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# Boundary controllability of time-discrete parabolic systems: a moments method approach

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## Abstract

In this work, we investigate the boundary controllability of time-discrete parabolic systems, uniformly with respect to the discretization parameter. To establish our main results, we adapt and extend the moment method first introduced by Fattorini and Russell, to the time-discrete setting. While this method has proven effective in the continuous framework and in the space-discrete case, its adaptation to time-discrete systems is challenging due to the fact that the more accurate results in the field are based on several complex analysis tools.

To overcome this, we introduce a new alternative proof for constructing biorthogonals to (generalized) exponential functions in the continuous setting, based on the original proof by Fattorini and Russell which avoids the use of these tools, and taking ideas from the block moment method introduced by Benabdallah, Morancey and the first author of this work. We then manage to adapt this strategy to the discrete setting, enabling the construction and estimation of biorthogonal families for some time-discrete functions that play the same role as the exponentials at the discrete level. Our results show that these biorthogonals can be uniformly estimated for a finite portion of the spectrum, determined by the discretization parameter  $\tau$ , and that converges to the whole spectrum as  $\tau$  goes to zero.

Using this tool, we prove a relaxed null-controllability result for discrete parabolic systems. It says that there exists a bounded sequence of time discrete controls that makes the solution reach a target at final time tending to zero exponentially fast as the discretization parameter  $\tau$  goes to zero. We will also study the effects of the existence of a minimal null-control time and how this phenomenon translates into the discrete world.

**Keywords:** Time-discrete heat equation, moment's method, biorthogonal families, minimal time  
**MSC2020:** 93C55, 93B05, 93C20, 30E05.

## 1 Introduction

In this paper, we are interested in studying null-controllability properties of boundary control systems at a time-discrete level. The prototypical example we shall focus on (see Section 4 for a discussion on more general systems) is the cascade system of parabolic equations given by

$$\begin{cases} \partial_t y_1(x, t) + \mathcal{A}y_1(x, t) = 0 & \text{in } (0, 1) \times (0, T), \\ \partial_t y_2(x, t) + \mathcal{A}y_2(x, t) + y_1(x, t) = 0 & \text{in } (0, 1) \times (0, T), \\ y_1(x, t) = \mathbf{1}_{\{x=1\}}v(t) & \text{on } \{0, 1\} \times (0, T), \\ y_2(x, t) = 0 & \text{on } \{0, 1\} \times (0, T), \\ y_1(x, 0) = y_{1,0}(x), \quad y_2(x, 0) = y_{2,0}(x). \end{cases} \quad (1.1)$$

where  $T > 0$  is fixed,  $y_{i,0} \in L^2(0, 1)$ ,  $i = 1, 2$ , are given initial data,  $v = v(t)$  is a boundary control function, and  $\mathcal{A}$  stands for a positive self-adjoint operator in  $L^2(0, 1)$  defined as

$$\mathcal{A} = -\partial_x(\gamma(x)\partial_x \cdot) \quad (1.2)$$

with domain  $D(\mathcal{A}) = H_0^1(0, 1) \cap H^2(0, 1)$  and where

$$\gamma \in C^1([0, 1]) \quad \text{verifying} \quad \gamma_{\min} := \inf_{x \in (0, 1)} \gamma > 0. \quad (1.3)$$

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The main difficulty to study the controllability of systems like (1.1) is that we have only one control available to steer both components  $(y_1, y_2)$  to 0 at the final time  $T$ . In fact, notice that  $v$  is being applied only at the boundary of the first component  $y_1$  and its action enters indirectly on the system through the coupling  $y_1$  in the second equation.

It is by now well-known, see e.g. [23], that for any  $y_{1,0}$  and  $y_{2,0}$ , system (1.1) is null controllable at time  $T$ , this is, there exists a control  $v \in L^2(0, T)$  such that

$$y_1(x, T) = y_2(x, T) = 0. \quad (1.4)$$

As mentioned in that work, the boundary controllability of coupled systems like (1.1) is more difficult to prove than for scalar equations or coupled systems with distributed control. The main reason is that the usual techniques based on Carleman strategies are mostly often inefficient for these kind of problems and, except for a few cases, this approach is useless, see [6, 10, 29].

To this day, the moment method, introduced by Fattorini and Russell in their seminal work [21], is the most successful tool for studying the boundary controllability of systems like (1.1). The results in the literature are very rich and varied, thus we refer the reader only to some of the most recent works for the application of the method to the controllability of coupled parabolic systems, see [2, 3, 4, 5, 9, 10, 19, 23, 25, 26, 33, 36]. This technique has allowed in particular to obtain Kalman-type conditions, see [3], for establishing the boundary controllability of more general coupled parabolic systems like the ones we shall study here. Also, this approach has been useful to study interesting phenomena, which were not initially expected for parabolic systems, such as the minimal time for controllability or the dependence on the geometry of the control domain, see [5] and the discussion in Section 4.

## 1.1 Problem statement

Let us fix  $d \in \mathbb{N}^*$ . Adopting the tensor product formalism used in [16] (see also [1]), we will study here the controllability of the general cascade system given by

$$\begin{cases} \partial_t y + \mathcal{A} \otimes \mathbb{I} y + \mathcal{I} \otimes \mathbf{C} y = 0 & \text{in } (0, 1) \times (0, T), \\ y = \mathbf{1}_{\{x=1\}} \otimes (\mathbf{B}v) & \text{on } \{0, 1\} \times (0, T), \\ y(x, 0) = y_0(x), & \text{in } (0, 1), \end{cases} \quad (1.5)$$

where  $y_0 \in L^2(\Omega) \otimes \mathbb{R}^d$  is a given initial datum, the control and coupling matrices  $\mathbf{B} \in \mathbb{R}^d$  and  $\mathbf{C} \in \mathbb{R}^{d \times d}$ , respectively, are given by

$$\mathbf{B} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & \text{---} & & & 0 \\ 1 & \diagdown & & & \\ 0 & \diagdown & & & \\ \vdots & \diagdown & & & \\ 0 & \text{---} & 0 & 1 & 0 \end{pmatrix}, \quad (1.6)$$

while  $\mathbb{I}$  and  $\mathcal{I}$  denote the identity operators in  $\mathbb{R}^d$  and  $L^2(0, 1)$ , respectively. Observe that the matrices  $\mathbb{I}$ ,  $\mathbf{C}$  and  $\mathbf{B}$  act pointwise on the components of the solution and the control in the system while  $\mathcal{A}$  and  $\mathcal{I}$  stand for differential operators acting on infinite dimensional function spaces. To simplify the notation, we shall define the complete (non self-adjoint) operator

$$\mathcal{L} := \mathcal{A} \otimes \mathbb{I} + \mathcal{I} \otimes \mathbf{C}. \quad (1.7)$$

Although it has not been written with this formalism, system (1.5) has been already studied in [3] from a control point of view. Due to the special structure of the matrices  $\mathbf{B}$  and  $\mathbf{C}$ , referred to as the cascade form, it is indeed possible to find a single scalar control  $v \in L^2(0, T)$  such that

$$y(\cdot, T) = 0.$$

With this in mind, the main goal of this paper is to study controllability properties for the time-discrete counterpart of (1.5). To be more precise, for any given  $M \in \mathbb{N}^*$ , we set  $\tau = T/M$  and introduce the following discretization of the time interval  $[0, T]$

$$0 = t_0 < t_1 < \dots < t_M = T, \quad (1.8)$$

with  $t_n = n\tau$  and  $n \in \llbracket 0, M \rrbracket$ . Hereinafter, we shall use the notation  $\llbracket a, b \rrbracket = [a, b] \cap \mathbb{N}$  for any real numbers  $a < b$ .

For any time-discrete scalar boundary control sequence  $v = \{v^n\}_{n \in \llbracket 1, M \rrbracket} \subset \mathbb{R}$ , consider the sequence of functions  $y = \{y^n\}_{n \in \llbracket 0, M \rrbracket} \subset L^2(0, 1) \otimes \mathbb{R}^d$  verifying the recursive formula

$$\begin{cases} y^{n+1} - y^n + \mathcal{L}y^{n+1} = 0, & n \in \llbracket 0, M-1 \rrbracket, \\ y_{|\partial\Omega}^{n+1} = \mathbf{1}_{\{x=1\}} \otimes (\mathbf{B}v^{n+1}), & n \in \llbracket 0, M-1 \rrbracket, \\ y^0 = y_0(x), \end{cases} \quad (1.9)$$

where  $y^n$  (resp.  $v^n$ ) denotes the approximation of  $y$  (resp.  $v$ ) at time  $t_n$ . Notice that system (1.9) is precisely an implicit Euler discretization of (1.5).

Analogous to the continuous case, we say that system (1.9) is null-controllable at time  $T$  if for any  $y_0 \in L^2(\Omega) \otimes \mathbb{R}^d$ , there exists a sequence  $\{v^n\}_{n \in \llbracket 1, M \rrbracket}$  such that the corresponding solution satisfies

$$y^M = 0. \quad (1.10)$$

It is straightforward to see that this requirement is impossible to achieve in our discrete setting since we only have a finite number  $M$  of degrees of freedom for the scalar control to achieve the constraint (1.10) which takes place in an infinite dimensional space. Note that, even though it is less clear at first sight, restrictions also hold in the distributed control case (in which case the controls live in an infinite dimensional space and the above argument does not apply). For instance, even for a simple scalar heat equation (i.e.  $d = 1$ ), for a distributed control applied on an open bounded set  $\omega \subset (0, 1)$ , it was pointed out in [37] that (1.9) is not even approximately controllable for any  $\tau > 0$ , except for the trivial case  $\omega = (0, 1)$ .

This obstruction leads to the natural question if whether relaxing condition (1.10), one is able to say something about the controllability of (1.9). As it turns out, this is indeed the case and several works have been devoted to study this problem. Still in the case when  $d = 1$  and the control is applied internally, in [37], it has been proved that the time-discrete heat equation can be controlled uniformly with respect to  $\tau > 0$  by projecting the solutions over a suitable class of low frequency Fourier components. Later, in [20], the authors have proved that for any controllable parabolic equation, the discretization in time preserves some controllability properties by applying an adequate filtering of high frequencies. Finally, in [14] the authors establish a Carleman-type estimate for time-discrete approximations of the parabolic operator  $-\partial_t - \partial_{xx}$  allowing them to obtain a  $\varphi(\tau)$ -controllability result where a small target is reached, that is,

$$|y^M|_{L^2(0,1)} \leq C \sqrt{\varphi(\tau)} |y_0|_{L^2(0,1)}, \quad (1.11)$$

where  $C > 0$  is uniform with respect to  $\tau$  and  $\tau \mapsto \varphi(\tau)$  is a suitable function decaying exponentially as  $\tau \rightarrow 0$ . We refer to [11, 28] for other works in this direction.

We remark that the works mentioned above assume that the control is applied in the interior of the domain and that we have a scalar system. Even though in [14], the authors study the possibility of controlling many equations with few controls, as we have mentioned, Carleman based techniques are mostly useless for boundary control problems. This important limitation is what motivate us to look for a systematic way to deal with problems like (1.9).

## 1.2 Notations and functional framework

Here, we shall introduce some notation and definitions that will help us to present our computations and results as closely as possible to the continuous framework.

Let  $\{X, |\cdot|_X\}$  be a real Banach space. We denote by  $L_\tau^p(0, T; X)$ ,  $1 \leq p < \infty$  the space of time-discrete functions  $\{u^n\}_{n \in \llbracket 1, M \rrbracket} \subset X$  endowed with the norm

$$\|u\|_{L_\tau^p(0, T; X)} := \left( \sum_{n=1}^M \tau |u^n|_X^p \right)^{1/p} \quad (1.12)$$

We also define the space  $L_\tau^\infty(0, T; X)$  by means of the norm

$$\|u\|_{L_\tau^\infty(0, T; X)} := \sup_{n \in \llbracket 1, M \rrbracket} |u^n|_X.$$

In the case where  $p = 2$  and  $X$  is replaced by a Hilbert space  $\{H, (\cdot, \cdot)_H\}$ , the space of functions  $\{u^n\}_{n \in \llbracket 1, M \rrbracket} \subset H$  becomes a Hilbert space for the norm induced by the inner product

$$(u, v)_{L_\tau^2(0, T; H)} := \sum_{n=1}^M \tau (u^n, v^n)_H. \quad (1.13)$$

In particular, if  $H = \mathbb{R}$ , we will simply denote the space  $L_\tau^2(0, T; \mathbb{R})$  by  $L_\tau^2(0, T)$ . We will also use the notation  $L_\tau^2(0, +\infty)$  to denote the space of discrete functions equipped with the norm

$$\|u\|_{L_\tau^2(0, +\infty)} := \left( \sum_{n=1}^{+\infty} \tau |u^n|^2 \right)^{1/2}. \quad (1.14)$$

## 1.3 Main results

Let us consider any given increasing function  $\tau \mapsto \varphi(\tau)$  of the discretization parameter verifying

$$\varphi(0) = 0, \text{ and } \limsup_{\tau \rightarrow 0} |\tau \log \varphi(\tau)| = 0. \quad (1.15)$$

This means that  $\varphi$  does not tend to zero faster than some function  $\tau \mapsto e^{-C/\tau}$ . In practice, polynomial functions such as  $\varphi(\tau) = \tau^p$  can be considered, where  $p$  corresponds to the accuracy of the numerical method being considered, see the discussion in [12].

Our main controllability result reads as follows.

**Theorem 1.1.** *Let us consider  $T > 0$  and a discretization parameter  $\tau \in (0, 1)$ . Then, for any  $y_0 \in L^2(\Omega)$  there exists a time-discrete control  $v \in L^2_\tau(0, T)$  such that*

$$\|v\|_{L^2_\tau(0, T)} \leq C|y^0|_{L^2(\Omega)} \quad (1.16)$$

and such that the associated controlled solution of (1.5) verifies

$$|y^M|_{L^2(\Omega)} \leq C\sqrt{\varphi(\tau)}|y^0|_{L^2(\Omega)}. \quad (1.17)$$

where the constant  $C > 0$  depends only on  $T$  and  $\phi$ .

As we have mentioned before, in general we cannot achieve a standard null controllability result as in (1.10). Instead, Theorem 1.1 indicates that we can reach a target  $y^M$  whose size goes to zero as  $\tau \rightarrow 0$  at the prescribed rate  $\sqrt{\varphi(\tau)}$  with controls that remain uniformly bounded with respect to  $\tau$ . Thus, we speak of  $\varphi(\tau)$ -controllability result. In this case, we see that up to a subsequence, we can recover the known result in the continuous setting.

To prove Theorem 1.1, we will follow closely the classical moment method introduced in [22] that we suitably adapt to the time-discrete world. In fact, the moment method has been used in [3] to prove the controllability of (1.5) in the continuous framework and therefore seems to be a natural approach. Nonetheless, the proof presented in [3] or in [7] relies on several complex analysis tools for proving the existence of biorthogonal families to some (generalized) exponential functions and, as far as we know, there is no easy translation of these results to the discrete setting. For this reason, we shall present here an alternative proof for constructing and estimating biorthogonal families inspired by the block moment method as presented in [8]. Although this result is evidently not new in the literature, see for instance [23], the approach we use here is original. It can be applied to a system with any number of components and, more importantly, it is well suited to an adaptation to the time-discrete case that we will present in Theorem 3.2 and which is one of the main original results of the paper.

At this point, it is important to mention that the moment method has proven effective for establishing controllability properties of other (space-) discrete systems; see, for instance, [2, 32, 35]. However, our approach differs significantly from these works. In those papers, the underlying equations are discretized in the space variable, leading to continuous ordinary differential equations. The difficulty comes from the fact that the spectrum of the problem is a finite approximation of the continuous spectrum but the moment problem retains its structure. Here, the eigenfunctions and eigenvalues of the problem are the same as in the continuous setting, but the moment problem must be solved for time-discrete functions approximating the continuous exponentials. This approximation is what makes our problem particularly challenging. See Section 3.1 for a precise formulation and further discussion on this.

Once we are able to estimate solutions to time-discrete moment problems, other applications are possible like the time-discrete controllability property in cases where the control system has a positive minimal null-control time, see Section 4. This will be done by looking at associated relaxed observability inequalities for the time-discrete adjoint system.

## 1.4 Outline

The rest of the paper is organized as follows. In Section 2, we recall some instrumental definitions and present various results of the moment method in the continuous setting. Specifically, we will focus on giving an alternative constructive proof for biorthogonal families to exponential functions in Section 2.2. These results will be translated to the time-discrete setting in Section 3, where we present the proof of our main theorem Theorem 1.1. Section 4 is devoted to discussing the phenomenon of minimal null control time for some general time-discrete control systems. We conclude this paper by presenting some final remarks and comments in Section 5. Some technical results are gathered in appendices.

## 2 The continuous setting

We begin this section by recalling briefly the classical moment's method as introduced in [22]. The ideas presented below form the basis of the results shown in the following sections.

For an operator defined as in (1.2)–(1.3), it is well-known that there exists an orthonormal basis of  $L^2(0, 1)$  made of eigenfunctions of  $\mathcal{A}$  and associated simple and real eigenvalues. We denote by  $\Lambda$  the set of eigenvalues  $\lambda$  and we denote by  $(\phi_\lambda)_{\lambda \in \Lambda}$  the corresponding Hilbert basis of  $L^2(\Omega)$ .

Let us fix  $T > 0$  and  $y^0 \in L^2(\Omega)$ . For the sake of exposition, let us consider first the scalar problem

$$\begin{cases} \partial_t y + \mathcal{A}y = 0 & \text{in } Q, \\ y(0, t) = 0, \quad y(1, t) = v & \text{in } (0, T), \\ y(x, 0) = y^0(x) & \text{in } (0, 1). \end{cases} \quad (2.1)$$

It is well-known (see, e.g., [18]) that for any given  $v \in L^2(0, T)$ , there exists a unique solution  $y \in C(0, T; H^{-1}(0, 1))$  to (2.1) verifying

$$\langle y(t), \phi_\lambda \rangle_{-1,1} - \langle y^0, e^{-\lambda t} \phi_\lambda \rangle_{-1,1} = - \left( e^{-\lambda(t-\cdot)} \partial_x \phi_\lambda(1), v \right)_{L^2(0, T)}, \quad \forall \lambda \in \Lambda, \quad t \in [0, T],$$

where  $\langle \cdot, \cdot \rangle_{-1,1}$  denotes the usual duality product between  $H^{-1}(0,1)$  and  $H_0^1(0,1)$ . Since we are interested in the null controllability at time  $T$  of system (2.1), the above formula evaluated at  $t = T$  gives

$$\langle y(T), \phi_\lambda \rangle_{-1,1} - \langle y^0, e^{-\lambda T} \phi_\lambda \rangle_{-1,1} = - \int_0^T \gamma(1) \partial_x \phi_\lambda(1) e^{-\lambda(T-t)} v(t) dt, \quad \forall \lambda \in \Lambda, \quad (2.2)$$

We see that the null-controllability problem at time  $T$  is reduced to find  $v \in L^2(0, T)$  such that

$$\int_0^T v(t) e^{-\lambda(T-t)} dt = \frac{\langle y^0, \phi_\lambda \rangle_{-1,1} e^{-\lambda T}}{\gamma(1) \partial_x \phi_\lambda(1)}, \quad \forall \lambda \in \Lambda. \quad (2.3)$$

This set of equations is called a moment problem. The moment's method hence consists in solving (2.3) by using a so-called biorthogonal family to the exponentials  $(e^{-\lambda(T-\bullet)})_{\lambda \in \Lambda}$ .

**Definition 2.1.** We say that a family  $(q_{\lambda'})_{\lambda' \in \Lambda}$  is biorthogonal to  $(e^{-\lambda(T-\bullet)})_{\lambda \in \Lambda}$  in  $L^2(0, T)$  if

$$\int_0^T e^{-\lambda(T-t)} q_{\lambda'}(t) dt = \delta_{\lambda, \lambda'}, \quad \forall \lambda, \lambda' \in \Lambda. \quad (2.4)$$

With this definition, it is not difficult to see that we may formally solve the moment problem by defining

$$v(t) := \sum_{\lambda \in \Lambda} \frac{\langle y^0, \phi_\lambda \rangle_{-1,1} e^{-\lambda T}}{\gamma(1) \partial_x \phi_\lambda(1)} q_\lambda(t). \quad (2.5)$$

It is well known that the quantities  $|\partial_x \phi_\lambda(1)|$  are uniformly bounded from below. For instance, one can find in [13, Theorem IV.1.3] the following lower bound

$$|\partial_x \phi_\lambda(1)| \geq C\sqrt{\lambda}, \quad \forall \lambda \in \Lambda. \quad (2.6)$$

Thus, in order to justify the methodology it remains to prove the existence of the biorthogonal family  $(q_\lambda)_{\lambda \in \Lambda}$ , as well as a suitable upper bound of the  $L^2$ -norms of  $q_\lambda$  for all  $\lambda \in \Lambda$ . This will let us obtain the convergence of the formal series (2.5) and conclude the proof.

In this context, those two steps summarize the essentials of the moment's method, as developed most notably, in [22], and will be at the heart of our analysis.

## 2.1 The moment's method for the controllability of coupled parabolic systems

The methodology presented above can be applied, with very few modifications, to address the controllability of the coupled system (1.5). Since Carleman estimates are useless in this particular case, this approach is of particular interest.

First, it is not difficult to see that the adjoint operator of  $\mathcal{L}$  (defined in (1.7)) is given by  $\mathcal{L}^* := \mathcal{A} \otimes \mathbb{1} + \mathcal{I} \otimes \mathbf{C}^*$  and has the same set of eigenvalues  $\Lambda$  as  $\mathcal{A}$ . Each of this eigenvalues is geometrically simple but has algebraic multiplicity equal to  $d$  and is associated to a  $d \times d$  Jordan block. More precisely, we have that

$$\mathcal{L}^*(\phi_\lambda \otimes \mathbf{e}_1) = \lambda \phi_\lambda \otimes \mathbf{e}_1 \quad (2.7)$$

and, iteratively,

$$\mathcal{L}^*(\phi_\lambda \otimes \mathbf{e}_j) = \lambda \phi_\lambda \otimes \mathbf{e}_j + \phi_\lambda \otimes \mathbf{e}_{j-1}, \quad j \in \llbracket 2, d \rrbracket. \quad (2.8)$$

where  $(\mathbf{e}_1, \dots, \mathbf{e}_d)$  is the canonical basis of  $\mathbb{R}^d$ . Thus,  $\phi_\lambda \otimes \mathbf{e}_1$  is an eigenfunction of  $\mathcal{L}^*$  and  $\phi_\lambda \otimes \mathbf{e}_j$ ,  $j \in \llbracket 2, d \rrbracket$ , are generalized eigenfunctions.

