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Time-space formulation of a conservative string subject to finite transformations [★]

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Abstract: This paper investigates power-balanced descriptions of the conservative vibrating string. First, the hypotheses of continuum mechanics are recalled for the case of infinitesimal transformations. The string is described by classical partial differential equations (PDEs) and reformulated as a port Hamiltonian system (PHS). Second, the case of finite (possibly large) transformations is considered, for which the time variation of the elastic energy appears to be no longer the elastic power. To naturally solve this difficulty and ensure the invariance of the elastic power with respect to the superposition of rigid body motion, the problem is here addressed in the framework of a time-space formulation. Eulerian and Lagrangian conservative formulations are proposed in the context of non relativistic velocities. This work also yields perspectives for a time-space representation of a port-Hamiltonian vibrating string invariant to the change of observer in a relativistic context.

Keywords: String vibrations, Finite-transformations, Conservation laws, Hamiltonian dynamics, Geometric mechanics, Time-space formulation

1. INTRODUCTION

Consider a string defined as a 1D geometric object embedded into (3D or) 2D space in the form of a line, exclusively resistant to traction (see Figure 1). We are interested in building intrinsic formulations of a conservative vibrating string under increasingly demanding hypotheses (from small perturbations, to finite transformations, and time-space covariant formulation). The string is equipped with an inertial mass and elastic behaviour, and vibrates around an equilibrium state imposed by a static pre-stress.

In classical mechanics, the kinematics is described by $x = \Phi(t, X) = X + W(t, X)$. In the case of the string model, the exclusive resistance to traction can be formulated by assuming that tangent application F of Φ may be decomposed into a in plane rotation of angle θ (local direction of the string) and a 1D-strain λ such that $F = R(\theta)U(\lambda)$. Note that this approach with a similar kinematics can be found in the work of Golo et al. (2003) for a beam under infinitesimal deformation within the port-Hamiltonian framework; see also Bideau et al. (2011) for finite transformations.

In this paper, we propose a transformation such as its tangent application can be identified with this decomposition. This leads to an in-plane displacement that depends on X^1, X^2 .

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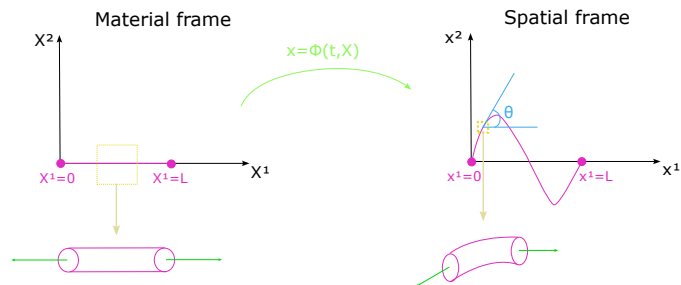


Fig. 1. The string kinematic hypothesis.

This paper investigates these hypotheses (geometry, inertia, kinematics, material behaviour) and proposes a model in three contexts: linear behaviour and infinitesimal transformations in Section 2, nonlinear behaviour under finite transformations in Section 3, time-space formulation invariant to change of observer (covariant formulation) in Section 4. The paper ends with some perspectives.

NOTATIONS

The coordinates of a point M is denoted by the n-uplet $x = (x^1, x^2, \dots)$ with:

$X^{i=1,2}$: spatial coordinates in the material 2D frame with absolute time t ,

$x^{i=1,2}$: spatial coordinates in the current 2D frame with absolute time t ,

$\hat{x}^{\nu=0,1,2,3}$: time-space coordinates for the proper observer,

$x^{\nu=0,1,2,3}$: time-space coordinates for the current observer. Vectors and tensors are denoted respectively: $\mathbf{u} = u^i \mathbf{e}_i$ with u^i : component and \mathbf{e}_i basis vector; $\mathbf{T} = T^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ with T^{ij} : component and \otimes the tensorial product.

2. CONSERVATIVE STRING UNDER INFINITESIMAL TRANSFORMATIONS

2.1 Hypotheses and representation

In this section, the hypothesis of infinitesimal transformations is assumed. The string motion is represented by the transverse displacement w (in [m]) in the current frame (t, x^1, x^2) . Its inertia and elastic behaviour is governed by linear mechanical laws.

