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Abstract

We investigate the quasi-limiting behavior of bisexual subcritical Galton-Watson branching processes. While classical subcritical Galton-Watson processes have been extensively analyzed, bisexual Galton-Watson branching processes present unique difficulties because of the lack of the branching property. To prove the existence of and convergence to one or several quasi-stationary distributions, we leverage on recent developments linking bisexual Galton-Watson branching processes extinction to the eigenvalue of a concave operator.

1 Introduction

We study the quasi-limiting behaviour of bisexual subcritical Galton-Watson branching processes (bGWbp). This problem has been studied and largely solved decades ago for classical sub-critical Galton-Watson processes (see e.g. [**?**, **?**, **?**]), but remains open for its bisexual counterpart. Recent developments in the theory of multi-dimensional bGWbp [**?**] show that, under appropriate super-additivity assumptions, the extinction of these processes can be characterised by the eigenvalue λ^* associated to a positively homogeneous concave operator and, more precisely, by its position relatively to 1: if $\lambda^* \leq 1$, then the process is eventually extinct almost surely, while, if $\lambda^* > 1$, then, with positive probability, the process grows exponentially fast with rate equal or close to λ^*. We refer the reader to the surveys [**?**, **?**, **?**] on bGWbp and [**?**, **?**, **?**, **?**, **?**] for further references. In this paper, we consider the sub-critical case $\lambda^* < 1$ and show that the process admits infinitely many quasi-stationary distributions. Under an additional polynomial moment assumption, we prove that the process admits a finitely generated family of quasi-stationary distributions and, under an additional irreducibility assumption, a unique quasi-limiting distribution for compactly supported initial distributions. Although this is similar to the classical Galton-Watson case, λ^* is not in general equal to the exponential survival rate of the process.

Let us introduce more formally our settings and assumptions. Consider $p, q \in \mathbb{N} = \{0, 1, 2, \ldots\}$ with $p, q \geq 1$, a non-negative vector function $\xi : \mathbb{N}^q \to \mathbb{N}^p$ such that $\xi(0) = 0$, and a family of integrable random vectors $V = (V_{i,\cdot})_{1 \leq i \leq p}$, taking value in $\mathbb{N}^q$ and whose expectation is denoted by $\mathbb{V}$. We assume that $\sum_{i=1}^{q} \mathbb{V}_{i,j} > 0$ for all $j \in \{1, \ldots, q\}$. We consider a process $Z = (Z_n)_{n \in \mathbb{N}}$ on $\mathbb{N}^p$ which represents the random sequence of the number of couples of each type in the population, and which evolves as follows: given $Z_{n-1} = (Z_{n-1,1}, \ldots, Z_{n-1,p})$, we define for $n \geq 1$ the vector of children of the $n-$th generation $(W_{n,1}, \ldots, W_{n,q})$ by

$$W_{n,j} = \sum_{i=1}^{p} \sum_{k=1}^{Z_{n-1,i}} V_{i,j}^{(k,n)}, \text{ for } 1 \leq j \leq q,$$

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where $(V^{(k,n)})_{k,n\in\mathbb{N}}$ is a family of i.i.d. copies of V. Then, we set the vector of couples in the $n-$th generation as

$$Z_n = (Z_{n,1},\dots,Z_{n,p}) = \xi(W_{n,1},\dots,W_{n,q}).$$

The function ξ is referred to as the *mating function*. For instance, if $p=1$, $q=2$ and if $W_{n,1}$ represents the number of females and $W_{n,2}$ the number of males in the $n-$th generation, classical choices for ξ are $\xi(x,y)=\min\{x,y\}$ (the *perfect fidelity* mating function) and $\xi(x,y)=x\min\{1,y\}$ (the *promiscuous* mating function). If $p=q=2$, then $\xi(x,y)=(x,y)$ corresponds to the classical bi-dimensional Galton-Watson process.

For any probability measure μ on $\mathbb{N}^p$, we use the notation $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ for the law and associated expectation of the bisexual Galton-Watson process with initial number of couples Z_0 distributed according to μ. As usual, we use the notations $\mathbb{P}_x$ and $\mathbb{E}_x$ when $\mu=\delta_x$ for some $x\in\mathbb{N}^p$. Note that $\xi(0)=0$ entails that 0 is absorbing for Z.

We are interested in conditions ensuring the existence of a quasi-stationary distribution for Z. We recall that a *quasi-stationary distribution* for Z is a probability measure ν_{QS} on $\mathbb{N}^p\setminus\{0\}$ such that, for all $n\geq 0$, $\mathbb{P}_{\nu_{QS}}(Z_n\neq 0)>0$ and

$$\mathbb{P}_{\nu_{QS}}(Z_n\in\cdot\mid Z_n\neq 0)=\nu_{QS}(\cdot).$$

It is known that (we refere the reader to the book [**?**] and to the surveys [**?**, **?**] for general properties and further examples), for any quasi-stationary distribution ν_{QS}, there exists $\theta_{QS}\in(0,1]$ such that

$$\mathbb{P}_{\nu_{QS}}(Z_n\neq 0)=\theta_{QS}^n \text{ and } \mathbb{P}_{\nu_{QS}}(Z_n\in\cdot,\ Z_n\neq 0)=\theta_{QS}^n\nu_{QS}(\cdot).$$

In what follows, θ_{QS} is referred to as the *absorption parameter* of ν_{QS}.

Throughout the paper, we make the assumption that ξ is super-additive and sub-affine:

Assumption (S). There exists $\alpha,\beta\in\mathbb{R}_+^p$ such that

$$\alpha|x_1+x_2|+\beta\geq\xi(x_1+x_2)\geq\xi(x_1)+\xi(x_2),\ \forall x_1,x_2\in\mathbb{N}^q, \tag{1}$$

where $|\cdot|$ is the ℓ^1-norm.

This assumption implies that the function $\mathfrak{M}:\mathbb{R}_+^p\mapsto\mathbb{R}_+^p$ given by

$$\mathfrak{M}(z)=\lim_{k\to\infty}\frac{\xi(\lfloor kz\mathbb{V}\rfloor)}{k}$$

is well defined, bounded over $\mathbb{S}:=\left\{z\in\mathbb{R}_+^p,\ |z|=1\right\}$, positively homogeneous (i.e. $\mathfrak{M}(az)=a\mathfrak{M}(z)$ for all $a>0$ and all $z\in\mathbb{R}_+^p$) and concave. We refer the reader to Section 3 in [**?**], where other properties of this functional are derived. In addition, we assume that

Assumption (P). $\mathfrak{M}$ is primitive, which means that there exists $n_0\geq 1$ such that $\mathfrak{M}^m(z)>0$ for all $m\geq n_0$ and $z\in\mathbb{R}_+^p\setminus\{0\}$, where $\mathfrak{M}^m$ is the $m-$th iterate of $\mathfrak{M}$ and $\mathfrak{M}^m(z)>0$ means that all the coordinates of $\mathfrak{M}^m(z)$ are strictly positive.

In particular, this entails that there exists $\lambda^*>0$ and $z^*\in\mathbb{S}^*:=\left\{z\in(\mathbb{R}_+\setminus\{0\})^p,\ |z|=1\right\}$ such that the limit

$$\lim_{n\to\infty}\frac{\mathfrak{M}^n(z)}{(\lambda^*)^n}=\mathscr{P}(z)z^* \tag{2}$$

exists, where $\mathscr{P}:\mathbb{R}_+^p\to\mathbb{R}_+$ is strictly positive on $\mathbb{R}_+^p\setminus\{0\}$, positively homogeneous and concave (see [**?**] for a general development on this theory and Section 4.1 of [**?**] for an application to the

context of bGWbp). In addition, according to Lemma 37 in [**?**], $((\lambda^*)^{-n}\mathscr{P}(Z_n))_{n\in\mathbb{N}}$ is a supermartingale. In particular, $\lambda^* < 1$ implies that $(Z_n)_{n\in\mathbb{N}}$ goes extinct almost surely for any initial distribution (see [**?**] for details).

Our first result states that, in the sub-critical case, the process admits infinitely many quasi-stationary distributions, indexed by $\theta \in [\upsilon_0, 1)$, where

$$\upsilon_0 := \inf\{\upsilon > 0, \text{ such that } \mathbb{E}_z(\upsilon^{-T_0}) < +\infty, \ \forall z \in \mathbb{N}^p\} \tag{3}$$

with T_0 the extinction time.

We also define the *exponential convergence parameter* θ_0 by

$$\theta_0 = \sup_{z\in\mathbb{N}^p\setminus\{0\}} \sup\{\theta > 0, \liminf_{n\to+\infty} \theta^{-n}\mathbb{P}_z(Z_n \neq 0) > 0\}, \tag{4}$$

with $\sup\emptyset = -\infty$.

Proposition 1.1. *If Assumptions (S) and (P) hold true, then we have* $\theta_0 \leq \upsilon_0 \leq \lambda^*$.

Theorem 1.2. *If Assumptions (S) and (P) hold true and* $\lambda^* < 1$*, then* Z *admits a continuum of linearly independent quasi-stationary distributions: for any* $\theta \in [\upsilon_0, 1)$*, there exists a quasi-stationary distribution with absorption parameter* θ.

The proofs of Proposition 1.1 and Theorem 1.2 are detailed in Section 2, where we first state and prove a general result ensuring the existence of infinitely many quasi-stationary distributions for Feller processes, and then apply it to the sub-critical bGWbp. We discovered during the finalization of this manuscript that the general result and its proof, which was part of the PhD thesis of one of the author [**?**], has also been proved independently for discrete state space processes in the recent preprint [**?**], using very similar methods.We emphasize that in the classical Galton-Watson case, much more precise results are known (we recommend to the interested reader the paper [**?**], where the author proves a complete result for Galton-Watson processes and gives, in Section 3, a detailed exposition of the history of the problem, with its link to the identification of the Martin boundary of the classical Galton-Watson branching process). We also refer the reader to [**?**], where a general existence result for quasi-stationary distributions is obtained by a renewal/compactness argument.

