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Local Finite-Time Stability for a Class of *Time-Delay* Systems

Xiaodi Li, Xinyi He, Dan Yang, Emmanuel Moulay and Qing Hui

Abstract—This paper studies the local finite-time stability for a class of time-delay systems. Some Lyapunov-based sufficient conditions are proposed in the framework of finite time, where a potential connection between time delay and system parameters for local *FTS* is established. An estimate of the upper bound of the settling time depending on the time delay is derived. Examples are given to illustrate the effectiveness of the proposed results. Especially, a finite-time stabilizing memoryless controller for a class of second-order systems with time-varying delay is proposed.

Index Terms—Finite-time stability, time-delay systems, Lyapunov method, settling time, memoryless controller.

I. INTRODUCTION

Over the past decades, the control community has witnessed extensive research on finite-time stability (*FTS*) of dynamical systems, see [1], [2], [3], [4]. Different from asymptotic stability, which implies the convergence of the system to a Lyapunov stable equilibrium state over infinite horizons, *FTS* can guarantee the finite-time convergence of the system. At first, the concept of *FTS* arose in time-optimal control, see [5], [6]. Then with the complement of homogeneous approach and Lyapunov theorem, the theory of *FTS* has been developed in the last decades. For instance, it has been shown in [7] that a system is *FTS* if it is globally asymptotically stable and homogeneous of negative degree; Based on Lyapunov method, a basic tool for the analysis of *FTS* has been given in [5]; Whereas a necessary and sufficient condition for *FTS* of continuous systems has been given in [8]. From the practical point of view, *FTS* plays an important role in many engineering problems such as proportional navigation guidance [9], [10], high-quality robot control [11], attitude tracking of spacecraft [12], [13], etc.

Note that time delays appear in many dynamical systems, including engineering systems, communication networks, chemical process, manufacturing systems and so on. Considerable research on the stability and stabilization of time-delay systems

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over infinite horizons can be found in [14], [15]. In the framework of finite time, however, the corresponding development is comparatively retarded due to the fact that *FTS* requires a much stronger performance than classical Lyapunov stability. Related results on *FTS* can be found in [16], [17], [18] and the references therein. It is well known that there are mainly two ways available for studying stability of time-delay systems. One is Lyapunov-Razumikhin (*L-R*) approach, which has been used for *FTS* analysis of time-delay systems in previous papers [17], [18]. However, as noticed in [19], the Razumikhin condition in [17] is insufficient for *FTS* of systems with time delays. To guarantee *FTS* of time-delay systems, there must be additional restrictions on the evolution of the system and one of necessary conditions has been presented in [19] for constant delay. More recently, by strengthening the classical Razumikhin condition, *L-R* approach has been employed in [18] to ensure finite-time convergence. Especially, the cases of exponential, finite-time and fixed-time (with respect to a ball) convergences have been extensively considered. But the proposed results in [18] are based on a class of strengthening Razumikhin conditions, which leads to a limited application. Another approach developing the Lyapunov method to *FTS* analysis of time-delay systems is Lyapunov-Krasovskii (*L-K*) approach, which has been successfully used in [16]. Although some general sufficient conditions have been given in [16] by finding an *L-K* functional whose derivative is upper bounded by a certain negative function, it is hard to reflect the effects of time delay on system dynamics. Moreover, it is difficult to find a suitable *L-K* functional to meet all assumptions in [16]. In general, the research on *FTS* of time-delay systems is still a very difficult and challenging topic.

In this paper, we shall study the *FTS* for a class of non-linear systems with time-varying delays, where a Halanay-like inequality acting on Lyapunov functions is given to ensure *FTS*. An estimate of the upper bound of the settling time depending on the time delay is derived. It shows that, when subject to some types of time-varying delays, the local *FTS* of systems can be achieved. Although this paper copes with the problem of *FTS* under an additional assumption of delays, it still covers a general case of systems. Especially, the proposed results can be employed to finite-time stabilization under memoryless control. The outline of this paper is as follows. In Section II, the problem is formulated and preliminaries are given. In Section III, the finite-time conditions are presented for time-delay systems. Two numerical examples are given in Section IV to illustrate the proposed results, where a stabilizing memoryless controller for a class of second-order time-delay systems is designed. Conclusions are given in Section V.

