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Circuits of ferromagnetic nanowires

Sergiy M. Bokoch^{*†}, Gilles Carbou[‡], Stéphane Labbé[§], Stéphane Despréaux[¶]

May 6, 2024

Abstract: in this paper we establish rigorously a one dimensional model of a junction of several ferromagnetic nanowires. Such structures appear in nano electronics or in new memory devices. We present also a numerical scheme adapted to this configuration and we compare our results with the 3d simulation obtained with the code EMicroM.

Keywords: ferromagnetism, nanowires, asymptotic analysis, Landau-Lifschitz equation

MSC: 35K55, 35Q60, 65M06, 65Z05

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1 Introduction

Since the beginning of the century, interest for ferromagnetic devices has increased, in particular with the development of solid-state magnetic random access memories (MRAMs) [27, 28] and magnetic logic devices [1]. The main advantages of MRAM is the non volatility, involving low energy consumption, and also their high reactivity [2, 31, 34]. The key idea is to exploit the multiplicity of possible magnetization configurations in nanostructures to store information, via ferromagnetic devices involving several geometry scales (nanowires, thin films, complex and bulk structures, see for

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example [27, 28]). This idea is also exploited in an alternative approach called DWs-logic (DW for domain wall), explained in [1], based upon complex nanowires structures (see [3, 4, 5, 6, 26, 32, 33] for experiments).

In this paper, our goal is to develop a reduced model of junctions of several ferromagnetic nanowires. As in [18], we aim to obtain an equivalent one dimensional model for junctions. On the one hand, the 1d simulations will be much more simple than the 3d ones. On the other hand, we hope that theoretical studies about DW formation, stability and dynamics will be possible in this model, as it is the case for simple devices [15, 16, 17, 18, 20, 21, 25].

Our study is based upon the 3d micromagnetic model introduced by W.-F. Brown [9, 10]. By asymptotic process, we aim to obtain reduced models of connected nanowires with whatever configuration, both in the static and the dynamic cases. First let us recall the 3d model of ferromagnetic material.

1.1 Three-dimensional model

We consider a ferromagnetic body confined in an open bounded set $\mathcal{O} \subset \mathbb{R}^3$. We denote by $\mathbf{M}(\mathbf{t}, \mathbf{x})$ the magnetic moment at time \mathbf{t} and position $\mathbf{x} \in \mathcal{O}$. We assume that the ferromagnetic material is saturated so that the magnetic moment \mathbf{M} satisfies the saturation constraint $|\mathbf{M}| = \mathbf{M}_s$. For static configurations, the energy associated to a magnetization $\mathbf{M} : \mathcal{O} \rightarrow \mathbb{R}^3$ is given by:

$$\mathbf{E}_{mic}(\mathbf{M}) = \frac{1}{2} \frac{A}{\mathbf{M}_s^2} \int_{\mathcal{O}} |\nabla_{\mathbf{x}} \mathbf{M}|^2 d\mathbf{x} + \frac{1}{2} \mu_0 \int_{\mathbb{R}^3} |h_d(\mathbf{M})|^2 d\mathbf{x},$$

where A is the exchange constant, and $\mu_0 = 4\pi \cdot 10^{-7} \text{ kg.m.s}^{-2}.\text{A}^{-2}$ is the vacuum permeability, and the demagnetizing field $h_d(\mathbf{M})$ is given by:

$$\text{curl } h_d(\mathbf{M}) = 0 \quad \text{and} \quad \text{div}(h_d(\mathbf{M}) + \overline{\mathbf{M}}) = 0 \quad \text{in } \mathbb{R}^3, \quad (1.1)$$

where $\overline{\mathbf{M}}$ is the extension of \mathbf{M} by zero outside \mathcal{O} .

In the stationary case, the observed configuration are critical points of \mathbf{E} under the constraint $|\mathbf{M}| = \mathbf{M}_s$. In the non stationary case, the dynamics of $\mathbf{M} : (\mathbf{t}, \mathbf{x}) \mapsto \mathbf{M}(\mathbf{t}, \mathbf{x})$ is described by the Landau Lischitz equation:

$$\frac{\partial \mathbf{M}}{\partial \mathbf{t}} = -\gamma \mathbf{M} \times \mathbf{H}_e - \frac{\alpha \gamma}{\mathbf{M}_s} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_e), \quad (1.2)$$

where γ is the gyromagnetic ratio, α is the damping coefficient, and \mathbf{H}_e is the effective field derived from the energy \mathbf{E} and given by:

$$\mathbf{H}_e = \frac{A}{\mathbf{M}_s^2} \Delta_{\mathbf{x}} \mathbf{M} + \mu_0 h_d(\mathbf{M}).$$

We obtain a dimensionless model by writing:

$$\mathbf{M}(\mathbf{t}, \mathbf{x}) = \mathbf{M}_s m\left(\frac{\mathbf{t}}{\tau}, \frac{\mathbf{x}}{\ell}\right), \quad \text{with } \ell^2 = \frac{A}{\mu_0 \mathbf{M}_s^2} \quad \text{and} \quad \tau = \frac{1}{\gamma \mathbf{M}_s \mu_0}.$$

We denote by $t = \frac{\mathbf{t}}{\tau}$ (resp. $x = \frac{\mathbf{x}}{\ell}$) the adimensioned time (resp. position). The adimensioned magnetic moment $: (t, x) \mapsto m(t, x)$ is defined on $\mathbb{R}^+ \times \Omega$, where $\Omega = \frac{1}{\ell} \mathcal{O}$. It satisfies the saturation constraint $|m(t, x)| = 1$ for *a. e.* (t, x) . We introduce the set $H^1(\Omega; S^2)$ defined by:

$$H^1(\Omega; S^2) = \{u \in H^1(\Omega; \mathbb{R}^3), \quad \text{such that } |u| = 1 \quad \textit{a.e.}\}.$$

The renormalized energy is given by:

$$\mathcal{E}_{mic}(m) = \frac{1}{A\ell} \mathbf{E}(\mathbf{M}) = \mathcal{E}_{exch}(m) + \mathcal{E}_{dem}(m), \quad (1.3)$$

where the exchange term is given by

$$\mathcal{E}_{exch}(m) = \frac{1}{2} \int_{\Omega} |\nabla m|^2 dx,$$

and the demagnetizing term is given by

$$\mathcal{E}_{dem}(m) = \frac{1}{2} \int_{\mathbb{R}^3} |h_d(m)|^2 dx,$$

where $h_d(m)$ is obtained from m by the relations:

$$\operatorname{curl} h_d(m) = 0 \text{ and } \operatorname{div} (h_d(m) + \bar{m}) = 0 \text{ in } \mathbb{R}^3, \quad (1.4)$$

where we denote by \bar{m} the extension of m by zero outside Ω . We remark that $-h_d(m)$ is the orthogonal projection of \bar{m} onto the fields of gradients, so that we have:

$$E_{dem}(m) = -\frac{1}{2} \int_{\Omega} h_d(m) \cdot m dx.$$

The static configurations m are the local minimizers of \mathcal{E}_{mic} in $H^1(\Omega; S^2)$, *i.e.* under the saturation constraint. Existence of minimizers is standard (see [30] for example). Partial regularity results for minimizers or critical points are established in [19] and [11].

In the time-dependent case, the adimensioned Landau-Lifschitz equation writes:

$$\begin{cases} \frac{\partial m}{\partial t} = -m \times H_e - \alpha m \times (m \times H_e) \text{ in } \mathbb{R}^+ \times \Omega, \\ H_e = -\partial_m \mathcal{E}_{mic} = \Delta m + h_d(m), \\ \frac{\partial m}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.5)$$

For regular solutions, the previous equation is equivalent to the Landau-Lifschitz-Gilberg equation we will deal with to construct weak solutions:

$$\frac{\partial m}{\partial t} - \alpha m \times \frac{\partial m}{\partial t} = -(1 + \alpha^2) m \times H_e. \quad (1.6)$$

Existence results for weak solutions of (1.6) are obtained in [7, 12, 22, 30]. Existence of strong solutions is established in [13, 14].

We will focus on the case of a ferromagnetic domain formed by a junction of several nanowires of characteristic thickness η . Let us describe such a domain.

1.2 Geometry of the junction

We consider a junction of n ferromagnetic nanowires described as follows. The i^{th} wire, denoted by W_i^η , is a straight round cylinder of length L_i and radius $\eta\beta_i$ directed by the unit vector \vec{u}_i . It is described by the change of coordinates:

$$\begin{aligned} \Psi_i^\eta : [0, L_i] \times B_2(0, 1) &\rightarrow \mathbb{R}^3 \\ (s, \xi_1, \xi_2) &\mapsto (\eta\gamma_i + s)\vec{u}_i + \beta_i\eta(\xi_1\vec{v}_i + \xi_2\vec{w}_i), \end{aligned} \quad (1.7)$$

where the vectors \vec{v}_i and \vec{w}_i are chosen so that $(\vec{u}_i, \vec{v}_i, \vec{w}_i)$ is a direct orthonormal basis of \mathbb{R}^3 , and where $\gamma_i > 0$. We assume that the \vec{u}_i , $i \in \{1, \dots, n\}$ are distinct two by two.

