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Crouzeix-Raviart elements on simplicial meshes in d dimensions

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Abstract

In this paper we introduce Crouzeix-Raviart elements of general polynomial order k and spatial dimension $d \geq 2$ for simplicial finite element meshes. We give explicit representations of the non-conforming basis functions and prove that the conforming companion space, i.e., the conforming finite element space of polynomial order k is contained in the Crouzeix-Raviart space. We prove a direct sum decomposition of the Crouzeix-Raviart space into (a subspace of) the conforming companion space and the span of the non-conforming basis functions.

Degrees of freedom are introduced which are bidual to the basis functions and give rise to the definition of a local approximation/interpolation operator. In two dimensions or for $k = 1$, these freedoms can be split into simplex and $(d-1)$ dimensional facet integrals in such a way that, in a basis representation of Crouzeix-Raviart functions, all coefficients which belong to basis functions related to lower-dimensional faces in the mesh are determined by these facet integrals. It will also be shown that such a set of degrees of freedom does **not** exist in higher space dimension and $k > 1$.

AMS-Classification: Primary 33C45, 33C50, 65N30; Secondary 65N12.

Keywords: finite elements; non-conforming; Crouzeix-Raviart, orthogonal polynomials on simplices

1 Introduction

The Crouzeix-Raviart (CR) finite elements spaces have been introduced in the seminal paper [7] and allow for a very economic non-conforming discretization

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of the Stokes equation. The original definition in [7] is general but implicit by imposing certain moment conditions across interelement facets. In the original paper, basis functions are presented for the lowest order CR elements in dimension $d \in \{2, 3\}$; this is the reason that most implementations and methodological developments of CR elements are restricted to lowest order. Explicit representations for $d = 2$ of the non-conforming CR functions exist in the literature, see [1], [2], [3], [11], [4, for $p = 4, 6.$], [5] while for spatial dimension $d \geq 3$ the lowest order CR element has been introduced in [7]. For *quadratic* CR elements in 3D, an explicit basis has been introduced in [9]. In [6], a spanning set of functions is presented in 3D for a *maximal* CR space of any polynomial order $k \in \mathbb{N}$ which allows for a *local basis*. However, the question of linear independence is subtle, in particular, the definition of a basis for a minimal CR space. In [10] basis functions of a CR space have been introduced for general space dimension $d \geq 2$ and polynomial order k which are the analoga of the 2D CR spaces. However, the corresponding space do not always include its conforming *companion space*, i.e., the conforming finite element space of degree k and suboptimal convergence rates must be expected.

In this paper we introduce a basis for CR elements for general polynomial degree k and space dimension d which contains the conforming companion space and its approximation properties are inherited. We introduce degrees of freedom which are bidual to the CR basis functions and allow to define local approximation operators.

The paper is structured as follows. In Section 2, we introduce the non-conforming CR space in d dimension and general polynomial order k. We define the non-conforming CR functions in such a way that the conforming companion space is contained in the CR space. For the construction of a full set of basis function we prove in Section 3 a direct sum decomposition of the CR space into (a subset of) the conforming companions space and the span of the linearly independent CR functions. This construction is given in Section 4 where a full basis of the CR space is defined which consists of the product of orthogonal polynomials with bubble functions on simplicial entities (vertices, edges, facets, etc.). This choice of basis functions has impact on the construction of degrees of freedom and an interpolation operator. In Section 5 we introduce degrees of freedom which are bidual to the CR basis functions and (mostly) defined by integrals over *simplices* for certain weight functions. In Section 6 we construct for the two-dimensional case degrees of freedom for the edge basis functions which are defined as integrals over the *edges* with appropriate weight functions. Interestingly, such a construction is possible only in two dimensions or for $k = 1$; the details and a proof will be given in Section 6.3.

2 Crouzeix-Raviart Finite Elements

In this section, we define Crouzeix-Raviart spaces in d dimensions, $d = 2, 3, \ldots$ and polynomial degree $k = 1, 2, \ldots$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz polytope. As usual $L^2(\Omega)$ is the space of

measurable square-integrable functions with scalar product $(u,v)_{L^2(\Omega)} := \int_\Omega uv$ and norm $\|u\|_{L^2(\Omega)} := (u,u)^{1/2}_{L^2(\Omega)}$. The Sobolev space $H^1(\Omega)$ contains all $L^2(\Omega)$ functions whose weak derivatives (exist and) are square integrable. All function spaces are considered over the field of real numbers.

Let $\mathbb{N} := \{1,2,3,\ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We employ the multi-index notation in the usual way: for a vector $\mathbf{x} = (x_\ell)_{\ell=1}^m \in \mathbb{R}^m_{\geq 0}$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^m$, we set

$$\begin{aligned} \mathbf{x}^{\boldsymbol{\alpha}} &:= \prod_{\ell=1}^m x_i^{\alpha_i}, \quad |\mathbf{x}| := \sum_{\ell=1}^m x_i, \quad \boldsymbol{\alpha}! := \prod_{\ell=1}^m \alpha_\ell!, \\ \mathbb{N}^m_{\leq k} &:= \{\boldsymbol{\alpha} \in \mathbb{N}_0^m \mid |\boldsymbol{\alpha}| \leq k\}. \end{aligned}$$

Let $\mathcal{T}$ be a conforming finite element mesh for Ω consisting of closed simplices $K \in \mathcal{T}$. By $\mathcal{F}(K)$ we denote the set of $(d-1)$-dimensional closed facets in ∂K. The set of vertices is denoted by $\mathcal{V}(K)$ and we employ this notation also for lower dimensional simplicial entities in ∂K, e.g., for $F \in \mathcal{F}(K)$, the set $\mathcal{V}(F)$ is the set of vertices of F.

We denote by $\hat{K}$ the (closed) reference simplex with vertices

$$\hat{\mathbf{z}}_0 := \mathbf{0}, \quad \text{for } 1 \leq i \leq d,\ \hat{\mathbf{z}}_i := \mathbf{e}_i, \tag{1}$$

where $\mathbf{e}_i \in \mathbb{R}^d$ is the i-th canonical unit vector in $\mathbb{R}^d$. Moreover, let $\mathcal{F}$ ($\mathcal{V}$, resp.) be the set of all $(d-1)$-dimensional facets (vertices, resp.) in the mesh and let

$$\begin{aligned} \mathcal{F}_{\partial\Omega} &:= \{F \in \mathcal{F} \mid F \subset \partial\Omega\}, \quad \mathcal{F}_\Omega := \mathcal{F} \backslash \mathcal{F}_{\partial\Omega}, \\ \mathcal{V}_{\partial\Omega} &:= \{\mathbf{z} \in \mathcal{V} \mid \mathbf{z} \in \partial\Omega\}, \quad \mathcal{V}_\Omega := \mathcal{V} \backslash \mathcal{V}_{\partial\Omega}. \end{aligned} \tag{2}$$

For $F \in \mathcal{F}$, $\mathbf{z} \in \mathcal{V}$, we define facet and nodal patches by

$$\begin{aligned} \mathcal{T}_{\mathbf{z}} &:= \{K \in \mathcal{T} : \mathbf{z} \in K\}, \quad \omega_{\mathbf{z}} := \textstyle\bigcup_{K \in \mathcal{T}_{\mathbf{z}}} K, \\ \mathcal{F}_{\mathbf{z}} &:= \{F \in \mathcal{F} : \mathbf{z} \in F\}, \\ \mathcal{T}_F &:= \{K \in \mathcal{T} : F \subset K\}, \quad \omega_F := \textstyle\bigcup_{K \in \mathcal{T}_F} K. \end{aligned} \tag{3}$$

For a measurable subset $D \subset \mathbb{R}^d$, we denote by $|D|_d$ the d-dimensional volume of D and skip the index d if the dimension is clear from the context, e.g., $|K|$ denotes the d-dimensional volume of a simplex $K \in \mathcal{T}$, $|F|$ the $(d-1)$-dimensional volume of a facet $F \in \mathcal{F}$, etc. For a finite discrete set $\mathcal{I}$ its cardinality is denoted by $|\mathcal{I}|$.

For a conforming simplicial mesh $\mathcal{T}$ of the domain Ω, let

$$H^1(\mathcal{T}) := \left\{ u \in L^2(\Omega) \mid \forall K \in \mathcal{T} : \ u|_{\overset{\circ}{K}} \in H^1\left(\overset{\circ}{K}\right) \right\}. \tag{4}$$

For $k \in \mathbb{N}_0$ and a simplex K, we denote by $\mathbb{P}_k(K)$ the space of polynomials of maximal degree k on K and set $\mathbb{P}_{-1}(K) := \{0\}$. For a $(d-1)$ dimensional facet F, we denote by $\mathbb{P}_k(F)$ the space of $(d-1)$-variate polynomials in the local variables of F. We also we need the subspace

$$\overset{\bullet}{\mathbb{P}}_k(K) := \{v \in \mathbb{P}_k(K) \mid \forall \mathbf{y} \in \mathcal{V}(K) : v(\mathbf{y}) = 0\} \tag{5}$$

and employ this notation also for lower dimensional simplicial entities in ∂K, e.g., for $F \in \mathcal{F}(K)$:

$$\overset{\bullet}{\mathbb{P}}_k(F) := \{v \in \mathbb{P}_k(F) \mid \forall \mathbf{y} \in \mathcal{V}(F) : v(\mathbf{y}) = 0\}. \tag{6}$$

For the definition of the non-conforming Crouzeix-Raviart space, the functions $B_k^{\mathrm{CR},K}$, $K \in \mathcal{T}$, and $B_k^{\mathrm{CR},F}$, $F \in \mathcal{F}_\Omega$, will play an important role; their definition require some preliminaries.

Notation 1 *For a simplex K and $\mathbf{z} \in \mathcal{V}(K)$, the affine function $\lambda_{K,\mathbf{z}} \in \mathbb{P}_1(K)$ is the* barycentric coordinate *for the vertex $\mathbf{z}$ and characterized by*

$$\lambda_{K,\mathbf{z}}(\mathbf{y}) = \delta_{\mathbf{y},\mathbf{z}} \quad \forall \mathbf{y} \in \mathcal{V}(K),$$

where $\delta_{\mathbf{y},\mathbf{z}}$ denotes the Kronecker delta.

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}_0$. The *Jacobi polynomial* $P_n^{(\alpha,\beta)}$ is a polynomial of degree n such that

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\; q(x)\,(1-x)^{\alpha}\,(1+x)^{\beta}\; dx = 0 \tag{7}$$

for all polynomials q of degree less than n, one has (cf. [8, Table 18.6.1]):

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}, \qquad P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)_n}{n!}. \tag{8}$$

Here, the *shifted factorial* is defined by $(a)_n := a(a+1)\dots(a+n-1)$ for $n > 0$ and $(a)_0 := 1$. Note that $P_n^{(0,0)}$ are the Legendre polynomials (see [8, 18.7.9]). In our application, the Jacobi polynomials with $\alpha = 0$ and $\beta = d-2$ are relevant. In this case, the endpoint values are given by

$$P_n^{(0,d-2)}(1) = 1 \quad \text{and} \quad P_n^{(0,d-2)}(-1) = (-1)^n \rho_n \text{ with } \rho_n := \binom{n+d-2}{n}. \tag{9}$$

Next, we define the non-conforming shape functions for the Crouzeix-Raviart space. They differ from the definition in [10, (3.8)] and lead to a Crouzeix-Raviart space which, in contrast to the space in [10], always contains the conforming space as a subspace (see Corollary 10).

Definition 2 *Let $\mathcal{T}$ be a conforming simplicial finite element mesh.*

1. *For any $K \in \mathcal{T}$, the non-conforming* simplex function $B_k^{\mathrm{CR},K} \in \mathbb{P}_k(\mathcal{T})$ *is given by*

$$B_k^{\mathrm{CR},K} := \begin{cases} \frac{1}{d}\left(-1 + \sum_{\mathbf{z}\in\mathcal{V}(K)} P_k^{(0,d-2)}(1-2\lambda_{K,\mathbf{z}})\right) & \text{on } K, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The space spanned by these functions is

$$V_k^{\mathrm{nc}}(\mathcal{T}) := \operatorname{span}\left\{B_k^{\mathrm{CR},K} : K \in \mathcal{T}\right\}.$$

2. *For any $F \in \mathcal{F}$, the non-conforming* facet function $B_k^{\mathrm{CR},F} \in \mathbb{P}_k(\mathcal{T})$ *is given by*

$$B_k^{\mathrm{CR},F} := \begin{cases} P_k^{(0,d-2)}(1-2\lambda_{K,F}) - B_k^{\mathrm{CR},K} & \text{for } K \in \mathcal{T}_F, \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

where $\lambda_{K,F}$ denotes the barycentric coordinate for a simplex K adjacent to F corresponding to the vertex $\mathbf{z} \in \mathcal{V}(K)$ opposite to F. The spaces spanned by these functions are

$$V_k^{\mathrm{nc}}(\mathcal{F}) := \operatorname{span}\left\{B_k^{\mathrm{CR},F} : F \in \mathcal{F}\right\},$$
$$V_k^{\mathrm{nc}}(\mathcal{F}_\Omega) := \operatorname{span}\left\{B_k^{\mathrm{CR},F} : F \in \mathcal{F}_\Omega\right\}.$$

Remark 3 *Note that* $\operatorname{supp} B_k^{\mathrm{CR},F} = \omega_F$ *and* $\operatorname{supp} B_k^{\mathrm{CR},K} = K$. *Since* $\lambda_{K,F} = 0$ *on F, the value of* $P_k^{(0,d-2)}(1-2\lambda_{K,F})\Big|_F$ *in (11) is equal to* $P_k^{(0,d-2)}(1) = 1$.

For even k, the span $V_k^{\mathrm{nc}}(\mathcal{T})$ of the non-conforming shape functions $B_k^{\mathrm{CR},K}$ is the same as the one in [10, before Def. 3.6]. For $k = 1$, the space $V_1^{\mathrm{nc}}(\mathcal{F}_\Omega)$ equals the one in [10, before Def. 3.6]. The reason is that $B_1^{\mathrm{CR},K} = 0$ which will be proved as Case 2 of Lemma 11.

For $d = 2$ and odd k the subtraction of $B_k^{\mathrm{CR},K}$ in (11) can be removed since $B_k^{\mathrm{CR},K}$ is continuous in this case. On a facet $F \subset \partial K$ it holds

$$\left(P_k^{(0,0)}(1-2\lambda_{K,F}) - B_k^{\mathrm{CR},K}\right)\Big|_F = \left(P_k^{(0,0)}(1-2\lambda_{K,F})\right)\Big|_F. \tag{12}$$

Next, we introduce the relevant finite element spaces. For $k \in \mathbb{N}$, let $S_k(\mathcal{T})$ denote the space of globally continuous, piecewise polynomials of maximal degree k:

$$S_k(\mathcal{T}) := \left\{v \in C^0(\overline{\Omega}) \mid v|_K \in \mathbb{P}_k(K) \quad \forall K \in \mathcal{T}\right\}$$

and the subspace with zero traces on the boundary:

$$S_{k,0}(\mathcal{T}) := S_k(\mathcal{T}) \cap H_0^1(\Omega).$$

The space of discontinuous polynomials of maximal degree k is

$$\mathbb{P}_k(\mathcal{T}) := \left\{p \in L^2(\Omega) \mid p|_{\overset{\circ}{K}} \in \mathbb{P}_k(\overset{\circ}{K}) \quad \forall K \in \mathcal{T}\right\},$$

where we employed the following notation.