We consider now the problem of existence of a finitely generated set of quasi-stationary distributions satisfying an integrability assumption and of convergence of conditional laws toward a quasi-stationary distribution. Since $\mathfrak{M}$ is concave on $\mathbb{R}_+^p$, it is locally Lipschitz on $\mathbb{S}^*$. We add the following regularity assumption on $\mathfrak{M}$, where we use the notation, for all $I \subset \{1, \dots, p\}$,

$$\mathscr{Z}_I := \{x \in \mathbb{S} \text{ such that } \ x_i > 0 \ \forall i \in I, \ x_i = 0 \ \forall i \notin I\},$$

where, here and across the paper, x_i denotes the i^{th} coordinate of x.

Assumption (C). For all $I \neq \emptyset$, $\mathfrak{M}$ is uniformly continuous over $\mathscr{Z}_I$.

We also consider the following moment condition, related to the exponential convergence parameter θ_0.

Assumption (M). There exists $\varsigma > 1$ such that $(\lambda^*)^\varsigma < \theta_0$ and such that $\mathbb{E}(V_{i,j}^\varsigma) < +\infty$ for all i, j.

We emphasize that Assumption (M) implies $\lambda^* < 1$, so that the following results apply in the setting of the sub-critical bGWbp. This assumption also requires implicitly that $\theta_0 > 0$, which is the case in many situations.

The proof of the following theorem is detailed in Section 4. It is based on [**?**] and makes use of the following function and measure spaces. Given a positive function φ on $\mathbb{N}^p \setminus \{0\}$, we set

$$L^\infty(\varphi) := \{f : \mathbb{N}^p \setminus \{0\} \to \mathbb{R},\ \|f/\varphi\|_\infty < +\infty\},\ \text{endowed with the norm } \|f\|_\varphi = \|f/\varphi\|_\infty$$

and

$$\mathscr{M}(\varphi) := \{\mu \text{ non-negative measure on } \mathbb{N}^p \setminus \{0\} \text{ such that } \mu(\varphi) < +\infty\}$$

where $\mu(\varphi) = \int_{\mathbb{N}^p\setminus\{0\}} \varphi(z)\mu(\mathrm{d}z)$.

Finally, when Assumption (M) is enforced, we define, for all $a \in (1,\varsigma)$ such that $(\lambda^*)^a < \theta_0$, the function $Q_a : \mathbb{N}^p \setminus \{0\} \longrightarrow \mathbb{R}_+$ by

$$Q_a(z) = \mathscr{P}(z)^a.$$

Note that $\inf Q_a > 0$.

In the following theorem, we say that a process Z is aperiodic if, for all $z \in \mathbb{N}^p$, either $\mathbb{P}_z(\exists n \geq 1, Z_n = z) = 0$ or there exists $n_0(z) \geq 0$ such that, $\forall\, n \geq n_0(z)$, $\mathbb{P}_z(Z_n = z) > 0$.

Theorem 1.3. *We assume that Assumptions (S), (P), (C) and (M) are in place and that the process Z is aperiodic. We fix $a \in (1,\varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then there exists $\ell \geq 1$ and a family of quasi-stationary distributions $\nu_1, \ldots, \nu_\ell \in \mathscr{M}(Q_a)$ with absorption parameter θ_0 for Z such that any quasi-stationary distribution $\nu \in \mathscr{M}(Q_a)$ with absorption parameter θ_0 for Z is a convex combination of $\nu_1, \ldots, \nu_\ell$. In addition, there exists a bounded function $j : \mathbb{N}^p \setminus \{0\} \to \mathbb{N}$ and there exists, for each $i \in \{1, \ldots, \ell\}$, a non-negative non-identically zero function $\eta_i \in L^\infty(Q_a)$ such that, for all $f \in L^\infty(Q_a)$, all $n \geq 1$ and all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\left| \theta_0^{-n} n^{-j(z)} \mathbb{E}_z(f(Z_n)\mathbf{1}_{Z_n \neq 0}) - \sum_{i=1}^{\ell} \eta_i(z)\nu_i(f) \right| \leq \alpha_n Q_a(z) \|f\|_{Q_a}, \tag{5}$$

where α_n goes to 0 when $n \to +\infty$ which does not depend on f nor on z (but may depend on a).

In addition, the set $E := \{z \in \mathbb{N}^p \setminus \{0\},\ \sum_{i=1}^{\ell} \eta_i(z) = 0\}$ is finite, and any quasi-stationary distributions for Z in $\mathscr{M}(Q_a)$ with absorption parameter strictly smaller than θ_0 is supported by E.

Finally, there is no quasi-stationary distribution for Z in $\mathscr{M}(Q_a)$ with absorption parameter strictly larger than θ_0 and, for all $z \in \mathbb{N}^p \setminus \{0\}$, the conditional distribution $\mathbb{P}_z(Z_n \in \cdot \mid Z_n \neq 0)$ converges in $\mathscr{M}(Q_a)$.

Remark 1.1. Note that, under the assumptions of Theorem 1.3, we have $\upsilon_0 = \theta_0$. In fact, taking $f \equiv 1$ in (5) leads to $\mathbb{P}_z(Z_n \neq 0) \sim_{n\to\infty} \theta_0^n n^{j(z)} \sum_{i=1}^{\ell} \eta_i(z)$. Then for all $\upsilon > \theta_0$, for all $z \in \mathbb{N}^p \setminus \{0\}$, we have $\mathbb{E}_z(\upsilon^{-T_0}) = \sum_{n\in\mathbb{N}} \upsilon^{-n} (\mathbb{P}_z(Z_{n-1} \neq 0) - \mathbb{P}_z(Z_n \neq 0)) < \infty$. Then $\upsilon_0 \leq \theta_0$, and Proposition 1.1 leads to the equality.

Under the additional assumption that the process Z is irreducible (meaning that $\forall\, x, y \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_x(Z_n = y) > 0$), one obtains that, for all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\theta_0 = \sup\{\theta > 0,\ \liminf_{n\to+\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq 0) > 0\},$$

does not depend on z. In the irreducible case, we have the following result, proved in Section 5.

Theorem 1.4. *Assume that Assumptions (S), (P), (C) and (M) hold true and that the process Z is aperiodic and irreducible. We fix $a \in (1,\varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then the process Z admits a unique quasi-stationary distribution ν_{QS} in $\mathscr{M}(Q_a)$. Its absorption parameter θ_{QS} is equal to θ_0 and there exists a unique positive function η_{QS} in $L^\infty(Q_a)$, such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_0^n \quad \textit{and} \quad \mathbb{E}_z(\eta_{QS}(Z_n)\mathbf{1}_{Z_n\neq 0}) = \theta_0^n \eta_{QS}(z), \quad \forall\, n \geq 0,\ z \in \mathbb{N}^p \setminus \{0\} \tag{6}$$

and η_{QS} is lower bounded away from 0. In addition, there exists $C>0$ and $\gamma\in(0,1)$ such that, for any probability measure $\mu\in\mathscr{M}(Q_a)$ and any function $f\in L^\infty(Q_a)$, we have

$$\left|\theta_0^{-n}\mathbb{E}_\mu\left(f(Z_n)\mathbf{1}_{Z_n\neq 0}\right)-\mu(\eta_{QS})\nu_{QS}(f)\right|\leq C\gamma^n\mu(Q_a)\,\|f\|_{Q_a},\quad\forall n\geq 0 \tag{7}$$

and

$$\left|\mathbb{E}_\mu\left(f(Z_n)\mid Z_n\neq 0\right)-\nu_{QS}(f)\right|\leq C\gamma^n\,\mu(Q_a)\,\|f\|_{Q_a},\quad\forall n\geq 0. \tag{8}$$

Finally, for all probability measure $\mu\in\mathscr{M}(\eta_{QS})$,

$$\mathbb{P}_\mu\left(Z_n\in\cdot\mid Z_n\neq 0\right)\xrightarrow[n\to+\infty]{TV}\nu_{QS}, \tag{9}$$

where TV refers to the total variation convergence.

Theorem 1.4 states the existence and the uniqueness of a quasi-stationary distribution ν_{QS} integrating the Lyapunov function Q_a. Moreover, (8) and (9) establish the convergence of the law of the process conditioning to the non-extinction towards ν_{QS} for all initial distribution μ integrating Q_a or the eigenfunction η_{QS}, with moreover an exponential speed of convergence for μ integrating Q_a. In particular, the convergence towards ν_{QS} holds for any Dirac mass $\mu=\delta_z$, with $z\in E$, meaning that ν_{QS} is a Yaglom limit. In addition, (7) gives the convergence of the law of the process without conditioning. In particular, taking $f\equiv 1$ in (7) leads to the speed of convergence of the extinction probability towards 0 at speed exactly θ_0.

The proofs of the last two results rely in part on recent advances in the theory of quasi-stationary distributions, which allow to derive several properties from Foster-Lyapunov type criteria, given in our case by the following proposition.

Proposition 1.5. *If Assumptions (S), (P), (C) and (M) hold true, then, for any $a\in(1,\varsigma)$ such that $(\lambda^*)^a<\theta_0$, we have*

$$\mathbb{E}(Q_a(Z_1)\mathbf{1}_{Z_1\neq 0}|Z_0=z)\leq\theta_a Q_a(z)+C_a,\ \forall z\in\mathbb{N}^p\setminus\{0\} \tag{10}$$

for some constants $\theta_a\in[0,\theta_0)$ and $C_a>0$.

Remark 1.2. We already observed that $\theta_0\leq\lambda^*$. One also checks that, at least under the assumptions of Theorem 1.4, if $\mathscr{P}$ is not linear, then $\theta_0<\lambda^*$. Indeed, under these assumptions, let ν_{QS} be the quasi-stationary distribution from Theorem 1.4, so that

$$\theta_0^n\nu_{QS}(\mathscr{P})=\sum_{z\in\mathbb{N}^p\setminus\{0\}}\nu_{QS}(\{z\})\mathbb{E}_z\left(\mathscr{P}(Z_n)\right).$$

Since under the assumptions of Theorem 1.4, the process is irreducible, the support of ν_{QS} is $\mathbb{N}^p\setminus\{0\}$ and hence $\mathscr{P}$ is concave not affine on the support of ν_{QS}. Using (8), we deduce that there exists $z_0\in\mathbb{N}^p\setminus\{0\}$ and $n\geq 1$ such that $\mathscr{P}$ is concave not affine on the support of $\mathbb{P}_{z_0}(Z_n\in\cdot)$, so that, by Jensen (strict) inequality,

$$\mathbb{E}_{z_0}\left(\mathscr{P}(Z_n)\right)<\mathscr{P}(\mathbb{E}_{z_0}(Z_n))\leq\mathscr{P}(\mathfrak{M}^n(z_0))=(\lambda^*)^n\mathscr{P}(z_0),$$

where we used $\mathbb{E}_{z_0}(Z_n)\leq\mathfrak{M}^n(z_0)$ (see Lemma 27 in [**?**]) and the fact that $\mathscr{P}$ is superadditive (since $\mathfrak{M}$ is superadditive by Assumption (S) and by definition of $\mathscr{P}$ given by (2)) for the last inequality, and the definition of $\mathscr{P}$ (see (2)) for the last equality. Similarly, we have $\mathbb{E}_z\left(\mathscr{P}(Z_n)\right)\leq(\lambda^*)^n\mathscr{P}(z)$ for all $z\in\mathbb{N}^p\setminus\{0\}$. We conclude that

$$\theta_0^n\nu_{QS}(\mathscr{P})<\sum_{z\in\mathbb{N}^p\setminus\{0\}}\nu_{QS}(\{z\})(\lambda^*)^n\mathscr{P}(z)=(\lambda^*)^n\nu_{QS}(\mathscr{P}).$$

This implies that $\theta_0<\lambda^*$ when $\mathscr{P}$ is not linear.

