



HAL
open science

Quasi-limiting behaviour of the sub-critical Multitype Bisexual Galton-Watson Branching Process

Coralie Fritsch, Denis Villemonais, Nicolás Zaldueño

► **To cite this version:**

Coralie Fritsch, Denis Villemonais, Nicolás Zaldueño. Quasi-limiting behaviour of the sub-critical Multitype Bisexual Galton-Watson Branching Process. 2024. hal-04628438v1

HAL Id: hal-04628438

<https://hal.science/hal-04628438v1>

Preprint submitted on 28 Jun 2024 (v1), last revised 6 Sep 2024 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Quasi-limiting behaviour of the sub-critical Multitype Bisexual Galton-Watson Branching Process

Coralie Fritsch¹, Denis Villemonais^{1,2} and Nicolás Zaldueño³

June 27, 2024

Abstract

We investigate the quasi-limiting behaviour of bisexual subcritical Galton-Watson branching processes. While classical subcritical Galton-Watson processes have been extensively analyzed, bisexual Galton-Watson branching processes present unique difficulties because of the lack of the branching property. To prove the existence of and convergence to one or several quasi-stationary distributions, we leverage on recent developments linking bisexual Galton-Watson branching processes extinction to the eigenvalue of a concave operator.

1 Introduction

We study the quasilimiting behaviour of bisexual subcritical Galton-Watson branching processes (bGWbp). This problem has been studied and largely solved decades ago for classical sub-critical Galton-Watson processes (see e.g. [19, 17, 2]), but remains open for its bisexual counterpart. Recent developments in the theory of multi-dimensional bGWbp [9] show that, under appropriate super-additivity assumptions, the extinction of these processes can be characterised by the eigenvalue λ^* associated to a positively homogeneous concave operator and, more precisely, by its position relatively to 1: if $\lambda^* \leq 1$, then the process is eventually extinct almost surely, while, if $\lambda^* > 1$, then, with positive probability, the process grows exponentially fast with rate equal or close to λ^* . We refer the reader to the surveys [1, 12, 16] on bGWbp and [6, 7, 10, 11] for further references. In this paper, we consider the sub-critical case $\lambda^* < 1$ and show that the process admits infinitely many quasi-stationary distributions. Under an additional polynomial moment assumption, we prove that the process admits a finitely generated family of quasi-stationary distributions and, under an additional irreducibility assumption, a unique quasi-limiting distribution for compactly supported initial distributions. Although this is similar to the classical Galton-Watson case, λ^* is not in general equal to the exponential survival rate of the process.

Let us introduce more formally our settings and assumptions. Consider $p, q \in \mathbb{N} = \{0, 1, 2, \dots\}$ with $p, q \geq 1$, a non-negative vector function $\xi : \mathbb{N}^q \rightarrow \mathbb{N}^p$ such that $\xi(0) = 0$, and a family of integrable random vectors $V = (V_{i,\cdot})_{1 \leq i \leq p}$, taking value in \mathbb{N}^q and whose expectation is denoted by \mathbb{V} . We assume that $\sum_{i=1}^q \mathbb{V}_{i,j} > 0$ for all $j \in \{1, \dots, q\}$. We consider a process $Z = (Z_n)_{n \in \mathbb{N}}$ on \mathbb{N}^p which represents the random sequence of the number of couples of each type in the population, and which evolves as follows: given $Z_{n-1} = (Z_{n-1,1}, \dots, Z_{n-1,p})$, we define for $n \geq 1$ the vector of children of the n -th generation $(W_{n,1}, \dots, W_{n,q})$ by

$$W_{n,j} = \sum_{i=1}^p \sum_{k=1}^{Z_{n-1,i}} V_{i,j}^{(k,n)}, \text{ for } 1 \leq j \leq q,$$

¹Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

²Institut universitaire de France (IUF)

³MISTEA, Université de Montpellier, INRAE, Institut Agro, Montpellier, France

where $(V^{(k,n)})_{k,n \in \mathbb{N}}$ is a family of i.i.d. copies of V . Then, we set the vector of couples in the n -th generation as

$$Z_n = (Z_{n,1}, \dots, Z_{n,p}) = \xi(W_{n,1}, \dots, W_{n,q}).$$

The function ξ is referred to as the *mating function*. For instance, if $p = 1$, $q = 2$ and if $W_{n,1}$ represents the number of females and $W_{n,2}$ the number of males in the n -th generation, classical choices for ξ are $\xi(x, y) = \min\{x, y\}$ (the *perfect fidelity* mating function) and $\xi(x, y) = x \min\{1, y\}$ (the *promiscuous* mating function). If $p = q = 2$, then $\xi(x, y) = (x, y)$ corresponds to the classical bi-dimensional Galton-Watson process.

For any probability measure μ on \mathbb{N}^p , we use the notation \mathbb{P}_μ and \mathbb{E}_μ for the law and associated expectation of the bisexual Galton-Watson process with initial number of couples Z_0 distributed according to μ . As usual, we use the notations \mathbb{P}_x and \mathbb{E}_x when $\mu = \delta_x$ for some $x \in \mathbb{N}^p$. Note that $\xi(0) = 0$ entails that 0 is absorbing for Z .

We are interested in conditions ensuring the existence of a quasi-stationary distribution for Z . We recall that a *quasi-stationary distribution* for Z is a probability measure ν_{QS} on $\mathbb{N}^p \setminus \{0\}$ such that, for all $n \geq 0$, $\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) > 0$ and

$$\mathbb{P}_{\nu_{QS}}(Z_n \in \cdot \mid Z_n \neq 0) = \nu_{QS}(\cdot).$$

It is known that (we refer the reader to the book [5] and to the surveys [18, 15] for general properties and further examples), for any quasi-stationary distribution ν_{QS} , there exists $\theta_{QS} \in (0, 1]$ such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_{QS}^n \text{ and } \mathbb{P}_{\nu_{QS}}(Z_n \in \cdot, Z_n \neq 0) = \theta_{QS}^n \nu_{QS}(\cdot).$$

In what follows, θ_{QS} is referred to as the *absorption parameter* of ν_{QS} .

Throughout the paper, we make the assumption that ξ is super-additive and sub-affine:

Assumption (S). There exists $\alpha, \beta \in \mathbb{R}_+^p$ such that

$$\alpha |x_1 + x_2| + \beta \geq \xi(x_1 + x_2) \geq \xi(x_1) + \xi(x_2), \quad \forall x_1, x_2 \in \mathbb{N}^q, \quad (1)$$

where $|\cdot|$ is the ℓ^1 -norm.