Notation 4 *A function $v \in H^1(\mathcal{T})$ may be discontinuous across simplex facets. In this way the values on a facet are not uniquely defined. For $K \in \mathcal{T}$ the function $v|_{\overset{\circ}{K}}$ is well defined and the function $v|_K$ equals $v|_{\overset{\circ}{K}}$ in the interior of K and is defined on ∂K as the continuous continuation of $v|_{\overset{\circ}{K}}$ to ∂K.*

In a similar fashion the sum of functions $v_1, v_2 \in H^1(\mathcal{T})$ on some $K \in \mathcal{T}$ is defined by

$$(v_1 + v_2)|_K := v_1|_K + v_2|_K .$$

If the functions v, v_1, v_2 belong to $C^0\left(\overline{\Omega}\right)$ these definitions coincide with the standard restriction to K.

Next we define the non-conforming Crouzeix-Raviart space in the original implicit way (see [7]). For a function $v \in H^1(\mathcal{T})$, we denote by $[v]_F$ the jump across the facet $F \in \mathcal{F}$. The Crouzeix-Raviart space of order k is defined implicitly by

$$\mathrm{CR}_k^{\max}(\mathcal{T}) := \{v \in \mathbb{P}_k(\mathcal{T}) \mid \forall F \in \mathcal{F}_\Omega \quad [v]_F \perp \mathbb{P}_{k-1}(F)\}$$

and the subspace with (approximate) zero boundary conditions by

$$\mathrm{CR}_{k,0}^{\max}(\mathcal{T}) := \{v \in \mathrm{CR}_k^{\max}(\mathcal{T}) \mid \forall F \in \mathcal{F}_{\partial\Omega} \quad v \perp \mathbb{P}_{k-1}(F)\}. \tag{13}$$

Here the symbol "$\perp$" always refers to L^2 orthogonality.

Lemma 5 *The functions $B_k^{\mathrm{CR},F}$ and $B_k^{\mathrm{CR},K}$ in Definition 2 belong to $\mathrm{CR}_k^{\max}(\mathcal{T})$.*

Proof. This follows from [10, Theorem 3.5] by showing that $P_k^{(0,d-2)}$ satisfies [10, Condition (A.1)]:

$$P_k^{(0,d-2)}(1 - 2\lambda_{K,\mathbf{z}})\Big|_{F_\mathbf{z}} = 1 \tag{14a}$$

$$\forall F \in \mathcal{F}(K) \setminus \{F_\mathbf{z}\} \quad \int_F P_k^{(0,d-2)}(1 - 2\lambda_{K,\mathbf{z}})\, q = 0 \quad \forall q \in \mathbb{P}_{k-1}(F) \tag{14b}$$

for all $K \in \mathcal{T}$, $\mathbf{z} \in \mathcal{V}(K)$, where $F_\mathbf{z}$ denotes the facet of K opposite to $\mathbf{z}$.

The first condition follows from $\lambda_{K,\mathbf{z}}|_{F_\mathbf{z}} = 0$ and $P_k^{(0,d-2)}(1) = 1$ (cf. (8)).

The second condition follows from [12, (2.5.21)] by choosing $\boldsymbol{\alpha} = (k, 0, \ldots, 0)$, $\kappa_1 = \ldots = \kappa_{d+1} = 1/2$ therein and by using

$$P_k^{(d-2,0)}(2\lambda_{K,\mathbf{z}} - 1) = (-1)^k P_k^{(0,d-2)}(1 - 2\lambda_{K,\mathbf{z}})$$

(see [8, Table 18.6.1]). Here, we present an elementary proof to give an intuition of this specific choice of the Jacobi weights.

By using the affine equivalence of K to the reference simplex it is sufficient to show the second condition for $\widehat{K}$, $\widehat{F} := \left\{\mathbf{x} = (x_j)_{j=1}^d \in \widehat{K} \mid x_d = 0\right\}$, and $\mathbf{z} = (1, 0, \ldots, 0)^T$: for all $\boldsymbol{\alpha} = (\alpha_i)_{i=1}^{d-1} \in \mathbb{N}_{\le k-1}^{d-1}$ and corresponding monomials

$$g(\mathbf{x}) := \prod_{j=1}^{d-1} x_j^{\alpha_j} \quad \forall \mathbf{x} = (x_j)_{j=1}^{d-1}$$

it holds

$$\int_{\widehat{F}} P_k^{(0,d-2)}(1-2x_1)\, g(\mathbf{x})\, d\mathbf{x} = \int_0^1 P_k^{(0,d-2)}(1-2x_1)\, G(x_1)\, dx_1,$$

where

$$G(x_1) = \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\ldots-x_{d-2}} g(\mathbf{x})\, dx_{d-1} dx_{d-2} \ldots dx_2.$$

Let $\boldsymbol{\alpha}' = (\alpha_j)_{j=2}^{d-1}$. From the proof of [10, Lemma A.1] it follows

$$G(x_1) = c_{d,\boldsymbol{\alpha}} x_1^{\alpha_1} (1-x_1)^{d-2+|\boldsymbol{\alpha}'|} \quad \text{for } c_{d,\boldsymbol{\alpha}} := \frac{\boldsymbol{\alpha}'!}{(d-2+|\boldsymbol{\alpha}'|)!}.$$

Hence,

$$\begin{aligned}\int_{\widehat{F}} P_k^{(0,d-2)}(1-2x_1)\, g(\mathbf{x})\, d\mathbf{x} &= c_{d,\boldsymbol{\alpha}} \int_0^1 P_k^{(0,d-2)}(1-2x_1)\, x_1^{\alpha_1} (1-x_1)^{d-2+|\boldsymbol{\alpha}'|}\, dx_1 \\ &= \frac{c_{d,\boldsymbol{\alpha}}}{2^{d-1}} \int_{-1}^1 \omega(t)\, P_k^{(0,d-2)}(t) \left(\frac{1-t}{2}\right)^{\alpha_1} \left(\frac{t+1}{2}\right)^{|\boldsymbol{\alpha}'|} dt\end{aligned}$$

for the weight function $\omega(t) = (t+1)^{d-2}$. Since $\left(\frac{1-t}{2}\right)^{\alpha_1} \left(\frac{t+1}{2}\right)^{|\boldsymbol{\alpha}'|} \in \mathbb{P}_{|\boldsymbol{\alpha}|} \subset \mathbb{P}_{k-1}$, the orthogonality properties of the Jacobi polynomials imply

$$\int_{\widehat{F}} P_k^{(0,d-2)}(1-2x_1)\, g(\mathbf{x})\, d\mathbf{x} = 0.$$

■

Next, we introduce canonical Crouzeix-Raviart functions on simplicial meshes in d dimensions by modifying the basic space in [10].

Definition 6 *The scalar* canonical Crouzeix-Raviart space *of order k for conforming simplicial finite element meshes $\mathcal{T}$ is given by*

$$\mathrm{CR}_k(\mathcal{T}) := \begin{cases} S_k(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{T}) & \text{if } k \text{ is even,} \\ S_k(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{F}) & \text{if } k \text{ is odd.} \end{cases}$$

and the subspace for approximating problems with zero boundary conditions by:

$$\mathrm{CR}_{k,0}(\mathcal{T}) := \begin{cases} S_{k,0}(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{T}) & \text{if } k \text{ is even,} \\ S_{k,0}(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{F}_\Omega) & \text{if } k \text{ is odd.} \end{cases} \tag{15}$$

We emphasize that the sums in (15), in general, are not direct.

3 A direct sum decomposition of the canonical Crouzeix-Raviart space

The Lagrange interpolation points for the reference element are given by

$$\mathcal{N}_k\left(\hat{K}\right) := \begin{cases} \frac{1}{d+1}\mathbf{1} & k=0, \\ \left\{\frac{\boldsymbol{\mu}}{k} : \boldsymbol{\mu} \in \mathbb{N}_k^d\right\} & k \geq 1, \end{cases} \tag{16}$$

where $\mathbf{1} \in \mathbb{R}^d$ is the vector with constant coefficients 1. As usual they are lifted to $K \in \mathcal{T}$ via an affine map $\varphi_K : \hat{K} \to K$ to obtain the interpolation points on the mesh element K via $\mathcal{N}_k(K) := \left\{\varphi_K(\hat{\mathbf{y}}) : \hat{\mathbf{y}} \in \mathcal{N}_k\left(\hat{K}\right)\right\}$. We also will need the subset

$$\overset{\bullet}{\mathcal{N}}_k(K) := \mathcal{N}_k(K) \setminus \mathcal{V}(K) \tag{17}$$

and the corresponding subspaces of $S_k(\mathcal{T})$, $S_{k,0}(\mathcal{T})$ consisting of functions which vanish at the vertices of the mesh are given by

$$\overset{\bullet}{S}_k(\mathcal{T}) := \{v \in S_k(\mathcal{T}) \mid \forall \mathbf{z} \in \mathcal{V} : \quad v(\mathbf{z}) = 0\}, \tag{18}$$

$$\overset{\bullet}{S}_{k,0}(\mathcal{T}) := \{v \in S_{k,0}(\mathcal{T}) \mid \forall \mathbf{z} \in \mathcal{V} : \quad v(\mathbf{z}) = 0\}. \tag{19}$$

The Lagrange interpolation points for $\mathcal{T}$ are given by $\mathcal{N}_k(\mathcal{T}) := \bigcup_{K\in\mathcal{T}} \mathcal{N}_k(K)$. Let $\mathcal{N}_{k,0}(\mathcal{T}) \subset \mathcal{N}_k(\mathcal{T})$ be the subset, where the Lagrange points on the boundary $\partial\Omega$ are removed. Further, let $\overset{\bullet}{\mathcal{N}}_k(\mathcal{T}) := \mathcal{N}_k(\mathcal{T}) \setminus \mathcal{V}$ and $\overset{\bullet}{\mathcal{N}}_{k,0}(\mathcal{T}) := \mathcal{N}_{k,0}(\mathcal{T}) \setminus \mathcal{V}_\Omega$. As usual, for $\mathbf{z} \in \mathcal{N}_k(\mathcal{T})$, the corresponding Lagrange basis $B_k^{\mathbf{z}} \in S_k(\mathcal{T})$ is defined implicitly by the conditions

$$B_k^{\mathbf{z}}(\mathbf{y}) = \delta_{\mathbf{z},\mathbf{y}} \quad \forall \mathbf{y}, \mathbf{z} \in \mathcal{N}_k(\mathcal{T}).$$

The basis functions on the reference element $\hat{K}$ are denoted by $\hat{B}_k^{\hat{\mathbf{z}}} \in \mathbb{P}_k\left(\hat{K}\right)$ for $\hat{\mathbf{z}} \in \mathcal{N}_k\left(\hat{K}\right)$, and characterized by: $\hat{B}_k^{\hat{\mathbf{z}}}(\hat{\mathbf{y}}) = \delta_{\hat{\mathbf{y}},\hat{\mathbf{z}}}$ for all $\hat{\mathbf{y}}, \hat{\mathbf{z}} \in \mathcal{N}_k\left(\hat{K}\right)$.

Next, some useful properties of the non-conforming functions $B_k^{\mathrm{CR},K}$ and $B_k^{\mathrm{CR},F}$ and corresponding spaces $V_k^{\mathrm{nc}}(\mathcal{T})$ $V_k^{\mathrm{nc}}(\mathcal{F})$, and $V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ are derived.

The endpoint properties of the Jacobi polynomials (cf. (9)) allow us to compute the values of the non-conforming functions at the simplex vertices. For $K \in \mathcal{T}$ it holds:

$$B_k^{\mathrm{CR},K}\Big|_K(\mathbf{y}) = \frac{d-1+(-1)^k\rho_k}{d} \quad \forall \mathbf{y} \in \mathcal{V}(K). \tag{20a}$$

For $F \in \mathcal{F}$, $K \in \mathcal{T}_F$ it holds:

$$B_k^{\mathrm{CR},F}\Big|_K(\mathbf{y}) = \begin{cases} \frac{1+(-1)^{k+1}\rho_k}{d} & \text{if } \mathbf{y} \in \mathcal{V}(F), \\ \frac{1+(-1)^{k+1}\rho_k}{d}(1-d) & \text{if } \mathbf{y} \in \mathcal{V}(K) \text{ is opposite to } F. \end{cases} \tag{20b}$$

We need the following technical lemma with regard to the vertex values in (20).

Lemma 7 *Let $d \geq 2$ and $k \geq 1$. It holds*

$$d - 1 + (-1)^k \rho_k = 0 \iff (k = 1) \vee (d = 2 \wedge k \text{ is odd}), \tag{21a}$$

$$d + (-1)^k \rho_k \neq 0, \tag{21b}$$

$$\rho_k = 1 \iff d = 2. \tag{21c}$$

Proof. We first prove (21a).

The direction " $\Longleftarrow$ " in (21a) follows by direct computation since $\rho_k = d-1$ for $k = 1$ and $\rho_k = 1$ for $d = 2$. In the right-hand side of (21a) k is odd so that $\theta(k) := d - 1 + (-1)^k \rho_k = 0$.

Next we prove " $\Longrightarrow$ " in (21a). For even k, we have $\theta(k) = d - 1 + \rho_k \geq d > 0$. Hence $\theta(k) = 0$ implies k is odd and $\theta(k) = d - 1 - \rho_k$.

For $k = 1$, we have $\rho_k = d - 1$ so that $\theta(1) = 0$ and it remains to consider odd $k \geq 3$. It is easy to verify that $\binom{k+d-2}{k}$ is monotonically growing with respect to k. For odd $k \geq 3$ it holds

$$\theta(k) = d - 1 - \binom{k+d-2}{k} \leq d - 1 - \binom{d+1}{3} = -\frac{1}{6}(d+3)(d-1)(d-2).$$

Since $d \geq 2$ the condition $\theta(k) = 0$ implies $d = 2$.

Next we prove (21b). For even k it holds $d + (-1)^k \rho_k = d + \rho_k > 0$. For $d = 2$ it holds $d + (-1)^k \rho_k = 2 + (-1)^k > 0$ while for $k = 1$ we get $d + (-1)^k \rho_k = 1$.

It remains to consider the case $d \geq 3$ and odd $k \geq 3$. A similar monotonicity argument leads to

$$d - \rho_k = d - \binom{k+d-2}{k} \overset{k\geq 3}{\leq} d - \binom{d+1}{3} = -\frac{1}{6} d\left(d^2 - 7\right) < 0.$$

Hence, (21a) is proved. The equivalence (21c) is trivial. ■

Lemma 8 *Let $k \in \mathbb{N}$. For $\mathbf{z} \in \mathcal{V}$, the function*

$$\psi_k^{\mathbf{z}} := \sum_{F \in \mathcal{F}_{\mathbf{z}}} B_k^{\mathrm{CR},F} \tag{22}$$

satisfies

$$\begin{aligned} &\psi_k^{\mathbf{z}} \in \begin{cases} S_k(\mathcal{T}) & \text{for any } \mathbf{z} \in \mathcal{V}, \\ S_{k,0}(\mathcal{T}) & \text{if } \mathbf{z} \in \mathcal{V}_\Omega, \end{cases} \\ &\forall \mathbf{y} \in \mathcal{V} : \psi_k^{\mathbf{z}}(\mathbf{y}) = \psi_k^{\mathbf{z}}|_K(\mathbf{y}) = \left(1 + (-1)^{k+1} \rho_k\right) \delta_{\mathbf{y},\mathbf{z}} \end{aligned} \tag{23}$$

Proof. Let $\mathbf{z} \in \mathcal{V}$. The function $\psi_k^{\mathbf{z}}$ has support in $\omega_{\mathbf{z}}$ and for any $F \in \mathcal{F}_{\mathbf{z}}$ and adjacent simplex $K \in \mathcal{T}$ it holds

$$\psi_k^{\mathbf{z}}|_K = \sum_{F \in \mathcal{F}_{\mathbf{z}}} B_k^{\mathrm{CR},F}\Big|_K = \sum_{\mathbf{y} \in \mathcal{V}(K) \setminus \{\mathbf{z}\}} \left(P_k^{(0,d-2)}(1 - 2\lambda_{K,\mathbf{y}}) - B_k^{\mathrm{CR},K}\right)\Big|_K.$$

For $K,\ K' \in \mathcal{T}_F$ and $\mathbf{y} \in \mathcal{V}(F)$ it holds $\lambda_{K,\mathbf{y}}|_F = \lambda_{K',\mathbf{y}}|_F$ so that the function $\psi_k^{\mathbf{z}}$ is continuous across F so that $\psi_k^{\mathbf{z}}|_{\omega_{\mathbf{z}}}$ is continuous.

