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# Fractional-step methods and PSPG stabilized finite elements for the transient Stokes equations

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## Abstract

We propose an error analysis for a numerical approximation of the the transient Stokes problem which combines an incremental pressure-correction fractional-step scheme in time with a PSPG (pressure stabilized Petrov–Galerkin) finite element method in space. Optimal velocity convergence is obtained for affine approximations, whereas an inverse CFL condition is required with high-order polynomials and for the pressure.

**Keywords:** Transient Stokes equations, fractional-step scheme, PSPG finite element method.

## 1 Introduction

The pressure stabilized Petrov–Galerkin (PSPG) method, is a widespread approach for the numerical approximation of the Stokes equations using equal-order velocity/pressure interpolations. Though the analysis of the method in the steady case has been well understood since its introduction in [1], the transient case in conjunction with an implicit time-stepping has been much more involved (see [2–5]).

The first studies (see [2, 3]) indicated that an inverse CFL-condition ( $h^2 \leq \tau$ , with  $\tau$  and  $h$  respectively denoting the time and space discretization parameters) is required for stability, notably for the pressure. In [4], unconditional optimal convergence of the velocity is obtained for piecewise affine approximations. For higher order

polynomials, the results holds, but under the above mentioned inverse CFL-condition. This condition is also required for the error estimate of the pressure, even for affine approximations. In [5], the inverse CFL-condition is circumvented by using a particular discrete initial velocity.

The aim of this work is to investigate numerical approximations of the transient Stokes problem which combine the PSPG method in space with a fractional-step time-marching, thus segregating the computation of the velocity and of the pressure. To the best of our knowledge no error analysis for this class of methods has been reported in the literature so far.

For a general overview of fractional-step methods for incompressible flow we refer to [6]. The analysis of the incremental fractional-step scheme in its time semi-discrete version can be found in [7]. The fully discrete method with a spatial discretization based on inf-sup stable finite elements has been analyzed in [8]. As regards non inf-sup stable approximations in space, all the existing results rely on symmetric stabilization methods. Energy stability estimates without error bounds are provided in [2] (see also [9]) and [10], with the Brezzi-Pitkäranta stabilization [11] and the orthogonal subscales stabilization [12], respectively. The error analysis for an incremental fractional-step scheme with a general class of symmetric finite element stabilizations has been reported in [13]. An error analysis with the Brezzi-Pitkäranta stabilization can also be found in [14, 15].

Non-symmetric stabilizations, such as the PSPG method considered in this paper, do not fit in the analysis framework of the above mentioned studies. In this paper, we propose a numerical method for the transient Stokes problem, which combines the PSPG method with an incremental pressure-correction fractional-step projection scheme. A priori error estimates are provided in the case of the first-order time-stepping variant. The results obtained for the velocity are similar to those reported in [4] with monolithic time-stepping schemes. Numerical results are discussed for both the first- and second-order time-stepping variant.

The rest of the paper is organized as follows. The considered setting together with some notation are introduced in Section 2. Section 3 describes the proposed numerical method. The error analysis of the first-order variant is carried out in Section 4. Some numerical experiments are shown in Section 5, and a summary of the results is provided in Section 6.

## 2 Problem setting

Let  $\Omega$  be a convex domain in  $\mathbb{R}^d$  ( $d = 2$  or  $3$ ) with a polyhedral boundary  $\partial\Omega$ . For  $T > 0$ , we consider the problem of solving, for  $\mathbf{u} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$  and  $p : \Omega \times (0, T) \rightarrow \mathbb{R}$ , the following time-dependent Stokes problem:

$$\left\{ \begin{array}{ll} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \times (0, T), \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \times (0, T), \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 & \text{in } \Omega. \end{array} \right. \quad (1)$$

Here,  $\mathbf{f} : \Omega \times (0, T) \rightarrow \mathbb{R}^d$  stands for the source term,  $\mathbf{u}_0 : \Omega \rightarrow \mathbb{R}^d$  for the initial velocity and  $\nu > 0$  for a given constant viscosity. In order to introduce a variational setting for (1) we consider the following standard velocity and pressure spaces

$$\mathbf{V} \stackrel{\text{def}}{=} [H_0^1(\Omega)]^d, \quad \mathbf{H} \stackrel{\text{def}}{=} [L^2(\Omega)]^d, \quad Q \stackrel{\text{def}}{=} L_0^2(\Omega),$$

normed with

$$\|\mathbf{v}\|_H \stackrel{\text{def}}{=} (\mathbf{v}, \mathbf{v})^{\frac{1}{2}}, \quad \|\mathbf{v}\|_V \stackrel{\text{def}}{=} \|\nu^{\frac{1}{2}} \nabla \mathbf{v}\|_H, \quad \|q\|_Q \stackrel{\text{def}}{=} \|\nu^{-\frac{1}{2}} q\|_H,$$

where  $(\cdot, \cdot)$  denotes the standard inner product in  $L^2(\Omega)$ . The standard norm of  $H^k(\Omega)$  will be denoted by  $\|\cdot\|_k$ .

The transient Stokes' problem may be formulated in weak form as follows: For all  $t \in (0, T)$ , find  $(\mathbf{u}(t), p(t)) \in \mathbf{V} \times Q$  such that

$$\begin{cases} (\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) & \text{in } (0, T), \\ b(q, \mathbf{u}) = 0 & \text{in } (0, T), \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0 & \text{in } \Omega \end{cases} \quad (2)$$

for all  $(\mathbf{v}, q) \in \mathbf{V} \times Q$ , and with the notations

$$a(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} (\nu \nabla \mathbf{u}, \nabla \mathbf{v}), \quad b(p, \mathbf{v}) \stackrel{\text{def}}{=} -(p, \nabla \cdot \mathbf{v}) = (\nabla p, \mathbf{v}).$$

Throughout this paper,  $C$  stands for a generic positive constant independent of the discretization parameters and of  $\nu$ , but not of the mesh-geometry. We also use the notation  $a \lesssim b$  meaning  $a \leq Cb$ .

### 3 Numerical methods: space and time discretizations

In this section we fully discretize problem (2), using the pressure stabilized PSPG method in space, and a fractional-step time-marching scheme which segregates the computation of the velocity and pressure approximations.

#### 3.1 PSPG finite element semi-discrete approximation

We introduce the approximation space  $X_h$  with optimal approximation properties. The approximation space consists of finite element functions, with  $X_h \subset C^0(\bar{\Omega})$ , where

$$X_h \stackrel{\text{def}}{=} \{v_h \in H^1(\Omega) : v_h|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{T}_h\}, \quad (3)$$

with  $\mathcal{T}_h$  denoting a conforming, shape-regular triangulation of  $\Omega$ . The discrete spaces for velocities and pressure respectively are given by  $\mathbf{V}_h \stackrel{\text{def}}{=} [X_h]^d \cap V$  and  $Q_h \stackrel{\text{def}}{=} X_h \cap Q$ . We denote by  $\pi_h : \mathbf{H} \rightarrow \mathbf{V}_h$  the standard  $L^2$ -projection operator onto  $\mathbf{V}_h$ .