Now, for any  $\lambda \in \mathbb{R}^+$  and any fixed  $t$ , we define  $e_t[\lambda]$  to be the exponential

$$e_t[\lambda] := e^{-\lambda t}. \quad (2.9)$$

and  $e[\lambda] : (0, +\infty) \rightarrow \mathbb{R}$  to be the associated function of the time variable

$$e[\lambda] : t \mapsto e_t[\lambda]. \quad (2.10)$$

With these notations and using the formalism of divided differences (see Section A.1), we can give a convenient and compact expression of the action of the semigroup on the (generalized) eigenfunctions as follows

$$\begin{aligned} e^{-t\mathcal{L}^*}(\phi_\lambda \otimes \mathbf{e}_r) &= \sum_{\ell=0}^{r-1} \frac{(-t)^\ell}{\ell!} e^{-\lambda t} (\phi_\lambda \otimes \mathbf{e}_{r-\ell}) \\ &= \sum_{\ell=0}^{r-1} e_t[\lambda^{(\ell+1)}] (\phi_\lambda \otimes \mathbf{e}_{r-\ell}), \quad \forall t \geq 0, \quad \forall r \in \llbracket 1, d \rrbracket. \end{aligned} \quad (2.11)$$

where, in particular, we have used Definition A.2.

Using formula (2.11) and arguing as previously, we can readily see that the boundary controllability problem for (1.5) reduces to find a control  $v \in L^2(0, T)$  such that for any  $\lambda \in \Lambda$  and any  $r \in \llbracket 1, d \rrbracket$  the following equality holds

$$\left\langle y^0, e^{-T\mathcal{L}^*}(\phi_\lambda \otimes \mathbf{e}_r) \right\rangle_{-1,1} = \left( (\partial_{x|x=1} \otimes B^*) e^{-(T-\bullet)\mathcal{L}^*}(\phi_\lambda \otimes \mathbf{e}_r), v \right)_{L^2(0,T)}.$$

Thanks to the special structure of matrix  $B$  and performing the change of variables  $T-t \mapsto t$ , we deduce using (2.11) that

$$\left\langle y^0, e^{-T\mathcal{L}^*}(\phi_\lambda \otimes \mathbf{e}_r) \right\rangle_{-1,1} = \gamma(1)\partial_x\phi_\lambda(1) \int_0^T e_t[\lambda^{(r)}]v(T-t)dt, \quad \forall \lambda \in \Lambda, \forall r \in \llbracket 1, d \rrbracket. \quad (2.12)$$

From this expression, it is clear what the moment problem to be solved is. Specifically, we look for a family of functions that is biorthogonal to  $(e[\lambda^{(i)}])_{\lambda \in \Lambda}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2(0, T)$ . The precise definition is as follows.

**Definition 2.2.** We say that a family  $(q_{\lambda', j})_{\lambda' \in \Lambda}^{j \in \llbracket 1, d \rrbracket}$  is biorthogonal to  $(e[\lambda^{(i)}])_{\lambda \in \Lambda}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2(0, T)$  if

$$\int_0^T e_t[\lambda^{(i)}]q_{\lambda', j}(t)dt = \delta_{\lambda, \lambda'}\delta_{i, j}, \quad \forall \lambda, \lambda' \in \Lambda, \forall i, j \in \llbracket 1, d \rrbracket. \quad (2.13)$$

With this definition, we can construct an auxiliary control function by setting

$$\tilde{v}(t) := \sum_{j=1}^d \sum_{\lambda \in \Lambda} \frac{\alpha_\lambda^j}{\gamma(1)\partial_x\phi_\lambda(1)} q_{\lambda, j}(t) \quad (2.14)$$

where we have denoted  $\alpha_\lambda^j := \left\langle y^0, e^{-T\mathcal{L}^*}(\phi_\lambda \otimes \mathbf{e}_j) \right\rangle_{-1,1}$  and set the change of variables  $v(t) = \tilde{v}(T-t)$ . As in the scalar case, to justify the method, we will prove the existence of the biorthogonal family and also give precise estimates on its norm. The controllability result will follow by observing that we have the bound

$$|\alpha_\lambda^j| \leq C_T e^{-\lambda T} |y_0|_{L^2},$$

as well as the bound from below for the normal derivatives given in (2.6).

## 2.2 Existence of biorthogonal families to exponential functions

### 2.2.1 Assumptions

Before stating the main result of this section, we need to introduce some instrumental definitions and assumptions on the family of eigenvalues  $\Lambda$ .

From now on, we will work with families of eigenvalues  $\Lambda \subset (0, +\infty)$  verifying

- Asymptotic behavior:

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty. \quad (2.15)$$

- Gap condition with parameter  $\rho$ :

$$|\lambda - \lambda'| > \rho, \quad \forall \lambda, \lambda' \in \Lambda, \quad \lambda \neq \lambda'. \quad (2.16)$$

For a family  $\Lambda$  verifying (2.15), we introduce the following notion.

**Definition 2.3.** A function  $R: \mathbb{R} \rightarrow [0, \infty)$  is called a remainder function for the family  $\Lambda$  if it satisfies

$$\lim_{r \rightarrow \infty} R(r) = 0$$

and

$$\sum_{\substack{\lambda \in \Lambda \\ \lambda > r}} \frac{1}{\lambda} \leq R(r), \quad \forall r \in \mathbb{R}. \quad (2.17)$$

Clearly, the family  $\Lambda$  of eigenvalues associated to the operator  $\mathcal{A}$  defined in (1.2)–(1.3) verify (2.15)–(2.16) and a non-increasing remainder function  $R$  can be computed such that  $R(r) \sim \frac{C}{\sqrt{r}}$  as  $r \rightarrow \infty$ . Also note that since (2.15) holds

$$\lambda \geq 1/R(0) \quad \text{for all } \lambda \in \Lambda. \quad (2.18)$$

## 2.2.2 Biorthogonal families

Now, we are in position to state the main result of this section. This result, which is already known in the literature (see Remark 2.5) gives the existence of a biorthogonal family to the family of functions  $(e[\lambda^{(i)}])_{\lambda \in \Lambda}^{i \in \llbracket 1, d \rrbracket}$ , as well as suitable estimates.

**Theorem 2.4.** *Let  $d \in \mathbb{N}^*$ ,  $T \in (0, +\infty]$  and  $\Lambda$  be a family of eigenvalues verifying (2.15)–(2.18). There exists a family  $(q_{\lambda, i})_{\lambda \in \Lambda}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2(0, T)$  satisfying*

$$\left( e[\lambda^{(i)}], q_{\lambda', j} \right)_{L^2(0, T)} = \delta_{\lambda, \lambda'} \delta_{i, j}, \quad \forall \lambda, \lambda' \in \Lambda, \quad \forall i, j \in \llbracket 1, d \rrbracket \quad (2.19)$$

and the estimate

$$\|q_{\lambda, i}\|_{L^2(0, +\infty)} \leq C e^{\delta(\lambda)\lambda}, \quad \forall \lambda \in \Lambda, \quad \forall i \in \llbracket 1, d \rrbracket, \quad (2.20)$$

where  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  is a non-increasing function such that  $\lim_{s \rightarrow +\infty} \delta(s) = 0$  that only depends on  $\rho$ ,  $R$ ,  $d$ . The constant  $C > 0$  depends at most on  $d$ ,  $R$  and  $T$ .

With this result at hand, it is not difficult to see that the control defined in (2.14) is bounded in  $L^2(0, T)$  and that the null controllability problem for system (1.5) is solved. Since similar computations will be performed at the discrete level, we skip the details and only focus on the original parts of the argument.

**Remark 2.5.** *As said previously this result is not new by itself but the strategy of proof we propose is original and, more importantly, can be adapted to the time-discrete setting (see Section 3), which is not the case for the other approaches.*

- In [3, 7, 13], such a result was proved, even with more accurate estimates in some cases, by using Laplace transform, Paley-Wiener theorems and the construction of suitable analytic functions called multipliers. Unfortunately, it is not known how to follow this way to obtain those results in the time-discrete setting.
- On the other hand, the proof given in [23], in the case  $d = 2$ , follows closely the method given for  $d = 1$  in [22]. However, this proof does not immediately translates to any value of  $d$  and seems to be delicate to translate into the time-discrete framework as it.

*This process can be simplified by following the ideas of [8] that consists in grouping the eigenvalues and using the concept of divided differences, which has the advantage to work for any value of  $d$ . This is the strategy we shall use in this paper.*

In the remainder of the section, we will give a proof of Theorem 2.4 for  $T = +\infty$  that relies on quite direct computations that we will manage to adapt later to the discrete setting. Then using a classical restriction argument (see e.g. [22, Theorem 1.3]), we will obtain the same result in a time interval  $(0, T)$  for any  $T > 0$  finite.

More precisely, our proof of Theorem 2.4 will be mainly based on the following result.

**Proposition 2.6.** *Let  $d \in \mathbb{N}^*$  and  $\Lambda$  be a family of eigenvalues verifying (2.15)–(2.18). There exists  $\epsilon_0 > 0$  only depending on  $d$ ,  $\rho$  and  $R$  such that for any  $\epsilon < \epsilon_0$ , there exists a family  $(q_{\lambda, i, \epsilon})_{\lambda \in \Lambda}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2(0, +\infty)$  satisfying*

$$\left( e[\lambda, \dots, \lambda + (i-1)\epsilon], q_{\lambda', j, \epsilon} \right)_{L^2(0, +\infty)} = \delta_{\lambda, \lambda'} \delta_{i, j}, \quad \forall \lambda, \lambda' \in \Lambda, \quad \forall i, j \in \llbracket 1, d \rrbracket \quad (2.21)$$

and the estimate

$$\|q_{\lambda, i, \epsilon}\|_{L^2(0, +\infty)} \leq C e^{\delta(\lambda)\lambda}, \quad \forall \lambda \in \Lambda, \quad \forall i \in \llbracket 1, d \rrbracket, \quad (2.22)$$

where  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  is a non-increasing function such that  $\lim_{s \rightarrow +\infty} \delta(s) = 0$  that only depends on  $\rho$ ,  $d$ ,  $R$ , and  $C > 0$  is a constant only depending on  $d$  and  $R$ .

This result says that we can find a biorthogonal family to divided differences of the exponentials  $e[\lambda]$ , which in addition satisfies good uniform estimates (w.r.t  $\epsilon$ ). Then, it can be seen that

$$e[\lambda, \dots, \lambda + (i-1)\epsilon] \rightarrow e[\lambda^{(i)}] \quad \text{in } L^2(0, +\infty)$$

and by a limit process one can deduce the proof of Theorem 2.4.

This same strategy will be employed at the discrete level, so we shall only focus on proving Proposition 2.6. The proof combines the classical methodology of Fattorini and Russell and some ideas of the block moment method presented in [8].

## 2.2.3 Reduction to divided differences

Here we prove the existence of biorthogonal families to divided differences of the exponentials.

*Proof of Proposition 2.6.* Let  $\epsilon \in \left(0, \frac{\rho}{4(d-1)}\right)$  and define

$$\tilde{\Lambda} := \bigcup_{r=0}^{d-1} (\Lambda + r\epsilon) = \bigcup_{\lambda \in \Lambda} G_\lambda \quad (2.23)$$



where  $G_\lambda$  is the group formed by the ordered elements

$$G_\lambda = \{\lambda, \lambda + \epsilon, \dots, \lambda + (d-1)\epsilon\}. \quad (2.24)$$

Those sets depend on  $\epsilon$  but we will not make this appear in the notation to ease as much as possible the reading. Defined in this way, we are gluing together  $d$  families of eigenvalues where each of them satisfy a gap condition (2.16). Note that our choice of  $\epsilon$  above ensures that  $G_\lambda \cap G_{\lambda'} = \emptyset$  for any  $\lambda, \lambda' \in \Lambda$ .

For the sake of exposition, we have divided the rest of the proof in three different steps that hold for each fixed  $\lambda \in \Lambda$ .

– **Step 1. Measuring the distance between some exponential functions.** Let us consider the closed subspace of  $L^2(0, +\infty)$  defined by

$$E_\lambda := \overline{\text{span} \{e[\sigma] : \sigma \in \tilde{\Lambda}_\lambda\}}^{L^2(0, +\infty)} \quad (2.25)$$

where

$$\tilde{\Lambda}_\lambda := \tilde{\Lambda} \setminus G_\lambda,$$

and let  $P_{E_\lambda}$  be the orthogonal projection onto  $E_\lambda$  in  $L^2(0, +\infty)$ .

Following the spirit of [22], the first step is to compute the distance in the  $L^2$ -norm of the functions  $e[\sigma]$  with  $\sigma$  in  $G_\lambda$  to the space  $E_\lambda$ . We have the following result

**Lemma 2.7.** *For any  $\sigma, \sigma' \in G_\lambda$*

$$(e[\sigma] - P_{E_\lambda}e[\sigma], e[\sigma'] - P_{E_\lambda}e[\sigma'])_{L^2(0, +\infty)} = \frac{1}{\sigma + \sigma'} W_\lambda(\sigma) W_\lambda(\sigma')$$

where

$$W_\lambda(\sigma) = \prod_{\mu \in \tilde{\Lambda}_\lambda} \frac{\mu - \sigma}{\mu + \sigma}.$$

The proof of this result is classical and we only made some minor adaptations for handling the grouping of eigenvalues. For the sake of completeness, we present the proof in Appendix B.1.

We observe also that, by definition of the orthogonal projection, we have

$$(e[\sigma] - P_{E_\lambda}e[\sigma], e[\gamma])_{L^2(0, +\infty)} = 0 \quad (2.26)$$

for all  $\sigma \in G_\lambda$  and  $\gamma \in \tilde{\Lambda}_\lambda$ .

– **Step 2. A biorthogonal for an auxiliary family.** For any  $\sigma \in G_\lambda$ , we define

$$\tilde{e}[\sigma] := e[\sigma] - P_{E_\lambda}e[\sigma], \text{ and } f[\sigma] := \frac{\tilde{e}[\sigma]}{W_\lambda[\sigma]}.$$

Note from Lemma 2.7 that

$$(f[\sigma], f[\sigma'])_{L^2(0, +\infty)} = \frac{1}{\sigma + \sigma'} = (e[\sigma], e[\sigma'])_{L^2(0, +\infty)}, \quad \forall \sigma, \sigma' \in G_\lambda \quad (2.27)$$

where the rightmost equality follows from a direct computation. Also notice that by linearity, the family  $\mathcal{F}_{\lambda, \epsilon} := ((2\lambda)^{i-1/2} f[\lambda, \dots, \lambda + (i-1)\epsilon])_{i \in \llbracket 1, d \rrbracket}$  spans the same space as

$$\mathcal{E}_{\lambda, \epsilon} := (\tilde{e}[\lambda, \dots, \lambda + (i-1)\epsilon])_{i \in \llbracket 1, d \rrbracket}.$$

Taking into account all of this, we will focus on building a biorthogonal family  $(q_{\lambda, i}^{\text{aux}})_{i \in \llbracket 1, d \rrbracket}$  to  $\mathcal{F}_{\lambda, \epsilon}$ , that is a family satisfying

$$\left( (2\lambda)^{i-1/2} f[\lambda, \dots, \lambda + (i-1)\epsilon], q_{\lambda, j}^{\text{aux}} \right)_{L^2(0, +\infty)} = \delta_{i, j}, \quad \forall i, j \in \llbracket 1, d \rrbracket, \quad (2.28)$$

together with suitable estimates. Since  $\lambda$  is fixed, and since  $\mathcal{F}_{\lambda, \epsilon}$  is linearly independent, we are faced with a finite dimensional problem that has a unique solution in the space spanned by  $\mathcal{F}_{\lambda, \epsilon}$ , which is the one with minimal  $L^2$  norm.

This solution can be found in the following form

$$q_{\lambda, i}^{\text{aux}} := \nu_{i,1} (2\lambda)^{1/2} f[\lambda] + \nu_{i,2} (2\lambda)^{3/2} f[\lambda, \lambda + \epsilon] + \dots + \nu_{i,d} (2\lambda)^{d-1/2} f[\lambda, \dots, \lambda + (d-1)\epsilon], \quad i \in \llbracket 1, d \rrbracket, \quad (2.29)$$

where  $\nu_{i,j} \in \mathbb{R}$ ,  $i, j \in \llbracket 1, d \rrbracket$  are suitable coefficients. By construction, the biorthogonal problem (2.28) can be reduced to finding the coefficients  $\boldsymbol{\nu}_i = (\nu_{i,1}, \dots, \nu_{i,d})^\top \in \mathbb{R}^d$  such that

$$\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\nu}_i = \mathbf{e}_i, \quad i \in \llbracket 1, d \rrbracket, \quad (2.30)$$

where  $\mathbf{e}_i$  is the  $i$ -th canonical vector of  $\mathbb{R}^d$  and  $\mathcal{G}_{\lambda,\epsilon}$  is the Gram matrix of the family  $\mathcal{F}_{\lambda,\epsilon}$  with entries

$$\mathcal{G}_{\lambda,\epsilon} = \left( (2\lambda)^{i+j-1} (f[\lambda, \dots, \lambda + (i-1)\epsilon], f[\lambda, \dots, \lambda + (j-1)\epsilon])_{L^2(0,+\infty)} \right)_{1 \leq i, j \leq d}. \quad (2.31)$$

The matrix  $\mathcal{G}_{\lambda,\epsilon}$  is invertible and the coefficients  $\boldsymbol{\nu}_i$  can be readily obtained from (2.30) by setting

$$\boldsymbol{\nu}_i = \mathcal{G}_{\lambda,\epsilon}^{-1} \mathbf{e}_i, \quad i \in \llbracket 1, d \rrbracket. \quad (2.32)$$

The inverse of the Gram matrix  $\mathcal{G}_{\lambda,\epsilon}$  can be estimated by the following result.

**Lemma 2.8.** *Under the assumptions of Proposition 2.6, there exists  $\epsilon_0 > 0$  only depending on  $\rho$ ,  $R$ , and  $d$  such that for any  $\epsilon < \epsilon_0$ , there exists a constant  $c > 0$  only depending on  $d$  such that*

$$(\mathcal{G}_{\lambda,\epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq c \|\boldsymbol{\xi}\|_{\infty}^2 \quad (2.33)$$

for any  $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^\top \in \mathbb{R}^d$  and any  $\lambda \in \Lambda$  and where  $(\cdot, \cdot)$  stands for the usual inner product in  $\mathbb{R}^d$ .

The proof of Lemma 2.8 can be found at the end of this section and it is based on the fact that for any  $\sigma, \sigma' \in G_\lambda$ , formula (2.27) gives a nice expression to compute explicitly the entries of (2.31).

To end this step, we shall estimate the  $L^2$ -norm of  $q_{\lambda,i}^{\text{aux}}$ ,  $i \in \llbracket 1, d \rrbracket$ . Using expressions (2.29) and (2.31), we get

$$\|q_{\lambda,i}^{\text{aux}}\|_{L^2(0,+\infty)}^2 = (\mathcal{G}_{\lambda,\epsilon} \boldsymbol{\nu}_i, \boldsymbol{\nu}_i), \quad i \in \llbracket 1, d \rrbracket. \quad (2.34)$$

Then, from (2.30), we have that

$$\|q_{\lambda,i}^{\text{aux}}\|_{L^2(0,+\infty)}^2 = (\mathbf{e}_i, \boldsymbol{\nu}_i), \quad i \in \llbracket 1, d \rrbracket,$$

whence

$$\|q_{\lambda,i}^{\text{aux}}\|_{L^2(0,+\infty)}^2 \leq \sqrt{d} \|\boldsymbol{\nu}_i\|_{\infty}. \quad (2.35)$$

Using formula (2.33) with  $\boldsymbol{\xi} = \boldsymbol{\nu}_i$  and (2.34), we deduce that

$$\|\boldsymbol{\nu}_i\|_{\infty} \leq \tilde{C}, \quad (2.36)$$

for some  $\tilde{C} > 0$  not depending on  $\epsilon$  and  $\lambda$ . Combining estimates (2.35) and (2.36) yield that there is a uniform constant  $C > 0$  such that

$$\|q_{\lambda,i}^{\text{aux}}\|_{L^2(0,+\infty)} \leq C \quad (2.37)$$

for all  $i \in \llbracket 1, d \rrbracket$  and  $\lambda \in \Lambda$ .

