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Quasi-limiting behaviour of the sub-critical Multitype Bisexual Galton-Watson Branching Process

Coralie Fritsch¹, Denis Villemonais^{1,2} and Nicolás Zaldueño³

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Abstract

We investigate the quasi-limiting behaviour of bisexual subcritical Galton-Watson branching processes. While classical subcritical Galton-Watson processes have been extensively analyzed, bisexual Galton-Watson branching processes present unique difficulties because of the lack of the branching property. To prove the existence of and convergence to one or several quasi-stationary distributions, we leverage on recent developments linking bisexual Galton-Watson branching processes extinction to the eigenvalue of a concave operator.

1 Introduction

We study the quasilimiting behaviour of bisexual subcritical Galton-Watson branching processes (bGWbp). This problem has been studied and largely solved decades ago for classical sub-critical Galton-Watson processes (see e.g. [19, 17, 2]), but remains open for its bisexual counterpart. Recent developments in the theory of multi-dimensional bGWbp [9] show that, under appropriate super-additivity assumptions, the extinction of these processes can be characterised by the eigenvalue λ^* associated to a positively homogeneous concave operator and, more precisely, by its position relatively to 1: if $\lambda^* \leq 1$, then the process is eventually extinct almost surely, while, if $\lambda^* > 1$, then, with positive probability, the process grows exponentially fast with rate equal or close to λ^* . We refer the reader to the surveys [1, 12, 16] on bGWbp and [6, 7, 10, 11] for further references. In this paper, we consider the sub-critical case $\lambda^* < 1$ and show that the process admits infinitely many quasi-stationary distributions. Under an additional polynomial moment assumption, we prove that the process admits a finitely generated family of quasi-stationary distributions and, under an additional irreducibility assumption, a unique quasi-limiting distribution for compactly supported initial distributions. Although this is similar to the classical Galton-Watson case, λ^* is not in general equal to the exponential survival rate of the process.

Let us introduce more formally our settings and assumptions. Consider $p, q \in \mathbb{N} = \{0, 1, 2, \dots\}$ with $p, q \geq 1$, a non-negative vector function $\xi : \mathbb{N}^q \rightarrow \mathbb{N}^p$ such that $\xi(0) = 0$, and a family of integrable random vectors $V = (V_{i,\cdot})_{1 \leq i \leq p}$, taking value in \mathbb{N}^q and whose expectation is denoted by \mathbb{V} . We assume that $\sum_{i=1}^q \mathbb{V}_{i,j} > 0$ for all $j \in \{1, \dots, q\}$. We consider a process $Z = (Z_n)_{n \in \mathbb{N}}$ on \mathbb{N}^p which represents the random sequence of the number of couples of each type in the population, and which evolves as follows: given $Z_{n-1} = (Z_{n-1,1}, \dots, Z_{n-1,p})$, we define for $n \geq 1$ the vector of children of the n -th generation $(W_{n,1}, \dots, W_{n,q})$ by

$$W_{n,j} = \sum_{i=1}^p \sum_{k=1}^{Z_{n-1,i}} V_{i,j}^{(k,n)}, \text{ for } 1 \leq j \leq q,$$

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where $(V^{(k,n)})_{k,n \in \mathbb{N}}$ is a family of i.i.d. copies of V . Then, we set the vector of couples in the n -th generation as

$$Z_n = (Z_{n,1}, \dots, Z_{n,p}) = \xi(W_{n,1}, \dots, W_{n,q}).$$

The function ξ is referred to as the *mating function*. For instance, if $p = 1$, $q = 2$ and if $W_{n,1}$ represents the number of females and $W_{n,2}$ the number of males in the n -th generation, classical choices for ξ are $\xi(x, y) = \min\{x, y\}$ (the *perfect fidelity* mating function) and $\xi(x, y) = x \min\{1, y\}$ (the *promiscuous* mating function). If $p = q = 2$, then $\xi(x, y) = (x, y)$ corresponds to the classical bi-dimensional Galton-Watson process.

For any probability measure μ on \mathbb{N}^p , we use the notation \mathbb{P}_μ and \mathbb{E}_μ for the law and associated expectation of the bisexual Galton-Watson process with initial number of couples Z_0 distributed according to μ . As usual, we use the notations \mathbb{P}_x and \mathbb{E}_x when $\mu = \delta_x$ for some $x \in \mathbb{N}^p$. Note that $\xi(0) = 0$ entails that 0 is absorbing for Z .

We are interested in conditions ensuring the existence of a quasi-stationary distribution for Z . We recall that a *quasi-stationary distribution* for Z is a probability measure ν_{QS} on $\mathbb{N}^p \setminus \{0\}$ such that, for all $n \geq 0$, $\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) > 0$ and

$$\mathbb{P}_{\nu_{QS}}(Z_n \in \cdot \mid Z_n \neq 0) = \nu_{QS}(\cdot).$$

It is known that (we refer the reader to the book [5] and to the surveys [18, 15] for general properties and further examples), for any quasi-stationary distribution ν_{QS} , there exists $\theta_{QS} \in (0, 1]$ such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_{QS}^n \text{ and } \mathbb{P}_{\nu_{QS}}(Z_n \in \cdot, Z_n \neq 0) = \theta_{QS}^n \nu_{QS}(\cdot).$$

In what follows, θ_{QS} is referred to as the *absorption parameter* of ν_{QS} .

Throughout the paper, we make the assumption that ξ is super-additive and sub-affine:

Assumption (S). There exists $\alpha, \beta \in \mathbb{R}_+^p$ such that

$$\alpha |x_1 + x_2| + \beta \geq \xi(x_1 + x_2) \geq \xi(x_1) + \xi(x_2), \quad \forall x_1, x_2 \in \mathbb{N}^q, \quad (1)$$

where $|\cdot|$ is the ℓ^1 -norm.

This assumption implies that

$$\mathfrak{M}(z) = \lim_{k \rightarrow \infty} \frac{\xi(\lfloor kz \rfloor)}{k}, \quad \forall z \in \mathbb{R}_+^p$$

is well defined, bounded over $\mathbb{S} := \{z \in \mathbb{R}_+^p, |z| = 1\}$, positively homogeneous (i.e. $\mathfrak{M}(az) = a\mathfrak{M}(z)$ for all $a > 0$ and all $z \in \mathbb{R}_+^p$) and concave. We refer the reader to Section 3 in [9], where other properties of this functional are derived. In addition, we assume that

Assumption (P). \mathfrak{M} is primitive, which means that there exists $n_0 \geq 1$ such that $\mathfrak{M}^m(z) > 0$ for all $m \geq n_0$ and $z \in \mathbb{R}_+^p \setminus \{0\}$, where \mathfrak{M}^m is the m -th iterate of \mathfrak{M} .

In particular, this entails that there exists $\lambda^* > 0$ and $z^* \in \mathbb{S}^* := \{z \in (\mathbb{R}_+ \setminus \{0\})^p, |z| = 1\}$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{M}^n(z)}{(\lambda^*)^n} = \mathcal{P}(z) z^* \quad (2)$$

exists, where $\mathcal{P} : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$ is strictly positive on $\mathbb{R}_+^p \setminus \{0\}$, positively homogeneous and concave (see [13] for a general development on this theory and Section 4.1 of [9] for an application to the context of bGWbp). In addition, according to Lemma 37 in [9], $((\lambda^*)^{-n} \mathcal{P}(Z_n))_{n \in \mathbb{N}}$ is a

supermartingale. In particular, $\lambda^* < 1$ implies that $(Z_n)_{n \in \mathbb{N}}$ goes extinct almost surely for any initial distribution (see [9] for details).

Our first result states that, in the sub-critical case, the process admits infinitely many quasi-stationary distributions, indexed by $\theta \in [\nu_0, 1)$, where

$$\nu_0 := \inf\{\nu > 0, \text{ such that } \mathbb{E}_z(\nu^{-T_0}) < +\infty, \forall z \in \mathbb{N}^p\} \quad (3)$$

with T_0 the extinction time.

We also define the *exponential convergence parameter* θ_0 by

$$\theta_0 = \sup_{z \in \mathbb{N}^p \setminus \{0\}} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq 0) > 0\}, \quad (4)$$

with $\sup \emptyset = -\infty$.

Proposition 1.1. *If Assumptions (S) and (P) hold true, then we have $\theta_0 \leq \nu_0 \leq \lambda^*$.*

Note that, under the assumptions of Theorem 1.3, we have $\nu_0 = \theta_0$ (take $f \equiv 1$ in (5)).

Theorem 1.2. *If Assumptions (S) and (P) hold true and $\lambda^* < 1$, then Z admits a continuum of linearly independent quasi-stationary distributions. More precisely, for any $\theta \in [\nu_0, 1)$, there exists a quasi-stationary distribution with absorption parameter θ .*

The proofs of Proposition 1.1 and Theorem 1.2 are detailed in Section 2, where we first state and prove a general result ensuring the existence of infinitely many quasi-stationary distributions for Feller processes, and then apply it to the sub-critical bGWbp. We emphasize that in the classical Galton-Watson case, much more precise results are known (we recommend to the interested reader the paper [14], where the authors prove a complete result for Galton-Watson processes and gives, in Section 3, a detailed exposition of the history of the problem, with its link to the identification of the Martin boundary of the classical Galton-Watson branching process). We also refer the reader to [8], where a general existence result for quasi-stationary distributions is obtained by a renewal/compactness argument.

