)}$  for s.c.m., one has the following result.

**Lemma 10.15.** *The two sequences  $\chi(-h^2 A_{\kappa,g})v_k$  and  $\chi(-a_{\kappa,g}(x, \xi^0))v_k$  have the same s.c.m., that is,  $|\chi(-a_{\kappa,g}(x^0, \xi^0))|^2 \|\psi\|_{L^2}^2 \delta_{(x^0, \xi^0)}$ .*

**Proof of Lemma 10.14.** For  $z \in \mathbb{C} \setminus \mathbb{R}$  and  $w \in \mathcal{S}(\mathbb{R}^d)$  one has

$$\theta \operatorname{Op}^h(b_z)(\tilde{\theta}w)(x) = (2\pi)^{-d} \int e^{ix \cdot \xi} \theta(x) b_z(x, h\xi) \widehat{\tilde{\theta}w}(\xi) d\xi.$$

With the form of the differential operator  $A_{\kappa,g}$  in local coordinates compute

$$\begin{aligned}
& h^2 A_{\kappa,g} \left( e^{ix \cdot \xi} \theta(x) b_z(x, h\xi) \right) \\
&= i\tilde{\kappa}^{-1} \sum_{i,j} h \partial_{x_i} \left( e^{ix \cdot \xi} \tilde{\kappa} g^{ij} \theta b_z(x, h\xi) h \xi_j \right) \\
&\quad + h\tilde{\kappa}^{-1} \sum_{i,j} h \partial_{x_i} \left( e^{ix \cdot \xi} \tilde{\kappa} g^{ij} \partial_{x_j} (\theta b_z)(x, h\xi) \right) \\
&= e^{ix \cdot \xi} (\theta a b_z)(x, h\xi) + ih\tilde{\kappa}^{-1} e^{ix \cdot \xi} m(x, h\xi) + h^2 \tilde{\kappa}^{-1} e^{ix \cdot \xi} \sum_i \partial_{x_i} \ell_i(x, h\xi),
\end{aligned}$$

where

$$m(x, \xi) = \sum_{i,j} \left( \partial_{x_i} (\tilde{\kappa} g^{ij} \theta b_z)(x, \xi) \xi_j + \tilde{\kappa} g^{ij} \partial_{x_j} (\theta b_z)(x, \xi) \xi_i \right),$$

with  $\ell_i = \sum_j \tilde{\kappa} g^{ij} \partial_{x_j} (\theta b_z)$ . We deduce that

$$(10.9) \quad (h^2 A_{\kappa,g} + z) \theta \text{Op}^h(b_z) \tilde{\theta} = \theta + ih\tilde{\kappa}^{-1} \text{Op}^h(m) \tilde{\theta} + h^2 \sum_i \tilde{\kappa}^{-1} \partial_{x_i} \text{Op}^h(\ell_i) \tilde{\theta}.$$

One checks that  $m$  fulfills the assumptions of Lemma 10.16 with  $\delta = 1$  and so do the symbols  $\ell_i$ ,  $1 \leq i \leq d$ , with  $\delta = 2$ , implying that  $\text{Op}^h(m)$ , and  $\text{Op}^h(\ell_i)$  are bounded on  $L^2(\mathbb{R}^d)$ .

The following bounds hold for the resolvent

$$\begin{aligned}
& \| (h^2 A_{\kappa,g} + z)^{-1} \|_{\mathcal{L}(L^2(\mathcal{M}))} \leq |\text{Im } z|^{-1}, \\
& \| (h^2 A_{\kappa,g} + z)^{-1} \|_{\mathcal{L}(L^2(\mathcal{M}), H_0^1(\mathcal{M}))} \leq |h \text{Im } z|^{-1} (|\text{Re } z| + |\text{Im } z|)^{1/2}.
\end{aligned}$$

From the second estimate one deduces also that

$$\| (h^2 A_{\kappa,g} + z)^{-1} \|_{\mathcal{L}(H^{-1}(\mathcal{M}), L^2(\mathcal{M}))} \leq |h \text{Im } z|^{-1} (|\text{Re } z| + |\text{Im } z|)^{1/2}.$$

One thus obtains

$$\| (h^2 A_{\kappa,g} + z)^{-1} \phi^* \text{Op}^h(m) \tilde{\theta} (\phi^{-1})^* \|_{\mathcal{L}(L^2(\mathcal{M}))} \lesssim |\text{Im}(z)|^{-1},$$

and

$$\begin{aligned}
& \| (h^2 A_{\kappa,g} + z)^{-1} \tilde{\kappa}^{-1} \phi^* \partial_{x_i} \text{Op}^h(\ell_i) \tilde{\theta} (\phi^{-1})^* \|_{\mathcal{L}(L^2(\mathcal{M}))} \\
& \lesssim |h \text{Im}(z)|^{-1} |z|^{1/2} \|\tilde{\kappa}\|_{W^{1,\infty}}, \quad i = 1, \dots, d.
\end{aligned}$$

If one applies the resolvent  $(h^2 A_{\kappa,g} + z)^{-1}$  to the left of identity (10.9), one then obtains

$$\begin{aligned} & \| (h^2 A_{\kappa,g} + z)^{-1} \hat{\theta} - \phi^* \theta \text{Op}^h(b_z) \tilde{\theta}(\phi^{-1})^* \|_{\mathcal{L}(L^2(\mathcal{M}))} \\ & \lesssim h |\text{Im}(z)|^{-1} (1 + |z|^{1/2} \|\tilde{\kappa}\|_{W^{1,\infty}}), \end{aligned}$$

which gives the result.  $\square$

For  $\delta \geq 0$  set

$$L_{0,d+1}^{-\delta}(a) = \max_{|\beta| \leq d+1} \sup_{(x,\xi)} |\partial_\xi^\beta a(x,\xi)| \langle \xi \rangle^{|\beta|+\delta}.$$

Compare  $L_{0,d+1}^{-\delta}$  and  $M_{0,d+1}^{-(d+1)}$ . Here, less decay is expected on  $a(x,\xi)$ ; yet decay improves with differentiations with respect to  $\xi$ .

**Lemma 10.16.** *Suppose  $a(x,\xi) \in L^\infty(\mathbb{R}^{2d})$  is smooth in  $\xi$  and  $L_{0,d+1}^{-\delta}(a) < \infty$  for some  $\delta > 0$ . Then,  $\text{Op}^h(a)$  is bounded on  $L^2(\mathbb{R}^d)$  and*

$$\| \text{Op}^h(a) \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_{\delta,d} L_{0,d+1}^{-\delta}(a).$$

Compare with Lemma 5.6.

**Proof.** Consider  $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  such that  $0 \leq \theta \leq 1$ ,  $\theta(\xi) = 1$  if  $|\xi| \leq 1/2$ , and  $\theta(\xi) = 0$  if  $|\xi| \geq 1$ . Set  $\psi_0 = \theta$  and

$$\psi(\xi) = \theta(\xi) - \theta(2\xi) \quad \text{and} \quad \psi_j(\xi) = \psi(2^{-j}\xi) \quad \text{for } j \in \mathbb{N}^*,$$

yielding a dyadic partition of unity  $1 = \sum_{j \in \mathbb{N}} \psi_j$ . Set  $a_j(x,\xi) = \psi_j(\xi) a(x,\xi)$ . With Lemma 5.6 one finds

$$\| \text{Op}^h(a_0) \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim M_{0,d+1}^{-(d+1)}(a_0) \lesssim L_{0,d+1}^0(a) \lesssim L_{0,d+1}^{-\delta}(a),$$

since  $\psi_0$  has compact support. Consider now  $j \geq 1$ . With  $\tilde{h}_j = 2^{-j} h_k$  one writes

$$\begin{aligned} \text{Op}^h(a_j)v(x) &= (2\pi)^{-d} \int e^{ix \cdot \xi} \psi(2^{-j} h_k \xi) a(x, h_k \xi) \hat{v}(\xi) d\xi \\ &= (2\pi)^{-d} \int e^{ix \cdot \xi} \psi(\tilde{h}_j \xi) a(x, \tilde{h}_j 2^j \xi) \hat{v}(\xi) d\xi \\ &= \text{Op}^{\tilde{h}_j}(b_j)v(x). \end{aligned}$$

with  $b_j(x, \xi) = \psi(\xi)a(x, 2^j\xi)$ . The symbol  $b_j$  is compactly supported in  $\xi$  and for  $\beta \in \mathbb{N}^d$ , with  $|\beta| \leq 1 + d$ , one finds

$$\begin{aligned} \langle \xi \rangle^{d+1} |\partial_\xi^\beta b_j(x, \xi)| &\lesssim \langle \xi \rangle^{d+1} \sum_{\beta'+\beta''=\beta} |\partial_\xi^{\beta'} \psi(\xi)| |\partial_\xi^{\beta''} a(x, 2^j\xi)| \\ &\lesssim L_{0,d+1}^{-\delta}(a) \langle \xi \rangle^{d+1} \sum_{\beta'+\beta''=\beta} 2^{|\beta''|j} \langle 2^j\xi \rangle^{-|\beta''|-\delta} |\partial_\xi^{\beta'} \psi(\xi)|. \end{aligned}$$

Since  $|\xi| \gtrsim 1$  in the compact supp  $\psi$ , one obtains

$$\langle \xi \rangle^{d+1} |\partial_\xi^\beta b_j(x, \xi)| \lesssim 2^{-\delta j} L_{0,d+1}^{-\delta}(a) \sum_{\beta'+\beta''=\beta} \langle \xi \rangle^{d+1} |\partial_\xi^{\beta'} \psi(\xi)| \lesssim 2^{-\delta j} L_{0,d+1}^{-\delta}(a).$$

Lemma 5.6 implies  $\|\text{Op}^{\tilde{h}_j}(b_j)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim 2^{-\delta j} L_{0,d+1}^{-\delta}(a)$ . Convergence of  $\sum_j 2^{-\delta j}$  gives the conclusion.  $\square$

In what follows we will also need the following results.

