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# Local Finite-Time Stability for a Class of *Time-Delay* Systems

Xiaodi Li, Xinyi He, Dan Yang, Emmanuel Moulay and Qing Hui

**Abstract**—This paper studies the local finite-time stability for a class of time-delay systems. Some Lyapunov-based sufficient conditions are proposed in the framework of finite time, where a potential connection between time delay and system parameters for local *FTS* is established. An estimate of the upper bound of the settling time depending on the time delay is derived. Examples are given to illustrate the effectiveness of the proposed results. Especially, a finite-time stabilizing memoryless controller for a class of second-order systems with time-varying delay is proposed.

**Index Terms**—Finite-time stability, time-delay systems, Lyapunov method, settling time, memoryless controller.

## I. INTRODUCTION

Over the past decades, the control community has witnessed extensive research on finite-time stability (*FTS*) of dynamical systems, see [1], [2], [3], [4]. Different from asymptotic stability, which implies the convergence of the system to a Lyapunov stable equilibrium state over infinite horizons, *FTS* can guarantee the finite-time convergence of the system. At first, the concept of *FTS* arose in time-optimal control, see [5], [6]. Then with the complement of homogeneous approach and Lyapunov theorem, the theory of *FTS* has been developed in the last decades. For instance, it has been shown in [7] that a system is *FTS* if it is globally asymptotically stable and homogeneous of negative degree; Based on Lyapunov method, a basic tool for the analysis of *FTS* has been given in [5]; Whereas a necessary and sufficient condition for *FTS* of continuous systems has been given in [8]. From the practical point of view, *FTS* plays an important role in many engineering problems such as proportional navigation guidance [9], [10], high-quality robot control [11], attitude tracking of spacecraft [12], [13], etc.

Note that time delays appear in many dynamical systems, including engineering systems, communication networks, chemical process, manufacturing systems and so on. Considerable research on the stability and stabilization of time-delay systems

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over infinite horizons can be found in [14], [15]. In the framework of finite time, however, the corresponding development is comparatively retarded due to the fact that *FTS* requires a much stronger performance than classical Lyapunov stability. Related results on *FTS* can be found in [16], [17], [18] and the references therein. It is well known that there are mainly two ways available for studying stability of time-delay systems. One is Lyapunov-Razumikhin (*L-R*) approach, which has been used for *FTS* analysis of time-delay systems in previous papers [17], [18]. However, as noticed in [19], the Razumikhin condition in [17] is insufficient for *FTS* of systems with time delays. To guarantee *FTS* of time-delay systems, there must be additional restrictions on the evolution of the system and one of necessary conditions has been presented in [19] for constant delay. More recently, by strengthening the classical Razumikhin condition, *L-R* approach has been employed in [18] to ensure finite-time convergence. Especially, the cases of exponential, finite-time and fixed-time (with respect to a ball) convergences have been extensively considered. But the proposed results in [18] are based on a class of strengthening Razumikhin conditions, which leads to a limited application. Another approach developing the Lyapunov method to *FTS* analysis of time-delay systems is Lyapunov-Krasovskii (*L-K*) approach, which has been successfully used in [16]. Although some general sufficient conditions have been given in [16] by finding an *L-K* functional whose derivative is upper bounded by a certain negative function, it is hard to reflect the effects of time delay on system dynamics. Moreover, it is difficult to find a suitable *L-K* functional to meet all assumptions in [16]. In general, the research on *FTS* of time-delay systems is still a very difficult and challenging topic.

In this paper, we shall study the *FTS* for a class of nonlinear systems with time-varying delays, where a Halanay-like inequality acting on Lyapunov functions is given to ensure *FTS*. An estimate of the upper bound of the settling time depending on the time delay is derived. It shows that, when subject to some types of time-varying delays, the local *FTS* of systems can be achieved. Although this paper copes with the problem of *FTS* under an additional assumption of delays, it still covers a general case of systems. Especially, the proposed results can be employed to finite-time stabilization under memoryless control. The outline of this paper is as follows. In Section II, the problem is formulated and preliminaries are given. In Section III, the finite-time conditions are presented for time-delay systems. Two numerical examples are given in Section IV to illustrate the proposed results, where a stabilizing memoryless controller for a class of second-order time-delay systems is designed. Conclusions are given in Section V.