2.2 Continuum mechanics formulation (PDE)

Under these hypotheses, function $w : (0, +\infty) \times (0, L) \rightarrow \mathbb{R}$ is governed in the current frame (t, x^1, x^2) by (see e.g. Chaigne and Kergomard (2016)), for all $(t, x^1) \in (0, +\infty) \times (0, L)$,

$$\mu(x^1) \partial_t^2 w(t, x^1) - \partial_{x^1} (T(x^1) \partial_{x^1} w(t, x^1)) = f(t, x^1), \quad (1)$$

where $\mu = \rho A$ [kg.m⁻¹] is the lineic density (also called linear density) of mass (for mass density ρ and cross-section area A), T [N] is the tension¹, and f [N/m] is the external lineic force distributed along the string. Typical Dirichlet boundary conditions ($w = 0$ on $(0, +\infty) \times \{0, L\}$) and zero initial conditions ($w = 0$ on $\{0\} \times (0, L)$) are considered.

The integral over $(0, L)$ of the force balance (1) multiplied by the velocity $\partial_t w$ yields the power balance. After integration by part, it leads to (omitting the independent variables t and x^1 for conciseness)

$$\partial_t \int_0^L \left(\frac{\mu(\partial_t w)^2}{2} + \frac{T(\partial_{x^1} w)^2}{2} \right) dx^1 = \int_0^L f \partial_t w dx^1. \quad (2)$$

This equation involves lineic densities of energy ($\frac{\mu(\partial_t w)^2}{2}$ for inertia and $\frac{T(\partial_{x^1} w)^2}{2}$ for elasticity) and of the external power received by the string ($f \partial_t w$).

This conservative problem naturally admits a PHS formulation as recalled in the following section.

2.3 Port-Hamiltonian Systems formulation (PHS)

This problem admits a port-Hamiltonian formulation (see van der Schaft and Maschke (2002))

$$\underbrace{\begin{pmatrix} \partial_t \alpha \\ y \end{pmatrix}}_{\text{flow } f} = \mathcal{J} \underbrace{\begin{pmatrix} \delta H(\alpha) \\ u \end{pmatrix}}_{\text{effort } e}. \quad (3)$$

where the quantities (effort e , flow f) and the equations stemming from operator \mathcal{J} are naturally interpretable (as detailed below) for the following physically relevant choice of α , input u and output y .

¹ The tension is usually considered to be homogeneous and due to pre-stressing. But a spatial dependency can be due for a vertical string subjected to gravity (Chaigne and Kergomard, 2016, §3.4.1).

State α (or configuration, or energy variable)

$$\alpha := \begin{bmatrix} \pi := \mu \partial_t w \\ \varepsilon := \partial_{x^1} w \end{bmatrix} \begin{bmatrix} [\text{kg/s}] \text{ lineic momentum} \\ [\text{adim}] \text{ (infinitesimal) strain} \end{bmatrix} \quad (4a)$$

with $x^1 \mapsto \pi(x^1, t) \in H_0^1$ (due to Dirichlet boundary conditions) and $x^1 \mapsto \varepsilon(x^1, t) \in H^1$ for strong solutions under Dirichlet boundary conditions².

Hamiltonian H (or energy functional) For all spatial function $\alpha : x^1 \mapsto [\pi(x^1), \varepsilon(x^1)]^\top$ in $L^2((0, L), \mathbb{R}^2)$,

$$\mathcal{H}(\alpha) := \int_0^L \frac{1}{2} \alpha(x^1)^\top W(x^1) \alpha(x^1) dx^1, \quad [\text{J}] \text{ energy} \quad (4b)$$

with $W = \text{diag}(1/\mu, T)$.

Flows, efforts and their relation with w or f

State α_i	Flow	Units	Effort	Units
$\alpha_1 = \pi$	$\partial_t \pi = \mu \partial_t^2 w$ lineic inertial force	[N/m]	$\pi/\mu = \partial_t w$ material point velocity	[m/s]
$\alpha_2 = \varepsilon$	$\partial_t \varepsilon = \partial_t \partial_{x^1} w$ strain rate	[s ⁻¹]	$T \partial_{x^1} w$ lineic elastic force	[N]
Port	$y := \partial_t w$ contact point velocity	[m/s]	$u := -f$ lineic force applied to the actuator	[N/m]

Operator $\mathcal{J} : H_0^1 \times H^1 \times L^2 \rightarrow (L^2)^3$

$$\mathcal{J} = \begin{bmatrix} 0 & \partial_{x^1} & -1 \\ \partial_{x^1} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (4c)$$

is skew-symmetric ($\mathcal{J} = -\mathcal{J}^*$), that is, $\langle a | \mathcal{J} b \rangle + \langle \mathcal{J} a | b \rangle = 0$ on the appropriate space, defining $\langle e | f \rangle := \int_0^L e(x^1)^\top f(x^1) dx^1$.