The PSPG finite element approximation of (2) read as follows: For all  $t > 0$ , find  $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$  such that

$$\begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \text{in } (0, T), \\ b(q_h, \mathbf{u}_h) = \delta(\partial_t \mathbf{u}_h - \nu \Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}, \nabla q_h)_h & \text{in } (0, T), \\ \mathbf{u}_h(\cdot, 0) = \mathbf{u}_{0,h} & \text{in } \Omega, \end{cases} \quad (4)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V} \times Q$ . Here,  $(\cdot, \cdot)_h$  denotes the element-wise  $L^2$ -scalar product and  $\delta > 0$  the stabilisation parameter given by

$$\delta \stackrel{\text{def}}{=} \frac{\gamma h^2}{\nu},$$

where  $\gamma > 0$  is a dimensionless algorithmic parameter (whose value will be specified in the analysis below).

The numerical analysis of the spatial semi-discrete approximation provided by (4) can be found in [5]. The case of fully discrete approximations been addressed in [4, 5] using monolithic time-stepping schemes. In the next section, we introduce a fractional-step time discretization of (4) which splits the computation of the velocity and the pressure.

### 3.2 Fully discrete method: fractional-step scheme

The fractional-step time-marching of (1) combined with finite elements and symmetric stabilization has been considered in [7, 9, 16] and analyzed in [13, 14] (see also [2, 10]). Unfortunately, nonsymmetric stabilizations such as (4) do not fit in the analysis framework therein. In this section, we propose an incremental pressure-correction fractional-step projection scheme for the time discretization of (4).

In what follows, the parameter  $\tau > 0$  denotes the time-step length and  $t_n \stackrel{\text{def}}{=} n\tau$ , for  $n = 0, \dots, N_T$ . We also introduce, for  $l \in \{1, 2\}$ , the following notations for the  $l$ -th order backward differences

$$\begin{aligned} \partial_\tau^\ell x^n &\stackrel{\text{def}}{=} \frac{(\ell + 1)x^n - 2\ell x^{n-1} + (\ell - 1)x^{n-2}}{2\tau}, \\ D_\tau^\ell \tilde{x}^n &\stackrel{\text{def}}{=} \frac{(\ell + 1)\tilde{x}^n - 2\ell \tilde{x}^{n-1} + (\ell - 1)\tilde{x}^{n-2}}{2\tau}. \end{aligned}$$

We assume that the approximations of the initial velocity and pressure,  $(\mathbf{u}_h^0, p_h^0) \in \mathbf{V}_h \times Q_h$ , are given (see Section 4.1). For  $n \geq \ell$ , with  $\ell \in \{1, 2\}$ , we propose to compute  $(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, p_h^n)$  as reported in Algorithm 1.

In the so-called pressure-Poisson problem (6), the PSPG stabilization operator  $S_h$  is defined as

$$S_h(\tilde{\mathbf{u}}_h^n, p_h^n, \mathbf{f}^n; q_h) \stackrel{\text{def}}{=} \delta(D_\tau^\ell \tilde{\mathbf{u}}_h^n - \nu \Delta \tilde{\mathbf{u}}_h^n + \nabla p_h^n - \mathbf{f}^n, \nabla q_h)_h. \quad (8)$$

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**Algorithm 1** Fractional-step scheme with PSPG stabilization.

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For  $n \geq \ell$ :

1. Find the intermediate velocity  $\tilde{\mathbf{u}}_h^n \in \mathbf{V}_h$  such that

$$(D_\tau^\ell \tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + a(\tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + b(p_h^{n-1}, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) \quad (5)$$

for all  $\mathbf{v}_h \in \mathbf{V}_h$ .

2. Find  $p_h^n \in Q_h$  such that

$$\tau(\nabla(p_h^n - p_h^{n-1}), \nabla q_h) + S_h(\tilde{\mathbf{u}}_h^n, p_h^n, \mathbf{f}^n; q_h) = \frac{\ell+1}{2} b(q_h, \tilde{\mathbf{u}}_h^n) \quad (6)$$

for all  $q_h \in Q_h$ .

3. Find the end-of-step velocity  $\mathbf{u}_h^n \in \mathbf{V}_h$  such that

$$\frac{\ell+1}{2\tau} (\mathbf{u}_h^n - \tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + (\nabla(p_h^n - p_h^{n-1}), \mathbf{v}_h) = 0 \quad (7)$$

for all  $\mathbf{v}_h \in \mathbf{V}_h$ .

---

Since step (7) can equivalently be rewritten as

$$\tilde{\mathbf{u}}_h^n = \mathbf{u}_h^n + \frac{2\tau}{\ell+1} \pi_h \nabla(p_h^n - p_h^{n-1}),$$

the PSPG stabilization operator (8) is also given by the relation

$$\begin{aligned} S_h(\tilde{\mathbf{u}}_h^n, p_h^n, \mathbf{f}^n; q_h) &= \delta(\partial_\tau^\ell \mathbf{u}_h^n - \nu \Delta \tilde{\mathbf{u}}_h^n + \nabla p_h^n - \mathbf{f}^n, \nabla q_h) \\ &\quad + \delta(\pi_h \nabla(p_h^n - p_h^{n-1}), \nabla q). \end{aligned} \quad (9)$$

This relation will be a fundamental ingredient in the numerical analysis of Section 4. The key feature of (8) is that the residual is fully computable without the need of simultaneously evaluating the unknown global projection  $\pi_h \nabla p_h^n$  (in contrast to some symmetric pressure stabilization methods [12]).

## 4 Error analysis with BDF1

This section is devoted to the energy-based error analysis of Algorithm 1 for the case  $\ell = 1$ . For any time dependent function  $x(t)$  we shall make use of the notation  $x^n \stackrel{\text{def}}{=} x(t_n)$ .

## 4.1 Preliminaries

For the convergence analysis below we introduce the following Stokes-PSPG like projection: Find  $(\mathbf{R}_h(\mathbf{u}, p), P_h(\mathbf{u}, p)) \in \mathbf{V}_h \times Q_h$ , such that

$$\begin{aligned} a(\mathbf{R}_h(\mathbf{u}, p), \mathbf{v}_h) + b(\mathbf{R}_h(\mathbf{u}, p), \mathbf{v}_h) &= a(\mathbf{u}, \mathbf{v}_h) + b(p, \mathbf{v}_h) \\ b(q_h, \mathbf{R}_h(\mathbf{u}, p)) - \delta(-\Delta \mathbf{R}_h(\mathbf{u}, p) + \nabla P_h(\mathbf{u}, p), \nabla q_h)_h &= b(q_h, \mathbf{u}) - \delta(-\Delta \mathbf{u} + \nabla p, \nabla q_h)_h \end{aligned} \quad (10)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ . It is known (see, e.g., [17]) that the above Ritz-projection satisfies the following a priori error estimate, for  $j = 0, 1$ :

$$\begin{aligned} \|\partial_t^j(\mathbf{R}_h(\mathbf{u}, p) - \mathbf{u})\|_H + h(\|\partial_t^j(\mathbf{R}_h(\mathbf{u}, p) - \mathbf{u})\|_V + \delta^{\frac{1}{2}}\|\partial_t^j \nabla(P_h(\mathbf{u}, p) - p)\|_H \\ + \nu^{-\frac{1}{2}}\|\partial_t^j(P_h(\mathbf{u}, p) - p)\|_H) \leq C_\nu h^{k+1}(\|\partial_t^j \mathbf{u}\|_{k+1} + \|\partial_t^j p\|_k), \end{aligned} \quad (11)$$

where  $C_\nu > 0$  is a constant that depends on  $\nu$ . We define  $\Pi_h : H \rightarrow Q_h$  as the standard  $L^2$ -projection onto  $Q_h$ . As an immediate consequence of (10) and of the  $H^1$ -stability of  $\Pi_h$ , we have the following stability result for the gradient of the pressure projection

$$\begin{aligned} \|\nabla P_h(\mathbf{u}, p)\|_H &\leq \|\nabla P_h(\mathbf{u}, p) - \nabla \Pi_h p\|_H + \|\nabla \Pi_h p\|_H \\ &\lesssim h^{-1}\|P_h(\mathbf{u}, p) - \Pi_h p\|_H + \|\nabla p\|_H \\ &\lesssim h^{-1}(\|P_h(\mathbf{u}, p) - p\|_H + \|p - \Pi_h p\|_H) + \|\nabla p\|_H \\ &\lesssim \|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}. \end{aligned} \quad (12)$$