*Notations.* Let  $\mathbb{Z}_+$  denote the set of positive integers,  $\mathbb{R}$  the set of real numbers,  $\mathbb{R}_+$  the set of nonnegative real numbers,  $\mathbb{R}^n$  the  $n$ -dimensional real space equipped with the Euclidean norm  $|\cdot|$ . A function  $\varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  is of class  $\mathcal{K}$  if  $\varphi$  is strictly increasing and  $\varphi(0) = 0$ . For given  $\tau > 0$ , we equip the linear space  $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$  (abr.  $\mathcal{C}_\tau$ ) with the norm  $\|\cdot\|_\tau$  defined by  $\|\phi\|_\tau = \sup_{s \in [-\tau, 0]} |\phi(s)|$ . For  $\sigma > 0$ ,  $\mathcal{C}_\tau^\sigma = \{\phi \in \mathcal{C}_\tau : \|\phi\|_\tau < \sigma\}$ . For real constants  $a, b$ , let  $a \vee b$  and  $a \wedge b$  denote the maximum and the minimum value of  $a$  and  $b$ .

## II. PRELIMINARIES

Consider the following nonlinear delay system

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t))), & t > 0, \\ x(t) = \phi(t), & t \in [-\bar{\tau}, 0], \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$ ,  $\dot{x}(t)$  denotes the upper right-hand derivative of  $x(t)$ ;  $\tau(t) : \mathbb{R}_+ \rightarrow [0, \bar{\tau}]$  is the time-varying delay, and  $\phi \in \mathcal{C}_{\bar{\tau}}$  represents the initial condition, where  $\bar{\tau}$  is a positive constant;  $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$  satisfying  $f(0, 0) = 0$ . We assume that function  $f$  satisfies suitable conditions so the solution  $x(t) := x(t, \phi)$  uniquely exists in forward time within relevant time intervals, see [20], [21]. For a locally Lipschitz continuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , the upper right-hand Dini derivative along system (1) is defined by

$$D^+V[x(t)] = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(x(t+h)) - V(x(t))].$$

When  $\tau(t) \equiv \tau^*$  is a constant, it has been shown in [16] that system (1) cannot be *FTS* if the origin is the only point of  $f(x, y) = 0$ . Thus, to guarantee *FTS* in the case of constant delay, the system needs to be equipped with some special structure,  $f(u, v) \equiv 0$  when  $u = 0$  for example, so that the effect of the constant delay disappears when the system reaches the origin [16]. In this paper, to study the *FTS* problem of time-delay system (1) without additional constraint on the system structure, the following assumption on  $\tau(t)$  is needed.

*Assumption 1:* There exist a positive constant  $b$  and a function  $a \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  satisfying  $a(t) \leq 1$  for all  $t > b$  such that

$$\tau(t) \leq a(t) \cdot |b - t|, \quad t \geq 0.$$

*Remark 1:* *Assumption 1* indicates that the time-varying delay  $\tau(t)$  vanishes at some instants, especially,  $\tau(b) = 0$ . Although it imposes a certain restriction on the time-varying delay, it still covers many types of time delays, such as the sinusoidal type  $\tau(t) = a(t) \cdot |\sin(\frac{\pi}{b}t + \varphi)|$ , the linear type  $\tau(t) = a(t) \cdot \prod_{i=1}^m |b_i - t|$  and so on, see Figure 1.

*Definition 1:* [16] System (1) is said to be

(a) Lyapunov stable (*LS*) if there exist a function  $\alpha \in \mathcal{K}$  and a constant  $\sigma > 0$  such that for any  $\phi \in \mathcal{C}_\tau^\sigma$ ,  $|x(t, \phi)| \leq \alpha(\|\phi\|_\tau)$  for all  $t \geq 0$ ;

(b) (locally) finite-time stable (*FTS*) if it is *LS* and for any  $\phi \in \mathcal{C}_\tau^\sigma$  there exists  $0 \leq T_\phi < +\infty$  such that  $x(t, \phi) = 0$  for all  $t \geq T_\phi$ . The functional  $T(\phi) =$