Note that reformulating (3) with respect to w reads

$$\text{momentum balance: } \partial_t (\mu \partial_t w) = \partial_{x^1} (T \partial_{x^1} w) + f, \quad (5a)$$

$$\text{kinematic concordance: } \partial_t (\partial_{x^1} w) = \partial_{x^1} (\partial_t w), \quad (5b)$$

$$\text{idem: } \partial_t w = \partial_t w, \quad (5c)$$

and $\langle e | f \rangle = 0$ ($\langle e | \mathcal{J} e \rangle = 0$ due to the skew-symmetry of \mathcal{J}) can be reformulated as (2), the power balance.

3. CONSERVATIVE STRING UNDER FINITE TRANSFORMATIONS

This section develops a model of a string in finite transformations following classical definitions (§3.1) and formulation (§3.2) of 3D-continuum mechanics, in order to review a number of issues (§3.3), that appear in both Euler and Lagrange representations or with classical port-Hamiltonian formulations.

3.1 Hypotheses (Hi) and definitions (Di)

(H1) String kinematics

The string kinematics can be seen as deformation $\mathbf{U}(\lambda)$ in the string longitudinal direction followed by a rotation $\mathbf{R}(\theta)$, this leads to the tangent linear application $\mathbf{F}(\theta, \lambda) = \mathbf{R}(\theta) \mathbf{U}(\lambda)$ on the string:

$$\mathbf{F}_j^i(\theta, \lambda) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (6)$$

² H^1 is a short notation for the Sobolev space $H^1((0, L), \mathbb{R}) := \{f \in L^2((0, L), \mathbb{R}) \text{ s.t. } f' \in L^2((0, L), \mathbb{R})\}$ and $H_0^1 := \{f \in H^1 \text{ s.t. } f(x^1 = 0) = f(x^1 = L) = 0\}$.

where λ is the axial elongation and θ the angle of the rotation of the axis of the string (see Figure 1).

(H2) Planar transformation

Consider a general space transformation defined by, for all $(t, X^1, X^2, X^3) \in \mathbb{R}^+ \times \mathbb{R}^3$,

$$x^1 = X^1 + W^1(t, X^1, X^2), \quad (7a)$$

$$x^2 = X^2 + W^2(t, X^1, X^2), \quad (7b)$$

$$x^3 = X^3, \quad (7c)$$

where $W^{1,2}$ accounts for the longitudinal and transverse displacement. The tangent linear application $T\Phi$ is noted $F_j^i = \partial_{X^j} x^i$ with

$$F_j^i = \begin{pmatrix} 1 + \partial_{X^1} W^1 & \partial_{X^2} W^1 & 0 \\ \partial_{X^1} W^2 & 1 + \partial_{X^2} W^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (8)$$

(H3) Matching the space transformation with the kinematics restricted by the string domain

The tangent linear application described by (8) has to coincide with the string kinematics (6), meaning that:

for all $(X^1, X^2) \in (0, L) \times \{0\}$,

$$\begin{pmatrix} 1 + \partial_{X^1} W^1 & \partial_{X^2} W^1 & 0 \\ \partial_{X^1} W^2 & 1 + \partial_{X^2} W^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda \cos \theta & -\sin \theta & 0 \\ \lambda \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (9)$$

leading to the following constraints:

$$1 + \partial_{X^1} W^1(t, X^1, 0) = \lambda \cos \theta, \quad (10a)$$

$$1 + \partial_{X^2} W^2(t, X^1, 0) = \cos \theta, \quad (10b)$$

$$\partial_{X^2} W^1(t, X^1, 0) = -\sin \theta, \quad (10c)$$

$$\partial_{X^1} W^2(t, X^1, 0) = \lambda \sin \theta. \quad (10d)$$

(D4) Strain tensor \mathbf{E}

The strain is described by the second order covariant tensor \mathbf{E} defined by