In all the paper, we denote

$$J^i = [0, L_i] \times B_2(0, 1).$$

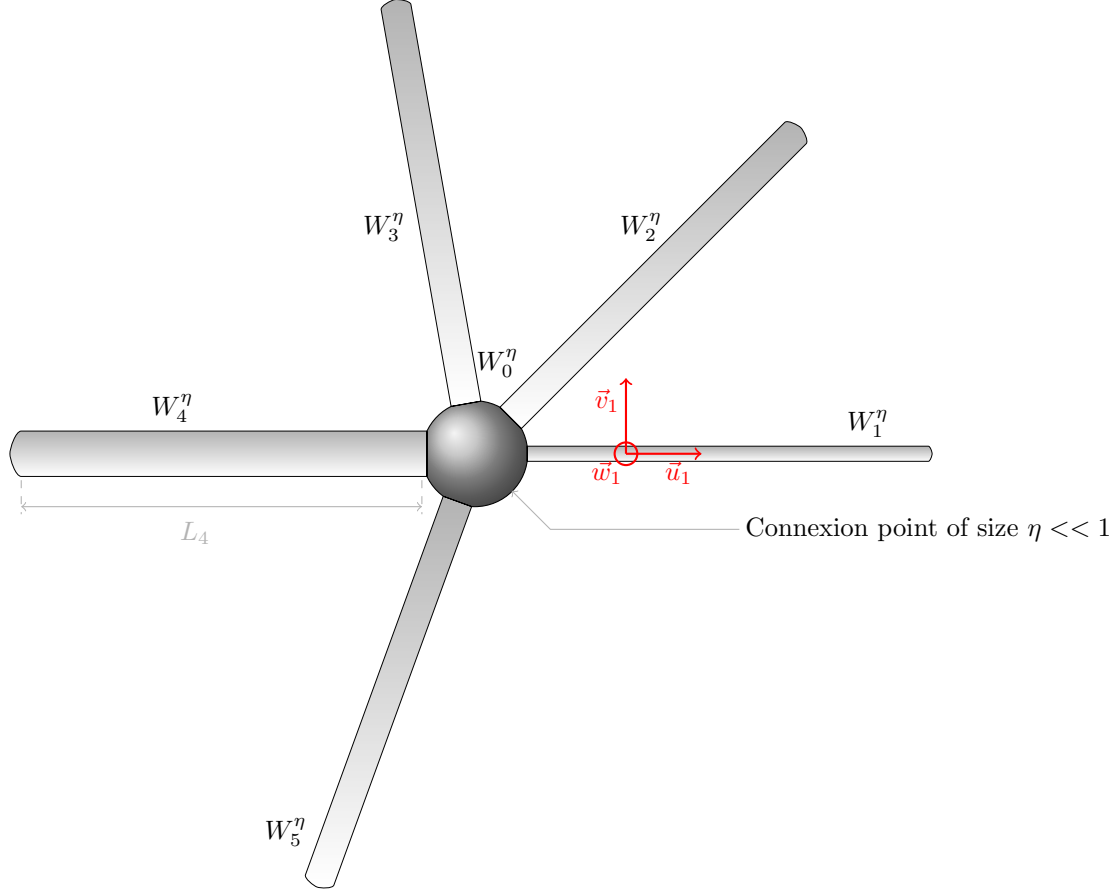


Figure 1: Connexion of 5 nanowires of size proportional to η .

The junction denoted by W_0^η is centered at zero, and its characteristic size is proportional to η . It is obtained by homothetic transformation of center 0 and factor η from a fixed pattern J^0 . It is described by the change of coordinates:

$$\begin{aligned} \Psi_0^\eta : J^0 &\rightarrow \mathbb{R}^3 \\ x &\mapsto \eta x. \end{aligned} \quad (1.8)$$

We assume that J^0 is star-shaped with center 0, and that for all $i \in \{1, \dots, n\}$, ∂J^0 contains the disk D_i of center $\gamma_i \vec{u}_i$, contained in $\{\vec{u}_i\}^\perp$, of radius β_i and that those disks are non-overlapping, that is

$$\overline{D_i} \cap \overline{D_j} = \emptyset \quad \text{for } i \neq j.$$

We denote by $\Omega^\eta = \bigcup_{i=0}^n W_i^\eta$ the ferromagnetic domain of characteristic thickness η . We assume that J^0 is designed so that the boundary of Ω^η is Lipschitz and \mathcal{C}^1 by parts. We will parametrize Ω^η using the previous scaling in the following way:

We define \mathcal{F} the set of the $M = (M_0, M_1, \dots, M_n)$ such that:

- $M_i \in H^1(J^i; S^2)$ for $i \in \{0, \dots, n\}$,
- $M_i(0, \xi_1, \xi_2) = M_0(\gamma_i \vec{u}_i + \beta_i(\xi_1 \vec{v}_i + \xi_2 \vec{w}_i))$ in the trace sense, for $i \in \{1, \dots, n\}$.

We endow \mathcal{F} with the norm:

$$\|M\|_{\mathcal{F}} = \sum_{i=0}^n \|M_i\|_{H^1(J^i)}.$$

For $M \in \mathcal{F}$, we define m^η in $H^1(\Omega^\eta; S^2)$ by

$$m^\eta(x) = M_i \left((\Psi_i^\eta)^{-1}(x) \right) \text{ for } x \in W_i^\eta. \quad (1.9)$$

The trace conditions ensure that m^η is in $H^1(\Omega^\eta)$.

1.3 Statement of the results in the stationary case

The micromagnetism energy on Ω^η is given for $m \in H^1(\Omega^\eta; S^2)$ by:

$$\mathcal{E}_{mic}^\eta(m) = \frac{1}{2} \int_{\Omega^\eta} |\nabla m|^2 dx - \frac{1}{2} \int_{\Omega^\eta} m \cdot h_d(m) dx, \quad (1.10)$$

where $h_d(m)$ is defined by

$$\text{curl } h_d(m) = 0 \text{ and } \text{div } (h_d(m) + \bar{m}) = 0 \text{ in } \mathbb{R}^3, \quad (1.11)$$

where we denote by \bar{m} the extension of m by zero outside Ω^η . The static configurations m^η are the local minimizers of the problem \mathcal{P}^η :

$$\mathcal{P}^\eta : \quad \text{find } m^\eta \in H^1(\Omega^\eta; S^2) \text{ such that } \mathcal{E}_{mic}^\eta(m^\eta) \leq \mathcal{E}_{mic}^\eta(v) \text{ for all } v \in H^1(\Omega^\eta; S^2).$$

Using the rescaling defined above, this problem is equivalent to the minimization problem set on \mathcal{F}

$$\tilde{\mathcal{P}}^\eta : \quad \text{find } M^\eta \in \mathcal{F} \text{ such that } \tilde{\mathcal{E}}^\eta(M^\eta) \leq \tilde{\mathcal{E}}^\eta(V) \text{ for all } V \in \mathcal{F}.$$

The rescaled energy $\tilde{\mathcal{E}}^\eta(M) = \frac{1}{\eta^2} \mathcal{E}_{mic}^\eta(m)$ is defined by

$$\tilde{\mathcal{E}}^\eta = \tilde{\mathcal{E}}_{exch}^\eta + \tilde{\mathcal{E}}_{dem}^\eta, \quad (1.12)$$

with

$$\tilde{\mathcal{E}}_{exch}^\eta(M) = \frac{1}{2\eta} \int_{J^0} |\nabla M_0|^2 + \sum_{i=1}^n \frac{1}{2} \int_{J^i} \left(\beta_i^2 \left| \frac{\partial M_i}{\partial s} \right|^2 + \frac{1}{\eta^2} \left| \frac{\partial M_i}{\partial \xi_1} \right|^2 + \frac{1}{\eta^2} \left| \frac{\partial M_i}{\partial \xi_2} \right|^2 \right),$$

and with

$$\tilde{\mathcal{E}}_{dem}^\eta = -\frac{\eta}{2} \int_{J^0} M_0 \mathcal{H}_0^\eta(M) - \sum_{i=1}^n \frac{1}{2} \int_{J^i} \beta_i^2 M_i \mathcal{H}_i^\eta(M),$$

where $\mathcal{H}_i^\eta(M)$ represents the rescaling of the restriction on W_i^η of the demagnetizing field generated by the magnetization m :

$$\mathcal{H}_i^\eta(M)(X) = h_d(m)(\Psi_i^\eta(X)).$$

We denote by $\mathcal{H}_{i,j}^\eta(M)$ the rescaling on W_i^η of the demagnetizing field generated by the restriction of m on W_j^η :

$$\mathcal{H}_{i,j}^\eta(M) = (h_d(M_i \circ (\Psi_i^\eta)^{-1})) \circ \Psi_j^\eta.$$

We introduce the limit minimization problem: we define \mathcal{F}^0 the set of the $M = (M_0, M_1, \dots, M_n) \in \mathcal{F}$ such that:

- $M_0 \in H^1(J^0; S^2)$ is constant,
- $M_i \in H^1(J^i; S^2)$ for $i \in \{1, \dots, n\}$, with $\frac{\partial M_i}{\partial \xi_1} = \frac{\partial M_i}{\partial \xi_2} = 0$ (M_i only depends on s),
- $M_i(0) = M_0$ (from the trace conditions satisfied by the elements of \mathcal{F}).

We introduce the problem \mathcal{P} :

$$\mathcal{P} \begin{cases} \text{find } M^0 \in \mathcal{F}^0, \\ \tilde{\mathcal{E}}(M^0) \leq \tilde{\mathcal{E}}(V) \text{ for all } V \in \mathcal{F}^0, \end{cases}$$

where $\tilde{\mathcal{E}}(M) = \tilde{\mathcal{E}}_{exch}(M) + \tilde{\mathcal{E}}_{dem}(M)$ with

$$\tilde{\mathcal{E}}_{exch}(M) = \frac{1}{2} \sum_{i=1}^n \int_{J^i} \beta_i^2 \left| \frac{\partial M_i}{\partial s} \right|^2 ds \quad \text{and} \quad \tilde{\mathcal{E}}_{dem}(M) = \frac{1}{4} \sum_{i=1}^n \int_{J^i} \beta_i^2 \left((M_i \cdot \vec{v}_i)^2 + (M_i \cdot \vec{w}_i)^2 \right) ds.$$

Remark 1.1. Using the saturation constraint $|M_i| = 1$ a. e., we should rewrite the limit demagnetizing field as:

$$\tilde{\mathcal{E}}_{dem}(M) = \frac{1}{4} \sum_{i=1}^n \int_{J^i} \beta_i^2 (1 - (M_i \cdot \vec{u}_i)^2) ds.$$

We establish in Section 2 the following Γ -convergence result:

Theorem 1.1. $\tilde{\mathcal{E}}^\eta$ Γ -converges weakly to $\tilde{\mathcal{E}}$, that is:

- (lower semi continuity) for all sequence $(M^\eta)_\eta$ of \mathcal{F} such that M^η weakly converges to M for the norm $\|\cdot\|_{\mathcal{F}}$, and such that the sequence $(\tilde{\mathcal{E}}^\eta(M^\eta))_\eta$ is bounded when η tends to zero, we have that the limit M is in \mathcal{F}^0 and

$$\tilde{\mathcal{E}}(M) \leq \liminf_{\eta \rightarrow 0} \tilde{\mathcal{E}}^\eta(M^\eta),$$

- (reconstruction) for all $U \in \mathcal{F}^0$, there exists a sequence $(U^\eta)_\eta$ such that

$$\begin{cases} U^\eta \in \mathcal{F}, \\ U^\eta \text{ tends to } U \text{ weakly in } \mathcal{F} \text{ when } \eta \rightarrow 0, \\ \limsup_{\eta \rightarrow 0} \tilde{\mathcal{E}}^\eta(U^\eta) \leq \tilde{\mathcal{E}}(U). \end{cases}$$

Taking into account the properties of the elements of \mathcal{F}^0 , the Γ -limit problem \mathcal{P} reduces to the following 1d problem: we define \mathcal{G} by

$$\mathcal{G} = \{ \mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_n), \quad \mathbf{m}_i \in H^1([0, L_i]; S^2) \text{ with } \mathbf{m}_1(0) = \mathbf{m}_2(0) = \dots = \mathbf{m}_n(0) \}.$$