We introduce the following error decomposition:

$$\begin{aligned} \mathbf{u}^n - \tilde{\mathbf{u}}_h^n &= \underbrace{\mathbf{u}^n - \mathbf{R}_h(\mathbf{u}^n, p^n)}_{\stackrel{\text{def}}{=} \boldsymbol{\theta}^n} + \underbrace{\mathbf{R}_h(\mathbf{u}^n, p^n) - \tilde{\mathbf{u}}_h^n}_{\stackrel{\text{def}}{=} \tilde{\boldsymbol{\theta}}_h^n} = \boldsymbol{\theta}^n + \tilde{\boldsymbol{\theta}}_h^n, \quad n \geq 1, \\ \mathbf{u}^n - \mathbf{u}_h^n &= \mathbf{u}^n - \mathbf{R}_h(\mathbf{u}^n, p^n) + \underbrace{\mathbf{R}_h(\mathbf{u}^n, p^n) - \mathbf{u}_h^n}_{\boldsymbol{\theta}_h^n} = \boldsymbol{\theta}^n + \boldsymbol{\theta}_h^n, \quad n \geq 0, \\ p^n - p_h^n &= \underbrace{p^n - P_h(\mathbf{u}^n, p^n)}_{\stackrel{\text{def}}{=} \xi^n} + \underbrace{P_h(\mathbf{u}^n, p^n) - p_h^n}_{\stackrel{\text{def}}{=} \xi_h^n} = \xi^n + \xi_h^n, \quad n \geq 0. \end{aligned} \quad (13)$$

Hence, owing to (10), we have

$$\begin{aligned} a(\boldsymbol{\theta}^n, \mathbf{v}_h) + b(\xi^n, \mathbf{v}_h) &= 0, \\ b(q_h, \boldsymbol{\theta}^n) - \delta(-\Delta \boldsymbol{\theta}^n + \nabla \xi^n, \nabla q_h)_h &= 0 \end{aligned} \quad (14)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$  and  $n \geq 0$ .

## 4.2 Velocity error estimate

For  $n \geq 1$ , we introduce the notations

$$\psi_h^{n-1} \stackrel{\text{def}}{=} \xi_h^{n-1} + \tau \partial_\tau P_h(\mathbf{u}^n, p^n), \quad \mathbf{g}^n \stackrel{\text{def}}{=} \partial_\tau \mathbf{u}^n - \partial_t \mathbf{u}^n - \partial_\tau \boldsymbol{\theta}^n. \quad (15)$$

By subtracting (5)-(6) from (2) with  $t = t_n$  and then by using (13)-(14), we get the following relations for  $(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n, \xi_h^n)$ :

$$\frac{1}{\tau} (\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}, \mathbf{v}_h) + a(\tilde{\boldsymbol{\theta}}_h^n, \mathbf{v}_h) + b(\psi_h^{n-1}, \mathbf{v}_h) = (\mathbf{g}^n, \mathbf{v}_h), \quad (16)$$

$$\tau (\nabla(\xi_h^n - \psi_h^{n-1}), \nabla q_h) + S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; q_h) = b(q_h, \tilde{\boldsymbol{\theta}}_h^n), \quad (17)$$

$$\frac{1}{\tau} (\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n, \mathbf{v}_h) + (\nabla(\xi_h^n - \psi_h^{n-1}), \mathbf{v}_h) = 0 \quad (18)$$

for  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ . In other words, the discrete errors  $(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n, \xi_h^n)$  satisfy a system similar to the approximations  $(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, p_h^n)$  but with the momentum right-hand side  $\mathbf{g}^n$  instead of  $\mathbf{f}^n$  and the pressure extrapolation  $\psi_h^{n-1}$  instead of  $p_h^{n-1}$ .

The following theorem provides an energy estimate for  $(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n, \xi_h^n)$ .

**Theorem 1** (Stability velocity) *Assume that  $\gamma \leq C_i/16$  and let  $\{(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n, \xi_h^n)\}_{n \geq 1}$  be discrete errors satisfying the relations (16)–(18) with the initial data  $\boldsymbol{\theta}_h^0, \xi_h^0$ . Then, the following estimate holds unconditionally if  $k = 1$  and, under the condition  $\delta \leq \tau$ , if  $k \geq 2$ :*

$$\begin{aligned} & \|\boldsymbol{\theta}_h^n\|_H^2 + \beta\delta \|\boldsymbol{\theta}_h^n\|_V^2 + \tau^2 \|\nabla \xi_h^n\|_H^2 + \tau\delta \|\pi_h \nabla \xi_h^n\|_H^2 + \tau \sum_{m=1}^n \|\tilde{\boldsymbol{\theta}}_h^m\|_V^2 + \tau\delta \sum_{m=1}^n \|\partial_\tau \boldsymbol{\theta}_h^m + \nabla \xi_h^m\|_H^2 \\ & + \frac{\delta}{\tau} \sum_{m=1}^n \|\boldsymbol{\theta}_h^m - \tilde{\boldsymbol{\theta}}_h^m\|_H^2 \lesssim \tau \left( \frac{C_P^2}{\nu} + \delta \right) \sum_{m=1}^n \|\mathbf{g}^m\|_H^2 + \tau^3 (\tau + T) \sum_{m=1}^n \|\nabla \partial_\tau P_h(\mathbf{u}^m, p^m)\|_H^2 \\ & \quad + \|\boldsymbol{\theta}_h^0\|_H^2 + \beta\delta \|\boldsymbol{\theta}_h^0\|_V^2 + \tau^2 \|\nabla \xi_h^0\|_H^2 + \tau\delta \|\pi_h \nabla \xi_h^0\|_H^2 \end{aligned} \quad (19)$$

for  $1 \leq n \leq N_T$ . Here,  $C_i, C_P > 0$  denote the constants in the inverse and Poincaré inequalities, respectively, and

$$\beta \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \delta \leq \tau, \\ 1 & \text{if } \delta > \tau. \end{cases}$$

Moreover, for  $n = 1$ , we also have

$$\begin{aligned} & \|\boldsymbol{\theta}_h^1\|_H^2 + \beta\delta \|\boldsymbol{\theta}_h^1\|_V^2 + \tau^2 \|\nabla \xi_h^1\|_H^2 + \tau\delta \|\pi_h \nabla \xi_h^1\|_H^2 + \tau \|\tilde{\boldsymbol{\theta}}_h^1\|_V^2 + \tau\delta \|\partial_\tau \boldsymbol{\theta}_h^1 + \nabla \xi_h^1\|_H^2 + \frac{\delta}{\tau} \|\boldsymbol{\theta}_h^1 - \tilde{\boldsymbol{\theta}}_h^1\|_H^2 \\ & \lesssim \tau(\tau + \delta) \|\mathbf{g}^1\|_H^2 + \tau^4 \|\nabla \partial_\tau P_h(\mathbf{u}^1, p^1)\|_H^2 + \|\boldsymbol{\theta}_h^0\|_H^2 + \beta\delta \|\boldsymbol{\theta}_h^0\|_V^2 + \tau^2 \|\nabla \xi_h^0\|_H^2 + \tau\delta \|\pi_h \nabla \xi_h^0\|_H^2. \end{aligned} \quad (20)$$