We consider now the problem of existence of a finitely generated set of quasi-stationary distributions satisfying an integrability assumption and of convergence of conditional laws toward a quasi-stationary distribution. Since \mathfrak{M} is concave on \mathbb{R}_+^p , it is locally Lipschitz on \mathbb{S}^* . We add the following regularity assumption on \mathfrak{M} , where we use the notation, for all $I \subset \{1, \dots, p\}$,

$$\mathcal{Z}_I := \{x \in \mathbb{S} \text{ such that } x_i > 0 \forall i \in I, x_i = 0 \forall i \notin I\},$$

where, here and across the paper, x_i denotes the i^{th} coordinate of x .

Assumption (C). For all $I \neq \emptyset$, \mathfrak{M} is uniformly continuous over \mathcal{Z}_I .

We also consider the following moment condition, based on the exponential convergence parameter θ_0 .

Assumption (M). There exists $\varsigma > 1$ such that $(\lambda^*)^\varsigma < \theta_0$ and such that $\mathbb{E}(V_{i,j}^\varsigma) < +\infty$ for all i, j .

We emphasize that Assumption (M) implies $\lambda^* < 1$, so that the following results applies in the setting of the sub-critical bGWbp. This assumption also requires implicitly that $\theta_0 > 0$, which is the case in many situations.

The proof of the following theorem is detailed in Section 4. It is based on [3] and make use of the following function and measure spaces. Given a positive function φ on $\mathbb{N}^p \setminus \{0\}$, we set

$$L^\infty(\varphi) := \{f : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}, \|f/\varphi\|_\infty < +\infty\}, \text{ endowed with the norm } \|f\|_\varphi = \|f/\varphi\|_\infty$$

and

$$\mathcal{M}(\varphi) := \{\mu \text{ non-negative measure on } \mathbb{N}^p \setminus \{0\} \text{ such that } \mu(\varphi) < +\infty\}$$

where $\mu(\varphi) = \int_{\mathbb{N}^p \setminus \{0\}} \varphi(z) \mu(dz)$.

Finally, when Assumption (M) is enforced, we define, for all $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$, the function $Q_a : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}_+$ by

$$Q_a(z) = \mathcal{P}(z)^a.$$

Note that $\inf Q_a > 0$.

In the following theorem, we say that a process Z is aperiodic if, for all $z \in \mathbb{N}^p$, either $\mathbb{P}_z(\exists n \geq 1, Z_n = z) = 0$ or there exists $n_0(z) \geq 0$ such that, $\forall n \geq n_0(z)$, $\mathbb{P}_z(Z_n = z) > 0$.

Theorem 1.3. *We assume that Assumptions (S), (P), (C) and (M) are in place and that the process Z is aperiodic. We fix $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then there exist $\ell \geq 1$ and a family of quasi-stationary distributions $\nu_1, \dots, \nu_\ell \in \mathcal{M}(Q_a)$ with absorption parameter θ_0 for Z such that any quasi-stationary distribution $\nu \in \mathcal{M}(Q_a)$ with absorption parameter θ_0 for Z is a convex combination of ν_1, \dots, ν_ℓ . In addition, there exists a bounded function $j : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{N}$ and there exists, for each $i \in \{1, \dots, \ell\}$, a non-negative non-identically zero function $\eta_i \in L^\infty(Q_a)$ such that, for all $f \in L^\infty(Q_a)$, all $n \geq 1$ and all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\left| \theta_0^{-n} n^{-j(z)} \mathbb{E}_z(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \sum_{i=1}^{\ell} \eta_i(z) \nu_i(f) \right| \leq \alpha_n Q_a(z) \|f\|_{Q_a}, \quad (5)$$

where α_n goes to 0 when $n \rightarrow +\infty$ which does not depend on f nor on z (but may depend on a).

In addition, the set $E := \{z \in \mathbb{N}^p \setminus \{0\}, \sum_{i=1}^{\ell} \eta_i(z) = 0\}$ is finite, and any quasi-stationary distributions for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly smaller than θ_0 is supported by E .

Finally, there is no quasi-stationary distribution for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly larger than θ_0 and, for all $z \in \mathbb{N}^p \setminus \{0\}$, the conditional distribution $\mathbb{P}_z(Z_n \in \cdot \mid Z_n \neq 0)$ converges in $\mathcal{M}(Q_a)$.

Under the additional assumption that the process Z is irreducible (meaning that $\forall x, y \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_x(Z_n = y) > 0$), one obtains that, for all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\theta_0 = \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq 0) > 0\},$$

does not depend on z . In the irreducible case, we have the following result, proved in Section 5.

Theorem 1.4. *Assume that Assumptions (S), (P), (C) and (M) hold true and that the process Z is aperiodic and irreducible. We fix $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then the process Z admits a unique quasi-stationary distribution ν_{QS} in $\mathcal{M}(Q_a)$. Its absorption parameter θ_{QS} is equal to θ_0 and there exists a unique positive function η_{QS} in $L^\infty(Q_a)$, such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_0^n \quad \text{and} \quad \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) = \theta_0^n \eta_{QS}(z), \quad \forall n \geq 0, z \in \mathbb{N}^p \setminus \{0\} \quad (6)$$

and η_{QS} is lower bounded away from 0. In addition, there exist $C > 0$ and $\gamma \in (0, 1)$ such that, for any probability measure $\mu \in \mathcal{M}(Q_a)$ and any function $f \in L^\infty(Q_a)$, we have

$$|\theta_0^{-n} \mathbb{E}_\mu(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \mu(\eta_{QS}) \nu_{QS}(f)| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0 \quad (7)$$

and

$$|\mathbb{E}_\mu(f(Z_n) \mid Z_n \neq 0) - \nu_{QS}(f)| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0. \quad (8)$$

Finally, for all probability measure $\mu \in \mathcal{M}(\eta_{QS})$,

$$\mathbb{P}_\mu(Z_n \in \cdot \mid Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{TV} \nu_{QS}, \quad (9)$$

where TV refers to the total variation convergence.

Theorem 1.4 states the existence and the uniqueness of a quasi-stationary distribution ν_{QS} integrating the Lyapunov function Q_a . Moreover, (8) and (9) establish the convergence of the law of the process conditioning to the non-extinction towards ν_{QS} for all initial distribution μ integrating Q_a or the eigenfunction η_{QS} , with moreover an exponential speed of convergence for μ integrating Q_a . In particular, the convergence towards ν_{QS} holds for any Dirac mass $\mu = \delta_z$, with $z \in E$, meaning that ν_{QS} is a Yaglom limit. In addition, (7) gives the convergence of the law of the process without conditioning. In particular, taking $f \equiv 1$ in (7) leads to the speed of convergence of the extinction probability towards 0 at speed exactly θ_0 .

The proofs of the last two results rely in part on recent advances in the theory of quasi-stationary distributions, which allow to derive several properties from Foster-Lyapunov type criteria, given in our case by the following proposition.

Proposition 1.5. *If Assumptions (S), (P), (C) and (M) hold true, then, for any $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$, we have*

$$\mathbb{E}(Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = z) \leq \theta_a Q_a(z) + C_a, \quad \forall z \in \mathbb{N}^p \setminus \{0\} \quad (10)$$

for some constants $\theta_a \in [0, \theta_0)$ and $C_a > 0$.

Remark 1.1. We already observed that $\theta_0 \leq \lambda^*$. One also checks that, at least under the assumptions of Theorem 1.4, if \mathcal{P} is not linear, then $\theta_0 < \lambda^*$. Indeed, under these assumptions, let ν_{QS} be the quasi-stationary distribution from Theorem 1.4, so that

$$\theta_0^n \nu_{QS}(\mathcal{P}) = \sum_{z \in \mathbb{N}^p \setminus \{0\}} \nu_{QS}(\{z\}) \mathbb{E}_z(\mathcal{P}(Z_n)).$$

Since under the assumptions of Theorem 1.4, the process is irreducible, the support of ν_{QS} is $\mathbb{N}^p \setminus \{0\}$ and hence \mathcal{P} is concave not affine on the support of ν_{QS} . Using (8), we deduce that there exist $z_0 \in \mathbb{N}^p \setminus \{0\}$ and $n \geq 1$ such that \mathcal{P} is concave not affine on the support of $\mathbb{P}_{z_0}(Z_n \in \cdot)$, so that, by Jensen (strict) inequality,

$$\mathbb{E}_{z_0}(\mathcal{P}(Z_n)) < \mathcal{P}(\mathbb{E}_{z_0}(Z_n)) \leq \mathcal{P}(\mathfrak{M}^n(z_0)) = (\lambda^*)^n \mathcal{P}(z_0),$$

where we used $\mathbb{E}_{z_0}(Z_n) \leq \mathfrak{M}^n(z_0)$ (see Lemma 27 in [9]) and the fact that \mathcal{P} is superadditive (since \mathfrak{M} is superadditive by Assumption (S) and by definition of \mathcal{P} given by (2)) for the last inequality, and the definition of \mathcal{P} (see (2)) for the last equality. Similarly, we have $\mathbb{E}_z(\mathcal{P}(Z_n)) \leq (\lambda^*)^n \mathcal{P}(z)$ for all $z \in \mathbb{N}^p \setminus \{0\}$. We conclude that

$$\theta_0^n \nu_{QS}(\mathcal{P}) < \sum_{z \in \mathbb{N}^p \setminus \{0\}} \nu_{QS}(\{z\}) (\lambda^*)^n \mathcal{P}(z) = (\lambda^*)^n \nu_{QS}(\mathcal{P}).$$

This implies that $\theta_0 < \lambda^*$ when \mathcal{P} is not linear.