For $\mathbf{m} \in \mathcal{G}$, we set

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2} \sum_{i=1}^n \int_0^{L_i} \beta_i^2 |\partial_s \mathbf{m}_i|^2 + \frac{1}{4} \sum_{i=1}^n \int_0^{L_i} \beta_i^2 (1 - (\mathbf{m}_i \cdot \vec{u}_i)^2).$$

Then Problem \mathcal{P} is equivalent to solve the following Problem (\mathcal{Q}) :

$$(\mathcal{Q}) \quad \text{find } \mathbf{m} \in \mathcal{G}, \quad \forall \mathbf{v} \in \mathcal{G}, \quad \mathcal{E}(\mathbf{m}) \leq \mathcal{E}(\mathbf{v}).$$

The critical points $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_n)$ of \mathcal{E} in \mathcal{G} satisfy the following Euler Equation:

$$\forall i \in \{1, \dots, n\}, \quad \mathbf{m}_i \times (\partial_{ss} \mathbf{m}_i + (\mathbf{m}_i \cdot \vec{u}_i) \vec{u}_i) = 0,$$

with the boundary conditions:

$$\forall i \in \{1, \dots, n\}, \quad \partial_s \mathbf{m}_i(L_i) = 0,$$

and with the junction conditions:

$$\begin{cases} \mathbf{m}_1(0) = \mathbf{m}_2(0) = \dots = \mathbf{m}_n(0), \\ \sum_{i=1}^n \beta_i^2 \partial_s \mathbf{m}_i(0) = 0. \end{cases}$$

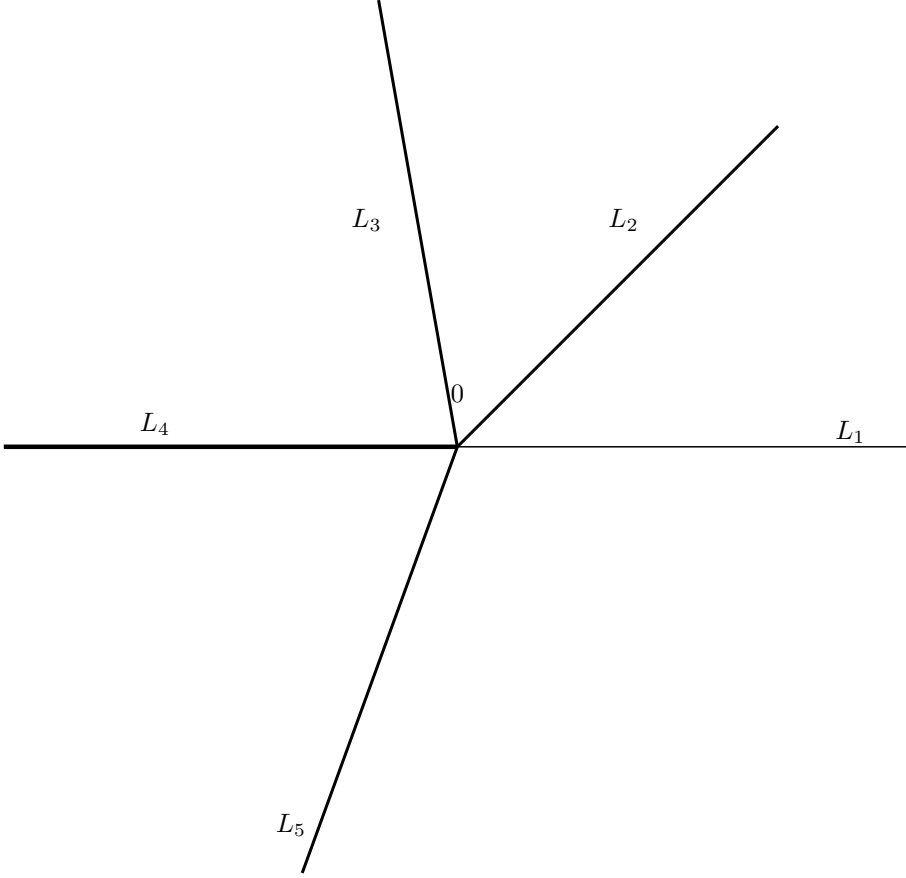


Figure 2: Limit domain for the junction of 5 nanowires.

1.4 Statement of the results in the time-dependent model

We consider now the time-dependent 3-dimensional model for the ferromagnetic domain Ω^η given by (1.6). Existence of weak solutions for (1.6) is given by the following proposition (see [30, 7, 22]):

Proposition 1.1. *Let $m_0^\eta \in H^1(\Omega^\eta; S^2)$. There exists $m^\eta : \mathbb{R}^+ \times \Omega^\eta \rightarrow \mathbb{R}^3$ satisfying:*

- $m^\eta \in L^\infty(\mathbb{R}^+; H^1(\Omega^\eta))$ and $\frac{\partial m^\eta}{\partial t} \in L^2(\mathbb{R}^+; L^2(\Omega^\eta))$,
- $|m^\eta(t, x)| = 1$ a.e. (saturation constraint),
- for all $\Psi \in C_c^\infty(\mathbb{R}^+; H^1(\Omega^\eta))$,

$$\int_{\mathbb{R}^+ \times \Omega^\eta} \left(\frac{\partial m^\eta}{\partial t} - \alpha m^\eta \times \frac{\partial m^\eta}{\partial t} \right) \cdot \Psi = (1 + \alpha^2) \int_{\mathbb{R}^+ \times \Omega^\eta} \sum_{i=1}^3 \left(m^\eta \times \frac{\partial m^\eta}{\partial x_i} \right) \cdot \frac{\partial \Psi}{\partial x_i} - (1 + \alpha^2) \int_{\mathbb{R}^+ \times \Omega^\eta} (m^\eta \times h_d(m^\eta)) \cdot \Psi, \quad (1.13)$$

- $m^\eta(0, \cdot) = m_0^\eta$ in the trace sense,
- for almost every $t > 0$, we have the following energy inequality:

$$\mathcal{E}_{mic}^\eta(m^\eta(t)) + \frac{\alpha}{1 + \alpha^2} \int_0^t \int_{\Omega^\eta} \left| \frac{\partial m^\eta}{\partial t} \right|^2 \leq \mathcal{E}_{mic}^\eta(m^\eta(0)). \quad (1.14)$$

When η tends to zero, we prove the following convergence theorem:

Theorem 1.2. *Let $M^0 = (M_0^0, \dots, M_n^0) \in \mathcal{F}^0$. We define $m^{0,\eta} \in H^1(\Omega^\eta; S^2)$ by*

$$m^{0,\eta}(x) = M_i^0 \left((\Psi_i^\eta)^{-1}(x) \right) \text{ for } x \in W_i^\eta.$$

We consider m^η a weak solution for the Landau-Lifschitz-Guilbert equation with initial data $m^{0,\eta}$ given by Proposition 1.1. We introduce $M^\eta : \mathbb{R}^+ \rightarrow \mathcal{F}$ the rescaling of m^η defined by

$$M^\eta = (M_0^\eta, \dots, M_n^\eta),$$

with $M_i^\eta(t, X) = m^\eta(t, \Psi_i^\eta(X))$ for $X \in J^i$. Then, there exists $M \in L^\infty(\mathbb{R}^+; \mathcal{F}^0)$ such that for all $i = 1, \dots, n$, up to a subsequence, M_i^η tends to M_i in $L^\infty(\mathbb{R}^+; H^1(J^i))$ weak \star and $\frac{\partial M_i^\eta}{\partial t}$ tends to $\frac{\partial M_i}{\partial t}$ in $L^2(\mathbb{R}^+ \times J^i)$ weak when η tends to zero. In addition, $M \in \mathcal{F}^0$ for almost every t so that we can write $M(t, s, \xi_1, \xi_2) = \mathbf{m}(t, s)$ for $i \geq 1$, and \mathbf{m} satisfies:

- $\mathbf{m}_i \in L^\infty(\mathbb{R}^+; H^1([0, L_i]; S^2))$ and $\frac{\partial \mathbf{m}_i}{\partial t} \in L^2(\mathbb{R}^+ \times [0, L_i])$,
- for almost every t , $\mathbf{m}_1(t, 0) = \mathbf{m}_2(t, 0) = \dots = \mathbf{m}_n(t, 0)$,
- $\mathbf{m}_i(0, s) = M_i^0(s)$ in the trace sense,
- for all $\varphi = (\varphi_1, \dots, \varphi_n)$ satisfying $\varphi \in C_c^\infty(\mathbb{R}^+; H^1([0, L_i]))$ for all i with $\varphi_1(t, 0) = \dots = \varphi_n(t, 0)$ for all t , then:

$$\begin{aligned} \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times [0, L_i]} \left(\frac{\partial \mathbf{m}_i}{\partial t} - \alpha \mathbf{m}_i \times \frac{\partial \mathbf{m}_i}{\partial t} \right) \cdot \varphi_i &= (1 + \alpha^2) \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times [0, L_i]} \mathbf{m}_i \times \frac{\partial \mathbf{m}_i}{\partial s} \cdot \frac{\partial \varphi_i}{\partial s} \\ &\quad - \frac{1 + \alpha^2}{2} \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times [0, L_i]} \mathbf{m}_i \times (\mathbf{m}_i \cdot \vec{u}_i) \vec{u}_i \cdot \varphi_i. \end{aligned}$$

We remark that the weak formulation obtained in Theorem 1.2 is equivalent to the following problem where the domain is described as presented in the Fig. 2:

$$\left\{ \begin{array}{l} \mathbf{m}_i \in L^\infty(\mathbb{R}^+; H^1([0, L_i]; S^2)), \\ \mathbf{m}_i(0, s) = M_i^0(s), \\ \frac{\partial \mathbf{m}_i}{\partial t} = -\mathbf{m}_i \times \mathbf{h}_i^e - \alpha \mathbf{m}_i \times (\mathbf{m}_i \times \mathbf{h}_i^e), \\ \mathbf{h}_i^e = \partial_{ss} \mathbf{m}_i + \frac{1}{2} (\mathbf{m}_i \cdot \vec{u}_i) \vec{u}_i, \\ \partial_s \mathbf{m}_i(t, L_i) = 0, \\ \mathbf{m}_1(t, 0) = \mathbf{m}_2(t, 0) = \dots = \mathbf{m}_n(t, 0), \\ \sum_{i=1}^n \beta_i^2 \partial_s \mathbf{m}_i(t, 0) = 0. \end{array} \right. \quad (1.15)$$

Remark 1.2. *Note that in the model above, the coupling between the wires, described by the last two equations, is only present at the junction. This model therefore lends itself well to parallelization.*

1.5 Organization of the paper

Sections 2 and 3 are respectively devoted to the proofs of Theorem 1.1 and 1.2. In Section 4 we present numerical simulations comparing the 3d case and the asymptotic 1d model. 3d computations have been performed with the simulation code EMicroM [22, 24]. 1d computations are performed with scilab [29] using a finite difference scheme adapted to the case of the connexion node.