$$\mathbf{E} = \frac{1}{2} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) \quad (11a)$$

$$E_{ij} = \frac{1}{2} \begin{pmatrix} (1 + \partial_{X^1} W^1)^2 + (\partial_{X^1} W^2)^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11b)$$

$$= \frac{1}{2} \begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11c)$$

or by the pull-back $\mathbf{E} = \mathbf{F}^T \mathbf{eF}$. The transformation 7a-7c has been chosen to allow identification of the the tangent linear application 8 with 6 (combination of a deformation in the axial axis of the string and a rotation). This transformation of the 3D space is built to support the kinematic model of the string obtained for $X^2 = 0$. This allows us to verify that only E_{11} is non zero as required by our hypothesis.

The push-forward \mathbf{e} of \mathbf{E} is defined by

$$\mathbf{e} = \frac{1}{2} (\mathbf{I} - (\mathbf{F}^{-1})^T \mathbf{F}^{-1}) = (\mathbf{F}^{-1})^T \mathbf{E} \mathbf{F}^{-1} \quad (12)$$

(H5) Elastic energy (volume) density ψ .

$$\psi = \frac{1}{2} \mathbf{E} : \mathbf{CE}, \quad (13)$$

where the fourth order tensor \mathbf{C} characterises the behaviour of the string, i.e. tension associated to

stiffness (Young's modulus E) along its longitudinal axis:

$$C^{ijkl} : \begin{pmatrix} E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (14)$$

with Voigt notation. This yields

$$\psi = \frac{E}{8} \left[\left((1 + \partial_{X^1} W^1)^2 + (\partial_{X^1} W^2)^2 - 1 \right)^2 \right] \quad (15)$$

$$= \frac{E}{8} \left[(\lambda^2 - 1)^2 \right] \quad (16)$$

The Piola-Kirchhoff 2 stress tensor $\Sigma = \frac{d\psi}{d\mathbf{E}} = \mathbf{CE}$ is equal to the derivative of the proposed energy ψ :

$$\Sigma^{ij} = \frac{d\psi}{dE_{ij}} = \frac{E}{2} \begin{pmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (17)$$

(D6) The Cauchy stress tensor $\boldsymbol{\sigma}$ is the push forward of the PK2 stress tensor Σ

$$\boldsymbol{\sigma} = J^{-1} \mathbf{F} \Sigma \mathbf{F}^T \quad (18)$$

where J is the determinant of \mathbf{F} .

Hypothesis (H5) allows to define the Lagrangian stress tensor Σ , whereas definition (D6) is the Cauchy (Eulerian) stress tensor $\boldsymbol{\sigma}$.

3.2 Continuum mechanics formulation

Consider the velocity \mathbf{v} defined by $v^i = \frac{dx^i}{dt}$. In the current inertial frame the principle of linear momentum (see for example Eringen (1962) p. 104) is:

$$\text{div}(\boldsymbol{\sigma}) + \mathbf{f} = \rho \frac{d\mathbf{v}}{dt} \quad (19)$$

where ρ is the mass density and div is the divergence operator in an orthonormal coordinate system. Note that this formulation is Eulerian; the unknown is the velocity field at observation point defined by spatial variables and time.

The integral of (19) over the volume Ω , multiplied by the velocity \mathbf{v} yields the power balance. After integration by part, this leads to

$$\int_{\Omega} \left(\rho \frac{d(\frac{\mathbf{v}^2}{2})}{dt} + \boldsymbol{\sigma} : \mathbf{d} \right) d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega. \quad (20)$$

where \mathbf{d} is the rate of deformation (symmetric part of the velocity gradient). This balance equation involves the power density $\rho \frac{d(\frac{\mathbf{v}^2}{2})}{dt}$ for inertia, the mechanical power density ($\boldsymbol{\sigma} : \mathbf{d}$) and the external power received by the string ($\mathbf{f} \cdot \mathbf{v}$). Note that (20) is the counterpart of (2) for finite transformations.

3.3 Issues

This model of a string in finite transformations based on classical 3D-continuum mechanics suffers several issues. First, note that the principle of linear momentum is stated in the Eulerian configuration and in an inertial frame. Moreover:

Euler representation (see Besson et al. (2010) pp. 295-299): The Cauchy stress tensor in (18) appears to depend on the rigid motion via the transport of the anisotropic elastic tensor \mathbf{C} . This property breaks material symmetries and is considered unsuitable with the objectivity principle.