It is also known that $\mathfrak{M}(z) = \sup_{k \geq 1} \frac{1}{k} \mathbb{E}_{kz}(Z_1)$ (see Corollary 7 in [9]), and the same calculation as above shows that, if $\mathbb{E}_{z_0}(Z_1) < \mathfrak{M}(z_0)$ for some $z_0 \in \mathbb{N}^p$, then $\theta_0 < \lambda^*$ (even if \mathcal{P} is linear).

Remark 1.2. When $\theta_0 = \lambda^*$, Assumption (M) requires polynomial moments for V , with exponents that can be arbitrarily close to 1. We leave as an open question whether, in this case,

$L \log L$ type criteria can be obtained, as it is the case in the classical Galton-Watson case (for which $\theta_0 = \lambda^*$).

In the situation where $\theta_0 < \lambda^*$, one can not choose ς arbitrarily close to 1 in Assumption (M), since it imposes $\varsigma > \frac{|\log \theta_0|}{|\log \lambda^*|}$. We leave as an open question whether $\mathbb{E}(V_{i,j}^\varsigma) < +\infty$ for some $\varsigma \leq \frac{|\log \theta_0|}{|\log \lambda^*|}$ may be sufficient to obtain the conclusion of Theorem 1.3 or 1.4, at least in some particular cases.

In Section 2, we state a general result for the existence of infinitely many quasi-stationary distributions, then we prove Proposition 1.1 and Theorem 1.2. Section 3 is devoted to the proof of Proposition 1.5. Theorems 1.3 and 1.4 are proved in Sections 4 and 5 respectively. For the proofs of the last two theorems, we make use of Assumption (E) of [4] which is recalled in Appendix. In Appendix we also state a general result implying Assumption (E) of [4].

2 A simple criterion for the existence of infinitely many quasi-stationary distributions and application to the bGWbp

In Section 2.1, we first state and prove a result on the existence of a continuum of quasi-stationary distributions for general sub-Markov kernels on Polish spaces. In Section 2, we apply this result to the bGWbp.

2.1 General setting

Let (X, d) be a Polish space endowed with a σ -algebra \mathcal{X} that contains the topology induced by d . Let K be a sub-Markov kernel on X , which means that K is a non-negative kernel on X such that $K(x, X) \leq 1$ for all $x \in X$. We assume that K has Feller regularity, meaning that the associated operator $K : f \mapsto Kf := \int_X K(\cdot, dy) f(y)$ is a well defined functional from $C_b(X)$ to $C_b(X)$ (the set of continuous bounded functions on X). We say that a probability measure μ on X is a quasi-stationary distribution for K if, for some $\lambda \in (0, 1]$,

$$\int_X \mu(dx) K(x, \cdot) = \lambda \mu(\cdot).$$

Our aim is to prove that K admits infinitely many quasi-stationary distributions under mild assumptions.

In the following theorem, we say that a sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ tends to infinity if it eventually leaves any compact set: for all compact subset L of X , there exists $n_L \geq 0$ such that, for all $n \geq n_L$, $x_n \notin L$. Similarly, we say that a real functional on X tends to infinity when $x \rightarrow +\infty$ if, for all $R > 0$, there exists a compact subset L_R of X such that the functional is bounded below by R on $X \setminus L_R$. We also define the iterated kernels K^n as usual by

$$K^0(x, A) = \mathbf{1}_{x \in A}, \quad K^{\ell+1}(x, A) = \int_X K^\ell(x, dy) K(y, A) \quad \forall x \in X, A \in \mathcal{X}, \ell \geq 0.$$

Theorem 2.1. *Assume that there exists $a > 0$ and $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ tending to infinity such that*

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^\ell(x_n, X) < +\infty, \quad \forall n \geq 0 \tag{11}$$

and let

$$a_0 := \inf\{a > 0, \text{ s.t. there exists } (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} \text{ tending to infinity such that (11) holds true.}\} \tag{12}$$

Assume also that there exists $a_1 > a_0$ such that

$$\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) \xrightarrow{x \rightarrow +\infty} +\infty. \quad (13)$$

Then K admits an infinite family of quasi-stationary distributions $(\mu_\lambda)_{\lambda \in [a_0, a_1]}$, with $\int_X \mu_\lambda(dx) K(x, \cdot) = \lambda \mu_\lambda$ for all $\lambda \in [a_0, a_1]$.

Proof of Theorem 2.1. Fix $\lambda \in (a_0, a_1)$. Since $\lambda > a_0$ then there exists $(x_n)_{n \in \mathbb{N}} \in X^\mathbb{N}$ tending to infinity such that, for all $n \geq 0$, the probability measure

$$\mu_{\lambda, n} = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)}$$

is well defined. We have, for all $n \geq 0$,

$$\begin{aligned} \mu_{\lambda, n} K(\cdot) &:= \int_X \mu_{\lambda, n}(dx) K(x, \cdot) = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-(\ell+1)} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot) - \lambda \delta_{x_n}(\cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &\leq \lambda \mu_{\lambda, n}(\cdot). \end{aligned} \quad (14)$$

We deduce that, for all $\ell \geq 1$,

$$\mu_{\lambda, n} K^\ell \leq \lambda^\ell \mu_{\lambda, n}.$$

In particular, we obtain that

$$\int_X \mu_{\lambda, n}(dx) \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) = \sum_{\ell=0}^{\infty} a_1^{-\ell} \mu_{\lambda, n} K^\ell(X) \leq \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{a_1} \right)^\ell = \frac{a_1}{a_1 - \lambda}. \quad (15)$$

Since we assumed that $\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X)$ goes to infinity when $x \rightarrow \infty$, this and Prokhorov's Theorem imply that the sequence of measures $(\mu_{\lambda, n})_{n \in \mathbb{N}}$ is relatively compact for the weak convergence topology.

Let μ_λ be any adherent point of $(\mu_{\lambda, n})_{n \in \mathbb{N}}$, and observe that μ_λ is a probability measure. Since K is assumed to preserve bounded continuous functions, we deduce that, up to a subsequence and for the weak convergence topology,

$$\mu_\lambda K = \lim_{n \rightarrow +\infty} \mu_{\lambda, n} K.$$

But we computed (see (14))

$$\mu_{\lambda, n} K = \lambda \mu_{\lambda, n} - \frac{\lambda \delta_{x_n}}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)},$$

where, by assumption,

$$\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X) \geq \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x_n, X) \xrightarrow{n \rightarrow +\infty} +\infty.$$

We deduce that $\lim_{n \rightarrow +\infty} \mu_{\lambda,n} K = \lim_{n \rightarrow +\infty} \lambda \mu_{\lambda,n}$, which is equal to $\lambda \mu_\lambda$. We thus proved that $\mu_\lambda K = \lambda \mu_\lambda$, which means that μ_λ is a quasi-stationary distribution for K with absorption parameter λ .

It remains to prove that there exists a quasi-stationary distribution with absorption parameter a_0 . Since the set of probability measures μ_λ , $\lambda \in (a_0, (a_0 + a_1)/2)$, satisfies (15) with a uniform upper bound, we deduce that it is relatively compact. One checks as above that any limit point, when $\lambda \rightarrow a_0$, of this family is a quasi-stationary distribution for K with absorption parameter a_0 . This concludes the proof of Theorem 2.1. \square

Remark 2.1. One easily checks from the proof that the condition that K preserves bounded continuous functions is stronger than necessary. Actually, it would be sufficient to assume that K sends a generating family of bounded continuous functions into a family of bounded continuous functions. For instance, it is sufficient to assume that K sends compactly supported continuous functions to bounded continuous functions.