On the facet $F_{\mathbf{z}}$ in K opposite to $\mathbf{z}$ it holds

$$\begin{aligned}\psi_k^{\mathbf{z}}|_K|_{F_{\mathbf{z}}} &= \sum_{\mathbf{y}\in\mathcal{V}(F_{\mathbf{z}})} P_k^{(0,d-2)}\left(1-2\,\lambda_{K,\mathbf{y}}|_{F_{\mathbf{z}}}\right) - \left(-1_{F_{\mathbf{z}}} + \sum_{\mathbf{y}\in\mathcal{V}(K)} P_k^{(0,d-2)}\left(1-2\,\lambda_{K,\mathbf{y}}|_{F_{\mathbf{z}}}\right)\right)\\ &= \sum_{\mathbf{y}\in\mathcal{V}(F_{\mathbf{z}})} P_k^{(0,d-2)}\left(1-2\,\lambda_{K,\mathbf{y}}|_{F_{\mathbf{z}}}\right) - \left(\sum_{\mathbf{y}\in\mathcal{V}(F_{\mathbf{z}})} P_k^{(0,d-2)}\left(1-2\,\lambda_{K,\mathbf{y}}|_{F_{\mathbf{z}}}\right)\right) = 0.\end{aligned}$$

Hence, in general $\psi_k^{\mathbf{z}} \in S_k(\mathcal{T})$ with $\operatorname{supp}\psi_k^{\mathbf{z}} \subset \omega_{\mathbf{z}}$ and for inner vertices $\mathbf{z} \in \mathcal{V}_\Omega$ it follows $\psi_k^{\mathbf{z}} \in S_{k,0}(\mathcal{T})$.

We use the vertex values of the non-conforming function (20) to evaluate $\psi_k^{\mathbf{z}}$ at the simplex vertices. ■

Lemma 9 *For any $k \in \mathbb{N}$, it holds*

$$\mathrm{CR}_k(\mathcal{T}) = \begin{cases} \overset{\bullet}{S}_k(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{F}) & k \text{ odd,} \\ S_k(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{T}) & k \text{ even,} \end{cases} \tag{24}$$

and

$$\mathrm{CR}_{k,0}(\mathcal{T}) = \begin{cases} \overset{\bullet}{S}_{k,0}(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{F}_\Omega) & k \text{ odd,} \\ S_{k,0}(\mathcal{T}) + V_k^{\mathrm{nc}}(\mathcal{T}) & k \text{ even.} \end{cases} \tag{25}$$

Proof. The second statements in (24) and (25) are simply repetitions of the definition.

We only prove the first statement in (25); the first one in (24) follows by the same arguments. We employ the function $\psi_k^{\mathbf{z}}$ as in Lemma 8. The combination of (21b) and (23) implies $\psi_k^{\mathbf{z}}(\mathbf{z}) \neq 0$ so that the scaled version $\tilde{\psi}_k^{\mathbf{z}} = \psi_k^{\mathbf{z}}/\psi_k^{\mathbf{z}}(\mathbf{z})$ is well defined, belongs to $S_{k,0}(\mathcal{T})$, and satisfies for $\mathbf{z} \in \mathcal{V}_\Omega$:

$$\tilde{\psi}_k^{\mathbf{z}}(\mathbf{y}) = \delta_{\mathbf{y},\mathbf{z}} \quad \forall \mathbf{y} \in \mathcal{V}.$$

Any $v \in \mathrm{CR}_{k,0}(\mathcal{T})$ can be written in the form

$$v = w^{\mathrm{c}} + w_{\mathcal{F}}^{\mathrm{nc}} \quad \text{for} \quad w^{\mathrm{c}} \in S_{k,0}(\mathcal{T}),\ w_{\mathcal{F}}^{\mathrm{nc}} \in V_k^{\mathrm{nc}}(\mathcal{F}_\Omega).$$

Then,

$$w^{\mathrm{c}} = w_1^{\mathrm{c}} + w_0^{\mathrm{c}} \quad \text{for } w_1^{\mathrm{c}} := \sum_{\mathbf{z}\in\mathcal{V}_\Omega} w^{\mathrm{c}}(\mathbf{z})\,\tilde{\psi}_k^{\mathbf{z}} \quad \text{and} \quad w_0^{\mathrm{c}} \in \overset{\bullet}{S}_{k,0}(\mathcal{T}).$$

Since $\tilde{\psi}_k^{\mathbf{z}} \in V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ it follows $w_1^{\mathrm{c}} \in V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ and we have derived the representation $v = w_0^{\mathrm{c}} + w_1^{\mathrm{c}} + w_{\mathcal{F}}^{\mathrm{nc}}$ with $w_0^{\mathrm{c}} \in \overset{\bullet}{S}_{k,0}(\mathcal{T})$ and $w_1^{\mathrm{c}} + w_{\mathcal{F}}^{\mathrm{nc}} \in V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$. ■

Corollary 10 *The conforming finite element space are contained in the Crouzeix-Raviart spaces for $k \geq 1$:*

$$S_k \subset \mathrm{CR}_k(\mathcal{T}) \quad \text{and } S_{k,0} \subset \mathrm{CR}_{k,0}(\mathcal{T}).$$

Lemma 11 *For $k \geq 2$, a basis for $V_k^{\mathrm{nc}}(\mathcal{T})$ is given by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{T}) := \left\{B_k^{\mathrm{CR},K}, K \in \mathcal{T}\right\}$ while for $k = 1$ it holds $V_1^{\mathrm{nc}}(\mathcal{T}) = \{0\}$.*

For odd $k \geq 1$, a basis for $V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ is given by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}_\Omega) := \left\{B_k^{\mathrm{CR},F}, F \in \mathcal{F}_\Omega\right\}$ and a basis for $V_k^{\mathrm{nc}}(\mathcal{F})$ by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}) := \left\{B_k^{\mathrm{CR},F}, F \in \mathcal{F}\right\}$.

Proof. From the definition of $V_k^{\mathrm{nc}}(\mathcal{T})$ and $V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ it follows that it is sufficient to prove that each of the sets $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{T})$, $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ consists of linearly independent functions.

Case 1, the space $V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$.

It is sufficient to prove that

$$\left(0 = \sum_{F \in \mathcal{F}_\Omega} \alpha_F B_k^{\mathrm{CR},F}\right) \implies (\forall F \in \mathcal{F}_\Omega \quad \alpha_F = 0). \tag{26}$$

To simplify the notation in the sequel we define $\alpha_F := 0$ for $F \in \mathcal{F}_{\partial\Omega}$ so that

$$\sum_{F \in \mathcal{F}_\Omega} \alpha_F B_k^{\mathrm{CR},F} = \sum_{F \in \mathcal{F}} \alpha_F B_k^{\mathrm{CR},F}.$$

Let $K \in \mathcal{T}$ and note that

$$\left(\sum_{F \in \mathcal{F}_\Omega} \alpha_F B_k^{\mathrm{CR},F}\right)\Big|_K = \sum_{F \in \mathcal{F}(K)} \alpha_F \, B_k^{\mathrm{CR},F}\Big|_K. \tag{27}$$

The vertex values in (20) combined with the assumption in (26) and (27) imply the following vertex conditions:

$$-d\alpha_{F_{\mathbf{y}}} + \sum_{F \in \mathcal{F}(K)} \alpha_F = 0 \quad \forall \mathbf{y} \in \mathcal{V}(K), \tag{28}$$

where $F_{\mathbf{y}}$ is the facet of K opposite to $\mathbf{y}$. We collect the coefficients in $\boldsymbol{\alpha} = (\alpha_F)_{F \in \mathcal{F}(K)}$ and define the matrix

$$\mathbf{Q}_d(s) = \begin{bmatrix} -s & 1 & \dots & 1 \\ 1 & -s & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -s \end{bmatrix} \in \mathbb{R}^{d \times d}, \tag{29}$$

so that (28) becomes equivalent to

$$\mathbf{Q}_{d+1}(d-1)\,\boldsymbol{\alpha} = \mathbf{0}. \tag{30}$$

In Appendix A we will prove that the determinant is given by

$$\det \mathbf{Q}_d(s) = (-1)^{d+1}(1+s)^{d-1}(d-1-s) \tag{31}$$

so that

$$\det \mathbf{Q}_{d+1}(d-1) = (-d)^d \neq 0. \tag{32}$$

Hence, the system (30) has the unique solution $\boldsymbol{\alpha} = \mathbf{0}$. Since $K \in \mathcal{T}$ was arbitrary we conclude that (26) holds.

The proof that the functions in $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F})$ are linearly independent is verbatim.

Case 2, the space $V_k^{\mathrm{nc}}(\mathcal{T})$.

The supports of the functions $B_k^{\mathrm{CR},K}$ have pairwise disjoint interior and so the functions in $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{T})$ are $L^2(\Omega)$-orthogonal. Hence, it suffices to show that $B_k^{\mathrm{CR},K}$ is not the zero function for $k \geq 2$ and $B_1^{\mathrm{CR},K} = 0$. For the reference element, it holds

$$d\, B_k^{\mathrm{CR},\hat{K}}\Big|_{\hat{K}}(\mathbf{x}) = \left(\sum_{\ell=1}^{d} P_k^{(0,d-2)}(1-2x_\ell)\right) + P_k^{(0,d-2)}(-(1-2|\mathbf{x}|)) - 1 \quad \forall \mathbf{x} = (x_\ell)_{\ell=1}^d \in \hat{K}.$$

We employ the representation (see [8, 18.5.7]):

$$\begin{aligned} P_k^{(0,d-2)}(1-2s) &= \sum_{\ell=0}^{k} (-1)^\ell \binom{k+\ell+d-2}{\ell}\binom{k}{\ell} s^\ell \\ P_k^{(0,d-2)}(-(1-2s)) &\overset{[8,\ \text{Table } 18.6.1]}{=} (-1)^k P_k^{(d-2,0)}(1-2s) \\ &= (-1)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k+\ell+d-2}{\ell}\binom{k+d-2}{\ell+d-2} s^\ell \end{aligned}$$

to obtain

$$d\, B_k^{\mathrm{CR},\hat{K}}\Big|_{\hat{K}}(\mathbf{x}) = -1 + \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell}\binom{k+\ell+d-2}{\ell} \sum_{j=1}^{d} x_j^\ell \tag{33}$$

$$+ (-1)^k \sum_{\ell=0}^{k} (-1)^\ell \binom{k+\ell+d-2}{\ell}\binom{k+d-2}{\ell+d-2} |\mathbf{x}|^\ell \tag{34}$$

$$= \binom{2k+d-2}{k}\left(\left((-1)^k \sum_{j=1}^{d} x_j^k\right) + |\mathbf{x}|^k\right) + p_{k-1}(\mathbf{x}),$$

where $p_{k-1} \in \mathbb{P}_{k-1}$. It is easy to see that for $k \geq 2$ the highest order term $\left(\left((-1)^k \sum_{j=1}^d x_j^k\right) + |\mathbf{x}|^k\right)$ is not the zero function and $B_k^{\mathrm{CR},\hat{K}}\Big|_{\hat{K}} \neq 0$.

For $k = 1$, we obtain from (33)

$$d\, B_1^{\mathrm{CR},\hat{K}}\Big|_{\hat{K}}(\mathbf{x}) = -1 + d - d|\mathbf{x}| - (d-1) + d|\mathbf{x}| = 0.$$

■

Lemma 12 *With the Notation as in Lemma 11, a basis for* $\mathrm{CR}_k(\mathcal{T})$ *is given by*

$$\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T}) := \begin{cases} \left\{ B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \overset{\bullet}{\mathcal{N}}_k(\mathcal{T}) \right\} \cup \mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}) & k \text{ odd}, \\ \{ B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_k(\mathcal{T}) \} \cup \mathcal{B}_k^{\mathrm{nc}}(\mathcal{T}') & k \text{ even}, \end{cases} \tag{35}$$

where $\mathcal{T}' \subset \mathcal{T}$ *is any submesh with* $|\mathcal{T}'| = |\mathcal{T}| - 1$.

A basis for the subspace $\mathrm{CR}_{k,0}(\mathcal{T})$ *is given by*

$$\mathcal{B}_{k,0}^{\mathrm{CR}}(\mathcal{T}) := \begin{cases} \left\{ B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \overset{\bullet}{\mathcal{N}}_{k,0}(\mathcal{T}) \right\} \cup \mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}_\Omega) & k \text{ odd}, \\ \{ B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_{k,0}(\mathcal{T}) \} \cup \mathcal{B}_k^{\mathrm{nc}}(\mathcal{T}) & k \text{ even}. \end{cases} \tag{36}$$

Proof. Case 1: $k \geq 1$ is odd.

For this case, we only prove that $\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T})$ is a basis of $\mathrm{CR}_k(\mathcal{T})$ while the proof of (36) is verbatim. From (25) and the definition of $V_k^{\mathrm{nc}}(\mathcal{F})$ and $V_k^{\mathrm{nc}}(\mathcal{T})$ we conclude that it is sufficient to prove

$$\begin{aligned} & \underbrace{\sum_{\mathbf{z} \in \overset{\bullet}{\mathcal{N}}_k(\mathcal{T})} \gamma_{\mathbf{z}} B_k^{\mathbf{z}}}_{=:w_{\mathrm{c}}} + \underbrace{\sum_{F \in \mathcal{F}} \alpha_F B_k^{\mathrm{CR},F}}_{=:w_{\mathcal{F}}} = 0 \\ \Longrightarrow \quad & \text{all coefficients } \gamma_{\mathbf{z}},\ \alpha_F \text{ are zero.} \end{aligned} \tag{37}$$

Let $K \in \mathcal{T}$ and

$$w_K := \sum_{F \in \mathcal{F}(K)} \alpha_F \, B_k^{\mathrm{CR},F}\Big|_K . \tag{38}$$

Since $w_{\mathrm{c}}(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{V}$, condition (37) implies the vertex condition $w_K(\mathbf{y}) = 0$ for all $\mathbf{y} \in \mathcal{V}(K)$. Reasoning as in Case 1 of the proof of Lemma 11 we conclude that all coefficients α_F are zero. It is well known that the conforming basis functions $B_k^{\mathbf{z}}$ are linearly independent so that also all coefficients $\gamma_{\mathbf{z}}$ in (37) are zero.

Case 2: $k \geq 2$ is even.

Part 1: We first prove that $\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T})$ in (35) defines a basis of $\mathrm{CR}_k(\mathcal{T})$. The statement is trivial if $\mathcal{T}$ contains only a single simplex since then $\mathrm{CR}_k(\mathcal{T}) = S_k(\mathcal{T})$ and $\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T}) = \{B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_k(\mathcal{T})\}$ is the standard Lagrange basis. It remains to consider the case $|\mathcal{T}| \geq 2$.