**Lemma 10.17.** *There exists  $C > 0$  such that  $\|h_k \nabla_g \chi(-h_k^2 A_{\kappa,g}) v_k\|_{L^2(\mathcal{M})} \leq C$ . If  $(x^0, \xi^0)$  in the definitions of  $w_k$  in (10.5) and  $v_k = \phi^* w_k$  is chosen such that  $\chi(-a_{\kappa,g}(x^0, \xi^0)) \neq 0$ , then there exists  $C' > 0$  such that*

$$1/C \leq \|\chi(-h_k^2 A_{\kappa,g}) v_k\|_{L^2(\mathcal{M})} \leq C \quad \text{and} \quad 1/C \leq \|h_k \nabla_g \chi(-h_k^2 A_{\kappa,g}) v_k\|_{L^2(\mathcal{M})} \leq C,$$

for  $k$  sufficiently large.

**Lemma 10.18.** *One has*

$$h_k \nabla_g \chi(-h_k^2 A_{\kappa,g}) v_k = h_k \nabla_g (\chi(-a_{\kappa,g}(x, \xi^0)) v_k) + O(h_k^{1/4})_{L^2(\mathcal{M})}.$$

**Proof of Lemma 10.17.** Set  $\tilde{v}_k = \chi(-h_k^2 A_{\kappa,g}) v_k$ . One writes

$$(10.10) \quad \|h_k \nabla_g \tilde{v}_k\|_{L^2(\mathcal{M})}^2 = (\tilde{\chi}(-h_k^2 A_{\kappa,g}) v_k, \tilde{v}_k)_{L^2(\mathcal{M})},$$

with  $\tilde{\chi}(\lambda) = \lambda \tilde{\chi}(\lambda)$ . The same analysis used for  $\tilde{v}_k$  applies to  $\tilde{\chi}(-h_k^2 A_{\kappa,g}) v_k$ . In particular  $\tilde{\chi}(-h_k^2 A_{\kappa,g}) v_k = \tilde{\chi}(-a_{\kappa,g}(x, \xi^0)) v_k + O(h_k^{1/2})_{L^2}$ . One thus obtains the first result

There exists a neighborhood  $V$  of  $x^0$  such that  $|\chi(-a_{\kappa,g}(x, \xi^0))| \gtrsim 1$  for  $x \in V$ . For  $k$  sufficiently large  $\text{supp } v_k \subset V$  implying

$$\|\chi(-a_{\kappa,g}(x, \xi^0)) v_k\|_{L^2(\mathcal{M})} \gtrsim \|v_k\|_{L^2(\mathcal{M})} \gtrsim 1.$$

With (10.8) one concludes that  $\|\tilde{v}_k\|_{L^2(\mathcal{M})} \gtrsim 1$ .

Arguing the same with (10.10) and using that  $|\tilde{\chi} \chi(-a_{\kappa,g}(x, \xi^0))| \gtrsim 1$  in a neighborhood of  $x^0$ , one obtains that  $\|h_k \nabla_g \tilde{v}_k\|_{L^2(\mathcal{M})} \gtrsim 1$ .  $\square$



**Proof of Lemma 10.18.** Set  $z_k = \chi(-h_k^2 A_{\kappa,g})v_k - \chi(-a_{\kappa,g}(x, \xi^0))v_k$ . With (10.8) one has  $\|z_k\|_{L^2(\mathcal{M})} = O(h_k^{1/2})$ . Lemma 10.17 gives a  $L^2$ -bound for the sequence  $h_k \nabla_g \chi(-h_k^2 A_{\kappa,g})v_k$  and a simple computation gives  $h_k \nabla_g \chi(-a_{\kappa,g}(x, \xi^0))v_k$  also  $L^2$ -bounded. Hence, a preliminary estimate is  $\|h_k \nabla_g z_k\|_{L^2(\mathcal{M})} = O(1)$ .

Compute  $\|h_k \nabla_g z_k\|_{L^2(\mathcal{M})}^2 = N_1 + N_2$  with

$$\begin{aligned} N_1 &= (-h_k^2 A_{\kappa,g} \chi(-h_k^2 A_{\kappa,g})v_k, z_k)_{L^2(\mathcal{M})}, \\ N_2 &= (h_k \nabla_g \chi(-a_{\kappa,g}(x, \xi^0))v_k, h_k \nabla_g z_k)_{L^2(\mathcal{M})}. \end{aligned}$$

Note that  $-h_k^2 A_{\kappa,g} \chi(-h_k^2 A_{\kappa,g})v_k = \tilde{\chi}(-h_k^2 A_{\kappa,g})v_k$ , with  $\tilde{\chi}(\lambda) = \lambda \tilde{\chi}(\lambda)$ , is  $L^2$ -bounded since the same analysis used for  $\chi(-h_k^2 A_{\kappa,g})v_k$  applies. Hence,  $N_1 = O(h_k^{1/2})$ . Writing

$$h_k \nabla_g \chi(-a_{\kappa,g}(x, \xi^0))v_k = h_k [\nabla_g, \chi(-a_{\kappa,g}(x, \xi^0))]v_k + \chi(-a_{\kappa,g}(x, \xi^0))h_k \nabla_g v_k,$$

one finds

$$N_2 = (\chi(-a_{\kappa,g}(x, \xi^0))h_k \nabla_g v_k, h_k \nabla_g z_k)_{L^2(\mathcal{M})} + O(h_k).$$

With a similar commutator computation one further obtains

$$N_2 = -(\chi(-a_{\kappa,g}(x, \xi^0))h_k^2 A_{\kappa,g} v_k, z_k)_{L^2(\mathcal{M})} + O(h_k).$$

With Lemma 10.10 one concludes that  $N_2 = O(h_k^{1/2})$ .  $\square$

**10.5. Proof of the necessary geometric control condition.** Here, we prove Theorem 10.3. Assume that observability holds and yet the condition of Definition 10.1 (resp. Definition 10.2) does not hold. This section aims to reach a contradiction.

If the weak interior geometric control condition does not hold, there exist  $\varrho^0 = (t^0, x^0, \tau^0, \xi^0) \in \text{Char } p \cap T^* \mathcal{L}$  and  $V$  an open neighborhood of  $[0, T] \times \bar{\omega}$  such that no generalized bicharacteristic going through  $\varrho^0$  reaches a point above  $V$ . If the weak boundary geometric control condition does not hold, there exist  $\varrho^0$  and  $V_\partial$  an open neighborhood of  $[0, T] \times \bar{\Gamma}$  such that no generalized bicharacteristic going through  $\varrho^0$  reaches a boundary escape point above  $V_\partial$ .

**10.5.1. Interior initial point.** We first treat the case  $\varrho^0 \in \text{Char}(p) \cap (T^* \mathcal{L} \setminus \partial T^* \mathcal{L})$ . The case  $\varrho^0 \in \text{Char}(p) \cap \partial T^* \mathcal{L}$  is treated in a second round.

One has  $\tau^0 \neq 0$ . With some scaling in the cotangent variables, one may assume  $|\tau^0| \in [\alpha, \alpha^{-1}]$  for some  $0 < \alpha < 1$ . One has  $(\tau^0)^2 = -a_{\kappa,g}(x^0, \xi^0) = g^{ij}(x^0) \xi_i^0 \xi_j^0$ .