Remark 2.2. One can easily extend this result to the general setting of non-conservative kernels K . More precisely, assume that there exists a positive function h on X such that $Kh \leq ch$ for some $c > 0$. Then the kernel defined by

$$\tilde{K}(x, A) = \frac{1}{ch(x)} \int_A K(x, dy) h(y), \quad \forall x \in X, \quad \forall A \in \mathcal{X},$$

is a sub-Markov kernel to which Theorem 2.1 may apply. Note that the assumptions would translate into: for any continuous function $f : X \rightarrow \mathbb{R}$, $x \in X \mapsto \frac{1}{h(x)} \int_X K(x, dy) h(y) f(y)$ is continuous; for all $n \geq 0$, $\sum_{\ell=0}^{\infty} a_0^{-\ell} \int_X K^\ell(x_n, dy) h(y) < +\infty$ and $\frac{1}{h(x)} \sum_{\ell=0}^{\infty} a_1^{-\ell} \int_X K^\ell(x, dy) h(y) \xrightarrow{x \rightarrow +\infty} +\infty$. In addition, the resulting quasi-stationary measures may not be a finite measure if h is not lower bounded (we only obtain that $\int_X \mu_\lambda(dy) h(y) < +\infty$ for all $\lambda \in [a_0, a_1]$). On the other hand, if h is lower bounded away from 0, then μ_λ is a finite measure.

2.2 Proof of Proposition 1.1 and Theorem 1.2

In the context of bGWbp, we choose $K(x, dy) = \mathbb{P}_x(Z_1 \in dy, Z_1 \neq 0)$ for all $x \in X = \mathbb{N}^p \setminus \{0\}$. We first prove Proposition 1.1, as well as that $a_0 \leq v_0$, where a_0 is defined in Theorem 2.1 and v_0 is defined in (3). Then we prove Theorem 1.2.

First, from their definitions, it is clear that $v_0 \leq 1$ and $\theta_0 \leq 1$. For all $\theta < \theta_0$, there exist $v \in (\theta, 1)$ and $z \in \mathbb{N}^p \setminus \{0\}$ such that $\liminf_{n \rightarrow \infty} v^n \mathbb{P}_z(Z_n \neq 0) > 0$. For such v and z , we have $\mathbb{E}_z(v^{-T_0}) = \sum_{\ell=0}^{\infty} v^{-\ell} \mathbb{P}_z(T_0 = \ell)$ with

$$\begin{aligned} \liminf_{\ell \rightarrow \infty} v^{-\ell} \mathbb{P}_z(T_0 = \ell) &= \liminf_{\ell \rightarrow \infty} v^{-\ell} (\mathbb{P}_z(Z_{\ell-1} \neq 0) - \mathbb{P}_z(Z_\ell \neq 0)) \\ &= (v^{-1} - 1) \liminf_{\ell \rightarrow \infty} v^{-\ell} \mathbb{P}_z(Z_\ell \neq 0) > 0. \end{aligned}$$

Then $\mathbb{E}_z(v^{-T_0}) = \infty$ and hence $v \geq v_0$. We deduce that $v_0 \geq \theta$ and hence $v_0 \geq \theta_0$.

In order to prove that $a_0 \leq v_0$, we first observe that, for all $a > v > v_0$, for all $z \in \mathbb{N}^p \setminus \{0\}$ and

$$\sum_{\ell \geq 0} a^{-\ell} K^\ell(z, X) = \sum_{\ell \geq 0} a^{-\ell} \mathbb{P}_z(T_0 \geq \ell + 1) \leq \sum_{\ell \geq 0} a^{-\ell} v^{\ell+1} \mathbb{E}_z(v^{-T_0}) \leq C_{a,v} \mathbb{E}_z(v^{-T_0}) < \infty.$$

This shows that $a_0 \leq v_0$.

Using the fact that \mathcal{P} is lower bounded away from 0 on $\mathbb{N}^p \setminus \{0\}$ (as \mathcal{P} is strictly positive and positively homogeneous) and that K vanishes at 0, we deduce that there exists a constant $c > 0$ such that, for all $\ell \geq 1$ and all $x \in \mathbb{N}^p$,

$$K^\ell(x, X) \leq \frac{1}{c} \mathbb{E}_x(\mathcal{P}(Z_\ell)).$$

In addition, $(\lambda^*)^{-n} \mathcal{P}(Z_n)$ is a supermartingale (see Lemma 37 in [9]), so that, for any $a \in (\lambda^*, 1)$,

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^{\ell}(x, X) \leq \frac{1}{c} \sum_{\ell=0}^{\infty} a^{-\ell} (\lambda^*)^{\ell} \mathcal{P}(x) < +\infty.$$

This shows that $v_0 \leq \lambda^*$, then Proposition 1.1 holds.

Since \mathfrak{M} is primitive by Assumption (P), Theorem 6 in [9] implies that, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $c_0 > 0$ and $k_0 \geq 1$ such that for all $k \geq k_0$

$$\mathbb{P}(Z_{n_0} \geq c_0(1, \dots, 1) \mid Z_0 = kz) > 0.$$

Applying this result for z taken in each element of the canonical basis of \mathbb{N}^p and using the super-additivity of ξ , one deduces that there exist uniform constants $r > 0$ and $c'_0 > 0$ such that

$$\inf_{|z| > r} \mathbb{P}(Z_{n_0} \geq c'_0(1, \dots, 1) \mid Z_0 = z) > 0.$$

Using the Markov property and by iteration, this implies that, for all $n \geq 1$,

$$\mathbb{P}_z(Z_n = 0) \xrightarrow{|z| \rightarrow +\infty} 0.$$

In particular, we deduce from Fatou's Lemma that, for any $a_1 \in (0, 1)$,

$$\sum_{\ell=0}^{\infty} a_1^{-\ell} \mathbb{P}_z(Z_{\ell} \neq 0) \xrightarrow{|z| \rightarrow +\infty} +\infty.$$

and hence (13) holds true for any $a_1 \in (a_0, 1) \supset (v_0, 1)$.

We deduce that Theorem 2.1 applies to the subcritical bGWbp. Since quasi-stationary distributions with different absorption parameters are linearly independent, this proves Theorem 1.2.

3 Proof of Proposition 1.5

We start with the following technical lemma on the regularity of \mathcal{P} .

Lemma 3.1. *Assume that Assumptions (S), (P), (C) and (M) hold true. Then, for all $I \subset \{1, \dots, p\}$, $I \neq \emptyset$, the operator \mathcal{P} restricted to \mathcal{Z}_I is uniformly continuous.*

Proof of Lemma 3.1. Since $\mathcal{P} \circ \mathfrak{M}^{n_0} = (\lambda^*)^{n_0} \mathcal{P}$, it is sufficient to prove that $\mathcal{P} \circ \mathfrak{M}^{n_0}$ is uniformly continuous on \mathcal{Z}_I . But $\mathfrak{M}^{n_0}(\mathcal{Z}_I) \subset \mathfrak{M}^{n_0}(\mathbb{S})$ which is relatively compact in $(\mathbb{R}_+ \setminus \{0\})^p$ (since \mathfrak{M} is primitive and bounded by assumption), and \mathcal{P} is concave and thus locally Lipschitz in $(\mathbb{R}_+ \setminus \{0\})^p$, thus it is sufficient to prove that \mathfrak{M}^{n_0} is uniformly continuous on \mathcal{Z}_I .

Let us first observe that \mathfrak{M} is locally uniformly continuous on $(\varepsilon + \mathbb{R}_+ \setminus \{0\}) \mathcal{Z}_I$ for any $\varepsilon > 0$, since \mathfrak{M} is uniformly continuous on \mathcal{Z}_I , bounded on $\mathbb{S} \supset \mathcal{Z}_I$, and positively homogeneous. In order to conclude, it remains to prove that there exist $J \subset \{1, \dots, p\}$ and $\varepsilon > 0$ such that $\mathfrak{M}(\mathcal{Z}_I) \subset (\varepsilon + \mathbb{R}_+ \setminus \{0\}) \mathcal{Z}_J$ (the result then follows by induction). Let $x \in \mathcal{Z}_I$ and let $J = \{i \in \{1, \dots, p\}, \mathfrak{M}(x)_i > 0\}$. Then, for all $y \in \mathcal{Z}_I$, there exists $\delta > 0$ such that $\delta x \leq y$, so that, using the fact that \mathfrak{M} is increasing and homogeneous, $\mathfrak{M}(y) \geq \delta \mathfrak{M}(x)$ and hence $J \subset \{i \in \{1, \dots, p\}, \mathfrak{M}(y)_i > 0\}$. Reciprocally, $\{i \in \{1, \dots, p\}, \mathfrak{M}(y)_i > 0\} \subset J$, so that $\mathfrak{M}(y) \in \mathbb{R}_+ \mathcal{Z}_J$. Since on \mathbb{S} , the norm of \mathfrak{M} is bounded from below away from 0 and from above, we have $\mathfrak{M}(\mathcal{Z}_I) \subset [\varepsilon_1, \varepsilon_2] \mathcal{Z}_J$ for some $\varepsilon_1, \varepsilon_2 > 0$. This implies that \mathfrak{M}^2 is uniformly continuous on \mathcal{Z}_I . We conclude by iteration. \square

Let us now proceed with the proof of Proposition 1.5 in two steps: first we show that, for all $z \in \mathbb{R}_+^p \setminus \{0\}$,

$$\limsup_{n \rightarrow +\infty} \frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] = (\lambda^*)^a Q_a(z); \quad (16)$$

Second, we extend this to a uniform convergence in z by using the regularity of \mathcal{P} .