Notations. Let \mathbb{Z}_+ denote the set of positive integers, \mathbb{R} the set of real numbers, \mathbb{R}_+ the set of nonnegative real numbers, \mathbb{R}^n the n -dimensional real space equipped with the Euclidean norm $|\cdot|$. A function $\varphi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ is of class \mathcal{K} if φ is strictly increasing and $\varphi(0) = 0$. For given $\tau > 0$, we equip the linear space $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ (abr. \mathcal{C}_τ) with the norm $\|\cdot\|_\tau$ defined by $\|\phi\|_\tau = \sup_{s \in [-\tau, 0]} |\phi(s)|$. For $\sigma > 0$, $\mathcal{C}_\tau^\sigma = \{\phi \in \mathcal{C}_\tau : \|\phi\|_\tau < \sigma\}$. For real constants a, b , let $a \vee b$ and $a \wedge b$ denote the maximum and the minimum value of a and b .

II. PRELIMINARIES

Consider the following nonlinear delay system

$$\begin{cases} \dot{x}(t) = f(x(t), x(t - \tau(t))), & t > 0, \\ x(t) = \phi(t), & t \in [-\bar{\tau}, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $\dot{x}(t)$ denotes the upper right-hand derivative of $x(t)$; $\tau(t) : \mathbb{R}_+ \rightarrow [0, \bar{\tau}]$ is the time-varying delay, and $\phi \in \mathcal{C}_{\bar{\tau}}$ represents the initial condition, where $\bar{\tau}$ is a positive constant; $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfying $f(0, 0) = 0$. We assume that function f satisfies suitable conditions so the solution $x(t) := x(t, \phi)$ uniquely exists in forward time within relevant time intervals, see [20], [21]. For a locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, the upper right-hand Dini derivative along system (1) is defined by

$$D^+V[x(t)] = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(x(t+h)) - V(x(t))].$$

When $\tau(t) \equiv \tau^*$ is a constant, it has been shown in [16] that system (1) cannot be *FTS* if the origin is the only point of $f(x, y) = 0$. Thus, to guarantee *FTS* in the case of constant delay, the system needs to be equipped with some special structure, $f(u, v) \equiv 0$ when $u = 0$ for example, so that the effect of the constant delay disappears when the system reaches the origin [16]. In this paper, to study the *FTS* problem of time-delay system (1) without additional constraint on the system structure, the following assumption on $\tau(t)$ is needed.

Assumption 1: There exist a positive constant b and a function $a \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ satisfying $a(t) \leq 1$ for all $t > b$ such that

$$\tau(t) \leq a(t) \cdot |b - t|, \quad t \geq 0.$$

Remark 1: *Assumption 1* indicates that the time-varying delay $\tau(t)$ vanishes at some instants, especially, $\tau(b) = 0$. Although it imposes a certain restriction on the time-varying delay, it still covers many types of time delays, such as the sinusoidal type $\tau(t) = a(t) \cdot |\sin(\frac{\pi}{b}t + \varphi)|$, the linear type $\tau(t) = a(t) \cdot \prod_{i=1}^m |b_i - t|$ and so on, see Figure 1.

Definition 1: [16] System (1) is said to be

(a) Lyapunov stable (*LS*) if there exist a function $\alpha \in \mathcal{K}$ and a constant $\sigma > 0$ such that for any $\phi \in \mathcal{C}_\tau^\sigma$, $|x(t, \phi)| \leq \alpha(\|\phi\|_\tau)$ for all $t \geq 0$;

(b) (locally) finite-time stable (*FTS*) if it is *LS* and for any $\phi \in \mathcal{C}_\tau^\sigma$ there exists $0 \leq T_\phi < +\infty$ such that $x(t, \phi) = 0$ for all $t \geq T_\phi$. The functional $T(\phi) =$

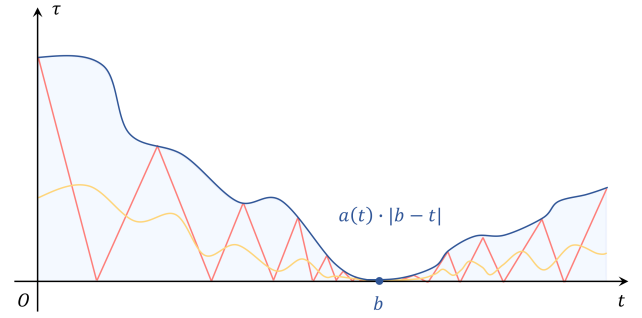


Figure 1. The bound of $\tau(t)$ and different types of delays.

$\inf \{ T_\phi \geq 0 \mid x(t, \phi) = 0, \forall t \geq T_\phi \}$ is called the settling time of system (1).

Moreover, for given $\sigma > 0$, if there exists a constant $\eta > 0$ such that $T(\phi) \leq \eta$ for any $\phi \in \mathcal{C}_\tau^\sigma$, then settling time $T(\phi)$ of system (1) is said to be uniformly bounded by η over \mathcal{C}_τ^σ .

III. MAIN RESULTS

In this section, some Lyapunov-based sufficient conditions are presented for the *FTS* of nonlinear time-delay system (1).