1st step: We show

$$V_k^{\mathrm{nc}}(\mathcal{T}) \cap S_k(\mathcal{T}) = \operatorname{span}\{\Psi_k\} \quad \text{with} \quad \Psi_k := \sum_{K \in \mathcal{T}} B_k^{\mathrm{CR},K}. \tag{39}$$

Let $w \in V_k^{\mathrm{nc}}(\mathcal{T})$. For any $K \in \mathcal{T}$ it holds $w|_K = \beta_K \, B_k^{\mathrm{CR},K}\Big|_K$. Let K' be an adjacent simplex: $F := K \cap K' \in \mathcal{F}_\Omega$. From (20a) and (21a) it follows $B_k^{\mathrm{CR},K}\Big|_K\Big|_F = B_k^{\mathrm{CR},K'}\Big|_{K'}\Big|_F \neq 0$ so that the condition $w \in S_k(\mathcal{T})$ implies that $\beta_{K'} = \beta_K$. By iterating this argument over adjacent simplices we obtain $\beta_{K''} = \beta_K$ for all $K'' \in \mathcal{T}$ and (39) follows.

2nd step: Since $\Psi_k \notin V_k^{\mathrm{nc}}(\mathcal{T}')$ we conclude that $V_k^{\mathrm{nc}}(\mathcal{T}') \cap S_k(\mathcal{T}) = \{0\}$ and the sum in $\widetilde{\mathrm{CR}_k} := V_k^{\mathrm{nc}}(\mathcal{T}') \oplus S_k(\mathcal{T})$ is direct. From Lemma 11 and the well-known fact that the functions in $\{B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_k(\mathcal{T})\}$ are linearly independent we conclude that $\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T})$ is a basis of $\widetilde{\mathrm{CR}_k}$. It remains to show $\widetilde{\mathrm{CR}_k} = \mathrm{CR}_k(\mathcal{T})$. Since $\widetilde{\mathrm{CR}_k} \subset \mathrm{CR}_k(\mathcal{T})$ it is sufficient to prove $B_k^{\mathrm{CR},K} \in \widetilde{\mathrm{CR}_k}$ for $\{K\} = \mathcal{T} \backslash \mathcal{T}'$. Let $\tilde{\Psi}_k := \sum_{K' \in \mathcal{T}'} B_k^{\mathrm{CR},K'} \in \widetilde{\mathrm{CR}_k}$. Then

$$\Psi_k = B_k^{\mathrm{CR},K} + \tilde{\Psi}_k \in S_k(\mathcal{T}),$$

i.e., $B_k^{\mathrm{CR},K} = \Psi_k - \tilde{\Psi}_k$ with $\Psi_k \in S_k(\mathcal{T}) \subset \widetilde{\mathrm{CR}_k}$ and $\tilde{\Psi}_k \in \widetilde{\mathrm{CR}_k}$.

Part 2: It remains to prove that $\mathcal{B}_{k,0}^{\mathrm{CR}}(\mathcal{T})$ in (36) defines a basis of $\mathrm{CR}_{k,0}(\mathcal{T})$. We generalize the proof [5, Thm. 22] in 2D to general dimension d. It is sufficient to prove the conclusion

$$\begin{aligned} &\underbrace{\sum_{\mathbf{z} \in \mathcal{N}_{k,0}(\mathcal{T})} \gamma_{\mathbf{z}} B_k^{\mathbf{z}}}_{=:w_{\mathrm{c}}} + \underbrace{\sum_{K \in \mathcal{T}} \beta_K B_k^{\mathrm{CR},K}}_{=:w_{\mathcal{T}}} = 0 \\ \Longrightarrow \quad &\text{all coefficients } \gamma_{\mathbf{z}},\ \beta_F \text{ are zero.} \end{aligned} \tag{40}$$

Consider some $K \in \mathcal{T}$ which has at least one facet, say F, on $\partial\Omega$. Then

$$(w_{\mathrm{c}} + w_{\mathcal{T}})|_F = w_{\mathcal{T}}|_F = \beta_K \left. B_k^{\mathrm{CR},K} \right|_F.$$

From (10), (20), and (21a) we conclude that for $\mathbf{y} \in \mathcal{V}(F)$:

$$\left. B_k^{\mathrm{CR},K} \right|_F (\mathbf{y}) = \frac{d - 1 + (-1)^k \rho_k}{d} \neq 0,$$

so that $(w_{\mathrm{c}} + w_{\mathcal{T}})|_F = 0$ implies $\beta_K = 0$ and, in turn

$$\left((w_{\mathrm{c}} + w_{\mathcal{T}})|_K = w_{\mathrm{c}}|_K = 0 \right) \implies \left(\gamma_{\mathbf{z}} = 0 \quad \forall \mathbf{z} \in \mathcal{N}_k \cap K \right). \tag{41}$$

This argument allows us to set up an induction step. We set $\Omega' := \Omega \backslash K$ and $\mathcal{T}' := \mathcal{T} \backslash \{K\}$. Then, the function $(w_{\mathrm{c}} + w_{\mathcal{T}})_{\Omega'}$ vanishes on $\partial\Omega'$; we again choose some $K' \in \mathcal{T}'$ with one facet on $\partial\Omega'$ and conclude as before that the corresponding coefficients $\beta_{K'}$ and $\gamma_{\mathbf{z}}$, $\mathbf{z} \in K'$ are zero. ■

These findings are collected as the following theorem.

Theorem 13 *The sums in (25) are direct: For any $k \in \mathbb{N}$, it holds*

$$\mathrm{CR}_k(\mathcal{T}) = \begin{cases} \overset{\bullet}{S}_k(\mathcal{T}) \oplus V_k^{\mathrm{nc}}(\mathcal{F}) & k \text{ odd,} \\ S_k(\mathcal{T}) \oplus V_k^{\mathrm{nc}}(\mathcal{T}') & k \text{ even,} \end{cases}$$

where $\mathcal{T}' \subset \mathcal{T}$ is any submesh with $|\mathcal{T}'| = |\mathcal{T}| - 1$ and

$$\mathrm{CR}_{k,0}(\mathcal{T}) = \begin{cases} \overset{\bullet}{S}_{k,0}(\mathcal{T}) \oplus V_k^{\mathrm{nc}}(\mathcal{F}_\Omega) & k \text{ odd,} \\ S_{k,0}(\mathcal{T}) \oplus V_k^{\mathrm{nc}}(\mathcal{T}) & k \text{ even.} \end{cases}$$

For odd $k \geq 1$, a basis for $V_k^{\mathrm{nc}}(\mathcal{F})$ is given by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}) = \left\{B_k^{\mathrm{CR},F}, F \in \mathcal{F}\right\}$ and for $V_k^{\mathrm{nc}}(\mathcal{F}_\Omega)$ by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{F}_\Omega) = \left\{B_k^{\mathrm{CR},F}, F \in \mathcal{F}_\Omega\right\}$ while, for $k \geq 2$, a basis for $V_k^{\mathrm{nc}}(\mathcal{T})$ is given by $\mathcal{B}_k^{\mathrm{nc}}(\mathcal{T}) = \left\{B_k^{\mathrm{CR},K}, K \in \mathcal{T}\right\}$. For $k = 1$ it holds $V_1^{\mathrm{nc}}(\mathcal{T}) = \{0\}$.

It holds

$$S_k(\mathcal{T}) \subset \mathrm{CR}_k(\mathcal{T}) \subset \mathrm{CR}_k^{\max}(\mathcal{T}) \tag{42}$$

and

$$S_{k,0}(\mathcal{T}) \subset \mathrm{CR}_{k,0}(\mathcal{T}) \subset \mathrm{CR}_{k,0}^{\max}(\mathcal{T}). \tag{43}$$

The first inclusion in (43) is strict if k is even or if k is odd and $\mathcal{T}$ consists of more than a single simplex. The second inclusion in (43) is strict, e.g., for $d = 3$ and $k \geq 2$.

Proof. The first claims are direct consequences of the previous lemmata. The first inclusion in (43) is strict if $\mathrm{CR}_{k,0}(\mathcal{T})$ contains at least one non-conforming shape function. This is always the case if $k \geq 2$ is even and for odd k if $\mathcal{T}$ contains at least one inner facet, i.e., if $\mathcal{T}$ consists of more than a single simplex. The second inclusion is strict, e.g., for $d = 3$ and any $k \geq 2$ (see [6, Sec. 5.3.1, Table 5.3.2]). For $d = 2$ however, the second inclusion becomes an equality (see [2], [5]). ■

4 A basis for Crouzeix-Raviart spaces $\mathrm{CR}_k(\mathcal{T})$ and $\mathrm{CR}_{k,0}(\mathcal{T})$

In this section we will present a basis for the Crouzeix-Raviart space which is composed of orthogonal polynomials multiplied by face bubble functions. This allows for a simple construction of the degrees of freedom for the face basis functions as facet and simplex integrals with weight functions such they are well defined on $\mathrm{CR}_{k,0}(\mathcal{T})$.

Theorem 13 directly give guidelines for a construction: any (standard) basis for $\overset{\bullet}{S}_k(\mathcal{T})$, $\overset{\bullet}{S}_{k,0}(\mathcal{T})$ appended by non conforming facet functions $B_k^{\mathrm{CR},F}$ for odd k and by non-conforming simplex functions $B_k^{\mathrm{CR},K}$ form a basis of the Crouzeix-Raviart spaces. However, the construction of the local bidual degrees of freedom is a non-trivial task and depends on the choice of basis for the conforming parts. Since the functions in $\mathrm{CR}_{k,0}(\mathcal{T})$ are in general discontinuous across facets while their moments up to a polynomial degree $k-1$ are continuous, the degrees of freedom associated to the inner facets must be moments of the form $J_\mu^F(u) = \int_F g_\mu^F u$ for polynomial weights g_μ^F up to a degree $k-1$. Orthogonality relations on facets will become important for the choice of g_μ^F and this is the reason to define a basis based on the products of face bubbles with orthogonal polynomials on the simplicial entities (edges, facets, etc.).

4.1 The simplicial complex of a simplicial finite element mesh

Let $\mathcal{T}$ be a conforming finite element mesh for Ω consisting of closed simplices $K \in \mathcal{T}$. Let $\mathcal{S}$ denote the associate simplicial complex, i.e., the set of ℓ-(dimensional) simplices (faces), $\ell \in \{0, 1, \ldots, d\}$, that satisfies the following conditions:

1. Every face[1] of a simplex $\tau \in \mathcal{S}$ is also in $\mathcal{S}$.
2. The non-empty intersection of any two simplices $\tau_1, \tau_2 \in \mathcal{S}$ is a face of both, τ_1 and τ_2.

We distinguish between the boundary faces $\mathcal{S}_{\partial\Omega} := \{\tau \in \mathcal{S} \mid \tau \in \partial\Omega\}$ and the inner faces $\mathcal{S}_\Omega := \mathcal{S} \backslash \mathcal{S}_{\partial\Omega}$. The subset of $\mathcal{S}$ which contains all ℓ-simplices in $\mathcal{S}$ is $\mathcal{S}_\ell$. Specifically, the set of vertices $\mathcal{V}$ is $\mathcal{S}_0$, the set of edges $\mathcal{E}$ is $\mathcal{S}_1$ and the set of facets $\mathcal{F}$ equals $\mathcal{S}_{d-1}$. The subset of $\mathcal{S}_\ell$ which contains all those ℓ-simplices which are *not* a subset of the boundary $\partial\Omega$ is denoted by $\mathcal{S}_{\ell,\Omega}$. The corresponding *skeletons* Σ_ℓ, $\Sigma_{\ell,\Omega}$ are given by

$$\Sigma_\ell := \bigcup_{\tau \in \mathcal{S}_\ell} \tau \quad \text{and} \quad \Sigma_{\ell,\Omega} := \bigcup_{\tau \in \mathcal{S}_{\ell,\Omega}} \tau. \tag{44}$$

In analogy to (3) we define for $\tau \in \mathcal{S}_\ell$ the adjacent simplex patch by

$$\mathcal{T}_\tau := \{K \in \mathcal{T} \mid \tau \subset K\} \quad \text{and} \quad \omega_\tau := \textstyle\bigcup_{K \in \mathcal{T}_\tau} K.$$

4.2 The face bubble functions

For $\mathbf{z} \in \mathcal{V}$, let $\varphi^1_{\mathbf{z}} \in S_1(\mathcal{T})$ be the "hat function" for the vertex $\mathbf{z}$ characterized by the condition $\varphi^1_{\mathbf{z}}(\mathbf{y}) = \delta_{\mathbf{z},\mathbf{y}}$ for all $\mathbf{z}, \mathbf{y} \in \mathcal{V}$. For a face τ, the set of vertices is denoted by $\mathcal{V}(\tau)$, the face bubble $W_\tau \in S_{k,0}(\mathcal{T})$ is given by

$$W_\tau = \prod_{\mathbf{z} \in \mathcal{V}(\tau)} \varphi^1_{\mathbf{z}} \qquad \text{and satisfies } \operatorname{supp} W_\tau = \mathcal{T}_\tau.$$

4.3 Orthogonal polynomials on the reference simplex

In [12, §2.5.2] orthogonal polynomials on the ℓ-dimensional reference element

$$\hat{K}_\ell := \left\{\mathbf{x} = (x_j)_{j=1}^\ell \in \mathbb{R}^\ell_{\geq 0} \mid \textstyle\sum_{j=1}^\ell x_j \leq 1\right\}$$

are defined which are generalizations of the univariate Jacobi polynomials as we will explain in the following. Let $\hat{\mathbf{z}}_0 := \mathbf{0}$, $\hat{\mathbf{z}}_j := \mathbf{e}_j$, $1 \leq j \leq \ell$ denote the vertices of $\hat{K}_\ell$ (see (1)), where $\mathbf{e}_j$ denotes the j-th canonical unit vector in $\mathbb{R}^\ell$,

[1] A *face* of a simplex is the convex hull of a non-empty subset of the simplex vertices. The *facets* of a simplex are the $(d-1$ dimensional) faces with exactly d vertices.

and let $\hat{\lambda}_j$ denote the barycentric coordinate in $\hat{K}_\ell$ for the vertex $\hat{\mathbf{z}}_j$. In [12, §2.5.2], the polynomials $P_{\boldsymbol{\alpha}}$ are defined for $\boldsymbol{\alpha} \in \mathbb{N}_0^\ell$, by first introducing the formal expression for $\mathbf{y} = (y_j)_{j=0}^\ell \in \mathbb{R}^{\ell+1}$

$$P_{\boldsymbol{\alpha}}^{\text{bary}}(\mathbf{y}) = \prod_{j=1}^{\ell} \left(y_0 + \sum_{m=j}^{\ell} y_m \right)^{\alpha_j} P_{\alpha_j}^{(s_j,1)} \left(\frac{2y_j}{y_0 + \sum_{m=j}^{\ell} y_m} - 1 \right)$$

with

$$s_j = 2 \left(\sum_{m=j+1}^{\ell} \alpha_m \right) + 2(\ell - j) + 1$$

and then inserting the barycentric coordinates $\widehat{\boldsymbol{\lambda}} = \left(\hat{\lambda}_j\right)_{j=0}^{\ell}$:

$$P_{\boldsymbol{\alpha}}(\hat{\mathbf{x}}) := P_{\boldsymbol{\alpha}}^{\text{bary}} \left(\widehat{\boldsymbol{\lambda}}(\hat{\mathbf{x}}) \right) \quad \forall \hat{\mathbf{x}} \in \hat{K}_\ell.$$

They are orthogonal with respect to the scalar product

$$(u, v)_{\hat{K}_\ell} := \int_{\hat{K}_\ell} W_{\hat{K}_\ell} uv \quad \text{with weight function } W_{\hat{K}_\ell} := \prod_{\ell=0}^{d} \hat{\lambda}_\ell,$$

i.e.,

$$(P_{\boldsymbol{\alpha}}, P_{\boldsymbol{\beta}})_{\hat{K}_\ell} = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} c_{\boldsymbol{\alpha}}$$

for some constants $c_{\boldsymbol{\alpha}} \neq 0$ (the value of $c_{\boldsymbol{\alpha}}$ is given in [12, §2.5.2]). In this way, the set $\left\{ P_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{N}_{\leq m}^d \right\}$ forms an orthogonal basis of the space $\mathbb{P}_m\left(\hat{K}_\ell\right)$ of $\ell-$variate polynomials on $\hat{K}_\ell$ of maximal total degree m.