This assumption implies that

$$\mathfrak{M}(z) = \lim_{k \rightarrow \infty} \frac{\xi(\lfloor kz \rfloor)}{k}, \quad \forall z \in \mathbb{R}_+^p$$

is well defined, bounded over $\mathbb{S} := \{z \in \mathbb{R}_+^p, |z| = 1\}$, positively homogeneous (i.e. $\mathfrak{M}(az) = a\mathfrak{M}(z)$ for all $a > 0$ and all $z \in \mathbb{R}_+^p$) and concave. We refer the reader to Section 3 in [9], where other properties of this functional are derived. In addition, we assume that

Assumption (P). \mathfrak{M} is primitive, which means that there exists $n_0 \geq 1$ such that $\mathfrak{M}^m(z) > 0$ for all $m \geq n_0$ and $z \in \mathbb{R}_+^p \setminus \{0\}$, where \mathfrak{M}^m is the m -th iterate of \mathfrak{M} .

In particular, this entails that there exists $\lambda^* > 0$ and $z^* \in \mathbb{S}^* := \{z \in (\mathbb{R}_+ \setminus \{0\})^p, |z| = 1\}$ such that the limit

$$\lim_{n \rightarrow \infty} \frac{\mathfrak{M}^n(z)}{(\lambda^*)^n} = \mathcal{P}(z) z^* \quad (2)$$

exists, where $\mathcal{P} : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$ is strictly positive on $\mathbb{R}_+^p \setminus \{0\}$, positively homogeneous and concave (see [13] for a general development on this theory and Section 4.1 of [9] for an application to the context of bGWbp). In addition, according to Lemma 37 in [9], $((\lambda^*)^{-n} \mathcal{P}(Z_n))_{n \in \mathbb{N}}$ is a

supermartingale. In particular, $\lambda^* < 1$ implies that $(Z_n)_{n \in \mathbb{N}}$ goes extinct almost surely for any initial distribution (see [9] for details).

Our first result states that, in the sub-critical case, the process admits infinitely many quasi-stationary distributions, indexed by $\theta \in [\nu_0, 1)$, where

$$\nu_0 := \inf\{\nu > 0, \text{ such that } \mathbb{E}_z(\nu^{-T_0}) < +\infty, \forall z \in \mathbb{N}^p\} \quad (3)$$

with T_0 the extinction time.

We also define the *exponential convergence parameter* θ_0 by

$$\theta_0 = \sup_{z \in \mathbb{N}^p \setminus \{0\}} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq 0) > 0\}, \quad (4)$$

with $\sup \emptyset = -\infty$.

Proposition 1.1. *If Assumptions (S) and (P) hold true, then we have $\theta_0 \leq \nu_0 \leq \lambda^*$.*

Note that, under the assumptions of Theorem 1.3, we have $\nu_0 = \theta_0$ (take $f \equiv 1$ in (5)).

Theorem 1.2. *If Assumptions (S) and (P) hold true and $\lambda^* < 1$, then Z admits a continuum of linearly independent quasi-stationary distributions. More precisely, for any $\theta \in [\nu_0, 1)$, there exists a quasi-stationary distribution with absorption parameter θ .*

The proofs of Proposition 1.1 and Theorem 1.2 are detailed in Section 2, where we first state and prove a general result ensuring the existence of infinitely many quasi-stationary distributions for Feller processes, and then apply it to the sub-critical bGWbp. We emphasize that in the classical Galton-Watson case, much more precise results are known (we recommend to the interested reader the paper [14], where the authors prove a complete result for Galton-Watson processes and gives, in Section 3, a detailed exposition of the history of the problem, with its link to the identification of the Martin boundary of the classical Galton-Watson branching process). We also refer the reader to [8], where a general existence result for quasi-stationary distributions is obtained by a renewal/compactness argument.

We consider now the problem of existence of a finitely generated set of quasi-stationary distributions satisfying an integrability assumption and of convergence of conditional laws toward a quasi-stationary distribution. Since \mathfrak{M} is concave on \mathbb{R}_+^p , it is locally Lipschitz on \mathbb{S}^* . We add the following regularity assumption on \mathfrak{M} , where we use the notation, for all $I \subset \{1, \dots, p\}$,

$$\mathcal{X}_I := \{x \in \mathbb{S} \text{ such that } x_i > 0 \forall i \in I, x_i = 0 \forall i \notin I\},$$

where, here and across the paper, x_i denotes the i^{th} coordinate of x .

Assumption (C). For all $I \neq \emptyset$, \mathfrak{M} is uniformly continuous over \mathcal{X}_I .

We also consider the following moment condition, based on the exponential convergence parameter θ_0 .

Assumption (M). There exists $\zeta > 1$ such that $(\lambda^*)^\zeta < \theta_0$ and such that $\mathbb{E}(V_{i,j}^\zeta) < +\infty$ for all i, j .

We emphasize that Assumption (M) implies $\lambda^* < 1$, so that the following results applies in the setting of the sub-critical bGWbp. This assumption also requires implicitly that $\theta_0 > 0$, which is the case in many situations.

The proof of the following theorem is detailed in Section 4. It is based on [3] and make use of the following function and measure spaces. Given a positive function φ on $\mathbb{N}^p \setminus \{0\}$, we set

$$L^\infty(\varphi) := \{f : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}, \|f/\varphi\|_\infty < +\infty\}, \text{ endowed with the norm } \|f\|_\varphi = \|f/\varphi\|_\infty$$

and

$$\mathcal{M}(\varphi) := \{\mu \text{ non-negative measure on } \mathbb{N}^p \setminus \{0\} \text{ such that } \mu(\varphi) < +\infty\}$$

where $\mu(\varphi) = \int_{\mathbb{N}^p \setminus \{0\}} \varphi(z) \mu(dz)$.

Finally, when Assumption (M) is enforced, we define, for all $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$, the function $Q_a : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}_+$ by

$$Q_a(z) = \mathcal{P}(z)^a.$$

Note that $\inf Q_a > 0$.

In the following theorem, we say that a process Z is aperiodic if, for all $z \in \mathbb{N}^p$, either $\mathbb{P}_z(\exists n \geq 1, Z_n = z) = 0$ or there exists $n_0(z) \geq 0$ such that, $\forall n \geq n_0(z)$, $\mathbb{P}_z(Z_n = z) > 0$.