*Proof* The first part of the proof follows the steps of [8, Theorem 5.5] (see also [13, Theorem 4.1]). We start by taking  $\mathbf{v}_h = \tau \tilde{\boldsymbol{\theta}}_h^n$  in (16), which yields

$$\frac{1}{2} (\|\tilde{\boldsymbol{\theta}}_h^n\|_H^2 - \|\boldsymbol{\theta}_h^{n-1}\|_H^2 + \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2) + \tau \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 + \tau (\nabla \psi_h^{n-1}, \tilde{\mathbf{u}}_h^n) = \tau (\mathbf{g}^n, \tilde{\boldsymbol{\theta}}_h^n). \quad (21)$$



In order to control the last term of the right-hand side, we take  $q_h = \tau\psi_h^{n-1}$  in (17), which gives

$$\tau(\nabla\psi_h^{n-1}, \tilde{\boldsymbol{\theta}}_h^n) = \frac{\tau^2}{2} \left( \|\nabla\xi_h^n\|_H^2 - \|\nabla\psi_h^{n-1}\|_H^2 - \|\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 \right) + \tau S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; \psi_h^{n-1}). \quad (22)$$

Now, owing to (18), we have the identity

$$\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n = -\tau\pi_h\nabla(\xi_h^n - \psi_h^{n-1}), \quad (23)$$

so that, taking  $\mathbf{v}_h = \tau\boldsymbol{\theta}_h^n$  in (18) yields

$$\frac{1}{2} (\|\boldsymbol{\theta}_h^n\|_H^2 - \|\tilde{\boldsymbol{\theta}}_h^n\|_H^2 + \tau^2 \|\pi_h\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2) + \tau(\nabla(\xi_h^n - \psi_h^{n-1}), \boldsymbol{\theta}_h^n) = 0. \quad (24)$$

On the other hand, by taking  $\mathbf{v}_h = \tau\pi_h\nabla(\xi_h^n - \psi_h^{n-1})$  in (18), we have

$$\tau(\tilde{\boldsymbol{\theta}}_h^n, \nabla(\xi_h^n - \psi_h^{n-1})) = \tau(\boldsymbol{\theta}_h^n, \nabla(\xi_h^n - \psi_h^{n-1})) + \tau^2 \|\pi_h\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 \quad (25)$$

and  $q_h = \tau(\xi_h^n - \psi_h^{n-1})$  in (17) yields

$$\tau^2 \|\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 + \tau S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; \xi_h^n - \psi_h^{n-1}) = \tau(\tilde{\boldsymbol{\theta}}_h^n, \nabla(\xi_h^n - \psi_h^{n-1})).$$

Hence, by inserting this last identity into (25), we get

$$\tau(\boldsymbol{\theta}_h^n, \nabla(\xi_h^n - \psi_h^{n-1})) = \tau^2 \|(I - \pi_h)\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 + \tau S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; \xi_h^n - \psi_h^{n-1}). \quad (26)$$

Owing to (26), the relation (24) yields

$$\begin{aligned} & \frac{1}{2} (\|\boldsymbol{\theta}_h^n\|_H^2 - \|\tilde{\boldsymbol{\theta}}_h^n\|_H^2 + \tau^2 \|\pi_h\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2) + \tau^2 \|(I - \pi_h)\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 \\ & + \tau S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; \xi_h^n - \psi_h^{n-1}) = 0. \end{aligned} \quad (27)$$

At last, by inserting (22) into (21) and by adding (27) to the resulting expression, we get the velocity energy estimate

$$\begin{aligned} & \frac{1}{2} (\|\boldsymbol{\theta}_h^n\|_H^2 - \|\boldsymbol{\theta}_h^{n-1}\|_H^2) + \frac{1}{2} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2 + \tau \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 \\ & + \frac{\tau^2}{2} (\|\nabla\xi_h^n\|_H^2 - \|\nabla\psi_h^{n-1}\|_H^2) + \frac{\tau^2}{2} \|(I - \pi_h)\nabla(\xi_h^n - \psi_h^{n-1})\|_H^2 \\ & + \underbrace{\tau S_h(\tilde{\boldsymbol{\theta}}_h^n, \xi_h^n, \mathbf{g}^n; \xi_h^n)}_{\stackrel{\text{def}}{=} T_1} = \tau(\mathbf{g}^n, \tilde{\boldsymbol{\theta}}_h^n). \end{aligned} \quad (28)$$

It should be noted the lack of telescoping sum on  $\|\nabla\xi_h^n\|_H^2 - \|\nabla\psi_h^{n-1}\|_H^2$  is not an issue here since, owing to the definition of  $\psi_h^{n-1}$  in (15), we have

$$\|\nabla\psi_h^{n-1}\|_H^2 \leq (1 + \tau/T) \|\nabla\xi_h^{n-1}\|_H^2 + (1 + T/\tau)\tau^2 \|\nabla\partial_\tau P_h(\mathbf{u}^n, p^n)\|_H^2, \quad (29)$$

so that this term can be treated via Gronwall's Lemma (see, e.g., [18, Lemma 5.1]).

The main difficulty lies in the estimation of the term coming from the PSPG stabilization, namely, the term  $T_1$  in (28). This is the fundamental contribution of the present analysis, notably with respect to [8, 13, 14].

We first note that, from the definition of  $S_h$  in (8) and from (23), we have

$$\begin{aligned} T_1 &= \tau\delta(\partial_\tau\boldsymbol{\theta}_h^n + \nabla\xi_h^n, \nabla\xi_h^n) - \tau\delta(\nu\Delta\tilde{\boldsymbol{\theta}}_h^n + \mathbf{g}^n, \nabla\xi_h^n)_h \\ &+ \frac{\tau\delta}{2} \left( \|\pi_h\nabla\xi_h^n\|_H^2 - \|\pi_h\nabla\psi_h^{n-1}\|_H^2 \right) + \frac{\delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2. \end{aligned} \quad (30)$$

On the other hand, the addition of (16) and (18) yields

$$(\partial_\tau \boldsymbol{\theta}_h^n, \mathbf{v}_h) + a(\tilde{\boldsymbol{\theta}}_h^n, \mathbf{v}_h) + (\nabla \xi_h^n, \mathbf{v}_h) = (\mathbf{g}^n, \mathbf{v}_h) \quad (31)$$

for all  $\mathbf{v}_h \in \mathbf{V}_h$ . By taking  $\mathbf{v}_h = \tau \delta \partial_\tau \boldsymbol{\theta}_h^n$  in this expression, we get

$$0 = \tau \delta (\partial_\tau \boldsymbol{\theta}_h^n + \nabla \xi_h^n, \partial_\tau \boldsymbol{\theta}_h^n) + \tau \delta a(\tilde{\boldsymbol{\theta}}_h^n, \partial_\tau \boldsymbol{\theta}_h^n) - \tau \delta (\mathbf{g}^n, \partial_\tau \boldsymbol{\theta}_h^n).$$