Remark 14 *Let $\ell \in \{1, 2, \ldots, d\}$. Let $m \geq \ell + 1$. The set*

$$\left\{ W_{\hat{K}_\ell} P_{\boldsymbol{\alpha}} \mid \boldsymbol{\alpha} \in \mathbb{N}_{\leq m-(\ell+1)}^{\ell} \right\}$$

forms a basis of

$$\mathbb{P}_{m,0}\left(\hat{K}_\ell\right) := \left\{ v \in \mathbb{P}_m\left(\hat{K}_\ell\right) \mid v|_{\partial \hat{K}_\ell} = 0 \right\}. \tag{45}$$

Example 15 *The orthogonal polynomials $P_{\boldsymbol{\alpha}}(\hat{\mathbf{x}})$ have the explicit form*

1. *for $\ell = 1$:*

$$P_\alpha(\hat{x}) = P_\alpha^{(1,1)}\left(2\hat{\lambda}_1(\hat{x}) - 1\right)$$

2. *and for $\ell = 2$:*

$$P_{(\alpha_1,\alpha_2)}(\hat{\mathbf{x}}) = P_{\alpha_1}^{(2\alpha_2+3,1)}\left(2\hat{\lambda}_1(\hat{\mathbf{x}}) - 1\right)\left(1 - \hat{\lambda}_1(\hat{\mathbf{x}})\right)^{\alpha_2} P_{\alpha_2}^{(1,1)}\left(\frac{2\hat{\lambda}_2(\hat{\mathbf{x}})}{1 - \hat{\lambda}_1(\hat{\mathbf{x}})} - 1\right).$$

4.4 Basis functions associated with simplex faces

To define basis functions associated to a face $\tau \in \mathcal{S}_\ell$ we fix a numbering of its vertices $\mathbf{A}_j(\tau)$, $0 \le j \le \ell$. For any adjacent simplex $K \in \mathcal{T}_\tau$, we choose an affine pullback $\chi_K : \hat{K}_d \to K$ such that the restriction to $\hat{K}_\ell$ satisfies

$$\chi_K(\hat{\mathbf{x}}) := \mathbf{A}_0(\tau) + \sum_{j=1}^{\ell} \hat{x}_j \left(\mathbf{A}_j(\tau) - \mathbf{A}_0(\tau)\right) \quad \forall \hat{\mathbf{x}} = (\hat{x}_j)_{j=1}^d \in \hat{K}_d \text{ with } (x_j)_{j=\ell+1}^d = \mathbf{0}.$$

It is important to note that the restrictions $\chi_K|_{\hat{K}_\ell}$ coincide for all $K \in \mathcal{T}_\tau$ due to the chosen fixed numbering of vertices in τ, i.e., τ is parametrized in the same way for all adjacent simplices. According to Remark 14 basis functions associated to a face $\tau \in \mathcal{S}_\ell$ exist only if $k \ge \ell + 1$ which we assume for the following. We employ the index sets

$$\mathcal{I}_{k,\ell} := \left\{(\tau, \boldsymbol{\alpha}) : \tau \in \mathcal{S}_\ell \quad \boldsymbol{\alpha} \in \mathbb{N}^\ell_{k-(\ell+1)}\right\}, \quad \mathcal{I}_{k,\ell,\Omega} := \left\{(\tau, \boldsymbol{\alpha}) \in \mathcal{I}_{k,\ell} : \tau \in \mathcal{S}_{\ell,\Omega}\right\},$$

$$\mathcal{I}_k := \begin{cases} \bigcup_{\ell=1}^{\min\{k-1,d\}} \mathcal{I}_{k,\ell} & \text{if } k \text{ is odd,} \\ \bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{I}_{k,\ell} & \text{if } k \text{ is even,} \end{cases} \qquad \mathcal{I}_{k,\Omega} := \begin{cases} \bigcup_{\ell=1}^{\min\{k-1,d\}} \mathcal{I}_{k,\ell,\Omega} & \text{if } k \text{ is odd,} \\ \bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{I}_{k,\ell,\Omega} & \text{if } k \text{ is even,} \end{cases}$$

$$\mathcal{I}_{k,\partial\Omega} := \mathcal{I}_k \backslash \mathcal{I}_{k,\Omega}.$$

Definition 16 *Let $k \in \mathbb{N}_{\ge 1}$. For $\ell \in \{0, 1, \ldots, k-1\}$, the conforming basis functions associated to a face $\tau \in \mathcal{S}_\ell$ are given for*

1. *$\ell = 0$ by*
$$B_{\mathbf{z},\boldsymbol{\alpha}} = \varphi^1_{\mathbf{z}} \quad \forall \mathbf{z} \in \mathcal{V} \quad \boldsymbol{\alpha} \in \mathbb{N}^0_{\le k-1} := \{0\},$$
2. *$\ell \ge 1$ by*
$$B_{\tau,\boldsymbol{\alpha}} := \left\{ \begin{array}{ll} W_\tau P_{\boldsymbol{\alpha}} \circ \chi_K^{-1} & K \in \mathcal{T}_\tau, \\ 0 & \textit{otherwise} \end{array} \right| \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^\ell_{\le k-(\ell+1)} \tag{46}$$

and collected in

$$\mathcal{B}_\ell(\mathcal{T}) := \{B_{\tau,\boldsymbol{\alpha}} \mid \forall (\tau, \boldsymbol{\alpha}) \in \mathcal{I}_{k,\ell}\},$$
$$\mathcal{B}_{\ell,0}(\mathcal{T}) := \{B_{\tau,\boldsymbol{\alpha}} \mid \forall (\tau, \boldsymbol{\alpha}) \in \mathcal{I}_{k,\ell,\Omega}\}.$$

Remark 17 *The function $B_{\tau,\boldsymbol{\alpha}}$ in (46) can be expressed without the pullback χ_K by setting $\boldsymbol{\varphi}^1_\tau := \left(\varphi^1_{\mathbf{A}_i(\tau)}\right)_{i=0}^{\ell}$ and*

$$B_{\tau,\boldsymbol{\alpha}} = W_\tau P^{\text{bary}}_{\boldsymbol{\alpha}}\left(\boldsymbol{\varphi}^1_\tau\right).$$

4.5 A basis for $\mathrm{CR}_k(\mathcal{T})$ and $\mathrm{CR}_{k,0}(\mathcal{T})$

In this section, we will prove that the union of the basis functions in $\mathcal{B}_\ell(\mathcal{T})$ with the non-conforming facet/simplex functions yields a basis of the Crouzeix-Raviart spaces.

Definition 18 *Let $k \in \mathbb{N}_{\geq 1}$. A basis for $\mathrm{CR}_k(\mathcal{T})$ is given by*

$$\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T}) := \begin{cases} \left(\bigcup_{\ell=1}^{\min\{k-1,d\}} \mathcal{B}_\ell(\mathcal{T}) \right) \cup \left\{ B_k^{\mathrm{CR},F} : F \in \mathcal{F} \right\} & \text{if } k \text{ is odd,} \\ \left(\bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{B}_\ell(\mathcal{T}) \right) \cup \left\{ B_k^{\mathrm{CR},K} : K \in \mathcal{T} \right\} & \text{if } k \text{ is even} \end{cases} \tag{47}$$

and for $\mathrm{CR}_{k,0}(\mathcal{T})$ by

$$\mathcal{B}_{k,0}^{\mathrm{CR}}(\mathcal{T}) := \begin{cases} \left(\bigcup_{\ell=1}^{\min\{k-1,d\}} \mathcal{B}_{\ell,0}(\mathcal{T}) \right) \cup \left\{ B_k^{\mathrm{CR},F} : F \in \mathcal{F}_\Omega \right\} & \text{if } k \text{ is odd,} \\ \left(\bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{B}_{\ell,0}(\mathcal{T}) \right) \cup \left\{ B_k^{\mathrm{CR},K} : K \in \mathcal{T} \right\} & \text{if } k \text{ is even.} \end{cases} \tag{48}$$

The following lemma states that the notion "basis" in Definition 18 indeed is justified.

Lemma 19 *The set $\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T})$ forms a basis of $\mathrm{CR}_k(\mathcal{T})$ and $\mathcal{B}_{k,0}^{\mathrm{CR}}(\mathcal{T})$ of $\mathrm{CR}_{k,0}(\mathcal{T})$.*

Proof. We only prove the first statement. Due to Theorem 13 it is sufficient to prove that

$$\tilde{\mathcal{B}}_k^{\mathrm{CR}}(\mathcal{T}) := \begin{cases} \bigcup_{\ell=1}^{\min\{k-1,d\}} \mathcal{B}_\ell(\mathcal{T}) & \text{if } k \text{ is odd,} \\ \bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{B}_\ell(\mathcal{T}) & \text{if } k \text{ is even} \end{cases}$$

is a basis for $\overset{\bullet}{S}_k(\mathcal{T})$ for odd k and for $S_k(\mathcal{T})$ for even k. We start with the linear independence and assume $0 \leq \ell \leq \min\{k-1, d\}$. The construction of the basis functions $B_{\tau,\boldsymbol{\alpha}}$ via the face bubbles W_τ imply that

$$B_{\tau,\boldsymbol{\alpha}}|_{\tau'} = 0 \quad \forall \tau \in \mathcal{S}_\ell \quad \forall \tau' \in \mathcal{S}_{\ell'} \quad \forall \ell' \leq \ell.$$

Since $B_{\tau,\boldsymbol{\alpha}}|_\tau$ is not the zero function the direct sum representation follows

$$\sum_{\ell=0}^{\min\{k-1,d\}} \sum_{\tau \in \mathcal{S}_\ell} \operatorname{span}\left\{ B_{\tau,\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{N}_{k-(\ell+1)}^\ell \right\} = \bigoplus_{\ell=0}^{\min\{k-1,d\}} \bigoplus_{\tau \in \mathcal{S}_\ell} \operatorname{span}\left\{ B_{\tau,\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{N}_{k-(\ell+1)}^\ell \right\}.$$

Remark 14 implies via a simple counting argument:

$$\dim P_{k-(\ell+1),0}(\tau) = \binom{k-1}{\ell} = \left|\mathbb{N}^{\ell}_{k-(\ell+1)}\right|$$

so that the functions $B_{\tau,\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathbb{N}^{\ell}_{k-(\ell+1)}$ are also linearly independent and, in turn, the set $\bigcup_{\ell=0}^{\min\{k-1,d\}} \mathcal{B}_{\ell}(\mathcal{T})$ consists of linearly independent functions. It remains to show that any function in $\overset{\bullet}{S}_k(\mathcal{T})$, $S_k(\mathcal{T})$ can be represented as a linear combination of $\tilde{\mathcal{B}}^{\mathrm{CR}}_k(\mathcal{T})$.

Case 1: k is even.

We employ the Lagrange basis as in Lemma 12. Recall the definition of the skeleton Σ_{ℓ} as in (44) and note that $\mathcal{N}_k(\mathcal{T}) \cap \Sigma_0 = \mathcal{V}$. Any function $u \in S_k(\mathcal{T})$ can be written in the form

$$u = u_0 + v_1 \quad \text{for } u_0 = \sum_{\mathbf{z} \in \mathcal{N}_k(\mathcal{T}) \cap \Sigma_0} u(\mathbf{z})\, B^{\mathbf{z}}_k \tag{49}$$

and v_1 vanishes in all simplex vertices. Remark 14 implies that there is a function $w_0 \in \operatorname{span} \tilde{\mathcal{B}}^{\mathrm{CR}}_k(\mathcal{T})$ such that

$$u_0(\mathbf{z}) = w_0(\mathbf{z}) \quad \forall \mathbf{z} \in \mathcal{V}.$$

Hence, the difference $u - w_0$ vanish in all simplex vertices and $u - w_0$ has the representation[2]

$$u - w_0 = u_1 + v_2 \quad \text{for } v_2 = \sum_{\mathbf{z} \in \mathcal{N}_k(\mathcal{T}) \cap (\Sigma_1 \setminus \Sigma_0)} u(\mathbf{z})\, B^{\mathbf{z}}_k.$$

Again Remark 14 implies that the restriction of u_1 to an edge can be represented by some $w_1 \in \operatorname{span} \tilde{\mathcal{B}}^{\mathrm{CR}}_k(\mathcal{T})$:

$$u_1|_E = w_1|_E \quad \forall E \in \mathcal{S}_1(\mathcal{T})$$

and $u - w_0 - w_1$ vanishes on Σ_1. In this way, an induction argument with respect to the dimension ℓ implies

$$u \in \operatorname{span} \tilde{\mathcal{B}}^{\mathrm{CR}}_k(\mathcal{T}).$$

Case 2: k is odd.

This case can be concluded by the same arguments as before by observing the $u_0 = 0$ in (49) for $u \in \overset{\bullet}{S}_k(\mathcal{T})$. ■

[2] Note that $\mathcal{N}_k(\mathcal{T}) \cap (\Sigma_1 \setminus \Sigma_0)$ is the set of nodal points which lie in the interiors of the edges $E \in \mathcal{E}$.

5 Degrees of freedom for the case $d \geq 2$ and k is odd

In this section, we will define degrees of freedom for the Crouzeix-Raviart space for general dimension $d \in \{2, 3, \ldots\}$ and odd k which form a dual basis. We recall the formal definition.