Theorem 1: Assume that Assumption 1 holds for some function $a \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and constant $b > 0$. If there exist functions $\omega_1, \omega_2 \in \mathcal{K}$, constants $\gamma \in (0, 1)$, $\mu_1 > 0$, $\sigma > 0$, $\mu_2 \geq 0$ and a locally Lipschitz continuous function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

- (i) $\omega_1(|x|) \leq V(x) \leq \omega_2(|x|)$, $\forall x \in \mathbb{R}^n$;
- (ii) the derivative of V along the solution $x(t) := x(t, \phi)$ of system (1) with $\phi \in \mathcal{C}_\tau^\sigma$ satisfies

$$D^+V[x(t)] \leq -\mu_1 V^\gamma(x(t)) + \mu_2 V^\gamma(x(t - \tau(t))); \quad (2)$$

- (iii) $\mu \geq \frac{\omega_2^{1-\gamma}(\sigma)}{(1-\gamma)b}$, where $\mu := \mu_1 - (1 + \bar{a})^{1-\gamma} \mu_2$ and $\bar{a} = \max_{s \in [0, b]} a(s)$.

Then system (1) is *FTS* with settling time $T(\phi)$ uniformly bounded by b over \mathcal{C}_τ^σ .

Proof: For any given initial data $\phi \in \mathcal{C}_\tau^\sigma$, let $x(t) := x(t, \phi)$ be the solution of system (1) through $(0, \phi)$. Without loss of generality, we assume that $\|\phi\|_\tau > 0$. Set

$$V(t) := V(x(t)), \quad \psi(t) := V^{1-\gamma}(t) + \epsilon(1-\gamma)(t \vee 0),$$

where $\epsilon := \frac{\omega_2^{1-\gamma}(\|\phi\|_\tau)}{(1-\gamma)b}$. First, we claim that

$$\psi(t) \leq \omega_2^{1-\gamma}(\|\phi\|_\tau), \quad \forall t \in [0, b]. \quad (3)$$

Obviously, $\psi(0) = V^{1-\gamma}(0) \leq \omega_2^{1-\gamma}(\|\phi\|_\tau)$. Suppose that (3) does not hold on $(0, b)$, then there exists $t^* \in (0, b)$ such that $\psi(t^*) = \omega_2^{1-\gamma}(\|\phi\|_\tau)$, $D^+\psi(t^*) \geq 0$, and $\psi(t) \leq \psi(t^*)$ for $t \in [0, t^*]$. Then there are two cases: $t^* - \tau(t^*) \geq 0$ and $t^* - \tau(t^*) < 0$. When $t^* - \tau(t^*) \geq 0$, it follows from $\psi(t) \leq \psi(t^*)$ for $t \in [0, t^*]$ that

$$V^{1-\gamma}(t^* - \tau(t^*)) + \epsilon(1-\gamma)(t^* - \tau(t^*)) \leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)t^*,$$

which implies that

$$V^{1-\gamma}(t^* - \tau(t^*)) \leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*). \quad (4)$$

When $t^* - \tau(t^*) < 0$, it follows from condition (i) that

$$\begin{aligned} V^{1-\gamma}(t^* - \tau(t^*)) &\leq \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) \\ &= V^{1-\gamma}(t^*) + \epsilon(1-\gamma)t^* \\ &= V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*) + \epsilon(1-\gamma)(t^* - \tau(t^*)) \\ &\leq V^{1-\gamma}(t^*) + \epsilon(1-\gamma)\tau(t^*). \end{aligned}$$

Hence, in both cases, (4) always holds. Moreover, it is easy to see that $V(t^*) \neq 0$. In fact, if $V(t^*) = 0$, then it follows from $\psi(t^*) = \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}})$ and the definition of ϵ that

$$\frac{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}})}{b}t^* = \epsilon(1-\gamma)t^* = \omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}),$$

i.e., $t^* = b$, which leads to a contradiction with $t^* < b$. Since $V(t^*) \neq 0$, there exists $\delta_0 > 0$ such that $V(t) > 0$ on the interval $(t^* - \delta_0, t^* + \delta_0)$. It then can be deduced that

$$\begin{aligned} D^+\psi(t) |_{t=t^*} &= (1-\gamma)\frac{D^+V[x(t^*)]}{V^\gamma(t^*)} + \epsilon(1-\gamma) \\ &\leq -\mu_1(1-\gamma) + \mu_2(1-\gamma)\frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)} + \epsilon(1-\gamma) \\ &= \mu_2(1-\gamma)\left(\frac{-\mu_1 + \epsilon}{\mu_2} + \frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)}\right). \end{aligned} \quad (5)$$