Hence, by adding this relation to (30) and by adding and subtracting suitable terms, there follows that

$$\begin{aligned} T_1 &= \tau \delta \|\partial_\tau \boldsymbol{\theta}_h^n + \nabla \xi_h^n\|_H^2 + \frac{\tau \delta}{2} \left( \|\pi_h \nabla \xi_h^n\|_H^2 - \|\pi_h \nabla \psi_h^{n-1}\|_H^2 \right) \\ &\underbrace{- \tau \delta (\nu \Delta \tilde{\boldsymbol{\theta}}_h^n + \mathbf{g}^n, \nabla \xi_h^n + \partial_\tau \boldsymbol{\theta}_h^n)}_{\stackrel{\text{def}}{=} T_2} + \underbrace{\delta \nu (\Delta \tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n - \boldsymbol{\theta}_h^{n-1})_h + \delta a(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n - \boldsymbol{\theta}_h^{n-1}) + \frac{\delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2}_{\stackrel{\text{def}}{=} T_3}, \end{aligned} \quad (32)$$

where the second term of the left-hand side is treated via Gronwall's Lemma using an argument similar to (29). The term  $T_2$  is standard and can be controlled via the PSPG numerical dissipation provided by the term  $\tau \delta \|\partial_\tau \boldsymbol{\theta}_h^n + \nabla \xi_h^n\|_H^2$  in (32), viz.,

$$T_2 \geq -\frac{\gamma \tau}{2\epsilon_1 C_i} \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \frac{\tau \delta}{2\epsilon_1} \|\mathbf{g}^n\|_H^2 - \epsilon_1 \tau \delta \|\partial_\tau \boldsymbol{\theta}_h^n + \nabla \xi_h^n\|_H^2. \quad (33)$$

with  $\epsilon_1 > 0$ . The estimation of the term  $T_3$  is more delicate, we will treat separately the cases  $\delta \leq \tau$  and  $\tau < \delta$ .

We first consider the case  $\delta \leq \tau$ . By adding and subtracting suitable terms, we have

$$\begin{aligned} T_3 &= \underbrace{\delta \nu (\Delta \tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n)_h + \delta \nu (\Delta \tilde{\boldsymbol{\theta}}_h^n, \tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1})_h}_{\stackrel{\text{def}}{=} T_{3,1}} + \underbrace{\delta a(\tilde{\boldsymbol{\theta}}_h^n, \boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n) + \delta a(\tilde{\boldsymbol{\theta}}_h^n, \tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1})}_{\stackrel{\text{def}}{=} T_{3,2}} \\ &\quad + \frac{\delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2. \end{aligned} \quad (34)$$

Using an inverse inequality and the assumption  $\delta \leq \tau$ , we get

$$T_{3,1} \geq -\frac{\gamma \tau}{\epsilon_2 C_i} \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \frac{\epsilon_2 \delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 - \frac{\epsilon_2}{2} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2,$$

with  $\epsilon_2 > 0$ . Similarly, for  $T_{3,2}$ , we have

$$\begin{aligned} T_{3,2} &\geq -\frac{\epsilon_3 \tau}{2} \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \frac{\delta^2 \nu}{2\epsilon_3 \tau} \|\nabla(\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n)\|_H^2 - \frac{\epsilon_3 \delta}{2} \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \frac{\delta \nu}{2\epsilon_3} \|\nabla(\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1})\|_H^2 \\ &\geq -\epsilon_3 \tau \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \frac{\gamma}{2\epsilon_3 C_i} \frac{\delta}{\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 - \frac{\gamma}{2\epsilon_3 C_i} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2, \end{aligned}$$

with  $\epsilon_3 > 0$ . Therefore, by collecting these two estimates into (34), we have

$$T_3 \geq -\tau \left( \frac{\gamma}{\epsilon_2 C_i} + \epsilon_3 \right) \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 + \left( 1 - \epsilon_2 - \frac{\gamma}{\epsilon_3 C_i} \right) \frac{\delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 - \left( \epsilon_2 + \frac{\gamma}{\epsilon_3 C_i} \right) \frac{1}{2} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2. \quad (35)$$

where the first and third terms can be controlled via the physical and numerical dissipation provided by the terms  $\frac{1}{2} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_H^2 + \tau \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2$  in (28). Indeed, it suffices to take  $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1/4$  and  $\gamma \leq C_i/16$ .

The energy estimate (19) with  $\beta = 0$  hence follows by inserting (33) and (35) into (28) and by summing over  $m = 1, \dots, n$ . It should be noted that this a priori energy bound is valid for all polynomial order  $k \geq 1$  under the condition  $\delta \leq \tau$ . This completes the proof.

We now consider the case  $\tau < \delta$  and  $k = 1$ . Since the local Laplacian contributions vanish, we have  $T_{3,1} = 0$ . The term  $T_3 = T_{3,2}$  can hence be estimated as follows, using an inverse inequality and the assumption  $\tau < \delta$ :

$$\begin{aligned} T_3 &= \frac{\delta}{2} \left( \|\boldsymbol{\theta}_h^n\|_V^2 - \|\tilde{\boldsymbol{\theta}}_h^n\|_H^2 - \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^n\|_V^2 \right) \\ &\quad + \frac{\delta}{2} \left( \|\tilde{\boldsymbol{\theta}}_h^n\|_V^2 - \|\boldsymbol{\theta}_h^{n-1}\|_V^2 + \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_V^2 \right) + \frac{\delta}{2\tau} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 \\ &> -\frac{\delta}{2} \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^n\|_V^2 + \frac{\delta}{2} (\|\boldsymbol{\theta}_h^n\|_V^2 - \|\boldsymbol{\theta}_h^{n-1}\|_V^2 + \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_V^2) + \frac{1}{2} \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 \\ &\geq \left( \frac{1}{2} - \frac{\gamma}{2C_1} \right) \|\boldsymbol{\theta}_h^n - \tilde{\boldsymbol{\theta}}_h^n\|_H^2 + \frac{\delta}{2} (\|\boldsymbol{\theta}_h^n\|_V^2 - \|\boldsymbol{\theta}_h^{n-1}\|_V^2 + \|\tilde{\boldsymbol{\theta}}_h^n - \boldsymbol{\theta}_h^{n-1}\|_V^2). \end{aligned} \quad (36)$$

The energy estimate (19) with  $\beta = 1$  then follows by applying (33) and (36) to (28) and by summing over  $m = 1, \dots, n$ .

The energy estimate (20) is derived using similar arguments. The sole difference is that, since we consider a single step  $n = 1$ , we do not need to rely on telescoping or Gronwall arguments. More precisely, instead of (21), we have

$$\|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \|\tilde{\boldsymbol{\theta}}_h^1 - \boldsymbol{\theta}_h^0\|_H^2 + \tau \|\tilde{\boldsymbol{\theta}}_h^1\|_V^2 \lesssim \|\boldsymbol{\theta}_h^0\|_H^2 + \tau^2 \left( \|\mathbf{g}^1\|_H^2 + \|\nabla \psi_h^0\|_H^2 \right), \quad (37)$$

where, for the last term (instead of (29)), we simply have

$$\|\nabla \psi_h^0\|_H^2 \lesssim \|\nabla \xi_h^0\|_H^2 + \tau^2 \|\nabla \partial_\tau P_h(\mathbf{u}^1, p^1)\|_H^2.$$

Taking  $\mathbf{v}_h = \tau \boldsymbol{\theta}_h^1$  in (18) with  $n = 1$ , yields

$$\frac{1}{2} \|\boldsymbol{\theta}_h^1\|_H^2 + \frac{\tau^2}{2} \|\pi_h \nabla(\xi_h^1 - \psi_h^0)\|_H^2 + \tau (\nabla \xi_h^1, \boldsymbol{\theta}_h^1) = \frac{1}{2} \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \tau (\nabla \psi_h^0, \boldsymbol{\theta}_h^1), \quad (38)$$