It is also known that $\mathfrak{M}(z)=\sup_{k\geq 1}\frac{1}{k}\mathbb{E}_{kz}(Z_1)$ (see Corollary 7 in [**?**]), and the same calculation as above shows that, if $\mathbb{E}_{z_0}(Z_1)<\mathfrak{M}(z_0)$ for some $z_0\in\mathbb{N}^p$, then $\theta_0<\lambda^*$ (even if $\mathscr{P}$ is linear).

Remark 1.3. When $\theta_0 = \lambda^*$, Assumption (M) requires polynomial moments for V, with exponents that can be arbitrarily close to 1. We leave as an open question whether, in this case, $L\log L$ type criteria can be obtained, as it is the case in the classical Galton-Watson case (for which $\theta_0 = \lambda^*$).

In the situation where $\theta_0 < \lambda^*$, one can not choose ς arbitrarily close to 1 in Assumption (M), since it imposes $\varsigma > \frac{|\log\theta_0|}{|\log\lambda^*|}$. We leave as an open question whether $\mathbb{E}(V_{i,j}^{\varsigma}) < +\infty$ for some $\varsigma \leq \frac{|\log\theta_0|}{|\log\lambda^*|}$ may be sufficient to obtain the conclusion of Theorem 1.3 or 1.4, at least in some particular cases.

In Section 2, we state a general result for the existence of infinitely many quasi-stationary distributions, then we prove Proposition 1.1 and Theorem 1.2. Section 3 is devoted to the proof of Proposition 1.5. Theorems 1.3 and 1.4 are proved in Sections 4 and 5 respectively. For the proofs of the last two theorems, we make use of Assumption (E) of [**?**] which is recalled in Appendix. In Appendix we also state a general result implying Assumption (E) of [**?**].

2 A simple criterion for the existence of infinitely many quasi-stationary distributions and application to the bGWbp

In Section 2.1, we first state and prove a result on the existence of a continuum of quasi-stationary distributions for general sub-Markov kernels on Polish spaces. In Section 2, we apply this result to the bGWbp.

2.1 General setting

Let (X,d) be a Polish space endowed with a σ-algebra $\mathscr{X}$ that contains the toplogy induced by d. Let K be a sub-Markov kernel on X, which means that K is a non-negative kernel on X such that $K(x,X) \leq 1$ for all $x \in X$. We assume that K has Feller regularity, meaning that the associated operator $K : f \mapsto Kf := \int_X K(\cdot, \mathrm{d}y) f(y)$ is a well defined functional from $C_b(X)$ to $C_b(X)$ (the set of continuous bounded functions on X). We say that a probability measure μ on X is a quasi-stationary distribution for K if, for some $\lambda \in (0,1]$,

$$\int_X \mu(\mathrm{d}x) K(x,\cdot) = \lambda\mu(\cdot).$$

Our aim is to prove that K admits infinitely many quasi-stationary distributions under mild assumptions.

In the following theorem, we say that a sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ tends to infinity if it eventually leaves any compact set: for all compact subset L of X, there exists $n_L \geq 0$ such that, for all $n \geq n_L$, $x_n \notin L$. Similarly, we say that a real functional on X tends to infinity when $x \to +\infty$ if, for all $R > 0$, there exists a compact subset L_R of X such that the functional is bounded below by R on $X \setminus L_R$. We also define the iterated kernels K^n as usual by

$$K^0(x,A) = \mathbf{1}_{x\in A},\ K^{\ell+1}(x,A) = \int_X K^{\ell}(x,\mathrm{d}y) K(y,A)\ \forall x \in X, A \in \mathscr{X}, \ell \geq 0.$$

Theorem 2.1. *Assume that there exists $a > 0$ and $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ tending to infinity such that*

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^{\ell}(x_n, X) < +\infty,\ \forall n \geq 0 \tag{11}$$

and let

$$a_0 := \inf\{a > 0,\ \textit{s.t. there exists } (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \textit{ tending to infinity such that } (11) \textit{ holds true.}\} \tag{12}$$

Assume also that there exists $a_1 > a_0$ such that

$$\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) \xrightarrow[x\to+\infty]{} +\infty. \tag{13}$$

Then K admits an infinite family of quasi-stationary distributions $(\mu_\lambda)_{\lambda\in[a_0,a_1)}$, with $\int_X \mu_\lambda(\mathrm{d}x)K(x,\cdot) = \lambda\mu_\lambda$ for all $\lambda \in [a_0, a_1)$.

Proof of Theorem 2.1. Fix $\lambda \in (a_0, a_1)$. Since $\lambda > a_0$ then there exists $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ tending to infinity such that, for all $n \geq 0$, the probability measure

$$\mu_{\lambda,n} = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)}$$

is well defined. We have, for all $n \geq 0$,

$$\begin{aligned}
\mu_{\lambda,n}K(\cdot) := \int_X \mu_{\lambda,n}(\mathrm{d}x)K(x,\cdot) &= \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\
&= \frac{\lambda\sum_{\ell=0}^{\infty} \lambda^{-(\ell+1)} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\
&= \frac{\lambda\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot) - \lambda\delta_{x_n}(\cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\
&\leq \lambda\mu_{\lambda,n}(\cdot).
\end{aligned} \tag{14}$$

We deduce that, for all $\ell \geq 1$,

$$\mu_{\lambda,n}K^\ell \leq \lambda^\ell \mu_{\lambda,n}.$$

In particular, we obtain that

$$\int_X \mu_{\lambda,n}(\mathrm{d}x) \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) = \sum_{\ell=0}^{\infty} a_1^{-\ell} \mu_{\lambda,n} K^\ell(X) \leq \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{a_1}\right)^\ell = \frac{a_1}{a_1 - \lambda}. \tag{15}$$

Since we assumed that $\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X)$ goes to infinity when $x \to \infty$, this and Prokhorov's Theorem imply that the sequence of measures $(\mu_{\lambda,n})_{n\in\mathbb{N}}$ is relatively compact for the weak convergence topology.

Let μ_λ be any adherent point of $(\mu_{\lambda,n})_{n\in\mathbb{N}}$, and observe that μ_λ is a probability measure. Since K is assumed to preserve bounded continuous functions, we deduce that, up to a subsequence and for the weak convergence topology,

$$\mu_\lambda K = \lim_{n\to+\infty} \mu_{\lambda,n} K.$$

But we computed (see (14))

$$\mu_{\lambda,n}K = \lambda\mu_{\lambda,n} - \frac{\lambda\delta_{x_n}}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)},$$

where, by assumption,

$$\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X) \geq \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x_n, X) \xrightarrow[n\to+\infty]{} +\infty.$$

We deduce that $\lim_{n\to+\infty}\mu_{\lambda,n}K = \lim_{n\to+\infty}\lambda\mu_{\lambda,n}$, which is equal to $\lambda\mu_\lambda$. We thus proved that $\mu_\lambda K = \lambda\mu_\lambda$, which means that μ_λ is a quasi-stationary distribution for K with absorption parameter λ.

It remains to prove that there exists a quasi-stationary distribution with absorption parameter a_0. Since the set of probability measures μ_λ, $\lambda \in (a_0,(a_0+a_1)/2)$, satisfies (15) with a uniform upper bound, we deduce that it is relatively compact. One checks as above that any limit point, when $\lambda \to a_0$, of this family is a quasi-stationary distribution for K with absorption parameter a_0. This concludes the proof of Theorem 2.1. □

Remark 2.1. One easily checks from the proof that the condition that K preserves bounded continuous functions is stronger than necessary. Actually, it would be sufficient to assume that K sends a generating family of bounded continuous functions into a family of bounded continuous functions. For instance, it sufficient to assume that K sends compactly supported continuous functions to bounded continuous functions.

Remark 2.2. One can easily extends this result to the general setting of non-conservative kernels K. More precisely, assume that there exists a positive function h on X such that $Kh \le ch$ for some $c > 0$. Then the kernel defined by

$$\widetilde{K}(x,A) = \frac{1}{ch(x)}\int_A K(x,\mathrm{d}y)h(y),\ \forall x\in X,\ \forall A\in\mathscr{X},$$

is a sub-Markov kernel to which Theorem 2.1 may apply. Note that the assumptions would translate into: for any continuous function $f : X \to \mathbb{R}$, $x \in X \mapsto \frac{1}{h(x)}\int_X K(x,\mathrm{d}y)h(y)f(y)$ is continuous; for all $n \ge 0$, $\sum_{\ell=0}^\infty a_0^{-\ell}\int_X K^\ell(x_n,\mathrm{d}y)h(y) < +\infty$ and $\frac{1}{h(x)}\sum_{\ell=0}^\infty a_1^{-\ell}\int_X K^\ell(x,\mathrm{d}y)h(y) \xrightarrow[x\to+\infty]{} +\infty$. In addition, the resulting quasi-stationary measures may not be finite measures if h is not lower bounded (we only obtain that $\int_X \mu_\lambda(\mathrm{d}y)h(y) < +\infty$ for all $\lambda \in [a_0,a_1)$). On the other hand, if h is lower bounded away from 0, then μ_λ is a finite measure.