On the contrary the elastic energy ψ (13) does not depend on rigid body motion and its expression is also valid in an Eulerian configuration, i.e. $\psi = \frac{1}{2} \mathbf{e} : \boldsymbol{\sigma}$. This seems to be a good basis to derive a port-Hamiltonian objective formulation but this will reveal another issue about the mechanical power.

About the strain rate in PHS: The power balance given by (20) reveals that the stress power density ($\boldsymbol{\sigma} : \mathbf{d}$) is not derived with the differentiation of the mechanical energy with respect to time because $\dot{\mathbf{e}} \neq \mathbf{d}$. Note that $\dot{\mathbf{E}} = \mathbf{D}$ is valid only in the Lagrange configuration (\mathbf{D} being the material rate of deformation). These issues prevent to write a proper PHS formulation involving the standard time derivative whether in the Lagrangian or the Eulerian point of view. Every other nonlinear model considered in the literature (Kirchhoff-Carrier, Timoschenko beam, etc.) should encounter the same difficulties.

4. TIME-SPACE COVARIANT FORMULATION

4.1 Definitions and notations for a time-space formulation

The principle of covariance due to Einstein (1921) states that the formulation of the laws of physics should be invariant to changes of observers, in other word, should be covariant. We here call such a formulation an *intrinsic* formulation to avoid the confusion with the covariant *vs* contravariant status of a tensor. In the theory of relativity, an energy-momentum tensor \mathbf{T} is defined, the generalization of the stress tensor. Its conservation encompasses the conservation of energy and the balance of momentum for *any observer*. The formulation of continuum mechanics within a time-space context should thus bring new insights in particular for the treatment of problems involving systems undergoing finite transformations as proposed in Rouhaud et al. (2013); Panicaud and Rouhaud (2014); Panicaud et al. (2016); Al Nahas et al. (2021): the invariance with respect to changes of observers and passivity is by construction insured with such a formulation. For these reasons, this formulation should be of interest even in the case of non-relativistic velocities, such as in musical acoustics, in order to address the issues mentioned in section 3.3.

Consider thus a space-time domain modeled with a four-dimensional Riemannian manifold \mathcal{M} . A point M in \mathcal{M} is called an event. The manifold \mathcal{M} is endowed with a pseudo-metric tensor field \mathbf{g} , a symmetric bilinear form of signature $(1, -1, -1, -1)$.

We define an observer as a 4D coordinate system chosen on \mathcal{M} . On practical bases, this definition is to be related to the point of view of a physicist, that is an observer, who has made a specific convenient choice for a 4D coordinate system, corresponding to a choice of coordinates for space and clock for time. Some geometric definitions are first proposed to precise these notions (see for example Kolev (2020)). First define a map of \mathcal{M} as a set (U_α, ϕ_α) , where

U_α is an open set of \mathcal{M} , homeomorphic to an open set of \mathbb{R}^4 via the homeomorphism noted ϕ_α . Define then the local coordinates of an event M as the set of four real numbers $(x^\mu)_{\mu \in \{0,1,2,3\}} \in S_\alpha$ associated to M via the map (U_α, ϕ_α) . Consider two maps (U_α, ϕ_α) and (U_β, ϕ_β) ; when their intersection is not empty, it is possible to define diffeomorphism associated to a change of coordinates such that

$$\varphi : \tilde{x}^\nu \in S_\alpha \mapsto x^\mu = \varphi^\mu(\tilde{x}^\nu) \in S_\beta. \quad (21)$$

Then, in a neighborhood of M endowed with the coordinate system x^μ , a vector \mathbf{v} in $T_M\mathcal{M}$, the tangent space of \mathcal{M} at M , may be projected locally as ³:

$$\mathbf{v} = \sum_{\mu=0}^3 v^\mu \frac{\partial}{\partial x^\mu} = v^\mu \mathbf{e}_\mu, \quad (22)$$

because the differentiations $\left\{ \frac{\partial}{\partial x^\mu} \right\}$ at M define a local basis \mathbf{e}_μ of $T_M\mathcal{M}$; the quantities $(v^\mu)_{\mu \in \{0,1,2,3\}}$ are the components of \mathbf{v} in the coordinate system x^μ . Under a local change of coordinates, the basis transforms as $\mathbf{e}_\mu \rightarrow \left(\frac{\partial x^\kappa}{\partial \tilde{x}^\mu} \right) \frac{\partial}{\partial x^\kappa} = \tilde{\mathbf{e}}_\mu$ where the non-singular matrix $\left(\frac{\partial x^\kappa}{\partial \tilde{x}^\mu} \right)$ is the Jacobian matrix of φ , defined in (21), and belongs to the group $GL(4, \mathbb{R})$.