Step 1. As \mathcal{P} is positively homogeneous, we have for any $z \in \mathbb{R}_+^p \setminus \{0\}$,

$$\begin{aligned} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] &= \mathbb{E} \left[\left(\mathcal{P} \circ \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)^a \right] \\ &= |n|^a \mathbb{E} \left[\left(\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right]. \end{aligned}$$

According to Lemma 30 in [9], we have

$$\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathfrak{M}(z).$$

Let $I \subset \{1, \dots, p\}$ be such that $\mathfrak{M}(z) \in \mathbb{R}_+^* \mathcal{Z}_I$, then the above convergence entails that, almost surely, for n large enough, $\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \in \mathbb{R}_+^* \mathcal{Z}_J$ for some $J \supset I$. Using Corollary 7 in [9], we have, for all $n \geq 1$,

$$\mathfrak{M}(z) \geq \frac{1}{n} \mathbb{E} [Z_1 \mid Z_0 = \lfloor nz \rfloor] = \frac{1}{n} \mathbb{E} \left[\xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right],$$

so that we also have $J \subset I$ almost surely.

Since \mathcal{P} restricted to $\mathbb{R}_+^* \mathcal{Z}_I$ is continuous since it is concave, and since $\mathfrak{M} \circ \mathcal{P} = \lambda^* \mathcal{P}$ by (2), we deduce that

$$\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathcal{P}(\mathfrak{M}(z)) = \lambda^* \mathcal{P}(z).$$

In addition, since ξ is sub-affine, since $\frac{1}{n} \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)}$ is bounded in L^ζ by Assumption (M) and \mathcal{P} being positively homogeneous, we deduce that $\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)$ is also bounded in L^ζ (where $\zeta > a$), and we deduce from the dominated convergence theorem that

$$\mathbb{E} \left[\left(\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right] \xrightarrow[n \rightarrow +\infty]{} (\lambda^*)^a \mathcal{P}(z)^a.$$

We deduce that (16) holds true. This concludes the first step of the proof.

Step 2. Fix a non-empty $I \subset \{1, \dots, p\}$. Let $\varepsilon_1 > 0$ be a small number that will be fixed later and let $z^1, \dots, z^{m_1} \in (1 + \varepsilon_1) \mathcal{Z}_I$, $m_1 = m(\varepsilon_1) \geq 1$, be such that, for all $z \in \mathcal{Z}_I$, there exists $k = k(z, \varepsilon_1) \in \{1, \dots, m_1\}$ such that $z \leq z^k$ and $|z - z^k| \leq 2\varepsilon_1$. The existence of this finite family z^1, \dots, z^{m_1} easily follows from a compactness argument. Let $n_1 = n_1(\varepsilon_1)$ be large enough so that, for all $n \geq n_1$, for all $k \in \{1, \dots, m_1\}$,

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz^k \rfloor] \leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^k)^a.$$

Then, since Q_a is non-decreasing and since Z_1 (with $Z_0 = \lfloor nz \rfloor$) is stochastically non-decreasing with respect to z , we deduce that, for all $n \geq n_1$ and all $z \in \mathcal{Z}_I$,

$$\begin{aligned} \frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] &\leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^{k(z, \varepsilon_1)})^a \\ &\leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z)^a \left(\sup_{x, y \in \mathcal{Z}_I, |(1 + \varepsilon_1)x - y| \leq 2\varepsilon_1} \frac{\mathcal{P}((1 + \varepsilon_1)x)}{\mathcal{P}(y)} \right)^a. \end{aligned} \quad (17)$$

Since \mathcal{P} is uniformly continuous on \mathcal{Z}_I and lower bounded away from 0 on \mathcal{Z}_I , we deduce that there exists ε_1 small enough so that

$$\theta_a := (1 + \varepsilon_1)^{a+1} (\lambda^*)^a \left(\sup_{x, y \in \mathcal{Z}_I, |(1 + \varepsilon_1)x - y| \leq 2\varepsilon_1} \frac{\mathcal{P}(x)}{\mathcal{P}(y)} \right)^a < \theta_0.$$

This and (17) implies that

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] \leq \theta_a \mathcal{P}(z)^a.$$

Since any $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})$ can be written as $n \frac{z}{|z|}$ with $n = |z|$ and $\frac{z}{|z|} \in \mathcal{Z}_I$, we deduce that, for any such z with $|z| \geq n_1$,

$$\frac{1}{|z|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = z] \leq \theta_a \mathcal{P}(z/|z|)^a.$$

Finally, the homogeneity of \mathcal{P} allows us to conclude that Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})$. Since there are only finitely many subsets I of $\{1, \dots, p\}$, Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\}$.

4 Proof of Theorem 1.3

The proof of the first statement relies on Theorem 4.1 [3] and in part on some arguments of the proof of Theorem 5.1 therein. The main additional difficulty is that our reference exponential parameter θ_0 is *a priori* larger than the one considered in this reference. We overcome this difficulty in Lemma 4.1 below. Another difficulty is that a key part of the argument relies on the assumption (E1-4) in [4], which does not apply directly when the Lyapunov parameter θ_a (in Proposition 1.5) is only assumed strictly smaller than θ_0 . To overcome this difficulty, we prove in Proposition A.1 that this last condition actually entails the assumption (E1-4) in [4].

Fix $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$ and set $H = Q_a / \inf Q_a$. According to Proposition 1.5 and since $H(z)$ tends to ∞ when $|z| \rightarrow +\infty$, there exists $\theta_a < \theta_0$ such that

$$\limsup_{|z| \rightarrow +\infty} \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} \leq \theta_a < \gamma := (\theta_a + \theta_0)/2 < \theta_0. \quad (18)$$

In particular, there exist only finitely many communication classes E_1, \dots, E_{k_0} , $k_0 \geq 1$, for the process $(Z_n)_{n \in \mathbb{N}}$ such that

$$\forall i \in \{1, \dots, k_0\}, \exists z \in E_i, \text{ such that } \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} > \gamma.$$

Note that $k_0 \neq 0$ (otherwise $\theta_0 \leq \gamma$). For all $i \in \{1, \dots, k_0\}$, we define $H_i = H \mathbf{1}_{E_i}$ and denote by $Y^{(i)}$ the process with state space $E_i \cup \{0\}$ defined by

$$Y_n^{(i)} = \begin{cases} Z_n & \text{if } Z_n \in E_i \\ 0 & \text{otherwise.} \end{cases}$$

For all $i \in \{1, \dots, k_0\}$, we denote by $\theta_{0,i}$ the absorption parameter of $Y^{(i)}$:

$$\theta_{0,i} = \sup_{z \in E_i} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Y_n^{(i)} \neq 0) > 0\}$$

and we set

$$\bar{\theta} = \max_{i \in \{1, \dots, k_0\}} \theta_{0,i}.$$

We assume that E_1, \dots, E_k , $k \in \{1, \dots, k_0\}$, are the communication classes such that $\theta_{0,i} = \bar{\theta}$, so that $\max_{i \geq k+1} \theta_{0,i} < \bar{\theta}$.

We define the set

$$E'_0 := \mathbb{N}^p \setminus \left(\{0\} \cup \bigcup_{i=1}^{k_0} E_i \right).$$

By definition of E_1, \dots, E_{k_0} , we have

$$\forall z \in E'_0, \frac{\mathbb{E}_z(H(Z_1)\mathbf{1}_{Z_1 \neq 0})}{H(z)} \leq \gamma. \quad (19)$$

In what follows, we make use of the following crucial lemma.

Lemma 4.1. *We have $\bar{\theta} = \theta_0$.*

Proof of Lemma 4.1. Our aim is to show that, for all $\rho > \bar{\theta} \vee \gamma$ and all $z \in \mathbb{N}^p$, we have

$$\liminf_{n \rightarrow +\infty} \rho^{-n} \mathbb{P}_z(Z_n \neq 0) = 0. \quad (20)$$

This shows that $\bar{\theta} \vee \gamma \geq \theta_0$ and hence $\bar{\theta} \geq \theta_0$, while the converse inequality is trivial, concluding the proof of Lemma 4.1.

For all $i \neq j \in \{1, \dots, k_0\}$, we have

$$\forall z \in E_j, \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) > 0 \text{ or } \forall z \in E_j, \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) = 0$$

If the condition on the left is satisfied, we write $E_i < E_j$, otherwise we write $E_i \not< E_j$. This defines a partial order on $\{1, \dots, k_0\}$. For all $i \in \{1, \dots, k_0\}$, we denote by $c(i)$ the maximal length of sets of distinct indices $i_1 = i, \dots, i_c$ such that $E_{i_c} < \dots < E_{i_2} < E_{i_1}$. We also define, for all $\ell \geq 1$, the set

$$E_0^\ell := \{x \in E'_0, \exists i \in \{1, \dots, k_0\} \text{ with } c(i) = \ell \text{ and } x \rightarrow E_i, \text{ and } x \not\rightarrow E_j \forall j \text{ such that } c(j) > \ell\},$$

where $x \rightarrow E_i$ means that $\mathbb{P}_x(T_{E_i} < \infty) > 0$, while $x \not\rightarrow E_j$ means that $\mathbb{P}_x(T_{E_j} < \infty) = 0$. We also set

$$E_0^0 := E'_0 \setminus \bigcup_{\ell=1}^{\max_i c(i)} E_0^\ell.$$

We obtain (20) for all $Z \in \mathbb{N}^p \setminus \{0\}$ by proving that, for any fixed $\rho > \bar{\theta} \vee \gamma$,

$$\mathbb{E}_z(H(Z_n)\mathbf{1}_{Z_n \neq 0}) \leq C_\rho \rho^n H(z), \quad \forall z \in \mathbb{N}^p \setminus \{0\} \quad (21)$$

where C_ρ does not depend on z . This last inequality is immediate for all $z \in E_0^0$ (see beginning of Step 1 below) and is proved for all $z \in E_i \cup E_0^{c(i)}$, $i \in \{1, \dots, k_0\}$, by induction on $c(i)$.