Suppose  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  is such that  $\chi \geq 0$ ,  $\text{supp } \chi \subset ]\alpha^2, \alpha^{-2}[$ , and  $\chi((\tau^0)^2) = \|\psi\|_{L^2}^{-1}$ , with  $\psi$  used in (10.5). With the sequence  $(v_k)_k \subset L^2(\mathcal{M})$  constructed above set

$$\underline{u}_k^0 = \chi(-h^2 A_{\kappa,g})v_k \quad \text{and} \quad \underline{u}_k^1 = ih_k^{-1}\tau^0 \underline{u}_k^0$$

and denote by  $u_k$  the solution to the *homogeneous* wave equation

$$(10.11) \quad \begin{cases} P_{\kappa,g} u_k = 0 & \text{in } \mathbb{R} \times \mathcal{M}, \\ u_k = 0 & \text{in } \mathbb{R} \times \partial\mathcal{M}, \\ u_k|_{t=t^0} = \underline{u}_k^0, \quad \partial_t u_k|_{t=t^0} = \underline{u}_k^1 & \text{in } \mathcal{M}. \end{cases}$$

Since  $\underline{u}_k^0 \subset E_k$ , with  $\underline{u}_k^0 = \sum_{\nu \in J_k} u_{k,\nu}^0 e_\nu$ , one finds

$$u_k = \sum_{\nu \in J_k} (e^{i(t-t^0)\sqrt{\lambda_\nu}} u_{k,\nu} + e^{-i(t-t^0)\sqrt{\lambda_\nu}} u_{k,-\nu}) e_\nu,$$

with  $u_{k,\pm\nu} = u_{k,\nu}^0 (1 \pm h_k^{-1} \lambda_\nu^{-1/2} \tau^0)/2$ . With Lemma 10.17 observe that

$$(10.12) \quad \mathcal{E}^h(u_k) = \frac{1}{2} (\|h_k \nabla_g \underline{u}_k^0\|_{L^2}^2 + \|h_k \underline{u}_k^1\|_{L^2}^2) = \frac{1}{2} (\|h_k \nabla_g \underline{u}_k^0\|_{L^2}^2 + (\tau^0)^2 \|\underline{u}_k^0\|_{L^2}^2) \approx 1.$$

The solution  $(u_k)_k$  is bounded in  $L_{\text{loc}}^2(\mathcal{L})$  as in Section 6.2.1, and can be associated with a s.c.m.  $\mu$  (up to a possible subsequence extraction). Associated with  $h_k \partial_n u_k|_{\partial\mathcal{L}}$  is a s.c.m.  $\nu$ . Arguing as in Proposition 6.4 one finds

$$(10.13) \quad \begin{aligned} \text{supp } \mu &\subset \text{Char } p \cap T^*\mathcal{L} \cap \{\alpha \leq |\tau| \leq \alpha^{-1}\}, \\ \text{supp } \nu &\subset T^*\partial\mathcal{L} \cap \{\alpha \leq |\tau| \leq \alpha^{-1}\}. \end{aligned}$$

Hence, Theorem 6.1 applies and both measure  $\mu$  and  $\nu$  satisfy the measure propagation equation (6.2). As in the proof of Proposition 6.4 one finds that no mass leaks at infinity at scale  $H$ , in the sense of Definition 5.16.

As the s.c.m. of  $(v_k)_k$  at scale  $H = (h_k)_k$  is  $\|\psi\|_{L^2}^2 \delta_{(x^0, \xi^0)}$ , with Lemma 10.15 one finds the following results.

**Lemma 10.19.** *The sequence  $(\underline{u}_k^0, h_k \underline{u}_k^1)_k$  admits the Hermitian s.c.m.*

$$\nu^0 = \begin{pmatrix} \nu_{0,0}^0 & \nu_{0,1}^0 \\ \nu_{1,0}^0 & \nu_{1,1}^0 \end{pmatrix} = \begin{pmatrix} 1 & -i\tau^0 \\ i\tau^0 & (\tau^0)^2 \end{pmatrix} \delta_{(x^0, \xi^0)}$$

on  $T^*\mathcal{M}$  at scale  $H = (h_k)_k$ .

As in Section 10.2 denote by  $\mu^+$  the s.c.m. associated with  $(\mathbf{1}_{t>t^0}u_k)_k$ . By Lemma 10.6 one has  $\mu^+ = \mathbf{1}_{t>t^0}\mu$ . Observe that

$$a\nu_{0,0}^0 - \nu_{1,1}^0 = -2(\tau^0)^2\delta_{(x^0,\xi^0)} \quad \text{and} \quad \text{Im}\nu_{0,1}^0 = -\tau^0\delta_{(x^0,\xi^0)}.$$

Theorem 10.7 applies and, as  $\tau^+ - \tau^- = 2|a_{\kappa,g}(x^0, \xi^0)|^{1/2} = 2|\tau^0|$ , one obtains

$$(10.14) \quad \mathbb{H}_p\mu^+ = -{}^t\mathbb{H}_p\mu^+ = -2\tau^0\delta_{\varrho^0},$$

away from  $\partial(T^*\mathcal{L})$ , using that for  $\varrho = (t^0, x^0, 0, \xi^0)$  one has  $\varrho^0 = \varrho^\oplus$  if  $\tau^0 > 0$  and  $\varrho^0 = \varrho^\ominus$  if  $\tau^0 < 0$ .

**Lemma 10.20.** *The measure  $\mu$  vanishes in a neighborhood of  $\{t = t^0\} \cap \partial(T^*\mathcal{L})$ .*

A proof is given below. With Lemma 10.20 and (10.14), one concludes that  $\text{supp}\mu^+ \cap \{t = t^0\} = \{\varrho^0\}$ . As  $\mu^+ = \mathbf{1}_{t>t^0}\mu$  one also has  $\text{supp}\mu \cap \{t = t^0\} = \{\varrho^0\}$ . With Theorem 2.14 one obtains the following lemma.

**Lemma 10.21.** *The support of  $\mu$  is a union of maximal generalized bicharacteristics that go through  $\varrho^0$ .*

**Case 1: interior observability.** If interior observability holds, then inequality (1.6) is valid for the sequence  $(u_k)_k$ . By (10.12) one has  $\|\mathbf{1}_{]0,T[ \times \omega} h_k \partial_t u_k\|_{L^2(\mathcal{L})} \gtrsim 1$ , implying

$$(10.15) \quad \text{supp}\mu \cap T^*V \neq \emptyset.$$

The open set  $V$  is introduced in the beginning of the proof. In fact, consider  $\varphi \in \mathcal{C}_c^\infty(\mathcal{L})$  nonnegative such that  $\text{supp}\varphi \subset V$  and  $\varphi = 1$  in a neighborhood of  $[0, T] \times \bar{\omega}$ . With Proposition 5.21 one finds  $\langle \mu, \varphi \tau^2 \rangle = \lim_{k \rightarrow +\infty} (\varphi h_k \partial_t u_k, h_k \partial_t u_k)_{L^2(\mathcal{L})} \gtrsim 1$ , yielding (10.15). With Lemma 10.21 however, the existence of a point in  $\text{supp}\mu \cap T^*V$  yields a contradiction with the choice of the point  $\varrho^0$  made at the beginning of the proof.

**Case 2: boundary observability.** If boundary observability holds, then inequality (1.7) is valid for  $(u_k)_k$ . With (10.12) one has  $\|\mathbf{1}_{]0,T[ \times \Gamma} \partial_n u|_{\mathbb{R} \times \partial\mathcal{M}}\|_{L^2(\partial\mathcal{L})} \gtrsim 1$ , implying that  $\text{supp}\nu \cap T^*V_\partial \neq \emptyset$ ; the open set  $V_\partial$  is introduced in the beginning of the proof. Suppose  $\varrho^1 = (t^1, x^1, \tau^1, \xi^1) \in T^*V_\partial$ .

**Case  $\varrho^1 \in \mathcal{H}_\partial$ , a hyperbolic point:** Denote by  $\varrho^{1,\pm} \in \mathcal{H}_\partial^\pm$  the points such that  $\pi_\parallel(\varrho^{1,\pm}) = \varrho^1$ . They are boundary escape points. With Lemma 10.21 the existence of such a point in  $\text{supp}\mu$  yields a contradiction. Thus  $\mu = 0$  locally near these points. With Theorem 6.1, near a

hyperbolic point one has  ${}^t\mathrm{H}_p \mu = \tilde{\mu} \otimes \delta_{z=0}$ , for  $\tilde{\mu}$  some measure on  $\partial(T^*\mathcal{L})$ . Here, one has  $\varrho^{1,\pm} \notin \mathrm{supp}(\tilde{\mu} \otimes \delta_{z=0})$  implying  $\|\varrho^1 \notin \mathrm{supp} \nu$ . One concludes that  $\|\mathcal{H}_\partial \cap \mathrm{supp} \nu = \emptyset$ .