Note that $\tau(t^*) \leq a(t^*)(b - t^*)$, which leads to

$$\frac{\epsilon(1-\gamma)\tau(t^*)}{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) - \epsilon(1-\gamma)t^*} = \frac{\tau(t^*)}{b - t^*} \leq a(t^*). \quad (6)$$

According to condition (iii), one may obtain that $\mu > \epsilon$, i.e.,

$$1 + a(t^*) \leq 1 + \max_{s \in [0, b]} a(s) < \left(\frac{\mu_1 - \epsilon}{\mu_2}\right)^{\frac{1-\gamma}{\gamma}}. \quad (7)$$

Combining (4), (6) and (7), we have

$$\begin{aligned} \frac{V^{1-\gamma}(t^* - \tau(t^*))}{V^{1-\gamma}(t^*)} &\leq 1 + \frac{\epsilon(1-\gamma)\tau(t^*)}{\omega_2^{1-\gamma}(\|\phi\|_{\bar{\tau}}) - \epsilon(1-\gamma)t^*} \\ &< \left(\frac{\mu_1 - \epsilon}{\mu_2}\right)^{\frac{1-\gamma}{\gamma}}, \end{aligned}$$

which implies that

$$\frac{V^\gamma(t^* - \tau(t^*))}{V^\gamma(t^*)} < \frac{\mu_1 - \epsilon}{\mu_2}.$$

It then follows from (5) that $D^+\psi(t) |_{t=t^*} < 0$, which is a contradiction with $D^+\psi(t^*) \geq 0$ and thus (3) holds.

Next, we shall show that $V(t) = 0$ for $t \geq b$. To this end, we first consider the instant $t = b$. If $V(b) \neq 0$, i.e., $V(b) > 0$, then in view of the continuity of function V , there exists $\delta_1 \in (0, b]$ such that $V(t) > \frac{1}{2}V(b)$ on the interval $(b - \delta_1, b)$. Take $\delta_2 = \delta_1 \wedge \frac{V^{1-\gamma}(b)}{2^{1-\gamma}\epsilon(1-\gamma)}$, then $b - t \leq \frac{V^{1-\gamma}(b)}{2^{1-\gamma}\epsilon(1-\gamma)}$ for any $t \in (b - \delta_2, b)$. From (3) and the definition of ϵ , one can derive that

$$V^{1-\gamma}(t) + \epsilon(1-\gamma)t \leq \epsilon(1-\gamma)b, \quad \forall t \in (b - \delta_2, b),$$

which gives $V^{1-\gamma}(t) \leq \epsilon(1-\gamma)(b - t)$. It indicates that $V(t) \leq \frac{1}{2}V(b)$ for $t \in (b - \delta_2, b)$, which is a contradiction with $V(t) > \frac{1}{2}V(b)$. Thus it holds that $V(b) = 0$. Then we show that $V(t) \equiv 0$ for all $t > b$. For any $\varepsilon > 0$, consider an auxiliary function

$F_\varepsilon(t) := V(t) - \varepsilon t$, $t \geq b$. We show that $F_\varepsilon(t) \leq 0$ for all $t > b$. It is obvious that $F_\varepsilon(b) < 0$. Suppose that there exists some $t > b$ such that $F_\varepsilon(t) > 0$, and define $t^* := \inf\{t > b \mid F_\varepsilon(t) > 0\}$. Then $F_\varepsilon(t^*) = 0$, $D^+F_\varepsilon(t^*) \geq 0$, and $F_\varepsilon(t) \leq 0$ for $t \in [b, t^*]$. Since $F_\varepsilon(b) < 0$, it holds that $t^* > b$. Moreover, one can obtain that $V(t^*) = \varepsilon t^*$, $V(t) \leq \varepsilon t$ for $t \in [b, t^*]$. In view of $a(t) \in [0, 1]$ for all $t > b$, it holds that

$$\begin{aligned} t^* - \tau(t^*) &\geq t^* - a(t^*) \cdot (t^* - b) \\ &= (1 - a(t^*))t^* + a(t^*)b \geq (1 - a(t^*))b + a(t^*)b = b, \end{aligned}$$

which leads to

$$V(t^* - \tau(t^*)) \leq \varepsilon(t^* - \tau(t^*)) \leq \varepsilon t^* = V(t^*). \quad (8)$$

It then can be derived from (8) and condition (iii) that

$$\begin{aligned} D^+F_\varepsilon(t) |_{t=t^*} &= D^+V[x(t^*)] - \varepsilon \\ &\leq -\mu_1V^\gamma(t^*) + \mu_2V^\gamma(t^* - \tau(t^*)) - \varepsilon \\ &\leq (-\mu_1 + \mu_2)V^\gamma(t^*) - \varepsilon < 0. \end{aligned}$$