Step 1. We first observe that, for all $\ell \geq 0$ and all $z \in E_0^\ell$, using (19),

$$\mathbb{E}_z \left(H(Z_n) \mathbf{1}_{Z_n \in E_0^\ell} \right) \leq \gamma^n H(z) \leq \rho^n H(z). \quad (22)$$

In particular, since $E_0^0 \cup \{0\}$ is a closed set, this shows (21) for all $z \in E_0^0$.

We now prove that, for all $i \in \{1, \dots, k_0\}$ and all $z \in E_i$,

$$\mathbb{E}_z(H(Z_n)1_{Z_n \in E_i}) \leq C\rho^n H(z). \quad (23)$$

In order to do so, we consider the set

$$K_i = \{z \in E_i, \frac{\mathbb{E}_z(H(Z_1)1_{Z_1 \neq 0})}{H(z)} > \gamma\}.$$

For all $z \in E_i$, since $\rho > \bar{\theta}$ and by definition of $\bar{\theta}$,

$$\liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Y_n^{(i)} \neq 0) = 0. \quad (24)$$

Since K_i is finite according to (18), we deduce that there exists a constant $C > 0$ such that

$$\mathbb{P}_z(Z_n \notin E_i) = \mathbb{P}_z(Y_n^{(i)} \neq 0) \leq C\rho^n, \forall z \in K_i. \quad (25)$$

In what follows, the constant C may change from line to line. Moreover, denoting $\tau_{K_i} = \min\{\ell \geq 0, Z_\ell \in K_i\}$, we deduce from the definition of H and of K_i that

$$\mathbb{P}_z(Z_n \notin E_i \text{ and } n < \tau_{K_i}) \leq \mathbb{E}_z(H(Z_n)1_{Z_n \notin E_i \text{ and } n < \tau_{K_i}}) \leq \gamma^n H(z), \forall z \in E_i. \quad (26)$$

Hence, for all $z \in E_i$ and all $n \geq 0$, we have, using the strong Markov property at time τ_{K_i} ,

$$\begin{aligned} \mathbb{P}_z(Z_n \notin E_i) &\leq \mathbb{P}_z(Z_n \notin E_i \text{ and } n < \tau_{K_i}) + \sum_{\ell=0}^n \mathbb{P}_z(\tau_{K_i} = \ell) \sup_{y \in K_i} \mathbb{P}_y(Z_{n-\ell} \notin E_i) \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \mathbb{P}_z(\tau_{K_i} = \ell) C\rho^{n-\ell} \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \mathbb{P}_z(Z_\ell \neq 0 \text{ and } \ell-1 < \tau_{K_i}) C\rho^{n-\ell} \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \gamma^{\ell-1} H(z) C\rho^{n-\ell} \\ &\leq C\rho^n H(z), \end{aligned}$$

where C may depend on ρ and γ . Finally, iterating (10) and using the last inequality, we deduce that

$$\begin{aligned} \mathbb{E}_z(H(Z_n)1_{Z_n \notin E_i}) &\leq \theta_a^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a} \theta_a^{n-\ell-1} \mathbb{P}_z(Z_\ell \neq 0) \\ &\leq C\gamma^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a} \gamma^{n-\ell-1} C\rho^\ell H(z) \\ &\leq C\rho^n H(z), \end{aligned}$$

where C may depend on ρ and γ . This proves (23) for all $z \in E_i$.

Step 2. Assume that there exists $c \geq 0$ such that (21) holds true for all $z \in E_0^0, \dots, E_0^c$ and all $z \in E_i$ with $c(i) \leq c$ (Steps 1 shows that this is true for $c = 0$). If $c = \max_{i \in \{1, \dots, k_0\}} c(i)$, then there is nothing to prove. Otherwise, let $i \in \{1, \dots, k_0\}$ such that $c(i) = c + 1$.

Denote by T_i the first exit time from E_i by $(Z_n)_{n \in \mathbb{N}}$ and let us use a similar computation as in the end of Step 1. For all $z \in E_i$ and all $n \geq 0$, we have

$$\begin{aligned}\mathbb{E}_z(H(Z_n)1_{Z_n \neq 0}) &= \mathbb{E}_z(H(Z_n)1_{Z_n \in E_i}) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i=\ell} \mathbb{E}_{X_\ell}(H(Z_{n-\ell})1_{Z_{n-\ell} \neq 0})) \\ &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i=\ell} C\rho \rho^{n-\ell} H(X_\ell))\end{aligned}$$

where we used (23), the fact that

$$Z_{T_i} \in \{0\} \cup \bigcup_{\ell \leq c} E_0^\ell \cup \bigcup_{c(j) \leq c} E_j \text{ almost surely,}$$

and the induction assumption. We then obtain

$$\begin{aligned}\mathbb{E}_z(H(Z_n)1_{Z_n \neq 0}) &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i > \ell-1} C\rho \rho^{n-\ell} (\theta_a H(X_{\ell-1}) + C_a)) \\ &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i > \ell-1} C\rho \rho^{n-\ell} (\theta_a + C_a) H(X_{\ell-1}))\end{aligned}$$

where we used the Markov property at time $\ell - 1$ and Proposition 1.5. We deduce from (23) that

$$\begin{aligned}\mathbb{P}_z(Z_n \neq 0) &\leq C\rho^n H(z) + C\rho(\theta_a + C_a)C \sum_{\ell=0}^n \rho^n H(z) \\ &\leq C(n+1)\rho^n H(z).\end{aligned}$$

Since this is true for any $\rho > \bar{\theta} \vee \gamma$, this proves (21) for all $z \in E_i$ with $c(i) = c + 1$.

The same arguments show that this also holds true for all $z \in E_0^{c+1}$.

This concludes the proof of Lemma 4.1. □

From now on, we assume without loss of generality that there exists $k \in \{1, \dots, k_0\}$ such that, for all $i \in \{k+1, \dots, k\}$, $\theta_{0,i} < \bar{\theta}$, and, for all $i \in \{1, \dots, k\}$, $\theta_{0,i} = \bar{\theta} = \theta_0$. For any $i \in \{k+1, \dots, k_0\}$, we have, fixing any $\rho \in (\max_{j \in \{k'+1, \dots, k\}} \theta_{0,j}, \bar{\theta})$ and using the same method as in Step 1 from the proof of Lemma 4.1,

$$\mathbb{E}_z(H(Z_n)1_{n < T_i}) \leq C\rho \rho^n H(z), \quad \forall z \in E_i,$$

where we recall that T_i is the first exit time from E_i by $(Z_n)_{n \in \mathbb{N}}$ and where $C\rho$ is a positive constant which may depend on ρ . Setting

$$E_0 := E_0' \cup \bigcup_{i=k+1}^{k_0} E_i,$$

one easily checks, using the last inequality and (19), that, for any $r \in (\gamma \vee \rho, \bar{\theta})$, there exists $C_r > 0$ such that

$$\mathbb{E}_z(H(Z_n)1_{Z_n \in E_0}) \leq C_r r^n H(z), \quad \forall z \in E_0.$$

In order to apply Theorem 4.1 in [3], which implies the first part of Theorem 1.3, it remains to prove that, for all $i \in \{1, \dots, k'\}$ and all $z \in E_i$, we have

$$|\theta_0^{-n} \mathbb{E}_z(f(Z_n)1_{Z_n \in E_i}) - h_i(z)\mu_i(f)| \leq C\alpha^n H_i(x) \|f\|_{H_i}, \quad \forall n \geq 0, \quad \forall f \in L^\infty(H_i), \quad (27)$$

for some $C > 0, \alpha \in (0, 1)$, some non-negative non-zero function h_i and some probability measure μ_i on E_i . In order to do so, it is sufficient to check Assumptions (E1-4) in Section 2 of [4]. These conditions are recalled in the Appendix and we will more precisely check Assumptions (E1), (E2'), (E3) and (E4), which is sufficient to obtain (E1-4) according to Proposition A.1. Fixing $i \in \{1, \dots, k'\}$, we define the set

$$K_i := \left\{ z \in E_i, \frac{\mathbb{E}_z(H_i(Z_1)\mathbf{1}_{Z_1 \neq 0})}{H_i(z)} > \gamma \right\},$$

which is finite according to (18).