2.2 Proof of Proposition 1.1 and Theorem 1.2

In the context of bGWbp, we choose $K(x,\mathrm{d}y) = \mathbb{P}_x(Z_1 \in \mathrm{d}y,\, Z_1 \neq 0)$ for all $x \in X = \mathbb{N}^p \setminus \{0\}$. We first prove Proposition 1.1, as well as that $a_0 \le \upsilon_0$, where a_0 is defined in Theorem 2.1 and υ_0 is defined in (3). Then we prove Theorem 1.2.

First, from their definitions, it is clear that $\upsilon_0 \le 1$ and $\theta_0 \le 1$. For all $\theta < \theta_0$, there exists $\upsilon \in (\theta,1)$ and $z \in \mathbb{N}^p \setminus \{0\}$ such that $\liminf_{n\to\infty}\upsilon^{-n}\mathbb{P}_z(Z_n \neq 0) > 0$. For such υ and z, we have $\mathbb{E}_z(\upsilon^{-T_0}) = \sum_{\ell=0}^\infty \upsilon^{-\ell}\mathbb{P}_z(T_0=\ell)$ with

$$\begin{aligned}\liminf_{\ell\to\infty}\upsilon^{-\ell}\mathbb{P}_z(T_0=\ell) &= \liminf_{\ell\to\infty}\upsilon^{-\ell}\left(\mathbb{P}_z(Z_{\ell-1}\neq 0)-\mathbb{P}_z(Z_\ell\neq 0)\right)\\ &= (\upsilon^{-1}-1)\liminf_{\ell\to\infty}\upsilon^{-\ell}\mathbb{P}_z(Z_\ell\neq 0) > 0.\end{aligned}$$

Then $\mathbb{E}_z(\upsilon^{-T_0}) = \infty$ and hence $\upsilon \ge \upsilon_0$. We deduce that $\upsilon_0 \ge \theta$ and hence $\upsilon_0 \ge \theta_0$.

In order to prove that $a_0 \le \upsilon_0$, we first observe that, for all $a > \upsilon_0$, there exists $\upsilon \le 1$, such that $\upsilon < a$ and for all $z \in \mathbb{N}^p \setminus \{0\}$, $\mathbb{E}_z(\upsilon^{-T_0}) < \infty$. For such υ we then obtain

$$\sum_{\ell\ge 0} a^{-\ell}K^\ell(z,X) = \sum_{\ell\ge 0} a^{-\ell}\mathbb{P}_z(T_0\ge \ell+1) \le \sum_{\ell\ge 0} a^{-\ell}\upsilon^{\ell+1}\mathbb{E}_z(\upsilon^{-T_0}) \le C_{a,\upsilon}\mathbb{E}_z(\upsilon^{-T_0}) < \infty.$$

This shows that $a_0 \le \upsilon_0$.

Using the fact that $\mathscr{P}$ is lower bounded away from 0 on $\mathbb{N}^p \setminus \{0\}$ (as $\mathscr{P}$ is strictly positive and positively homogeneous) and that $K(\cdot,X)$ is bounded by 1 and vanishes at 0 , we deduce that

there exists a constant $c>0$ such that, for all $\ell \geq 1$ and all $x \in \mathbb{N}^p$,

$$K^\ell(x,X) \leq \frac{1}{c}\mathbb{E}_x\left(\mathscr{P}(Z_\ell)\right).$$

In addition, $(\lambda^*)^{-n}\mathscr{P}(Z_n)$ is a supermartingale (see Lemma 37 in [**?**]), so that, for any $a \in (\lambda^*,1)$,

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^\ell(x,X) \leq \frac{1}{c}\sum_{\ell=0}^{\infty} a^{-\ell}(\lambda^*)^\ell \mathscr{P}(x) < +\infty.$$

This shows that $v_0 \leq \lambda^*$, then Proposition 1.1 holds.

Since $\mathfrak{M}$ is primitive by Assumption (P), Theorem 6 in [**?**] implies that, for all $z \in \mathbb{N}^p \setminus \{0\}$ and for all $n \geq n_0$, there exists $c_0>0$ and $k_0 \geq 1$ such that for all $k \geq k_0$

$$\mathbb{P}(Z_n \geq c_0(1,\ldots,1) \mid Z_0 = kz) > 0.$$

Applying this result for z taken in each element of the canonical basis of $\mathbb{N}^p$ and using the super-additivity of ξ, one deduces that, for any $n \geq n_0$, there exist constants $r>0$ and $c_0'>0$ such that

$$\inf_{|z|>r} \mathbb{P}(Z_n \geq c_0'(1,\ldots,1) \mid Z_0 = z) > 0.$$

Using again the superadditivity of the model, we obtain that , for all $z_1,\ldots,z_\ell$, with $\ell \geq 1$ and $|z_i|>r$,

$$\mathbb{P}_{z_1+\cdots+z_\ell}(Z_n=0) \leq \mathbb{P}_{z_1}(Z_n=0)\cdots\mathbb{P}_{z_\ell}(Z_n=0) \leq \left(1-\inf_{|z|>r}\mathbb{P}(Z_n \geq c_0'(1,\ldots,1) \mid Z_0=z)\right)^\ell.$$

and hence

$$\mathbb{P}_z(Z_n=0) \xrightarrow[|z|\to+\infty]{} 0.$$

In particular, we deduce from Fatou's Lemma that, for any $a_1 \in (0,1)$,

$$\sum_{\ell=0}^{\infty} a_1^{-\ell}\mathbb{P}_z(Z_\ell \neq 0) \xrightarrow[|z|\to+\infty]{} +\infty.$$

and hence (13) holds true for any $a_1 \in (a_0,1) \supset (v_0,1)$.

We deduce that Theorem 2.1 applies to the subcritical bGWbp. Since quasi-stationary distributions with different absorption parameters are linearly independent, this proves Theorem 1.2.

3 Proof of Proposition 1.5

We start with the following technical lemma on the regularity of $\mathscr{P}$.

Lemma 3.1. *Assume that Assumptions (S), (P), (C) and (M) hold true. Then, for all $I \subset \{1,\ldots,p\}$, $I \neq \emptyset$, the operator $\mathscr{P}$ restricted to $\mathcal{Z}_I$ is uniformly continuous.*

Proof of Lemma 3.1. Since $\mathscr{P}\circ\mathfrak{M}^{n_0} = (\lambda^*)^{n_0}\mathscr{P}$, it is sufficient to prove that $\mathscr{P}\circ\mathfrak{M}^{n_0}$ is uniformly continuous on $\mathcal{Z}_I$. But $\mathfrak{M}^{n_0}(\mathcal{Z}_I) \subset \mathfrak{M}^{n_0}(\mathbb{S})$ which is relatively compact in $(\mathbb{R}_+\setminus\{0\})^p$ (since $\mathfrak{M}$ is primitive and bounded by assumption), and $\mathscr{P}$ is concave and hence it is locally Lipschitz in $(\mathbb{R}_+\setminus\{0\})^p$. It is thus sufficient to prove that $\mathfrak{M}^{n_0}$ is uniformly continuous on $\mathcal{Z}_I$.

Let us first observe that $\mathfrak{M}$ is locally uniformly continuous on $(\varepsilon+\mathbb{R}_+\setminus\{0\})\mathcal{Z}_I$ for any $\varepsilon>0$, since $\mathfrak{M}$ is uniformly continuous on $\mathcal{Z}_I$, bounded on $\mathbb{S}\supset\mathcal{Z}_I$, and positively homogeneous. In order to conclude, it remains to prove that there exists $J\subset\{1,\dots,p\}$ and $\varepsilon>0$ such that $\mathfrak{M}(\mathcal{Z}_I)\subset(\varepsilon+\mathbb{R}_+\setminus\{0\})\mathcal{Z}_J$ (the result then follows by induction). Let $x\in\mathcal{Z}_I$ and let $J=\{i\in\{1,\dots,p\},\mathfrak{M}(x)_i>0\}$. Then, for all $y\in\mathcal{Z}_I$, there exists $\delta>0$ such that $\delta x\le y$, so that, using the fact that $\mathfrak{M}$ is increasing and homogeneous, $\mathfrak{M}(y)\ge\delta\mathfrak{M}(x)$ and hence $J\subset\{i\in\{1,\dots,p\},\mathfrak{M}(y)_i>0\}$. Reciprocally, $\{i\in\{1,\dots,p\},\mathfrak{M}(y)_i>0\}\subset J$, so that $\mathfrak{M}(y)\in\mathbb{R}_+\mathcal{Z}_J$. Since on $\mathbb{S}$, the norm of $\mathfrak{M}$ is bounded from below away from 0 and from above, we have $\mathfrak{M}(\mathcal{Z}_I)\subset[\varepsilon_1,\varepsilon_2]\mathcal{Z}_J$ for some $\varepsilon_1,\varepsilon_2>0$. This implies that $\mathfrak{M}^2$ is uniformly continuous on $\mathcal{Z}_I$. We conclude by iteration. □

Let us now proceed with the proof of Proposition 1.5 in two steps: first we show that, for all $z\in\mathbb{R}_+^p\setminus\{0\}$,

$$\limsup_{n\to+\infty}\frac{1}{|n|^a}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq0}\mid Z_0=\lfloor nz\rfloor\right]=(\lambda^*)^aQ_a(z);\tag{16}$$

second, we extend this to a uniform convergence in z by using the regularity of $\mathscr{P}$.

Step 1. As $\mathscr{P}$ is positively homogeneous, we have for any $z\in\mathbb{R}_+^p\setminus\{0\}$,

$$\begin{aligned}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq0}\mid Z_0=\lfloor nz\rfloor\right]&=\mathbb{E}\left[\left(\mathscr{P}\circ\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right)^a\right]\\&=|n|^a\mathbb{E}\left[\left(\mathscr{P}\left(\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right)\right)^a\right].\end{aligned}$$

According to Lemma 30 in [**?**], we have

$$\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\xrightarrow[n\to+\infty]{a.s}\mathfrak{M}(z).$$

Let $I\subset\{1,\dots,p\}$ be such that $\mathfrak{M}(z)\in(\mathbb{R}_+\setminus\{0\})\mathcal{Z}_I$, then the above convergence entails that, almost surely, for n large enough, $\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\in(\mathbb{R}_+\setminus\{0\})\mathcal{Z}_J$ for some $J\supset I$. Using Corollary 7 in [**?**], we have, for all $n\ge1$,

$$\mathfrak{M}(z)\ge\frac{1}{n}\mathbb{E}\left[Z_1\mid Z_0=\lfloor nz\rfloor\right]=\frac{1}{n}\mathbb{E}\left[\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right],$$

so that we also have $J\subset I$ almost surely.