**Case  $\|\varrho^1 = \varrho^1 \in \mathcal{G}_\partial \cap \mathcal{B}_{esc}$ , a glancing escape point:**  $\varrho^1 \in \|\mathcal{G}_\partial = \mathcal{G}_\partial$ . If  $\varrho^1 \in \mathrm{supp} \mu$  one reaches a contradiction with Lemma 10.21 as  $\varrho^1$  is boundary escape point. Thus, locally  $\mu = 0$ . In local coordinates, in a neighborhood  $W$  of  $\varrho^1$ , Theorem 6.1 and Remark 2.15 give

$$\langle \nu, \partial_\zeta q|_{z=\zeta=0} \rangle = 0,$$

for any  $q \in \mathcal{C}_c^\infty(\mathbb{R}^{2d+2})$  supported in  $W$ , since there is no hyperbolic point in  $\mathrm{supp} \nu \cap W$ . As any compactly supported function  $\tilde{q}$  on  $\{z = \zeta = 0\}$  can be written in the form  $\partial_\zeta q|_{z=\zeta=0}$ , this implies that  $\nu$  vanishes in a neighborhood of  $\varrho^1$ . One concludes that  $\mathcal{G}_\partial \cap \mathcal{B}_{esc} \cap \mathrm{supp} \nu = \emptyset$ .

With Proposition 9.10, Lemma 2.12, and the two cases above, one concludes that  $\mathrm{supp} \nu \subset \mathcal{G}_\partial \setminus \mathcal{B}_{esc} \subset \mathcal{G}_\partial^d \cup \mathcal{G}_\partial^3$ . Yet, the measure  $\nu$  has no mass on this set by Proposition 3.5 in the companion article [5], that is,  $\langle \nu, \mathbf{1}_{\mathcal{G}_\partial^d \cup \mathcal{G}_\partial^3} \rangle = 0$ , implying that  $\nu$  vanishes; a contradiction.

10.5.2. *Boundary initial point.* We now treat the case  $\varrho^0 \in \mathrm{Char}(p) \cap \partial T^*\mathcal{L}$ .

**Case 3:  $\varrho^0 \in \mathrm{Char} p \cap \partial T^*\mathcal{L}$  for a interior observation.** Suppose that  $V$  is a neighborhood of  $[0, T] \times \bar{\omega}$  in  $\partial T^*\mathcal{L}$  such that no generalized bicharacteristic going through  $\varrho^0$  reaches a point above  $V$ . Consider  $\tilde{V}$  a neighborhood of  $[0, T] \times \bar{\omega}$  in  $\partial T^*\mathcal{L}$  such that  $\tilde{V} \Subset V$  and  $\varepsilon = \mathrm{dist}(\tilde{V}, V^c)$ . For  $\underline{\varrho} = (\underline{t}, \underline{x}, \underline{\tau}, \underline{\xi}) \in T^*\mathcal{L}$  and  $T > 0$ , set

$$\Gamma^T(\underline{y}) = \{|t - \underline{t}| \leq T\} \cap \bigcup_{\underline{y} \in \mathcal{G}_{\tilde{\gamma}}} \mathcal{G}_{\tilde{\gamma}},$$

that is, the union of all generalized bicharacteristic that pass through  $\underline{y}$ , restricted to the time interval  $[\underline{t} - T, \underline{t} + T]$ .

With the continuity result of Proposition 2.11 in the companion article [5], for  $T > 0$  there exists  $\delta > 0$  such for any  $\tilde{\varrho}^0 \in T^*\mathcal{L}$  one has

$$\mathrm{dist}(\tilde{\varrho}^0, \varrho^0) \leq \delta \text{ and } \varrho \in \Gamma^T(\tilde{\varrho}^0) \Rightarrow \mathrm{dist}(\varrho, \Gamma^T(\varrho^0)) \leq \varepsilon/2.$$

Thus, for  $T$  chosen sufficiently large there exists  $\tilde{\varrho}^0 \in \mathrm{Char} p \cap (T^*\mathcal{L} \setminus \partial T^*\mathcal{L})$  such that no generalized bicharacteristic going through  $\tilde{\varrho}^0$  reaches a point above  $\tilde{V}$ , meaning we are back to the configuration considered above.

**Case 4:  $\varrho^0 \in \text{Char } p \cap \partial T^* \mathcal{L}$  for a boundary observation.** Note that one cannot argue as in the case of an interior observation since  $\mathcal{G}_\partial \setminus \mathcal{B}_{esc} \subset \overline{\mathcal{H}_\partial}$ . However, the method used here applies to the case of an interior observation; the argument is yet much more involved. Note also that the argument simplifies if generalized bicharacteristics are uniquely defined, that is, in the presence of a generalized bicharacteristic flow.

Suppose  $V_\partial$  is a neighborhood of  $[0, T] \times \overline{\Gamma}$  in  $\partial T^* \mathcal{L}$  such that no generalized bicharacteristic going through  $\varrho^0$  reaches a boundary escape point above  $V_\partial$ . Write  $\varrho^0 = (t^0, x^0, \tau^0, \xi^0)$ , where  $x^0 = (x'^0, z^0)$  with  $z^0 = 0$ . One considers a sequence  $(\varrho^n)_n \subset \text{Char } p \cap T^* \mathcal{L} \setminus \partial(T^* \mathcal{L})$  such that  $\varrho^n = (t^0, x'^0, z^n, \tau^n, \xi^n) \rightarrow \varrho^0$  as  $n \rightarrow +\infty$ , that is,  $z^n \rightarrow 0^+$  and  $(\tau^n, \xi^n) \rightarrow (\tau^0, \xi^0)$ . With each  $\varrho^n$ , construct a sequence of solutions  $(u_{n,k})_k$  to the wave equation as done above, that is, with a s.c.m.  $\mu_n$  whose support is a union of maximal generalized bicharacteristics that go through  $\varrho^n$ . One has  $\text{supp } \mu_n \subset \text{Char } p \cap \{\alpha \leq |\tau| \leq \alpha^{-1}\}$ . With this construction, the mass of  $\mu_n$  on  $T^*((-T, T) \times \hat{\mathcal{M}})$  is uniformly bounded for any  $T > 0$ . This implies, that  $(\mu_n)_n$  is a bounded sequence of measure in the sense of the measure topology. Consequently, there exists a measure  $\mu$  such that  $\mu_n \rightarrow \mu$  for a subsequence, still denoted by  $\mu_n$ , in the measure topology on  $T^* \hat{\mathcal{L}}$ . One has  $\text{supp } \mu \subset \text{Char } p \cap \{\alpha \leq |\tau| \leq \alpha^{-1}\}$ .

Consider  $\tilde{\mathcal{M}}$  a bounded neighborhood of  $\overline{\mathcal{M}}$  in  $\hat{\mathcal{M}}$  and set  $\tilde{\mathcal{L}} = \mathbb{R} \times \tilde{\mathcal{M}}$ . Denote by  $U$  a neighborhood of  $\text{Char } p \cap T^* \tilde{\mathcal{L}} \cap \{\alpha \leq |\tau| \leq \alpha^{-1}\}$  in  $T^* \tilde{\mathcal{L}}$ , such that  $U_T = U \cap \{|t - t^0| \leq T\}$  is compact. There exists a increasing function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$|\langle \mu - \mu_{\varphi(n)}, a \rangle| \leq \frac{1}{n} \|a\|_{L^\infty}, \quad a \in \mathcal{C}_c^0(U_n),$$

recalling that the strong topology is equivalent to the weak topology for a converging sequence of measures; see [30, Section 34.4]. Denote by  $\mu_n$  this extracted sequence for concision:

$$(10.16) \quad |\langle \mu - \mu_n, a \rangle| \leq \frac{1}{n} \|a\|_{L^\infty}, \quad a \in \mathcal{C}_c^0(U_n).$$

Consider a sequence  $(\psi_n)_n \subset \mathcal{C}_c^\infty(\mathbb{R})$  such that  $\psi_n = 1$  on a neighborhood of  $[-n, n]$  and  $\text{supp } \psi_n \subset [-n - 1, n + 1]$ . We write  $\psi_n$  in place of  $\psi_n(t)$  for concision. The measure  $|\psi_n|^2 \mu_n$  is the limit of  $\mu_{n,k} = |W \psi_n u_{n,k}|^2$  as  $k \rightarrow +\infty$  in the measure sense, where  $W \psi_n u_{n,k}$  is (a variant of) the Wiegner transform of  $\psi_n(t) u_{n,k}(t, x)$ ; see [16]. Thus, there exists a increasing function  $\tilde{\varphi} : \mathbb{N} \rightarrow \mathbb{N}$  such

that

(10.17)

$$|\langle \mu_{n, \hat{\varphi}(n)} - |\psi_n|^2 \mu_n, a \rangle| = |\langle \mu_{n, \hat{\varphi}(n)} - \mu_n, a \rangle| \leq \frac{1}{n} \|a\|_{L^\infty}, \quad a \in \mathcal{C}_c^0(U_n).$$

From (10.16)–(10.17), one finds that  $\mu_{n, \hat{\varphi}(n)} \rightarrow \mu$  on  $U$  in the measure sense as  $n \rightarrow \infty$ . It follows that  $\mu$  is the s.c.m. of  $v_n = u_{n, \varphi(n)}$  on  $\hat{\mathcal{L}}$  at scale  $h_{\varphi(n)}$  by [16, Proposition 1.4]. Denote by  $\nu$  the s.c.m. of  $h_{\varphi(n)} \partial_\nu v_n$ , by potentially performing yet another subsequence extraction. One has  $\text{supp } \nu \subset \{\alpha \leq |\tau| \leq \alpha^{-1}\}$ . Theorems 6.1 and 2.14 apply, implying that  $\text{supp } \mu$  is a union of maximal generalized bicharacteristics.