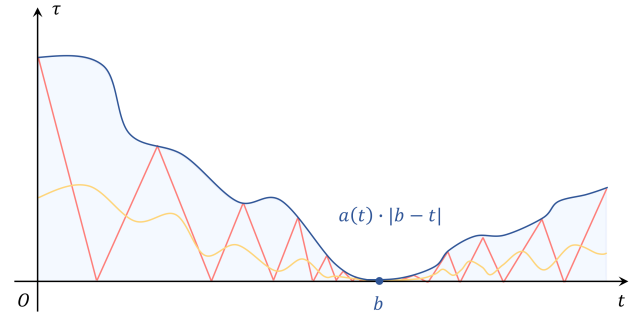


Figure 1. The bound of  $\tau(t)$  and different types of delays.

$\inf \{ T_\phi \geq 0 \mid x(t, \phi) = 0, \forall t \geq T_\phi \}$  is called the settling time of system (1).

Moreover, for given  $\sigma > 0$ , if there exists a constant  $\eta > 0$  such that  $T(\phi) \leq \eta$  for any  $\phi \in \mathcal{C}_\tau^\sigma$ , then settling time  $T(\phi)$  of system (1) is said to be uniformly bounded by  $\eta$  over  $\mathcal{C}_\tau^\sigma$ .

## III. MAIN RESULTS

In this section, some Lyapunov-based sufficient conditions are presented for the *FTS* of nonlinear time-delay system (1).

*Theorem 1:* Assume that Assumption 1 holds for some function  $a \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$  and constant  $b > 0$ . If there exist functions  $\omega_1, \omega_2 \in \mathcal{K}$ , constants  $\gamma \in (0, 1)$ ,  $\mu_1 > 0$ ,  $\sigma > 0$ ,  $\mu_2 \geq 0$  and a locally Lipschitz continuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that

(i)  $\omega_1(|x|) \leq V(x) \leq \omega_2(|x|)$ ,  $\forall x \in \mathbb{R}^n$ ;

(ii) the derivative of  $V$  along the solution  $x(t) := x(t, \phi)$  of system (1) with  $\phi \in \mathcal{C}_\tau^\sigma$  satisfies

$$D^+V[x(t)] \leq -\mu_1 V^\gamma(x(t)) + \mu_2 V^\gamma(x(t - \tau(t))); \quad (2)$$

(iii)  $\mu \geq \frac{\omega_2^{1-\gamma}(\sigma)}{(1-\gamma)b}$ , where  $\mu := \mu_1 - (1 + \bar{a})^{1-\gamma} \mu_2$  and  $\bar{a} = \max_{s \in [0, b]} a(s)$ .

Then system (1) is *FTS* with settling time  $T(\phi)$  uniformly bounded by  $b$  over  $\mathcal{C}_\tau^\sigma$ .

*Proof:* For any given initial data  $\phi \in \mathcal{C}_\tau^\sigma$ , let  $x(t) := x(t, \phi)$  be the solution of system (1) through  $(0, \phi)$ . Without loss of generality, we assume that  $\|\phi\|_\tau > 0$ . Set

$$V(t) := V(x(t)), \quad \psi(t) := V^{1-\gamma}(t) + \epsilon(1-\gamma)(t \vee 0),$$

where  $\epsilon := \frac{\omega_2^{1-\gamma}(\|\phi\|_\tau)}{(1-\gamma)b}$ . First, we claim that

$$\psi(t) \leq \omega_2^{1-\gamma}(\|\phi\|_\tau), \quad \forall t \in [0, b]. \quad (3)$$