Since $\mathscr{P}$ restricted to $(\mathbb{R}_+\setminus\{0\})\mathcal{Z}_I$ is continuous, and since $\mathfrak{M}\circ\mathscr{P}=\lambda^*\mathscr{P}$ by (2), we deduce that

$$\mathscr{P}\left(\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right)\xrightarrow[n\to+\infty]{a.s}\mathscr{P}\left(\mathfrak{M}(z)\right)=\lambda^*\mathscr{P}(z).$$

In addition, since ξ is sub-affine, since $\frac{1}{n}\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}$ is bounded in L^ς by Assumption (M) and $\mathscr{P}$ being positively homogeneous, we deduce that $\mathscr{P}\left(\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right)$ is also bounded in L^ς (where $\varsigma>a$), and we deduce from the dominated convergence theorem that

$$\mathbb{E}\left[\left(\mathscr{P}\left(\frac{1}{n}\xi\left(\sum_{i=1}^p\sum_{k=1}^{\lfloor nz_i\rfloor}V_{i,\cdot}^{(k,1)}\right)\right)\right)^a\right]\xrightarrow[n\to+\infty]{}(\lambda^*)^a\mathscr{P}(z)^a.$$

We deduce that (16) holds true. This concludes the first step of the proof.

Step 2. Fix a non-empty $I \subset \{1,\dots,p\}$ and let $\varepsilon_1 > 0$ be a small number that will be fixed later. Let $z^1,\dots,z^{m_1} \in (1+\varepsilon_1)\mathcal{Z}_I$ and $m_1 = m(\varepsilon_1) \geq 1$ be such that, for all $z \in \mathcal{Z}_I$, there exists $k = k(z,\varepsilon_1) \in \{1,\dots,m_1\}$ such that $z \leq z^k$ and $|z - z^k| \leq 2\varepsilon_1$. The existence of this finite family $z^1,\dots,z^{m_1}$ easily follows from a compactness argument. Let $n_1 = n_1(\varepsilon_1)$ be large enough so that, for all $n \geq n_1$, for all $k \in \{1,\dots,m_1\}$,

$$\frac{1}{|n|^a}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq 0} \mid Z_0 = \lfloor nz^k \rfloor\right] \leq (1+\varepsilon_1)(\lambda^*)^a \mathscr{P}(z^k)^a.$$

Then, since Q_a is non-decreasing and since Z_1 (with $Z_0 = \lfloor nz \rfloor$) is stochastically non-decreasing with respect to z, we deduce that, for all $n \geq n_1$ and all $z \in \mathcal{Z}_I$,

$$\begin{aligned}\frac{1}{|n|^a}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq 0} \mid Z_0 = \lfloor nz \rfloor\right] &\leq (1+\varepsilon_1)(\lambda^*)^a \mathscr{P}(z^{k(z,\varepsilon_1)})^a \\ &\leq (1+\varepsilon_1)(\lambda^*)^a \mathscr{P}(z)^a \left(\sup_{x,y\in\mathcal{Z}_I, |(1+\varepsilon_1)x-y|\leq 2\varepsilon_1} \frac{\mathscr{P}((1+\varepsilon_1)x)}{\mathscr{P}(y)}\right)^a. \end{aligned} \tag{17}$$

Since $\mathscr{P}$ is uniformly continuous on $\mathcal{Z}_I$ and lower bounded away from 0 on $\mathcal{Z}_I$, we deduce that there exists ε_1 small enough so that

$$\theta_a := (1+\varepsilon_1)^{a+1}(\lambda^*)^a \left(\sup_{x,y\in\mathcal{Z}_I, |(1+\varepsilon_1)x-y|\leq 2\varepsilon_1} \frac{\mathscr{P}(x)}{\mathscr{P}(y)}\right)^a < \theta_0.$$

This and (17) implies that

$$\frac{1}{|n|^a}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq 0} \mid Z_0 = \lfloor nz \rfloor\right] \leq \theta_a \mathscr{P}(z)^a.$$

Since any $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})\,\mathcal{Z}_I$ can be written as $n\frac{z}{|z|}$ with $n = |z|$ and $\frac{z}{|z|} \in \mathcal{Z}_I$, we deduce that, for any such z with $|z| \geq n_1$,

$$\frac{1}{|z|^a}\mathbb{E}\left[Q_a(Z_1)\mathbf{1}_{Z_1\neq 0} \mid Z_0 = z\right] \leq \theta_a \mathscr{P}(z/|z|)^a.$$

Finally, the homogeneity of $\mathscr{P}$ allows us to conclude that Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})\,\mathcal{Z}_I$. Since there are only finitely many subsets I of $\{1,\dots,p\}$, Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\}$.

4 Proof of Theorem 1.3

The proof of the first statement relies on Theorem 4.1 [**?**] and in part on some arguments of the proof of Theorem 5.1 therein. The main additional difficulty is that our reference exponential parameter θ_0 is *a priori* larger than the one considered in this reference. We overcome this difficulty in Lemma 4.1 below. Another difficulty is that a key part of the argument relies on the assumption (E1-4) in [**?**], which does not apply directly when the Lyapunov parameter θ_a (in Proposition 1.5) is only assumed strictly smaller than θ_0. To overcome this difficulty, we prove in Proposition A.1 that this last condition actually entails the assumption (E1-4) in [**?**].

Fix $a \in (1,\varsigma)$ such that $(\lambda^*)^a < \theta_0$ and set $H = Q_a / \inf Q_a$. According to Proposition 1.5 and since $H(z)$ tends to ∞ when $|z| \to +\infty$, there exists $\theta_a < \theta_0$ such that

$$\limsup_{|z|\to+\infty} \frac{\mathbb{E}_z(H(Z_1)\mathbf{1}_{Z_1\neq 0})}{H(z)} \leq \theta_a < \gamma := (\theta_a + \theta_0)/2 < \theta_0. \tag{18}$$

In particular, there exist only finitely many communication classes $E_1, \dots, E_{k_0}$, $k_0 \geq 1$, for the process $(Z_n)_{n\in\mathbb{N}}$ such that

$$\forall i \in \{1,\dots,k_0\}, \exists z \in E_i, \text{ such that } \frac{\mathbb{E}_z(H(Z_1)\mathbf{1}_{Z_1\neq 0})}{H(z)} > \gamma.$$

Note that $k_0 \neq 0$ (otherwise $\theta_0 \leq \gamma$). For all $i \in \{1,\dots,k_0\}$, we define $H_i = H\mathbf{1}_{E_i}$ and denote by $Y^{(i)}$ the process with state space $E_i \cup \{0\}$ defined by

$$Y_n^{(i)} = \begin{cases} Z_n & \text{if } Z_n \in E_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that 0 is an absorbing state for the process $Y^{(i)}$ as E_i is a communication class. For all $i \in \{1,\dots,k_0\}$, we denote by $\theta_{0,i}$ the absorption parameter of $Y^{(i)}$:

$$\theta_{0,i} = \sup_{z\in E_i} \sup\{\theta > 0,\ \liminf_{n\to+\infty} \theta^{-n}\mathbb{P}_z(Y_n^{(i)} \neq 0) > 0\}$$

and we set

$$\bar{\theta} = \max_{i\in\{1,\dots,k_0\}} \theta_{0,i}.$$

We define the set

$$E_0' := \mathbb{N}^p \setminus \left(\{0\} \cup \bigcup_{i=1}^{k_0} E_i\right).$$

By definition of $E_1, \dots, E_{k_0}$, we have

$$\forall z \in E_0',\ \frac{\mathbb{E}_z(H(Z_1)\mathbf{1}_{Z_1\neq 0})}{H(z)} \leq \gamma. \tag{19}$$

In what follows, we make use of the following crucial lemma.

Lemma 4.1. *We have $\bar{\theta} = \theta_0$.*

Proof of Lemma 4.1. Our aim is to show that, for all $\rho > \bar{\theta} \vee \gamma$ and all $z \in \mathbb{N}^p$, we have

$$\liminf_{n\to+\infty} \rho^{-n}\mathbb{P}_z(Z_n \neq 0) = 0. \tag{20}$$

This shows that $\bar{\theta} \vee \gamma \geq \theta_0$ and hence, as by definition $\gamma < \theta_0$, that $\bar{\theta} \geq \theta_0$. The converse inequality is trivial, concluding the proof of Lemma 4.1.

For all $i \neq j \in \{1,\dots,k_0\}$, we have

$$\forall z \in E_j,\ \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) > 0 \ \text{ or } \forall z \in E_j,\ \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) = 0.$$

If the condition on the left is satisfied, we write $E_i \prec E_j$, otherwise we write $E_i \not\prec E_j$. This defines a partial order on $\{1,\dots,k_0\}$. For all $i \in \{1,\dots,k_0\}$, we denote by $c(i)$ the maximal length of sets of distinct indices $i_1 = i, i_2, \dots, i_{c(i)}$ such that $E_{i_{c(i)}} \prec \cdots \prec E_{i_2} \prec E_{i_1}$. We also define, for all $\ell \geq 1$, the set

$$E_0^\ell := \{x \in E_0',\ \exists i \in \{1,\dots,k_0\} \text{ with } c(i) = \ell \text{ and } x \to E_i, \text{ and } x \not\to E_j\ \forall j \text{ such that } c(j) > \ell\},$$

where $x \to E_i$ means that $\mathbb{P}_x(\exists n \geq 0,\ Z_n \in E_i) > 0$, while $x \not\to E_j$ means that $\mathbb{P}_x(\exists n \geq 0,\ Z_n \in E_i) = 0$. We also set

$$E_0^0 := E_0' \setminus \cup_{\ell=1}^{\max_i c(i)} E_0^\ell.$$

We obtain (20) by proving that, for any fixed $\rho > \bar{\theta} \vee \gamma$, for all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\mathbb{E}_z(H(Z_n)1_{Z_n \neq 0}) \leq C_\rho \rho^n H(z) \tag{21}$$

where C_ρ does not depend on z.