2 Γ -convergence for the stationary model

Proof of the lower semi-continuity

Let $M^\eta = (M_0^\eta, M_1^\eta, \dots, M_n^\eta)$ in \mathcal{F} such that M^η weakly converges to M in \mathcal{F} (that is M_i^η weakly converges in H^1 to M_i), and such that there exists C such that

$$\tilde{\mathcal{E}}^\eta(M^\eta) \leq C \text{ for all } \eta.$$

By compactness of the Sobolev embedding in Lipschitzian domains, we can assume that M^η tends to M strongly in L^2 and almost everywhere (by extracting a subsequence), so that M satisfies the saturation constraint $|M| = 1$ almost everywhere, i.e. $M \in \mathcal{F}$. In addition, since the demagnetizing energy is non negative, we obtain that

$$\|\nabla M_0^\eta\|_{L^2(J^0)}^2 \leq C\eta,$$

and for $i \in \{1, \dots, n\}$,

$$\left\| \frac{\partial M_i^\eta}{\partial \xi_1} \right\|_{L^2(J^i)}^2 + \left\| \frac{\partial M_i^\eta}{\partial \xi_2} \right\|_{L^2(J^i)}^2 \leq C\eta^2,$$

so that M_0 is constant in J^0 and M_i only depends on s . Therefore we obtain that $M \in \mathcal{F}^0$.

From classical convexity arguments, it is clear that

$$\tilde{\mathcal{E}}_{exch}(M) \leq \liminf \tilde{\mathcal{E}}_{exch}^\eta(M^\eta).$$

Concerning the rescaled demagnetizing field, we split it in 4 parts:

- the effect of the demagnetizing field on the junction will be neglected by application of the following lemma:

Lemma 2.1. *There exists a constant C such that for all $\eta > 0$, for all $V = (V_0, V_1, \dots, V_n) \in \mathcal{F}$,*

$$\left| \eta \int_{J^0} V_0 \cdot \mathcal{H}_0^\eta(V) \right| \leq C\sqrt{\eta}.$$

Proof. Let V be in \mathcal{F} . We denote by v^η the rescaling of V defined by (1.9). We have

$$\eta \int_{J^0} V_0 \cdot \mathcal{H}_0^\eta(V) = \frac{1}{\eta^2} \int_{W_0^\eta} v^\eta \cdot h^d(v^\eta),$$

so

$$\begin{aligned} \left| \eta \int_{J^0} V_0 \cdot \mathcal{H}_0^\eta(V) \right| &\leq \frac{1}{\eta^2} \|v^\eta\|_{L^2(W_0^\eta)} \|h^d(v^\eta)\|_{L^2(\mathbb{R}^3)}, \\ &\leq \frac{1}{\eta^2} \|v^\eta\|_{L^2(W_0^\eta)} \|v^\eta\|_{L^2(\Omega^\eta)}, \\ &\leq \frac{1}{\eta^2} (\text{meas}(W_0^\eta))^{\frac{1}{2}} (\text{meas}(\Omega^\eta))^{\frac{1}{2}}, \\ &\leq \frac{1}{\eta^2} C (\eta^3 \cdot \eta^2)^{\frac{1}{2}}, \\ &\leq C\sqrt{\eta}. \end{aligned}$$

□

- The demagnetizing field generated by the junction will be neglected thanks to the following lemma:

Lemma 2.2. *There exists a constant C such that for all $\eta > 0$ for all $V \in \mathcal{F}$ and all $i \in \{1, \dots, n\}$,*

$$\left| \beta_i^2 \int_{J^i} V_i \cdot \mathcal{H}_{0,i}^\eta(V) \right| \leq C\sqrt{\eta}.$$

Proof. Let V be in \mathcal{F} . We denote by v^η the rescaling of V defined by (1.9). We have

$$\beta_i^2 \int_{J^i} V_i \cdot \mathcal{H}_{0,i}^\eta(V) = \frac{1}{\eta^2} \int_{W_i^\eta} v^\eta \cdot h^d(v^\eta \chi_{W_0^\eta}),$$

so

$$\begin{aligned} \left| \beta_i^2 \int_{J^i} V_i \cdot \mathcal{H}_{0,i}^\eta(V) \right| &\leq \frac{1}{\eta^2} \|v^\eta \chi_{W_i^\eta}\|_{L^2(W_i^\eta)} \|h^d(v^\eta \chi_{W_0^\eta})\|_{L^2(\mathbb{R}^3)}, \\ &\leq \frac{1}{\eta^2} (\text{meas}(W_i^\eta))^{\frac{1}{2}} \|v^\eta \chi_{W_0^\eta}\|_{L^2(\mathbb{R}^3)}, \\ &\leq \frac{1}{\eta^2} (C\eta^2)^{\frac{1}{2}} (\text{meas}(W_0^\eta))^{\frac{1}{2}}, \\ &\leq \frac{1}{\eta} C(\eta^3)^{\frac{1}{2}} \leq C\sqrt{\eta}. \end{aligned}$$

□

- The demagnetizing field generated by the wire i on the wire j will vanish:

Lemma 2.3. *For all $i \geq 1$ and $j \geq 1$, with $i \neq j$, for all $\eta > 0$ and all $M \in \mathcal{F}$,*

$$\left| \beta_i^2 \int_{J^j} M_j \cdot \mathcal{H}_{i,j}^\eta(M) \right| \leq C\eta.$$

Proof. Let V be in \mathcal{F} . We denote by v^η the rescaling of V defined by (1.9). The demagnetizing field generated by $v_i^\eta \chi_{W_i^\eta}$ is described by the formula:

$$h_d(m\mathbf{1}_{W_i^\eta})(y) = \frac{1}{4\pi} \int_{W_i^\eta} \nabla_y \text{div}_y \frac{v^\eta(z) \mathbf{1}_{W_i^\eta}(z)}{|z-y|} dz,$$

so that

$$|h_d(m\mathbf{1}_{W_i^\eta})(y)| \leq C \int_{W_i^\eta} \frac{1}{|z-y|^3} dz.$$

We claim the following inequality:

Claim: *there exists $K > 0$ such that for all $\eta > 0$, if $i \neq j$, if $y = \Psi_i^\eta(s, \xi)$ and $z = \Psi_j^\eta(s', \xi')$, then*

$$|z - y| \geq K(\eta + s + s').$$

Proof of the Claim: let us assume that the claim is false. For all n we find $\eta_n \in]0, 1]$, ξ_n and ξ'_n in $B_2(0, 1)$, $s_n \in [0, L_i]$ and $s'_n \in [0, L_j]$ such that

$$|\Psi_j^{\eta_n}(s_n, \xi_n) - \Psi_i^{\eta_n}(s'_n, \xi'_n)| < \frac{1}{n}(\eta_n + s_n + s'_n),$$

that is

$$|\eta_n [X_n - X'_n] + s_n \vec{u}_i - s'_n \vec{u}_j| < \frac{1}{n}(\eta_n + s_n + s'_n), \quad (2.16)$$

with $X_n = (\gamma_i \vec{u}_i + \beta_i (\xi_{n,1} \vec{v}_i + \xi_{n,2} \vec{w}_i)) \in D_i$ and $X'_n = (\gamma_j \vec{u}_j + \beta_j (\xi'_{n,1} \vec{v}_j + \xi'_{n,2} \vec{w}_j)) \in D_j$. By extracting a subsequence, we assume that $X_n \rightarrow X$ in D_i and $X'_n \rightarrow X'$ in D_j .

First case: if $\frac{s_n}{\eta_n}$ and $\frac{s'_n}{\eta_n}$ are bounded, dividing (2.16) by η_n , we obtain that $|X - X'| = 0$ which is contradictory to the fact that $\overline{D_i} \cap \overline{D_j} = \emptyset$.

Second case: if $\frac{s_n}{\eta_n}$ is unbounded and if we can extract a subsequence (still denoted by the index n) such that $s'_n \leq s_n$ and $\frac{s_n}{\eta_n} \rightarrow +\infty$, then we divide (2.16) by $\frac{s_n}{\eta_n}$, extracting a subsequence so that $\frac{s'_n}{s_n} \rightarrow \tau \in [0, 1]$, we obtain that

$$|\vec{u}_i - \tau \vec{u}_j| = 0,$$

with is impossible since \vec{u}_i and \vec{u}_j are distinct unitary vectors, and since $\tau \geq 0$.

Third case: if $\frac{s'_n}{\eta_n}$ is unbounded and if we can extract a subsequence (still denoted by the index n) such that $s_n \leq s'_n$ and $\frac{s'_n}{\eta_n} \rightarrow +\infty$, by the same argument, we obtain a contradiction.

This concludes the proof of the Claim. \square

So from the Claim, if $y \in W_j^\eta$ is given by $y = \Psi_j^\eta(s, \xi)$, we have $\text{dist}(y, W_i^\eta) \geq \alpha\eta + Cs$ so

$$\left| \int_{J^j} M_j \cdot \mathcal{H}_{i,j}(M) \right| \leq \int_0^{L_j} \frac{K\eta^2}{(\alpha\eta + Cs)^2} ds \leq K\eta,$$

so this term tends to zero when η tends to zero. \square

- The behavior of the demagnetizing field generated by the wire i onto itself is described in [15]. For the convenience of the reader we explicit the limit of this contribution in the following lemma:

Lemma 2.4. *Let $i \in \{1, \dots, n\}$ to be fixed. If M_i^η tends strongly to M_i in $L^2(J^i)$, then $\mathcal{H}_{i,i}^\eta(M^\eta)$ tends to $-\frac{1}{2}((M_i \cdot \vec{v}_i)\vec{v}_i + (M_i \cdot \vec{w}_i)\vec{w}_i)$ in $L^2(J^i)$ strongly, so that*

$$\lim_{\eta \rightarrow 0} \beta_i^2 \int_{J^i} M_i^\eta \cdot \mathcal{H}_{i,i}^\eta(M_i^\eta) = -\frac{1}{2} \beta_i^2 \int_{J^i} ((M_i \cdot \vec{v}_i)^2 + (M_i \cdot \vec{w}_i)^2).$$

Proof. We rewrite M_i^η and $\mathcal{H}_{i,i}^\eta(M^\eta)$ in the basis $(\vec{u}_i, \vec{v}_i, \vec{w}_i)$ in the following way: $M_i^\eta = \mathbf{m}_1^\eta \vec{u}_i + \mathbf{m}_2^\eta \vec{v}_i + \mathbf{m}_3^\eta \vec{w}_i$ and $\mathcal{H}_{i,i}^\eta(M^\eta) = \mathbf{H}_1^\eta \vec{u}_i + \mathbf{H}_2^\eta \vec{v}_i + \mathbf{H}_3^\eta \vec{w}_i$.