Obviously,  $\psi(0) = V^{1-\gamma}(0) \leq \omega_2^{1-\gamma}(\|\phi\|_\tau)$ . Suppose that (3) does not hold on  $(0, b)$ , then there exists  $t^* \in (0, b)$  such that  $\psi(t^*) = \omega_2^{1-\gamma}(\|\phi\|_\tau)$ ,  $D^+\psi(t^*) \geq 0$ , and  $\psi(t) \leq \psi(t^*)$  for  $t \in [0, t^*]$ . Then there are two cases:  $t^* - \tau(t^*) \geq 0$  and  $t^* - \tau(t^*) < 0$ . When  $t^* - \tau(t^*) \geq 0$ , it follows from  $\psi(t) \leq \psi(t^*)$  for  $t \in [0, t^*]$  that

$$V^{1-\gamma}(t^* - \tau(t^*)) + \epsilon(1-\gamma)(t^* - \tau(t^*)) \leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)t^*,$$

which implies that

$$V^{1-\gamma}(t^* - \tau(t^*)) \leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*). \quad (4)$$

When  $t^* - \tau(t^*) < 0$ , it follows from condition (i) that

$$\begin{aligned} V^{1-\gamma}(t^* - \tau(t^*)) &\leq \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) \\ &= V^{1-\gamma}(t^*) + \epsilon(1-\gamma)t^* \\ &= V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*) + \epsilon(1-\gamma)(t^* - \tau(t^*)) \\ &\leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*). \end{aligned}$$

Hence, in both cases, (4) always holds. Moreover, it is easy to see that  $V(t^*) \neq 0$ . In fact, if  $V(t^*) = 0$ , then it follows from  $\psi(t^*) = \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}})$  and the definition of  $\epsilon$  that

$$\frac{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}})}{b}t^* = \epsilon(1-\gamma)t^* = \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}),$$

i.e.,  $t^* = b$ , which leads to a contradiction with  $t^* < b$ . Since  $V(t^*) \neq 0$ , there exists  $\delta_0 > 0$  such that  $V(t) > 0$  on the interval  $(t^* - \delta_0, t^* + \delta_0)$ . It then can be deduced that

$$\begin{aligned} D^+\psi(t) |_{t=t^*} &= (1-\gamma)\frac{D^+V[x(t^*)]}{V^\gamma(t^*)} + \epsilon(1-\gamma) \\ &\leq -\mu_1(1-\gamma) + \mu_2(1-\gamma)\frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)} + \epsilon(1-\gamma) \\ &= \mu_2(1-\gamma)\left(\frac{-\mu_1 + \epsilon}{\mu_2} + \frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)}\right). \end{aligned} \quad (5)$$

Note that  $\tau(t^*) \leq a(t^*)(b - t^*)$ , which leads to

$$\frac{\epsilon(1-\gamma)\tau(t^*)}{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) - \epsilon(1-\gamma)t^*} = \frac{\tau(t^*)}{b - t^*} \leq a(t^*). \quad (6)$$

According to condition (iii), one may obtain that  $\mu > \epsilon$ , i.e.,

$$1 + a(t^*) \leq 1 + \max_{s \in [0, b]} a(s) < \left(\frac{\mu_1 - \epsilon}{\mu_2}\right)^{\frac{1-\gamma}{\gamma}}. \quad (7)$$

Combining (4), (6) and (7), we have

$$\begin{aligned} \frac{V^{1-\gamma}(t^* - \tau(t^*))}{V^{1-\gamma}(t^*)} &\leq 1 + \frac{\epsilon(1-\gamma)\tau(t^*)}{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) - \epsilon(1-\gamma)t^*} \\ &< \left(\frac{\mu_1 - \epsilon}{\mu_2}\right)^{\frac{1-\gamma}{\gamma}}, \end{aligned}$$

which implies that

$$\frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)} < \frac{\mu_1 - \epsilon}{\mu_2}.$$

It then follows from (5) that  $D^+\psi(t) |_{t=t^*} < 0$ , which is a contradiction with  $D^+\psi(t^*) \geq 0$  and thus (3) holds.