Theorem 1.3. *We assume that Assumptions (S), (P), (C) and (M) are in place and that the process Z is aperiodic. We fix $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then there exist $\ell \geq 1$ and a family of quasi-stationary distributions $\nu_1, \dots, \nu_\ell \in \mathcal{M}(Q_a)$ with absorption parameter θ_0 for Z such that any quasi-stationary distribution $\nu \in \mathcal{M}(Q_a)$ with absorption parameter θ_0 for Z is a convex combination of ν_1, \dots, ν_ℓ . In addition, there exists a bounded function $j : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{N}$ and there exists, for each $i \in \{1, \dots, \ell\}$, a non-negative non-identically zero function $\eta_i \in L^\infty(Q_a)$ such that, for all $f \in L^\infty(Q_a)$, all $n \geq 1$ and all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\left| \theta_0^{-n} n^{-j(z)} \mathbb{E}_z(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \sum_{i=1}^{\ell} \eta_i(z) \nu_i(f) \right| \leq \alpha_n Q_a(z) \|f\|_{Q_a}, \quad (5)$$

where α_n goes to 0 when $n \rightarrow +\infty$ which does not depend on f nor on z (but may depend on a).

In addition, the set $E := \{z \in \mathbb{N}^p \setminus \{0\}, \sum_{i=1}^{\ell} \eta_i(z) = 0\}$ is finite, and any quasi-stationary distributions for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly smaller than θ_0 is supported by E .

Finally, there is no quasi-stationary distribution for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly larger than θ_0 and, for all $z \in \mathbb{N}^p \setminus \{0\}$, the conditional distribution $\mathbb{P}_z(Z_n \in \cdot \mid Z_n \neq 0)$ converges in $\mathcal{M}(Q_a)$.

Under the additional assumption that the process Z is irreducible (meaning that $\forall x, y \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_x(Z_n = y) > 0$), one obtains that, for all $z \in \mathbb{N}^p \setminus \{0\}$,

$$\theta_0 = \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq 0) > 0\},$$

does not depend on z . In the irreducible case, we have the following result, proved in Section 5.

Theorem 1.4. *Assume that Assumptions (S), (P), (C) and (M) hold true and that the process Z is aperiodic and irreducible. We fix $a \in (1, \varsigma)$ such that $(\lambda^*)^a < \theta_0$.*

Then the process Z admits a unique quasi-stationary distribution ν_{QS} in $\mathcal{M}(Q_a)$. Its absorption parameter θ_{QS} is equal to θ_0 and there exists a unique positive function η_{QS} in $L^\infty(Q_a)$, such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_0^n \quad \text{and} \quad \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) = \theta_0^n \eta_{QS}(z), \quad \forall n \geq 0, z \in \mathbb{N}^p \setminus \{0\} \quad (6)$$

and η_{QS} is lower bounded away from 0. In addition, there exist $C > 0$ and $\gamma \in (0, 1)$ such that, for any probability measure $\mu \in \mathcal{M}(Q_a)$ and any function $f \in L^\infty(Q_a)$, we have

$$\left| \theta_0^{-n} \mathbb{E}_\mu(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \mu(\eta_{QS}) \nu_{QS}(f) \right| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0 \quad (7)$$

and

$$\left| \mathbb{E}_\mu(f(Z_n) \mid Z_n \neq 0) - \nu_{QS}(f) \right| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0. \quad (8)$$

Finally, for all probability measure $\mu \in \mathcal{M}(\eta_{QS})$,

$$\mathbb{P}_\mu(Z_n \in \cdot \mid Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{TV} \nu_{QS}, \quad (9)$$

where TV refers to the total variation convergence.

Theorem 1.4 states the existence and the uniqueness of a quasi-stationary distribution ν_{QS} integrating the Lyapunov function Q_a . Moreover, (8) and (9) establish the convergence of the law of the process conditioning to the non-extinction towards ν_{QS} for all initial distribution μ integrating Q_a or the eigenfunction η_{QS} , with moreover an exponential speed of convergence for μ integrating Q_a . In particular, the convergence towards ν_{QS} holds for any Dirac mass $\mu = \delta_z$, with $z \in E$, meaning that ν_{QS} is a Yaglom limit. In addition, (7) gives the convergence of the law of the process without conditioning. In particular, taking $f \equiv 1$ in (7) leads to the speed of convergence of the extinction probability towards 0 at speed exactly θ_0 .

The proofs of the last two results rely in part on recent advances in the theory of quasi-stationary distributions, which allow to derive several properties from Foster-Lyapunov type criteria, given in our case by the following proposition.

Proposition 1.5. *If Assumptions (S), (P), (C) and (M) hold true, then, for any $a \in (1, \zeta)$ such that $(\lambda^*)^a < \theta_0$, we have*

$$\mathbb{E}(Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = z) \leq \theta_a Q_a(z) + C_a, \quad \forall z \in \mathbb{N}^p \setminus \{0\} \quad (10)$$

for some constants $\theta_a \in [0, \theta_0)$ and $C_a > 0$.

Remark 1.1. We already observed that $\theta_0 \leq \lambda^*$. One also checks that, at least under the assumptions of Theorem 1.4, if \mathcal{P} is not linear, then $\theta_0 < \lambda^*$. Indeed, under these assumptions, let ν_{QS} be the quasi-stationary distribution from Theorem 1.4, so that

$$\theta_0^n \nu_{QS}(\mathcal{P}) = \sum_{z \in \mathbb{N}^p \setminus \{0\}} \nu_{QS}(\{z\}) \mathbb{E}_z(\mathcal{P}(Z_n)).$$

Since under the assumptions of Theorem 1.4, the process is irreducible, the support of ν_{QS} is $\mathbb{N}^p \setminus \{0\}$ and hence \mathcal{P} is concave not affine on the support of ν_{QS} . Using (8), we deduce that there exist $z_0 \in \mathbb{N}^p \setminus \{0\}$ and $n \geq 1$ such that \mathcal{P} is concave not affine on the support of $\mathbb{P}_{z_0}(Z_n \in \cdot)$, so that, by Jensen (strict) inequality,

$$\mathbb{E}_{z_0}(\mathcal{P}(Z_n)) < \mathcal{P}(\mathbb{E}_{z_0}(Z_n)) \leq \mathcal{P}(\mathfrak{M}^n(z_0)) = (\lambda^*)^n \mathcal{P}(z_0),$$

where we used $\mathbb{E}_{z_0}(Z_n) \leq \mathfrak{M}^n(z_0)$ (see Lemma 27 in [9]) and the fact that \mathcal{P} is superadditive (since \mathfrak{M} is superadditive by Assumption (S) and by definition of \mathcal{P} given by (2)) for the last inequality, and the definition of \mathcal{P} (see (2)) for the last equality. Similarly, we have $\mathbb{E}_z(\mathcal{P}(Z_n)) \leq (\lambda^*)^n \mathcal{P}(z)$ for all $z \in \mathbb{N}^p \setminus \{0\}$. We conclude that

$$\theta_0^n \nu_{QS}(\mathcal{P}) < \sum_{z \in \mathbb{N}^p \setminus \{0\}} \nu_{QS}(\{z\}) (\lambda^*)^n \mathcal{P}(z) = (\lambda^*)^n \nu_{QS}(\mathcal{P}).$$

This implies that $\theta_0 < \lambda^*$ when \mathcal{P} is not linear.