We first remark that \mathbf{H}^η is uniformly bounded in $L^2(\mathbb{R}^3)$ since

$$\begin{aligned} \|\mathbf{H}^\eta\|_{L^2(\mathbb{R}^3)}^2 &= \frac{1}{\eta^2} \|h^d(m_i^\eta)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{1}{\eta^2} \|m_i^\eta\|_{L^2(\mathbb{R}^3)}^2, \\ &\leq \frac{1}{\eta^2} \text{meas}(\Omega_i^\eta) \leq C, \end{aligned}$$

so we can assume (by extracting a subsequence) that $\mathbf{H}^\eta \rightharpoonup \mathbf{H}$ in $L^2(\mathbb{R}^3)$ weakly.

Rewriting (1.4) in the coordinates system (s, ξ_1, ξ_2) (using that the basis $(\vec{u}_i, \vec{v}_i, \vec{w}_i)$ is orthonormal and direct), we obtain that:

$$\begin{cases} \eta\beta_i\partial_s(\mathbf{H}_1^\eta + \overline{\mathbf{m}}_1^\eta) + \partial_{\xi_1}(\mathbf{H}_2^\eta + \overline{\mathbf{m}}_2^\eta) + \partial_{\xi_2}(\mathbf{H}_3^\eta + \overline{\mathbf{m}}_3^\eta) = 0, \\ \partial_{\xi_1}\mathbf{H}_3^\eta - \partial_{\xi_2}\mathbf{H}_2^\eta = 0, \\ \partial_{\xi_1}\mathbf{H}_1^\eta - \eta\beta_i\partial_s\mathbf{H}_2^\eta = 0, \\ \eta\beta_i\partial_s\mathbf{H}_3^\eta - \partial_{\xi_2}\mathbf{H}_1^\eta = 0. \end{cases}$$

Taking the weak limit we obtain that:

$$\begin{cases} \partial_{\xi_1}(\mathbf{H}_2 + \overline{\mathbf{m}}_2) + \partial_{\xi_2}(\mathbf{H}_3 + \overline{\mathbf{m}}_3) = 0, \\ \partial_{\xi_1}\mathbf{H}_3 - \partial_{\xi_2}\mathbf{H}_2 = 0, \\ \partial_{\xi_1}\mathbf{H}_1 = 0, \\ \partial_{\xi_2}\mathbf{H}_1 = 0. \end{cases}$$

Therefore since $\mathbf{H} \in L^2(\mathbb{R}^3)$, we obtain that $\mathbf{H}_1 = 0$ and that the transversal part of \mathbf{H} is the 2-dimensional (in the plane $\{s\} \times \mathbb{R}^2$) demagnetizing field associated to the transversal part of \mathbf{m} (that is $(\mathbf{m}_2, \mathbf{m}_3)(s, \cdot)$). Since the section of the wire is a ball of \mathbb{R}^2 , since $(\mathbf{m}_2, \mathbf{m}_3)$ is constant in the section of the wire, the calculation of the two-dimensional demagnetizing field generated by the constant configuration is classical (see [15] for an elementary proof in 2d), and we obtain that \mathbf{H} is given by:

$$\mathbf{H}(t, s, \xi_1, \xi_2) = \begin{cases} -\frac{1}{2} \begin{pmatrix} 0 \\ \mathbf{m}_2(t, s) \\ \mathbf{m}_3(t, s) \end{pmatrix} & \text{for } (s, \xi_1, \xi_2) \in J^i, \\ \frac{1}{2} \frac{1}{((\xi_1)^2 + (\xi_2)^2)^2} \begin{pmatrix} 0 \\ \mathbf{m}_2(t, s)((\xi_1)^2 - (\xi_2)^2) + 2\mathbf{m}_3(t, s)\xi_1\xi_2 \\ -\mathbf{m}_3(t, s)((\xi_1)^2 - (\xi_2)^2) - 2\mathbf{m}_2(t, s)\xi_1\xi_2 \end{pmatrix} \\ & \text{for } s \in [0, L_i] \text{ and } (\xi_1)^2 + (\xi_2)^2 > 1, \\ 0 & \text{for } s \notin [0, L_i]. \end{cases}$$

Since for all $\eta > 0$,

$$\int_{\mathbb{R}^3} |\mathbf{H}^\eta|^2 = - \int_{J^i} \mathbf{H}^\eta \cdot \mathbf{m}^\eta,$$

since $\mathbf{H}^\eta \rightharpoonup \mathbf{H}$ in L^2 weak and $\mathbf{m}^\eta \rightarrow \mathbf{m}^0$ in L^2 strong, we obtain that

$$\lim_{\eta \rightarrow 0} \int_{\mathbb{R}^3} |\mathbf{H}^\eta|^2 = \frac{1}{2} \int_{\Omega_1} (|\mathbf{H}_2|^2 + |\mathbf{H}_3|^2) = \int_{\mathbb{R}^3} |\mathbf{H}|^2,$$

so that \mathbf{H}^η tends strongly to \mathbf{H} in $L^2(\mathbb{R}^3)$ when η tends to zero. □

Using the previous lemmas, we obtain that $\tilde{\mathcal{E}}_{dem}^\eta(M^\eta)$ tends to $\tilde{\mathcal{E}}_{dem}(M^0)$.

This concludes the proof of the lower semi continuity.

Proof of the reconstruction.

Let M be fixed in \mathcal{F}^0 . For $\eta > 0$ we set $M^\eta = M$. Let us study the limit of $\tilde{\mathcal{E}}^\eta(M^\eta)$ when η tends to zero.

We have:

$$\tilde{\mathcal{E}}^\eta(M^\eta) = \frac{1}{2} \sum_{i=1}^n \int_{J^i} \beta_i^2 \left| \frac{\partial M_i}{\partial s} \right|^2 - \frac{\eta}{2} \int_{J^0} M_0 \cdot \mathcal{H}_0^\eta(M) - \frac{1}{2} \sum_{i=1}^n \beta_i^2 \int_{J^i} M_i \cdot \mathcal{H}_i^\eta(M).$$

As in the previous step, we obtain on the one hand that

$$-\eta \int_{J^0} M_0 \cdot \mathcal{H}_0^\eta(M) \longrightarrow 0 \text{ when } \eta \longrightarrow 0,$$

and on the other hand that

$$-\int_{J^i} M_i \cdot \mathcal{H}_i^\eta(M) \longrightarrow \frac{\pi}{2} \int_{[0, L_i]} ((M_i^0 \cdot \vec{v}_i)^2 + (M_i^0 \cdot \vec{w}_i)^2) ds,$$

that is

$$\tilde{\mathcal{E}}^\eta(M^\eta) \longrightarrow \tilde{\mathcal{E}}(M) \text{ when } \eta \longrightarrow 0.$$

This concludes the proof of Theorem 1.1.

3 Asymptotic process in the dynamic case

First step: estimates

By rescaling of the energy formula (1.14), we obtain that for almost every t ,

$$\tilde{\mathcal{E}}^\eta(M^\eta(t)) + \frac{\alpha}{1 + \alpha^2} \int_0^t \left(\sum_{i=1}^n \beta_i^2 \int_{J^i} \left| \frac{\partial M_i^\eta}{\partial t} \right|^2 + \eta \int_{J^0} \left| \frac{\partial M_0^\eta}{\partial t} \right|^2 \right) \leq \tilde{\mathcal{E}}^\eta(M^0). \quad (3.17)$$

We remark that since $M^0 \in \mathcal{F}^0$,

$$\tilde{\mathcal{E}}^\eta(M^0) = \frac{1}{2} \sum_{i=1}^n \pi \beta_i^2 \int_0^{L_i} \left| \frac{dM_i^0}{ds} \right|^2 - \frac{\eta}{2} \int_{J^0} M_0^0 \cdot \mathcal{H}_0^\eta(M^0) - \frac{1}{2} \sum_{i=1}^n \int_{J^i} \beta_i^2 M_i^0 \cdot \mathcal{H}_i^\eta(M^0).$$

Applying lemmas 2.1-2.4, we obtain that when η tends to zero:

$$\tilde{\mathcal{E}}^\eta(M^0) \longrightarrow \frac{1}{2} \sum_{i=1}^n \pi \beta_i^2 \left(\int_0^{L_i} \left| \frac{dM_i^0}{ds} \right|^2 + \frac{1}{2} \int_0^{L_i} ((M_i^0 \cdot \vec{v}_i)^2 + (M_i^0 \cdot \vec{w}_i)^2) \right). \quad (3.18)$$

In particular, we obtain that there exists a constant C such that for almost every t ,

$$\begin{aligned} \frac{1}{\eta} \int_{J^0} |\nabla M_0^\eta|^2 + \sum_{i=1}^n \int_{J^i} \left(\beta_i^2 \left| \frac{\partial M_i^\eta}{\partial s} \right|^2 + \frac{1}{\eta^2} \left| \frac{\partial M_i^\eta}{\partial \xi_1} \right|^2 + \frac{1}{\eta^2} \left| \frac{\partial M_i^\eta}{\partial \xi_2} \right|^2 \right) \\ + \int_0^t \left(\sum_{i=1}^n \beta_i^2 \int_{J^i} \left| \frac{\partial M_i^\eta}{\partial t} \right|^2 + \eta \int_{J^0} \left| \frac{\partial M_0^\eta}{\partial t} \right|^2 \right) \leq C, \end{aligned}$$

so by extracting a subsequence (still denoted by M^η), when η tends to zero, we have:

- $\nabla M_0^\eta \longrightarrow 0$ in $L^\infty(\mathbb{R}^+; L^2(J^0))$,
- $\frac{\partial M_i^\eta}{\partial \xi_1}$ and $\frac{\partial M_i^\eta}{\partial \xi_2}$ tends strongly to zero in $L^\infty(\mathbb{R}^+; L^2(J^i))$ for $i = 1, \dots, n$,

- $M_i^\eta \rightharpoonup M_i$ in $L^\infty(\mathbb{R}^+; H^1(J^i))$ weak $*$,
- $M_0^\eta \rightharpoonup M_0$ in $L^\infty(\mathbb{R}^+; H^1(J^0))$ weak $*$,
- $\frac{\partial M_i^\eta}{\partial t} \rightharpoonup \frac{\partial M_i}{\partial t}$ in $L^2(\mathbb{R}^+ \times J^i)$ weak.

In particular, we see that M_0 does not depend on the space variable, so that we write

$$M_0(t, X) = \mathbf{m}_0(t).$$

In addition, for $i \neq 0$, M_i does not depend on the transverse variables, and we denote

$$M_i(t, s, \xi_1, \xi_2) = \mathbf{m}_i(t, s).$$

Applying Aubin's lemma, M_i^η tends to M_i strongly in $L^\infty(\mathbb{R}^+; L^2(J^i))$, so we obtain that $|\mathbf{m}_i| = 1$ almost everywhere.