This is a contradiction. Thus $F_\varepsilon(t) \leq 0$ for all $t > b$. That is $V(t) \leq \varepsilon t$ for all $t > b$. Considering the arbitrary of ε , we have that $V(t) \equiv 0$ for all $t > b$. In fact, if there exists $t' > b$ such that $V(t') > 0$, one can choose $\varepsilon = \frac{1}{2t'}V(t') > 0$, then $V(t) \leq \frac{t}{2t'}V(t')$ for all $t > b$. When $t = t' > b$, $V(t') \leq \frac{1}{2}V(t')$, which is impossible. Consequently, it holds that $V(t) \equiv 0$ for all $t \geq b$. Combining (3) with condition (i), it then follows that $|x(t, \phi)| \leq \omega_1^{-1} \circ \omega_2(\|\phi\|_{\bar{\tau}})$ for any $t \geq 0$ and $\phi \in \mathcal{C}_{\bar{\tau}}^\sigma$, which ensures that system (1) is Lyapunov stable. Hence, the *FTS* of system (1) can be concluded. This completes the proof of *Theorem 1*. \square

Remark 2: It is worth mentioning that the classical Halanay's inequality only requires $\mu_1 > \mu_2$ for asymptotic stability over an infinite horizon, see [22]. In *Theorem 1*, however, one can see that such condition is strengthened by condition (iii) in order to achieve *FTS*. Moreover, with the help of *Assumption 1*, any trajectory of system (1) will reach the equilibrium point before the instant $t = b$ and stay there thereafter. But we mention that, although the uniform upper bound of the settling time is estimated by $T(\phi) \leq b$, the proposed *FTS* result in *Theorem 1* is different from the conventional fixed-time stability [23], [24], [25]. Because estimation $T(\phi) \leq b$ of the settling time only holds for the solution $x(t) = x(t, \phi)$ of system (1) with initial state ϕ within $\mathcal{C}_{\bar{\tau}}^\sigma$. Thus the *FTS* considered in this paper is of local feature, while fixed-time stability is generally related to the global property of the system. In addition, it should be noted that, although the upper bound of the settling time is estimated by $T(\phi) \leq b$, it does not mean that b is the settling time of the system. This assertion is illustrated in Section IV.

Remark 3: When applying *Theorem 1*, one needs to find a Lyapunov function of system (1) with parameters μ_1, μ_2 satisfying Halanay-like inequality (2), which together with condition (iii) then gives an upper bound of σ to restrict the initial function for the *FTS*. It shows that to relax the constraint of initial function ϕ , a bigger μ would be needed for the *FTS* of system (1), which implies that μ_1 should be relatively bigger than μ_2 . The construction of such a Lyapunov function

satisfying conditions of Theorem 1 is given in Section IV for a second-order time-delay system.

Remark 4: Comparing with the existing *FTS* results in [16], [18], where the *FTS* problem for general time-delay systems has been studied via strengthening Lyapunov functional or Razumikhin method, this paper focuses on the *FTS* problem subjects to a class of time-varying delays. Although the restriction on time delay is strengthened to certain extent, construction of Lyapunov function is easier than that in the existing results, which can be seen from *Example 2*.

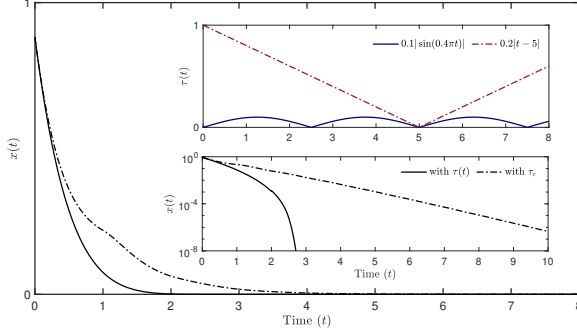


Figure 2. State trajectory of system (9)-(10) with $\phi = 0.9$.

IV. EXAMPLES

In this section, two examples are given to demonstrate the efficiency of the obtained results.