Definition 20 *Let $\{b_i : i \in \{1, 2, \ldots, N\}\}$ be a basis of a finite dimensional vector space V_N. A* bidual basis *is a set of functionals J_i, $i \in \{1, 2, \ldots, N\}$, in the dual space V_N' such that*

$$J_i(b_j) = \delta_{i,j} \quad \forall i, j \in \{1, 2, \ldots, N\}.$$

As a very mild assumption on the mesh we require that

$$\mathcal{T} \text{ contains more than only one tetrahedron.} \tag{50}$$

The construction starts with some preliminaries for a single simplex $K \in \mathcal{T}$. We set

$$L^2(\Omega)\big|_K := \left\{ f|_K : f \in L^2(\Omega) \right\}$$

and use this notation also for subspaces of $L^2(\Omega)$. We introduce index sets for $K \in \mathcal{T}$ and $F \in \mathcal{F}$ by

$$\begin{aligned}
&\mathcal{S}_\ell(K) := \{\tau \in \mathcal{S}_\ell \mid \tau \subset K\}, && \mathcal{S}(K) := \bigcup_{\ell=1}^{d} \mathcal{S}_\ell(K), \\
&\mathcal{S}_\ell(F) := \{\tau \in \mathcal{S}_\ell \mid \tau \subset F\}, && \mathcal{S}(F) := \bigcup_{\ell=1}^{d} \mathcal{S}_\ell(F), \\
&\mathcal{I}_{k,\ell}(K) := \left\{ (\tau, \boldsymbol{\alpha}) : \tau \in \mathcal{S}_\ell(K) \quad \boldsymbol{\alpha} \in \mathbb{N}^\ell_{\leq k-(\ell+1)} \right\}, && \mathcal{I}_k(K) := \bigcup_{\ell=1}^{d} \mathcal{I}_{k,\ell}(K), \\
&\mathcal{I}_{k,\ell}(F) := \left\{ (\tau, \boldsymbol{\alpha}) : \tau \in \mathcal{S}_\ell(F) \quad \boldsymbol{\alpha} \in \mathbb{N}^\ell_{\leq k-(\ell+1)} \right\}, && \mathcal{I}_k(F) := \bigcup_{\ell=1}^{d-1} \mathcal{I}_{k,\ell}(F), \\
&\mathcal{B}_k^{\mathrm{CR}}(K) := \left\{ B_{\tau,\boldsymbol{\alpha}}|_K : (\tau, \boldsymbol{\alpha}) \in \mathcal{I}_k(K) \right\} \cup \left\{ B_k^{\mathrm{CR},F}\Big|_K : F \in \mathcal{F}(K) \right\}.
\end{aligned} \tag{51}$$

Since $S_k(\mathcal{T}) \subset \mathrm{CR}_k(\mathcal{T})$ it holds

$$\mathbb{P}_k(K) = S_k(\mathcal{T})|_K = \mathrm{CR}_k(\mathcal{T})|_K = \operatorname{span}\left\{ \mathcal{B}_k^{\mathrm{CR}}(K) \right\}. \tag{52}$$

Since the number $d+1$ of vertices of a simplex equals the number of facets it follows from the well known fact (cf. (16), (51))

$$\left| \mathcal{N}_k\left(\hat{K}\right) \right| = \left| \bigcup_{\ell=0}^{d} \mathcal{I}_{k,\ell}(K) \right| = \dim \mathbb{P}_k(K)$$

that $\left|\mathcal{B}_k^{\mathrm{CR}}(K)\right| = \dim \mathbb{P}_k(K)$ and the combination with Theorem 13 yields that the functions in $\mathcal{B}_k^{\mathrm{CR}}(K)$ are linearly independent. From elementary linear

algebra we conclude that there exists a bidual basis for $\mathcal{B}_k^{\mathrm{CR}}(K)$, induced by $g_{\tau,\boldsymbol{\alpha}}^K \in L^2(K)$, $(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(K)$ and $g_F^{\mathrm{CR},K} \in L^2(K)$, $F \in \mathcal{F}(K)$ so that the associated functionals

$$J_{\tau,\boldsymbol{\alpha}}^K u := (g_{\tau,\boldsymbol{\alpha}}, u)_{L^2(K)} \quad \text{and} \quad J_F^{\mathrm{CR},K} u := \left(g_F^{\mathrm{CR},K}, u\right)_{L^2(K)}$$

satisfy

$$\begin{aligned} &J_{\tau,\boldsymbol{\alpha}}^K\left(B_{t,\boldsymbol{\beta}}|_K\right) = \delta_{\tau,t}\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \quad && J_{\tau,\boldsymbol{\alpha}}^K\left(B_k^{\mathrm{CR},F}\Big|_K\right) = 0, \\ &J_F^{\mathrm{CR},K}\left(B_{t,\boldsymbol{\beta}}|_K\right) = 0, && J_F^{\mathrm{CR},K}\left(B_k^{\mathrm{CR},F'}\Big|_K\right) = \delta_{F,F'}. \end{aligned}$$

A second ingredient for the definition of degrees of freedom is the selection of an assignment function $\mathrm{mark}_{\mathcal{T}} : \mathcal{S} \to \mathcal{T} \cup \mathcal{F}_{\partial\Omega}$ which satisfies

$$\mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{T} \quad \text{and } \tau \subset \mathrm{mark}_{\mathcal{T}}(\tau) \quad \forall \tau \in \mathcal{S}_\Omega, \tag{53a}$$

$$\mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{F}_{\partial\Omega} \quad \text{and } \tau \subset \mathrm{mark}_{\mathcal{T}}(\tau) \quad \forall \tau \in \mathcal{S}_{\partial\Omega}. \tag{53b}$$

For the final definition of the degrees of freedom for Crouzeix-Raviart spaces we need the following lemma. For $K \in \mathcal{T}$ and $F, F' \in \mathcal{F}(K)$, the restrictions of the basis functions in $\mathcal{B}_k^{\mathrm{CR}}(K)$ (see (51)) to a simplex facet are denoted by

$$B_{F,\tau,\boldsymbol{\alpha}}^K := B_{F,\boldsymbol{\alpha}}|_K\big|_F \quad \text{and} \quad B_{F,k}^{\mathrm{CR},F'} := B_k^{\mathrm{CR},F'}\Big|_K\Big|_F.$$

Lemma 21 *Let k be odd. For $K \in \mathcal{T}$ and $F \in \mathcal{F}(K)$, let $\overset{\bullet}{\mathcal{F}}(K) \subset \mathcal{F}(K)$ be a strict subset, i.e., $\left|\overset{\bullet}{\mathcal{F}}(K)\right| < |\mathcal{F}(K)|$, and $F \in \overset{\bullet}{\mathcal{F}}(K)$. Then the functions in*

$$\overset{\bullet}{\mathcal{B}}{}_k^{\mathrm{CR}}(K,F) := \left\{B_{F,\tau,\boldsymbol{\alpha}}^K : (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(F)\right\} \cup \left\{B_{F,k}^{\mathrm{CR},F'} : F' \in \overset{\bullet}{\mathcal{F}}(K)\right\} \tag{54}$$

are linearly independent in $L^2(F)$.

Proof. We first prove that the sum of the spans of the two sets in (54) is direct while in the second part of the proof we show the functions in each of the two sets of (54) are linearly independent.

Let $u \in \operatorname{span} \overset{\bullet}{\mathcal{B}}{}_k^{\mathrm{CR}}(K,F)$. Then, there is a splitting $u = u_0 + u_1$ with $u_1 \in \operatorname{span}\left\{B_{F,k}^{\mathrm{CR},F'} : F' \in \overset{\bullet}{\mathcal{F}}(K)\right\}$ and

$$u_0 \in \operatorname{span}\left\{B_{F,\tau,\boldsymbol{\alpha}}^K : (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(F)\right\}.$$

First, we prove $u = 0 \implies u_0 = u_1 = 0$. Since u_0 vanishes in all simplex vertices, the condition $u = 0$ implies

$$u_1(\mathbf{z}) = \sum_{F' \in \overset{\bullet}{\mathcal{F}}(K)} \gamma_{F'} B_{F,k}^{\mathrm{CR},F'}(\mathbf{z}) = 0 \quad \forall \mathbf{z} \in \mathcal{V}(F). \tag{55}$$

In a similar fashion as in (30) this condition can be formulated as a linear system for the coefficients $\boldsymbol{\gamma}_K := (\gamma_{F'})_{F' \in \overset{\bullet}{\mathcal{F}}(K)}$. We choose a numbering of the facets in $\overset{\bullet}{\mathcal{F}}(K)$ such that $F_1 = F$ and number the vertices in $\mathcal{V}(K)$ such that $\mathbf{z}_j$ is opposite to the facet F_j. From $F \in \overset{\bullet}{\mathcal{F}}(K)$ and $B_{F,k}^{\mathrm{CR},F} = 1$ we conclude that the implication "condition (55) $\Longrightarrow$ $\boldsymbol{\gamma}_K = \mathbf{0}$" is equivalent to the condition

$$\det \mathbf{R}_d (d-1) \neq 0 \tag{56}$$

with $\mathbf{R}_d$ as in (77). The relation (56) is proved in the appendix (Lemma 36). and $\boldsymbol{\gamma}_K = \mathbf{0}$ follows. Hence, $u_1 = 0$ which implies $u_0 = 0$.

From (56) it follows that the functions $B_{F,k}^{\mathrm{CR},F'}$, $F' \in \overset{\bullet}{\mathcal{F}}(K)$, are linearly independent. Finally, we show that the functions $\left\{ B_{F,\tau,\boldsymbol{\alpha}}^K : (\tau, \boldsymbol{\alpha}) \in \bigcup_{\ell=1}^{d-1} \mathcal{I}_{k,\ell}(K) \right\}$ are linearly independent. The proof is standard and follows by induction over $\ell = 1, 2, \ldots$, from

1. the support properties of $B_{F,\tau,\boldsymbol{\alpha}}^K$, i.e.: $B_{F,\tau,\boldsymbol{\alpha}}^K\big|_{\tau'} = 0$ for all $\tau \in \mathcal{S}_\ell(K)$ and $\tau' \in \left(\bigcup_{r=1}^{\ell} \mathcal{S}_r(K) \right) \setminus \{\tau\}$ and
2. the linear independence of $\left(B_{F,\tau,\boldsymbol{\alpha}}^K\big|_\tau \right)_{\boldsymbol{\alpha} \in \mathbb{N}_{\leq k-\ell-1}^\ell}$; see the proof of Lemma 19. ■

For $F \in \mathcal{F}_{\partial\Omega}$ and adjacent $K \in \mathcal{T}$, let

$$\overset{\bullet}{\mathcal{F}}(K) := \{F' \in \mathcal{F}(K) \cap \mathcal{F}_{\partial\Omega}\}. \tag{57}$$

Clearly, $F \in \overset{\bullet}{\mathcal{F}}(K)$ and assumption (50) imply $\left|\overset{\bullet}{\mathcal{F}}(K)\right| < |\mathcal{F}(K)|$ so that Lemma 21 becomes applicable. Elementary linear algebra tells us that there exists a bidual basis for the functions in $\overset{\bullet}{\mathcal{B}}_k^{\mathrm{CR}}(K,F)$ (see (54)), induced by $g_{\tau,\boldsymbol{\alpha}}^F \in L^2(F)$, $(\tau, \boldsymbol{\alpha}) \in \mathcal{I}_k(F)$ and $g_{F'}^{\mathrm{CR},F} \in L^2(F)$, $F' \in \overset{\bullet}{\mathcal{F}}(K)$ such that the associated functionals

$$J_{\tau,\boldsymbol{\alpha}}^F u := \left(g_{\tau,\boldsymbol{\alpha}}^F, u\right)_{L^2(F)} \quad \text{and} \quad J_{F'}^{\mathrm{CR},F} u := \left(g_{F'}^{\mathrm{CR},F}, u\right)_{L^2(F)} \tag{58}$$

satisfy

$$\begin{aligned} &J_{\tau,\boldsymbol{\alpha}}^F \left(B_{F,t,\boldsymbol{\beta}}^K\right) = \delta_{\tau,t}\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}, \quad && J_{\tau,\boldsymbol{\alpha}}^F \left(B_{F'',k}^{\mathrm{CR},K}\right) = 0, \\ &J_{F'}^{\mathrm{CR},F} \left(B_{F,t,\boldsymbol{\beta}}^K\right) = 0, && J_{F'}^{\mathrm{CR},F} \left(B_{F'',k}^{\mathrm{CR},K}\right) = \delta_{F',F''} \end{aligned} \tag{59}$$

for all $(\tau, \boldsymbol{\alpha}), (t, \boldsymbol{\beta}) \in \mathcal{I}_k(F)$ and $F', F'' \in \overset{\bullet}{\mathcal{F}}(K)$.

Definition 22 *Let k be odd and an assignment function $\mathrm{mark}_{\mathcal{T}} : \mathcal{S} \to \mathcal{T}$ be chosen which satisfies (53). The* degrees of freedom for the Crouzeix-Raviart spaces *are given*

1. *for $\mathrm{CR}_k(\mathcal{T})$ by*

$$\mathcal{J}_k := \left\{ J^K_{\tau,\boldsymbol{\alpha}} \mid \forall (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_{k,\Omega} \quad \textit{with } \mathrm{mark}_{\mathcal{T}}(\tau) = K \in \mathcal{T} \right\} \tag{60a}$$

$$\cup \left\{ J^F_{\tau,\boldsymbol{\alpha}} : \forall (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_{k,\partial\Omega} \quad \textit{with } \mathrm{mark}_{\mathcal{T}}(\tau) = F \in \mathcal{F}_{\partial\Omega} \right\} \tag{60b}$$

$$\cup \left\{ J^{\mathrm{CR},K}_F \mid \forall F \in \mathcal{F}_\Omega \quad \textit{with } \mathrm{mark}_{\mathcal{T}}(F) = K \in \mathcal{T} \right\} \tag{60c}$$

$$\cup \left\{ J^{\mathrm{CR},F}_F : \forall F \in \mathcal{F}_{\partial\Omega} \right\} \tag{60d}$$

2. *and for $\mathrm{CR}_{k,0}(\mathcal{T})$ by*

$$\mathcal{J}_{k,0} := \left\{ J^K_{\tau,\boldsymbol{\alpha}} \mid \forall (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_{k,\Omega} \quad \textit{with } \mathrm{mark}_{\mathcal{T}}(\tau) = K \in \mathcal{T} \right\} \tag{61a}$$

$$\cup \left\{ J^{\mathrm{CR},K}_F \mid \forall F \in \mathcal{F}_\Omega \quad \textit{with } \mathrm{mark}_{\mathcal{T}}(F) = K \in \mathcal{T} \right\}. \tag{61b}$$

Theorem 23 *Let k be odd and assume that $\mathcal{T}$ contains more than one simplex. Then, the degrees of freedom (60) are bidual to the basis in (47) and the set of freedoms (61) are bidual to set of basis function (48).*

Proof. We prove the statement only for the space $\mathrm{CR}_k(\mathcal{T})$ more concretely we show first that for any $u \in \mathrm{CR}_k(\mathcal{T})$ the implication

$$(J(u) = 0 \quad \forall J \in \mathcal{J}_k) \implies (u = 0) \tag{62}$$

holds. Let $J(u) = 0$ for all $J \in \mathcal{J}_k$. Then,

$$u|_K = \sum_{(t,\boldsymbol{\beta}) \in \mathcal{I}_k(K)} \gamma^K_{(t,\boldsymbol{\beta})}\, B_{t,\boldsymbol{\beta}}|_K + \sum_{F' \in \mathcal{F}(K)} \delta_{F'}\, B^{\mathrm{CR},F'}_k \Big|_K .$$

The biduality property of $J^K_{\tau,\boldsymbol{\alpha}}$ imply

$$\gamma^K_{(\tau,\boldsymbol{\alpha})} = 0 \quad \text{and} \quad \delta_F = 0$$

for all $(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_{k,\Omega}$ with $K = \mathrm{mark}_{\mathcal{T}}(\tau)$ and $F \in \mathcal{F}_\Omega$ with $K = \mathrm{mark}_{\mathcal{T}}(F)$. By applying this argument for all $K \in \mathcal{T}$ it follows that u has the representation

$$u = \sum_{(t,\boldsymbol{\beta}) \in \mathcal{I}_{k,\partial\Omega}} \gamma^F_{t,\boldsymbol{\beta}} B^F_{t,\boldsymbol{\beta}} + \sum_{F \in \mathcal{F}_{\partial\Omega}} \gamma_F B^{\mathrm{CR},F}_F . \tag{63}$$

Now, let $F \in \mathcal{F}_{\partial\Omega}$ and $K \in \mathcal{T}$ be the adjacent simplex. As stated after (57), $F \in \overset{\bullet}{\mathcal{F}}(K)$ and $\left|\overset{\bullet}{\mathcal{F}}(K)\right| < |\mathcal{F}(K)|$ so that Lemma 21 is applicable and we can use the bidual properties (59) and

$$u|_F = \sum_{(t,\boldsymbol{\beta}) \in \mathcal{I}_k(F)} \gamma^F_{t,\boldsymbol{\beta}} B^F_{t,\boldsymbol{\beta}} + \sum_{F' \in \overset{\bullet}{\mathcal{F}}(K) \cap \mathcal{F}_{\partial\Omega}} \gamma_{F'} B^{\mathrm{CR},F'}_{F'}$$

to see that all the coefficients in (63) are zero.