Taking the weak limit on the traces at $x = 0$, we obtain that for $i = 1, \dots, n$, $\mathbf{m}_i(t, 0) = \mathbf{m}_0(t)$ for almost every t . Taking the weak limit on the trace in time, we obtain that for $i = 1, \dots, n$, $\mathbf{m}_i(0, s) = M_i^0(s)$.

Second step: energy formula

Now, we aim to take the limit in the energy formula. We remark that for almost every t , we have:

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^n \beta_i^2 \int_{J^i} \left| \frac{\partial M_i^\eta}{\partial s}(t) \right|^2 - \frac{\eta}{2} \int_{J^0} M_0^\eta \cdot \mathcal{H}_0^\eta(M^\eta) - \frac{1}{2} \sum_{i=0}^n \beta_i^2 \int_{J^i} M_i^\eta \cdot \mathcal{H}^\eta(M^\eta) \\ + \frac{\alpha}{1 + \alpha^2} \int_0^t \sum_{i=1}^n \beta_i^2 \int_{J^i} \left| \frac{\partial M_i^\eta}{\partial t} \right|^2 \leq \tilde{\mathcal{E}}^\eta(M^0). \end{aligned}$$

By using lemmas 2.1 and 2.4 for the demagnetizing terms, we know that

$$\mathcal{H}_i^\eta(M^\eta) \text{ tends to } -\frac{1}{2} \left((\mathbf{m}_i \cdot \vec{v}_i) \vec{v}_i + (\mathbf{m}_i \cdot \vec{w}_i) \vec{w}_i \right) \text{ strongly in } L^\infty(\mathbb{R}^+; L^2(J^i)).$$

Taking the limit when η tends to zero, by convexity for the derivative terms, we obtain that:

$$\mathcal{E}(\mathbf{m}(t)) + \frac{\alpha}{1 + \alpha^2} \int_0^t \sum_{i=1}^n \beta_i^2 \int_0^{L_i} \left| \frac{\partial \mathbf{m}_i}{\partial t} \right|^2 \leq \mathcal{E}(\mathbf{m}^0).$$

Third step: limit equation

Let us rewrite the weak equation satisfied by m^η in the rescaled coordinates. We consider a test function $\Phi \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbf{F})$, with $\Phi = (\Phi_0, \dots, \Phi_n)$, and we define Ψ by

$$\Psi(t, x) = \Phi_i(t, (\Psi_i^\eta)^{-1}(x)) \quad \text{if } x \in \Omega_i^\eta.$$

Taking $\Psi \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1(\Omega^\eta))$ as a test function in (1.13), we obtain that:

$$\begin{aligned} \sum_{i=1}^n \int_{\mathbb{R}^+ \times J^i} \beta_i^2 \left(\frac{\partial M_i^\eta}{\partial t} - \alpha M_i^\eta \times \frac{\partial M_i^\eta}{\partial t} \right) \cdot \Phi_i + \eta \int_{J^0} \left(\frac{\partial M_0^\eta}{\partial t} - \alpha M_0^\eta \times \frac{\partial M_0^\eta}{\partial t} \right) \cdot \Phi_0 = \\ (1 + \alpha^2) \sum_{i=1}^n \int_{\mathbb{R}^+ \times J^i} \left(\beta_i^2 M_i^\eta \times \frac{\partial M_i^\eta}{\partial s} \cdot \frac{\partial \Phi_i}{\partial s} + \frac{1}{\eta^2} M_i^\eta \times \frac{\partial M_i^\eta}{\partial \xi_1} \cdot \frac{\partial \Phi_i}{\partial \xi_1} + \frac{1}{\eta^2} M_i^\eta \times \frac{\partial M_i^\eta}{\partial \xi_2} \cdot \frac{\partial \Phi_i}{\partial \xi_2} \right) \\ + (1 + \alpha^2) \frac{1}{\eta} \int_{J^0} \sum_{j=1}^3 M_0^\eta \times \frac{\partial M_0^\eta}{\partial X_j} \cdot \frac{\partial \Phi_0}{\partial X_j} - (1 + \alpha^2) \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times J^i} M_i^\eta \times \mathcal{H}_i^\eta(M^\eta) \cdot \Phi_i \\ - (1 + \alpha^2) \eta \int_{J^0} M_0^\eta \times \mathcal{H}_0^\eta(M^\eta) \cdot \Phi_0. \end{aligned} \tag{3.19}$$

In order to take the limit in the equations, we consider $\varphi = (\varphi_1, \dots, \varphi_n)$ such that for all $i \in \{1, \dots, n\}$, $\varphi_i \in \mathcal{C}_c^\infty(\mathbb{R}^+; H^1([0, L_i]))$ with $\varphi_1(t, 0) = \varphi_2(t, 0) = \dots = \varphi_n(t, 0)$ for all t . We define $\Phi = (\Phi_0, \Phi_1, \dots, \Phi_n)$ by

- $\Phi_0(t, X) = \varphi_1(t, 0)$ ($= \varphi_2(t, 0) = \dots = \varphi_n(t, 0)$ by hypothesis),
- $\Phi_i(t, s, \xi_1, \xi_2) = \varphi_i(t, s)$ for $i \geq 1$.

We obtain that $\Phi \in \mathcal{C}_c^\infty(\mathbb{R}^+; \mathbf{F})$. Since Φ_i does not depend on the transverse variables and since Φ_0 is constant, taking this test function, (3.19) yields

$$\begin{aligned} & \sum_{i=1}^n \int_{\mathbb{R}^+ \times J^i} \beta_i^2 \left(\frac{\partial M_i^\eta}{\partial t} - M_i^\eta \times \frac{\partial M_i^\eta}{\partial t} \right) \cdot \Phi_i + \eta \int_{J^0} \left(\frac{\partial M_0^\eta}{\partial t} - M_0^\eta \times \frac{\partial M_0^\eta}{\partial t} \right) \cdot \Phi_0 = \\ & 2 \sum_{i=1}^n \int_{\mathbb{R}^+ \times J^i} \beta_i^2 M_i^\eta \times \frac{\partial M_i^\eta}{\partial s} \cdot \frac{\partial \Phi_i}{\partial s} - 2 \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times J^i} M_i^\eta \times \mathcal{H}_i^\eta(M_i^\eta) \cdot \Phi_i - 2\eta \int_{J^0} M_0^\eta \times \mathcal{H}_0^\eta(M_0^\eta) \cdot \Phi_0. \end{aligned}$$

We take now the limit of the previous equality when η tends to zero. Since

- $M_i^\eta \rightarrow \mathbf{m}_i$ strongly in $L^\infty(\mathbb{R}^+; L^2(J^i))$,
- $\frac{\partial M_i^\eta}{\partial t} \rightharpoonup \frac{\partial \mathbf{m}_i}{\partial t}$ weakly in $L^2(\mathbb{R}^+ \times J^i)$,
- $\frac{\partial M_i^\eta}{\partial s} \rightharpoonup \frac{\partial \mathbf{m}_i}{\partial s}$ in $L^\infty(\mathbb{R}^+; L^2(J^i))$ weak $*$,
- $H_i^\eta(M_i^\eta) \rightarrow -\frac{1}{2} \left((\mathbf{m}_i \cdot \vec{v}_i) \vec{v}_i + (\mathbf{m}_i \cdot \vec{w}_i) \vec{w}_i \right) = \frac{1}{2} (\mathbf{m}_i - (\mathbf{m}_i \cdot \vec{u}_i) \vec{u}_i)$ strongly in $L^\infty(\mathbb{R}^+; L^2(J^i))$,

we obtain that:

$$\begin{aligned} & \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times J^i} \left(\frac{\partial \mathbf{m}_i}{\partial t} - \alpha \mathbf{m}_i \times \frac{\partial \mathbf{m}_i}{\partial t} \right) \cdot \varphi_i = (1 + \alpha^2) \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times J^i} \mathbf{m}_i \times \frac{\partial \mathbf{m}_i}{\partial s} \cdot \frac{\partial \varphi_i}{\partial s} \\ & - \frac{1 + \alpha^2}{2} \sum_{i=1}^n \beta_i^2 \int_{\mathbb{R}^+ \times J^i} \mathbf{m}_i \times (\mathbf{m}_i \cdot \vec{u}_i) \vec{u}_i \cdot \varphi_i. \end{aligned}$$

This concludes the proof of Theorem 1.2.

4 Numerical simulation

In the framework of this article, we will give a description of the numerical scheme developed in order to simulate the dynamical evolution of the magnetization in the nanowires network; we will not focus on the convergence of the scheme, we only give the discretization formulas with a swift explanation of the computations performed in order to build it.

4.1 Discretization

Each nanowire is discretized, in this article, with a regular mesh. Then, for each i in $\{1, \dots, n\}$, we set

$$\forall j \in \{1, \dots, n_j\}, s_j^i = \frac{j-1}{n_j-1} L_i = (j-1)h_i.$$

For each j in $\{1, \dots, n_i\}$, we set also control domains associated to discretization points:

$$\begin{aligned} \text{for } j \in \{2, \dots, n_i\}, \omega_j^i &= \left] s_j^i - \frac{h_j}{2}, s_j^i + \frac{h_j}{2} \right[, \\ \omega_{n_i}^i &= \left] L_i - \frac{h_i}{2}, L_i \right[, \\ \omega_1^i &= \left] 0, \frac{h_i}{2} \right[. \end{aligned}$$