Since E_i is a communication class and the process is assumed to be aperiodic, there exists $n_1^i \geq 0$ such that

$$c_1^i := \inf_{x, y \in K_i} \mathbb{P}_x(Z_{n_1^i} = y) > 0.$$

Note that $K_i \neq \emptyset$, since it would imply that $\theta_{0,i} = \theta_0 \leq \gamma$. Fix $z_i \in K_i$. Then, for all $z \in K_i$,

$$\mathbb{P}_z(Z_{n_1^i} \in \cdot) \geq \mathbb{P}_z(Z_{n_1^i} = z_i) \delta_{z_i}(\cdot) \geq c_1^i \nu_i(\cdot \cap K),$$

for $\nu_i = \delta_{z_i}$. This entails (E1). In addition, taking $\varphi_1 = H_i$ and $\theta_1 = \gamma < \theta_0 = \theta_{0,i}$, the condition (E'2) is an immediate consequence of the definition of K_i .

For all $x, y \in K_i$ and $n \geq 0$, we have, using the Markov property at time n_1^i and the fact that $\mathbb{P}_x(Z_n \neq 0)$ is non-increasing in n (as $\{0\}$ is absorbing),

$$\mathbb{P}_x(Z_n \neq 0) \geq \mathbb{P}_x(Z_{n \vee n_1^i} \neq 0) \geq \mathbb{P}_x(Z_{n_1^i} = y) \mathbb{P}_y(Z_{n \vee n_1^i - n_1^i} \neq 0) \geq c_1^i \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic. This concludes the proof of the first part of Theorem 1.3.

To conclude the proof of Theorem 1.3, it remains to prove that the set $E = \{x \in \mathbb{N}^p \setminus \{0\}, \sum_{i \in I} \eta_i(x) = 0\}$ is finite and that any quasi-stationary distribution with absorption parameter strictly smaller than θ_0 is supported by E .

We first observe that Theorem 4.1 in [3] also implies that $j \equiv 0$ on the set E . Assume on the contrary that E is not finite and fix $z_0 \in \mathbb{N}^p \setminus \{0\}$ such that $\sum_{i=1}^\ell \eta_i(z_0) > 0$ and $j(z_0) = 0$ (existence of such a point z_0 is guaranteed by Proposition 2.3 in [3]). We observe that

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_{z_0}(Z_n \neq 0) > 0.$$

Then, by primitivity of \mathfrak{M} , there exists n_0 such that for all $x \in \mathbb{N}^p$ we have $\mathfrak{M}^{n_0}(x) > 0$. This implies that for $k > 0$ large enough we have $\mathbb{P}_{kx}(Z_{n_0} \geq z_0) > 0$. Indeed, if it is not the case, we can conclude using Theorem 6 in [9] that

$$\mathfrak{M}^{n_0}(x) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_{n_0} | Z_0 = kx)}{k} \geq \lim_{k \rightarrow +\infty} \frac{z_0}{k} = 0,$$

which is a contradiction. Then, we have for any z large enough $\mathbb{P}_z(Z_{n_0} \geq z_0) > 0$. Hence, using the fact that E is not finite and hence contains arbitrarily large points, there exists $z \in E$ and $z'_0 \geq z_0$ such that $\mathbb{P}_z(Z_{n_0} = z'_0) > 0$. By super-additivity of ξ , we have $\mathbb{P}_{z'_0}(Z_n \neq 0) \geq \mathbb{P}_z(Z_n \neq 0)$ and hence

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_z(Z_{n_0+n} \neq 0) > 0.$$

This is not compatible with the definition of E and (5). We have thus proved by contradiction that E is finite.

Let ν_{QS} be a quasi-stationary distribution for Z in $\mathcal{M}(Q_a)$ with absorption parameter $\theta_{QS} < \theta_0$. Then (5) implies that $\sum_{i=1}^{\ell} \nu_{QS}(\eta_i) = 0$ and hence the support of ν_{QS} is included in E . Since E is finite and since E is a closed set for the Markov chain, we deduce, by usual finite dimensional linear algebra arguments, that ν_{QS} belongs to the convex hull of any maximal family of positive left eigenvector of the transition matrix of Z restricted to E . This concludes the proof of the penultimate assertion of Theorem 1.3.

To conclude, it remains to prove that there are no quasi-stationary distributions for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly larger than θ_0 . This is a direct consequence of (5). Indeed, since j is bounded, integrating this inequality with respect to a quasi-stationary distribution ν_{QS} with absorption parameter θ_{QS} , and taking $f \equiv 1$, shows that

$$\limsup_{n \rightarrow +\infty} \theta_0^{-n} n^{-\|j\|_{\infty}} \theta_{QS}^n \mathbb{P}_z(Z_n \neq 0) < +\infty,$$

and hence $\theta_{QS} \leq \theta_0$.

5 Proof of Theorem 1.4

Fix $a \in (1, \zeta)$ such that $(\lambda^*)^a < \theta_0$, and consider the Lyapunov type property with constants θ_a and C_a from Proposition 1.5. We make use of Section 2 in [4]. In a first step, we check that Assumption E therein (recalled in the Appendix below) holds true for the process under consideration. In a second step, we prove (6), (7) and (8) and (9) using the results of [4].

Step 1. Assumption (E) holds true. Fix $\theta_1 \in (\theta_a, \theta_0)$. As \mathcal{P} is strictly positive on $\mathbb{R}_+^p \setminus \{0\}$ and positively homogeneous, we can fix $r_1 \geq p$ large enough so that $Q_a(z) \geq \frac{C_a}{\theta_1 - \theta_a}$ for all $|z| \geq r_1$, so that

$$\mathbb{E}_z(Q_a(Z_1)) \leq \theta_a Q_a(z) + C_a \leq \theta_1 Q_a(z), \quad \forall z \in \mathbb{N}^p \text{ such that } |z| \geq r_1. \quad (28)$$

Let us set $K = \{z \in \mathbb{N}^p \setminus \{0\}, |z| \leq r_1\}$ and $\varphi_1 = Q_a / \inf Q_a$ (note that $\inf Q_a > 0$ by Assumption (P)).

We deduce from the irreducibility and aperiodicity properties that there exists $n_1 \geq 0$ such that

$$c_1 := \inf_{x, y \in K} \mathbb{P}_x(Z_{n_1} = y) > 0. \quad (29)$$

Setting $\nu = \delta_{(1, \dots, 1)}$, this entails property (E1).

We have $\inf \varphi_1 = 1$, so that the first line of (E2) is satisfied. Moreover, according to (28), the third line of (E2) also holds true.

Choosing $\theta_2 \in (\theta_1, \theta_0)$, we deduce from the definition of θ_0 and the irreducibility of Z that

$$\theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty,$$

where e_i is the i^{th} element of the canonical basis of \mathbb{N}^p . Since, by the super-additivity of ξ , the process Z is stochastically non-decreasing in the initial condition and since, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $i \in \{1, \dots, p\}$ such that $z \geq e_i$, we deduce that

$$\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \geq \theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty.$$

Let $n \geq 1$ be large enough so that $\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \geq 1$, and set, for all $z \in \mathbb{N}^p \setminus \{0\}$

$$\varphi_2(z) = \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0).$$

We have, for all $z \in \mathbb{N}^p \setminus \{0\}$ and using the Markov property at time 1,

$$\begin{aligned}
\left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq 0}) &= \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_{k+1} \neq 0) \\
&= \theta_2 \sum_{k=1}^n \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0) \\
&= \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \varphi_2(z) + \theta_2 (\theta_2^{-n} \mathbb{P}_z(Z_n \neq 0) - 1) \\
&\geq \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \varphi_2(z).
\end{aligned}$$

As a consequence the second and fourth lines of (E2) are satisfied. Note also that we have $\inf_{\mathbb{N}^p \setminus \{0\}} \varphi_2 \geq \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}}$.

For all $x, y \in K$ and $n \geq 0$, we have, using the Markov property at time n_1 and the fact that $\mathbb{P}_x(Z_n \neq 0)$ is non-increasing in n (as $\{0\}$ is absorbing),

$$\mathbb{P}_x(Z_n \neq 0) \geq \mathbb{P}_x(Z_{n \vee n_1} \neq 0) \geq \mathbb{P}_x(Z_{n_1} = y) \mathbb{P}_y(Z_{n \vee n_1 - n_1} \neq 0) \geq c_1 \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic and irreducible.

We have thus proved that Assumption (E) holds true for Z absorbed when it leaves $\mathbb{N}^p \setminus \{0\}$. This concludes the first step.