It is also known that $\mathfrak{M}(z) = \sup_{k \geq 1} \frac{1}{k} \mathbb{E}_{kz}(Z_1)$ (see Corollary 7 in [9]), and the same calculation as above shows that, if $\mathbb{E}_{z_0}(Z_1) < \mathfrak{M}(z_0)$ for some $z_0 \in \mathbb{N}^p$, then $\theta_0 < \lambda^*$ (even if \mathcal{P} is linear).

Remark 1.2. When $\theta_0 = \lambda^*$, Assumption (M) requires polynomial moments for V , with exponents that can be arbitrarily close to 1. We leave as an open question whether, in this case,

$L \log L$ type criteria can be obtained, as it is the case in the classical Galton-Watson case (for which $\theta_0 = \lambda^*$).

In the situation where $\theta_0 < \lambda^*$, one can not choose ζ arbitrarily close to 1 in Assumption (M), since it imposes $\zeta > \frac{|\log \theta_0|}{|\log \lambda^*|}$. We leave as an open question whether $\mathbb{E}(V_{i,j}^\zeta) < +\infty$ for some $\zeta \leq \frac{|\log \theta_0|}{|\log \lambda^*|}$ may be sufficient to obtain the conclusion of Theorem 1.3 or 1.4, at least in some particular cases.

In Section 2, we state a general result for the existence of infinitely many quasi-stationary distributions, then we prove Proposition 1.1 and Theorem 1.2. Section 3 is devoted to the proof of Proposition 1.5. Theorems 1.3 and 1.4 are proved in Sections 4 and 5 respectively. For the proofs of the last two theorems, we make use of Assumption (E) of [4] which is recalled in Appendix. In Appendix we also state a general result implying Assumption (E) of [4].

2 A simple criterion for the existence of infinitely many quasi-stationary distributions and application to the bGWbp

In Section 2.1, we first state and prove a result on the existence of a continuum of quasi-stationary distributions for general sub-Markov kernels on Polish spaces. In Section 2, we apply this result to the bGWbp.

2.1 General setting

Let (X, d) be a Polish space endowed with a σ -algebra \mathcal{X} that contains the topology induced by d . Let K be a sub-Markov kernel on X , which means that K is a non-negative kernel on X such that $K(x, X) \leq 1$ for all $x \in X$. We assume that K has Feller regularity, meaning that the associated operator $K : f \mapsto Kf := \int_X K(\cdot, dy) f(y)$ is a well defined functional from $C_b(X)$ to $C_b(X)$ (the set of continuous bounded functions on X). We say that a probability measure μ on X is a quasi-stationary distribution for K if, for some $\lambda \in (0, 1]$,

$$\int_X \mu(dx) K(x, \cdot) = \lambda \mu(\cdot).$$

Our aim is to prove that K admits infinitely many quasi-stationary distributions under mild assumptions.

In the following theorem, we say that a sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ tends to infinity if it eventually leaves any compact set: for all compact subset L of X , there exists $n_L \geq 0$ such that, for all $n \geq n_L$, $x_n \notin L$. Similarly, we say that a real functional on X tends to infinity when $x \rightarrow +\infty$ if, for all $R > 0$, there exists a compact subset L_R of X such that the functional is bounded below by R on $X \setminus L_R$. We also define the iterated kernels K^n as usual by

$$K^0(x, A) = \mathbf{1}_{x \in A}, \quad K^{\ell+1}(x, A) = \int_X K^\ell(x, dy) K(y, A) \quad \forall x \in X, A \in \mathcal{X}, \ell \geq 0.$$

Theorem 2.1. *Assume that there exists $a > 0$ and $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ tending to infinity such that*

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^\ell(x_n, X) < +\infty, \quad \forall n \geq 0 \tag{11}$$

and let

$$a_0 := \inf\{a > 0, \text{ s.t. there exists } (x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}} \text{ tending to infinity such that (11) holds true.}\} \tag{12}$$

Assume also that there exists $a_1 > a_0$ such that

$$\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) \xrightarrow{x \rightarrow +\infty} +\infty. \quad (13)$$

Then K admits an infinite family of quasi-stationary distributions $(\mu_\lambda)_{\lambda \in [a_0, a_1]}$, with $\int_X \mu_\lambda(dx) K(x, \cdot) = \lambda \mu_\lambda$ for all $\lambda \in [a_0, a_1]$.

Proof of Theorem 2.1. Fix $\lambda \in (a_0, a_1)$. Since $\lambda > a_0$ then there exists $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ tending to infinity such that, for all $n \geq 0$, the probability measure

$$\mu_{\lambda, n} = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)}$$

is well defined. We have, for all $n \geq 0$,

$$\begin{aligned} \mu_{\lambda, n} K(\cdot) &:= \int_X \mu_{\lambda, n}(dx) K(x, \cdot) = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-(\ell+1)} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot) - \lambda \delta_{x_n}(\cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)} \\ &\leq \lambda \mu_{\lambda, n}(\cdot). \end{aligned} \quad (14)$$

We deduce that, for all $\ell \geq 1$,

$$\mu_{\lambda, n} K^\ell \leq \lambda^\ell \mu_{\lambda, n}.$$

In particular, we obtain that

$$\int_X \mu_{\lambda, n}(dx) \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X) = \sum_{\ell=0}^{\infty} a_1^{-\ell} \mu_{\lambda, n} K^\ell(X) \leq \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{a_1} \right)^\ell = \frac{a_1}{a_1 - \lambda}. \quad (15)$$

Since we assumed that $\sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x, X)$ goes to infinity when $x \rightarrow \infty$, this and Prokhorov's Theorem imply that the sequence of measures $(\mu_{\lambda, n})_{n \in \mathbb{N}}$ is relatively compact for the weak convergence topology.