Next, we shall show that  $V(t) = 0$  for  $t \geq b$ . To this end, we first consider the instant  $t = b$ . If  $V(b) \neq 0$ , i.e.,  $V(b) > 0$ , then in view of the continuity of function  $V$ , there exists  $\delta_1 \in (0, b]$  such that  $V(t) > \frac{1}{2}V(b)$  on the interval  $(b - \delta_1, b)$ . Take  $\delta_2 = \delta_1 \wedge \frac{V^{1-\gamma}(b)}{2^{1-\gamma}\epsilon(1-\gamma)}$ , then  $b - t \leq \frac{V^{1-\gamma}(b)}{2^{1-\gamma}\epsilon(1-\gamma)}$  for any  $t \in (b - \delta_2, b)$ . From (3) and the definition of  $\epsilon$ , one can derive that

$$V^{1-\gamma}(t) + \epsilon(1-\gamma)t \leq \epsilon(1-\gamma)b, \quad \forall t \in (b - \delta_2, b),$$

which gives  $V^{1-\gamma}(t) \leq \epsilon(1-\gamma)(b - t)$ . It indicates that  $V(t) \leq \frac{1}{2}V(b)$  for  $t \in (b - \delta_2, b)$ , which is a contradiction with  $V(t) > \frac{1}{2}V(b)$ . Thus it holds that  $V(b) = 0$ . Then we show that  $V(t) \equiv 0$  for all  $t > b$ . For any  $\varepsilon > 0$ , consider an auxiliary function

$F_\varepsilon(t) := V(t) - \varepsilon t$ ,  $t \geq b$ . We show that  $F_\varepsilon(t) \leq 0$  for all  $t > b$ . It is obvious that  $F_\varepsilon(b) < 0$ . Suppose that there exists some  $t > b$  such that  $F_\varepsilon(t) > 0$ , and define  $t^* := \inf\{t > b \mid F_\varepsilon(t) > 0\}$ . Then  $F_\varepsilon(t^*) = 0$ ,  $D^+F_\varepsilon(t^*) \geq 0$ , and  $F_\varepsilon(t) \leq 0$  for  $t \in [b, t^*]$ . Since  $F_\varepsilon(b) < 0$ , it holds that  $t^* > b$ . Moreover, one can obtain that  $V(t^*) = \varepsilon t^*$ ,  $V(t) \leq \varepsilon t$  for  $t \in [b, t^*]$ . In view of  $a(t) \in [0, 1]$  for all  $t > b$ , it holds that

$$\begin{aligned} t^* - \tau(t^*) &\geq t^* - a(t^*) \cdot (t^* - b) \\ &= (1 - a(t^*))t^* + a(t^*)b \geq (1 - a(t^*))b + a(t^*)b = b, \end{aligned}$$

which leads to

$$V(t^* - \tau(t^*)) \leq \varepsilon(t^* - \tau(t^*)) \leq \varepsilon t^* = V(t^*). \quad (8)$$

It then can be derived from (8) and condition (iii) that

$$\begin{aligned} D^+F_\varepsilon(t) |_{t=t^*} &= D^+V[x(t^*)] - \varepsilon \\ &\leq -\mu_1V^\gamma(t^*) + \mu_2V^\gamma(t^* - \tau(t^*)) - \varepsilon \\ &\leq (-\mu_1 + \mu_2)V^\gamma(t^*) - \varepsilon < 0. \end{aligned}$$

This is a contradiction. Thus  $F_\varepsilon(t) \leq 0$  for all  $t > b$ . That is  $V(t) \leq \varepsilon t$  for all  $t > b$ . Considering the arbitrary of  $\varepsilon$ , we have that  $V(t) \equiv 0$  for all  $t > b$ . In fact, if there exists  $t' > b$  such that  $V(t') > 0$ , one can choose  $\varepsilon = \frac{1}{2t'}V(t') > 0$ , then  $V(t) \leq \frac{t}{2t'}V(t')$  for all  $t > b$ . When  $t = t' > b$ ,  $V(t) \leq \frac{1}{2}V(t')$ , which is impossible. Consequently, it holds that  $V(t) \equiv 0$  for all  $t \geq b$ . Combining (3) with condition (i), it then follows that  $|x(t, \phi)| \leq \omega_1^{-1} \circ \omega_2(\|\phi\|_{\bar{\tau}})$  for any  $t \geq 0$  and  $\phi \in \mathcal{C}_{\bar{\tau}}^\sigma$ , which ensures that system (1) is Lyapunov stable. Hence, the *F**T**S* of system (1) can be concluded. This completes the proof of *Theorem 1*.  $\square$