Tensors fields defined on \mathcal{M} are thus used to describe physical entities because tensors are by construction invariant to changes of observers, in other words are intrinsic. As an example of this invariance, consider a vector \mathbf{V} and a second rank covariant tensor \mathbf{A} : they are said to be invariant to changes of observers because, by definition:

$$\begin{aligned} \mathbf{V} &= V^\mu \mathbf{e}_\mu = \tilde{V}^\mu \tilde{\mathbf{e}}_\mu, \\ \mathbf{A} &= A_{\mu\nu} \mathbf{e}^\mu \otimes \mathbf{e}^\nu = \tilde{A}_{\mu\nu} \tilde{\mathbf{e}}^\mu \otimes \tilde{\mathbf{e}}^\nu, \end{aligned} \quad (23)$$

where V^μ , \tilde{V}^μ , $A_{\mu\nu}$ and $\tilde{A}_{\mu\nu}$, are the respective components of \mathbf{V} and \mathbf{A} in the coordinate systems x^μ and \tilde{x}^μ and \mathbf{e}_μ and $\tilde{\mathbf{e}}_\mu$ are the basis vectors associated to the coordinate system x^μ and \tilde{x}^μ respectively. Then, through this change of coordinates, the components transform as:

$$\tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\lambda} V^\lambda, \quad (24)$$

$$\tilde{A}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu} \frac{\partial x^\kappa}{\partial \tilde{x}^\nu} A_{\lambda\kappa}. \quad (25)$$

Such transformation rules may be established for tensor fields of other ranks and variances.

Define a world-line as a curve in \mathcal{M} described by $\gamma : \mathbb{R} \rightarrow \mathcal{M}$, $s \mapsto \gamma(s)$; in a chart of local coordinates $\gamma(s)$ is described by $\{x^0(s), x^1(s), x^2(s), x^3(s)\}$. The parameter s is related to the proper time τ by the relation $s = c\tau$ converted in length (time multiplied by the speed of light c). The four-velocity, vector tangent to the world-line at a point $P = \gamma(s_P)$, has the components

$$u^\mu = \frac{dx^\mu}{ds} \quad \text{with} \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{and} \quad u_\mu u^\mu = 1. \quad (26)$$

The four-velocity \mathbf{u} is a time-like future oriented unit vector.

³ We use an intrinsic notation and switch to indices notation along with Einstein's convention for summation on repeated indices when useful to clarify the operations and the covariant/contravariant status of the quantities.

A *proper* observer, or proper coordinate system, $\hat{\mathcal{R}}$ is defined as a coordinate system such that the components of the four-velocity are:

$$\hat{u}^\mu = (1, 0, 0, 0) \quad (27)$$

for all events. In the following, all quantities (coordinates and components) observed in a proper coordinate system are marked by a hat symbol. An *inertial* observer, or inertial coordinate system, \mathcal{R} is defined as a coordinate system such that the components of the metric tensor are the components of Minkowski's metric. In an inertial coordinate system, the first coordinate is such that $x^0 = ct$ where t is the absolute time. Now define a specific change of observer, from the proper observer $\hat{\mathcal{R}}$ to the current inertial observer \mathcal{R} :

$$\Phi : \hat{x}^\nu \in \hat{S} \mapsto x^\mu = \Phi^\mu(\hat{x}^\nu) \in S. \quad (28)$$

The Jacobian matrix of this change of observer and its inverse are respectively noted $\frac{\partial x^\mu}{\partial \hat{x}^\nu}$ and $\frac{\partial \hat{x}^\mu}{\partial x^\nu}$.