Suppose  $\mathsf{T} > 0$  and  $a \in \mathcal{C}_c^0(U_{\mathsf{T}})$  is such that  $\text{supp } a \cap \Gamma^{\mathsf{T}}(\varrho^0) = \emptyset$ . In particular, set  $\varepsilon = \text{dist}(\text{supp } a, \Gamma^{\mathsf{T}}(\varrho^0))$ . There exists  $N \in \mathbb{N}^*$  such that

$$n \geq N \text{ and } \varrho \in \Gamma^{\mathsf{T}}(\varrho^n) \Rightarrow \text{dist}(\varrho, \Gamma^{\mathsf{T}}(\varrho^0)) \leq \varepsilon/2,$$

by Proposition 2.11 in the companion article [5]. Because of the description of  $\text{supp } \mu_n$  given above one finds that  $\langle \mu_n, a \rangle = 0$  if  $n \geq N$ . With (10.16) one obtains  $|\langle \mu, a \rangle| \leq \frac{1}{n} \|a\|_{L^\infty}$  if  $n \geq N$  thus giving  $\langle \mu, a \rangle = 0$ . Hence,

$$\text{supp } \mu \cap U_{\mathsf{T}} \subset \Gamma^{\mathsf{T}}(\varrho^0).$$

One concludes that  $\text{supp } \mu$  is a union of maximal generalized bicharacteristics that all go through  $\varrho^0$ .

One is now in the same position as in the proof of the case of a boundary observation where  $\varrho^0 \notin \partial T^* \mathcal{L}$ . The proof can be carried out *mutatis mutandis*: first,  $\text{supp } \nu \cap \|\mathcal{H}_\partial = \emptyset$ , second,  $\mathcal{G}_\partial \cap \mathcal{B}_{esc} \cap \text{supp } \nu = \emptyset$  implying that  $\text{supp } \nu \subset \mathcal{G}_\partial^d \cup \mathcal{G}_\partial^3$  yielding a contradiction.

This concludes the proof of Theorem 10.3.  $\square$

**Proof of Lemma 10.20.** Consider  $y_k$  solution to the homogeneous wave equation (10.11) with  $y_k|_{t=t^0} = \underline{y}_k^0 = \chi(-a_{\kappa, g}(x, \xi^0))v_k$  and  $\partial_t y_k|_{t=t^0} = \underline{y}_k^1 = ih_k^{-1} \tau^0 \underline{y}_k^0$ . Since  $\text{supp}(y_k|_{t=t^0})$  and  $\text{supp}(\partial_t y_k|_{t=t^0})$  are away from  $\partial \mathcal{M}$ , by finite-speed propagation  $y_k$  vanishes in a fixed open neighborhood  $W$  of  $\{t = t^0\} \cap \partial \mathcal{L}$ . By (10.8) and Lemma 10.18 one has

$$\|h_k \nabla_g(\underline{u}_k^0 - \underline{y}_k^0)\|_{L^2(\mathcal{M})} \rightarrow 0 \quad \text{and} \quad \|h_k \underline{u}_k^1 - h_k \underline{y}_k^1\|_{L^2(\mathcal{M})} \rightarrow 0.$$

One concludes that the semi-classical energy of  $u_k - y_k$  converges to 0. Hence, one finds

$$\|h_k \nabla_g u_k\|_{L^2(W)}^2 + \|h_k \partial_t u_k\|_{L^2(W)}^2 \rightarrow 0,$$

yielding  $(|\tau|^2 + |\xi|_x^2)\mu = 0$  in  $T^*W$ . In particular, this implies that the support of  $\mu$  is restricted to the null section in  $T^*W$ . With (10.13) one obtains that  $\mu$  vanishes in  $T^*W$ .  $\square$

## 11. PROOF OF THE MEASURE EQUATION AT AN ISOCHRONE

Here, we prove Theorem 10.7. We treat the case  $\underline{t} = 0$  without any loss of generality. At the hypersurface  $t = 0$ , there is no boundary condition. The two traces  $u_k|_{t=0} = \underline{u}_k^0$  and  $\partial_t u_k|_{t=0} = \underline{u}_k^1$  have to be taken into account in the analysis. Proceeding as is done for the measure equations at the boundary  $\partial\mathcal{L}$  in Sections 7 and 9 makes a double-layer potential appear, and it cannot be handled by the method used therein. We choose to proceed differently here, letting the measure act on tangential symbols. This approach can for instance be found in [11] for the treatment of Zarembo boundary conditions at a boundary.

**11.1. Preliminary filtering.** As in the proof of the measure equation of Theorem 6.1 we first apply some filtering to reduce the support of the measures. The principle is very similar to what is done in the beginning of Section 7.1, yet more technical because of the low regularity of the coefficients of the operator  $A_{\kappa,g}$ .

Consider  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  with  $0 \notin \text{supp } \chi$ . Set  $\tilde{u}_k = \chi(-h^2 A_{\kappa,g})u_k$ ,  $\tilde{f}_k = \chi(-h^2 A_{\kappa,g})f_k$ ,  $\tilde{\underline{u}}_k^0 = \chi(-h^2 A_{\kappa,g})\underline{u}_k^0$ , and  $\tilde{\underline{u}}_k^1 = \chi(-h^2 A_{\kappa,g})\underline{u}_k^1$ . One has

$$\begin{cases} P_{\kappa,g} \tilde{u}_k = \tilde{f}_k & \text{in } \mathbb{R} \times \mathcal{M}, \\ \tilde{u}_k = 0 & \text{in } \mathbb{R} \times \partial\mathcal{M}, \\ \tilde{u}_k|_{t=0} = \tilde{\underline{u}}_k^0, \partial_t \tilde{u}_k|_{t=0} = \tilde{\underline{u}}_k^1 & \text{in } \mathcal{M}. \end{cases}$$

**Proposition 11.1.** *The sequence  ${}^t(\tilde{u}_k, h_k \tilde{f}_k)_k$  admits  $|\chi(-a_{\kappa,g})|^2 M$  as its Hermitian s.c.m. on  $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$  at scale  $H$ . The sequence  ${}^t(\tilde{\underline{u}}_k^0, h_k \tilde{\underline{u}}_k^1)_k$  admits  $|\chi(-a_{\kappa,g})|^2 \nu^0$  as its Hermitian s.c.m. on  $T^*\mathcal{M} \setminus \partial(T^*\mathcal{M})$  at scale  $H$ .*

The proof of this intuitive result is given in Section 11.1.1 below. Note that the s.c.m. of  $\tilde{u}_k$  also reads  $|\chi(\tau^2)|^2 \mu$  from the assumed support properties.

With Lemma 10.6, the sequence  ${}^t(\mathbf{1}_{t>0} \tilde{u}_k, \mathbf{1}_{t>0} h_k \tilde{f}_k)_k$  has  $|\chi(-a_{\kappa,g})|^2 M^+$  for measure. If we prove that the measure equation (10.3) holds for  $M^+$  and  $\nu^0$  replaced by  $|\chi(-a_{\kappa,g})|^2 M^+$  and  $|\chi(-a_{\kappa,g})|^2 \nu^0$ , then using (10.2) one finds that (10.3) holds also for  $\mu^+$  and  $\nu^0$  by the dominated-convergence theorem. Without any loss of generality we may thus replace  $u_k$  by  $\tilde{u}_k$ ,  $f_k$  by  $\tilde{f}_k$ ,  $\underline{u}_k^0$  by  $\tilde{\underline{u}}_k^0$ , and  $\underline{u}_k^1$  by  $\tilde{\underline{u}}_k^1$ . Then, there exists  $0 < C_{\mu,0} < 1 < C_{\mu,1} < \infty$  such that

$$(11.1) \quad \text{supp } \mu^+ \subset \text{Char } p \cap T^*\mathcal{L} \cap \{C_{\mu,0} \leq |\xi| \leq C_{\mu,1}\},$$

and

$$\text{supp } \nu^0 \subset T^*\mathcal{M} \cap \{C_{\mu,0} \leq |\xi| \leq C_{\mu,1}\}.$$