Example 1: Consider a scalar time-delay system

$$\dot{x}(t) = -|x(t)|^{\frac{1}{2}} \text{sign}(x(t))g(x(t), x(t - \tau(t))), \quad t \geq 0, \quad (9)$$

where $\tau(t)$ satisfies *Assumption 1* for $a(t) = 0.2, b = 5$, and

$$g(x(t), x(t - \tau(t))) = 3x^{\frac{2}{3}}(t) - x^{\frac{2}{3}}(t - \tau(t)). \quad (10)$$

The *FTS* of system (9) with $g = 1 + x^2(t - \tau(t))$ has been studied in Example 5 of [16]. It is easy to see that the effect of time delay vanishes during the differentiation of the Lyapunov functional. For some general cases such as (10), the theoretical result proposed in [16] becomes invalid because it is difficult to find a suitable *L-K* functional to meet those assumptions. In addition, one may find that the proposed result in [18] is also invalid for (10) due to its conservative Razumikhin condition. To determine the dynamics of system (9)-(10) following the proposed result, one can choose Lyapunov function $V(x(t)) = |x(t)|$. Then it is easy to verify that conditions in *Theorem 1* hold with $\mu_1 = 2.82, \mu_2 = 0.83$ and $\gamma = 17/21$. Note that $\mu = 1.0186$. Thus, according to *Theorem 1*, any solution $x(t, \phi)$ of system (9)-(10) with $\phi \in \mathcal{C}_\tau^\sigma$ is *FTS*, where $\sigma = 0.83$ satisfying condition (iii). In simulations, we take $\tau(t) = 0.1|\sin(0.4\pi t)|$, $t \neq 2k$, and $\tau(2k) = 0.2|2k - 5|$, $k \in \mathbb{Z}_+$, then Figure 2 (solid black) illustrates the trajectories of system (9)-(10) and the discontinuous delay $\tau(t)$. It shows that the state trajectory maintains at the equilibrium point after arriving it and moreover, the settling time satisfies $T(\phi) < 3 < b$. Moreover, if we take

$\tau(t) = \tau_c = 1$ as a constant delay, then it is shown in Figure 2 (dashed black) that system (9)-(10) has no *FTS* proposition. It verifies the discussion before *Assumption 1*.

Example 2: Consider second-order system

$$\begin{cases} \dot{x}_1(t) = x_2(t) + cx_1^\lambda(t - \tau(t)), \\ \dot{x}_2(t) = u(t), \end{cases} \quad (11)$$

where $c \geq 0, \lambda = \frac{q}{p} \in (\frac{1}{2}, 1)$ with odd constants $p, q > 0$, $\tau(t)$ is the time delay, $u \in \mathbb{R}$ is the control input. We now give the following proposition, and the detailed computation of Lyapunov functions is presented in the proof.

Proposition 1: Given $\sigma > 0$, if there exist constants $\alpha_1 > \frac{c+2^{1-\lambda}}{\lambda+1}$ and $\alpha_2 > \frac{2^{1-\lambda}\lambda}{\lambda+1}$ such that condition (iii) in *Theorem 1* holds with $\mu_1 = 2^{\frac{\lambda+1}{2}}(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}) \wedge (\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1})$, $\mu_2 = \frac{\lambda+1}{2} \frac{c\lambda}{\lambda+1}$ and $\omega_2(s) = \bar{\alpha}s$, where $\bar{\alpha} = (\frac{1}{2} + \alpha_1^{\frac{2}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}) \vee (1 + \alpha_1^{\frac{1}{\lambda}})$.

Then system (11) is *FTS* with memoryless controller

$$u = -\alpha_2(x_2^{\frac{1}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}x_1)^{2\lambda-1}, \quad (12)$$

for any $\tau(t)$ satisfying *Assumption 1* and $\phi \in \mathcal{C}_\tau^\sigma$.

Proof: The computation of a Lyapunov function satisfying conditions in *Theorem 1* consists of three steps.

Step 1. Choose $V_1 = \frac{1}{2}x_1^2$. According to (2), we have

$$\dot{V}_1 = x_1\dot{x}_1 = -\alpha_1x_1^{\lambda+1} + x_1(x_2 + \alpha_1x_1^\lambda) + cx_1x_1^\lambda(t - \tau(t)),$$

which together with Lemma 2.4 in [26] deduces that

$$\begin{aligned} \dot{V}_1 \leq & -\left(\alpha_1 - \frac{c}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ & + x_1(x_2 - x_2^*), \end{aligned}$$

where $x_2^* = -\alpha_1x_1^\lambda$, $\alpha_1 > \frac{c+2^{1-\lambda}}{\lambda+1}$.

Step 2. Let

$$V_2 = V_1 + \int_{x_2^*}^{x_2} (s^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}})^{2-\lambda} ds. \quad (13)$$

It can be verified that V_2 is positive definite, and it follows from Lemmas 2.3 and 2.4 in [26] that the derivative of V_2 satisfies

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot \dot{x}_2 \\ &= \dot{V}_1 + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u \\ &\leq -\left(\alpha_1 - \frac{c}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ &\quad + x_1(x_2 - x_2^*) + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u \\ &\leq -\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)x_1^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t - \tau(t)) \\ &\quad + \frac{2^{1-\lambda}\lambda}{\lambda+1} \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{\lambda+1} + \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{2-\lambda} \cdot u. \end{aligned}$$