On the other hand, by taking  $\mathbf{v}_h = \tau \pi_h \nabla \xi_h^1$  in (18) with  $n = 1$ , we have

$$\tau (\tilde{\boldsymbol{\theta}}_h^1, \nabla \xi_h^1) = \tau (\boldsymbol{\theta}_h^1, \nabla \xi_h^1) + \tau^2 (\pi_h \nabla(\xi_h^1 - \psi_h^0), \nabla \xi_h^1)$$

and  $q_h = \tau \xi_h^1$  in (17) with  $n = 1$  yields

$$\tau^2 (\nabla(\xi_h^1 - \psi_h^0), \nabla \xi_h^1) + \tau S_h(\tilde{\boldsymbol{\theta}}_h^1, \xi_h^1, \mathbf{g}^1; \xi_h^1) = \tau (\tilde{\boldsymbol{\theta}}_h^1, \nabla \xi_h^1). \quad (39)$$

Therefore,

$$\tau (\boldsymbol{\theta}_h^1, \nabla \xi_h^1) = \tau^2 ((I - \pi_h) \nabla(\xi_h^1 - \psi_h^0), \nabla \xi_h^1) + \tau S_h(\tilde{\boldsymbol{\theta}}_h^1, \xi_h^1, \mathbf{g}^1; \xi_h^1).$$

By inserting this expression into (38), we get the relation

$$\begin{aligned} &\frac{1}{2} \|\boldsymbol{\theta}_h^1\|_H^2 + \frac{\tau^2}{2} \|\pi_h \nabla(\xi_h^1 - \psi_h^0)\|_H^2 + \tau^2 ((I - \pi_h) \nabla(\xi_h^1 - \psi_h^0), \nabla \xi_h^1) + \tau S_h(\tilde{\boldsymbol{\theta}}_h^1, \xi_h^1, \mathbf{g}^1; \xi_h^1) \\ &\leq \frac{1}{2} \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \tau^2 \|\nabla \psi_h^0\|_H^2. \end{aligned}$$

The estimate (20) then follows from (37) and by applying the same arguments as above to the PSPG stabilization term  $\tau S_h(\tilde{\boldsymbol{\theta}}_h^1, \xi_h^1, \mathbf{g}^1; \xi_h^1)$ .

We now assume that  $\tau \leq \delta$  and  $k = 1$ . We start by multiplying (39) by  $1/(\tau\delta)$ , which yields

$$\frac{\tau}{\delta} (\nabla(\xi_h^1 - \psi_h^0), \nabla \xi_h^1) + (D_\tau \tilde{\boldsymbol{\theta}}_h^1 + \nabla \xi_h^1 - \mathbf{g}^1, \nabla \xi_h^1) = \frac{1}{\delta} (\tilde{\boldsymbol{\theta}}_h^1, \nabla \xi_h^1).$$

We then have

$$\frac{\tau}{2\delta} \|\nabla \xi_h^1\|_H^2 + \frac{1}{2} \|\nabla \xi_h^1\|_H^2 \leq \frac{1}{2\tau^2} \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \frac{\tau^2}{2\delta^2} \|\nabla \xi_h^1\|_H^2 + \frac{\tau}{2\delta} \|\nabla \psi_h^0\|_H^2 + \frac{1}{2} \|D_\tau \tilde{\boldsymbol{\theta}}_h^1 - \mathbf{g}^1\|_H^2.$$

As a result,

$$\frac{\tau}{2\delta} \left(1 - \frac{\tau}{\delta}\right) \|\nabla \xi_h^1\|_H^2 + \frac{1}{2} \|\nabla \xi_h^1\|_H^2 \leq \frac{1}{2\tau^2} \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \frac{\tau}{2\delta} \|\nabla \psi_h^0\|_H^2 + \frac{1}{2} \|D_\tau \tilde{\boldsymbol{\theta}}_h^1 - \mathbf{g}^1\|_H^2.$$

Using the assumption  $\tau \leq \delta$ , we hence have

$$\|\nabla \xi_h^1\|_H^2 \lesssim \frac{1}{\tau^2} \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \|\nabla \psi_h^0\|_H^2 + \|D_\tau \tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \|\mathbf{g}^1\|_H^2.$$

On the other hand, from (38) we infer that

$$\|\boldsymbol{\theta}_h^1\|_H^2 + \frac{\tau^2}{2} \|\pi_h \nabla (\xi_h^1 - \psi_h^0)\|_H^2 \lesssim \|\tilde{\boldsymbol{\theta}}_h^1\|_H^2 + \tau^2 (\|\nabla \psi_h^0\|_H^2 + \|\nabla \xi_h^1\|_H^2).$$

□

In order to derive an error estimate for the velocity in the energy norm, we assume that the discrete initial data  $\mathbf{u}_h^0, p_h^0$  are generated in such a way that

$$\|\boldsymbol{\theta}_h^0\|_H \lesssim C_{\nu, \mathbf{u}_0} h^{k+1}, \quad \|\boldsymbol{\theta}_h^0\|_V \lesssim C_{\nu, \mathbf{u}_0} h^k, \quad \|\nabla \xi_h^0\|_H \lesssim C_{\nu, \mathbf{u}_0}. \quad (40)$$

**Corollary 2** (Velocity error estimate) *Let  $\{(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, p_h^n)\}_{n \geq 1}$  be given by Algorithm 1 initialized in such a way that (40) holds. Under the assumptions of Theorem 1, we have the following error estimates:*

$$\begin{aligned} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^\infty(0, T; H)} &\lesssim C_{\nu, \mathbf{u}, p} (\tau + h^{k+1}), \\ \|\mathbf{u}^n - \tilde{\mathbf{u}}_h^n\|_{L^2(0, T; V)} &\lesssim C_{\nu, \mathbf{u}, p} (\tau + h^k). \end{aligned} \quad (41)$$

*Proof* It suffices to note that from (11), (15) and using Taylor expansions, in (19) we have

$$\|\mathbf{g}^m\|_H^2 \leq \tau \|\partial_{tt} \mathbf{u}\|_{L^2(t_{m-1}, t_m; L^2(\Omega))}^2 + \tau^{-1} h^{2k+2} \|\partial_t \mathbf{u}\|_{L^2(t_{m-1}, t_m; H^{k+1}(\Omega))}^2$$

and similarly, using (12),

$$\begin{aligned} \|\nabla \partial_\tau P_h(\mathbf{u}^m, p^m)\|_H^2 &= \|\nabla P_h(\partial_\tau \mathbf{u}^m, \partial_\tau p^m)\|_H^2 \\ &\lesssim \|\partial_\tau \mathbf{u}^m\|_{2, \Omega}^2 + \|\partial_\tau p^m\|_{1, \Omega}^2 \\ &\leq \tau^{-1} \left( \|\partial_t \mathbf{u}\|_{L^2(t_{m-1}, t_m; H^2(\Omega))}^2 + \|\partial_t p\|_{L^2(t_{m-1}, t_m; H^1(\Omega))}^2 \right). \end{aligned}$$

The desired error estimates (41) then follow in a standard fashion, from (13), (11), (19) and (40). □

Some observations are in order. Theorem 1 guarantees the unconditional convergence of the velocity approximations provided by Algorithm 1 for  $k = 1$ . For  $k \geq 2$  convergence is obtained under the condition  $\delta \leq \tau$ . Similar results were obtained in [4, Lemma 1] with a monolithic BDF1 time-stepping scheme. Unconditional error estimates for the monolithic scheme with  $k \geq 2$  has been reported in [5] (using a specific construction of the initial velocity approximation  $\mathbf{u}_{0, h}$ ), but the arguments therein do not seem to fit the time-marching framework of Algorithm 1.