Suppose  $I$  is a time interval. With the filtering used above, one has

$$(11.2) \quad \|u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 A_{\kappa,g} u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 \partial_t^2 u_k - h_k^2 f_k\|_{L^2(I \times \mathcal{M})}.$$

Assume that a subsequence of  $u_k$  converges to 0 in  $L^2(I \times \mathcal{M})$ . This gives  $\mu = 0$  on  $T^*(I \times \mathcal{M})$ . With (11.2), one finds that  $\|h_k^2 \partial_t^2 u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0$  and  $\|h_k^2 A_{\kappa,g} u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0$  also, using that  $h_k f_k$  is  $L^2_{\text{loc}}$ -bounded. Then, ellipticity up to the boundary gives  $\|h_k^2 u_k\|_{H^2(I \times \mathcal{M})} \rightarrow 0$  and interpolation gives

$$\|h_k \partial_t u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0 \quad \text{and} \quad \|h_k \nabla_{g_k} u_k\|_{L^2(I \times \mathcal{M})} \rightarrow 0.$$

Since (10.1) implies the time continuity of the semi-classical energy uniformly in  $k$  one obtains that

$$\|h_k \partial_t u_k(0, \cdot)\|_{L^2(\mathcal{M})} \rightarrow 0 \quad \text{and} \quad \|h_k \nabla_{g_k}(0, \cdot)\|_{L^2(\mathcal{M})} \rightarrow 0.$$

One concludes that  $\nu^0 = 0$ . Hence, all terms in the measure equation vanish, in this case. One may thus assume that  $\|u_k\|_{L^2(I \times \mathcal{M})} \gtrsim 1$ , for any interval  $I$ . Then, one finds that

$$1 \approx \|u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 A_{\kappa,g} u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k^2 \partial_t^2 u_k\|_{L^2(I \times \mathcal{M})},$$

and one further obtains

$$(11.3) \quad \|h_k^{\ell+2\ell} D_t^\ell A_{\kappa,g}^{\ell'} u_k\|_{L^2(I \times \mathcal{M})} \approx 1, \quad \ell, \ell' \in \mathbb{N},$$

and

$$\|h_k \partial_t u_k\|_{L^2(I \times \mathcal{M})} \approx \|h_k \nabla_{g_k} u_k\|_{L^2(I \times \mathcal{M})} \approx 1.$$

By (11.3) one finds that  $\mathbf{1}_{t>0} \psi(t) u_k \in H_{sc}^s(\mathcal{L})$  for any  $s \in [0, 1/2[$  and  $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ . From Lemma 5.17 one deduces the following result.

**Lemma 11.2.** *No mass leaks at infinity at scale  $H$  for  $\mathbf{1}_{t>0}(\psi(t)u_k)_k$ .*



11.1.1. *Proof of Proposition 11.1.* We prove that the s.c.m. of  $\tilde{u}_k$  is  $|\chi(-a_{\kappa,g})|^2\mu$  on  $T^*\mathcal{L} \setminus \partial(T^*\mathcal{L})$ . The proof for the other sequences and measures are the same.

Suppose  $j \in \mathcal{C}_c^\infty(T^*\mathcal{L})$  and  $\psi \in \mathcal{C}_c^\infty(\mathcal{L})$  with  $\psi = 1$  on the  $(t, x)$ -projection of  $\text{supp } j$  and  $\text{supp}(\psi) \cap \partial\mathcal{L} = \emptyset$ . Arguing as in Proposition 7.2 one proves the following lemma.

**Lemma 11.3.** *One has  $[[\text{Op}^h](j)\psi, h^2A_{\kappa,g}] \in h\mathcal{L}(L_{\text{loc}}^2(\mathcal{L}))$ .*

One deduces the following result.

**Lemma 11.4.** *One has  $[[\text{Op}^h](j)\psi, (z + h^2A_{\kappa,g})^{-1}] \in h|\text{Im } z|^{-2}\mathcal{L}(L_{\text{loc}}^2(\mathcal{L}))$ .*

**Proof.** Suppose  $\text{Im } z \neq 0$ . With Lemma 11.3 one writes

$$(z + h^2A_{\kappa,g})[\text{Op}^h](j)\psi - [\text{Op}^h](j)\psi(z + h^2A_{\kappa,g}) \in h\mathcal{L}(L_{\text{loc}}^2(\mathcal{L})).$$

Letting  $(z + h^2A_{\kappa,g})^{-1}$  act, both from the left and the right one obtains the result using that  $\|(z + h^2A_{\kappa,g})^{-1}\|_{\mathcal{L}(L^2(\mathcal{M}))} \lesssim |\text{Im } z|^{-1}$ .  $\square$

With the Helffer-Sjöstrand formula (10.6) the result extends to  $\chi(-h^2A_{\kappa,g})$ .

**Lemma 11.5.** *Suppose  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ . One has*

$$[[\text{Op}^h](j)\psi, \chi(-h^2A_{\kappa,g})] \in h\mathcal{L}(L_{\text{loc}}^2(\mathcal{L})).$$

With the previous lemma, one writes

$$\begin{aligned} & ([\text{Op}^h](j)\psi\chi(-h^2A_{\kappa,g})u_k, \chi(-h^2A_{\kappa,g})u_k)_{L^2(\mathcal{L})} \\ &= (\chi(-h^2A_{\kappa,g})[\text{Op}^h](j)\psi\chi(-h^2A_{\kappa,g})u_k, u_k)_{L^2(\mathcal{L})} \\ &= (\chi(-h^2A_{\kappa,g})^2[\text{Op}^h](j)\psi u_k, u_k)_{L^2(\mathcal{L})} + O(h_k) \\ &= (\chi^2(-h^2A_{\kappa,g})[\text{Op}^h](j)\psi u_k, u_k)_{L^2(\mathcal{L})} + O(h_k) \\ &= ([\text{Op}^h](\chi^2(-a_{\kappa,g}))[\text{Op}^h](j)\psi u_k, u_k)_{L^2(\mathcal{L})} + O(h_k), \end{aligned}$$

using (10.7).

It suffices to prove the result of the proposition with a test function supported in a local chart. Moreover, it can be chosen of the form  $j(t, x, \tau, \xi) = j_1(t, x)j_2(\tau, \xi)$  with  $j_1, j_2 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ . Then,  $\text{Op}^h(j) = j_1 \text{Op}^h(j_2)$ . With a partition of unity and using Proposition 5.10 one has  $[[\text{Op}^h](\chi^2(-a_{\kappa,g})), j_1] \in h\mathcal{L}(L^2(\mathcal{M}))$  yielding

$$\begin{aligned} & ([\text{Op}^h](j)\psi\chi(-h^2A_{\kappa,g})u_k, \chi(-h^2A_{\kappa,g})u_k)_{L^2(\mathcal{L})} \\ &= ([\text{Op}^h](j_1j_2\chi^2(-a_{\kappa,g}))\psi u_k, u_k)_{L^2(\mathcal{L})} + O(h_k). \end{aligned}$$

One thus obtains

$$\lim_{k \rightarrow +\infty} ([\text{Op}^h](j)\chi(-h^2 A_{\kappa,g})\psi u_k, \chi(-h^2 A_{\kappa,g})u_k)_{L^2(\mathcal{L})} = \langle \mu, \chi^2(-a_{\kappa,g})j \rangle,$$

which is the result of Proposition 11.1.  $\square$

**11.2. Symbol decomposition.** The measure equation is local. Consequently, the remaining of the proof can be carried out in local charts. Suppose  $\mathcal{C} = (\mathcal{O}, \phi_{\mathcal{L}})$  is such a local chart, with  $\mathcal{O}$  neighborhood of a point  $\varrho^0 \in \mathcal{L} \setminus \partial\mathcal{L}$ .

As in Section 8.1 we consider  $b(\varrho) \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  and proceed with a Euclidean symbol division. The symbol  $b$  has compact support in the  $y = (t, x)$  variables,  $\text{supp } b \subset K \times \mathbb{R}^{d+1}$ ,  $K$  a compact subset of  $\phi_{\mathcal{L}}(\mathcal{O})$  and fast decay in the  $\eta = (\tau, \xi)$  variables.

Consider  $0 < C_0 < C'_0 < C'_1 < C_1$  and  $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$  such that  $\chi(\xi) = 1$  in a neighborhood of  $\{C'_0 \leq |\xi| \leq C'_1\}$  and supported in  $\{C_0 \leq |\xi| \leq C_1\}$ . One has

$$\chi(\xi)b(\varrho) = b_0(t, x, \xi) + b_1(t, x, \xi)\tau + qp(\varrho),$$

by Proposition 8.2, here, with the role of the variable  $\xi_d$  played by  $\tau$ . The symbols  $b_0$  and  $b_1$  fulfill (an adapted version of) Property (8.3), that is,

$$(11.4) \quad |\partial_y^\alpha \partial_\xi^\beta b_j(y, \xi)| \leq C_{N,\beta} \langle \xi \rangle^{-N},$$

for  $N \in \mathbb{N}$ ,  $\alpha \in \mathbb{N}^{d+1}$ ,  $|\alpha| \leq 1$ ,  $\beta \in \mathbb{N}^d$ ,  $j = 0, 1$ ,  $y \in \mathbb{R}^{d+1}$ ,  $\xi \in \mathbb{R}^d$ ,

and  $q$  fulfills (an adapted version of) Property (8.4).