*Remark 2:* It is worth mentioning that the classical Halanay's inequality only requires  $\mu_1 > \mu_2$  for asymptotic stability over an infinite horizon, see [22]. In *Theorem 1*, however, one can see that such condition is strengthened by condition (iii) in order to achieve *F**T**S*. Moreover, with the help of *Assumption 1*, any trajectory of system (1) will reach the equilibrium point before the instant  $t = b$  and stay there thereafter. But we mention that, although the uniform upper bound of the settling time is estimated by  $T(\phi) \leq b$ , the proposed *F**T**S* result in *Theorem 1* is different from the conventional fixed-time stability [23], [24], [25]. Because estimation  $T(\phi) \leq b$  of the settling time only holds for the solution  $x(t) = x(t, \phi)$  of system (1) with initial state  $\phi$  within  $\mathcal{C}_{\bar{\tau}}^\sigma$ . Thus the *F**T**S* considered in this paper is of local feature, while fixed-time stability is generally related to the global property of the system. In addition, it should be noted that, although the upper bound of the settling time is estimated by  $T(\phi) \leq b$ , it does not mean that  $b$  is the settling time of the system. This assertion is illustrated in Section IV.

*Remark 3:* When applying *Theorem 1*, one needs to find a Lyapunov function of system (1) with parameters  $\mu_1, \mu_2$  satisfying Halanay-like inequality (2), which together with condition (iii) then gives an upper bound of  $\sigma$  to restrict the initial function for the *F**T**S*. It shows that to relax the constraint of initial function  $\phi$ , a bigger  $\mu$  would be needed for the *F**T**S* of system (1), which implies that  $\mu_1$  should be relatively bigger than  $\mu_2$ . The construction of such a Lyapunov function

satisfying conditions of Theorem 1 is given in Section IV for a second-order time-delay system.

*Remark 4:* Comparing with the existing *FTS* results in [16], [18], where the *FTS* problem for general time-delay systems has been studied via strengthening Lyapunov functional or Razumikhin method, this paper focuses on the *FTS* problem subjects to a class of time-varying delays. Although the restriction on time delay is strengthened to certain extent, construction of Lyapunov function is easier than that in the existing results, which can be seen from *Example 2*.

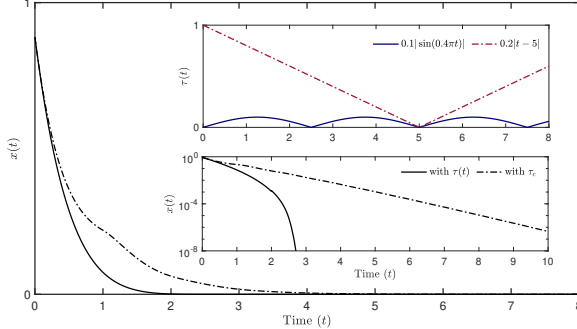


Figure 2. State trajectory of system (9)-(10) with  $\phi = 0.9$ .

#### IV. EXAMPLES

In this section, two examples are given to demonstrate the efficiency of the obtained results.

*Example 1:* Consider a scalar time-delay system

$$\dot{x}(t) = -|x(t)|^{\frac{1}{2}} \text{sign}(x(t))g(x(t), x(t - \tau(t))), \quad t \geq 0, \quad (9)$$

where  $\tau(t)$  satisfies *Assumption 1* for  $a(t) = 0.2, b = 5$ , and

$$g(x(t), x(t - \tau(t))) = 3x^{\frac{2}{3}}(t) - x^{\frac{2}{3}}(t - \tau(t)). \quad (10)$$