Projectors enable to identify the time-like or space-like contributions of a given tensor (see for example Landau and Lifshitz (1975)). The *time projector* is defined as a projection on the four-velocity \mathbf{u} . The *spatial projector* $\underline{\underline{\mathbf{g}}}$ is defined such that:

$$\underline{\underline{\mathbf{g}}} = \mathbf{g} - \mathbf{u} \otimes \mathbf{u}. \quad (29)$$

The time-space deformation \mathbf{b} is next defined. It is described by a second order covariant tensor \mathbf{b}

$$\mathbf{b} = \eta_{\mu\nu} \hat{\mathbf{e}}^\mu \otimes \hat{\mathbf{e}}^\nu \text{ with } \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (30)$$

Note that \mathbf{b} is an intrinsic quantity. Its components in $\hat{\mathcal{R}}$ and \mathcal{R} are detailed in the following table:

$\hat{\mathcal{R}}$	\mathcal{R}
$\hat{b}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$b_{\mu\nu} = \frac{\partial \hat{x}^\lambda}{\partial x^\mu} \frac{\partial \hat{x}^\kappa}{\partial x^\nu} \eta_{\lambda\kappa}$

The time-space strain is described by the second order tensor \mathbf{e} defined by

$$\mathbf{e} = \frac{1}{2} (\mathbf{g} - \mathbf{b}). \quad (31)$$

Its components in $\hat{\mathcal{R}}$ and \mathcal{R} are detailed in the following table:

$\hat{\mathcal{R}}$	\mathcal{R}
$\hat{e}_{\mu\nu} = \frac{1}{2} \left(\frac{\partial x^\alpha}{\partial \hat{x}^\mu} \frac{\partial x^\beta}{\partial \hat{x}^\nu} \eta_{\alpha\beta} - \hat{b}_{\mu\nu} \right)$	$e_{\mu\nu} = \frac{1}{2} \left(\eta_{\mu\nu} - \frac{\partial \hat{x}^\alpha}{\partial x^\mu} \frac{\partial \hat{x}^\beta}{\partial x^\nu} \hat{b}_{\alpha\beta} \right)$

The energy momentum tensor \mathbf{T} related to matter is defined by:

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} \quad (32)$$

where \mathcal{L}_M describes the matter field for the Hilbert action (see Hilbert (1915); Misner et al. (1973)). This tensor represents the flux of energy through a 3D volume, a surface of the 4D hyper-volume. The decomposition of \mathbf{T} on time and space leads to the definition of three tensors (see Landau and Lifshitz (1975)):

- a scalar quantity \mathcal{U} , the projection of \mathbf{T} twice on time:
$$\mathcal{U} = T_{\kappa\lambda} u^\kappa u^\lambda, \quad (33)$$

- a vector noted $\underline{\mathbf{T}}$, its projection on time and space,
$$\underline{T}^\mu = T_{\kappa\lambda} \underline{g}^{\mu\kappa} u^\lambda, \quad (34)$$

- a second order contravariant tensor noted $\underline{\underline{\mathbf{T}}}$, its projection twice on space:

$$\underline{\underline{T}}^{\mu\nu} = T_{\kappa\lambda} \underline{g}^{\kappa\mu} \underline{g}^{\lambda\nu}. \quad (35)$$

which leads to:

$$T^{\mu\nu} = \mathcal{U} u^\mu u^\nu \underline{T}^\mu u^\nu + \underline{T}^\nu u^\mu + \underline{\underline{T}}^{\mu\nu}. \quad (36)$$

As proposed for example by Grot and Eringen (1966), on the bases of physical considerations in the proper coordinate system, it is possible to identify each of these terms with

- the energy density for the scalar $\mathcal{U} = T_{\kappa\lambda} u^\kappa u^\lambda$,
- the energy density (heat) flux for the vector $\underline{\mathbf{T}}$,
- a stress tensor for the tensor $\underline{\underline{\mathbf{T}}}$.

The conservation of energy and momentum is written (see Einstein (1921):

$$\forall M \in \mathcal{M}, \nabla_\nu T^{\mu\nu} = 0 \quad (37)$$

where ∇ denotes the covariant derivative. Note that this conservation may also be written for any non inertial observer. Using the projectors, this conservation may be projected on time:

$$u_\mu \nabla_\nu T^{\mu\nu} = 0 \quad (38)$$

and space:

$$\nabla_\nu T^{\mu\nu} - u^\mu (u_\alpha \nabla_\nu T^{\alpha\nu}) = 0. \quad (39)$$

With some derivation (see for example Grot and Eringen (1966)), it may be demonstrated that the projection on time (38) of the conservation of the energy-momentum (37) is the equivalent of the 3D equation of conservation of energy; its projection on space (39) is the equivalent of the 3D balance of momentum. The fact that the conservation of energy-momentum (37) is valid for any observer, added to the fact that this equation combines the conservation of energy and of momentum constitutes the reason why it seems interesting to investigate the use of a 4D formalism for non relativistic motions (that is motions for which the speed of any points of the system is small compared to the velocity of light) like for example the dynamics of non-linear systems like beams and shells. We take here the example of the vibrations of an elastic string undergoing finite transformations.