Let μ_λ be any adherent point of $(\mu_{\lambda, n})_{n \in \mathbb{N}}$, and observe that μ_λ is a probability measure. Since K is assumed to preserve bounded continuous functions, we deduce that, up to a subsequence and for the weak convergence topology,

$$\mu_\lambda K = \lim_{n \rightarrow +\infty} \mu_{\lambda, n} K.$$

But we computed (see (14))

$$\mu_{\lambda, n} K = \lambda \mu_{\lambda, n} - \frac{\lambda \delta_{x_n}}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X)},$$

where, by assumption,

$$\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, X) \geq \sum_{\ell=0}^{\infty} a_1^{-\ell} K^\ell(x_n, X) \xrightarrow{n \rightarrow +\infty} +\infty.$$

We deduce that $\lim_{n \rightarrow +\infty} \mu_{\lambda, n} K = \lim_{n \rightarrow +\infty} \lambda \mu_{\lambda, n}$, which is equal to $\lambda \mu_{\lambda}$. We thus proved that $\mu_{\lambda} K = \lambda \mu_{\lambda}$, which means that μ_{λ} is a quasi-stationary distribution for K with absorption parameter λ .

It remains to prove that there exists a quasi-stationary distribution with absorption parameter a_0 . Since the set of probability measures μ_{λ} , $\lambda \in (a_0, (a_0 + a_1)/2)$, satisfies (15) with a uniform upper bound, we deduce that it is relatively compact. One checks as above that any limit point, when $\lambda \rightarrow a_0$, of this family is a quasi-stationary distribution for K with absorption parameter a_0 . This concludes the proof of Theorem 2.1. \square

Remark 2.1. One easily checks from the proof that the condition that K preserves bounded continuous functions is stronger than necessary. Actually, it would be sufficient to assume that K sends a generating family of bounded continuous functions into a family of bounded continuous functions. For instance, it is sufficient to assume that K sends compactly supported continuous functions to bounded continuous functions.

Remark 2.2. One can easily extend this result to the general setting of non-conservative kernels K . More precisely, assume that there exists a positive function h on X such that $Kh \leq ch$ for some $c > 0$. Then the kernel defined by

$$\tilde{K}(x, A) = \frac{1}{ch(x)} \int_A K(x, dy) h(y), \quad \forall x \in X, \forall A \in \mathcal{X},$$

is a sub-Markov kernel to which Theorem 2.1 may apply. Note that the assumptions would translate into: for any continuous function $f : X \rightarrow \mathbb{R}$, $x \in X \mapsto \frac{1}{h(x)} \int_X K(x, dy) h(y) f(y)$ is continuous; for all $n \geq 0$, $\sum_{\ell=0}^{\infty} a_0^{-\ell} \int_X K^{\ell}(x_n, dy) h(y) < +\infty$ and $\frac{1}{h(x)} \sum_{\ell=0}^{\infty} a_1^{-\ell} \int_X K^{\ell}(x, dy) h(y) \xrightarrow{x \rightarrow +\infty} +\infty$. In addition, the resulting quasi-stationary measures may not be a finite measure if h is not lower bounded (we only obtain that $\int_X \mu_{\lambda}(dy) h(y) < +\infty$ for all $\lambda \in [a_0, a_1)$). On the other hand, if h is lower bounded away from 0, then μ_{λ} is a finite measure.

2.2 Proof of Proposition 1.1 and Theorem 1.2

In the context of bGWbp, we choose $K(x, dy) = \mathbb{P}_x(Z_1 \in dy, Z_1 \neq 0)$ for all $x \in X = \mathbb{N}^p \setminus \{0\}$. We first prove Proposition 1.1, as well as that $a_0 \leq v_0$, where a_0 is defined in Theorem 2.1 and v_0 is defined in (3). Then we prove Theorem 1.2.

First, from their definitions, it is clear that $v_0 \leq 1$ and $\theta_0 \leq 1$. For all $\theta < \theta_0$, there exist $v \in (\theta, 1)$ and $z \in \mathbb{N}^p \setminus \{0\}$ such that $\liminf_{n \rightarrow \infty} v^n \mathbb{P}_z(Z_n \neq 0) > 0$. For such v and z , we have $\mathbb{E}_z(v^{-T_0}) = \sum_{\ell=0}^{\infty} v^{-\ell} \mathbb{P}_z(T_0 = \ell)$ with

$$\begin{aligned} \liminf_{\ell \rightarrow \infty} v^{-\ell} \mathbb{P}_z(T_0 = \ell) &= \liminf_{\ell \rightarrow \infty} v^{-\ell} (\mathbb{P}_z(Z_{\ell-1} \neq 0) - \mathbb{P}_z(Z_{\ell} \neq 0)) \\ &= (v^{-1} - 1) \liminf_{\ell \rightarrow \infty} v^{-\ell} \mathbb{P}_z(Z_{\ell} \neq 0) > 0. \end{aligned}$$

Then $\mathbb{E}_z(v^{-T_0}) = \infty$ and hence $v \geq v_0$. We deduce that $v_0 \geq \theta$ and hence $v_0 \geq \theta_0$.

In order to prove that $a_0 \leq v_0$, we first observe that, for all $a > v > v_0$, for all $z \in \mathbb{N}^p \setminus \{0\}$ and

$$\sum_{\ell \geq 0} a^{-\ell} K^{\ell}(z, X) = \sum_{\ell \geq 0} a^{-\ell} \mathbb{P}_z(T_0 \geq \ell + 1) \leq \sum_{\ell \geq 0} a^{-\ell} v^{\ell+1} \mathbb{E}_z(v^{-T_0}) \leq C_{a,v} \mathbb{E}_z(v^{-T_0}) < \infty.$$

This shows that $a_0 \leq v_0$.

Using the fact that \mathcal{P} is lower bounded away from 0 on $\mathbb{N}^p \setminus \{0\}$ (as \mathcal{P} is strictly positive and positively homogeneous) and that K vanishes at 0, we deduce that there exists a constant $c > 0$ such that, for all $\ell \geq 1$ and all $x \in \mathbb{N}^p$,

$$K^{\ell}(x, X) \leq \frac{1}{c} \mathbb{E}_x(\mathcal{P}(Z_{\ell})).$$

In addition, $(\lambda^*)^{-n} \mathcal{P}(Z_n)$ is a supermartingale (see Lemma 37 in [9]), so that, for any $a \in (\lambda^*, 1)$,

$$\sum_{\ell=0}^{\infty} a^{-\ell} K^\ell(x, X) \leq \frac{1}{c} \sum_{\ell=0}^{\infty} a^{-\ell} (\lambda^*)^\ell \mathcal{P}(x) < +\infty.$$

This shows that $v_0 \leq \lambda^*$, then Proposition 1.1 holds.