### 4.3 Pressure error estimate

In what follows, we assume that Algorithm 1 is initialized with

$$\mathbf{u}_h^0 = \mathbf{R}_h(\mathbf{u}^0, p^0), \quad p_h^0 = P_h(\mathbf{u}^0, p^0), \quad (42)$$

where the initial pressure  $p^0$  is determined (up to a constant) from (1) at  $t = 0$ , by solving the following elliptic problem (see, e.g., [5, 19]):

$$\begin{cases} -\Delta p^0 = -\nabla \cdot \mathbf{f}^0 & \text{in } \Omega, \\ \nabla p^0 \cdot \mathbf{n} = (\mathbf{f}^0 + \nu \Delta \mathbf{u}^0) \cdot \mathbf{n} & \text{on } \partial\Omega. \end{cases}$$

It should be noted that this initialisation procedure yields

$$\boldsymbol{\theta}_h^0 = \mathbf{0}, \quad \xi_h^0 = 0, \quad (43)$$

so that the assumption (40) is fulfilled.

The a priori analysis of the pressure error builds on the following modified inf-sup condition.

**Lemma 3** For all  $\boldsymbol{\xi}_h \in \mathbf{V}_h$  and  $q_h \in Q_h$  we have

$$\|q_h\|_Q \lesssim \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{|(q_h, \nabla \cdot \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{\mathbf{V}}} + (\delta/\gamma)^{\frac{1}{2}} \|\nabla q_h + \boldsymbol{\xi}_h\|_H. \quad (44)$$

*Proof* See [4, Lemma 3]. □

Owing to (44) and (31), we have

$$\begin{aligned} \tau \sum_{m=2}^n \|\xi_h^m\|_Q^2 &\lesssim \delta \tau \sum_{m=2}^n \|\nabla \xi_h^m + \partial_\tau \boldsymbol{\theta}_h^m\|_H^2 + \tau \sum_{m=2}^n \|\boldsymbol{\theta}_h^m\|_{\mathbf{V}}^2 \\ &\quad + \frac{C_P^2 \tau}{\nu} \sum_{m=2}^n \|\mathbf{g}^m\|_H^2 + \frac{C_P^2 \tau}{\nu} \sum_{m=2}^n \|\partial_\tau \boldsymbol{\theta}_h^m\|_H^2. \end{aligned} \quad (45)$$

We hence need to estimate the acceleration error

$$\frac{\tau}{\nu} \sum_{m=2}^n \|\partial_\tau \boldsymbol{\theta}_h^m\|_H^2.$$

This is the purpose of the next result.

**Theorem 4** (Stability acceleration) *Under the assumptions of Theorem 1 and with the initialisation procedure (42), the following estimate holds:*

$$\|\partial_\tau \boldsymbol{\theta}_h^n\|_{l^\infty(0,T;H)} + \|\partial_\tau \tilde{\boldsymbol{\theta}}_h^n\|_{l^2(0,T;V)} \lesssim \left(1 + \sqrt{\frac{\delta}{\tau}}\right) (\tau + h^k).$$

*Proof* By introducing the notations

$$\mathbf{y}_h^n \stackrel{\text{def}}{=} \partial_\tau \boldsymbol{\theta}_h^n, \quad \tilde{\mathbf{y}}_h^n \stackrel{\text{def}}{=} \partial_\tau \tilde{\boldsymbol{\theta}}_h^n, \quad s_h^n \stackrel{\text{def}}{=} \partial_\tau \xi_h^n, \quad r_h^n \stackrel{\text{def}}{=} \partial_\tau \psi_h^n, \quad (46)$$

and by applying the operator  $\partial_\tau$  to (16)–(18) for  $n \geq 2$ , we get that

$$\frac{1}{\tau} (\tilde{\mathbf{y}}_h^n - \mathbf{y}_h^{n-1}, \mathbf{v}_h) + a(\tilde{\mathbf{y}}_h^n, \mathbf{v}_h) + b(r_h^{n-1}, \mathbf{v}_h) = (\partial_\tau \mathbf{g}^n, \mathbf{v}_h), \quad (47)$$

$$\tau (\nabla (s_h^n - r_h^{n-1}), \nabla q_h) + S_h(\tilde{\mathbf{y}}_h^n, s_h^n, \partial_\tau \mathbf{g}^n; q_h) = b(q_h, \tilde{\mathbf{y}}_h^n), \quad (48)$$

$$\frac{1}{\tau} (\mathbf{y}_h^n - \tilde{\mathbf{y}}_h^n, \mathbf{v}_h) + (\nabla (s_h^n - r_h^{n-1}), \mathbf{v}_h) = 0 \quad (49)$$

for all  $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times Q_h$ . We also note that, from the definition (15), we have

$$\|\nabla r_h^{n-1}\|_H^2 \leq (1 + \tau/T) \|\nabla s_h^{n-1}\|_H^2 + (1 + T/\tau) \tau^2 \|\nabla \partial_\tau P_h(\partial_\tau \mathbf{u}^n, \partial_\tau p^n)\|_H^2.$$

Therefore, by applying to (47)–(49) the same arguments as in the proof of Theorem 1, the following estimate is inferred

$$\begin{aligned} & \|\mathbf{y}_h^n\|_H^2 + \tau \sum_{m=2}^n \|\tilde{\mathbf{y}}_h^m\|_V^2 + \tau \delta \sum_{m=2}^n \|\partial_\tau \mathbf{y}_h^m + \nabla s_h^m\|_H^2 \\ & \lesssim \tau \left( \frac{C_P^2}{\nu} + \delta \right) \sum_{m=2}^n \|\partial_\tau \mathbf{g}^m\|_{\Omega}^2 + \tau^3 (\tau + T) \sum_{m=2}^n \|\nabla \partial_\tau P_h(\partial_\tau \mathbf{u}^m, \partial_\tau p^m)\|_H^2 \\ & \quad + \underbrace{\|\mathbf{y}_h^1\|_H^2 + \beta \delta \|\mathbf{y}_h^1\|_V^2 + \tau^2 \|\nabla s_h^1\|_H^2 + \tau \delta \|\pi_h \nabla s_h^1\|_H^2}_{\stackrel{\text{def}}{=} T_1}. \end{aligned} \quad (50)$$

The last terms, corresponding to  $n = 1$ , can be bounded as follows. From (43) and (46), we have

$$T_1 = \tau^{-2} \left( \|\boldsymbol{\theta}_h^1\|_H^2 + \beta \delta \|\boldsymbol{\theta}_h^1\|_V^2 + \tau^2 \|\nabla \xi_h^1\|_H^2 + \tau \delta \|\pi_h \nabla \xi_h^1\|_H^2 \right).$$

so that, owing to (20) and (43), we get

$$T_1 \lesssim \left(1 + \frac{\delta}{\tau}\right) \|\mathbf{g}^1\|_H^2 + \tau^2 \|\nabla \partial_\tau P_h(\mathbf{u}^1, p^1)\|_H^2.$$

This completes the proof.  $\square$

**Corollary 5** (Pressure error estimate) *Let  $\{(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, p_h^n)\}_{n \geq 1}$  be given by Algorithm 1 initialized with (42). Under the assumptions of Theorem 1, the following error estimate holds:*

$$\|p^n - p_h^n\|_{l^2(0,T;Q)} \lesssim C_{\nu, \mathbf{u}, p} \left(1 + \sqrt{\frac{\delta}{\tau}}\right) (\tau + h^k). \quad (51)$$

## 5 Numerical experiment

The main purpose of this section is to provide numerical evidence on the results of Corollaries 2 and 5 for the BDF1 variant of Algorithm 1 ( $\ell = 1$ ), and for different values of the polynomial degree  $k \in \{1, 2, 3\}$ . We also assess numerically the convergence properties of the BDF2 variant ( $\ell = 2$ ) of Algorithm 1, though this case is not covered by the theory of Section 4.