4.2 String model

We focus here on a time-space formulation of the same vibrating string as the one described in section 3, i.e. a string whose speed is small (compared to the speed of light) and no gravitation involved. To propose such a model for a vibrating string, we make the following hypotheses:

- (H1) The manifold is flat, in other words, there is no gravitation. Then the inertial coordinate system exists and the components of the metric tensor for this coordinate system are $g^{\mu\nu} = \eta^{\mu\nu}$.
- (H2) The manifold is occupied by a material continuum composed of elastic matter and the motion is due to phenomena that are limited to mechanical effects; we hence do not consider any effects due to electromagnetism, exchange of molecules by diffusion, thermal

effects, chemical or nuclear reactions and we focus on motions with no dissipation.

To model such a case, it is imposed that:

- Thermodynamics effects are not considered in the model, then the flux of energy vector defined in (34) vanishes:

$$\underline{T} = \mathbf{0} \quad (40)$$

- the material contribution \mathcal{L}_M corresponds to the energy density \mathcal{U} defined in (33).

(H3) To model a hyper-elastic behaviour, \mathcal{U} depends only on the rest mass density $\tilde{\rho}_c$, the metric tensor \mathbf{g} , the deformation tensor \mathbf{b} and a tensor \mathbf{C} containing the characteristics of material, then

$$\mathcal{U}(\mathbf{g}, \tilde{\rho}_c, \mathbf{C}, \mathbf{b}) = \tilde{\rho}_c c^2 + \sqrt{g} W(\mathbf{g}, \mathbf{C}, \mathbf{b}) \quad (41)$$

where g is the opposite of the determinant of the metric tensor. The elastic energy density W is given by:

$$W = \frac{1}{2} \mathcal{C}^{\alpha\beta\kappa\lambda} \underline{e}_{\alpha\beta} \underline{e}_{\kappa\lambda} \quad (42)$$

where $\underline{e} = \frac{1}{2} (\underline{\mathbf{g}} - \underline{\mathbf{b}})$ is the strain projected twice on space. The components of the fourth order tensor \mathbf{C} are measured in the proper coordinate system, in Voigt notation, to give :

$$\hat{\mathcal{C}}^{\alpha\beta\kappa\lambda} : \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (43)$$

where E is a constant scalar. This choice has been made to obtain a behaviour similar to the one proposed with (14) in section 3.1.

4.3 String formulation

The choice for \mathcal{L}_M given by (H2), implies that, with the definition of the energy-momentum tensor given in (32):

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} = 2 \sqrt{g} \frac{\partial W}{\partial g_{\mu\nu}} + (\tilde{\rho}_c c^2 + \sqrt{g} W) g^{\mu\nu}. \quad (44)$$

The formulation above is generic for any purely mechanical system with an hyper-elastic behaviour, and the string hypothesis lies in the description of the energy (42) with the elasticity tensor (43). Note that, when the string is at rest, and for the proper observer, $g_{\mu\nu} = b_{\mu\nu} = \eta_{\mu\nu}$, and the energy momentum tensor takes the specific form:

$$T^{\mu\nu} = \tilde{\rho}_c c^2 \eta^{\mu\nu}, \quad (45)$$

which is the mass energy density.

Then, the conservation of energy-momentum (37) may be written, for example in the proper coordinate system to solve the problem. Numerical methods such as FEM have been used in Al Nahas et al. (2021) in the static case and will be further extended to dynamics.

5. PERSPECTIVES

Following the approach introduced in van der Schaft and Maschke (2002), future work will be devoted to formulating a time-space representation of a port-Hamiltonian model for a conservative vibrating string that is invariant

to the change of observer in a relativistic context. The next objective will be to propose numerical schemes that benefit from such an invariance.

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