– **Step 3. Final arrangements and conclusion.** With the biorthogonal functions  $(q_{\lambda,i}^{\text{aux}})_{i \in \llbracket 1, d \rrbracket}$ , let us set

$$q_{\lambda,i} = \sum_{k=i}^d \left( \frac{1}{W_\lambda} \right) [\lambda + (i-1)\epsilon, \lambda + (k-1)\epsilon] (2\lambda)^{k-1/2} q_{\lambda,k}^{\text{aux}}, \quad i \in \llbracket 1, d \rrbracket.$$

We will see that these  $q_{\lambda,i}$  verify (2.21) and (2.22). Recalling that  $\tilde{e}[\sigma] = f[\sigma] W_\lambda[\sigma]$  for  $\sigma \in G_\lambda$ , by Leibniz rule (see Proposition A.4), we get

$$\tilde{e}[\lambda, \dots, \lambda + (\ell-1)\epsilon] = \sum_{r=1}^{\ell} f[\lambda, \dots, \lambda + (r-1)\epsilon] W_\lambda[\lambda + (r-1)\epsilon, \dots, \lambda + (\ell-1)\epsilon], \quad \ell \in \llbracket 1, d \rrbracket,$$

whence, for  $k, \ell \in \llbracket 1, d \rrbracket$ , we have

$$\begin{aligned} & (\tilde{e}[\lambda, \dots, \lambda + (\ell-1)\epsilon], q_{\lambda,k})_{L^2(0,+\infty)} \\ &= \sum_{r=1}^{\ell} W_\lambda[\lambda + (r-1)\epsilon, \dots, \lambda + (\ell-1)\epsilon] (f[\lambda, \dots, \lambda + (r-1)\epsilon], q_{\lambda,k})_{L^2(0,+\infty)} \\ &= \sum_{r=1}^{\ell} \sum_{j=k}^d W_\lambda[\lambda + (r-1)\epsilon, \dots, \lambda + (\ell-1)\epsilon] \left( \frac{1}{W_\lambda} \right) [\lambda + (k-1)\epsilon, \dots, \lambda + (j-1)\epsilon] \\ & \quad \times (2\lambda)^{j-r} \left( (2\lambda)^{r-1/2} f[\lambda, \dots, \lambda + (r-1)\epsilon], q_{\lambda,j}^{\text{aux}} \right)_{L^2(0,+\infty)}, \end{aligned}$$

and in view of (2.28), the only terms that remain are the ones when  $r = j$ , that is

$$\begin{aligned} & (\tilde{e}[\lambda, \dots, \lambda + (\ell-1)\epsilon], q_{\lambda,k})_{L^2(0,+\infty)} \\ &= \sum_{r=1}^{\ell} \sum_{j=k}^d W_\lambda[\lambda + (r-1)\epsilon, \dots, \lambda + (\ell-1)\epsilon] \left( \frac{1}{W_\lambda} \right) [\lambda + (k-1)\epsilon, \dots, \lambda + (j-1)\epsilon] (2\lambda)^{j-r} \delta_{j,r}. \end{aligned}$$

Observe that for  $\ell < k$ , the above sum is zero since the case  $j = r$  is not possible. On the other hand, for  $\ell \geq k$ , by Leibniz rule (see Proposition A.4), we obtain

$$\begin{aligned}
& (\tilde{e}[\lambda, \dots, \lambda + (\ell - 1)\epsilon], q_{\lambda, k})_{L^2(0, +\infty)} \\
&= \sum_{r=k}^{\ell} W_{\lambda} [\lambda + (r - 1)\epsilon, \dots, \lambda + (\ell - 1)\epsilon] \left( \frac{1}{W_{\lambda}} \right) [\lambda + (k - 1)\epsilon, \dots, \lambda + (r - 1)\epsilon] \\
&= \left( W_{\lambda} \frac{1}{W_{\lambda}} \right) [\lambda + (k - 1)\epsilon, \dots, \lambda + (\ell - 1)\epsilon] \\
&= \delta_{k, \ell}.
\end{aligned} \tag{2.38}$$

since  $W_{\lambda} \frac{1}{W_{\lambda}}$  is a constant function.

We can now check that the family  $(q_{\lambda, k})_{\lambda \in \Lambda}^{k \in \llbracket 1, d \rrbracket}$  we have just built satisfies (2.21):

- First, formula (2.38) exactly gives (2.21) in the case  $\lambda = \lambda'$ .
- Next, if  $\lambda' \in \Lambda$  is such that  $\lambda' \neq \lambda$  then  $\lambda' + (i - 1)\epsilon \in \tilde{\Lambda}_{\lambda}$  for all  $i \in \llbracket 1, d \rrbracket$  and by construction (recall (2.26) and that  $q_{\lambda, \epsilon} \in \mathcal{F}_{\lambda, \epsilon}$ ) we have

$$(e[\lambda' + (i - 1)\epsilon], q_{\lambda, \epsilon})_{L^2(0, +\infty)} = 0$$

which, by linear combinations, yields

$$(e[\lambda', \dots, \lambda' + (i - 1)\epsilon], q_{\lambda, \epsilon})_{L^2(0, +\infty)} = 0, \quad \forall i \in \llbracket 1, d \rrbracket. \tag{2.39}$$

This proves the claim.

Finally, from (2.37), we readily obtain

$$\|q_{\lambda, k}\|_{L^2(0, +\infty)} \leq C \lambda^{d-1/2} \max_{j \in \llbracket k, d \rrbracket} \left| \left( \frac{1}{W_{\lambda}} \right) [\lambda + (k - 1)\epsilon, \dots, \lambda + (j - 1)\epsilon] \right|.$$

From Proposition A.3, for any  $k, j \in \llbracket 1, d \rrbracket$  we have

$$\frac{1}{W_{\lambda}} [\lambda + (k - 1)\epsilon, \dots, \lambda + (j - 1)\epsilon] = \frac{1}{(j - k)!} \left( \frac{1}{W_{\lambda}} \right)^{(j-k)} (\theta)$$

for some  $\theta \in [\lambda + (k - 1)\epsilon, \lambda + (j - 1)\epsilon]$ . Thus by Proposition A.8, we obtain

$$\|q_{\lambda, k}\|_{L^2(0, +\infty)} \leq C \lambda^{d-1/2} e^{\delta(\lambda)\lambda}. \tag{2.40}$$

for some  $C > 0$  only depending on  $d$ . From (2.18), we get

$$\lambda^{d-1/2} \leq C_{R, d} (1 + \lambda)^d = C_{R, d} e^{\frac{d \log(1 + \lambda)}{\lambda} \lambda} \tag{2.41}$$

thus, noting that the function  $\lambda \mapsto \log(1 + \lambda)/\lambda$  is decreasing and tends to 0 as  $\lambda \rightarrow +\infty$ , we obtain the desired estimate (2.22) by combining estimates (2.40)–(2.41) and redefining the function  $\delta$ . This ends the proof of Proposition 2.6.  $\square$

## 2.2.4 Estimate of the inverse of the Gram matrix $\mathcal{G}_{\lambda, \epsilon}$

We devote this section to prove Lemma 2.8. For any  $\xi \in \mathbb{R}^d$ , we have

$$(\mathcal{G}_{\lambda, \epsilon} \xi, \xi) = \left\| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} f[\lambda, \dots, \lambda + (i - 1)\epsilon] \right\|_{L^2(0, +\infty)}^2.$$

By the linearity of divided differences and using (2.27), we readily have that

$$\begin{aligned}
& (f[\lambda, \dots, \lambda + (k - 1)\epsilon], f[\lambda, \dots, \lambda + (\ell - 1)\epsilon])_{L^2(0, +\infty)} \\
&= (e[\lambda, \dots, \lambda + (k - 1)\epsilon], e[\lambda, \dots, \lambda + (\ell - 1)\epsilon])_{L^2(0, +\infty)}, \quad k, \ell \in \llbracket 1, d \rrbracket,
\end{aligned}$$

thus

$$(\mathcal{G}_{\lambda, \epsilon} \xi, \xi) = \left\| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} e[\lambda, \dots, \lambda + (i - 1)\epsilon] \right\|_{L^2(0, +\infty)}^2. \tag{2.42}$$

Now, we proceed to estimate (2.42). Using Proposition A.3 we see that

$$\begin{aligned}
(\mathcal{G}_{\lambda, \epsilon} \xi, \xi) &= \int_0^{+\infty} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} e_t[\lambda, \dots, \lambda + (i - 1)\epsilon] \right|^2 dt \\
&= \int_0^{+\infty} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} \frac{(-t)^{i-1}}{(i - 1)!} e^{-(\lambda + \theta_i)t} \right|^2 dt,
\end{aligned}$$

for some  $\theta_i \in [0, (i-1)\epsilon]$  (depending on  $t, \lambda$  and  $\epsilon$ ) with  $i \in \llbracket 1, d \rrbracket$  and using the change of variable  $t \mapsto t/\lambda$ , we get

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) = \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \xi_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} e^{-\frac{t}{\lambda} \theta_i} \right|^2 dt.$$

Using that  $(a-b)^2 \geq \frac{a^2}{2} - b^2$ , we readily deduce

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \frac{1}{2} \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \xi_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt - \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \xi_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} \left(1 - e^{-\frac{t}{\lambda} \theta_i}\right) \right|^2 dt.$$

Using the inequality  $1 - e^{-x} \leq x$  for  $x > 0$  in the second term of the above inequality, we deduce

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \frac{1}{2} \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \xi_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt - \int_0^{+\infty} e^{-2t} \sum_{i=1}^d |\xi_i|^2 2^{d-1} \frac{(-t)^{2(i-1)}}{[(i-1)!]^2} \left(\frac{t\theta_i}{\lambda}\right)^2 dt$$

and since  $\theta_i \leq (d-1)\epsilon$  for any  $i \in \llbracket 1, d \rrbracket$  and (2.18) holds, we obtain

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \xi_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt - C\epsilon^2 \|\boldsymbol{\xi}\|_\infty^2, \quad (2.43)$$

for some positive constant  $C$  only depending on  $d$  and  $R$ . The first term in the right-hand side can be seen as the square of a Euclidean norm (depending only on  $d$ ) of  $\boldsymbol{\xi}$  and therefore, there exists a  $c_d > 0$  such that

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq c_d \|\boldsymbol{\xi}\|_\infty^2 - C\epsilon^2 \|\boldsymbol{\xi}\|_\infty^2.$$

Taking  $\epsilon_0 := \min \left\{ \sqrt{\frac{\rho}{4(d-1)}}, \sqrt{\frac{c_d}{2C}} \right\}$  where  $C > 0$  is the constant appearing in (2.43) we get for every  $\epsilon < \epsilon_0$ , and every  $\boldsymbol{\xi}$ ,

$$(\mathcal{G}_{\lambda, \epsilon} \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \frac{c_d}{2} \|\boldsymbol{\xi}\|_\infty^2.$$

This concludes the proof of Lemma 2.8.

## 3 A time-discrete moment's method

### 3.1 Motivation

In this section, we shall present in the time-discrete setting, a result analogous to the one given in the previous section. The motivation and methodology follows the same spirit as in the continuous case, so we aim to obtain an equation similar to (2.12) and state a moment problem for the discrete setting.

To this end, the first thing we need to check is how the time-discrete semigroup acts on the (generalized) eigenfunctions of our problem. Let us define for any  $\lambda \in \mathbb{R}^+$ ,  $\tau > 0$ , and any fixed  $n \in \mathbb{N}^*$  the quantity

$$p_n[\lambda] := (1 + \tau\lambda)^{-n} \quad (3.1)$$

and the associated time-discrete function

$$p[\lambda] : n \mapsto p_n[\lambda]. \quad (3.2)$$

A straightforward computation shows that for any  $j \geq 1$

$$p_n[\lambda^{(j)}] = \frac{(-\tau)^{j-1}}{(j-1)!} \langle\langle n \rangle\rangle_{j-1} (1 + \tau\lambda)^{-(n+j-1)}, \quad (3.3)$$

where, for convenience, we have introduced the notation

$$\langle\langle n \rangle\rangle_i := \frac{(n+i-1)!}{(n-1)!}. \quad (3.4)$$

Note that for any  $T > 0$ , and  $j \geq 1$ , we have that

$$\sup_{n \leq T/\tau} \left| p_n[\lambda^{(j)}] - e_{n\tau}[\lambda^{(j)}] \right| \rightarrow 0 \quad \text{as } \tau \rightarrow 0,$$

hence it is expected that the functions  $p[\lambda^{(j)}]$  will play the role of the exponentials  $e[\lambda^{(j)}]$  in the discrete setting. Indeed, using identities (2.7)–(2.8) and formula (3.3), it can be seen after a long but straightforward computation that, for each fixed  $r \in \llbracket 1, d \rrbracket$ , the sequence  $z = (z^{n+1})_{n \in \llbracket 0, M-1 \rrbracket}$  with elements given by

$$z^{n+1} = \sum_{\ell=0}^{r-1} p_{M-n}[\lambda^{(\ell+1)}] (\phi_\lambda \otimes \mathbf{e}_{r-\ell}), \quad n \in \llbracket 0, M-1 \rrbracket$$

solves the backward system

$$\begin{cases} z^n - z^{n+1} + \mathcal{L}^* z^n = 0, & \forall n \in \llbracket 1, M \rrbracket \\ (z^{n+1})|_{\partial\Omega} = 0, & \forall n \in \llbracket 1, M \rrbracket \\ z^{M+1} = \phi_\lambda \otimes \mathbf{e}_r. \end{cases} \quad (3.5)$$

This proves that the action of the time-discrete semigroup on the (generalized) eigenfunctions of the adjoint operator reads

$$(\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L}^*)^{-(M-n)} (\phi_\lambda \otimes \mathbf{e}_r) = \sum_{\ell=0}^{r-1} p_{M-n}[\lambda^{(\ell+1)}] (\phi_\lambda \otimes \mathbf{e}_{r-\ell}), \quad n \in \llbracket 0, M-1 \rrbracket \quad (3.6)$$

for all  $r \in \llbracket 1, d \rrbracket$  and any  $\lambda \in \Lambda$ .

Given this computation, we can multiply in  $L^2(\Omega)$  the system (1.9) by the adjoint semi-group given by (3.6) and we sum for  $n \in \llbracket 0, M-1 \rrbracket$ . After integration by parts in space and taking into account the boundary conditions, we obtain

$$\begin{aligned} \left( y^M, \phi_\lambda \otimes \mathbf{e}_r \right)_{L^2(\Omega)} &= \left( y_0, (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L}^*)^{-M} (\phi_\lambda \otimes \mathbf{e}_r) \right)_{L^2(\Omega)} \\ &\quad - \sum_{n=0}^{M-1} \tau (\partial_x|_{x=1} \otimes \mathbf{B}^t) (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L}^*)^{-(M-n)} (\phi_\lambda \otimes \mathbf{e}_r) v^{n+1}. \end{aligned}$$

Using the structure of matrix  $\mathbf{B}$  (see (1.6)) and formula (3.6) in the right-hand side of the previous expression, we deduce that

$$\left( y^M, \phi_\lambda \otimes \mathbf{e}_r \right)_{L^2(\Omega)} = \left( y_0, (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L}^*)^{-M} (\phi_\lambda \otimes \mathbf{e}_r) \right)_{L^2(\Omega)} - \gamma(1) \partial_x \phi_\lambda(1) \sum_{n=0}^{M-1} \tau p_{M-n}[\lambda^{(r)}] v^{n+1} \quad (3.7)$$

for all  $\lambda \in \Lambda$  and  $r \in \llbracket 1, d \rrbracket$ .

From here, it is easy to see that the null controllability problem for the time-discrete system (1.9) amounts to find a control sequence  $(v^{n+1})_{n \in \llbracket 0, M-1 \rrbracket} \subset \mathbb{R}$  such that for all  $\lambda \in \Lambda$  and  $r \in \llbracket 1, d \rrbracket$ , we have

$$\sum_{n=0}^{M-1} \tau p_{M-n}[\lambda^{(r)}] v^{n+1} = \frac{\alpha_{\lambda,r}}{\gamma(1) \partial_x \phi_\lambda(1)}, \quad (3.8)$$

where

$$\alpha_{\lambda,r} := \left( y_0, (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L}^*)^{-M} (\phi_\lambda \otimes \mathbf{e}_r) \right)_{L^2(\Omega)}. \quad (3.9)$$

As in the continuous case, we can reverse the order of time in (3.8) by setting the change of variables  $M-n \mapsto n$  to obtain the formula

$$\sum_{n=1}^M \tau p_n[\lambda^{(r)}] v^{M-n+1} = \frac{\alpha_{\lambda,r}}{\gamma(1) \partial_x \phi_\lambda(1)} \quad (3.10)$$

for all  $\lambda \in \Lambda$  and  $r \in \llbracket 1, d \rrbracket$ .

As we have mentioned in the introduction, we cannot expect to obtain a control function  $v$  such that (3.10) holds for all  $\lambda \in \Lambda$ , since  $\#\Lambda = +\infty$ . Instead, we shall look for a control  $v$  such that the corresponding solution  $y = (y^n)_{n \in \llbracket 0, M \rrbracket}$  verifies

$$\left( y^M, \phi_\lambda \otimes \mathbf{e}_r \right)_{L^2(\Omega)} = 0, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \quad \forall r \in \llbracket 1, d \rrbracket, \quad (3.11)$$

where  $\Lambda^{\tau, \varpi} \subset \Lambda$  is a suitable finite subfamily of eigenvalues determined by  $\tau$  and a parameter  $\varpi$  that we will describe below in (3.14). This is equivalent to solve the equations (3.10) for every  $r \in \llbracket 1, d \rrbracket$  and every  $\lambda \in \Lambda^{\tau, \varpi}$  instead of the whole spectrum  $\Lambda$ . Note that this family of eigenvalues will have the property that  $\Lambda^{\tau, \varpi} \rightarrow \Lambda$  as  $\tau \rightarrow 0$ , so that we shall recover the usual null-controllability property at the limit  $\tau \rightarrow 0$ .

Following the ideas of the continuous case, we will solve (3.11) by looking for a family of biorthogonal functions. The precise definition is as follows.

**Definition 3.1.** *We say that a family  $(q_{\lambda',j})_{\lambda' \in \Lambda^{\tau, \varpi}}^{j \in \llbracket 1, d \rrbracket}$  of time-discrete functions is biorthogonal to the family  $(p[\lambda^{(i)}])_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2_\tau(0, T)$  if*

$$\left( q_{\lambda',j}, p[\lambda^{(i)}] \right)_{L^2_\tau(0,T)} = \delta_{\lambda, \lambda'} \delta_{i,j}, \quad \forall \lambda, \lambda' \in \Lambda^{\tau, \varpi}, \quad \forall i, j \in \llbracket 1, d \rrbracket. \quad (3.12)$$

With this definition, the control problem can be formally solved by setting

$$\tilde{v}^n := \sum_{j=1}^d \sum_{\lambda \in \Lambda^{\tau, \varpi}} \frac{\alpha_{\lambda,j}}{\gamma(1) \partial_x \phi_\lambda(1)} q_{\lambda,j}^n, \quad n \in \llbracket 1, M \rrbracket,$$

and then taking  $v^n = \tilde{v}^{M-n+1}$  (remember that we have reversed the sense of the time in (3.10)). Thus, our task reduces to prove the existence of  $(q_{\lambda,j})_{\lambda \in \Lambda^{\tau, \varpi}}^{j \in \llbracket 1, d \rrbracket}$  and give precise uniform estimates. This is exactly what says the main result of this section.

**Theorem 3.2.** Let  $d \in \mathbb{N}^*$ ,  $T \in (0, +\infty]$  and  $\Lambda$  be a family of eigenvalues verifying (2.15)–(2.18).

There exist  $\varpi \in (0, 1)$  only depending on  $d$ , and  $\tau_0 \in (0, 1)$  only depending on  $\varpi$ ,  $\Lambda$ , and  $T$ , such that for any  $\tau \in (0, \tau_0)$ , there exists a time-discrete biorthogonal family  $(q_{\lambda,i})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2_\tau(0, T)$  satisfying the estimate

$$\|q_{\lambda,i}\|_{L^2_\tau(0,T)} \leq C e^{\delta(\lambda)\lambda}, \quad \forall i \in \llbracket 1, d \rrbracket, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \quad (3.13)$$

where

$$\Lambda^{\tau, \varpi} := \{\lambda \in \Lambda : \tau\lambda < \varpi\} \quad (3.14)$$

and  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  is a non-increasing function such that  $\lim_{s \rightarrow +\infty} \delta(s) = 0$  that only depends on  $d$ ,  $\rho$ , and  $R$  and  $C > 0$  is a constant only depending on  $d$ ,  $R$  and  $T$ .

Note that the biorthogonal family clearly depends on  $T$ , even though we do not explicitly mention it in the notation to ease the reading.

The rest of the section will consist in proving this result, first when  $T = +\infty$ , then for finite values of  $T$ .

### 3.2 The biorthogonal family for an infinite time horizon

We present here the construction of the biorthogonal in the time-discrete framework for the infinite time horizon.

The proof of this result relies on an auxiliary result concerning the construction of biorthogonals to divided differences of the functions  $p[\lambda]$  and on a limit process. This result is in fact the time-discrete counter part of Proposition 2.6 and reads as follows.

**Proposition 3.3.** Let  $d \in \mathbb{N}^*$  and  $\Lambda$  be a family of eigenvalues verifying (2.15)–(2.18). There exist constants  $\epsilon_0 > 0$  only depending on  $d$ ,  $\rho$ , and  $R$ , and  $\varpi \in (0, 1)$  only depending on  $d$  such that for any  $\tau \in (0, 1)$  and any  $\epsilon < \epsilon_0$ , there exists a family  $(\tilde{q}_{\lambda,i,\epsilon})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  in  $L^2_\tau(0, +\infty)$  satisfying

$$\left( \tilde{q}_{\lambda',i,\epsilon}, p[\lambda, \dots, \lambda + (j-1)\epsilon] \right)_{L^2_\tau(0,+\infty)} = \delta_{\lambda',\lambda} \delta_{i,j}, \quad \forall \lambda, \lambda' \in \Lambda^{\tau, \varpi}, \quad \forall i, j \in \llbracket 1, d \rrbracket, \quad (3.15)$$

and the estimate

$$\|\tilde{q}_{\lambda,i,\epsilon}\|_{L^2_\tau(0,+\infty)} \leq C e^{\delta(\lambda)\lambda}, \quad \forall i \in \llbracket 1, d \rrbracket, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \quad (3.16)$$

with  $\Lambda^{\tau, \varpi}$  as in (3.14) and where  $\delta : \mathbb{R} \rightarrow \mathbb{R}$  is a non-increasing function such that  $\lim_{s \rightarrow +\infty} \delta(s) = 0$  that only depends on  $\varpi$ ,  $d$ ,  $\rho$ , and  $R$ , and  $C > 0$  is a constant only depending on  $d$  and  $R$ .

The proof of Proposition 3.3 follows very closely the one of Proposition 2.6. We will crucially use the fact that we are only working with a finite portion of the spectrum to obtain the uniform bound (3.16). For the sake of readability, we will postpone it to the following section.

Assuming that Proposition 3.3 holds, we can prove our main theorem for an infinite time horizon.

*Proof of Theorem 3.2 in the case  $T = +\infty$ .* For each fixed  $\epsilon \in (0, \epsilon_0)$ , let us consider the biorthogonal family  $(\tilde{q}_{\lambda,i,\epsilon})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  provided by Proposition 3.3.