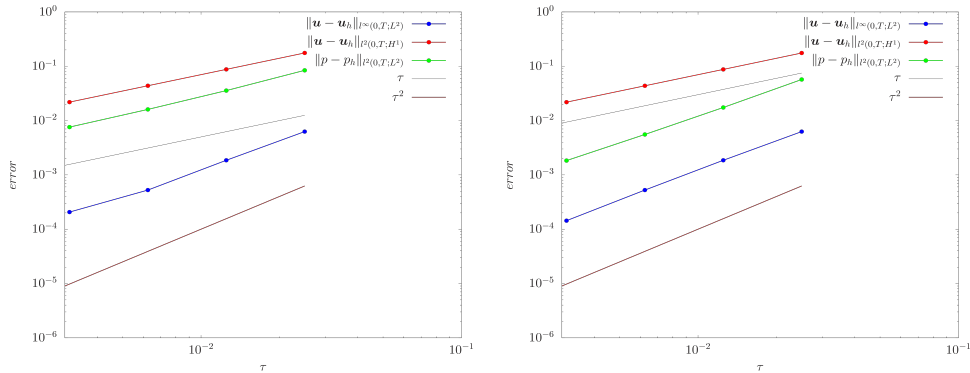
We consider problem (1) with  $\Omega = (0, 1)^2$  and where the source  $\mathbf{f}$  and the boundary conditions are chosen such that the exact solution is given by

$$\mathbf{u} = \cos(2\pi t) \begin{pmatrix} \sin(\pi x - 0.7) \sin(\pi y + 0.2) \\ \cos(\pi x - 0.7) \cos(\pi y + 0.2) \end{pmatrix},$$

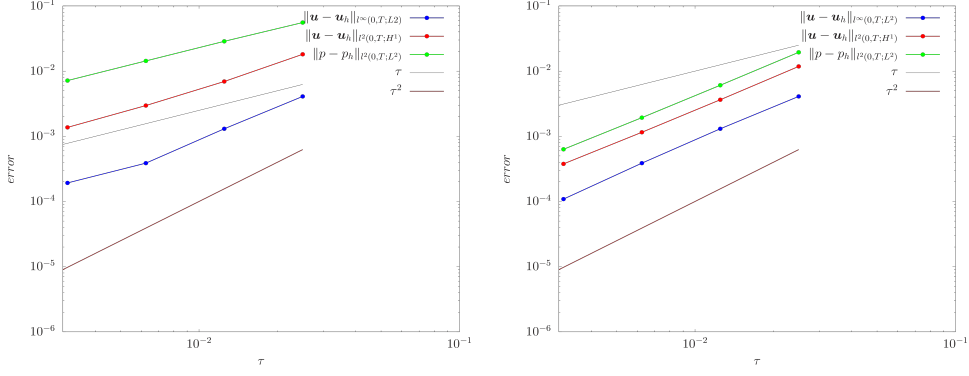
$$p = \cos(2\pi t)(\sin(x) \cos(y) + (\cos(1) - 1) \sin(1)).$$

Figures 1–3 report the convergence histories obtained with the two variants of Algorithm 1 ( $\ell = 1, 2$ ) and a finite element approximation in space based on continuous piece-wise polynomials of order  $k = 1, 2$  and 3, respectively. All the numerical computations have been performed using FreeFem++ [20], with a similar rate of refinement in space and in time, namely,  $\tau = \mathcal{O}(h)$ .

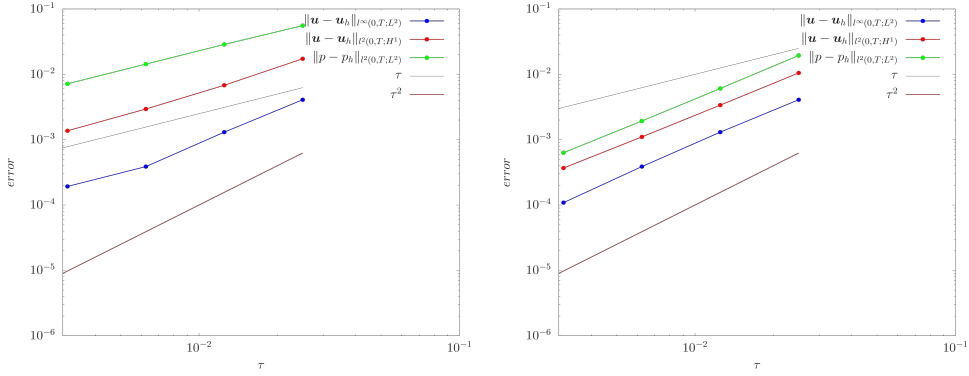
The results of Figures 1–3 with  $\ell = 1$  (left) are in agreement with the convergence rates obtained in Section 4. We can clearly observe that the global convergence rate is limited by the first-order accuracy of the time discretization. It is also worth noting that, for  $k = 2$  and  $k = 3$ , optimal accuracy is obtained without the need of the inverse CFL-condition  $\delta \leq \tau$ . In Figures 1–3 with  $\ell = 2$  we can notice the benefit of the second-order time-stepping, which yields an overall second order accuracy for  $k = 2, 3$ .



**Fig. 1** Simulation using  $\mathbb{P}^1/\mathbb{P}^1$ ,  $\tau = 0.05/2^n$ ,  $h = 0.1/2^n$ ,  $\delta = 0.01h^2$ , with  $n = 1, 2, 3, 4$ . Algorithm 1 with  $\ell = 1$  (left) and  $\ell = 2$  (right).



**Fig. 2** Simulation using  $\mathbb{P}^2/\mathbb{P}^2$ ,  $\tau = 0.05/2^n$ ,  $h = 0.1/2^n$ ,  $\delta = 0.01h^2$ , with  $n = 1, 2, 3, 4$ . Algorithm 1 with  $\ell = 1$  (left) and  $\ell = 2$  (right).



**Fig. 3** Simulation using  $\mathbb{P}^3/\mathbb{P}^3$ ,  $\tau = 0.05/2^n$ ,  $h = 0.1/2^n$ ,  $\delta = 0.01h^2$ , with  $n = 1, 2, 3, 4$ . Algorithm 1 with  $\ell = 1$  (left) and  $\ell = 2$  (right).

## 6 Conclusion

We have introduced and analyzed a numerical method for the transient Stokes problem which combines a fractional-step time-marching in time with a PSPG finite element method in space. To the best of our knowledge, this is the first time that this kind of non-symmetric stabilization is analyzed in such a time-stepping framework. The results obtained for the velocity are similar to those reported in [4] with monolithic time-stepping schemes.



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