Remarking that, by definition, $E_0^0 \cup \{0\}$ is an absorbing set, then (21) is immediate for all $z \in E_0^0$ by (19). In the following, we first obtain in Step 1 two useful inequalities (see (22) and (23) below). Then, we prove (21) for all $z \in E_i \cup E_0^{c(i)}$, $i \in \{1, \dots, k_0\}$, by induction on $c(i)$ in Step 2.

Step 1. We first observe that, for all $\ell \geq 1$ and all $z \in E_0^\ell$, using (19),

$$\mathbb{E}_z\left(H(Z_n)\mathbf{1}_{Z_n \in E_0^\ell}\right) \leq \gamma^n H(z) \leq \rho^n H(z). \tag{22}$$

We now prove that, for all $i \in \{1, \dots, k_0\}$ and all $z \in E_i$,

$$\mathbb{E}_z(H(Z_n)1_{Z_n \in E_i}) \leq C\rho^n H(z). \tag{23}$$

In order to do so, we consider the set

$$K_i = \left\{ z \in E_i,\ \frac{\mathbb{E}_z(H(Z_1)\mathbf{1}_{Z_1 \neq 0})}{H(z)} > \gamma \right\}.$$

For all $z \in E_i$, since $\rho > \bar{\theta}$ and by definition of $\bar{\theta}$,

$$\liminf_{n \to +\infty} \rho^{-n} \mathbb{P}_z(Y_n^{(i)} \neq 0) = 0. \tag{24}$$

Since K_i is finite according to (18), we deduce that there exists a constant $C > 0$ such that

$$\mathbb{P}_z(Z_n \in E_i) = \mathbb{P}_z(Y_n^{(i)} \neq 0) \leq C\rho^n,\ \forall z \in K_i. \tag{25}$$

In what follows, the constant C may change from line to line. Moreover, denoting $\tau_{K_i} = \min\{\ell \geq 0,\ Z_\ell \in K_i\}$, we deduce from the definition of H and of K_i that

$$\mathbb{P}_z(Z_n \in E_i \text{ and } n < \tau_{K_i}) \leq \mathbb{E}_z(H(Z_n)1_{Z_n \in E_i \text{ and } n < \tau_{K_i}}) \leq \gamma^n H(z),\ \forall z \in E_i. \tag{26}$$

Hence, for all $z \in E_i$ and all $n \geq 0$, we have, using the strong Markov property at time τ_{K_i},

$$\begin{aligned}
\mathbb{P}_z(Z_n \in E_i) &\leq \mathbb{P}_z(Z_n \in E_i \text{ and } n < \tau_{K_i}) + \sum_{\ell=0}^{n} \mathbb{P}_z(\tau_{K_i} = \ell) \sup_{y \in K_i} \mathbb{P}_y(Z_{n-\ell} \in E_i) \\
&\leq \gamma^n H(z) + \sum_{\ell=0}^{n} \mathbb{P}_z(\tau_{K_i} = \ell) C\rho^{n-\ell} \\
&\leq \gamma^n H(z) + \sum_{\ell=0}^{n} \mathbb{E}_z(1_{\tau_{K_i} > \ell-1} H(Z_{\ell-1})) C\rho^{n-\ell} \\
&\leq \gamma^n H(z) + \sum_{\ell=0}^{n} \gamma^{\ell-1} H(z) C\rho^{n-\ell} \\
&\leq C\rho^n H(z),
\end{aligned} \tag{27}$$

where C may depend on ρ and γ. Finally, from (10), we obtain for all $z \in E_i$

$$\mathbb{E}_z(H(Z_1)1_{Z_1 \in E_i}) \leq \theta_a H(z) + \frac{C_a}{\inf Q_a}.$$

Then, iterating this property and using (27), we deduce that

$$\begin{aligned}\mathbb{E}_z(H(Z_n)1_{Z_n\in E_i}) &\le \theta_a^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a}\theta_a^{n-\ell-1}\mathbb{P}_z(Z_\ell \in E_i)\\ &\le \gamma^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a}\gamma^{n-\ell-1} C\rho^\ell H(z)\\ &\le C\rho^n H(z),\end{aligned}$$

where C may depend on ρ and γ. This proves (23) for all $z \in E_i$.

Step 2. Assume that there exists $c \ge 0$ such that (21) holds true for all $z \in E_0^0, \dots, E_0^c$ and all $z \in E_i$ with $c(i) \le c$ (this is true for $c = 0$ as we already saw that (21) holds true for $z \in E_0^0$). If $c = \max_{i\in\{1,\dots,k_0\}} c(i)$, then there is nothing to prove. Otherwise, let $i \in \{1,\dots,k_0\}$ such that $c(i) = c+1$.

Denote by T_i the first exit time from E_i by $(Z_n)_{n\in\mathbb{N}}$ and let us use a similar computation as in the end of Step 1. For all $z \in E_i$ and all $n \ge 0$, we have

$$\begin{aligned}\mathbb{E}_z(H(Z_n)1_{Z_n\neq 0}) &= \mathbb{E}_z(H(Z_n)1_{Z_n\in E_i}) + \sum_{\ell=1}^{n} \mathbb{E}_z\left(1_{T_i=\ell}1_{Z_\ell\neq 0}\mathbb{E}_{Z_\ell}\left(H(Z_{n-\ell})1_{Z_{n-\ell}\neq 0}\right)\right)\\ &\le C\rho^n H(z) + \sum_{\ell=1}^{n} \mathbb{E}_z\left(1_{T_i=\ell}1_{Z_\ell\neq 0} C_\rho \rho^{n-\ell} H(Z_\ell)\right)\end{aligned}$$

where we used (23), the Markov property, the fact that, conditioning on $Z_0 \in E_i$,

$$Z_{T_i} \in \{0\} \cup \bigcup_{\ell\le c} E_0^\ell \cup \bigcup_{c(j)\le c} E_j \text{ almost surely},$$

and the induction assumption. Then, there exists $C_a > 0$ such that

$$\begin{aligned}\mathbb{E}_z(H(Z_n)1_{Z_n\neq 0}) &\le C\rho^n H(z) + \sum_{\ell=1}^{n} \mathbb{E}_z\left(1_{T_i>\ell-1} C_\rho \rho^{n-\ell}(\theta_a H(Z_{\ell-1}) + C_a)\right)\\ &\le C\rho^n H(z) + \sum_{\ell=1}^{n} \mathbb{E}_z\left(1_{T_i>\ell-1} C_\rho \rho^{n-\ell}(\theta_a + C_a) H(Z_{\ell-1})\right)\end{aligned}$$

where we used the Markov property at time $\ell-1$ and Proposition 1.5. We deduce from (23) that

$$\begin{aligned}\mathbb{P}_z(Z_n \neq 0) &\le C\rho^n H(z) + C_\rho(\theta_a + C_a)C\sum_{\ell=1}^{n} \rho^{n-1} H(z)\\ &\le \tilde{C}(n+1)\rho^n H(z)\end{aligned}$$

for some $\tilde{C} > 0$, which depends on ρ but not on z neither n. Since this is true for any $\rho > \bar{\theta} \vee \gamma$, this proves (21) for all $z \in E_i$ with $c(i) = c+1$.

By the same arguments, using (22) instead of (23) and the fact that, starting from E_0^{c+1}, at the exit time $\tilde{T}_i$ of E_0^{c+1}, we have

$$Z_{\tilde{T}_i} \in \{0\} \cup \bigcup_{\ell\le c} E_0^\ell \cup \bigcup_{c(j)\le c+1} E_j \text{ almost surely},$$

one shows that this also holds true for all $z \in E_0^{c+1}$.

This concludes the proof of Lemma 4.1. □

From now on, we assume without loss of generality that the communication classes $E_1, \dots, E_{k_0}$ are indexed in the following way : there exists $k \in \{1, \dots, k_0\}$ such that, for all $i \in \{k+1, \dots, k_0\}$, $\theta_{0,i} < \bar{\theta}$, and, for all $i \in \{1, \dots, k\}$, $\theta_{0,i} = \bar{\theta} = \theta_0$. For any $i \in \{k+1, \dots, k_0\}$, we have, fixing any $\rho \in (\max_{j \in \{k+1,\dots,k_0\}} \theta_{0,j}, \bar{\theta})$ and using the same method as in Step 1 from the proof of Lemma 4.1 (see (23)),

$$\mathbb{E}_z(H(Z_n)1_{Z_n \in E_i}) \leq C_\rho \rho^n H(z), \ \forall z \in E_i,$$

where C_ρ is a positive constant which may depend on ρ. Setting

$$E_0 := E_0' \cup \bigcup_{i=k+1}^{k_0} E_i,$$

one easily checks, using the last inequality and (19), that, for any $r \in (\gamma \vee \rho, \bar{\theta})$, there exists $C_r > 0$ such that

$$\mathbb{E}_z(H(Z_n)1_{n < T_{E_0}}) \leq C_r r^n H(z), \ \forall z \in E_0$$

with T_{E_0} the first exit time of E_0.

Note also that, from Proposition 1.5 since $\inf H = 1$, there exists $C_W > 0$ (possibly larger than θ_0) such that, for all $z \in \mathbb{N}^p \setminus \{0\}$, $\mathbb{E}_z(H(Z_1)1_{Z_1 \neq 0}) \leq C_w H(z)$.

In order to apply Theorem 4.1 in [**?**], which implies the first part of Theorem 1.3, it then remains to prove that, for all $i \in \{1, \dots, k\}$ and all $z \in E_i$, we have

$$\left|\theta_0^{-n} \mathbb{E}_z\left(f(Z_n)\mathbf{1}_{Z_n \in E_i}\right) - h_i(z)\mu_i(f)\right| \leq C\alpha^n H_i(z) \|f\|_{H_i}, \quad \forall n \geq 0, \ \forall f \in L^\infty(H_i), \tag{28}$$

for some $C > 0, \alpha \in (0,1)$, some non-negative non-zero function h_i and some probability measure μ_i on E_i. In order to do so, its is sufficient to check Assumptions (E1-4) in Section 2 of [**?**]. These conditions are recalled in the Appendix and we will more precisely check Assumptions (E1), (E2'), (E3) and (E4), which is sufficient to obtain (E1-4) according to Proposition A.1. Fixing $i \in \{1, \dots, k\}$, we define the set

$$K_i := \left\{ z \in E_i, \ \frac{\mathbb{E}_z(H_i(Z_1)\mathbf{1}_{Z_1 \neq 0})}{H_i(z)} > \gamma \right\},$$

which is finite according to (18).