Since the estimate (3.16) is uniform with respect to  $\epsilon$ , we can find a sequence  $(\epsilon_m)_m \subset (0, \epsilon_0)$  such that  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$  and such that  $(\tilde{q}_{\lambda,i,\epsilon_m})_m$  that converges weakly towards some  $q_{\lambda,i} \in L^2_\tau(0, +\infty)$  for each  $\lambda \in \Lambda^{\tau, \varpi}$  and  $i \in \llbracket 1, d \rrbracket$  verifying

$$\|q_{\lambda,i}\|_{L^2_\tau(0,+\infty)} \leq C e^{\delta(\lambda)\lambda}, \quad (3.17)$$

where  $C$  and  $\delta$  do not depend on  $\tau$ .

Now, it is not difficult to see that for any fixed  $\lambda$ , and any  $i \in \llbracket 1, d \rrbracket$ , we have the pointwise convergence

$$p_n[\lambda, \dots, \lambda + (i-1)\epsilon_m] \xrightarrow{m \rightarrow \infty} p_n[\lambda^{(i)}], \quad \forall n \in \mathbb{N}^*.$$

Furthermore, from Lagrange theorem (see Proposition A.3), we have that the following identity holds

$$p_n[\lambda, \dots, \lambda + (i-1)\epsilon_m] = p_n[\theta_{i,n,\epsilon_m}^{(i)}],$$

for some  $\theta_{i,n,\epsilon_m} \in [\lambda, \lambda + (i-1)\epsilon_m]$ , which together with estimate (A.3) yields

$$|p_n[\lambda, \dots, \lambda + (i-1)\epsilon_m]| \leq C_i (n\tau)^{i-1} p_n[\theta_{i,n,\epsilon_m}] \leq C_i (n\tau)^{i-1} p_n[\lambda], \quad (3.18)$$

for all  $n \in \mathbb{N}^*$  and  $i \in \llbracket 1, d \rrbracket$ . The right-hand side of the above expression is independent of  $\epsilon_m$  and square summable for each  $i \in \llbracket 1, d \rrbracket$ . Indeed, by Lemma A.6, we readily get

$$\sum_{n=1}^{\infty} \tau (n\tau)^{2(i-1)} |p_n[\lambda]|^2 \leq \frac{C_i}{\lambda^{2i-1}}, \quad i \in \llbracket 1, d \rrbracket, \quad (3.19)$$

for some constants  $C_i > 0$  uniform with respect to  $\tau$  and  $\lambda$ .

Therefore, using (3.18), (3.19) and dominated convergence theorem, we get that

$$p[\lambda, \dots, \lambda + (i-1)\epsilon_m] \xrightarrow{m \rightarrow \infty} p[\lambda^{(i)}] \quad \text{strongly in } L^2_\tau(0, +\infty).$$

Finally, the result follows by a weak-strong convergence argument in (3.15) and the bound (3.17). This ends the proof.  $\square$

### 3.2.1 Proof of Proposition 3.3

Here, we will give the construction of a biorthogonal family to divided differences the functions  $p[\lambda]$ . We shall perform the same steps as in the proof of Proposition 2.6, but taking into account the discrete nature of our problem.

– **Step 1. Measuring the distance between some discrete functions.** We begin by noting that the map (3.2) is well defined in  $L^2_\tau(0, +\infty)$  for any  $\lambda \in \Lambda$  and  $\tau > 0$ . From (1.13), we see that for any  $\lambda, \lambda' \in \Lambda$ ,

$$(p[\lambda], p[\lambda'])_{L^2_\tau(0, +\infty)} = \sum_{n=1}^{\infty} \tau (1 + \tau\lambda)^{-n} (1 + \tau\lambda')^{-n} = \frac{1}{\lambda + \lambda' + \tau\lambda\lambda'}, \quad (3.20)$$

and in particular

$$\|p[\lambda]\|_{L^2_\tau(0, +\infty)}^2 = \sum_{n=1}^{\infty} \tau (1 + \tau\lambda)^{-2n} = \frac{1}{2\lambda + \tau\lambda^2},$$

which resembles very much the continuous case except that we have an additional term depending on  $\tau$  in the denominator.

In what follows, we shall work only with a finite portion of the spectrum of the problem that was introduced in (3.14). For  $\epsilon \in (0, \frac{\rho}{4(d-1)})$ , we construct the finite family

$$\tilde{\Lambda}^{\tau, \varpi} = \bigcup_{r=0}^{d-1} (\Lambda^{\tau, \varpi} + r\epsilon) = \bigcup_{\lambda \in \Lambda^{\tau, \varpi}} G_\lambda, \quad (3.21)$$

where we recall that  $G_\lambda$  is the group defined in (2.24).

We will now compute the distance of the functions  $p[\sigma]$  with  $\sigma \in G_\lambda$  to the space generated by the others, namely, the space

$$E_\lambda^\tau := \text{span} \left\{ p[\lambda] : \lambda \in \tilde{\Lambda}_\lambda^{\tau, \varpi} \right\},$$

where  $\tilde{\Lambda}_\lambda^{\tau, \varpi} := \tilde{\Lambda}^{\tau, \varpi} \setminus G_\lambda$ . As before,  $P_{E_\lambda^\tau}$  stands for the orthogonal projection in  $L^2_\tau(0, +\infty)$  on  $E_\lambda^\tau$ .

The precise result is the following.

**Lemma 3.4.** *For any  $\sigma, \sigma' \in G_\lambda$*

$$(p[\sigma] - P_{E_\lambda^\tau} p[\sigma], p[\sigma'] - P_{E_\lambda^\tau} p[\sigma'])_{L^2_\tau(0, T)} = \frac{1}{\sigma + \sigma' + \tau\sigma\sigma'} W_\lambda^\tau(\sigma) W_\lambda^\tau(\sigma') \quad (3.22)$$

where

$$W_\lambda^\tau(\sigma) = \prod_{\mu \in \tilde{\Lambda}_\lambda^{\tau, \varpi}} \frac{\sigma - \mu}{\sigma + \mu + \tau\sigma\mu}. \quad (3.23)$$

The proof is analogous to the one of Lemma 2.7 and can be found on Appendix B.2. By construction, since we are removing the projection of  $p[\sigma]$  to the space  $E_\lambda^\tau$ , we have from Lemma 3.4 that

$$(p[\sigma] - P_{E_\lambda^\tau} p[\sigma], p[\gamma])_{L^2_\tau(0, +\infty)} = 0, \quad (3.24)$$

for all  $\sigma \in G_\lambda$  and  $\gamma \in \tilde{\Lambda}_\lambda^{\tau, \varpi}$ .

– **Step 2. A biorthogonal for an auxiliary family.** Let us define  $\tilde{p}[\sigma] := p[\sigma] - P_{E_\lambda^\tau} p[\sigma]$  with  $\sigma \in G_\lambda$  and set  $w[\sigma] := \frac{\tilde{p}[\sigma]}{W_\lambda^\tau(\sigma)}$ . This  $w[\sigma]$  will play a similar role to that of  $f[\sigma]$  in the continuous case (cf. eq. (2.27)).

By Lemma 3.4 and formula (3.20) we note that

$$(w[\sigma], w[\sigma'])_{L^2_\tau(0, +\infty)} = \frac{1}{\sigma + \sigma' + \tau\sigma\sigma'} = (p[\sigma], p[\sigma'])_{L^2_\tau(0, +\infty)}. \quad (3.25)$$

Thus, following the approach of the continuous case, we aim to build, for every  $\lambda \in \Lambda^{\tau, \varpi}$  a family  $(q_{\lambda, i}^{\text{aux}})_{i \in \llbracket 1, d \rrbracket}$  verifying

$$\left( q_{\lambda, i}^{\text{aux}}, (2\lambda)^{j-1/2} w[\lambda, \dots, \lambda + (j-1)\epsilon] \right)_{L^2_\tau(0, +\infty)} = \delta_{i, j}, \quad \forall i, j \in \llbracket 1, d \rrbracket, \quad (3.26)$$

Arguing as in Proposition 2.6, we know that problem (3.26) reduces to studying the inverse of the Gram matrix

$$\mathcal{G}_{\lambda, \epsilon}^\tau = \left( (2\lambda)^{i+j-1} (w[\lambda, \dots, \lambda + (i-1)\epsilon], w[\lambda, \dots, \lambda + (j-1)\epsilon])_{L^2_\tau(0, +\infty)} \right)_{1 \leq i, j \leq d}.$$

and, as in the continuous case, the invertibility can be ensured by obtaining a suitable lower bound. The precise result is the following.

**Lemma 3.5.** *Under the assumptions of Proposition 3.3, there exist  $\epsilon_0 > 0$  only depending on  $d, \rho$  and  $R$  and  $\varpi \in (0, 1)$  only depending on  $d$  such that for any  $\epsilon \leq \epsilon_0$ , there exists a constant  $C > 0$  uniform with respect to  $\epsilon, \lambda$  and  $\tau$ , such that*

$$(\mathcal{G}_{\lambda, \epsilon}^\tau, \boldsymbol{\xi}, \boldsymbol{\xi}) \geq C \|\boldsymbol{\xi}\|_\infty^2 \quad (3.27)$$

for any  $\boldsymbol{\xi} \in \mathbb{R}^d$  and any  $\lambda \in \Lambda^{\tau, \varpi}$ .

We postpone the proof of this result to the end of this section since it is technical but, before continuing, we shall remark that it is precisely at this point that a smallness condition on the parameter  $\varpi$  is imposed to ensure that the constant  $C > 0$  appearing in (3.27) is uniform with respect to  $\tau$ .

Thus, in view of Lemma 3.5 and arguing as we did for (2.37), we can obtain that

$$\|q_{\lambda,i}^{\text{aux}}\|_{L^2_\tau(0,+\infty)}^2 \leq C \quad (3.28)$$

for all  $i \in \llbracket 1, d \rrbracket$ ,  $\lambda \in \Lambda^{\tau,\varpi}$  and a constant  $C > 0$  uniform with respect to  $\lambda$ ,  $\epsilon$  and  $\tau$ .

– **Step 3. Final arrangements and conclusion.** We define

$$q_{\lambda,i} = \sum_{k=i}^d \left( \frac{1}{W_\lambda^\tau} \right) \left[ \lambda + (i-1)\epsilon, \lambda + (k-1)\epsilon \right] (2\lambda)^{k-1/2} q_{\lambda,k}^{\text{aux}}, \quad i \in \llbracket 1, d \rrbracket.$$

We claim that these functions satisfy (3.15) and (3.16). Indeed, by Proposition A.4 and the definition of  $\tilde{p}$

$$\tilde{p}[\lambda, \dots, \lambda + (\ell-1)\epsilon] = \sum_{r=1}^{\ell} w[\lambda, \dots, \lambda + (r-1)\epsilon] W_\lambda^\tau \left[ \lambda + (r-1)\epsilon, \dots, \lambda + (\ell-1)\epsilon \right], \quad \ell \in \llbracket 1, d \rrbracket,$$

whence, arguing as in the continuous case (just changing  $W_\lambda$  by  $W_\lambda^\tau$  everywhere), we have

$$(\tilde{p}[\lambda, \dots, \lambda + (\ell-1)\epsilon], q_{\lambda,k})_{L^2(0,+\infty)} = \delta_{k,\ell}. \quad (3.29)$$

Recalling that by construction (3.24) holds and by linearity of divided differences, we have

$$(\tilde{p}[\lambda', \dots, \lambda' + (\ell-1)\epsilon], q_{\lambda,k})_{L^2(0,+\infty)} = 0, \quad \forall k, l \in \llbracket 1, d \rrbracket, \quad \forall \lambda, \lambda' \in \Lambda^{\tau,\varpi}, \lambda \neq \lambda'.$$

This, together with (3.29) prove that  $(q_{\lambda,k})_{\lambda \in \Lambda^{\tau,\varpi}}^{k \in \llbracket 1, d \rrbracket}$  satisfies (3.15). Moreover, from (3.28), we readily obtain

$$\|q_{\lambda,k}\|_{L^2(0,+\infty)} \leq C \lambda^{d-1/2} \max_{j \in \llbracket k, d \rrbracket} \left| \left( \frac{1}{W_\lambda^\tau} \right) \left[ \lambda + (k-1)\epsilon, \dots, \lambda + (j-1)\epsilon \right] \right|.$$

Using Propositions A.3 and A.9, we can argue as in the end of Proposition 2.6 to obtain the desired estimate. The proof is complete.

### 3.2.2 Invertibility of $\mathcal{G}_{\lambda,\epsilon}^\tau$

It remains to prove the invertibility of the Gram matrix with a suitable uniform estimate of its inverse.

*Proof of Lemma 3.5.* In this proof, the eigenvalue  $\lambda \in \Lambda^{\tau,\varpi}$  is fixed. We can write, by (3.25),

$$\begin{aligned} (\mathcal{G}_{\lambda,\epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) &= \left\| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} w[\lambda, \dots, \lambda + (i-1)\epsilon] \right\|_{L^2_\tau(0,+\infty)}^2 \\ &= \left\| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} p[\lambda, \dots, \lambda + (i-1)\epsilon] \right\|_{L^2_\tau(0,+\infty)}^2. \end{aligned} \quad (3.30)$$

Noting that

$$p_n[\lambda, \dots, \lambda + (i-1)\epsilon] = (1 + \tau\lambda)^{-(i-1)} p_n[\lambda] p_n \left[ 0, \dots, \frac{\epsilon(i-1)}{1 + \tau\lambda} \right], \quad i \in \llbracket 1, d \rrbracket,$$

we have from (3.30)

$$(\mathcal{G}_{\lambda,\epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) = \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} p_n \left[ 0, \dots, \frac{(i-1)\epsilon}{1 + \tau\lambda} \right] \right|^2$$

and using the Lagrange Theorem (see Proposition A.3), we get

$$(\mathcal{G}_{\lambda,\epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) = \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} p_n \left[ \left( \frac{\theta_i}{1 + \tau\lambda} \right)^{(i)} \right] \right|^2$$

for some  $\theta_i \in [0, (i-1)\epsilon]$ .

Adding and subtracting  $p_n[0^{(i)}]$  in the above expression and using the inequality  $(a-b)^2 \geq \frac{1}{2}a^2 - b^2$ , we obtain

$$\begin{aligned} (\mathcal{G}_{\lambda,\epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) &\geq \frac{1}{2} \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} p_n[0^{(i)}] \right|^2 \\ &\quad - \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} \left( p_n[0^{(i)}] - p_n \left[ \left( \frac{\theta_i}{1 + \tau\lambda} \right)^{(i)} \right] \right) \right|^2. \end{aligned}$$



Then, using the explicit expression for the divided differences for  $p_n$  (see eq. (3.3)), we get

$$p_n \left[ \left( \frac{\theta_i}{1 + \tau\lambda} \right)^{(i)} \right] = p_n[0^{(i)}] \left( 1 + \frac{\tau\theta_i}{1 + \tau\lambda} \right)^{-(n+(i-1))}, \quad i \in \llbracket 1, d \rrbracket,$$

and thus

$$\begin{aligned} & (\mathcal{G}_{\lambda, \epsilon}^{\tau} \boldsymbol{\xi}, \boldsymbol{\xi}) \\ & \geq \frac{1}{2} \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} p_n[0^{(i)}] \right|^2 \\ & \quad - \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i (2\lambda)^{i-1/2} (1 + \tau\lambda)^{-(i-1)} p_n[0^{(i)}] \left\{ 1 - \left( 1 + \frac{\tau\theta_i}{1 + \tau\lambda} \right)^{-(n+(i-1))} \right\} \right|^2 \\ & =: \frac{1}{2} I_1 - I_2. \end{aligned} \quad (3.31)$$

We proceed to estimate the terms  $I_1$  and  $I_2$ . For  $I_2$ , we have

$$|I_2| \leq \sqrt{d} \|\boldsymbol{\xi}\|_{\infty}^2 \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \sum_{i=1}^d (2\lambda)^{2i-1} (1 + \tau\lambda)^{-2(i-1)} |f_i|^2 |p_n[0^{(i)}]|^2,$$

where  $f_i = 1 - e^{-(n+(i-1)) \log(1 + \frac{\tau\theta_i}{1 + \tau\lambda})}$ . From the inequalities  $\log(1 + x) < x$  and  $1 - e^{-x} < x$  for  $x > 0$ , we have that  $0 < f_i < (n + (i - 1)) \frac{\tau\theta_i}{1 + \tau\lambda}$ , whence

$$\begin{aligned} |I_2| & \leq \sqrt{d} \|\boldsymbol{\xi}\|_{\infty}^2 \\ & \quad \times \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} \sum_{i=1}^d (2\lambda)^{2i-1} (1 + \tau\lambda)^{-2(i-1)} \left[ (n + (i - 1)) \frac{\tau\theta_i}{1 + \tau\lambda} \right]^2 |p_n[0^{(i)}]|^2. \end{aligned} \quad (3.32)$$

Using that  $(n + (i - 1))^2 \leq C_i n^2$  for all  $i \in \llbracket 1, d \rrbracket$  and  $n \in \mathbb{N}^*$ , estimate (A.3), and recalling that  $\theta_i \leq (d - 1)\epsilon$ , we deduce that there exists a constant  $C > 0$  only depending on  $d$  such that

$$|I_2| \leq C \|\boldsymbol{\xi}\|_{\infty}^2 \epsilon^2 \sum_{i=1}^d (2\lambda)^{2i-1} (1 + \tau\lambda)^{-2i} \sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} (n\tau)^{2i}. \quad (3.33)$$

The infinite sum in the above inequality can be readily estimated. By Lemma A.6

$$\sum_{n=1}^{+\infty} \tau (1 + \tau\lambda)^{-2n} (n\tau)^{2i} \leq \frac{C_i}{\lambda^{2i+1}}, \quad i \in \llbracket 1, d \rrbracket, \quad (3.34)$$

for some  $C_i > 0$  uniform with respect to  $\tau$  and  $\lambda$ . Putting together (3.33)–(3.34), and the fact that  $(1 + \tau\lambda)^{2i} > 1$  for all  $i \in \llbracket 1, d \rrbracket$ , we get

$$|I_2| \leq C \|\boldsymbol{\xi}\|_{\infty}^2 \frac{\epsilon^2}{\lambda^2},$$

for a constant  $C > 0$  only depending on  $d$ . Recalling (2.18) we get

$$|I_2| \leq C' \|\boldsymbol{\xi}\|_{\infty}^2 \epsilon^2, \quad (3.35)$$

for some  $C' > 0$  depending on  $d$  and  $R$  but uniform with respect to  $\lambda$ ,  $\epsilon$  and  $\tau$ .

Let us turn our attention to  $I_1$ . Notice that this term can be rewritten as

$$\begin{aligned} I_1 & = \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1 + \tau\lambda)^{i-1}} p^n[0^{(i)}] \right|^2 dt \\ & = \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1 + \tau\lambda)^{i-1}} \left( \frac{(-t)^{i-1}}{(i-1)!} - \left\{ \frac{(-t)^{i-1}}{(i-1)!} - p^n[0^{(i)}] \right\} \right) \right|^2 dt. \end{aligned}$$

Using once again the inequality  $(a - b)^2 \geq \frac{1}{2}a^2 - b^2$ , we can bound the above expression as follows

$$\begin{aligned} I_1 & \geq \frac{1}{2} \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1 + \tau\lambda)^{i-1}} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt \\ & \quad - \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} (1 + \tau\lambda)^{-2n} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1 + \tau\lambda)^{i-1}} \left\{ \frac{(-t)^{i-1}}{(i-1)!} - p^n[0^{(i)}] \right\} \right|^2 dt \\ & =: \frac{1}{2} H_1 - H_2. \end{aligned}$$

To estimate the first term in the above inequality, we note that since  $\log(1+x) < x$  for  $x > 0$ , then  $\log(1+\tau\lambda)/\tau \leq \lambda$  uniformly with respect to  $\tau$ . Therefore,

$$H_1 \geq \frac{1}{2} \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} e^{-2\lambda t_n} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1+\tau\lambda)^{i-1}} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt$$

where we recall that  $t_n = n\tau$ . Since  $t_n \leq t + \tau$  for  $t \in (t_{n-1}, t_n)$  with  $n \in \mathbb{N}$ , we obtain

$$H_1 \geq \frac{1}{2} e^{-\tau\lambda} \int_0^{+\infty} e^{-2\lambda t} \left| \sum_{i=1}^d \xi_i \frac{(2\lambda)^{i-1/2}}{(1+\tau\lambda)^{i-1}} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt.$$

and setting the change of variables  $t \mapsto t/\lambda$

$$H_1 \geq \frac{1}{2} e^{-\tau\lambda} \int_0^{+\infty} e^{-2t} \left| \sum_{i=1}^d \tilde{\xi}_i 2^{i-1/2} \frac{(-t)^{i-1}}{(i-1)!} \right|^2 dt,$$

where  $\tilde{\xi}_i := \xi_i (1+\tau\lambda)^{-(i-1)}$ . Observe that this new expression has the same structure as in the continuous case (cf. equation (2.43)), therefore using the equivalence of norms in  $\mathbb{R}^d$ , we get

$$H_1 \geq c_d e^{-\tau\lambda} \|\tilde{\boldsymbol{\xi}}\|_\infty^2.$$

for a constant  $c_d > 0$ . Since  $\lambda \in \Lambda^{\tau, \varpi}$  with  $\varpi \in (0, 1)$ , we have  $\tau\lambda < 1$  so that  $e^{-\tau\lambda} \geq e^{-1}$  and  $\|\tilde{\boldsymbol{\xi}}\|_\infty \geq c_0 \|\boldsymbol{\xi}\|_\infty$ , for some  $c_0 > 0$  only depending on  $d$ , thus

$$H_1 \geq c \|\boldsymbol{\xi}\|_\infty^2, \quad (3.36)$$

for a constant  $c > 0$  only depending on  $d$ .