Then, if we designate the Lagrange interpolation polynomial at points X for values Y (where X and Y are two real vectors of same size p) by $\mathcal{I}_Y(X) \in \mathbb{R}_p[X]$, the space of finite dimension where discrete solutions are sought is W_N , with $N = \{n_1, \dots, n_n\}$, and is defined by

$$W_N = R_N((\mathbb{R}^3)^{n_1 + \dots + n_n}),$$

where, for all $U = (u_j^i)_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, n_i\}}$ in $(\mathbb{R}^3)^{n_1 + \dots + n_n}$ we set

$$\begin{aligned} R_N(U) &= \sum_{j \in \{1, \dots, n\}} \mathcal{I}_{\{s_1^i, s_2^i, s_3^i\}}(\{U_1^i, U_2^i, U_3^i\}) \chi_{\omega_1^i} \\ &+ \sum_{\substack{i \in \{1, \dots, n\} \\ j \in \{2, \dots, n_i - 1\}}} \mathcal{I}_{\{s_{j-1}^i, s_j^i, s_{j+1}^i\}}(\{U_j^i, U_j^i, U_{j+1}^i\}) \chi_{\omega_j^i} \\ &+ \sum_{i \in \{1, \dots, n\}} \mathcal{I}_{\{s_{n_i-1}^i, s_{n_i}^i\}}(\{U_{n_i-1}^i, U_{n_i}^i\}) \chi_{\omega_{n_i}^i}, \end{aligned}$$

where χ_ω designates the function equal to 1 in ω and to 0 outside. This gives, for all s in $\prod_{i \in \{1, \dots, n\}} [0, L_i]$:

$$\begin{aligned} R_N(U)(s) &= \sum_{i \in \{1, \dots, n\}} \left(U_1^i \frac{(s - h_j)(s - 2h_i)}{2h_i^2} - U_2^i \frac{s(s - 2h_i)}{h_i^2} + U_3^i \frac{s(s - h_i)}{2h_i^2} \right) \chi_{\omega_1^i}(s) \\ &+ \sum_{\substack{i \in \{1, \dots, n\} \\ j \in \{2, \dots, n_i - 1\}}} \left(U_j^i \frac{(s - s_j^i)(s - s_{j+1}^i)}{2h_i^2} - U_j^i \frac{(s - s_{j-1}^i)(s - s_{j+1}^i)}{h_i^2} + U_{j+1}^i \frac{(s - s_{j-1}^i)(s - s_j^i)}{2h_i^2} \right) \chi_{\omega_j^i} \\ &+ \sum_{i \in \{1, \dots, n\}} - \left(U_{n_i-1}^i \frac{s - s_{n_i}^i}{h_i} + U_{n_i}^i \frac{s - s_{n_i-1}^i}{h_i} \right) \chi_{\omega_{n_i}^i}. \end{aligned}$$

Then, the concentration phase performs a mean of the equation on each discretization domains. So the finite difference scheme is written as follows: find u in $C^1([0, T]; W_N)$ such that, for all (i, j) in $\{1, \dots, n\} \times \{2, \dots, n_i - 1\}$, we set

$$\int_{\omega_j^i} \frac{\partial u}{\partial t} dx = \int_{\omega_j^i} -u \times h_i^e(u) - \alpha u \times (u \times h_i^e(u)) dx.$$

For this scheme, the integral on each subdomain is approximated via a one point quadrature formula: the middle point approximation. Then we obtain, for all (i, j) in $\{1, \dots, n\} \times \{2, \dots, n_i - 1\}$:

$$\frac{dU_j^i}{dt} = -U_j^i \times h_i^e(u)(s_j^i) - \alpha U_j^i \times (U_j^i \times h_i^e(u)(s_j^i)).$$

Hence, we have for the approximated magnetic field: for all (i, j) in $\{1, \dots, n\} \times \{2, \dots, n_i - 1\}$

$$h_i^e(u)(s_j^i) = \frac{1}{h_i^2} (U_{j-1}^i - 2U_j^i + U_{j+1}^i) + \frac{1}{2} (U_j^i \cdot \vec{u}_i) \vec{u}_i.$$

On the extremal domain, we have: for all i in $\{1, \dots, n\}$,

$$\frac{\partial u}{\partial n}(t, Li) = \frac{1}{h_i}(U_{n_{i-1}}^i - U_{n_i}^i) = 0, \text{ then } U_{n_{i-1}}^i = U_{n_i}^i,$$

then, we have for each i in $\{1, \dots, n\}$

$$\begin{aligned} h_i^e(u)(s_{n_{i-1}}^i) &= \frac{1}{h_i^2}(U_{n_{i-2}}^i - 2U_{n_{i-1}}^i + U_{n_i}^i) + \frac{1}{2}(U_{n_{i-1}}^i \cdot \vec{u}_i)\vec{u}_i, \\ &= \frac{1}{h_i^2}(U_{n_{i-2}}^i - U_{n_{i-1}}^i) + \frac{1}{2}(U_{n_{i-1}}^i \cdot \vec{u}_i)\vec{u}_i. \end{aligned}$$

On the central domain $\omega_1 = \bigcup_{i \in \{1, \dots, n\}} \omega_1^i$, we have first the continuity condition

$$U_1^1 = \dots = U_n^1 = U^1.$$

In addition, on ω_1^i , one has:

$$u_i(s) = U_1^i \frac{(s - h_j)(s - 2h_i)}{2h_i^2} - U_2^i \frac{s(s - 2h_i)}{h_i^2} + U_3^i \frac{s(s - h_i)}{2h_i^2},$$

so that

$$\frac{\partial u_i}{\partial s}(t, 0) = U_1^i \frac{3}{2h_i} - U_2^i \frac{2}{h_i} + U_3^i \frac{1}{2h_i}.$$

Therefore the Neumann condition yields

$$0 = \sum_{i=1}^n \beta_i^2 \frac{\partial u_i}{\partial s}(t, 0) = \sum_{i \in \{1, \dots, n\}} \beta_i^2 \left(U_1^i \frac{3}{2h_i} - U_2^i \frac{2}{h_i} + U_3^i \frac{1}{2h_i} \right).$$

Then we obtain

$$U^1 = \frac{1}{\sum_{i=1}^n \beta_i^2} \sum_{i=1}^n \beta_i^2 \left(-\frac{1}{3}U_3^i + \frac{4}{3}U_2^i \right).$$

Hence, in the particular case of three nanowires (see Fig. 3), we have

$$U^1 = \sum_{i=1}^3 \left(-\frac{1}{9}U_3^i + \frac{4}{9}U_2^i \right).$$

In this simple case, we then give the formula of the effective field when the value of U^1 is injected

$$\begin{aligned} \forall i \in \{1, 2, 3\}, h_i^e(u)(s_2^i) &= \frac{1}{h_i^2}(U_1^i - 2U_2^i + U_3^i) + \frac{1}{2}(U_2^i \cdot \vec{u}_i)\vec{u}_i, \\ &= \frac{1}{h_i^2} \left(\sum_{k=1}^3 \left(-\frac{1}{9}U_3^k + \frac{4}{9}U_2^k \right) - 2U_2^i + U_3^i \right) + \frac{1}{2}(U_2^i \cdot \vec{u}_i)\vec{u}_i, \\ &= \frac{1}{h_i^2} \left(-\frac{14}{9}U_2^i + \frac{8}{9}U_3^i + \sum_{k=1, k \neq i}^3 \left(-\frac{1}{9}U_3^k + \frac{4}{9}U_2^k \right) \right) + \frac{1}{2}(U_2^i \cdot \vec{u}_i)\vec{u}_i. \end{aligned}$$

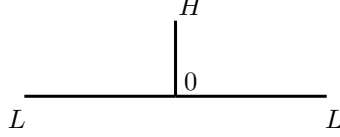


Figure 3: Three nanowires connexion.

4.2 Simulation of the reduced model

We consider the in-plane junction at the point 0 of the vertical wire of length H $[0, H]e_2$ with two horizontal nanowires of length L , $[0, L]e_1$ and $-[0, L]e_1$ (see Fig. 4.2). The wires are supposed to have the same dimensionless radius equal to 1 (so that $\beta_i = 1$ for all wires).

The program has been developed with the software Scilab [29]. The number of discretisation points is equal to 30 per unit length (*i. e.* $dx = 1/30$). In order to ensure stability of our explicit scheme, we take $dt = dx^2/10$, that is $dt = 0.0001111$. We consider 3 initial states:

- i) M_0 uniformly equal to e_1 ,
- ii) $M_0 = e_1$ on $[0, L]e_1$, $M_0 = -e_1$ on $[-L, 0]e_1$ and $M_0 = e_2$ on $[0, H]e_2$,
- iii) M_0 uniformly equal to $\frac{\sqrt{2}}{2}(e_1 + e_2)$.

For initial states i) and ii), in the case $L = 19, 2$ and $H = 9, 6$, we represent the magnetisation along the nanowires after relaxation to the equilibrium, at final time equal to 2:

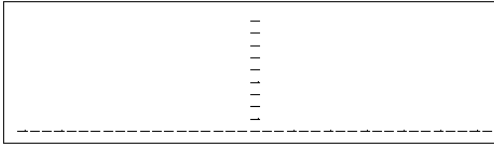


Figure 4: Initial data i)

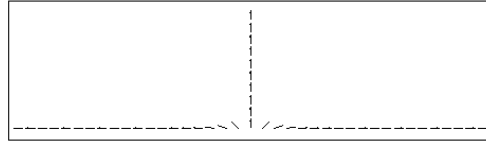


Figure 5: Initial data ii)

For initial data iii), we consider two geometries:

- A) $L = 19, 2$ and $H = 9, 6$,
- B) $L = 9, 6$ and $H = 0, 96$.

After relaxation, we observe that the configurations is almost constant equal to e_1 for short vertical wire, which is not the case for long vertical wire:

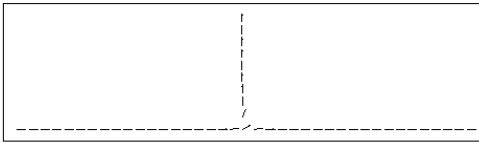


Figure 6: Initial data iii), geometry A)

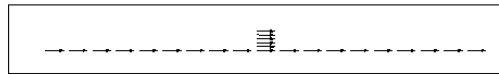


Figure 7: Initial data iii), geometry B)

4.3 Full 3D simulation

In order to compare the results obtained with the reduced 1d model to those obtained with the 3d model, in order also to observe the convergence of the 3d model to the 1d model when the radius of the wire becomes small, we performed 3d computation with the EMicroM code (see for example [24, 23, 8]) taking the radius of the wire smaller and smaller.

The EMicroM code is a full 3D simulator for the magnetization evolution in the context of the micromagnetic model. For comparison, we perform computation on a system of three connected nanowires whose configuration is similar to the geometry adopted for the reduced model. The mesh of the domain is regular and the simulation based upon a finite volume method; in our case we used 786 432 (mesh $16 \times 128 \times 128$) degrees of freedom. The parameters are the following: saturated magnetization set to $1,710^6$ A/m, exchange constant set to $A = 10^{-6}$ J/m, no anisotropy. The resulting exchange length is $\ell = 0,52\mu\text{m}$. We first consider the 3 following geometries:

1. $\mathbf{L} = 10\mu\text{m}$, $\mathbf{H} = 5\mu\text{m}$, $D = 2\mu\text{m}$,
2. $\mathbf{L} = 10\mu\text{m}$, $\mathbf{H} = 5\mu\text{m}$, $D = 1\mu\text{m}$,
3. $\mathbf{L} = 10\mu\text{m}$, $\mathbf{H} = 5\mu\text{m}$, $D = 0,5\mu\text{m}$.