The *FTS* of system (9) with  $g = 1 + x^2(t - \tau(t))$  has been studied in Example 5 of [16]. It is easy to see that the effect of time delay vanishes during the differentiation of the Lyapunov functional. For some general cases such as (10), the theoretical result proposed in [16] becomes invalid because it is difficult to find a suitable *L-K* functional to meet those assumptions. In addition, one may find that the proposed result in [18] is also invalid for (10) due to its conservative Razumikhin condition. To determine the dynamics of system (9)-(10) following the proposed result, one can choose Lyapunov function  $V(x(t)) = |x(t)|$ . Then it is easy to verify that conditions in *Theorem 1* hold with  $\mu_1 = 2.82, \mu_2 = 0.83$  and  $\gamma = 17/21$ . Note that  $\mu = 1.0186$ . Thus, according to *Theorem 1*, any solution  $x(t, \phi)$  of system (9)-(10) with  $\phi \in \mathcal{C}_\tau^\sigma$  is *FTS*, where  $\sigma = 0.83$  satisfying condition (iii). In simulations, we take  $\tau(t) = 0.1|\sin(0.4\pi t)|$ ,  $t \neq 2k$ , and  $\tau(2k) = 0.2|2k - 5|$ ,  $k \in \mathbb{Z}_+$ , then Figure 2 (solid black) illustrates the trajectories of system (9)-(10) and the discontinuous delay  $\tau(t)$ . It shows that the state trajectory maintains at the equilibrium point after arriving it and moreover, the settling time satisfies  $T(\phi) < 3 < b$ . Moreover, if we take

$\tau(t) = \tau_c = 1$  as a constant delay, then it is shown in Figure 2 (dashed black) that system (9)-(10) has no *FTS* proposition. It verifies the discussion before *Assumption 1*.

*Example 2:* Consider second-order system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + cx_1^\lambda(t - \tau(t)), \\ \dot{x}_2(t) = u(t), \end{cases} \quad (11)$$

where  $c \geq 0, \lambda = \frac{q}{p} \in (\frac{1}{2}, 1)$  with odd constants  $p, q > 0$ ,  $\tau(t)$  is the time delay,  $u \in \mathbb{R}$  is the control input. We now give the following proposition, and the detailed computation of Lyapunov functions is presented in the proof.

*Proposition 1:* Given  $\sigma > 0$ , if there exist constants  $\alpha_1 > \frac{c+2^{1-\lambda}}{\lambda+1}$  and  $\alpha_2 > \frac{2^{1-\lambda}\lambda}{\lambda+1}$  such that condition (iii) in *Theorem 1* holds with  $\mu_1 = 2^{\frac{\lambda+1}{2}}(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}) \wedge (\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1})$ ,  $\mu_2 = \frac{\lambda+1}{2} \frac{c\lambda}{\lambda+1}$  and  $\omega_2(s) = \bar{\alpha}s$ , where  $\bar{\alpha} = (\frac{1}{2} + \alpha_1^{\frac{2}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}) \vee (1 + \alpha_1^{\frac{1}{\lambda}})$ .

Then system (11) is *FTS* with memoryless controller

$$u = -\alpha_2(x_2^{\frac{1}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}x_1)^{2\lambda-1}, \quad (12)$$

for any  $\tau(t)$  satisfying *Assumption 1* and  $\phi \in \mathcal{C}_\tau^\sigma$ .

*Proof:* The computation of a Lyapunov function satisfying conditions in *Theorem 1* consists of three steps.

*Step 1.* Choose  $V_1 = \frac{1}{2}x_1^2$ . According to (2), we have

$$\dot{V}_1 = x_1\dot{x}_1 = -\alpha_1x_1^{\lambda+1} + x_1(x_2 + \alpha_1x_1^\lambda) + cx_1x_1^\lambda(t - \tau(t)),$$

which together with Lemma 2.4 in [26] deduces that

$$\begin{aligned} \dot{V}_1 \leq & -\left(\alpha_1 - \frac{c}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ & + x_1(x_2 - x_2^*), \end{aligned}$$

where  $x_2^* = -\alpha_1x_1^\lambda$ ,  $\alpha_1 > \frac{c+2^{1-\lambda}}{\lambda+1}$ .