Recall that  $\varrho = (t, x, \xi, \tau) \in \text{supp } \mu^+$  implies  $|\xi| \in [C_{\mu,0}, C_{\mu,1}]$ ; see (11.1). Without loss of generality one can assume  $C_{\mu,0} < 1 < C_{\mu,1}$ . The constants  $C_0, C'_0, C'_1, C_1$  are chosen such that

$$\text{supp } \chi \subset \{C_{\mu,0}^2 < |\xi| < C_{1,\mu}^2\}$$

and  $\chi = 1$  in a neighborhood of  $\{C_{\mu,0} \leq |\xi| \leq C_{\mu,1}\}$ .

One writes

$$\langle {}^t\text{H}_p \mu, b \rangle = \langle \mu, \text{H}_p b \rangle = \langle \mu, \chi \text{H}_p b \rangle = \langle \mu, \text{H}_p(\chi b) \rangle$$

since  $\chi = 1$  on  $\text{supp } \mu$  and  $\text{supp } d_\xi \chi \cap \text{supp } \mu = \emptyset$ . One has

$$\text{H}_p(\chi b) = \text{H}_p b_0 + (\text{H}_p b_1)\tau + (\text{H}_p q)p,$$

implying

$$(11.5) \quad \langle {}^t\text{H}_p \mu, b \rangle = \langle \mu, \text{H}_p b_0 + (\text{H}_p b_1)\tau \rangle.$$

With the support properties of  $\mu$  given in (11.1) this last duality bracket makes sense.

**11.3. Commutator and limits.** As in Section 7.2 we consider the commutator given by  $[h_k^2 P_{\kappa,g}, \text{Op}^h(b)\psi]$ . With (11.5), the symbol  $b$  can be chosen of the form  $b(\varrho) = b_0(y, \xi) + b_1(y, \xi)\tau$ , with  $\text{supp } b_j \subset K \times \mathbb{R}^{d+1}$  for  $K$  a compact of  $\phi_{\mathcal{L}}(\mathcal{O})$  and  $b_j$  fulfilling Property (11.4),  $j = 0, 1$ , meaning fast decay in the  $\xi$  variable and only one derivative in the  $x$  variable. Recall that  $y = (t, x)$  and  $\varrho = (t, x, \tau, \xi)$ . The symbols  $b_j$  are tangential with respect to time  $t$ .

Suppose  $\psi \in \mathcal{C}_c^\infty(\phi_{\mathcal{L}}(\mathcal{O}))$  is equal to 1 in a bounded neighborhood of  $K$ . Arguing as in the proof of Proposition 7.2 one obtains the following result.

**Lemma 11.6.** *One has*

$$[h_k^2 P_{\kappa,g}, \text{Op}^h(b)\psi] = -ih_k \text{Op}^h(\mathbb{H}_p b)\psi + o(h_k)_{\mathcal{L}(L^2)}.$$

One computes

$$\mathbb{H}_p b(\varrho) = \{p, b\}(\varrho) = \tilde{b}_0(y, \xi) + \tilde{b}_1(y, \xi)\tau + \tilde{b}_2(y, \xi)\tau^2.$$

From (11.4) one has the estimations

$$|\partial_\xi^\beta \tilde{b}_j(y, \xi)| \leq C_{N,\beta} \langle \xi \rangle^{-N}, \quad \text{for } N \in \mathbb{N}, \beta \in \mathbb{N}^d, \quad y \in \mathbb{R}^{d+1}, \xi \in \mathbb{R}^d,$$

that is,  $\tilde{b}_j \in \Sigma_{\mathbb{T},0}^{0,n}(\langle \eta' \rangle^{-N}; \mathbb{R}^{d+1} \times \mathbb{R}^d)$ , for any  $n$  and  $N$ . Note that  $\psi = 1$  on a neighborhood of the  $(t, x)$ -projection of  $\text{supp } \tilde{b}_j$ ,  $j = 0, 1, 2$ . By Lemma 5.19 in the tangential case (see Remark 5.22) and Proposition 5.21, one has

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_0)\psi u_k, u_k)_{L^2(t>0)} = \lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_0)\psi \mathbf{1}_{t>0} u_k, \mathbf{1}_{t>0} u_k)_{L^2(\mathbb{R}^d)} = \langle \mu^+, \tilde{b}_0 \rangle,$$

using that that no mass leaks at infinity at scale  $H$  by Lemma 11.2.

The following two results also hold.

**Lemma 11.7.** *One has*

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_1)h_k D_t \psi u_k, u_k)_{L^2(t>0)} = \langle \mu^+, \tilde{b}_1 \tau \rangle$$

and

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_2)h_k^2 D_t^2 \psi u_k, u_k)_{L^2(t>0)} = \langle \mu^+, \tilde{b}_2 \tau^2 \rangle.$$

With Lemmata 11.6 and 11.7 one concludes that

$$(11.6) \quad \lim_{k \rightarrow +\infty} i([h_k P_{\kappa,g}, \text{Op}^h(b)\psi]u_k, u_k)_{L^2(t>0)} = \langle \mu^+, \mathbb{H}_p b \rangle = \langle {}^t \mathbb{H}_p \mu^+, b \rangle.$$

**Proof of Lemma 11.7.** We treat the second limit. As  $[hD_t^2, \psi] \in h\mathcal{L}(L^2)hD_t + h\mathcal{L}(L^2)$ , with (11.3) one has

$$\begin{aligned} (\text{Op}^h(\tilde{b}_2)h_k^2D_t^2\psi u_k, u_k)_{L^2(t>0)} &= (\text{Op}^h(\tilde{b}_2)\psi h_k^2D_t^2u_k, u_k)_{L^2(t>0)} + O(h_k) \\ &= -(\text{Op}^h(\tilde{b}_2)\psi h_k^2A_{\kappa,g}u_k, u_k)_{L^2(t>0)} + O(h_k) \\ &= -(\text{Op}^h(\tilde{b}_2)\psi h_k^2A_{\kappa,g}\mathbf{1}_{t>0}u_k, \mathbf{1}_{t>0}u_k)_{L^2} + O(h_k). \end{aligned}$$

Then, one obtains

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_2)h_k^2D_t^2\psi u_k, u_k)_{L^2(t>0)} = -\langle \mu^+, \tilde{b}_2 a_{\kappa,g} \rangle,$$

arguing as in Proposition 7.2 and using Lemma 5.19 in the tangential case (see Remark 5.22) and Proposition 5.21, using that that no mass leaks at infinity at scale  $H$  by Lemma 11.2. Then, using that  $\text{supp } \mu^+ \subset \text{supp } \mu \subset \text{Char } p$  where  $\tau^2 = -a_{\kappa,g}$ , one obtains

$$\lim_{k \rightarrow +\infty} (\text{Op}^h(\tilde{b}_2)h_k^2D_t^2\psi u_k, u_k)_{L^2(t>0)} = \langle \mu^+, \tilde{b}_2 \tau^2 \rangle.$$

Second, suppose  $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$  with  $\varphi = 1$  in a neighborhood of 0. One writes

$$(\text{Op}^h(\tilde{b}_1)h_kD_t\psi u_k, u_k)_{L^2(t>0)} = (\text{Op}^h(\tilde{b}_1)\mathbf{1}_{t>0}h_kD_t\psi u_k, \mathbf{1}_{t>0}u_k)_{L^2} = I_1 + I_2,$$

with

$$I_1 = (\text{Op}^h(\tilde{b}_1)\text{Op}^h(1 - \varphi(\tau/R))\mathbf{1}_{t>0}h_kD_t\psi u_k, \mathbf{1}_{t>0}u_k)_{L^2},$$

and

$$I_2 = (\text{Op}^h(\tilde{b}_1)\text{Op}^h(\varphi(\tau/R))\mathbf{1}_{t>0}h_kD_t\psi u_k, \mathbf{1}_{t>0}u_k)_{L^2},$$