Since the number of basis functions and the number of degrees of freedom assigned to the same geometric entity $\tau \in \mathcal{S}$ are the same, it follows $|\mathcal{J}_k| = \left|\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T})\right|$ and $\mathcal{J}_k$ is well-defined as set of degrees of freedom for $\mathrm{CR}_k(\mathcal{T})$. The biduality follows by construction. ■

The dual basis directly gives rise to the definition of a (quasi-)interpolation operator.

Definition 24 *Let k be odd and assume that $\mathcal{T}$ contains more than one simplex. A* quasi-interpolation operator $I_{\mathcal{T},k} : H^1(\Omega) + \mathrm{CR}_k(\mathcal{T}) \to \mathrm{CR}_k(\mathcal{T})$ *is given by*

$$\begin{aligned} I_{\mathcal{T},k} u = &\sum_{(\tau,\boldsymbol{\alpha})\in\mathcal{I}_{k,\Omega}} \left(J^K_{\tau,\boldsymbol{\alpha}} u\right) B_{\tau,\boldsymbol{\alpha}} + \sum_{F\in\mathcal{F}_\Omega} \left(J_F^{\mathrm{CR},K} u\right) B_k^{\mathrm{CR},F} \\ &+ \sum_{(\tau,\boldsymbol{\alpha})\in\mathcal{I}_{k,\partial\Omega}} \left(J^F_{\tau,\boldsymbol{\alpha}} u\right) B_{\tau,\boldsymbol{\alpha}} + \sum_{F\in\mathcal{F}_{\partial\Omega}} \left(J_F^{\mathrm{CR},F} u\right) B_k^{\mathrm{CR},F} \end{aligned}$$

with the convention *that $K = \mathrm{mark}_{\mathcal{T}}(\tau)$ for $\tau \in \mathcal{F}_\Omega$ and $F = \mathrm{mark}_{\mathcal{T}}(\tau)$ for $\tau \in \mathcal{F}_{\partial\Omega}$.*

Lemma 25 *Let k be odd and assume that $\mathcal{T}$ contains more than one simplex. The quasi-interpolation operator $I_{\mathcal{T},k}$ is a projection and the restriction to $H^1_0(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T})$ satisfies*

$$I_{\mathcal{T},k} : H^1_0(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T}) \to \mathrm{CR}_{k,0}(\mathcal{T}). \tag{64}$$

For $u \in H^1_0(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T})$ it holds

$$I_{\mathcal{T},k} u = \sum_{(\tau,\boldsymbol{\alpha})\in\mathcal{I}_{k,\Omega}} \left(J^K_{\tau,\boldsymbol{\alpha}} u\right) B_{\tau,\boldsymbol{\alpha}} + \sum_{F\in\mathcal{F}_\Omega} \left(J_F^{\mathrm{CR},K} u\right) B_k^{\mathrm{CR},F}.$$

Proof. The projection property of $I_{\mathcal{T},k}$ is a direct consequence of the biduality of the degrees of freedom. The property (64) of the restriction simply follows since all functionals related to geometric entities on the boundary $\partial\Omega$ are defined via integrals over facets on the boundary and hence vanish for functions in $H^1_0(\Omega)$. The boundary facet functionals $J^F_{\tau,\boldsymbol{\alpha}}$, $(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_{k,\partial\Omega}$, and $J_F^{\mathrm{CR},F}$, $F \in \mathcal{F}_{\partial\Omega}$, vanish for functions $u \in \mathrm{CR}_{k,0}(\Omega)$ due to their biduality. ■

The following remark explains why, in general, local degrees of freedom *do not exist for even polynomial degree k.*

Remark 26 *Let k be even. From Lemma 12 it follows that a basis for $\mathrm{CR}_k(\mathcal{T})$ is given by*

$$\mathcal{B}_k^{\mathrm{CR}}(\mathcal{T}) = \{B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_k(\mathcal{T})\} \cup \left\{B_k^{\mathrm{CR},K} \mid \forall K \in \mathcal{T}'\right\},$$

where $\mathcal{T}' \subset \mathcal{T}$ is any submesh with $|\mathcal{T}'| = |\mathcal{T}| - 1$. Let $\widetilde{\mathcal{T}} \subset \mathcal{T}'$ and set $\widetilde{\Omega} := \bigcup_{K\in\widetilde{\mathcal{T}}} K$. Let

$$\widetilde{\mathcal{B}_k^{\mathrm{CR}}} = \left\{B_k^{\mathbf{z}} \mid \forall \mathbf{z} \in \mathcal{N}_k(\mathcal{T}) \ \text{ with } \ B_k^{\mathbf{z}}|_{\widetilde{\Omega}} \neq 0\right\} \cup \left\{B_k^{\mathrm{CR},K} \mid \forall K \in \widetilde{\mathcal{T}}\right\}$$

be the set of basis functions whose restrictions to $\widetilde{\Omega}$ *are non-trivial. By contradiction, assume that there is a set* $\widetilde{\mathcal{J}_k^{\mathrm{CR}}}$ *of basis functions for the dual space* $\left(\operatorname{span}\widetilde{\mathcal{B}_k^{\mathrm{CR}}}\right)'$ *with local support property:* $\operatorname{supp} J \subset \widetilde{\Omega}$ *for all* $J \in \widetilde{\mathcal{J}_k^{\mathrm{CR}}}$*. Then the corresponding Gram's matrix* $\left(J\left(B\right)\right)_{\substack{J\in\widetilde{\mathcal{J}_k^{\mathrm{CR}}}\\ B\in\widetilde{\mathcal{B}_k^{\mathrm{CR}}}}}$ *is regular. However, as in the proof (**Case 2, Part 1, 1st step**) of Lemma 12 it follows that*

$$S_k\left(\mathcal{T}\right)|_{\widetilde{\Omega}} \cap \operatorname{span}\left\{B_k^{\mathrm{CR},K} \mid \forall K \in \widetilde{\mathcal{T}}\right\} = \operatorname{span}\left\{\Psi_k|_{\widetilde{\Omega}}\right\}$$

with Ψ_k *as in (39). This implies that for the space* $\mathrm{CR}_k\left(\mathcal{T}\right)$ *and even* k *there exists* no *basis of the dual space which has the local support property.* ■

6 Degrees of freedom and an approximation operator for the case $d = 2$ and k is odd

The degrees of freedom associated to the inner faces for general dimension $d \geq 2$ are defined by integrals over adjacent volume simplices $K \in \mathcal{T}$. For the important case of two spatial dimensions $d = 2$, these degrees of freedom can be defined alternatively by facet (edge) integrals with weight functions which are polynomials of maximal degree $k - 1$. In this section we present their construction.

6.1 Degrees of freedom

For the following construction we assume that $k \geq 1$ is odd and that $\mathcal{T}$ is a triangulation of a two-dimensional polygonal domain. Recall the Σ_1 denotes the mesh skeleton (the union of all edges). We will need the space

$$\overset{\bullet\bullet}{S}_{k,0}\left(\mathcal{T}\right) := \left\{u \in \overset{\bullet}{S}_{k,0}\left(\mathcal{T}\right) \mid u|_{\Sigma_1} = 0\right\} \tag{65}$$

which is non-trivial for $k \geq 3$ and then consists of polynomial triangle bubbles.

We present a partial basis along degrees of freedom for $\mathrm{CR}_{k,0}\left(\mathcal{T}\right)$ which consists of conforming and non-conforming edge functions and the direct sum of their spans with $\overset{\bullet\bullet}{S}_{k,0}\left(\mathcal{T}\right)$ yields $\mathrm{CR}_{k,0}\left(\mathcal{T}\right)$.

This needs some notation. For $d = 2$ the set of edges and the set of facets coincide and we write $\mathcal{E}_\Omega$ for $\mathcal{F}_\Omega$ (cf. (2)). For $E \in \mathcal{E}_\Omega$, we fix a numbering of the endpoints of E: $\mathbf{A}_1\left(E\right), \mathbf{A}_2\left(E\right)$ and define the edge bubble by $b_E = \varphi^1_{\mathbf{A}_1(E)}\varphi^1_{\mathbf{A}_2(E)}$ which belongs to $S_{2,0}\left(\mathcal{T}\right)$.

The edge basis functions are given by

$$\left.\begin{array}{l}\forall E \in \mathcal{E}_\Omega\\ \forall \mu \in \{0, 1, \ldots, k-1\}\end{array}\right\} \quad B_\mu^E := \begin{cases} b_E P_\mu^{(1,1)}\left(2\varphi^1_{\mathbf{A}_2(E)} - 1\right) & \mu \leq k-2,\\ B_k^{\mathrm{CR},E} & \mu = k-1.\end{cases} \tag{66}$$

Corollary 27 *The basis functions have the following support properties*

$$\operatorname{supp} B_\mu^E \subset \mathcal{T}_E \qquad \forall \mu \in \{0, 1, \ldots, k-1\}.$$

Next we define the degrees of freedom for these functions. Since the jumps of $v \in \mathrm{CR}_{k,0}(\mathcal{T})$ satisfy moment conditions up to an order $k-1$ across each edge, edge functionals of the type $J_\nu^E(u) := \int_E g_\nu^E u$ defined via polynomials $g_\nu^E \in \mathbb{P}_{k-1}(E)$ are well-defined for Crouzeix-Raviart functions.

Definition 28 *The edge functionals J_ν^E are given for any $E \in \mathcal{E}_\Omega$ and $\nu \in \{1, 2, \ldots, k-1\}$ by*

$$J_\nu^E u := \frac{2}{|E|} \int_E g_\nu^E u \tag{67a}$$

with weight function

$$g_\nu^E := \gamma_\nu \left(P_\nu^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E)} - 1\right) - c_{\nu,k} P_{k-1}^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E)} - 1\right)\right) \tag{67b}$$

and constants[3]

$$\gamma_\nu := \frac{(2\nu+3)(\nu+2)}{8(\nu+1)} \quad \text{and } c_{\nu,k} := \begin{cases} \frac{1-(-1)^{\nu+1}}{2}\frac{k+1}{\nu+2} & \nu \in \{0, 1, \ldots, k-2\}, \\ \frac{1}{2k+1} & \nu = k-1. \end{cases} \tag{67c}$$

Lemma 29 *Let $k \geq 1$ be odd. For $\nu, \mu \in \{0, 1, \ldots, k-1\}$ and $E, E' \in \mathcal{E}_\Omega$ and $u \in \overset{\bullet\bullet}{S}_{k,0}(\mathcal{T})$ it holds*

$$J_\nu^E\left(B_\mu^{E'}\right) = \delta_{\nu,\mu}\delta_{E,E'}, \quad J_\nu^E(u) = 0. \tag{68}$$

Proof. First we consider the case $\mu \in \{0, 1, \ldots, k-2\}$.

The support properties (Corollary 27) imply that for $E' \in \mathcal{E}_\Omega \backslash \{E\}$ it holds $B_\mu^{E'}\Big|_E = 0$ and hence $J_\nu^E\left(B_\mu^{E'}\right) = 0$ for such E'. Next, let $E = E'$. We choose an affine bijection $\varphi_E : [-1, 1] \to E$ such that $\varphi_E(-1) = \mathbf{A}_1(E)$. Then

$$\left(\left(2\varphi_{\mathbf{A}_2(E)} - 1\right)\Big|_E \circ \varphi_E\right)(x) = x.$$

For $\mu \in \{0, 1, \ldots, k-2\}$ we get

$$J_\nu^E\left(B_\mu^E\right) = \gamma_\nu \int_{-1}^{1} (1-x)(1+x)\left(P_\nu^{(1,1)}(x) - c_\nu P_{k-1}^{(1,1)}(x)\right) P_\mu^{(1,1)}(x)\, dx = \delta_{\nu,\mu}.$$

It remains to consider the case $\mu = k-1$. Note that the function $B_{k-1}^E\Big|_{E'}$ is zero for all edges $E' \subset \mathcal{E}_\Omega$ which are not subset of the edge patch ω_E and hence $J_\nu^{E'}\left(B_{k-1}^E\right) = 0$ for those edges.

[3] The constant γ_ν takes into account the relation (cf. [8, Table 18.3.1]):

$$\int_{-1}^{1} (1-x)(1+x)\left(P_\nu^{(1,1)}(x)\right)^2 dx = \gamma_\nu^{-1}.$$

For $E' = E$, it holds

$$J_\nu^E\left(B_{k-1}^E\right) = \frac{2}{|E|}\int_E \gamma_\nu\left(P_\nu^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E)}-1\right) - c_{\nu,k}P_{k-1}^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E)}-1\right)\right)P_k^{(0,0)}(1)$$
$$= \int_{-1}^1 \gamma_\nu\left(P_\nu^{(1,1)}(x) - c_{\nu,k}P_{k-1}^{(1,1)}(x)\right)dx.$$

From [8, 18.9.15] it follows that $\frac{2}{\nu+2}\left(P_{\nu+1}^{(0,0)}\right)' = P_\nu^{(1,1)}$ so that

$$\int_{-1}^1 P_\nu^{(1,1)}(x)\,dx = \frac{2}{\nu+2}\int_{-1}^1\left(P_{\nu+1}^{(0,0)}\right)' = \frac{2\left(1-(-1)^{\nu+1}\right)}{\nu+2},$$
$$\int_{-1}^1 P_{k-1}^{(1,1)}(x)\,dx \overset{k\ \mathrm{odd}}{=} \frac{4}{k+1}.$$

The coefficient $c_{\nu,k}$ in (67c) is chosen such that

$$J_\nu^E\left(B_{k-1}^E\right) = \gamma_\nu\left(\frac{2\left(1-(-1)^{\nu+1}\right)}{\nu+2} - c_{\nu,k}\frac{4}{k+1}\right) = \delta_{k-1,\nu} \qquad \forall \nu \in \{0,1,\ldots,k-1\}.$$

It remains to consider those edges $E' \in \mathcal{E}_\Omega$ with $E' \subset \partial\omega_E$. Then,

$$J_\nu^{E'}\left(B_{k-1}^E\right) = \frac{2}{|E|}\int_E\left(P_\nu^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E')}-1\right) - c_{\nu,k}P_{k-1}^{(1,1)}\left(2\varphi_{\mathbf{A}_2(E')}-1\right)\right)P_k^{(0,0)}(1-2\lambda_{K,E}).$$

Since $P_k^{(0,0)}(1-2\lambda_{K,E})\Big|_{E'}$ is the (lifted) Legendre polynomial of degree k, the orthogonality properties of Legendre polynomials imply that this integral is zero.