Since \mathfrak{M} is primitive by Assumption (P), Theorem 6 in [9] implies that, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $c_0 > 0$ and $k_0 \geq 1$ such that for all $k \geq k_0$

$$\mathbb{P}(Z_{n_0} \geq c_0(1, \dots, 1) \mid Z_0 = kz) > 0.$$

Applying this result for z taken in each element of the canonical basis of \mathbb{N}^p and using the super-additivity of ξ , one deduces that there exist uniform constants $r > 0$ and $c'_0 > 0$ such that

$$\inf_{|z| > r} \mathbb{P}(Z_{n_0} \geq c'_0(1, \dots, 1) \mid Z_0 = z) > 0.$$

Using the Markov property and by iteration, this implies that, for all $n \geq 1$,

$$\mathbb{P}_z(Z_n = 0) \xrightarrow{|z| \rightarrow +\infty} 0.$$

In particular, we deduce from Fatou's Lemma that, for any $a_1 \in (0, 1)$,

$$\sum_{\ell=0}^{\infty} a_1^{-\ell} \mathbb{P}_z(Z_\ell \neq 0) \xrightarrow{|z| \rightarrow +\infty} +\infty.$$

and hence (13) holds true for any $a_1 \in (a_0, 1) \supset (v_0, 1)$.

We deduce that Theorem 2.1 applies to the subcritical bGWbp. Since quasi-stationary distributions with different absorption parameters are linearly independent, this proves Theorem 1.2.

3 Proof of Proposition 1.5

We start with the following technical lemma on the regularity of \mathcal{P} .

Lemma 3.1. *Assume that Assumptions (S), (P), (C) and (M) hold true. Then, for all $I \subset \{1, \dots, p\}$, $I \neq \emptyset$, the operator \mathcal{P} restricted to \mathcal{Z}_I is uniformly continuous.*

Proof of Lemma 3.1. Since $\mathcal{P} \circ \mathfrak{M}^{n_0} = (\lambda^*)^{n_0} \mathcal{P}$, it is sufficient to prove that $\mathcal{P} \circ \mathfrak{M}^{n_0}$ is uniformly continuous on \mathcal{Z}_I . But $\mathfrak{M}^{n_0}(\mathcal{Z}_I) \subset \mathfrak{M}^{n_0}(\mathbb{S})$ which is relatively compact in $(\mathbb{R}_+ \setminus \{0\})^p$ (since \mathfrak{M} is primitive and bounded by assumption), and \mathcal{P} is concave and thus locally Lipschitz in $(\mathbb{R}_+ \setminus \{0\})^p$, thus it is sufficient to prove that \mathfrak{M}^{n_0} is uniformly continuous on \mathcal{Z}_I .

Let us first observe that \mathfrak{M} is locally uniformly continuous on $(\varepsilon + \mathbb{R}_+ \setminus \{0\}) \mathcal{Z}_I$ for any $\varepsilon > 0$, since \mathfrak{M} is uniformly continuous on \mathcal{Z}_I , bounded on $\mathbb{S} \supset \mathcal{Z}_I$, and positively homogeneous. In order to conclude, it remains to prove that there exist $J \subset \{1, \dots, p\}$ and $\varepsilon > 0$ such that $\mathfrak{M}(\mathcal{Z}_I) \subset (\varepsilon + \mathbb{R}_+ \setminus \{0\}) \mathcal{Z}_J$ (the result then follows by induction). Let $x \in \mathcal{Z}_I$ and let $J = \{i \in \{1, \dots, p\}, \mathfrak{M}(x)_i > 0\}$. Then, for all $y \in \mathcal{Z}_I$, there exists $\delta > 0$ such that $\delta x \leq y$, so that, using the fact that \mathfrak{M} is increasing and homogeneous, $\mathfrak{M}(y) \geq \delta \mathfrak{M}(x)$ and hence $J \subset \{i \in \{1, \dots, p\}, \mathfrak{M}(y)_i > 0\}$. Reciprocally, $\{i \in \{1, \dots, p\}, \mathfrak{M}(y)_i > 0\} \subset J$, so that $\mathfrak{M}(y) \in \mathbb{R}_+ \mathcal{Z}_J$. Since on \mathbb{S} , the norm of \mathfrak{M} is bounded from below away from 0 and from above, we have $\mathfrak{M}(\mathcal{Z}_I) \subset [\varepsilon_1, \varepsilon_2] \mathcal{Z}_J$ for some $\varepsilon_1, \varepsilon_2 > 0$. This implies that \mathfrak{M}^2 is uniformly continuous on \mathcal{Z}_I . We conclude by iteration. \square

Let us now proceed with the proof of Proposition 1.5 in two steps: first we show that, for all $z \in \mathbb{R}_+^p \setminus \{0\}$,

$$\limsup_{n \rightarrow +\infty} \frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] = (\lambda^*)^a Q_a(z); \quad (16)$$

Second, we extend this to a uniform convergence in z by using the regularity of \mathcal{P} .

Step 1. As \mathcal{P} is positively homogeneous, we have for any $z \in \mathbb{R}_+^p \setminus \{0\}$,

$$\begin{aligned} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] &= \mathbb{E} \left[\left(\mathcal{P} \circ \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)^a \right] \\ &= |n|^a \mathbb{E} \left[\left(\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right]. \end{aligned}$$

According to Lemma 30 in [9], we have

$$\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathfrak{M}(z).$$

Let $I \subset \{1, \dots, p\}$ be such that $\mathfrak{M}(z) \in \mathbb{R}_+^* \mathcal{Z}_I$, then the above convergence entails that, almost surely, for n large enough, $\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \in \mathbb{R}_+^* \mathcal{Z}_J$ for some $J \supset I$. Using Corollary 7 in [9], we have, for all $n \geq 1$,

$$\mathfrak{M}(z) \geq \frac{1}{n} \mathbb{E} [Z_1 \mid Z_0 = \lfloor nz \rfloor] = \frac{1}{n} \mathbb{E} \left[\xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right],$$

so that we also have $J \subset I$ almost surely.

Since \mathcal{P} restricted to $\mathbb{R}_+^* \mathcal{Z}_I$ is continuous since it is concave, and since $\mathfrak{M} \circ \mathcal{P} = \lambda^* \mathcal{P}$ by (2), we deduce that

$$\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathcal{P}(\mathfrak{M}(z)) = \lambda^* \mathcal{P}(z).$$

In addition, since ξ is sub-affine, since $\frac{1}{n} \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)}$ is bounded in L^ζ by Assumption (M) and \mathcal{P} being positively homogeneous, we deduce that $\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)$ is also bounded in L^ζ (where $\zeta > a$), and we deduce from the dominated convergence theorem that

$$\mathbb{E} \left[\left(\mathcal{P} \left(\frac{1}{n} \xi \left(\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right] \xrightarrow[n \rightarrow +\infty]{} (\lambda^*)^a \mathcal{P}(z)^a.$$

We deduce that (16) holds true. This concludes the first step of the proof.