Obviously, with control input (12), we have

$$\begin{aligned} \dot{V}_2 &\leq -\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)x_1^{\lambda+1} - \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right) \\ &\quad \cdot \left(x_2^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}}\right)^{\lambda+1} + \frac{c\lambda}{\lambda+1}x_1^{\lambda+1}(t-\tau(t)) \\ &\leq -2^{\frac{\lambda+1}{2}}\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right)\left(\frac{1}{2}x_1^2\right)^{\frac{\lambda+1}{2}} \\ &\quad - \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right)\left(\int_{x_2^*}^{x_2}(s^{\frac{1}{\lambda}} - (x_2^*)^{\frac{1}{\lambda}})^{2-\lambda}ds\right)^{\frac{\lambda+1}{2}} \\ &\quad + \frac{c\lambda 2^{\frac{\lambda+1}{2}}}{\lambda+1}\left(\frac{1}{2}x_1^2(t-\tau(t))\right)^{\frac{\lambda+1}{2}} \\ &\leq -\mu_1 V_2^{\frac{\lambda+1}{2}} + \mu_2 V_2^{\frac{\lambda+1}{2}}(t-\tau(t)), \end{aligned}$$

where $\mu_1 = 2^{\frac{\lambda+1}{2}}\left(\alpha_1 - \frac{c+2^{1-\lambda}}{\lambda+1}\right) \wedge \left(\alpha_2 - \frac{2^{1-\lambda}\lambda}{\lambda+1}\right) > 0$, $\mu_2 = \frac{2^{\frac{\lambda+1}{2}}c\lambda}{\lambda+1} > 0$.

Step 3. Note that α_1 and α_2 are selected such that μ_1 and μ_2 satisfy condition (iii) with $\omega_2(s) = \bar{\alpha}s$ and $\bar{\alpha} = \left(\frac{1}{2} + \alpha_1^{\frac{2}{\lambda}} + \alpha_1^{\frac{1}{\lambda}}\right) \vee (1 + \alpha_1^{\frac{1}{\lambda}})$. Thus, V_2 is a Lyapunov function of time-delay system (11). Applying *Theorem 1*, system (11) is *FTS*. \square

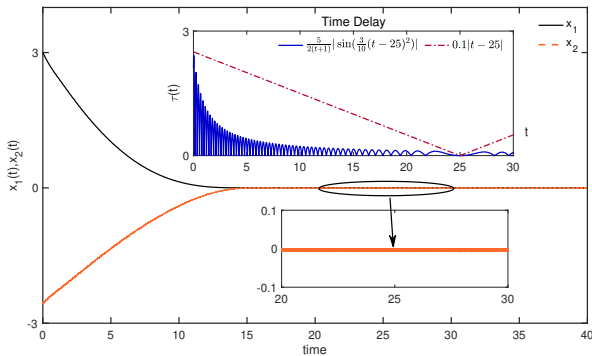


Figure 3. State trajectories of system (11) with (12) and $\phi(t) = (3, -2.55)^T$.

In simulation, take $c = 1, \lambda = \frac{3}{5}, \sigma = 3.27$ and $\tau(t) = \frac{5}{2(t+1)}|\sin(\frac{3}{10}(t-25)^2)|$. Then we can choose $\alpha_1 = 2$ and $\alpha_2 = 46$ so that condition (ii) holds with $\mu_1 = 1.49$ and $\mu_2 = 0.82$ and condition (iii) holds with $\mu = 0.2894$. According to *Proposition 1*, system (11)-(12) is *FTS* for any $\phi \in \mathcal{C}_{\bar{\tau}}^{\sigma}$, see Figure 3 (blue curve). It shows that the trajectories of system (11) are stabilized by (12) in finite time. Moreover, it is obvious that the settling time satisfies $T(\phi) < b = 25$, which verifies our previous assertion in *Remark 2*.

V. CONCLUSIONS

In this paper, the problem of local *FTS* for general time delay systems are studied, where a class of time-varying delays is fully considered. Some sufficient conditions are proposed based on Lyapunov method. It shows that the *FTS* of systems involving some types of time-varying delays can be achieved. In addition,

an estimate of the settling time depending on time delays is given. Future research topics include relaxing the proposed *FTS* conditions and exploring the fixed-time stability of more general time-delay systems.