Since  $h_k^{\ell+2\ell'}D_t^\ell A_{\kappa,g}^{\ell'}\psi u_k$  is bounded in  $L^2$  by (11.3) for any  $\ell, \ell' \in \mathbb{N}$ , one finds that  $\mathbf{1}_{t>0}h_kD_t\psi u_k \in H_{sc}^s(\mathcal{L})$  for any  $s \in [0, 1/2[$ . With Lemma 4.3 in [11] one obtains, for such  $s$ ,

$$\|\text{Op}^h(1 - \varphi(\tau/R))\mathbf{1}_{t>0}h_kD_t\psi u_k\|_{L^2} \leq C_s R^{-s},$$

uniformly in  $k$ . Thus, for any  $s \in [0, 1/2[$ , one has

$$(11.7) \quad |I_1| \leq C'_s R^{-s}.$$

Next, to treat the second term one writes

$$\mathbf{1}_{t>0}h_kD_t\psi u_k = h_kD_t(\psi\mathbf{1}_{t>0}u_k) + ih_k\psi|_{t=0}u_k^0 \otimes \delta_{t=0},$$

yielding  $I_2 = I'_2 + I''_2$  with

$$I'_2 = (\text{Op}^h(\tilde{b}_1) \text{Op}^h(\varphi(\tau/R)) h_k D_t(\psi \mathbf{1}_{t>0} u_k), \mathbf{1}_{t>0} u_k)_{L^2},$$

and

$$\begin{aligned} I''_2 &= i h_k (\text{Op}^h(\tilde{b}_1) \psi|_{t=0} \underline{u}_k^0 \otimes \text{Op}^h(\varphi(\tau/R)) \delta_{t=0}, \mathbf{1}_{t>0} u_k)_{L^2} \\ &= i h_k (\text{Op}^h(\tilde{b}_1) \psi|_{t=0} \underline{u}_k^0 \otimes \text{Op}^h(\varphi(\tau/R)) \delta_{t=0}, \mathbf{1}_{t>0} \psi u_k)_{L^2}. \end{aligned}$$

Set  $\hat{b}_1(t, x, \tau, \xi) = \tau \varphi(\tau/R) \tilde{b}_1(x, \xi)$ ; it is smooth, with compact support in  $(t, x)$  and has fast decay in  $(\tau, \xi)$ . One has  $I'_2 = (\text{Op}^h(\hat{b}_1)(\psi \mathbf{1}_{t>0} u_k), \mathbf{1}_{t>0} u_k)_{L^2}$  and

$$\lim_{k \rightarrow \infty} I'_2 = \langle \mu^+, \hat{b}_1 \rangle \xrightarrow{R \rightarrow \infty} \langle \mu^+, \tau \tilde{b}_1 \rangle,$$

by dominated convergence as  $\text{supp } \mu^+$  is compact in the  $(\tau, \xi)$  variables by (11.1).

One has  $\text{Op}^h(\varphi(\tau/R)) \delta_{t=0} = h^{-1} R \check{\varphi}(h^{-1} R t)$ , with  $\check{\varphi}$  the inverse Fourier transform of  $\varphi$ . Thus one obtains

(11.8)

$$|I''_2| \lesssim R \|\underline{u}_k^0\|_{L^2(\mathcal{M})} \|\check{\varphi}(h^{-1} R \cdot)\|_{L^2(\mathbb{R})} \|\psi \mathbf{1}_{t>0} u_k\|_{L^2(\mathcal{L})} \lesssim h_k^{1/2} R^{1/2} \|\check{\varphi}\|_{L^2(\mathbb{R})} \lesssim h_k^{1/2} R^{1/2}.$$

With (11.7)–(11.8) one concludes the proof.  $\square$

**11.4. End of the proof of the measure equation at an isochrone.** With integrations by parts and one computes

$$i([h_k P_{\kappa, g}, \text{Op}^h(b)\psi] u_k, u_k)_{L^2(t>0)} = I_k + J_k,$$

with

$$\begin{aligned} I_k &= i(\text{Op}^h(b)\psi u_k, h_k f_k)_{L^2(t>0)} - i(\text{Op}^h(b)\psi h_k f_k, u_k)_{L^2(t>0)}, \\ J_k &= -i h_k (\partial_t \text{Op}^h(b)\psi u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} + i(\text{Op}^h(b)\psi u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})}. \end{aligned}$$

With Remark 10.9 one obtains

$$(11.9) \quad \lim_{k \rightarrow \infty} I_k = i\langle M_{0,1}^+, b \rangle - i\langle M_{1,0}^+, b \rangle = 2\langle \text{Im } M_{0,1}^+, b \rangle.$$

One writes  $\partial_t \text{Op}^h(b)\psi = \text{Op}^h(b)\psi \partial_t + (\text{Op}^h(\partial_t b)\psi + \text{Op}^h(b)(\partial_t \psi))$ . As  $\partial_t b = \partial_t b_0 + (\partial_t b_1)\tau$ , this gives

$$\begin{aligned} -i h_k (\partial_t \text{Op}^h(b)\psi u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} &= -i(\text{Op}^h(b)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + O(h_k), \end{aligned}$$

using that  $(u_k|_{t=0})_k$  and  $(h_k \partial_t u_k|_{t=0})_k$  are bounded sequences in  $L^2(\mathcal{M})$ . One thus finds

$$J_k = -i(\text{Op}^h(b)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} + i(\text{Op}^h(b)\psi u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} + O(h_k).$$

One further writes

$$\begin{aligned} \text{Op}^h(b)\psi &= \text{Op}^h(b_0)\psi + \text{Op}^h(b_1)D_t\psi \\ &= \text{Op}^h(b_0)\psi + \text{Op}^h(b_1)\psi D_t - ih_k \text{Op}^h(b_1)(\partial_t\psi), \end{aligned}$$

implying

$$\begin{aligned} &(\text{Op}^h(b)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} \\ &= (\text{Op}^h(b_0)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} - i(\text{Op}^h(b_1)\psi h_k^2 \partial_t^2 u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + O(h_k) \\ &= (\text{Op}^h(b_0)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} - i(\text{Op}^h(b_1)\psi h_k^2 A_{\kappa,g} u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + O(h_k), \end{aligned}$$

and

$$\begin{aligned} &(\text{Op}^h(b)\psi u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} \\ &= (\text{Op}^h(b_0)\psi u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} - i(\text{Op}^h(b_1)\psi h_k \partial_t u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + O(h_k). \end{aligned}$$

This yields

$$\begin{aligned} J_k &= -i(\text{Op}^h(b_0)\psi h_k \partial_t u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} - (\text{Op}^h(b_1)\psi h_k^2 A_{\kappa,g} u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + i(\text{Op}^h(b_0)\psi u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} + (\text{Op}^h(b_1)\psi h_k \partial_t u_k|_{t=0}, h_k \partial_t u_k|_{t=0})_{L^2(\mathcal{M})} \\ &\quad + O(h_k). \end{aligned}$$

Arguing as in the proof of Proposition 7.2 one finds

$$(\text{Op}^h(b_1)\psi h_k^2 A_{\kappa,g} u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} = (\text{Op}^h(b_1 a_{\kappa,g})\psi u_k|_{t=0}, u_k|_{t=0})_{L^2(\mathcal{M})} + O(h_k).$$

With (11.6) and (11.9), this gives

$$\begin{aligned} (11.10) \quad \langle {}^t\text{H}_p \mu^+, b \rangle &= 2\langle \text{Im } M_{0,1}^+, b \rangle - i\langle \nu_{1,0}^0, b_0 \rangle - \langle \nu_{0,0}^0, b_1 a_{\kappa,g} \rangle + i\langle \nu_{0,1}^0, b_0 \rangle + \langle \nu_{1,1}^0, b_1 \rangle \\ &= 2\langle \text{Im } M_{0,1}^+, b \rangle - 2\langle \text{Im } \nu_{0,1}^0, b_0 \rangle + \langle \nu_{1,1}^0 - a_{\kappa,g} \nu_{0,0}^0, b_1 \rangle. \end{aligned}$$

With the analysis of Section 11.2, Equation (11.10) holds for a symbol  $b(\varrho) \in \Sigma_0^{\mathcal{H}}(\mathbb{R}^{2d+2})$  that we write  $b(\varrho) = b_0(t, x, \xi) + b_1(t, x, \xi)\tau + qp_{\kappa, g}(\varrho)$ . Suppose  $(x, \xi) \in T^*\mathcal{M}$  and set  $\varrho = (t = 0, x, \xi, 0)$ , and

$$\varrho^{\oplus} = (0, x, \tau^+(\varrho), \xi), \quad \varrho^{\ominus} = (0, x, \tau^-(\varrho), \xi).$$

as in Section 10.2. Since  $\varrho^{\oplus}, \varrho^{\ominus} \in \text{Char } p$  and  $\tau^+(\varrho) = -\tau^-(\varrho)$ , one has

$$b_0(t = 0, x, \xi) = \frac{1}{2}(b(\varrho^{\oplus}) + b(\varrho^{\ominus})), \quad b_1(t = 0, x, \xi) = \frac{b(\varrho^{\oplus}) - b(\varrho^{\ominus})}{\tau^+(\varrho) - \tau^-(\varrho)}.$$

Plugged in (11.10), this gives the result of Theorem 10.7.  $\square$

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