The second statement in (68) simply follows from $u|_E = 0$ for $u \in \overset{\bullet\bullet}{S}_{k,0}(\mathcal{T})$ and any edge $E \in \mathcal{E}$. ■

The linearly independent functions B_μ^E are collected in the set

$$\mathcal{B}_\mathcal{E}(\mathcal{T}) := \left\{B_\mu^E : \forall E \in \mathcal{E}_\Omega \quad \forall \mu \in \{0,1,\ldots,k-1\}\right\}$$

and span the interelement space

$$S_{k,0}^\mathcal{E}(\mathcal{T}) := \operatorname{span}\mathcal{B}_\mathcal{E}(\mathcal{T}).$$

Corollary 30 *Let $k \geq 1$ be odd. Then*

$$\mathrm{CR}_{k,0}(\mathcal{T}) = \overset{\bullet\bullet}{S}_{k,0}(\mathcal{T}) \oplus S_{k,0}^\mathcal{E}(\mathcal{T}).$$

6.2 An approximation operator

In this section, we will introduce a class of local approximation operators $I_{\mathcal{T},k} : H_0^1(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T}) \to \mathrm{CR}_{k,0}(\mathcal{T})$. The construction starts by defining the edge related interpolation

$$\mathcal{I}_{\mathcal{T},k}^\mathcal{E} u := \sum_{E\in\mathcal{E}_\Omega}\sum_{\nu=0}^{k-1}\left(J_\nu^E u\right)B_\nu^E. \tag{69}$$

Next, one chooses a bounded linear operator $\overset{\bullet\bullet}{\Pi}_{\mathcal{T},k} : H_0^1(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T}) \to \overset{\bullet\bullet}{S}_{k,0}(\mathcal{T})$ which is local in the sense that for all $K \in \mathcal{T}$

$$\left(\overset{\bullet\bullet}{\Pi}_{\mathcal{T},k} u\right)\Big|_K = \Pi_{k,0}^K\left(u|_K\right) \quad \forall u \in H_0^1(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T}),$$

for given local projections $\Pi_{k,0}^K : L^2(K) \to S_{k,0}(K)$. A typical choice of $\Pi_{k,0}^K$ is the $L^2(K)$-orthogonal projection onto $S_{k,0}(K)$.

Definition 31 *A class of* local approximation operators $I_{\mathcal{T},k} : H_0^1(\Omega) + \mathrm{CR}_{k,0}(\mathcal{T}) \to \mathrm{CR}_{k,0}(\mathcal{T})$ *is given by*

$$I_{\mathcal{T},k} u := \mathcal{I}_{\mathcal{T},k}^{\mathcal{E}} u + \overset{\bullet\bullet}{\Pi}_{\mathcal{T},k}\left(u - \mathcal{I}_{\mathcal{T},k}^{\mathcal{E}} u\right). \tag{70}$$

Remark 32 *A basis and bidual basis for the Crouzeix-Raviart space* $\mathrm{CR}_k(\mathcal{T})$ *for* $d = 2$ *and odd* k *is obtained by defining basis functions and degrees of freedom for the boundary edges analogously as in (66), (28) for inner edges. A local approximation operator in* $\mathrm{CR}_k(\mathcal{T})$ *is then obtained by summing in (69) over all* $E \in \mathcal{E}$ *and using the resulting edge interpolation operator in (70).*

6.3 Non-existence of split facet/simplex degrees of freedom for $d \geq 3$ and $k \neq 1$

We have already explained in Remark 26 that there exists no set of local degrees of freedom for even k. In this section we will prove that a construction as in Section 6.1 is possible only for $k = 1$ or $d = 2$ and k odd.

First, we consider the case $k = 1$.

Lemma 33 *For* $d \geq 2$ *and* $k = 1$ *the Crouzeix-Raviart basis functions are given by (cf. Definition 2, Remark 3):*

$$B_1^{\mathrm{CR},F} := \begin{cases} P_1^{(0,d-2)}(1 - 2\lambda_{K,F}) & \text{for } K \in \mathcal{T}_F, \\ 0 & \text{otherwise,} \end{cases}$$

and the degrees of freedom by

$$J_0^F(u) := \frac{1}{|F|}\int_F u.$$

They form a bidual basis:

$$J_0^F\left(B_1^{\mathrm{CR},F'}\right) = \delta_{F,F'} \quad \forall F, F' \in \mathcal{E}.$$

Proof. It holds

$$J_0^F\left(B_1^{\mathrm{CR},F}\right) = \frac{1}{|F|}\int_F P_1^{(0,d-2)}(1 - 2\lambda_{K,F}) = \frac{1}{|F|}\int_F 1 = 1.$$

For $F \in \mathcal{E}$, $K \in \mathcal{T}_F$, and $F' \subset \partial K \backslash \overset{\circ}{F}$ we get

$$J_0^{F'}\left(B_1^{\mathrm{CR},F}\right) = \frac{1}{|F'|}\int_{F'} P_1^{(0,d-2)}\left(1-2\lambda_{K,F}\right) \overset{(14b)}{=} 0.$$

If F' is outside ω_F the functional $J_0^{F'}\left(B_1^{\mathrm{CR},F}\right)$ vanishes since $\operatorname{supp} B_1^{\mathrm{CR},F} = \omega_F$. ■

Next, we consider the case that $d \geq 3$ and $k > 1$ is odd. Section 6 introduces a construction for a bidual basis for $\mathrm{CR}_k(\mathcal{T})$ ($\mathrm{CR}_{k,0}(\mathcal{T})$ resp.), where for all $(\tau, \boldsymbol{\alpha}) \in \mathcal{I}_{k,\ell}$, $\ell \in \{1, 2, ..., d-1\}$, the functionals $J_{\tau,\boldsymbol{\alpha}}$ are of the form

$$J_{\tau,\boldsymbol{\alpha}} := (g_{\tau,\boldsymbol{\alpha}}, \cdot)_{L^2(F)} \qquad g_{\tau,\boldsymbol{\alpha}} \in \mathbb{P}_{k-1}(F),\ \tau \subseteq F.$$

In this section we prove that this is not possible for $d \geq 3$ and $k \geq 3$ odd. Let the assignement function $\mathrm{mark}_{\mathcal{T}} : \mathcal{S} \to \mathcal{T} \cup \mathcal{F}$ satisfy

$$\mathrm{mark}_{\mathcal{T}}(K) = K \qquad \forall K \in \mathcal{T} \tag{71a}$$

$$\mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{F} \text{ and } \tau \subseteq \mathrm{mark}_{\mathcal{T}}(\tau) \qquad \forall \tau \in \mathcal{S} \setminus \mathcal{T}. \tag{71b}$$

Further let $J_{\tau,\boldsymbol{\alpha}}, J_F^{\mathrm{CR}} : H^1(\mathcal{T}) \to \mathbb{R}$ be given by

$$J_{\tau,\boldsymbol{\alpha}} u := (g_{\tau,\boldsymbol{\alpha}}, u)_{L^2(\mathrm{mark}_{\mathcal{T}}(\tau))} \quad g_{\tau,\boldsymbol{\alpha}} \in \begin{cases} L^2(\mathrm{mark}_{\mathcal{T}}(\tau)) & \mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{T}, \\ \mathbb{P}_{k-1}(\mathrm{mark}_{\mathcal{T}}(\tau)) & \mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{F}, \end{cases} \tag{72a}$$

$$J_F^{\mathrm{CR}} u := (g_F, u)_{L^2(F)} \quad g_F \in \mathbb{P}_{k-1}(F), \tag{72b}$$

and collect them in the sets

$$\mathcal{J}_k^{\mathrm{entity}}(\mathcal{T}) := \left\{ J_{\tau,\boldsymbol{\alpha}} \mid (\tau, \boldsymbol{\alpha}) \in \mathcal{I}_k \right\} \cup \left\{ J_F^{\mathrm{CR}} \;\middle|\; F \in \mathcal{F} \right\}, \tag{73a}$$

$$\mathcal{J}_{k,0}^{\mathrm{entity}}(\mathcal{T}) := \left\{ J_{\tau,\boldsymbol{\alpha}} \mid (\tau, \boldsymbol{\alpha}) \in \mathcal{I}_{k,\Omega} \right\} \cup \left\{ J_F^{\mathrm{CR}} \;\middle|\; F \in \mathcal{F}_\Omega \right\}. \tag{73b}$$

Theorem 34 *Let $d \geq 3$ and $k \geq 3$ odd. If the assignment function $\mathrm{mark}_{\mathcal{T}} : \mathcal{S} \to \mathcal{T} \cup \mathcal{F}$ satisfies (71), then there exists* no *set of functionals of the form (72) such that*

1. *$\mathcal{J}_k^{\mathrm{entity}}(\mathcal{T})$ is a bidual basis for $\mathrm{CR}_k(\mathcal{T})$.*
2. *$\mathcal{J}_{k,0}^{\mathrm{entity}}(\mathcal{T})$ is a bidual basis for $\mathrm{CR}_{k,0}(\mathcal{T})$ if there exists $K \in \mathcal{T}$ such that $K \cap \partial\Omega = \emptyset$.*

Proof. @1. Let us assume by contradiction that there exists a choice of functionals of the form (72), such that $\mathcal{J}_k^{\mathrm{entity}}(\mathcal{T})$ is a bidual basis. First observe that, due to the inclusion (42), the constant function $1 \in \mathrm{CR}_k(\mathcal{T})$. Hence for any functional $J \in \mathcal{J}_k^{\mathrm{entity}}(\mathcal{T})$, $J(1)$ is well defined. Thus we choose any simplex

$K \in \mathcal{T}$ and define the set $\mathcal{I}_k(\partial K) := \bigcup_{\ell=1}^{d-1} \mathcal{I}_{k,\ell}(K)$ as well as

$$\psi_{\partial K} := \sum_{(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(\partial K)} J_{\tau,\boldsymbol{\alpha}}(1)\, B_{\tau,\boldsymbol{\alpha}} + \sum_{F \in \mathcal{F}(K)} J_F^{\mathrm{CR}}(1)\, B_k^{\mathrm{CR},F},$$
$$\phi_{\partial K} := \psi_{\partial K} - \sum_{F \in \mathcal{F}(K)} B_k^{\mathrm{CR},F}. \tag{74}$$

Following the arguments of the proof of Lemma 19, we deduce that the functions $\{B_{\tau,\boldsymbol{\alpha}} \mid (\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(\partial K)\} \cup \left\{ B_k^{\mathrm{CR},F} \;\middle|\; F \in \mathcal{F}(K) \right\}$ are linearly independent on ∂K. Thus using the biduality property of $\mathcal{J}_k(\mathcal{T})$ we deduce that $\psi_{\partial K}|_{\partial K} = 1$. This and Definition 2 implies that $\phi_{\partial K}|_{\partial K} = B_k^{\mathrm{CR},K}\Big|_{\partial K}$ and hence

$$\phi_{\partial K} \perp \mathbb{P}_{k-1}(F) \qquad \forall F \in \mathcal{F}(K)$$

by construction. Hence it follows that $J_{\tau,\boldsymbol{\alpha}}^F(\phi_{\partial K}) = J_F^{\mathrm{CR}}(\phi_{\partial K}) = 0$ for all $F \in \mathcal{F}(K)$ and all $(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k^{d-1}(K)$ with $\mathrm{mark}_{\mathcal{T}}(\tau) = F$. Therefore $\phi_{\partial K}$ can be written as

$$\phi_{\partial K} = \sum_{(\tau,\boldsymbol{\alpha}) \in \mathcal{R}_k(\partial K)} J_{\tau,\boldsymbol{\alpha}}(1)\, B_{\tau,\boldsymbol{\alpha}},$$

where $\mathcal{R}_k(\partial K) := \{(\tau,\boldsymbol{\alpha}) \in \mathcal{I}_k(\partial K) \mid \mathrm{mark}_{\mathcal{T}}(\tau) \in \mathcal{F} \setminus \mathcal{F}(K)\}$. As a consequence of Lemma 19 this yields that $\phi_{\partial K} \in \overset{\bullet}{S}_k(\mathcal{T})$ and, in turn,

$$\phi_{\partial K}|_K(\mathbf{z}) = 0 \qquad \mathbf{z} \in \mathcal{V}(K). \tag{75}$$

However the combination of (21a) and (74) gives $\phi_{\partial K}|_{\partial K}(\mathbf{z}) \neq 0$, which contradicts (75).
@2. Set $\mathcal{T}_\Omega := \{K \in \mathcal{T} \mid K \cap \partial\Omega = \emptyset\}$. By assumption $|\mathcal{T}_\Omega| \geq 1$. By taking some $K \in \mathcal{T}_\Omega$ and repeating the previous construction the second claim follows.
■

■

Remark 35 *The reason why this proof does not apply for $k = 1$ or $d = 2$ is that then* (74) *and (75) do not contradict each other as a consequence of* (21a).

A Proof of the determinant formula (31)

In this section, we prove formula (31).

Lemma 36 *The determinant of the matrix $\mathbf{Q}_d(s)$ in (29) is given by*

$$\det \mathbf{Q}_d(s) = (-1)^{d+1}(1+s)^{d-1}(d-1-s) \tag{76}$$

Proof. We define the auxiliary matrix

$$\mathbf{R}_d(s) := \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & -s & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -s \end{bmatrix} \tag{77}$$

and claim

$$\det \mathbf{R}_d(s) = (-1-s)^{d-1}. \tag{78}$$

We prove (76) and (78) simultaneously by induction.

The Laplace expansion of the determinant of $\mathbf{Q}_d$ with respect to the first row yields

$$\det \mathbf{Q}_d(s) = -s \det \mathbf{Q}_{d-1}(s) + \sum_{\ell=1}^{d-1} (-1)^{\ell} \det \mathbf{R}_{d-1,\ell}(s),$$

where the matrix $\mathbf{R}_{d-1,\ell}(s)$ arises by moving the first row of $\mathbf{R}_{d-1}(s)$ to the ℓ-th column and shifting all rows $2,\dots,\ell-1$ one row up. In other word, an $(\ell-1)$-fold interchange of appropriate rows of $\mathbf{R}_{d-1,\ell}(s)$ yields the matrix $\mathbf{R}_{d-1}(s)$. Then, well-known properties of determinants imply

$$\det \mathbf{Q}_d(s) = -s \det \mathbf{Q}_{d-1}(s) - (d-1) \det \mathbf{R}_{d-1}(s).$$

We employ the same reasoning for $\mathbf{R}_d(s)$ and obtain

$$\det \mathbf{R}_d(s) = \det \mathbf{Q}_{d-1}(s) + \sum_{\ell=1}^{d-1} (-1)^{\ell} \det \mathbf{R}_{d-1,\ell}(s) = \det \mathbf{Q}_{d-1}(s) - (d-1) \det \mathbf{R}_{d-1}(s).$$

For the induction start we explicitly compute

$$\det \mathbf{Q}_1(s) = -s, \quad \det \mathbf{R}_1(s) = 1$$

and for the induction step we assume that (76) and (78) hold up to $d-1$. We then get

$$\begin{aligned} \det \mathbf{Q}_d(s) &= -s \det \mathbf{Q}_{d-1}(s) - (d-1) \det \mathbf{R}_{d-1,\ell}(s) \\ &= -s \left((-1)^d (1+s)^{d-2} (d-2-s) \right) - (d-1)(-1-s)^{d-2} \\ &= (-1)^d (1+s)^{d-2} (s+1)(s-d+1) \end{aligned}$$

and this is (76). In a similar way we get

$$\begin{aligned} \det \mathbf{R}_d(s) &= \det \mathbf{Q}_{d-1}(s) - (d-1) \det \mathbf{R}_{d-1}(s) \\ &= (-1)^d (1+s)^{d-2} (d-2-s) - (d-1)(-1-s)^{d-2} \\ &= (1+s)^{d-2} (-1)^d (-s-1) \end{aligned}$$

and this is (78). ■

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