Since E_i is a communication class and the process is assumed to be aperiodic, there exists $n_1^i \geq 0$ such that

$$c_1^i := \inf_{x,y \in K_i} \mathbb{P}_x\left(Z_{n_1^i} = y\right) > 0.$$

Note that $K_i \neq \emptyset$, by definition of E_i. Fix $z_i \in K_i$. Then, for all $z \in K_i$,

$$\mathbb{P}_z\left(Z_{n_1^i} \in \cdot\right) \geq \mathbb{P}_z\left(Z_{n_1^i} = z_i\right)\delta_{z_i}(\cdot) \geq c_1^i \nu_i(\cdot \cap K),$$

for $\nu_i = \delta_{z_i}$. This entails (E1). In addition, taking $\varphi_1 = H_i$ and $\theta_1 = \gamma < \theta_0 = \theta_{0,i}$, the condition (E'2) is an immediate consequence of the definition of K_i and the fact that $z \mapsto \mathbb{E}_z(H_i(Z_1)\mathbf{1}_{Z_1 \in E_i})$ is bounded on K_i by Proposition 1.5.

For all $x, y \in K_i$ and $n \geq 0$, we have, using the Markov property at time n_1^i and the fact that $\mathbb{P}_x(Z_n \in E_i)$ is non-increasing in n (as $\{0\}$ is absorbing for the process $Y^{(i)}$),

$$\mathbb{P}_x(Z_n \in E_i) \geq \mathbb{P}_x(Z_{n \vee n_1^i} \in E_i) \geq \mathbb{P}_x(Z_{n_1^i} = y)\mathbb{P}_y(Z_{n \vee n_1^i - n_1^i} \in E_i) \geq c_1^i \mathbb{P}_y(Z_n \in E_i).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic and E_i is a communication class. This concludes the proof of (5). In addition, any quasi-stationary distribution $\nu \in \mathscr{M}(Q_a)$ with absorption parameter θ_0 satisfies $\nu(\{x,\ j(x)=0\})=1$ (see Proposition 2.3 in [**?**]). Integrating (5) with respect to ν implies that ν is a convex combination of $\nu_1,\dots,\nu_\ell$. This concludes the first part of Theorem 1.3.

To conclude the proof of Theorem 1.3, it remains to prove that the set $E=\{z\in\mathbb{N}^p\setminus\{0\}, \sum_{i\in I}\eta_i(z)=0\}$ is finite and that any quasi-stationary distribution with absorption parameter strictly smaller than θ_0 is supported by E.

We first observe that Theorem 4.1 in [**?**] also implies that $j\equiv 0$ on the set E. Assume on the contrary that E is not finite and fix $z_0\in\mathbb{N}^p\setminus\{0\}$ such that $\sum_{i=1}^{\ell}\eta_i(z_0)>0$ and $j(z_0)=0$ (existence of such a point z_0 is guaranteed by Proposition 2.3 in [**?**]). We observe that

$$\liminf_{n\to+\infty}\theta_0^{-n}\mathbb{P}_{z_0}(Z_n\neq 0)>0.$$

Then, by primitivity of $\mathfrak{M}$, there exists n_0 such that for all $x\in\mathbb{N}^p\setminus\{0\}$ we have $\mathfrak{M}^{n_0}(x)>0$. This implies that for $k>0$ large enough we have $\mathbb{P}_{kx}(Z_{n_0}\geq z_0)>0$. Indeed, if it is not the case, we can conclude using Theorem 6 in [**?**] that

$$\mathfrak{M}^{n_0}(x)=\lim_{k\to+\infty}\frac{\mathbb{E}(Z_{n_0}\mid Z_0=kx)}{k}\leq\lim_{k\to+\infty}\frac{z_0}{k}=0,$$

which is a contradiction. Then, we have for any z large enough $\mathbb{P}_z(Z_{n_0}\geq z_0)>0$. Hence, using the fact that E is not finite and hence contains arbitrarily large points, there exists $z\in E$ and $z_0'\geq z_0$ such that $\mathbb{P}_z(Z_{n_0}=z_0')>0$. By super-additivity of ξ, we have $\mathbb{P}_{z_0'}(Z_n\neq 0)\geq\mathbb{P}_{z_0}(Z_n\neq 0)$ and hence

$$\liminf_{n\to+\infty}\theta_0^{-n}\mathbb{P}_z(Z_{n_0+n}\neq 0)>0.$$

As we noticed that $j\equiv 0$ on the set E, this is not compatible with the definition of E and (5). We have thus proved by contradiction that E is finite.

Let ν_{QS} be a quasi-stationary distribution for Z in $\mathscr{M}(Q_a)$ with absorption parameter $\theta_{QS}<\theta_0$. Then (5) implies that $\sum_{i=1}^{\ell}\nu_{QS}(\eta_i)=0$ and hence the support of ν_{QS} is included in E. This concludes the proof of the penultimate assertion of Theorem 1.3.

To conclude, it remains to prove that there are no quasi-stationary distributions for Z in $\mathscr{M}(Q_a)$ with absorption parameter strictly larger than θ_0. This is a direct consequence of (5). Indeed, since j is bounded, integrating this inequality with respect to a quasi-stationary distribution ν_{QS} with absorption parameter θ_{QS}, and taking $f\equiv 1$, shows that

$$\limsup_{n\to+\infty}\theta_0^{-n}n^{-\|j\|_\infty}\theta_{QS}^n<+\infty,$$

and hence $\theta_{QS}\leq\theta_0$.

5 Proof of Theorem 1.4

Fix $a\in(1,\varsigma)$ such that $(\lambda^*)^a<\theta_0$, and consider the Lyapunov type property with constants θ_a and C_a from Proposition 1.5. We make use of Section 2 in [**?**]. In a first step, we check that Assumption E therein (recalled in the Appendix below) holds true for the process under consideration. In a second step, we prove (6), (7) and (8) and (9) using the results of [**?**].

Step 1. Assumption (E) holds true. Fix $\theta_1\in(\theta_a,\theta_0)$. As $\mathscr{P}$ is strictly positive on $\mathbb{R}_+^p\setminus\{0\}$ and positively homogeneous, we can fix $r_1\geq p$ large enough so that $Q_a(z)\geq\frac{C_a}{\theta_1-\theta_a}$ for all $|z|\geq r_1$, so that

$$\mathbb{E}_z(Q_a(Z_1))\leq\theta_aQ_a(z)+C_a\leq\theta_1Q_a(z),\quad\forall z\in\mathbb{N}^p\text{ such that }|z|\geq r_1. \tag{29}$$

Let us set $K = \{z \in \mathbb{N}^p \setminus \{0\},\ |z| \le r_1\}$ and $\varphi_1 = Q_a / \inf Q_a$ (note that $\inf Q_a > 0$ by Assumption (P)).

We deduce from the irreducibility and aperiodicity properties that there exists $n_1 \ge 0$ such that

$$c_1 := \inf_{x,y \in K} \mathbb{P}_x(Z_{n_1} = y) > 0. \tag{30}$$

Setting $\nu = \delta_{(1,\dots,1)}$, this entails property (E1).

We have $\inf \varphi_1 = 1$, so that the first line of (E2) is satisfied. Moreover, according to (29), the third line of (E2) also holds true.

Choosing $\theta_2 \in (\theta_1, \theta_0)$, we deduce from the definition of θ_0 and the irreducibility of Z that

$$\theta_2^{-n} \min_{i \in \{1,\dots,p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow[n \to +\infty]{} +\infty,$$

where e_i is the i^{th} element of the canonical basis of $\mathbb{N}^p$. Since, by the super-additivity of ξ, the process Z is stochastically non-decreasing in the initial condition and since, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $i \in \{1,\dots,p\}$ such that $z \ge e_i$, we deduce that

$$\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \ge \theta_2^{-n} \min_{i \in \{1,\dots,p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow[n \to +\infty]{} +\infty.$$

Let $n \ge 1$ be large enough so that $\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \ge 1$, and set, for all $z \in \mathbb{N}^p \setminus \{0\}$

$$\varphi_2(z) = \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0).$$

We have, for all $z \in \mathbb{N}^p \setminus \{0\}$ and using the Markov property at time 1,

$$\begin{aligned}
\left(\sum_{k=0}^{n-1} \theta_2^{-k}\right) \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq 0}) &= \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_{k+1} \neq 0) \\
&= \theta_2 \sum_{k=1}^{n} \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0) \\
&= \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k}\right) \varphi_2(z) + \theta_2 \left(\theta_2^{-n} \mathbb{P}_z(Z_n \neq 0) - 1\right) \\
&\ge \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k}\right) \varphi_2(z).
\end{aligned}$$

As a consequence the second and fourth lines of (E2) are satisfied. Note also that we have $\inf_{\mathbb{N}^p \setminus \{0\}} \varphi_2 \ge \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}}$.

For all $x, y \in K$ and $n \ge 0$, we have, using the Markov property at time n_1 and the fact that $\mathbb{P}_x(Z_n \neq 0)$ is non-increasing in n (as $\{0\}$ is absorbing),

$$\mathbb{P}_x(Z_n \neq 0) \ge \mathbb{P}_x(Z_{n \vee n_1} \neq 0) \ge \mathbb{P}_x(Z_{n_1} = y)\, \mathbb{P}_y(Z_{n \vee n_1 - n_1} \neq 0) \ge c_1\, \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic and irreducible.

We have thus proved that Assumption (E) holds true for Z absorbed when it leaves $\mathbb{N}^p \setminus \{0\}$. This concludes the first step.