To estimate the term  $H_2$ , we proceed as follows. Let  $n \in \mathbb{N}^*$  and recalling (3.4), we define the function  $f_i(t) := t^{i-1} - \tau^{i-1} \langle n \rangle_{i-1}$  for  $i \in \llbracket 1, d \rrbracket$  and  $t \in (t_{n-1}, t_n)$ . Note that

$$f_1(t) \equiv 0 \quad (3.37)$$

for all  $t \in (t_{n-1}, t_n)$ . On the other hand,

$$\langle n \rangle_{i-1} = \prod_{k=1}^{i-1} (n+k-1) \leq (n+d-1)^{i-1}, \quad i \in \llbracket 2, d \rrbracket,$$

whence  $|f_i(t)| \leq \tau^{i-1} (n+d-1)^{i-1} - t_{n-1}^{i-1}$  for all  $i \in \llbracket 2, d \rrbracket$  and  $t \in (t_{n-1}, t_n)$ . Recalling that  $t_n = n\tau$ , we can further write

$$|f_i(t)| \leq \tau^{i-1} \left[ (n+d-1)^{i-1} - (n-1)^{i-1} \right]. \quad (3.38)$$

Applying Lemma A.7 in the right-hand side of (3.38), we get

$$|f_i(t)| \leq C_i \left( (1+(d-1)^{i-1}) \tau (\tau n)^{i-2} \right) \quad (3.39)$$

for all  $t \in (t_{n-1}, t_n)$  and  $i \in \llbracket 2, d \rrbracket$ . In view of (3.37), (3.39), and recalling the definition of  $p_n[0^{(i)}]$

$$\begin{aligned} |H_2| &\leq C_d \|\boldsymbol{\xi}\|_\infty^2 \sum_{n=1}^{+\infty} \int_{t_{n-1}}^{t_n} \mathbf{1}_{(t_{n-1}, t_n)} (1+\tau\lambda)^{-2n} \sum_{i=2}^d (2\lambda)^{2i-1} (1+\tau\lambda)^{-2(i-1)} |f_i(t)|^2 dt \\ &\leq C_d \|\boldsymbol{\xi}\|_\infty^2 \tau^2 \sum_{i=2}^d (2\lambda)^{2i-1} (1+\tau\lambda)^{-2(i-1)} \sum_{n=1}^{\infty} \tau (1+\tau\lambda)^{-2n} (n\tau)^{2(i-2)}. \end{aligned} \quad (3.40)$$

Using once again Lemma A.6, we see that for any  $\lambda \in \Lambda^{\tau, \varpi}$  and  $i \in \llbracket 2, d \rrbracket$

$$\sum_{n=1}^{\infty} \tau (1+\tau\lambda)^{-2n} (n\tau)^{2(i-2)} \leq \frac{C_i}{\lambda^{2i-3}}, \quad (3.41)$$

and putting (3.41) in (3.40) together with  $(1+\tau\lambda)^{2(i-1)} > 1$  for  $i \in \llbracket 2, d \rrbracket$ , we deduce

$$|H_2| \leq C'' \|\boldsymbol{\xi}\|_\infty^2 (\tau\lambda)^2 \quad (3.42)$$

for a constant  $C'' > 0$  only depending on  $d$ .

To conclude, combining estimates (3.31), (3.35), (3.36), and (3.42) yields

$$(\mathcal{G}_{\lambda, \epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \|\boldsymbol{\xi}\|_\infty^2 [c - \epsilon^2 C' - (\tau\lambda)^2 C''] .$$

Taking  $\epsilon_0 = \min \left\{ \sqrt{\frac{\rho}{4(d-1)}}, \sqrt{\frac{c}{4C'}} \right\}$  (which depends only on  $R, d$ , and  $\rho$ ) and reducing (if necessary) the value of  $\varpi \in (0, 1)$  such that  $\varpi \leq \sqrt{\frac{c}{4C''}}$  (a quantity that only depends on  $d$ ) we have that for any  $\epsilon \leq \epsilon_0$  and any  $\lambda \in \Lambda^{\tau, \varpi}$

$$(\mathcal{G}_{\lambda, \epsilon}^\tau \boldsymbol{\xi}, \boldsymbol{\xi}) \geq \frac{c}{2} \|\boldsymbol{\xi}\|_\infty^2,$$

where  $c > 0$  is a constant only depending on  $d$ . This concludes the proof.  $\square$

### 3.3 The biorthogonal family in the finite time horizon

We are now in position to prove Theorem 3.2 when  $T$  is finite. For this, we shall adapt to the time-discrete setting the classical restriction argument presented in [22].

For any  $T \in (0, +\infty]$ , we set

$$\mathcal{E}_\tau(\Lambda^{\tau, \varpi}, T) := \text{span}\{p[\lambda^{(i)}]_{\llbracket [1, T/\tau] \rrbracket} : \lambda \in \Lambda^{\tau, \varpi}, i \in \llbracket [1, d] \rrbracket\},$$

where we recall that  $\Lambda^{\tau, \varpi}$  is taken as in (3.14) for a parameter  $\varpi > 0$ . The goal of this section will be to analyze the restriction operator defined by

$$R_{T, \varpi, \tau} : g \in \mathcal{E}_\tau(\Lambda^{\tau, \varpi}, +\infty) \mapsto g|_{\llbracket [1, T/\tau] \rrbracket} \in \mathcal{E}_\tau(\Lambda^{\tau, \varpi}, T).$$

By using a very simple argument, we can first prove that this operator is invertible, as soon as  $\tau$  is small enough. More precisely, we have the following proposition.

**Proposition 3.6.** *There exists  $\tau_0 > 0$  depending only on  $\varpi, \Lambda$ , and  $T$  such that for any  $\tau \in (0, \tau_0)$ , the operator  $R_{T, \varpi, \tau}$  is invertible.*

*Proof.* Let us take some  $g \in \ker R_{T, \varpi, \tau}$  that we write as

$$g = \sum_{\lambda \in \Lambda^{\tau, \varpi}} \sum_{i=1}^d a_{\lambda, i} p[\lambda^{(i)}],$$

for some coefficients  $a_{\lambda, i} \in \mathbb{R}$ . By assumption, we have  $g_n = 0$  for any  $n \in \llbracket [1, T/\tau] \rrbracket$ . Introducing the polynomial  $Q_n(X) = X^n$  and the function  $\Phi_{Q_n}(x) = Q_n(1/x)$  for  $x > 0$ , it is easily seen from (3.3) that the condition  $g_n = 0$  can be written as follows

$$\sum_{\lambda \in \Lambda^{\tau, \varpi}} \sum_{i=1}^d a_{\lambda, i} \frac{(-\tau)^{i-1}}{(i-1)!} \Phi_{Q_n}^{(i-1)}(1 + \tau\lambda) = 0.$$

By linearity of the map  $Q \mapsto \Phi_Q$ , we deduce that for any polynomial  $Q$  with degree smaller than  $T/\tau$  and such that  $Q(0) = 0$ , the function  $\Phi_Q(x) = Q(1/x)$  satisfies

$$\sum_{\lambda \in \Lambda^{\tau, \varpi}} \sum_{i=1}^d a_{\lambda, i} \frac{(-\tau)^{i-1}}{(i-1)!} \Phi_Q^{(i-1)}(1 + \tau\lambda) = 0. \quad (3.43)$$

If we assume that

$$\frac{T}{\tau} \geq d \times \#\Lambda^{\tau, \varpi}, \quad (3.44)$$

we know by Hermite interpolation at the points

$$\left\{ \frac{1}{1 + \tau\lambda} : \lambda \in \Lambda^{\tau, \varpi} \right\} \cup \{0\},$$

that there exists a polynomial  $Q \in \mathbb{R}[X]$ , with degree less than  $T/\tau$ , satisfying  $Q(0) = 0$  and such that  $\Phi_Q$  satisfies

$$\Phi_Q^{(i-1)}(1 + \tau\lambda) = (-\tau)^{i-1} a_{\lambda, i}, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \forall i \in \llbracket [1, d] \rrbracket.$$

Therefore, (3.43) leads to

$$\sum_{\lambda \in \Lambda^{\tau, \varpi}} \sum_{i=1}^d |a_{\lambda, i}|^2 \frac{|\tau|^{2(i-1)}}{(i-1)!} = 0,$$

and thus  $a_{\lambda, i} = 0$  for every  $\lambda$  and  $i$ , that is  $g = 0$ .

It remains to prove that (3.44) holds true for  $\tau$  small enough. If we denote by  $N$  the counting function of the family  $\Lambda$ , we have by (3.14) that  $\#\Lambda^{\tau, \varpi} = N\left(\frac{\varpi}{\tau}\right)$  and therefore condition (3.44) reads as

$$\frac{T}{\tau} \geq dN\left(\frac{\varpi}{\tau}\right),$$

and this is actually valid for  $\tau$  small enough since we know, by the summability assumption (2.15), that  $N(r)/r \rightarrow 0$  when  $r \rightarrow \infty$  (see [13, Proposition A.5.38]).  $\square$

To conclude the proof of Theorem 3.2 and even if we are in a finite dimensional framework we need a more intricate argument based on complex analysis. This argument is adapted from the continuous case as presented in [22, Theorem 1.3]. Specifically, we aim to obtain an estimate on  $R_{T, \varpi, \tau}^{-1}$  that is not provided by Proposition 3.6.

**Proposition 3.7.** *There exist  $\tau_0 > 0$  and a constant  $C > 0$  only depending on  $\varpi, \Lambda$ , and  $T$  such that, for any  $\tau \in (0, \tau_0)$ , we have*

$$\|g\|_{L^2_\tau(0, +\infty)} \leq C \|g\|_{L^2_\tau(0, T)}, \quad \forall g \in \mathcal{E}_\tau(\Lambda^{\tau, \varpi}, +\infty). \quad (3.45)$$

*Equivalently this means that the uniform bound  $\|R_{T, \varpi, \tau}^{-1}\| \leq C$  holds.*

**Remark 3.8.** A careful inspection of the following proof shows that, in fact, the constant  $C$  only depends on the remainder function  $R$  associated to  $\Lambda$  and not on  $\Lambda$  itself. This observation is important to prove results that are uniform with respect to any parameters on which  $\Lambda$  may depend. A typical example is the fully discrete case where  $\Lambda$  depends on the space discretization parameter. For further discussion, we refer to Section 5.1.

Assuming Proposition 3.7 for the moment, we can conclude the proof of our main theorem.

*Proof of Theorem 3.2.* Let us denote by  $(q_{\infty, \lambda, i})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$ , the biorthogonal family given by Theorem 3.2 in the case  $T = +\infty$ . Note that we can always assume, by orthogonal projection, that each element of this family belongs to  $\mathcal{E}_\tau(\Lambda^{\tau, \varpi}, \infty)$  without modifying the uniform estimate we have.

We can now classically set

$$q_{T, \lambda, i} := (R_{T, \varpi, \tau}^*)^{-1} q_{\infty, \lambda, i}, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \forall i \in \llbracket 1, d \rrbracket, \quad (3.46)$$

and observe that

$$(q_{T, \lambda', j}, p[\lambda^{(i)}])_{L^2_\tau(0, T)} = (q_{\infty, \lambda, i}, (R_{T, \varpi, \tau})^{-1} p[\lambda^{(i)}])_{L^2_\tau(0, +\infty)} = (q_{\infty, \lambda, i}, p[\lambda^{(i)}])_{L^2_\tau(0, +\infty)} = \delta_{\lambda, \lambda'} \delta_{i, j}.$$

Therefore the family defined in (3.46) is indeed a biorthogonal family in  $L^2_\tau(0, T)$  that we are looking for. The bound on the inverse of the restriction operator given in Proposition 3.7 let us conclude on the final uniform bound on  $(q_{T, \lambda, i})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  and the proof is complete.  $\square$

It remains to prove Proposition 3.7.

*Proof of Proposition 3.7.* The proof is done by contradiction. Let  $\eta \in (0, T/4)$  be a fixed parameter. If we assume that (3.45) does not hold for some uniform  $C$ , then we can find a sequence of time steps  $(\tau^m)_{m \geq 1} \subset (0, \eta^2)$  such that  $\tau^m \rightarrow 0$  as  $m \rightarrow \infty$  and, for each  $m$ , a function  $g^m \in \mathcal{E}_{\tau^m}(\Lambda^{\tau^m, \varpi}, T)$  satisfying

$$\|g^m\|_{L^2_{\tau^m}(0, T)} < \frac{1}{m}. \quad (3.47)$$

while

$$\|g^m\|_{L^2_{\tau^m}(0, +\infty)} = 1, \quad \forall m \geq 1. \quad (3.48)$$

By definition of  $\mathcal{E}_{\tau^m}(\Lambda^{\tau^m, \varpi}, T)$ ,  $g^m$  can be written as

$$g^m = \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} \sum_{j=1}^d a_{\lambda, j}^m p^m[\lambda^{(j)}] \quad (3.49)$$

for some coefficients  $a_{\lambda, j}^m \in \mathbb{R}$ . Since we are dealing with a sequence of time steps, we have used here the notation

$$p^m[\lambda] : n \mapsto (1 + \tau^m \lambda)^{-n} \in L^2_{\tau^m}(0, +\infty).$$

In what follows,  $C$  will denote a generic positive constant uniform with respect to  $\lambda$  and  $m$ , which may vary from line to line.

For each  $m$ , we denote by  $(q_{\lambda, i, m})_{\lambda \in \Lambda^{\tau^m, \varpi}}^{i \in \llbracket 1, d \rrbracket}$  the biorthogonal family in  $L^2_{\tau^m}(0, +\infty)$  provided by Theorem 3.2. For this family, it can be readily seen that

$$(q_{\lambda', i, m}, g^m)_{L^2_{\tau^m}(0, +\infty)} = a_{\lambda', i}^m, \quad \forall \lambda' \in \Lambda^{\tau^m, \varpi}, \quad \forall i \in \llbracket 1, d \rrbracket.$$

This expression, together with estimate (3.13) in Theorem 3.2, the bound (3.48), and Cauchy-Schwarz inequality yields the uniform bound

$$|a_{\lambda, i}^m| \leq C e^{\delta(\lambda)\lambda} \quad (3.50)$$

for all  $\lambda \in \Lambda^{\tau^m, \varpi}$ ,  $i \in \llbracket 1, d \rrbracket$  and  $m \geq 1$ . Note that the function  $\delta$ , given by Theorem 3.2, does not depend on  $m$ .

Now, for each  $m \geq 1$ , let us consider the holomorphic function defined by

$$e_z^m[\lambda] := e^{-z \frac{\log(1 + \tau^m \lambda)}{\tau^m}}, \quad z \in \mathbb{C}. \quad (3.51)$$

Observe that in this way, the function  $p_n^m[\lambda] = (1 + \tau^m \lambda)^{-n}$  with  $n \in \mathbb{N}^*$  is nothing but a sampled version of (3.51) at the real discrete points  $n\tau^m$ , that is,

$$p_n^m[\lambda] = e_{n\tau^m}^m[\lambda]. \quad (3.52)$$

Using the notation of divided differences, we can define the function

$$G^m(z) := \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} \sum_{j=1}^d a_{\lambda, j}^m e_z^m[\lambda^{(j)}], \quad (3.53)$$

where we have seen that the coefficients  $a_{\lambda,j}^m$  verify (3.50). Our goal is now to derive some properties for the holomorphic function (3.53) that will be transferred later to  $g^m$  in (3.49).

A direct computation gives the explicit expression of  $e_z^m[\lambda^{(j)}]$ , this is,

$$e_z^m[\lambda^{(j)}] = \frac{(-1)^{j-1}}{(j-1)!} \prod_{i=1}^{j-1} (z - (i-j+1)\tau^m) e^{-(z+(j-1)\tau^m) \frac{\log(1+\tau^m \lambda)}{\tau^m}}. \quad (3.54)$$

Since  $\lambda \in \Lambda^{\tau^m, \varpi}$ , using that  $\log(1+x)/x \geq 1/2$  for  $x \in [0, 1]$ , we can deduce that for each  $m \geq 1$ , the following bound holds

$$\left| e^{-(z+(j-1)\tau^m) \frac{\log(1+\tau^m \lambda)}{\tau^m}} \right| \leq e^{-\frac{\Re(z)}{2} \lambda}, \quad (3.55)$$

for every  $j \in \llbracket 1, d \rrbracket$ . Moreover, by Young inequality and since  $\tau^m \leq 1$ , there is a constant  $C_j > 0$  uniform with respect to  $m$  such that

$$\left| \prod_{i=1}^{j-1} (z - (i-j+1)\tau^m) \right| \leq C_j (1 + |z|^{j-1}) \quad (3.56)$$

for all  $j \in \llbracket 1, d \rrbracket$ . Therefore, using estimates (3.50) and (3.55)–(3.56) in expression (3.53) we deduce

$$|G^m(z)| \leq C(1 + |z|^{d-1}) \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} e^{-\left(\frac{\Re(z)}{2} - \delta(\lambda)\right) \lambda}. \quad (3.57)$$

Since  $\eta < T/4$  is a fixed parameter, we have that

$$\begin{aligned} \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} e^{-\left(\frac{\Re(z)}{2} - \delta(\lambda)\right) \lambda} &\leq \sum_{\lambda \in \Lambda} e^{-\left(\frac{\Re(z)}{2} - \delta(\lambda)\right) \lambda} \\ &= e^{-R(0) \left(\frac{\Re(z)}{2} - \eta\right)} \sum_{\lambda \in \Lambda} e^{-(\lambda - R(0)) \frac{\Re(z)}{2}} e^{\delta(\lambda) \lambda} e^{-R(0) \eta}, \end{aligned}$$

where we recall (2.18). Since the function  $\lambda \mapsto \delta(\lambda)$  is non-increasing and verifies  $\lim_{s \rightarrow +\infty} \delta(s) = 0$ , there exists  $\lambda_0 > 0$  such that  $\delta(\lambda) \leq \eta$  for all  $\lambda > \lambda_0$ . Thus,

$$\begin{aligned} &\sum_{\lambda \in \Lambda^{\tau^m}} e^{-\left(\frac{\Re(z)}{2} - \delta(\lambda)\right) \lambda} \\ &\leq e^{-R(0) \left(\frac{\Re(z)}{2} - \eta\right)} \left( \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq \lambda_0}} e^{-(\lambda - R(0)) \frac{\Re(z)}{2}} e^{\delta(\lambda) \lambda} + \sum_{\substack{\lambda \in \Lambda \\ \lambda > \lambda_0}} e^{-(\lambda - R(0)) \left(\frac{\Re(z)}{2} - \eta\right)} \right). \end{aligned}$$

Let us introduce the truncated sector  $\mathcal{P}_\eta = \{z \in \mathbb{C} : \Re(z) \geq 4\eta, |\Im(z)| \leq \Re(z)\}$  (see Figure 1). Then, for all  $z \in \mathcal{P}_\eta$  we have

$$\begin{aligned} &\sum_{\lambda \in \Lambda^{\tau^m}} e^{-\left(\frac{\Re(z)}{2} - \delta(\lambda)\right) \lambda} \\ &\leq e^{-R(0) \left(\frac{\Re(z)}{2} - \eta\right)} \left( e^{\delta(R(0)) \lambda_0} \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq \lambda_0}} e^{-2\eta(\lambda - R(0))} + \sum_{\substack{\lambda \in \Lambda \\ \lambda > \lambda_0}} e^{-\eta(\lambda - R(0))} \right), \quad (3.58) \end{aligned}$$

and noting that the first sum is finite and the second one converges by (2.15), we can use estimate (3.58) in (3.57) to deduce the existence of a constant  $C > 0$  depending at most on  $d, R, \lambda_0$  and  $\eta$  such that

$$|G^m(z)| \leq C|z|^{d-1} e^{-R(0) \left(\frac{\Re(z)}{2} - \eta\right)}, \quad \forall z \in \mathcal{P}_\eta. \quad (3.59)$$

From (3.59) we deduce that  $(G^m)_m$  is a sequence of holomorphic functions uniformly bounded in  $L^\infty(\mathcal{P}_\eta)$ . Montel's theorem on normal families of holomorphic functions (see e.g. [17, Chapter 5]) implies that we can extract a subsequence, still denoted by  $(G^m)_m$  that converges locally uniformly towards a holomorphic function  $G$  in  $\mathcal{P}_\eta$ . In particular,  $G^m(t) \rightarrow G(t)$  for all  $t \in (4\eta, +\infty)$  and from estimate (3.59) we have

$$|G^m(t)| \leq C e^{-R(0) \left(\frac{t}{2} - \eta\right)} t^{d-1}, \quad \forall t \in (4\eta, +\infty).$$

The right-hand side being square integrable allows us to use dominated convergence theorem to deduce that

$$G^m \rightarrow G \quad \text{strongly in } L^2(4\eta, +\infty). \quad (3.60)$$

Let us come back to the discrete setting. For any function  $u = (u^n)_{n \in \llbracket 1, M \rrbracket} \in L_\tau^2(0, T)$ , we denote by  $\mathcal{F}[u]$  the element of  $L^2(0, T)$  defined as the piecewise constant interpolator

$$\mathcal{F}[u](t) = \sum_{n=1}^M \mathbf{1}_{(t_{n-1}, t_n)} u^n. \quad (3.61)$$

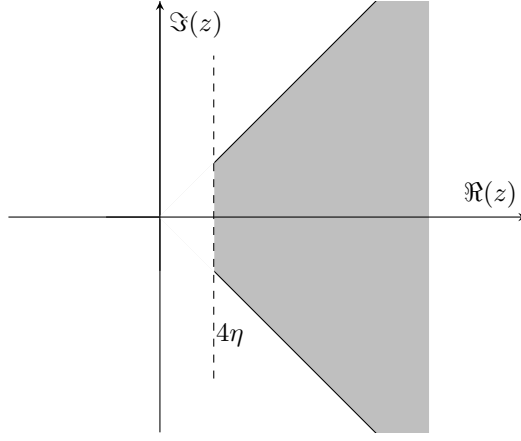


Figure 1: The sector  $\mathcal{P}_\eta$ .

Note that  $\|\mathcal{F}[u]\|_{L^2(0,T)}^2 = \|u\|_{L^2_\tau(0,T)}^2$ . Also notice that this notation equally makes sense if we replace  $M$  and  $T$  by  $\infty$  everywhere. Moreover, we will add the subscript  $m$  to indicate that we consider this interpolator for the time step  $\tau^m$ .

We see that the assumption (3.47) can be rephrased as

$$\|\mathcal{F}_m[g^m]\|_{L^2(0,T)} < \frac{1}{m},$$

and by letting  $m$  tend to infinity, we obtain

$$\mathcal{F}_m[g^m] \rightarrow 0 \quad \text{in } L^2(0,T). \quad (3.62)$$

Now, we claim the following:

**Claim 3.9.**  $\mathcal{F}_m[g^m] \rightarrow G$  strongly in  $L^2(4\eta, \infty)$ , where  $G$  is the holomorphic function found as the limit in (3.60).