REFERENCES

- [1] W. M. Haddad and J. Lee, "Finite-time stabilization and optimal feedback control for nonlinear discrete-time systems," *IEEE Transactions on Automatic Control*, 2022. DOI: 10.1109/TAC.2019.2928767.
- [2] H. Min, S. Xu, and Z. Zhang, "Adaptive finite-time stabilization of stochastic nonlinear systems subject to full-state constraints and input saturation," *IEEE Transactions on Automatic Control*, vol. 66, no. 3, pp. 1306–1313, 2020.
- [3] W. M. Haddad and J. Lee, "Finite-time stability of discrete autonomous systems," *Automatica*, vol. 122, p. 109282, 2020.
- [4] X. Li and Y. Zhao, "Sliding mode control for linear impulsive systems with matched disturbances," *IEEE Transactions on Automatic Control*, 2021. DOI: 10.1109/TAC.2021.3129735.
- [5] S. Bhat and D. Bernstein, "Finite-time stability of continuous autonomous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 3, pp. 751–766, 2000.
- [6] B. Zhou and Y. Shi, "Prescribed-time stabilization of a class of nonlinear systems by linear time-varying feedback," *IEEE Transactions on Automatic Control*, vol. 66, no. 12, pp. 6123–6130, 2021.
- [7] S. P. Bhat and D. S. Bernstein, "Finite-time stability of homogeneous systems," in *Proceedings of the 1997 American Control Conference (Cat. No. 97CH36041)*, vol. 4, pp. 2513–2514, IEEE, 1997.
- [8] E. Moulay and W. Perruquetti, "Finite time stability and stabilization of a class of continuous systems," *Journal of Mathematical analysis and applications*, vol. 323, no. 2, pp. 1430–1443, 2006.
- [9] H.-S. Shin, A. Tsourdos, and K.-B. Li, "A new three-dimensional sliding mode guidance law variation with finite time convergence," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 53, no. 5, pp. 2221–2232, 2017.
- [10] J. Zhou and J. Yang, "Guidance law design for impact time attack against moving targets," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 54, no. 5, pp. 2580–2589, 2018.
- [11] C. Yang, Y. Jiang, W. He, J. Na, Z. Li, and B. Xu, "Adaptive parameter estimation and control design for robot manipulators with finite-time convergence," *IEEE Transactions on Industrial Electronics*, vol. 65, no. 10, pp. 8112–8123, 2018.
- [12] K. Lu, Y. Xia, M. Fu, and C. Yu, "Adaptive finite-time attitude stabilization for rigid spacecraft with actuator faults and saturation constraints," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 1, pp. 28–46, 2016.
- [13] Y. Xia, J. Zhang, K. Lu, and N. Zhou, *Finite time and cooperative control of flight vehicles*. Springer, 2019.
- [14] E. Fridman, *Introduction to time-delay systems: Analysis and control*. Springer, 2014.
- [15] K. Gu, J. Chen, and V. L. Kharitonov, *Stability of time-delay systems*. Springer Science & Business Media, 2003.
- [16] E. Moulay, M. Dambrine, N. Yeganefer, and W. Perruquetti, "Finite-time stability and stabilization of time-delay systems," *Systems & Control Letters*, vol. 57, no. 7, pp. 561–566, 2008.
- [17] R. Yang and Y. Wang, "Finite-time stability analysis and H_{∞} control for a class of nonlinear time-delay hamiltonian systems," *Automatica*, vol. 49, no. 2, pp. 390–401, 2013.
- [18] D. Efimov and A. Aleksandrov, "On estimation of rates of convergence in Lyapunov–Razumikhin approach," *Automatica*, vol. 116, p. 108928, 2020.
- [19] D. Efimov, A. Polyakov, E. Fridman, W. Perruquetti, and J.-P. Richard, "Comments on finite-time stability of time-delay systems," *Automatica*, vol. 50, no. 7, pp. 1944–1947, 2014.
- [20] A. F. Filippov, *Differential equations with discontinuous righthand sides: control systems*, vol. 18. Springer Science & Business Media, 2013.
- [21] R. P. Agarwal and V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, vol. 6. World Scientific, 1993.
- [22] A. Halanay, *Differential equations: Stability, oscillations, time lags*, vol. 6. Elsevier, 1966.
- [23] A. Polyakov, "Nonlinear feedback design for fixed-time stabilization of linear control systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 8, pp. 2106–2110, 2011.
- [24] F. Lopez-Ramirez, D. Efimov, A. Polyakov, and W. Perruquetti, "Conditions for fixed-time stability and stabilization of continuous autonomous systems," *Systems & Control Letters*, vol. 129, pp. 26–35, 2019.

- [25] E. Moulay, V. Léchappé, E. Bernuau, and F. Plestan, "Robust fixed-time stability: application to sliding mode control," *IEEE Transactions on Automatic Control*, 2021. DOI: 10.1109/TAC.2021.3069667.
- [26] X. Huang, W. Lin, and B. Yang, "Global finite-time stabilization of a class of uncertain nonlinear systems," *Automatica*, vol. 41, no. 5, pp. 881–888, 2005.