Step 2. Fix a non-empty $I \subset \{1, \dots, p\}$. Let $\varepsilon_1 > 0$ be a small number that will be fixed later and let $z^1, \dots, z^{m_1} \in (1 + \varepsilon_1) \mathcal{Z}_I$, $m_1 = m(\varepsilon_1) \geq 1$, be such that, for all $z \in \mathcal{Z}_I$, there exists $k = k(z, \varepsilon_1) \in \{1, \dots, m_1\}$ such that $z \leq z^k$ and $|z - z^k| \leq 2\varepsilon_1$. The existence of this finite family z^1, \dots, z^{m_1} easily follows from a compactness argument. Let $n_1 = n_1(\varepsilon_1)$ be large enough so that, for all $n \geq n_1$, for all $k \in \{1, \dots, m_1\}$,

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz^k \rfloor] \leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^k)^a.$$

Then, since Q_a is non-decreasing and since Z_1 (with $Z_0 = \lfloor nz \rfloor$) is stochastically non-decreasing with respect to z , we deduce that, for all $n \geq n_1$ and all $z \in \mathcal{Z}_I$,

$$\begin{aligned} \frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] &\leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^{k(z, \varepsilon_1)})^a \\ &\leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z)^a \left(\sup_{x, y \in \mathcal{Z}_I, |(1 + \varepsilon_1)x - y| \leq 2\varepsilon_1} \frac{\mathcal{P}((1 + \varepsilon_1)x)}{\mathcal{P}(y)} \right)^a. \end{aligned} \quad (17)$$

Since \mathcal{P} is uniformly continuous on \mathcal{Z}_I and lower bounded away from 0 on \mathcal{Z}_I , we deduce that there exists ε_1 small enough so that

$$\theta_a := (1 + \varepsilon_1)^{a+1} (\lambda^*)^a \left(\sup_{x, y \in \mathcal{Z}_I, |(1 + \varepsilon_1)x - y| \leq 2\varepsilon_1} \frac{\mathcal{P}(x)}{\mathcal{P}(y)} \right)^a < \theta_0.$$

This and (17) implies that

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = \lfloor nz \rfloor] \leq \theta_a \mathcal{P}(z)^a.$$

Since any $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})$ can be written as $n \frac{z}{|z|}$ with $n = |z|$ and $\frac{z}{|z|} \in \mathcal{Z}_I$, we deduce that, for any such z with $|z| \geq n_1$,

$$\frac{1}{|z|^a} \mathbb{E} [Q_a(Z_1) \mathbf{1}_{Z_1 \neq 0} \mid Z_0 = z] \leq \theta_a \mathcal{P}(z/|z|)^a.$$

Finally, the homogeneity of \mathcal{P} allows us to conclude that Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\} \cap (\mathbb{R}_+ \setminus \{0\})$. Since there are only finitely many subsets I of $\{1, \dots, p\}$, Proposition 1.5 holds true for all $z \in \mathbb{N}^p \setminus \{0\}$.

4 Proof of Theorem 1.3

The proof of the first statement relies on Theorem 4.1 [3] and in part on some arguments of the proof of Theorem 5.1 therein. The main additional difficulty is that our reference exponential parameter θ_0 is *a priori* larger than the one considered in this reference. We overcome this difficulty in Lemma 4.1 below. Another difficulty is that a key part of the argument relies on the assumption (E1-4) in [4], which does not apply directly when the Lyapunov parameter θ_a (in Proposition 1.5) is only assumed strictly smaller than θ_0 . To overcome this difficulty, we prove in Proposition A.1 that this last condition actually entails the assumption (E1-4) in [4].

Fix $a \in (1, \zeta)$ such that $(\lambda^*)^a < \theta_0$ and set $H = Q_a / \inf Q_a$. According to Proposition 1.5 and since $H(z)$ tends to ∞ when $|z| \rightarrow +\infty$, there exists $\theta_a < \theta_0$ such that

$$\limsup_{|z| \rightarrow +\infty} \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} \leq \theta_a < \gamma := (\theta_a + \theta_0)/2 < \theta_0. \quad (18)$$

In particular, there exist only finitely many communication classes E_1, \dots, E_{k_0} , $k_0 \geq 1$, for the process $(Z_n)_{n \in \mathbb{N}}$ such that

$$\forall i \in \{1, \dots, k_0\}, \exists z \in E_i, \text{ such that } \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} > \gamma.$$

Note that $k_0 \neq 0$ (otherwise $\theta_0 \leq \gamma$). For all $i \in \{1, \dots, k_0\}$, we define $H_i = H \mathbf{1}_{E_i}$ and denote by $Y^{(i)}$ the process with state space $E_i \cup \{0\}$ defined by

$$Y_n^{(i)} = \begin{cases} Z_n & \text{if } Z_n \in E_i \\ 0 & \text{otherwise.} \end{cases}$$

For all $i \in \{1, \dots, k_0\}$, we denote by $\theta_{0,i}$ the absorption parameter of $Y^{(i)}$:

$$\theta_{0,i} = \sup_{z \in E_i} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Y_n^{(i)} \neq 0) > 0\}$$

and we set

$$\bar{\theta} = \max_{i \in \{1, \dots, k_0\}} \theta_{0,i}.$$

We assume that E_1, \dots, E_k , $k \in \{1, \dots, k_0\}$, are the communication classes such that $\theta_{0,i} = \bar{\theta}$, so that $\max_{i \geq k+1} \theta_{0,i} < \bar{\theta}$.

We define the set

$$E'_0 := \mathbb{N}^p \setminus \left(\{0\} \cup \bigcup_{i=1}^{k_0} E_i \right).$$

By definition of E_1, \dots, E_{k_0} , we have

$$\forall z \in E'_0, \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} \leq \gamma. \quad (19)$$

In what follows, we make use of the following crucial lemma.

Lemma 4.1. *We have $\bar{\theta} = \theta_0$.*

Proof of Lemma 4.1. Our aim is to show that, for all $\rho > \bar{\theta} \vee \gamma$ and all $z \in \mathbb{N}^p$, we have

$$\liminf_{n \rightarrow +\infty} \rho^{-n} \mathbb{P}_z(Z_n \neq 0) = 0. \quad (20)$$

This shows that $\bar{\theta} \vee \gamma \geq \theta_0$ and hence $\bar{\theta} \geq \theta_0$, while the converse inequality is trivial, concluding the proof of Lemma 4.1.