Let us assume for the moment that the above statement holds true. From (3.62), Claim 3.9, and the uniqueness of the limit, we deduce from (3.60) that  $G \equiv 0$  in  $(4\eta, T)$  and since  $G$  is a holomorphic function, this implies that

$$G \equiv 0 \quad \text{in } (4\eta, +\infty). \quad (3.63)$$

Thus, (3.63) together with (3.62) and Claim 3.9 allow us to conclude that

$$\mathcal{F}_m[g^m] \rightarrow 0 \quad \text{in } L^2(0 + \infty)$$

which is contradiction with our initial hypothesis (3.48) that says  $\|\mathcal{F}_m[g^m]\|_{L^2(0,+\infty)} = 1$ . This concludes the proof.  $\square$

We present the proof of Claim 3.9.

*Proof of Claim 3.9.* Fix  $m \geq 1$  and set  $n_\eta = \lfloor \frac{5\eta}{\tau^m} \rfloor$ . Using (3.49), (3.52), and (3.53), for any  $n \geq n_\eta$  and  $t \in [t_{n-1}^m, t_n^m]$ , we can readily compute

$$\begin{aligned} |G^m(t) - g_n^m| &= \left| \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} \sum_{j=1}^d a_{\lambda,j}^m \left( e_t^m[\lambda^{(j)}] - e_{n\tau^m}^m[\lambda^{(j)}] \right) \right| \\ &\leq \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} \sum_{j=1}^d |a_{\lambda,j}^m| \left| \frac{d}{dt} \Big|_{t=\zeta} e_t^m[\lambda^{(j)}] \right| |t_n^m - t| \end{aligned}$$

for some  $\zeta \in (t_{n-1}^m, t_n^m)$ . Using (3.50) and recalling that  $t \in [t_{n-1}^m, t_n^m]$ , we obtain

$$|G^m(t) - g_n^m| \leq C\tau^m \sum_{\lambda \in \Lambda^{\tau^m, \varpi}} \sum_{i=1}^d e^{\delta(\lambda)\lambda} \left| \frac{d}{dt} \Big|_{t=\zeta} e_t^m[\lambda^{(j)}] \right|. \quad (3.64)$$

The following result allow us to estimate in the right-hand side of (3.64).

**Lemma 3.10.** Assume that  $\lambda \in \Lambda^{\tau^m, \varpi}$  for some  $\varpi \in (0, 1)$ . For any  $n \geq n_\eta$  and  $\zeta \in (t_{n-1}^m, t_n^m)$ , we have

$$\left| \frac{d}{dt} \Big|_{t=\zeta} e_t^m[\lambda^{(j)}] \right| \leq \frac{C_{j,\eta}}{\sqrt{\tau^m}} |t_n^m|^{j-1} e^{-\frac{t_n^m}{4}\lambda}, \quad \forall j \in \llbracket 1, d \rrbracket. \quad (3.65)$$

The proof of this result can be found in Appendix B.3. Using (3.64), estimate (3.65), and arguing as we did for obtaining (3.59), we get

$$|G^m(t) - g_n^m| \leq C\sqrt{\tau^m} |t_n^m|^{d-1} e^{-R(0)\left(\frac{t_n^m}{4} - \eta\right)}$$

for some constant  $C > 0$  uniform with respect to  $m$ .

Squaring and then integrating over  $(t_{n-1}^m, t_n^m)$  on both sides of the previous inequality yields

$$\int_{t_{n-1}^m}^{t_n^m} |G^m(t) - \mathbf{1}_{(t_{n-1}^m, t_n^m)}(t)g_n^m|^2 dt \leq C(\tau^m)^2 |t_n^m|^{2(d-1)} e^{-2R(0)\left(\frac{t_n^m}{4} - \eta\right)}.$$

Now, we sum over  $n$  on both sides, more precisely

$$\begin{aligned} \sum_{n=n_\eta}^{+\infty} \int_{t_{n-1}^m}^{t_n^m} |G_m(t) - \mathbf{1}_{(t_{n-1}^m, t_n^m)}(t)g_n^m|^2 dt &\leq C\tau^m \left[ \sum_{n=n_\eta}^{+\infty} \tau^m |t_n^m|^{2(d-1)} e^{-R(0)\left(\frac{t_n^m}{4} - \eta\right)} \right] \\ &\leq C_\eta \tau^m \left[ \sum_{n=1}^{+\infty} \tau^m |t_n^m|^{2(d-1)} e^{-R(0)\frac{t_n^m}{4}} \right]. \end{aligned} \quad (3.66)$$

We emphasize that the infinite sum in the right-hand side of the above expression is convergent and is uniformly bounded with respect to  $m$ . Indeed,

$$\begin{aligned} \sum_{n=1}^{+\infty} \tau^m |t_n^m|^{2(d-1)} e^{-R(0)\frac{t_n^m}{4}} &= \sum_{n=1}^{+\infty} \int_{t_n^m}^{t_{n+1}^m} \mathbf{1}_{(t_n^m, t_{n+1}^m)} |t_n^m|^{2(d-1)} e^{-R(0)\frac{t_n^m}{4}} dt \\ &\leq \sum_{n=1}^{+\infty} \int_{t_n^m}^{t_{n+1}^m} \mathbf{1}_{(t_n^m, t_{n+1}^m)} t^{2(d-1)} e^{\tau^m} e^{-R(0)\frac{t}{4}} dt \\ &\leq e \int_0^{+\infty} t^{2(d-1)} e^{-R(0)\frac{t}{4}} dt = e 4^{2d-1} R(0)^{1-2d} \Gamma(2d-1) < +\infty. \end{aligned} \quad (3.67)$$

On the other hand, from the construction of  $\mathcal{F}$  and its properties and the choice of  $n_\eta$ , observe that the term in the left-hand side of (3.66) is bounded from below by  $\|G_m - \mathcal{F}_m[g_m]\|_{L^2(4\eta, \infty)}$ , that is,

$$\|G_m - \mathcal{F}_m[g_m]\|_{L^2(4\eta, \infty)}^2 \leq C\tau^m \left[ \sum_{n=1}^{+\infty} \tau^m |t_n^m|^{2(d-1)} e^{-R(0)\frac{t_n^m}{4}} \right]. \quad (3.68)$$

Putting together (3.67) and (3.68), we see that  $\|G_m - \mathcal{F}_m[g_m]\|_{L^2(4\eta, \infty)}^2 \rightarrow 0$  as  $m \rightarrow \infty$  since there is an extra factor  $\tau^m$  in the right-hand side. At this point, we note that

$$\|g - \mathcal{F}_m[g_m]\|_{L^2(4\eta, \infty)} \leq \|G - G_m\|_{L^2(4\eta, \infty)} + \|G_m - \mathcal{F}_m[g_m]\|_{L^2(4\eta, \infty)}$$

and using (3.60) on the first term, we can take the limit as  $m \rightarrow 0$  to obtain the desired result. This concludes the proof.  $\square$

### 3.4 Proof of the main theorem

We are now in position to prove our main  $\varphi(\tau)$ -controllability result, that is Theorem 1.1, which is a direct construction based on the time discrete biorthogonal families we have obtained above, as follows.

Let us fix  $T > 0$  and the initial data  $y^0 \in L^2(\Omega)$ . Let  $\varpi \in (0, 1)$  and  $\tau_0 > 0$  be the parameters provided by Theorem 3.2. Take  $\tau \in (0, \tau_0)$  and set

$$T_1 = \frac{T}{2} \quad \text{and} \quad \widetilde{M} = \left\lfloor \frac{T_1}{\tau} \right\rfloor.$$

From Theorem 3.2, we know that there exists a biorthogonal family  $(q_{\lambda, i})_{\lambda \in \Lambda^{\tau, \varpi}}^{i \in \llbracket 1, d \rrbracket} \subset L_\tau^2(0, T_1)$  verifying

$$\left( p[\lambda^{(i)}], q_{\lambda', j} \right)_{L_\tau^2(0, T_1)} = \delta_{\lambda, \lambda'} \delta_{i, j}, \quad \forall i, j \in \llbracket 1, d \rrbracket, \quad \forall \lambda, \lambda' \in \Lambda^{\tau, \varpi}$$

and

$$\|q_{\lambda, i}\|_{L_\tau^2(0, T_1)} \leq C e^{\delta(\lambda)\lambda}, \quad \forall i \in \llbracket 1, d \rrbracket, \quad \forall \lambda \in \Lambda^{\tau, \varpi}, \quad (3.69)$$

where  $C > 0$  is a constant independent of  $\tau$ .

Let us consider the control  $v_0 = (v_0^n)_{n \in \llbracket 1, \widetilde{M} \rrbracket}$  with elements given by

$$v_0^n = \sum_{i=1}^d \sum_{\lambda \in \Lambda^{\tau, \varpi}} \beta_{\lambda, i} q_{\lambda, i}^n, \quad n \in \llbracket 1, \widetilde{M} \rrbracket,$$

where

$$\beta_{\lambda,i} = \frac{\alpha_{\lambda,i}}{\gamma(1)\partial_x\phi_\lambda(1)}, \text{ with } \alpha_{\lambda,i} = \left( y_0, (\mathcal{I} \otimes \mathbf{I} + \tau\mathcal{L}^*)^{-\widetilde{M}}(\phi_\lambda \otimes \mathbf{e}_i) \right)_{L^2(\Omega)}. \quad (3.70)$$

Then, using formula (3.7) at  $M = \widetilde{M}$  and due to the special structure of  $v_0$  together with the properties of the biorthogonal, we easily deduce that

$$\left( y^{\widetilde{M}}, \phi_\lambda \otimes \mathbf{e}_r \right) = 0, \quad \forall \lambda \in \Lambda^{\tau,\varpi}, \quad \forall r \in \llbracket 1, d \rrbracket. \quad (3.71)$$

Since the sums defining  $v_0$  are finite, we do not have any convergence issues. Moreover, the condition  $\partial_x\phi_\lambda(1) \neq 0$  is guaranteed by (2.6). Thus, it remains to verify the uniform bound (1.16).

Using definition (3.6) and estimate (A.3) with  $n = \widetilde{M}$ , we obtain

$$\begin{aligned} \left| (\mathcal{I} \otimes \mathbf{I} + \tau\mathcal{L}^*)^{-\widetilde{M}}(\phi_\lambda \otimes \mathbf{e}_i) \right|_{L^2(\Omega)} &= \left| \sum_{\ell=0}^{i-1} p_{\widetilde{M}}[\lambda^{(\ell+1)}](\phi_\lambda \otimes \mathbf{e}_{r-\ell}) \right|_{L^2(\Omega)} \\ &\leq C_i \sum_{\ell=0}^{i-1} (\widetilde{M}\tau)^\ell p_{\widetilde{M}}[\lambda] |\phi_\lambda \otimes \mathbf{e}_{r-\ell}|_{L^2(\Omega)} \end{aligned}$$

for all  $i \in \llbracket 1, d \rrbracket$ . Using the definition of  $\widetilde{M}$  and since  $\lambda \in \Lambda^{\tau,\varpi}$ , we further estimate

$$\begin{aligned} \left| (\mathcal{I} \otimes \mathbf{I} + \tau\mathcal{L}^*)^{-\widetilde{M}}(\phi_\lambda \otimes \mathbf{e}_i) \right|_{L^2(\Omega)} &\leq C_i \sum_{\ell=0}^{i-1} T_1^\ell e^{-\lambda T_1 \frac{\log(1+\tau\lambda)}{\tau\lambda}} |\phi_\lambda \otimes \mathbf{e}_{r-\ell}|_{L^2(\Omega)} \\ &\leq C_i \sum_{\ell=0}^{i-1} T_1^\ell e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}} |\phi_\lambda \otimes \mathbf{e}_{r-\ell}|_{L^2(\Omega)}, \quad \forall i \in \llbracket 1, d \rrbracket, \end{aligned} \quad (3.72)$$

uniformly with respect to  $\tau$ . By Cauchy-Schwarz inequality and the fact that  $(\phi_\lambda \otimes \mathbf{e}_i)$  are normalized in the space  $L^2(\Omega) \otimes \mathbb{R}^d$ , we deduce

$$|\alpha_{\lambda,i}| \leq C_T |y_0|_{L^2(\Omega)} e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}}, \quad (3.73)$$

for some constant  $C_T$  only depending on  $T$ . Then, from (2.6), we obtain

$$|\beta_{\lambda,i}| \leq C_T |y_0|_{L^2(\Omega)} e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}}. \quad (3.74)$$

Hence, by the estimate on the biorthogonal family (3.69), we can now easily bound the norm of the control  $v_0$ , this is,

$$\|v_0\|_{L^2_\tau(0,T/2)} \leq C_T |y_0|_{L^2(\Omega)} \sum_{\lambda \in \Lambda^{\tau,\varpi}} e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}} e^{\delta(\lambda)\lambda}.$$

Using the properties of the function  $\lambda \mapsto \delta(\lambda)$ , there exists  $\lambda_0 > 0$  such that  $\delta(\lambda) \leq T_1 \frac{\log(1+\varpi)}{2\varpi}$  for all  $\lambda > \lambda_0$ , thus

$$\begin{aligned} \|v_0\|_{L^2_\tau(0,T/2)} &\leq C_T |y_0|_{L^2(\Omega)} \left( \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq \lambda_0}} e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}} e^{\delta(\lambda)\lambda} + \sum_{\substack{\lambda \in \Lambda \\ \lambda > \lambda_0}} e^{-\lambda T_1 \frac{\log(1+\varpi)}{2\varpi}} \right) \\ &\leq C_T |y_0|_{L^2(\Omega)} \left( e^{\delta(R(0))\lambda_0} \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq \lambda_0}} e^{-\lambda T_1 \frac{\log(1+\varpi)}{\varpi}} + \sum_{\substack{\lambda \in \Lambda \\ \lambda > \lambda_0}} e^{-\lambda T_1 \frac{\log(1+\varpi)}{2\varpi}} \right). \end{aligned}$$

Since the above series are convergent thanks to (2.15), we have thus found a control  $v_0$  in  $L^2_\tau(0, T/2)$  satisfying

$$\|v_0\|_{L^2_\tau(0,T/2)} \leq C |y_0|_{L^2(\Omega)} \quad (3.75)$$

with a constant  $C > 0$  uniform with respect to  $\tau$  and where the associated controlled solution verifies (3.71).

Using that system (1.9) is in cascade form and applying classical elliptic regularity estimates (component by component), after summing for  $n \in \llbracket 0, \widetilde{M} - 1 \rrbracket$  we obtain

$$\begin{aligned} \sup_{n \in \llbracket 1, \widetilde{M} \rrbracket} |y^n|_{H^{-1}(\Omega)} + \|y\|_{L^2_\tau(0,T_1;L^2(\Omega))} &\leq C_T \left( |y_0|_{H^{-1}(\Omega)} + \|v_0\|_{L^2_\tau(0,T/2)} \right) \\ &\leq C |y_0|_{L^2(\Omega)}, \end{aligned}$$

where we have used estimate (3.75) in the second line. From here, we deduce in particular that

$$|y^{\widetilde{M}}|_{H^{-1}(\Omega)} \leq C |y_0|_{L^2(\Omega)}. \quad (3.76)$$



To conclude, we will use the fact that the solution  $y$  verifies (3.71). We begin by extending the control  $v_0$  by zero in the interval  $[\widetilde{M} + 1, M]$  and consider the uncontrolled system

$$\begin{cases} \frac{y^{n+1} - y^n}{\tau} + \mathcal{L}y^{n+1} = 0, & n \in [\widetilde{M}, M - 1], \\ y|_{\partial\Omega}^{\tau} = 0, & n \in [\widetilde{M}, M - 1], \end{cases}$$

with initial datum  $y^{\widetilde{M}}$  coming from the previous step. For  $n = \widetilde{M}$ , the system of elliptic equations verified by  $y^{\widetilde{M}+1}$  is

$$\begin{cases} \tau \mathcal{A} \otimes \mathbf{I} y^{\widetilde{M}+1} + (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{I} \otimes \mathbf{C}) y^{\widetilde{M}+1} = y^{\widetilde{M}}, & \text{in } \Omega \\ y|_{\partial\Omega}^{\widetilde{M}+1} = 0. \end{cases}$$

Using once again the cascade form of the system and using well-known regularity estimates for elliptic PDEs, we obtain

$$|y^{\widetilde{M}+1}|_{H^{-1}(\Omega)} + \tau |y^{\widetilde{M}+1}|_{H_0^1(\Omega)} \leq C |y^{\widetilde{M}}|_{H^{-1}(\Omega)},$$

where  $C > 0$  only depends on  $d$ . This, together with estimate (3.76) yield

$$\sqrt{\tau} |y^{\widetilde{M}+1}|_{L^2(\Omega)} \leq C |y_0|_{L^2(\Omega)}. \quad (3.77)$$

For  $n \in [\widetilde{M} + 2, M - 1]$ , we will use the fact that we have an explicit expression for the solution. Indeed, at each step, we have

$$\begin{aligned} y^{n+1} &= \sum_{i=1}^d \sum_{\lambda \in \Lambda} (y^n, \phi_\lambda \otimes \mathbf{e}_i)_{L^2(\Omega)} (\mathcal{I} \otimes \mathbf{I} + \tau \mathcal{L})^{-1} (\phi_\lambda \otimes \mathbf{e}_i) \\ &= \sum_{\lambda \in \Lambda \setminus \Lambda^{\tau, \varpi}} \sum_{i=1}^d (y^n, \phi_\lambda \otimes \mathbf{e}_i)_{L^2(\Omega)} \sum_{\ell=1}^{i-1} p_1[\lambda^{(\ell+1)}] (\phi_\lambda \otimes \mathbf{e}_{i-\ell}), \end{aligned} \quad (3.78)$$

where we have used (3.71) in the second line. Taking the  $L^2$ -norm in both sides of the above expression and using estimate (A.3) together with the fact that  $\lambda \in \Lambda \setminus \Lambda^{\tau, \varpi}$ , we get

$$|y^{n+1}|_{L^2(\Omega)} \leq C e^{-\log(1+\varpi)} |y^n|_{L^2(\Omega)}. \quad (3.79)$$

Iterating estimate (3.79) for indices  $n \in [\widetilde{M} + 2, M - 1]$  and using and (3.77), we obtain

$$|y^M|_{L^2(\Omega)} \leq C e^{-(M - (\widetilde{M} + 1)) \log(1+\varpi)} |y_0|_{L^2(\Omega)},$$

whence, using the definition of  $\widetilde{M}$ , we deduce

$$|y^M|_{L^2(\Omega)} \leq \tilde{C} e^{-\frac{1}{\tau}(T - T_1 - \tau) \log(1+\varpi)} |y_0|_{L^2(\Omega)}, \quad (3.80)$$

for some new constant  $\tilde{C} > 0$  uniform with respect to  $\tau$ . This gives the desired result since  $T > T_1$  and thanks to (1.15).

## 4 On the minimal time for null-controllability of time-discrete systems

The minimal time for null controllability, a phenomenon which at first glance seems counterintuitive in the parabolic setting due to the classical results [24] and [30], has been addressed in many recent works, see, for instance, [4, 5, 19, 33, 36]. In such works, the authors exhibit different settings leading to the appearance of a minimal null-control time. In this section, we are interested on the consequences of such an effect on time-discretization.

Since the model system (1.5) we have studied up to now does not exhibit such phenomenon, we shall consider here a slightly more general setting to illustrate our discussion. However, we will see that our result on time discrete biorthogonal families Theorem 3.2 will be also applicable to this new setting.

### 4.1 A general framework

We consider an abstract time invariant linear control system with a scalar control

$$\begin{cases} y'(t) + \mathcal{L}y(t) = \mathcal{B}u(t), \\ y(0) = y_0, \end{cases} \quad (4.1)$$

where  $(\mathcal{L}, D(\mathcal{L}))$  is an unbounded operator in an Hilbert space  $X$  such that  $-\mathcal{L}$  generates a  $C^0$ -semigroup in  $X$ . We assume that  $\mathcal{L}$  has a spectrum included in  $(0, +\infty)$  and we define  $X_1^* = D(\mathcal{L}^*)$ , equipped with the norm  $|z|_1 = |\mathcal{L}^* z|_X$ . We can then define a dual norm on  $X$  as follows

$$|y|_{-1} := \sup_{z \in X_1^*} \frac{(y, z)_X}{|z|_1}, \quad \forall y \in X,$$

and then define the space  $X_{-1}$  to be the completion of  $X$  for this norm. We assume that the (scalar) control operator  $\mathcal{B}$  is continuous from  $\mathbb{R}$  into  $X_{-1}$ , so that, by duality  $\mathcal{B}^*$  is continuous from  $X_1^*$  into  $\mathbb{R}$ .

Under this assumptions, it is well-known that for any  $y_0 \in X_{-1}$  and any  $u \in L^2(0, T; \mathbb{R})$ , system (4.1) has a unique solution  $y \in C^0([0, T]; X_{-1})$  verifying the following identity

$$\langle y(t), q_t \rangle_{-1,1} - \langle y_0, e^{-t\mathcal{L}^*} q_t \rangle_{-1,1} = \int_0^t u(s) \mathcal{B}^*(e^{-(t-s)\mathcal{L}^*} q_t) ds,$$

for any  $t \in [0, T]$  and any  $q_t \in X_1^*$ .

It is well-known that under this functional setting, system (4.1) is null controllable at time  $T$  if and only the following observability inequality holds for some  $C > 0$

$$|e^{-T\mathcal{L}^*} q_T|_1 \leq C \left( \int_0^T |\mathcal{B}^* e^{-(T-t)\mathcal{L}^*} q_T|^2 dt \right)^{1/2}, \quad \forall q_T \in X_1^*. \quad (4.2)$$

We assume now that  $\mathcal{L}^*$  satisfies the following spectral assumptions: its spectrum  $\Lambda$  is made only of positive eigenvalues, each of them being geometrically simple (we denote by  $\Phi_{\lambda,1}$  an associated eigenfunction) and has a (constant) algebraic multiplicity  $d$ . The Jordan chains are denoted by  $\Phi_{\lambda,r}$  for  $r \in \llbracket 2, d \rrbracket$ . We additionally assume that the family  $(\Phi_{\lambda,r})_{\lambda \in \Lambda}^{r \in \llbracket 1, d \rrbracket}$  is complete in  $X_1^*$ . Note that the case where the algebraic multiplicity of all the eigenvalues are different (but still bounded by some  $d > 0$ ) can be treated exactly in the same way.