Step 2. Conclusion of the proof. Theorem 2.1 in [**?**] implies that there exists a unique quasi-stationary distribution ν_{QS} in $\mathscr{M}(Q_a)$ and that (8) holds true (using the fact that in our case, φ_2 is lower bounded away from 0). Theorem 2.3 in this reference also implies that there exists an

associated non-negative eigenfunction $\eta_{QS} \in L^\infty(Q_a)$ with eigenvalue $\theta_{QS} > 0$, where θ_{QS} is the absorption parameter associated to ν_{QS}, such that $\inf_K \eta_{QS} > 0$ and

$$\theta_{QS}^{-n} \mathbb{P}_\cdot(Z_n \neq 0) \xrightarrow[n\to+\infty]{L^\infty(Q_a)} \eta_{QS}(\cdot). \tag{31}$$

Since the process is irreducible, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_z(Z_n = (1,\dots,1)) > 0$ and hence

$$\theta_{QS}^n \eta_{QS}(z) = \mathbb{E}_z(\eta_{QS}(Z_n)\mathbf{1}_{Z_n \neq 0}) \geq \eta_{QS}((1,\dots,1))\mathbb{P}_z(Z_n = (1,\dots,1)) > 0,$$

so that η_{QS} is positive. We thus proved (6) (for the absorption parameter θ_{QS} which we prove below to be equal to θ_0). Finally, since ξ is super-additive, $\theta_{QS}^{-n}\mathbb{P}_z(Z_n \neq 0)$ and hence η_{QS} increases with z, so that η_{QS} is lower bounded away from 0. Corollary 2.7 in [**?**] shows that (7) holds true, and, taking $f \equiv 1$, also implies that $\theta_{QS} = \theta_0$. Finally, (9) is an immediate consequence of Corollary 2.11 in [**?**]. This concludes the proof of Theorem 1.4.

A Sufficient criterion for the exponential convergence to a quasi-stationary distribution

In Section 2 of [**?**], the authors state that Assumption (E) is sufficient to prove exponential convergence toward a quasi-stationary distribution. Let us recall this assumption, with the notations and settings of the present paper.

In what follows, we consider a process $(Z_n)_{n\in\mathbb{N}}$ evolving in a measurable state space $E \cup \{\partial\}$, where $\partial \notin E$ is an absorbing point.

Assumption (E). There exists a positive integer n_1, positive real constants $\theta_1, \theta_2, c_1, c_2, c_3$, two functions $\varphi_1, \varphi_2 : E \to \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that

(E1) *(Local Dobrushin coefficient).* $\forall z \in K$,

$$\mathbb{P}_z(Z_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(E2) *(Global Lyapunov criterion).* We have $\theta_1 < \theta_2$ and

$$\begin{aligned}
&\inf_{z\in E} \varphi_1(z) \geq 1, \ \sup_{z\in K} \varphi_1(z) < \infty \\
&\inf_{z\in K} \varphi_2(z) > 0, \ \sup_{z\in E} \varphi_2(z) \leq 1, \\
&\mathbb{E}_z(\varphi_1(Z_1)\mathbf{1}_{Z_1\neq\partial}) \leq \theta_1\varphi_1(z) + c_2\mathbf{1}_K(z), \ \forall z \in E \\
&\mathbb{E}_z(\varphi_2(Z_1)\mathbf{1}_{Z_1\neq\partial}) \geq \theta_2\varphi_2(z), \ \forall z \in E.
\end{aligned}$$

(E3) *(Local Harnack inequality).* We have

$$\sup_{n\in\mathbb{Z}_+} \frac{\sup_{y\in K} \mathbb{P}_y(Z_n \neq \partial)}{\inf_{y\in K} \mathbb{P}_y(Z_n \neq \partial)} \leq c_3.$$

(E4) *(Aperiodicity).* For all $z \in K$, there exists $n_4(z)$ such that, for all $n \geq n_4(z)$,

$$\mathbb{P}_z(Z_n \in K) > 0.$$

Our aim is to show that the condition involving the function φ_2 can be replaced by a condition involving θ_0, where

$$\theta_0 = \sup_{z\in E} \sup\{\theta > 0,\ \liminf_{n\to+\infty} \theta^{-n}\mathbb{P}_z(Z_n \neq \partial) > 0\}, \tag{32}$$

More precisely, we consider the following assumption.

(E2') We have $\theta_1 < \theta_0$ and

$$\begin{aligned} &\inf_{z\in E} \varphi_1(z) \geq 1,\ \sup_{z\in K} \varphi_1(z) < \infty \\ &\mathbb{E}_z(\varphi_1(Z_1)\mathbf{1}_{Z_1\neq\partial}) \leq \theta_1\varphi_1(z) + c_2\mathbf{1}_K(z),\ \forall z \in E. \end{aligned} \tag{33}$$

Proposition A.1. *Assume that there exists a positive integer n_1, positive real constants θ_1, c_1, c_2, c_3, a function $\varphi_1 : E \to \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that (E1), (E'2), (E3) and (E4) hold true. Then, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists φ_2 such that Assumption (E2) holds true.*

Proof. In a first step, we prove that, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z\in K} \theta_2^{-n}\mathbb{P}_z(Z_n \in K) \geq 1$. In a second step, we conclude by building a function φ_2 which satisfies Assumption (E2).

Step 1. Assume on the contrary that there exists $\theta_2 \in (\theta_1, \theta_0)$ such that, for all $n \geq 1$, $\inf_{z\in K} \theta_2^{-n}\mathbb{P}_z(Z_n \in K) < 1$. Then, using (E1), we deduce that, for all $n \geq n_1$,

$$1 > \inf_{z\in K} \theta_2^{-n}\mathbb{P}_z(Z_n \in K) \geq \theta_2^{-n} c_1 \mathbb{P}_\nu(Z_{n-n_1} \in K)$$

and hence that, for all $n \geq 0$,

$$\mathbb{P}_\nu(Z_n \in K) < \frac{\theta_2^{n_1}}{c_1}\theta_2^n.$$

Applying (33) iteratively, we deduce that, for all $z \in E$, for all $n \geq 1$,

$$\mathbb{E}_z(\varphi_1(Z_n)\mathbf{1}_{Z_n\neq\partial}) \leq \theta_1^n\varphi_1(z) + c_2\sum_{k=1}^{n} \theta_1^{n-k}\mathbb{P}_z(Z_{k-1} \in K).$$

Integrating with respect to ν and using the two previous inequalities, we get

$$\mathbb{E}_\nu(\varphi_1(Z_n)\mathbf{1}_{Z_n\neq\partial}) \leq \theta_1^n\nu(\varphi_1) + c_2\sum_{k=1}^{n} \theta_1^{n-k}\mathbb{P}_\nu(Z_{k-1} \in K) \leq \theta_1^n\nu(\varphi_1) + c_2\sum_{k=1}^{n} \theta_1^{n-k}\frac{\theta_2^{n_1-1}}{c_1}\theta_2^k \leq C'\theta_2^n, \tag{34}$$

for some constant C' where we used the fact that $\nu(\varphi_1) \leq \sup_K \varphi_1 < +\infty$ and $\theta_1 < \theta_2$.

For all $z \in E$ and all $n \geq 0$, we have, denoting $\tau_K = \min\{k \geq 0,\ Z_k \in K\}$ and using the strong Markov property at time τ_K,

$$\begin{aligned} \mathbb{P}_z(Z_n \neq \partial) &\leq \mathbb{E}_z(\varphi_1(Z_n)1_{Z_n\neq\partial}1_{n<\tau_K}) + \sum_{k=0}^{n} \mathbb{P}_z(\tau_K = k)\sup_{y\in K}\mathbb{P}_y(Z_{n-k} \neq \partial) \\ &\leq \theta_1^n\varphi_1(z) + \sum_{k=0}^{n} \mathbb{P}_z(\tau_K = k)c_3\mathbb{P}_\nu(Z_{n-k} \neq \partial), \end{aligned}$$

where we used (33) for the first term in the right hand side, and (E3) for the second term. Since we also have, by (33) for all $k \geq 2$ and trivially for $k = 1$,

$$\mathbb{P}_z(\tau_K = k) \leq \mathbb{P}_z(Z_{k-1} \neq \partial,\ k-1 < \tau_K) \leq \mathbb{E}_z(\varphi_1(Z_{k-1})1_{Z_{k-1}\neq\partial}1_{k-1<\tau_K}) \leq \theta_1^{k-1}\varphi_1(z),$$

and since $\mathbb{P}_\nu(Z_{n-k} \neq \partial) \leq \mathbb{E}_\nu(\varphi_1(Z_{n-k})\mathbf{1}_{Z_{n-k}\neq\partial})$, we deduce from the previous inequality and from (34) that

$$\mathbb{P}_z(Z_n \neq \partial) \leq \left(\theta_1^n + \sum_{k=0}^{n} \theta_1^{k-1} c_3 C' \theta_2^{n-k}\right)\varphi_1(z).$$

In particular, for all $z \in E$, for all $\theta > \theta_2 > \theta_1$,

$$\theta^{-n}\mathbb{P}_z(Z_n \neq \partial) \xrightarrow[n\to+\infty]{} 0,$$

then, by definition of θ_0, we get $\theta_0 \leq \theta_2$, which contradicts assumption $\theta_2 \in (\theta_1, \theta_0)$.

This concludes the proof of our first step: for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z\in K} \theta_2^{-n}\mathbb{P}_z(Z_n \in K) \geq 1$.

Step 2. Fix $\theta_2 \in (\theta_1, \theta_0)$ and let $n \geq 1$ such that $\inf_{z\in K} \theta_2^{-n}\mathbb{P}_z(Z_n \in K) \geq 1$, and define the function

$$\varphi_2 : z \in E \mapsto C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k}\mathbb{P}_z(Z_k \in K)$$

with $C_{\theta_2} := \left(\frac{1-\theta_2^n}{1-\theta_2}\right)^{-1}$. Then, $\inf_{z\in K} \varphi_2(z) > 0$ and $\sup_{z\in E} \varphi_2(z) \leq 1$. In addition,

$$\begin{aligned}\mathbb{E}_z(\varphi_2(Z_1)\mathbf{1}_{Z_1\neq\partial}) &= C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k}\mathbb{P}_z(Z_{k+1} \in K) = C_{\theta_2}\theta_2 \sum_{k=1}^{n} \theta_2^{-k}\mathbb{P}_z(Z_k \in K) \\ &= \theta_2\varphi_2(z) + C_{\theta_2}\theta_2\left(\theta_2^{-n}\mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z)\right).\end{aligned}$$

Since $\theta_2^{-n}\mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z) \geq 0$ for all $z \in K$ (by definition of n) and for all $z \in E \setminus K$ (since for such z, we have $1_K(z) = 0$), we deduce that

$$\mathbb{E}_z(\varphi_2(Z_1)\mathbf{1}_{Z_1\neq\partial}) \geq \theta_2\varphi_2(z),\ \forall z \in E.$$

This concludes the proof of Proposition A.1. □

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