*Step 2.* Let

$$V_2 = V_1 + \int_{x_2^*}^{x_2} (s^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}})^{2-\lambda} ds. \quad (13)$$

It can be verified that  $V_2$  is positive definite, and it follows from Lemmas 2.3 and 2.4 in [26] that the derivative of  $V_2$  satisfies

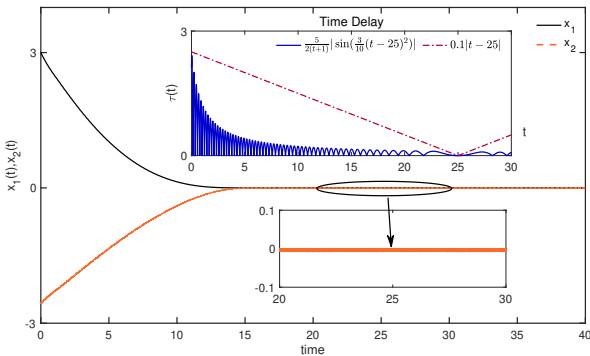
$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot \dot{x}_2 \\ &= \dot{V}_1 + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u \\ &\leq -\left(\alpha_1 - \frac{c}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ &\quad + x_1(x_2 - x_2^*) + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u \\ &\leq -\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ &\quad + \frac{2^{1-\lambda}\lambda}{\lambda+1} \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{\lambda+1} + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u. \end{aligned}$$

Obviously, with control input (12), we have

$$\begin{aligned} \dot{V}_2 &\leq -\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)x_1^{\lambda+1} - \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right) \\ &\quad \cdot \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t-\tau(t)) \\ &\leq -2^{\frac{\lambda+1}{2}}\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)\left(\frac{1}{2}x_1^2\right)^{\frac{\lambda+1}{2}} \\ &\quad - \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right)\left(\int_{x_2^*}^{x_2}(s^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}})^{2-\lambda}ds\right)^{\frac{\lambda+1}{2}} \\ &\quad + \frac{c\lambda 2^{\frac{\lambda+1}{2}}}{\lambda+1}\left(\frac{1}{2}x_1^2(t-\tau(t))\right)^{\frac{\lambda+1}{2}} \\ &\leq -\mu_1 V_2^{\frac{\lambda+1}{2}} + \mu_2 V_2^{\frac{\lambda+1}{2}}(t-\tau(t)), \end{aligned}$$

where  $\mu_1 = 2^{\frac{\lambda+1}{2}}\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right) \wedge \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right) > 0$ ,  $\mu_2 = \frac{2^{\frac{\lambda+1}{2}}c\lambda}{\lambda+1} > 0$ .

*Step 3.* Note that  $\alpha_1$  and  $\alpha_2$  are selected such that  $\mu_1$  and  $\mu_2$  satisfy condition (iii) with  $\omega_2(s) = \bar{\alpha}s$  and  $\bar{\alpha} = \left(\frac{1}{2} + \alpha_1^{\frac{2}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}\right) \vee (1 + \alpha_1^{\frac{1}{\lambda}})$ . Thus,  $V_2$  is a Lyapunov function of time-delay system (11). Applying *Theorem 1*, system (11) is *FTS*.  $\square$



**Figure 3.** State trajectories of system (11) with (12) and  $\phi(t) = (3, -2.55)^T$ .

In simulation, take  $c = 1, \lambda = \frac{3}{5}, \sigma = 3.27$  and  $\tau(t) = \frac{5}{2(t+1)}|\sin(\frac{3}{10}(t-25)^2)|$ . Then we can choose  $\alpha_1 = 2$  and  $\alpha_2 = 46$  so that condition (ii) holds with  $\mu_1 = 1.49$  and  $\mu_2 = 0.82$  and condition (iii) holds with  $\mu = 0.2894$ . According to *Proposition 1*, system (11)-(12) is *FTS* for any  $\phi \in \mathcal{C}_{\bar{\tau}}^{\sigma}$ , see Figure 3 (blue curve). It shows that the trajectories of system (11) are stabilized by (12) in finite time. Moreover, it is obvious that the settling time satisfies  $T(\phi) < b = 25$ , which verifies our previous assertion in *Remark 2*.

## V. CONCLUSIONS

In this paper, the problem of local *FTS* for general time delay systems are studied, where a class of time-varying delays is fully considered. Some sufficient conditions are proposed based on Lyapunov method. It shows that the *FTS* of systems involving some types of time-varying delays can be achieved. In addition,

an estimate of the settling time depending on time delays is given. Future research topics include relaxing the proposed *FTS* conditions and exploring the fixed-time stability of more general time-delay systems.

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