After renormalisation performed in Section 1.1, the exchange length ℓ equals $0,52\mu\text{m}$, so that the corresponding dimensionless quantities are $L = \frac{\mathbf{L}}{\ell} = 19,2$ and $H = \frac{\mathbf{H}}{\ell} = 9,6$. The dimensionless radius is given by $\eta = \frac{D}{2\ell}$, so that η equal 1,9 in the case a), 0,95 in case b) and 0,48 in case c). For initial data equal to e_1 , we represent for each geometry a slice view of the 3d magnetization, and an extraction of the solution along the axes of the nanowires. These extraction are to be compared with Fig. 4

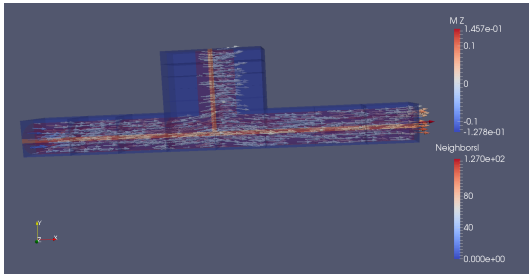


Figure 8: Slice view, geometry 1)

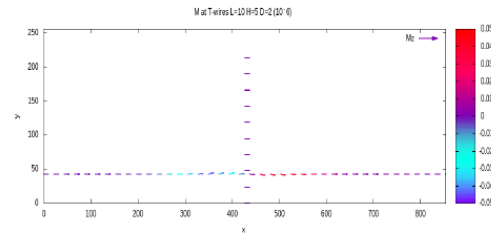


Figure 9: extraction, geometry 1)

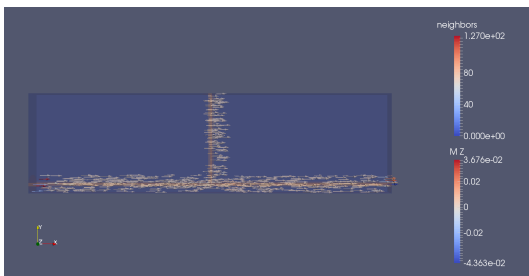


Figure 10: Slice view, geometry 2)

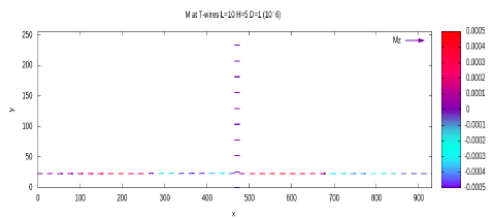


Figure 11: extraction, geometry 2)

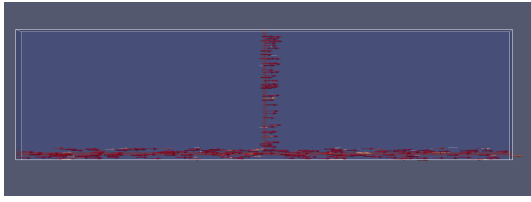


Figure 12: Slice view, geometry 3)

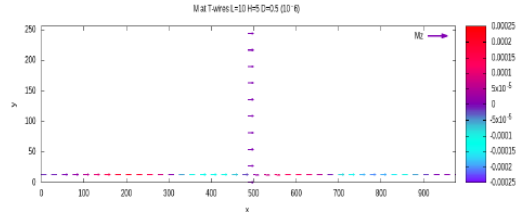


Figure 13: extraction, geometry 3)

In the same way, for initial data oriented along the wires, we represent for each geometry a slice view of the 3d magnetization, and an extraction of the solution along the axes of the nanowires. These extractions are to be compared with Fig. 5.

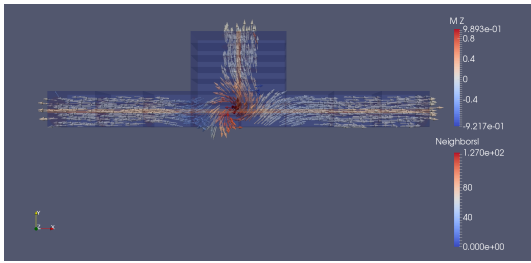


Figure 14: Slice view, geometry 1)

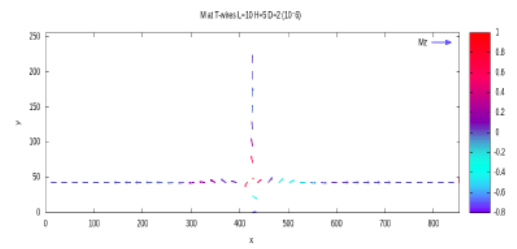


Figure 15: extraction, geometry 1)

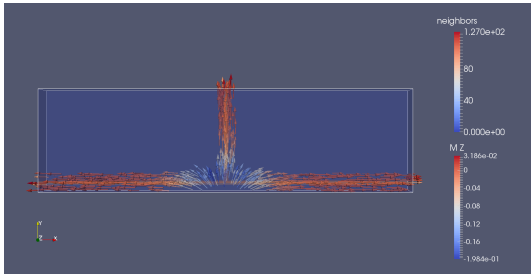


Figure 16: Slice view, geometry 2)

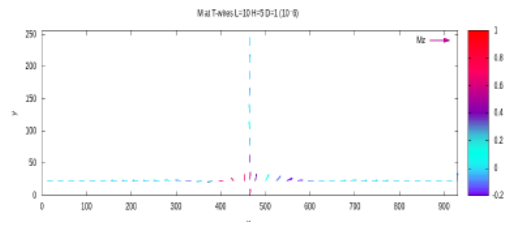


Figure 17: extraction, geometry 2)

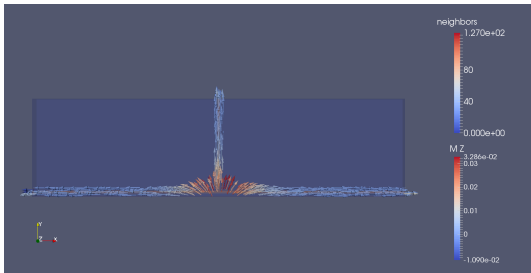


Figure 18: Slice view, geometry 3)

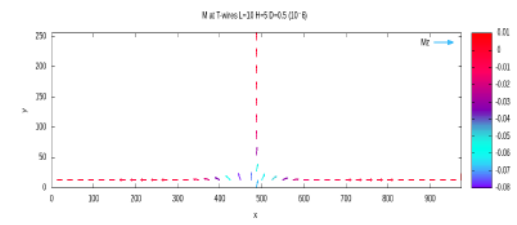


Figure 19: extraction, geometry 3)

We notice, in particular in Figure 14, that when the wire diameter is not small, the magnetization presents singularities at the junction which disappear when the diameter decreases (cf. Figure 18).

For initial data equal to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, we consider two geometries:

- a) $\mathbf{L} = 10\mu\text{m}$, $\mathbf{H} = 5\mu\text{m}$, $D = 0,5\mu\text{m}$, close to 1d geometry a.
- b) $\mathbf{L} = 5\mu\text{m}$, $\mathbf{H} = 0.5\mu\text{m}$, $D = 0.2\mu\text{m}$, close to 1d geometry b.

After relaxation, we observe in the first case the following configuration, to be compared with Fig. 6:

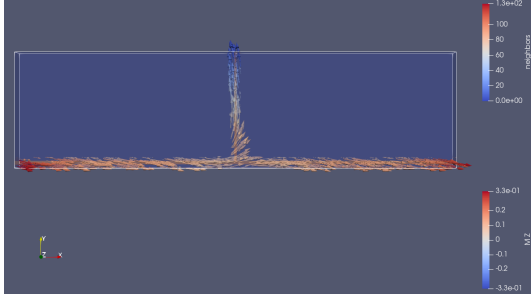


Figure 20: Slice view, geometry a)

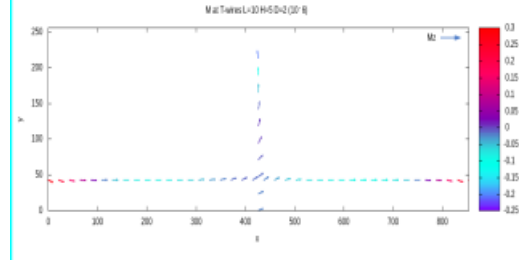


Figure 21: Extraction, geometry a)

In the second case, we observe the following configuration, to be compared with Fig. 7:

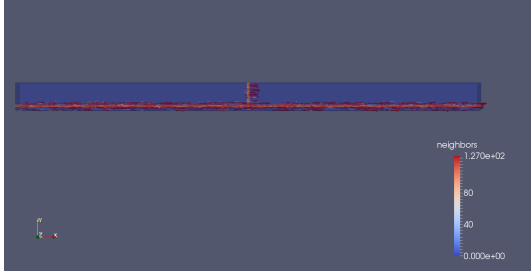


Figure 22: Slice view, geometry b)

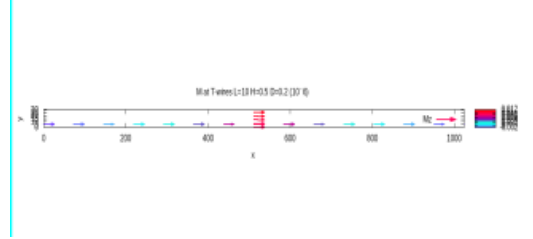


Figure 23: Extraction, geometry b)

In the two previous cases, the 1d model relaxes to the same configurations as the 3d model when the wire diameter is small.

4.4 Comparison of performances

The comparison we are able to perform here is a complexity comparison. The codes (reduced and full 3D) have not been developed on equivalent computation platforms and languages.

The discretization space steps are similar in 3D and 1D. Furthermore, the thickness of the 3D mesh, in order to capture the variations, is at least equal to \sqrt{N} where N is the number of cells in the nanowire length. Then, the number of degrees of freedom is equal to

| | 3D | reduced |
|------------------------|------------------|-----------|
| degrees of freedom | $N^{5/2}$ | $p N$ |
| time steps | $(1/N)^2$ | $(1/N)^2$ |
| complexity for $T = 1$ | $N^{9/2} \ln(N)$ | pN^3 |

where p is the number of nanowires. The complexity ratio between the approaches is proportional to $N^{3/2} \ln(N)$.

5 Conclusion

In this article we focused on the elaboration and analysis of a new model describing the behavior of the magnetization in ferromagnetic materials. Two versions have been developed, the static and the dynamical version, based upon an asymptotic analysis of the micromagnetic model developed by W.F. Brown [9] for materials in a bulk. The originality of the model is not the nanowire approximation, yet developed for infinite and isolated wires, but in the management of the connexions between wires. Thanks to this new model we are able to manage simulation of complex net of nanowires as those developed in the framework of the nano-electronic device elaboration. In order to illustrate the theoretical results, we expose an adapted finite difference scheme and perform a test on a configuration composed of three connected nanowires. The results obtained with the asymptotic model are compared to those obtained thanks to a full 3D code of micromagnetism, EMicroM. The qualitative comparison of results is totally satisfactory. Last point: the advantage of the asymptotic model for nanowires, in its domain of validity, is clearly illustrated by the comparison of the complexity of the two algorithms. Thank's to this approach we are now able to simulate realistic net of nanowires.

Acknowledgements

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