Step 2. Conclusion of the proof. Theorem 2.1 in [4] implies that there exists a unique quasi-stationary distribution ν_{QS} in $\mathcal{M}(Q_a)$ and that (8) holds true (using the fact that in our case, φ_2 is lower bounded away from 0). Theorem 2.3 in this reference also implies that there exists an associated non-negative eigenfunction $\eta_{QS} \in L^\infty(Q_a)$ with eigenvalue $\theta_{QS} > 0$, where θ_{QS} is the absorption parameter associated to ν_{QS} , such that $\inf_K \eta_{QS} > 0$ and

$$\theta_{QS}^{-n} \mathbb{P}_\cdot(Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{L^\infty(Q_a)} \eta_{QS}(\cdot). \quad (30)$$

Since the process is irreducible, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_z(Z_n = (1, \dots, 1)) > 0$ and hence

$$\theta_{QS}^n \eta_{QS}(z) = \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) \geq \eta_{QS}((1, \dots, 1)) \mathbb{P}_z(Z_n = (1, \dots, 1)) > 0,$$

so that η_{QS} is positive. We thus proved (6) (for the absorption parameter θ_{QS} which we prove below to be equal to θ_0). Finally, since ξ is super-additive, $\theta_{QS}^{-n} \mathbb{P}_z(Z_n \neq 0)$ and hence η_{QS} increases with z , so that η_{QS} is lower bounded away from 0. Corollary 2.7 in [4] shows that (7) holds true, and, taking $f \equiv 1$, also implies that $\theta_{QS} = \theta_0$. Finally, (9) is an immediate consequence of Corollaire 2.11 in [4]. This concludes the proof of Theorem 1.4.

A Sufficient criterion for the exponential convergence to a quasi-stationary distribution

In Section 2 of [4], the authors state that Assumption (E) is sufficient to prove exponential convergence toward a quasi-stationary distribution. Let us recall this assumption, with the notations and settings of the present paper.

In what follows, we consider a process $(Z_n)_{n \in \mathbb{N}}$ evolving in a measurable state space $E \cup \{\partial\}$, where $\partial \notin E$ is an absorbing point.

Assumption (E). There exist a positive integer n_1 , positive real constants $\theta_1, \theta_2, c_1, c_2, c_3$, two functions $\varphi_1, \varphi_2 : E \rightarrow \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that

(E1) (*Local Dobrushin coefficient*). $\forall z \in K$,

$$\mathbb{P}_z(Z_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(E2) (*Global Lyapunov criterion*). We have $\theta_1 < \theta_2$ and

$$\begin{aligned} \inf_{z \in E} \varphi_1(z) &\geq 1, \sup_{z \in K} \varphi_1(z) < \infty \\ \inf_{z \in K} \varphi_2(z) &> 0, \sup_{z \in E} \varphi_2(z) \leq 1, \\ \mathbb{E}_z(\varphi_1(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\leq \theta_1 \varphi_1(z) + c_2 \mathbf{1}_K(z), \forall z \in E \\ \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\geq \theta_2 \varphi_2(z), \forall z \in E. \end{aligned}$$

(E3) (*Local Harnack inequality*). We have

$$\sup_{n \in \mathbb{Z}_+} \frac{\sup_{y \in K} \mathbb{P}_y(Z_n \neq \partial)}{\inf_{y \in K} \mathbb{P}_y(Z_n \neq \partial)} \leq c_3.$$

(E4) (*Aperiodicity*). For all $z \in K$, there exists $n_4(z)$ such that, for all $n \geq n_4(z)$,

$$\mathbb{P}_z(Z_n \in K) > 0.$$

Our aim is to show that the condition involving the function φ_2 can be replaced by a condition involving θ_0 , where

$$\theta_0 = \sup_{z \in E} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq \partial) > 0\}, \quad (31)$$

More precisely, we consider the following assumption.

(E2') We have $\theta_1 < \theta_0$ and

$$\begin{aligned} \inf_{z \in E} \varphi_1(z) &\geq 1, \sup_{z \in K} \varphi_1(z) < \infty \\ \mathbb{E}_z(\varphi_1(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\leq \theta_1 \varphi_1(z) + c_2 \mathbf{1}_K(z), \forall z \in E. \end{aligned} \quad (32)$$

Proposition A.1. Assume that there exist a positive integer n_1 , positive real constants θ_1, c_1, c_2, c_3 , a function $\varphi_1 : E \rightarrow \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that (E1), (E2'), (E3) and (E4) hold true. Then, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists φ_2 such that Assumption (E2) holds true.

Proof. In a first step, we prove that, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$. In a second step, we conclude by building a function φ_2 which satisfies Assumption (E2).

Step 1. Assume on the contrary that there exists $\theta_2 \in (\theta_1, \theta_0)$ such that, for all $n \geq 1$, $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) < 1$. Then, using (E1), we deduce that, for all $n \geq n_1$,

$$1 > \inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq \theta_2^{-n} c_1 \mathbb{P}_\nu(Z_{n-n_1} \in K)$$

and hence that, for all $n \geq 0$,

$$\mathbb{P}_v(Z_n \in K) < \frac{\theta_2^{n_1}}{c_1} \theta_2^n.$$

Applying (32) iteratively, we deduce that, for all $z \in E$, for all $n \geq 1$,

$$\mathbb{E}_z(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial}) \leq \theta_1^n \varphi_1(z) + c_2 \sum_{k=1}^n \theta_1^{n-k} \mathbb{P}_z(Z_{k-1} \in K).$$

Integrating with respect to v and using the two previous inequalities, we get

$$\mathbb{E}_v(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial}) \leq \theta_1^n v(\varphi_1) + c_2 \sum_{k=1}^n \theta_1^{n-k} \mathbb{P}_v(Z_{k-1} \in K) \leq \theta_1^n v(\varphi_1) + c_2 \sum_{k=1}^n \theta_1^{n-k} \frac{\theta_2^{n_1-1}}{c_1} \theta_2^k \leq C' \theta_2^n, \quad (33)$$

for some constant C' where we used the fact that $v(\varphi_1) \leq \sup_K \varphi_1 < +\infty$ and $\theta_1 < \theta_2$.

For all $z \in E$ and all $n \geq 0$, we have, denoting $\tau_K = \min\{k \geq 0, Z_k \in K\}$ and using the strong Markov property at time τ_K ,

$$\begin{aligned} \mathbb{P}_z(Z_n \neq \partial) &\leq \mathbb{E}_z(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial} \mathbf{1}_{n < \tau_K}) + \sum_{k=0}^n \mathbb{P}_z(\tau_K = k) \sup_{y \in K} \mathbb{P}_y(Z_{n-k} \neq \partial) \\ &\leq \theta_1^n \varphi_1(z) + \sum_{k=0}^n \mathbb{P}_z(\tau_K = k) c_3 \mathbb{P}_v(Z_{n-k} \neq \partial), \end{aligned}$$

where we used (32) for the first term in the right hand side, and (E3) for the second term. Since we also have, by (32) and for all $k \geq 1$ and trivially for $k = 1$,

$$\mathbb{P}_z(\tau_K = k) \leq \mathbb{P}_z(Z_{k-1} \neq \partial, k-1 < \tau_K) \leq \mathbb{E}_z(\varphi_1(Z_{k-1}) \mathbf{1}_{Z_{k-1} \neq \partial} \mathbf{1}_{k-1 < \tau_K}) \leq \theta_1^{k-1} \varphi_1(z),$$

and since $\mathbb{P}_v(Z_{n-k} \neq \partial) \leq \mathbb{E}_v(\varphi_1(Z_{n-k}) \mathbf{1}_{Z_{n-k} \neq \partial})$, we deduce from the previous inequality and from (33) that

$$\mathbb{P}_z(Z_n \neq \partial) \leq \left(\theta_1^n + \sum_{k=0}^n \theta_1^{k-1} c_3 C' \theta_2^{n-k} \right) \varphi_1(z).$$

In particular, for all $z \in E$, for all $\theta > \theta_2 > \theta_1$,

$$\theta^{-n} \mathbb{P}_z(Z_n \neq \partial) \xrightarrow{n \rightarrow +\infty} 0,$$

then, by definition of θ_0 , we get $\theta_0 \leq \theta_2$, which contradicts assumption $\theta_2 \in (\theta_1, \theta_0)$.

This concludes the proof of our first step: for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$.

Step 2. Fix $\theta_2 \in (\theta_1, \theta_0)$ and let $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$, and define the function

$$\varphi_2 : z \in E \mapsto C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_k \in K)$$

with $C_{\theta_2} := \left(\frac{1-\theta_2^n}{1-\theta_2} \right)^{-1}$. Then, $\inf_{z \in K} \varphi_2(z) > 0$ and $\sup_{z \in E} \varphi_2(z) \leq 1$. In addition,

$$\begin{aligned} \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &= C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_{k+1} \in K) = C_{\theta_2} \theta_2 \sum_{k=1}^n \theta_2^{-k} \mathbb{P}_z(Z_k \in K) \\ &= \theta_2 \varphi_2(z) + C_{\theta_2} \theta_2 (\theta_2^{-n} \mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z)). \end{aligned}$$

Since $\theta_2^{-n} \mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z) \geq 0$ for all $z \in K$ (by definition of n) and for all $z \in E \setminus K$ (since for such z , we have $\mathbf{1}_K(z) = 0$), we deduce that

$$\mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) \geq \theta_2 \varphi_2(z), \quad \forall z \in E.$$

This concludes the proof of Proposition A.1. \square

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