It is clear from (4.2) that a necessary controllability condition is

$$\mathcal{B}^* \Phi_{\lambda,1} \neq 0, \quad \forall \lambda \in \Lambda. \quad (4.3)$$

Assuming this additional assumption, we can then uniquely specify the (generalized) eigenfunctions by imposing that

$$\mathcal{B}^* \Phi_{\lambda,1} = 1, \quad \text{and} \quad \mathcal{B}^* \Phi_{\lambda,i} = 0, \quad \forall i \in \llbracket 2, d \rrbracket, \forall \lambda \in \Lambda. \quad (4.4)$$

Finally, in order to be able to apply the moment's method, we need to assume in addition that  $\Lambda$  satisfies the summability condition (2.15) and the gap condition (2.16).

In this framework, the following result is a particular case of the results in [8] (see Section 1.4 of this reference).

**Theorem 4.1** (Minimal null-control time). *We define*

$$T_0 := \limsup_{\lambda \in \Lambda} \frac{\log \left( \max_{i \in \llbracket 1, d \rrbracket} |\Phi_{\lambda,i}|_1 \right)}{\lambda}. \quad (4.5)$$

*Then we have:*

- For  $T > T_0$ , the system (4.1) is null-controllable (that is (4.2) holds).
- For  $0 < T < T_0$ , the system (4.1) is not null-controllable (that is (4.2) does not hold).

Note that the cascade system (1.5) that we previously studied satisfies the assumptions above with  $T_0 = 0$ . However, if we assume that the coupling matrix  $C$  in (1.5) depends on the space variable  $x$ , a positive value of  $T_0$  can be found. We refer to [4, Section 6] and [8, Section 5] for concrete examples illustrating this fact.

## 4.2 The time-discrete system

Now, consider the time-discrete system

$$\begin{cases} \frac{y^{n+1} - y^n}{\tau} + \mathcal{L}y^{n+1} = \mathcal{B}u^{n+1}, & \forall n \in \llbracket 0, M-1 \rrbracket \\ y^0 = y_0. \end{cases} \quad (4.6)$$

As we have seen before we cannot hope a time-discrete version of the usual observability inequality (4.2) to hold, but we may expect relaxed observability inequality of the following form (note the extra term in the right-hand side)

$$\left| (\mathcal{I} + \tau\mathcal{L}^*)^{-M} q_T \right|_1 \leq C_{\varphi, T, \tau} \left( \sum_{n=0}^{M-1} \tau |\mathcal{B}^*(\mathcal{I} + \tau\mathcal{L}^*)^{-(M-n)} q_T|^2 + \varphi(\tau) |q_T|_1^2 \right)^{1/2}, \quad \forall q_T \in X_1^*. \quad (4.7)$$

We denote by  $C_{\varphi,T,\tau} > 0$  the smallest constant for which this estimate holds, for a given  $T$ ,  $\varphi$  and  $\tau$ . It is clear that, we can basically obtain  $C_{\varphi,T,\tau} \leq \varphi(\tau)^{-1/2}$  which does not give useful information.

We would like now to investigate whether or not this constant remains bounded when  $\tau \rightarrow 0$ . More precisely, we will show that  $C_{\varphi,T,\tau}$  is bounded when  $T > T_0$ , whereas it is not bounded when  $T < T_0$ . This is the time-discrete counterpart of the minimal null-control time property.

It is interesting to note that if, for a given value of  $T$ , one manages to actually compute this constant for different choices of discretization steps  $M$  and observe its behavior as  $M \rightarrow \infty$ , it may be possible to deduce whether  $T$  is below or above the actual minimal time  $T_0$  for the considered system. At least in the fully discrete setting (see Section 5.1), it is possible to implement this strategy, since it consists essentially of computing eigenvalues of matrices.

#### 4.2.1 Large time analysis

In this section we will consider a control time larger than the minimal time given by (4.5). In that case, we obtain a uniform relaxed time-discrete observability estimate.

**Theorem 4.2.** *Assume that  $T > T_0 \geq 0$ , and let  $\varphi$  be any given function satisfying (1.15). Then we have*

$$\limsup_{\tau \rightarrow 0} C_{\varphi,T,\tau} < +\infty.$$

We will only give the sketch of proof since it is in fact very similar to the one we did in Section 3. More precisely, we shall first consider some  $T_1 \in (T_0, T)$ , and then we will fix  $\varpi \in (0, 1)$  in such a way that

$$T_1 \log(1 + \varpi)/\varpi > T_0. \quad (4.8)$$

We apply the same strategy than in Section 3.4 in this slightly different framework and with this value of  $T_1$  instead of  $T/2$ . Everything works exactly the same, up to formulas (3.70) that becomes, thanks to (4.4),

$$\beta_{\lambda,i} = \left\langle y_0, (\mathcal{I} + \tau \mathcal{L}^*)^{-\tilde{M}} \Phi_{\lambda,i} \right\rangle_{-1,1}. \quad (4.9)$$

Note that in the present setting formula (3.6) reads

$$(\mathcal{I} + \tau \mathcal{L}^*)^{-n} \Phi_{\lambda,r} = \sum_{\ell=0}^{r-1} p_n[\lambda^{(\ell+1)}] \Phi_{\lambda,r-\ell}, \quad n \in \llbracket 1, M \rrbracket. \quad (4.10)$$

Therefore, the bound (3.74) needs to be changed because  $\Phi_{\lambda,i}$  can be exponentially large. Indeed, by (4.5), we have for any  $\epsilon > 0$  the estimate

$$|\Phi_{\lambda,i}|_1 \leq C_\epsilon e^{(T_0+\epsilon)\lambda},$$

which leads to

$$|\beta_{\lambda,i}| \leq C_\epsilon |y_0|_{-1} e^{(T_0+\epsilon)\lambda} e^{-\lambda T_1 \log(1+\varpi)/\varpi},$$

and, by (4.8), we see that we can choose  $\epsilon$  so that  $|\beta_{\lambda,i}|$  can be estimated by some  $C_T |y_0|_X e^{-\lambda \tilde{\epsilon}}$ , with  $\tilde{\epsilon} > 0$  which is enough to conclude the proof exactly as before.

#### 4.2.2 Small time analysis

Let us now consider the case where the control time is below the minimal null-control time. By Theorem 4.1 we know that the continuous problem is not null-controllable and in particular the observability inequality (4.2) does not hold. However, for every given  $\varphi$  and  $\tau$  the time-discrete relaxed observability estimate (4.7) holds but we will prove, in that case, that it cannot be uniform.

**Theorem 4.3.** *Assume that  $0 < T < T_0$ , and let  $\varphi$  be any given function satisfying (1.15). Then we have*

$$\limsup_{\tau \rightarrow 0} C_{\varphi,T,\tau} = +\infty.$$

*Proof.* By definition of  $T_0$ , given in (4.5), we see that there exists a unique integer  $r \in \llbracket 1, d \rrbracket$  such that

$$\limsup_{\lambda \in \Lambda} \frac{\log |\Phi_{\lambda,r}|_1}{\lambda} = T_0,$$

and

$$\limsup_{\lambda \in \Lambda} \frac{\log \left( \max_{i \in \llbracket 1, r-1 \rrbracket} |\Phi_{\lambda,i}|_1 \right)}{\lambda} < T_0.$$

Those two properties imply that, there exists  $\epsilon > 0$ , constants  $C_1, C_2 > 0$  and a sequence  $(\lambda_k)_k \subset \Lambda$  such that  $\lambda_k \rightarrow +\infty$  and

$$\begin{cases} |\Phi_{\lambda_k,r}|_1 \geq C_1 e^{(T_0-\epsilon)\lambda_k}, \\ |\Phi_{\lambda_k,r}|_1 \leq C_2 e^{(T_0+\epsilon)\lambda_k}, \\ |\Phi_{\lambda_k,i}|_1 \leq C_2 |\Phi_{\lambda_k,r}|_1 e^{-\epsilon\lambda_k}, \quad \forall i \in \llbracket 1, r-1 \rrbracket. \end{cases} \quad (4.11)$$

Since  $T < T_0$  by assumption, we will in addition choose  $\epsilon$  such that

$$T_0 - T - \epsilon > 0. \quad (4.12)$$

By using (4.4) and (4.10), it follows that

$$\mathcal{B}^* (\mathcal{I} + \tau \mathcal{L}^*)^{-n} \Phi_{\lambda_k, r} = p_n[\lambda_k^{(r)}],$$

and by (4.10), with (A.3) and (A.2), we get

$$\begin{aligned} |(\mathcal{I} + \tau \mathcal{L}^*)^{-n} \Phi_{\lambda_k, r}|_1 &\geq p_n[\lambda_k] \left( |\Phi_{\lambda_k, 1}|_1 - \sum_{\ell=1}^{r-1} T^{r-\ell} |\Phi_{\lambda_k, \ell}|_1 \right) \\ &\geq C_1 e^{(T_0 - \epsilon)\lambda_k} p_n[\lambda_k] \left( 1 - C_2 \sum_{\ell=1}^{r-1} T^{r-\ell} e^{-\epsilon\lambda_k} \right) \\ &\geq C_1 e^{(T_0 - T - \epsilon)\lambda_k} \left( 1 - C_2 \sum_{\ell=1}^{r-1} T^{r-\ell} e^{-\epsilon\lambda_k} \right). \end{aligned}$$

We can now test the weak observability inequality (4.7) with  $q_T = \Phi_{\lambda_k, r}$  for any  $k$  and any  $\tau$ . We obtain

$$C_1 e^{(T_0 - T - \epsilon)\lambda_k} \left( 1 - C_2 \sum_{\ell=1}^{r-1} T^{r-\ell} e^{-\epsilon\lambda_k} \right) \leq C_{\varphi, T, \tau} \left( \sum_{n=1}^M \tau \left| p_n[\lambda_k^{(r)}] \right|^2 + \varphi(\tau) C_2^2 e^{2(T_0 + \epsilon)\lambda_k} \right)^{1/2}.$$

Extending the sum in the right-hand side up to  $+\infty$  and bounding its value by (A.3) and (A.5), we get for  $k$  large enough (depending on  $T$ ) and for some  $C_3$ , depending only on  $C_1, C_2$  and  $\Lambda$ ,

$$C_3 e^{(T_0 - T - \epsilon)\lambda_k} \leq C_{\varphi, T, \tau} \left( 1 + \sqrt{\varphi(\tau)} e^{(T_0 + \epsilon)\lambda_k} \right), \quad \forall \tau \leq \frac{1}{\lambda_k}. \quad (4.13)$$

Since the function  $\varphi$  is continuous and strictly increasing, we know that for every  $k$  large enough there exists a number  $\delta_k > 0$  such that

$$\delta_k \leq \frac{1}{\lambda_k}, \quad \text{and} \quad \sqrt{\varphi(\delta_k)} \leq e^{-\lambda_k(T_0 + \epsilon)}. \quad (4.14)$$

Now, set  $M_k = \lceil \frac{T}{\delta_k} \rceil$ . Obviously, we have

$$M_k - 1 \leq \frac{T}{\delta_k} \leq M_k,$$

and defining  $\tau_k = \frac{T}{M_k}$  we have  $\tau_k \leq \delta_k$  and in particular  $\sqrt{\varphi(\tau_k)} \leq \sqrt{\varphi(\delta_k)} = e^{-\lambda_k(T_0 + \epsilon)}$ . It follows from (4.13) that

$$C_3 e^{(T_0 - T - \epsilon)\lambda_k} \leq 2C_{\varphi, T, \tau_k}.$$

Since we have taken  $\epsilon$  satisfying (4.12), the claim follows letting  $k$  goes to infinity.  $\square$

**Remark 4.4.** *In some cases, for instance if  $\varphi(\tau) = \tau^p$  for some  $p > 0$ , this behavior can be precised as follows*

$$\limsup_{\tau \rightarrow 0} \left( \varphi(\tau)^{\frac{T_0 - T}{T_0}} C_{\varphi, T, \tau} \right) > 0.$$

## 5 Concluding remarks

We devote this section to present additional discussion and concluding remarks regarding the controllability of time-discrete parabolic systems.

### 5.1 Controllability of the fully discrete case

The controllability of fully-discrete parabolic systems has been rarely studied and, as far as our knowledge, only few results are available in the literature (see [15, 20, 27, 31]). In particular, in [15] and [27], the authors address the controllability of some general class of parabolic equations and prove a relaxed observability inequality which yields a  $\varphi(h)$ -controllability result (in the same spirit as our work) by connecting appropriately the space-discrete parameter  $h$  with the time-step  $\tau$ . Nevertheless, all of these works are restricted to the scalar case and interior control.

Using the classical finite difference method and adopting the tensor product formalism (see for instance [1]), we can discretize system (1.9) and write the fully-discrete system as

$$\begin{cases} \frac{y_h^{n+1} - y_h^n}{\tau} + \mathcal{A}_h \otimes I y_h^{n+1} + \mathcal{I}_h \otimes C y_h^{n+1} = \mathcal{B}_h \otimes B v_h^{n+1}, & n \in \llbracket 0, M-1 \rrbracket, \\ y_h^0 = y_{0, h}(x). \end{cases} \quad (5.1)$$

where the state  $y_h^n$  is an element of  $\mathbb{R}^N \otimes \mathbb{R}^d$  and the matrices  $\mathcal{A}_h \in \mathbb{R}^{N \times N}$ ,  $\mathcal{B}_h \in \mathbb{R}^N$  and  $\mathcal{I}_h \in \mathbb{R}^{N \times N}$  are the classical ones coming from the discretization procedure.

The results proved in Section 3 apply for a quite general class of families  $\Lambda$  associated with the underlying problem and as long as the general assumptions (2.15)–(2.18) are fulfilled, the final results hold uniformly. In fact, the fully-discrete control system (5.1) fits into this framework by making a careful analysis of the interplay between a finite family of eigenvalues  $\Lambda_h$  associated to the fully-discrete problem and our methodology.

For instance, [2, Theorem 3.2], guarantees the existence of a constant  $h_1 > 0$  only depending on  $\gamma$  such that for any  $h < h_1$ , the gap property

$$|\lambda_h - \lambda'_h| > \rho$$

holds at the fully-discrete level with a constant  $\rho > 0$  uniform with respect to  $h$  for all  $\lambda_h, \lambda'_h \in \Lambda_{h, \max}$  where

$$\Lambda_{h, \max} := \{\lambda \in \Lambda_h : \lambda h^2 < \kappa_0\}$$

for some  $\kappa_0 > 0$  only depending on  $\gamma_{\min}$ .

By adapting the proofs of Lemma 3.5, Theorem 3.2, and Proposition 3.7 (see, in particular, Remark 3.8), it is possible to prove that for a parameter  $\varpi \in (0, \kappa_0]$  and a  $h_0 > 0$  (only depending on  $d$  and  $\kappa_0$ ) there exists, for every value of  $h < h_0$ , a time-discrete biorthogonal family  $(q_{\lambda, i}) \in L^2_\tau(0, T)$  associated with the space-discrete eigenvalues  $\Lambda_h$ , that is verifying

$$\left( p[\lambda_h^{(i)}], q_{\lambda'_h, j} \right)_{L^2_\tau(0, T)} = \delta_{\lambda_h, \lambda'_h} \delta_{i, j}, \quad \forall i, j \in \llbracket 1, d \rrbracket, \quad \forall \lambda_h, \lambda'_h \in \Lambda_h^{\tau, \varpi}$$

and

$$|q_{\lambda_h, i}|_{L^2_\tau(0, T)} \leq C e^{\delta(\lambda_h)\lambda_h}, \quad \forall i \in \llbracket 1, d \rrbracket, \quad \forall \lambda_h \in \Lambda_h^{\tau, \varpi},$$

uniformly with respect to  $\tau$  and  $h$ , where  $\Lambda_h^{\tau, \varpi} = \{\lambda_h \in \Lambda_h : \lambda_h \tau < \varpi\}$ .

As seen in Section 3.4, this biorthogonal family is actually the main ingredient for obtaining a control result. Due to the paper's length and the additional notation and complexity that such a detailed exposition would introduce, we omit further details.

## 5.2 Other time-discrete schemes

We have developed our methodology by taking as a starting point an implicit Euler scheme for the discretization of the time variable, however, in the spirit of [15, 20, 37], it is possible to consider the more general  $\theta$ -scheme for discretizing the system.

For the sake of exposition, let us consider the scalar control system given by

$$\begin{cases} \frac{y^{n+1} - y^n}{\tau} + \mathcal{A}\theta y^{n+1} + \mathcal{A}(1 - \theta)y^n = 0, & n \in \llbracket 0, M - 1 \rrbracket, \\ y_{|\partial\Omega}^{n+1} = \mathbf{1}_{\{x=1\}} v^{n+1} & n \in \llbracket 0, M - 1 \rrbracket, \\ y^0 = y_0, \end{cases} \quad (5.2)$$

where  $\theta \in (1/2, 1]$ . Observe that the case,  $\theta = 1$  corresponds to the implicit Euler scheme while  $\theta = 1/2$  is the Crank-Nicolson scheme.

Introducing the associated adjoint system (see [15]) for every eigenfunction of  $\mathcal{A}$ , that is

$$\begin{cases} \frac{q^M - q^{M+1}}{\tau} + \theta \mathcal{A}q^M = 0, \\ \frac{q^n - q^{n+1}}{\tau} + \theta \mathcal{A}q^n + (1 - \theta)\mathcal{A}q^{n+1} = 0, & n \in \llbracket 0, M - 1 \rrbracket \\ q_{|\partial\Omega}^n = 0, & n \in \llbracket 0, M \rrbracket, \\ q^{M+1} = \phi_\lambda. \end{cases} \quad (5.3)$$

and following the procedure of Section 3.1, it is not difficult to see that the appropriate moment problem formulation for the  $\theta$ -scheme is given by

$$\sum_{n=1}^M \tau v^{M-n+1} p_n^\theta[\lambda] = \frac{\alpha_{\lambda, \theta}}{\gamma(1)\partial_x \phi_\lambda(1)},$$

where we have defined

$$p_n^\theta[\lambda] := \left( \frac{1 + \tau\lambda\theta}{1 - (1 - \theta)\tau\lambda} \right)^{-n} \frac{1}{1 - (1 - \theta)\tau\lambda}$$

and  $\alpha_{\lambda, \theta}$  are suitable coefficients only depending on  $T$ ,  $\lambda$  and  $\theta$  (i.e., an analogous to equation (3.9)). Note that it immediately appears that this function is only well-defined and non negative for eigenvalues belonging to the set  $\Lambda_\theta^\tau := \{\lambda \in \Lambda : \tau\lambda < \frac{1}{1 - \theta}\}$ , which corresponds to well-known stability issues for those numerical methods.

Following our methodology, one of the most important changes to study this numerical scheme can be seen already while adapting Proposition 3.3. A straightforward computation allows to show that in this case, we have the formula

$$\left(p^\theta[\lambda], p^\theta[\lambda']\right)_{L^2_\tau(0, +\infty)} = \frac{1}{\lambda + \lambda' + (2\theta - 1)\tau\lambda\lambda'}, \quad (5.4)$$

instead of (3.20). Using (5.4), we can adapt all of our results without major changes, taking into account the set  $\Lambda_\theta^\tau$  instead of the whole  $\Lambda$ .

## A Some useful results

In this appendix, we gather various definitions and auxiliary results used along this paper.

### A.1 Basics of divided differences

Here, we make a brief presentation about the concept of divided differences and introduce two useful results employed in our methodology. This notion simplifies the presentation and facilitates the computations. The definition and results presented below are classical and for a complete discussion on this topic we refer to [34].

Let  $V$  be a real vector space,  $n \in \mathbb{N}$  and  $x_1, \dots, x_n \in \mathbb{R}$ . Assume that  $x_1, \dots, x_n$  are pairwise distinct. Let  $f_1, \dots, f_n \in V$  be given.

**Definition A.1.** *The divided differences are defined by*

$$f[x_i] := f_i, \quad \forall i \in \llbracket 1, n \rrbracket,$$

and then by induction for any  $k \in \llbracket 2, n \rrbracket$ , for any pairwise distinct  $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$ , by

$$f[x_{i_1}, \dots, x_{i_k}] := \frac{f[x_{i_1}, \dots, x_{i_{k-1}}] - f[x_{i_2}, \dots, x_{i_k}]}{x_{i_1} - x_{i_k}}.$$

Along the manuscript, if  $f : \mathbb{R} \rightarrow V$  is a given function, it is implicitly assumed that  $f_i = f[x_i] = f(x_i)$ . The following definition is instrumental in our manuscript.

**Definition A.2.** *For any function  $f : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^n$ , we define the divided difference  $f[x^{(n+1)}]$  as*

$$f[x^{(n+1)}] := \lim_{\epsilon \rightarrow 0} f[x, x + \epsilon, \dots, x + n\epsilon] = \frac{1}{n!} \frac{d^n f}{dx^n}(x). \quad (A.1)$$

The next result is key to obtain different estimates needed in our analysis.

**Proposition A.3** (Lagrange theorem). *Assume that  $V = \mathbb{R}$  and that  $f \in C^{n-1}(\text{Conv}\{x_1, \dots, x_n\})$ . For any  $k \in \llbracket 1, n \rrbracket$ , for any pairwise distinct  $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$ , there exists  $z \in \text{Conv}\{x_{i_1}, \dots, x_{i_k}\}$  such that*

$$f[x_{i_1}, \dots, x_{i_k}] = \frac{f^{(k-1)}(z)}{(k-1)!}.$$

The following result, known as Leibniz rule, gives a simple rule to compute the divided differences of a product of functions.

**Proposition A.4.** *Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be given functions. For any  $k \in \llbracket 1, n \rrbracket$ , for any pairwise distinct  $i_1, \dots, i_k \in \llbracket 1, n \rrbracket$ , we have*

$$(gf)[x_{i_1}, \dots, x_{i_k}] = \sum_{j=1}^k g[x_{i_1}, \dots, x_{i_j}] f[x_{i_j}, \dots, x_{i_k}].$$

### A.2 Divided differences of the discrete function $p$ and auxiliary results

The following results cover some useful estimates for estimating the divided differences of the discrete function (3.1) and its norm.

**Lemma A.5.** *Let  $n \in \mathbb{N}^*$ ,  $\lambda \in \mathbb{R}^+$ , and  $\tau > 0$  be given. We have*

$$p_n[\lambda] \geq e^{-\lambda n \tau}, \quad (A.2)$$

and

$$\left|p_n[\lambda^{(j)}]\right| \leq C_j |n\tau|^{j-1} p_n[\lambda], \quad \forall j \in \llbracket 1, d \rrbracket, \quad (A.3)$$

for some positive constant  $C_j$  uniform with respect to  $\tau$ .

*Proof.* The inequality (A.2) simply comes from the fact that  $x \in \mathbb{R}^+ \mapsto \log(1+x)/x$  is monotonically decreasing and has limit 1 at  $x = 0$ .

From definition (3.4), we observe that

$$\langle\langle n \rangle\rangle_{j-1} = n(n+1) \dots (n+j-2) \leq n \times (2n) \times \dots \times ((j-1)n) \leq C_j n^{j-1} \quad (\text{A.4})$$

for all  $n \in \mathbb{N}^*$ . On the other hand, recalling (3.1) we get

$$(1 + \tau\lambda)^{-(n+(j-1))} = \frac{1}{(1 + \tau\lambda)^{j-1}} p_n[\lambda] \leq p_n[\lambda] \quad (\text{A.5})$$

since  $(1 + \tau\lambda) > 1$  for any  $\tau > 0$  and  $\lambda \in \mathbb{R}^+$ . Using estimates (A.4) and (A.5) on expression (3.3) yields the desired result.  $\square$

**Lemma A.6.** *Let  $r \in \mathbb{N}$  and  $\lambda \in \Lambda^{\tau,1}$  and  $\tau > 0$ . Then, there is a constant  $C > 0$  only depending on  $r$  such that*

$$\lambda^{2r+1} \sum_{n=1}^{\infty} \tau(n\tau)^{2r} |p_n[\lambda]|^2 \leq C. \quad (\text{A.6})$$

*Proof.* For fixed  $\tau > 0$  and  $\lambda \in \Lambda^{\tau,1}$ , we define  $x = \tau\lambda$  and recalling (3.1) we see that our task is to prove that the quantity

$$x^{2r+1} \sum_{n=1}^{\infty} n^{2r} (1+x)^{-2n},$$

is uniformly bounded with respect to  $x \in (0, 1]$ .

For the case  $r = 0$ , we readily compute

$$x \sum_{n=1}^{\infty} (1+x)^{-2n} = \frac{1}{2+x} \leq \frac{1}{2},$$

since  $x > 0$  by definition.

For  $r \geq 1$ , we note that

$$\begin{aligned} x^{2r+1} \sum_{n=1}^{\infty} n^{2r} (1+x)^{-2n} &= x^{2r+1} \sum_{n=1}^{\infty} \int_n^{n+1} \mathbf{1}_{(n,n+1)}(s) n^{2r-1} (1+x)^{-2n} ds \\ &= x^{2r+1} \sum_{n=1}^{\infty} \int_n^{n+1} \mathbf{1}_{(n,n+1)}(s) n^{2r-1} e^{-2nx \frac{\log(1+x)}{x}} ds, \end{aligned}$$

and using that  $\log(1+x)/x \geq 1/2$  uniformly for  $x \in (0, 1]$ , we have

$$x^{2r+1} \sum_{n=1}^{\infty} n^{2r} (1+x)^{-2n} \leq x^{2r+1} \sum_{n=1}^{\infty} \int_n^{n+1} \mathbf{1}_{(n,n+1)}(s) n^{2r-1} e^{-nx} ds.$$

Since  $s \in [n, n+1]$ , the above expression yields

$$\begin{aligned} x^{2r+1} \sum_{n=1}^{\infty} n^{2r} (1+x)^{-2n} &\leq x^{2r+1} e^x \sum_{n=1}^{\infty} \int_n^{n+1} \mathbf{1}_{(n,n+1)}(s) s^{2r-1} e^{-sx} ds \\ &= x^{2r+1} e^x \int_1^{\infty} s^{2r-1} e^{-sx} ds \leq x^{2r+1} e^x \int_0^{\infty} s^{2r-1} e^{-sx} ds \\ &= x e^x \Gamma(2r), \end{aligned}$$

where  $\Gamma$  is the Gamma function. This quantity is bounded with respect to  $x \in (0, 1]$ , which proves the claim.  $\square$

**Lemma A.7.** *Let  $x \geq 1$ ,  $n \in \mathbb{N}$  and  $l \in \mathbb{N}^*$  be given. Then,*

$$(x+n)^l - (x-1)^l \leq C_l (1+n)^l x^{l-1} \quad (\text{A.7})$$

for some constant  $C_l > 0$ .

*Proof.* The case  $l = 1$  follows easily. For the general case, we have that

$$(x+n)^l - (x-1)^l = \sum_{k=0}^l \binom{l}{k} x^{l-k} n^k - \sum_{k=0}^l \binom{l}{k} x^{l-k} (-1)^k = \sum_{k=1}^l \binom{l}{k} x^{l-k} (n^k - (-1)^k).$$

Since  $x \geq 1$ , we readily get

$$(x+n)^l - (x-1)^l \leq x^{l-1} \sum_{k=1}^l \binom{l}{k} |n^k - (-1)^k| \leq x^{l-1} (1+n)^l \underbrace{\sum_{k=1}^l \binom{l}{k}}_{=: C_l}$$

as claimed.  $\square$

### A.3 Estimates on Blaschke-type products

**Proposition A.8.** *For any  $\ell \in \llbracket 0, d-1 \rrbracket$ , there exists a decreasing function  $r \mapsto \delta(r)$  such that  $\lim_{r \rightarrow +\infty} \delta(r) = 0$  depending only on  $\ell, \rho, d$  and  $R$  such that*

$$\left| \left( \frac{1}{W_\lambda} \right)^{(\ell)}(\theta) \right| \leq e^{\delta(\lambda)\lambda}. \quad (\text{A.8})$$

for all  $\lambda \in \Lambda$  and  $\theta \in \text{Conv}\{G_\lambda\}$ .

*Proof.* Noting that  $\theta \leq \lambda + \rho/2$  for any  $\theta \in \text{Conv}\{G_\lambda\}$ , the proof of this result is a direct application of [13, Proposition A.7.44] to the family  $L = \tilde{\Lambda}_\lambda$  and a redefinition of the function  $\epsilon$  in such result.  $\square$

**Proposition A.9.** *For any  $\tau \in (0, 1)$  and any  $\ell \in \llbracket 0, d-1 \rrbracket$ , there exists a decreasing function  $r \mapsto \delta(r)$  such that  $\lim_{r \rightarrow +\infty} \delta(r) = 0$  depending only on  $\ell, \rho, d$  and  $R$  such that*

$$\left| \left( \frac{1}{W_\lambda^\tau} \right)^{(\ell)}(\theta) \right| \leq e^{\delta(\lambda)\lambda}, \quad (\text{A.9})$$

for all  $\lambda \in \Lambda^{\tau, \varpi}$  with  $\varpi > 0$  and  $\theta \in \text{Conv}\{G_\lambda\}$ .

*Proof.* The proof of this result can be obtained by following closely the proofs of [13, Proposition A.7.44 and Corollary A.7.45]. For brevity, we will present only the main differences introduced by the time discretization.

Let  $\lambda \in \Lambda^{\tau, \varpi}$ . Recalling definition (3.23), the first goal is to bound the quantity

$$\left( \frac{1}{W_\lambda^\tau} \right)(\theta) = \prod_{\mu \in \tilde{\Lambda}_\lambda^{\tau, \varpi}} \frac{\theta + \mu + \tau\theta\mu}{\theta - \mu} \quad (\text{A.10})$$

for any  $\theta \in \text{Conv}\{G_\lambda\}$ . Note that (A.10) can be rewritten as

$$\left( \frac{1}{W_\lambda^\tau} \right)(\theta) = \frac{Q(\theta)}{D(\theta)} \quad (\text{A.11})$$

where

$$Q(\theta) := \prod_{\mu \in \tilde{\Lambda}_\lambda^{\tau, \varpi}} \left| 1 + \frac{\theta(1 + \tau\mu)}{\mu} \right| \quad \text{and} \quad D(\theta) := \prod_{\mu \in \tilde{\Lambda}_\lambda^{\tau, \varpi}} \left| 1 - \frac{\theta}{\mu} \right|.$$

Thus, we are reduced to find a bound from above for  $Q$  and a bound by below for  $D$ .

To obtain a bound for  $Q(\theta)$  we argue as follows. By definition of  $\tilde{\Lambda}_\lambda^{\tau, \varpi}$ , we have that

$$Q(\theta) = \prod_{i=1}^d \prod_{\substack{\lambda' \in \Lambda^{\tau, \varpi} \\ \lambda' \neq \lambda}} \left| 1 + \frac{\theta(1 + \tau(\lambda' + (i-1)\epsilon))}{\lambda'} \right|.$$

Using that  $\tau\lambda' < \varpi$  for all  $\lambda' \in \Lambda^{\tau, \varpi}$  and that  $\epsilon < \rho/4(d-1)$ , we have  $1 + \tau(\lambda' + (i-1)\epsilon) \leq 1 + \varpi + \rho/4 =: c_{d, \rho, \varpi}$  for all  $i \in \llbracket 1, d \rrbracket$ , thus using that  $\theta < (1 + R(0)\rho/4)\lambda$  for all  $\theta \in \text{Conv}\{G_\lambda\}$

$$Q(\theta) \leq \left( \prod_{\substack{\lambda' \in \Lambda^{\tau, \varpi} \\ \lambda' \neq \lambda}} \left| 1 + c_{d, \rho, \varpi} \frac{\theta}{\lambda'} \right| \right)^d \leq \left( \prod_{\substack{\lambda' \in \Lambda^{\tau, \varpi} \\ \lambda' \neq \lambda}} \left| 1 + c \frac{\lambda}{\lambda'} \right| \right)^d, \quad (\text{A.12})$$

for a constant  $c > 0$  only depending on  $d, \varpi, \rho$  and  $R(0)$ .

With (A.12) at hand, we can argue exactly as in the proof of [13, Proposition A.7.44] to obtain

$$\log(Q(\theta)) \leq C\lambda \left[ R(0) \frac{\log(1 + 2cR(0)\lambda)}{[\log(1 + \lambda)]^2} + R \left( c \frac{\lambda}{2(\log \lambda)^2} \right) \right], \quad (\text{A.13})$$

for some constants  $c, C > 0$  depending at most on  $d, \varpi, \rho$  and  $R(0)$ .

To estimate  $D(\theta)$ , we just have to notice that  $\text{dist}(\theta, \tilde{\Lambda}_\lambda) > \rho/2$  for any  $\theta \in \text{Conv}\{G_\lambda\}$  and repeat the same steps as in the proof of [13, Proposition A.7.44] to obtain

$$\log(D(\theta)) \geq -\theta \left[ (2 + \log 2)R(\theta/2) + C_{d, \rho} \sqrt{R(\theta/2)} \right].$$

Since the function  $R$  is non-increasing, we have that  $R(\theta/2) \leq R(\lambda/4)$  since  $\lambda/2 < \theta$  for all  $\theta \in \text{Conv}\{G_\lambda\}$ , thus

$$\log(D(\theta)) \geq -c_{\rho, R(0)}\lambda \left[ (2 + \log 2)R(\lambda/4) + C_{d, \rho} \sqrt{R(\lambda/4)} \right] \quad (\text{A.14})$$

where again we have used that  $\theta \leq c_{\rho, R(0)}\lambda$  for all  $\theta \in \text{Conv}\{G_\lambda\}$ .



Using (A.13) and (A.14), we can estimate (A.11) as

$$\log \left( \frac{1}{W_\lambda^\tau}(\theta) \right) \leq C_{d,\varpi,\rho,R(0)} \lambda \underbrace{\left( R(0) \frac{\log(1+2cR(0)\lambda)}{[\log(1+\lambda)]^2} + R \left( c \frac{\lambda}{2(\log \lambda)^2} \right) + R(\lambda/4) + \sqrt{R(\lambda/4)} \right)}_{:=\delta(\lambda)} \quad (\text{A.15})$$

which is the desired estimate for  $\ell = 0$  by recalling that  $R$  is a non-increasing function and a redefinition (up to a constant) of the function  $\delta$ .

For the cases  $\ell \in \llbracket 1, d-1 \rrbracket$ , noting that  $\text{dist}(\theta, \tilde{\Lambda}_\lambda^\tau) > 3\rho/4$  for any  $\theta \in \text{Conv}\{G_\lambda\}$ , a straightforward adaptation of [13, Corollary A.7.45] and estimate (A.15) yields the desired result. This ends the proof.  $\square$

## B Proof of some technical lemmas

### B.1 Proof of Lemma 2.7

Let us fix  $K \in \mathbb{N}^*$  and consider the finite family of eigenvalues

$$\Lambda^K = \{\lambda \in \Lambda : \lambda < K\}.$$

We construct the corresponding family

$$\tilde{\Lambda}^K = \bigcup_{\lambda \in \Lambda^K} G_\lambda$$

where we recall that  $G_\lambda$  is given in (2.24). Additionally, consider the set

$$\tilde{\Lambda}_\lambda^K = \tilde{\Lambda}^K \setminus G_\lambda.$$

Let  $P_{E_\lambda^K}$  be the orthogonal projection onto  $E_\lambda^K$  in  $L^2(0, +\infty)$ . Then, a direct application of [13, Proposition A.3.30] with  $E_\lambda^K := \text{span}\{p[\sigma] : \sigma \in \tilde{\Lambda}_\lambda^K\}$  and  $H = L^2(0, +\infty)$  gives

$$\left( e[\sigma] - P_{E_\lambda^K} e[\sigma], e[\sigma'] - P_{E_\lambda^K} e[\sigma'] \right)_{L^2(0, +\infty)} = \frac{1}{\sigma + \sigma'} W_\lambda^K(\sigma) W_\lambda^K(\sigma')$$

where

$$W_\lambda(\sigma) = \prod_{\mu \in \tilde{\Lambda}_\lambda^K} \frac{\mu - \sigma}{\mu + \sigma}.$$

Since  $\Lambda = \bigcup_{K \geq 1} \Lambda^K$ , [13, Lemma A.3.23] allows us to pass to the limit as  $K \rightarrow \infty$  and obtain the desired result. This ends the proof.

### B.2 Proof of Lemma 3.4

As in the continuous case, the proof of this result is based on [13, Proposition A.3.30] which in turn employs the useful result [13, Proposition A.3.31] for the computation of determinants of Cauchy matrices of the form

$$C_{A,B} := \left( \frac{1}{a_i + b_j} \right)_{i,j \in \llbracket 1, n \rrbracket}, \quad (\text{B.1})$$

where  $A = \{a_1, \dots, a_n\} \subset \mathbb{R}$  and  $B = \{b_1, \dots, b_n\} \subset \mathbb{R}$  are two given families.

In view of [13, Proposition A.3.30] and identity (3.20), our task is thus reduced to compute the determinant of the Cauchy matrix

$$C_{A,B,\tau} := \left( \frac{1}{a_i + b_j + \tau a_i b_j} \right)_{i,j \in \llbracket 1, n \rrbracket}$$

for any  $\tau > 0$  and two given families  $A$  and  $B$ . Note however that  $C_{A,B,\tau}$  can be rewritten as

$$C_{A,B,\tau} = \text{diag} \left( \frac{1}{b_1}, \dots, \frac{1}{b_n} \right) \times \left( \frac{1}{\left( \frac{1}{a_i} + \frac{\tau}{2} \right) + \left( \frac{1}{b_j} + \frac{\tau}{2} \right)} \right)_{i,j \in \llbracket 1, n \rrbracket} \times \text{diag} \left( \frac{1}{a_1}, \dots, \frac{1}{a_n} \right),$$

where the matrix in the middle is again of the form (B.1). Thus, a direct application of [13, Proposition A.3.31] and some straightforward simplifications yield

$$\det C_{A,B,\tau} = \prod_{i=1}^n \frac{1}{a_i + b_i + \tau a_i b_i} \times \prod_{\substack{i,j \in \llbracket 1, n \rrbracket \\ i < j}} \frac{(a_j - a_i)(b_j - b_i)}{(a_i + b_j + \tau a_i b_j)(b_i + a_j + \tau b_i a_j)}. \quad (\text{B.2})$$

Let  $\tilde{\Lambda}^{\tau,\varpi}$  be as in (3.21) and set  $\tilde{\Lambda}_\lambda^{\tau,\varpi} := \tilde{\Lambda}^{\tau,\varpi} \setminus G_\lambda$  where we recall that  $G_\lambda$  is defined in (2.24). Let

$$E_\lambda^\tau := \text{span} \left\{ p[\lambda] : \lambda \in \tilde{\Lambda}_\lambda^{\tau,\varpi} \right\},$$

and  $P_{E_\lambda^\tau}$  be the orthogonal projection in  $L_\tau^2(0, +\infty)$  on  $E_\lambda^\tau$ . A direct adaptation of [13, Proposition A.3.30] by using formula (B.2) (instead of the shown there) yields for any  $\sigma, \sigma' \in G_\lambda$  that

$$\begin{aligned} & (p[\sigma] - P_{E_\lambda^\tau} p[\sigma], p[\sigma'] - P_{E_\lambda^\tau} p[\sigma'])_{L_\tau^2(0, +\infty)} \\ &= \frac{\det C_{\{\sigma\} \cup \bar{\Lambda}_\lambda^{\tau, \varpi}, \{\sigma'\} \cup \Lambda_\lambda^{\tau, \varpi}, \tau}}{\det C_{\bar{\Lambda}_\lambda^{\tau, \varpi}, \Lambda_\lambda^{\tau, \varpi}, \tau}} \\ &= \frac{1}{\sigma + \sigma' + \tau\sigma\sigma'} \prod_{\mu \in \bar{\Lambda}_\lambda^{\tau, \varpi}} \frac{(\sigma - \mu)(\sigma' - \mu)}{(\sigma + \mu + \tau\sigma\mu)(\sigma' + \mu + \tau\sigma'\mu)}, \end{aligned}$$

which is exactly the desired result. This ends the proof.

### B.3 Proof of Lemma 3.10

Since  $t \mapsto e_t^m[\lambda^{(j)}]$  is a holomorphic function in the set  $\mathcal{P}_\eta$ , we can use expressions (3.54) and Cauchy's formula to compute for all  $j \in \llbracket 1, d \rrbracket$  and  $\zeta \in (t_{n-1}^m, t_n^m)$

$$\left| \frac{d}{dt} \Big|_{t=\zeta} e_t^m[\lambda^{(j)}] \right| = \left| \frac{1}{2\pi i} \oint_{|z-\zeta|=\gamma} \frac{\mathfrak{G}_j(z) e^{-(z+(j-1)\tau^m) \frac{\log(1+\tau^m \lambda)}{\tau^m}}}{(z-\zeta)^2} dz \right|, \quad (\text{B.3})$$

where  $\gamma$  is a well-chosen radius to be fixed later and  $\mathfrak{G}_j(z) := \frac{(-1)^{j-1}}{(j-1)!} \prod_{i=1}^{j-1} (z - (i-j+1)\tau^m)$ .

Since  $\eta > \sqrt{\tau^m}$ , it can be seen that as soon as  $\zeta \geq 5\eta$ , the circle defined by  $|z-\zeta| = \sqrt{\tau^m}$  is fully contained within the set  $\mathcal{P}_\eta$ . Moreover, for any  $j \in \llbracket 1, d \rrbracket$ , we obtain from estimates (3.55) and (3.56)

$$\left| \mathfrak{G}_j(z) e^{-(z+(i-1)\tau) \frac{\log(1+\tau^m \lambda^m)}{\tau^m}} \right| \leq C_j (1 + |z|^{j-1}) e^{-\frac{\Re(z)}{2} \lambda},$$

uniformly with respect to  $m$ . Using the above estimate and the parametrization  $z(\theta) = \zeta + \sqrt{\tau^m} e^{i\theta}$  where  $0 \leq \theta \leq 2\pi$ , we can bound the right-hand side of (B.3) as follows

$$\begin{aligned} & \left| \frac{1}{2\pi i} \oint_{|z-\zeta|=\sqrt{\tau^m}} \frac{\mathfrak{G}_j(z) e^{-(z+(j-1)\tau^m) \frac{\log(1+\tau^m \lambda)}{\tau^m}}}{(z-\zeta)^2} dz \right| \\ & \leq \frac{C_j}{2\pi} \int_0^{2\pi} \frac{(1 + |t_n + \sqrt{\tau^m} e^{i\theta}|^{j-1}) e^{-\frac{\Re(t_n + \sqrt{\tau^m} e^{i\theta})}{2} \lambda}}{\sqrt{\tau^m}} d\theta. \end{aligned}$$

Since

$$e^{-\frac{\Re(t_n + \sqrt{\tau^m} e^{i\theta})}{2} \lambda} \leq e^{-\frac{t_n}{2} \lambda + \frac{\sqrt{\tau^m}}{2} \lambda} \leq e^{-\frac{t_n}{2} \lambda + \frac{\eta}{2} \lambda}$$

and

$$\left| t_n + \sqrt{\tau^m} e^{i\theta} \right|^{j-1} \leq \left( |t_n| + |\sqrt{\tau^m}| \right)^{j-1} \leq (|t_n| + \eta)^{j-1} \leq C_{j,\eta} |t_n|^{j-1}$$

for all  $j \in \llbracket 1, d \rrbracket$ , we deduce that

$$\left| \frac{1}{2\pi i} \oint_{|z-\zeta|=\sqrt{\tau^m}} \frac{\mathfrak{G}_j(z) e^{-(z+(j-1)\tau^m) \frac{\log(1+\tau^m \lambda)}{\tau^m}}}{(z-\zeta)^2} dz \right| \leq \frac{C_{j,\eta}}{\sqrt{\tau^m}} |t_n|^{j-1} e^{-\frac{t_n}{2} \lambda + \frac{\eta}{2} \lambda}, \quad \forall j \in \llbracket 1, d \rrbracket. \quad (\text{B.4})$$

Putting together estimates (B.3), (B.4) and noting that  $t_n^m/2 \geq \eta + t_n^m/4$  yields the desired result. This ends the proof.

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