For all $i \neq j \in \{1, \dots, k_0\}$, we have

$$\forall z \in E_j, \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) > 0 \text{ or } \forall z \in E_j, \mathbb{P}_z(\exists n \geq 0, Z_n \in E_i) = 0$$

If the condition on the left is satisfied, we write $E_i < E_j$, otherwise we write $E_i \not< E_j$. This defines a partial order on $\{1, \dots, k_0\}$. For all $i \in \{1, \dots, k_0\}$, we denote by $c(i)$ the maximal length of sets of distinct indices $i_1 = i, \dots, i_c$ such that $E_{i_c} < \dots < E_{i_2} < E_{i_1}$. We also define, for all $\ell \geq 1$, the set

$$E_0^\ell := \{x \in E'_0, \exists i \in \{1, \dots, k_0\} \text{ with } c(i) = \ell \text{ and } x \rightarrow E_i, \text{ and } x \not\rightarrow E_j \forall j \text{ such that } c(j) > \ell\},$$

where $x \rightarrow E_i$ means that $\mathbb{P}_x(T_{E_i} < \infty) > 0$, while $x \not\rightarrow E_j$ means that $\mathbb{P}_x(T_{E_j} < \infty) = 0$. We also set

$$E_0^0 := E'_0 \setminus \bigcup_{\ell=1}^{\max_i c(i)} E_0^\ell.$$

We obtain (20) for all $Z \in \mathbb{N}^p \setminus \{0\}$ by proving that, for any fixed $\rho > \bar{\theta} \vee \gamma$,

$$\mathbb{E}_z(H(Z_n) \mathbf{1}_{Z_n \neq 0}) \leq C_\rho \rho^n H(z), \quad \forall z \in \mathbb{N}^p \setminus \{0\} \quad (21)$$

where C_ρ does not depend on z . This last inequality is immediate for all $z \in E_0^0$ (see beginning of Step 1 below) and is proved for all $z \in E_i \cup E_0^{c(i)}$, $i \in \{1, \dots, k_0\}$, by induction on $c(i)$.

Step 1. We first observe that, for all $\ell \geq 0$ and all $z \in E_0^\ell$, using (19),

$$\mathbb{E}_z \left(H(Z_n) \mathbf{1}_{Z_n \in E_0^\ell} \right) \leq \gamma^n H(z) \leq \rho^n H(z). \quad (22)$$

In particular, since $E_0^0 \cup \{0\}$ is a closed set, this shows (21) for all $z \in E_0^0$.

We now prove that, for all $i \in \{1, \dots, k_0\}$ and all $z \in E_i$,

$$\mathbb{E}_z(H(Z_n) \mathbf{1}_{Z_n \in E_i}) \leq C \rho^n H(z). \quad (23)$$

In order to do so, we consider the set

$$K_i = \{z \in E_i, \frac{\mathbb{E}_z(H(Z_1) \mathbf{1}_{Z_1 \neq 0})}{H(z)} > \gamma\}.$$

For all $z \in E_i$, since $\rho > \bar{\theta}$ and by definition of $\bar{\theta}$,

$$\liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Y_n^{(i)} \neq 0) = 0. \quad (24)$$

Since K_i is finite according to (18), we deduce that there exists a constant $C > 0$ such that

$$\mathbb{P}_z(Z_n \notin E_i) = \mathbb{P}_z(Y_n^{(i)} \neq 0) \leq C \rho^n, \forall z \in K_i. \quad (25)$$

In what follows, the constant C may change from line to line. Moreover, denoting $\tau_{K_i} = \min\{\ell \geq 0, Z_\ell \in K_i\}$, we deduce from the definition of H and of K_i that

$$\mathbb{P}_z(Z_n \notin E_i \text{ and } n < \tau_{K_i}) \leq \mathbb{E}_z(H(Z_n) \mathbf{1}_{Z_n \notin E_i \text{ and } n < \tau_{K_i}}) \leq \gamma^n H(z), \forall z \in E_i. \quad (26)$$

Hence, for all $z \in E_i$ and all $n \geq 0$, we have, using the strong Markov property at time τ_{K_i} ,

$$\begin{aligned} \mathbb{P}_z(Z_n \notin E_i) &\leq \mathbb{P}_z(Z_n \notin E_i \text{ and } n < \tau_{K_i}) + \sum_{\ell=0}^n \mathbb{P}_z(\tau_{K_i} = \ell) \sup_{y \in K_i} \mathbb{P}_y(Z_{n-\ell} \notin E_i) \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \mathbb{P}_z(\tau_{K_i} = \ell) C \rho^{n-\ell} \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \mathbb{P}_z(Z_\ell \neq 0 \text{ and } \ell - 1 < \tau_{K_i}) C \rho^{n-\ell} \\ &\leq \gamma^n H(z) + \sum_{\ell=0}^n \gamma^{\ell-1} H(z) C \rho^{n-\ell} \\ &\leq C \rho^n H(z), \end{aligned}$$

where C may depend on ρ and γ . Finally, iterating (10) and using the last inequality, we deduce that

$$\begin{aligned} \mathbb{E}_z(H(Z_n) \mathbf{1}_{Z_n \notin E_i}) &\leq \theta_a^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a} \theta_a^{n-\ell-1} \mathbb{P}_z(Z_\ell \neq 0) \\ &\leq C \gamma^n H(z) + \sum_{\ell=0}^{n-1} \frac{C_a}{\inf Q_a} \gamma^{n-\ell-1} C \rho^\ell H(z) \\ &\leq C \rho^n H(z), \end{aligned}$$

where C may depend on ρ and γ . This proves (23) for all $z \in E_i$.

Step 2. Assume that there exists $c \geq 0$ such that (21) holds true for all $z \in E_0^0, \dots, E_0^c$ and all $z \in E_i$ with $c(i) \leq c$ (Steps 1 shows that this is true for $c = 0$). If $c = \max_{i \in \{1, \dots, k_0\}} c(i)$, then there is nothing to prove. Otherwise, let $i \in \{1, \dots, k_0\}$ such that $c(i) = c + 1$.

Denote by T_i the first exit time from E_i by $(Z_n)_{n \in \mathbb{N}}$ and let us use a similar computation as in the end of Step 1. For all $z \in E_i$ and all $n \geq 0$, we have

$$\begin{aligned} \mathbb{E}_z(H(Z_n)1_{Z_n \neq 0}) &= \mathbb{E}_z(H(Z_n)1_{Z_n \in E_i}) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i=\ell} \mathbb{E}_{X_\ell}(H(Z_{n-\ell})1_{Z_{n-\ell} \neq 0})) \\ &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i=\ell} C_\rho \rho^{n-\ell} H(X_\ell)) \end{aligned}$$

where we used (23), the fact that

$$Z_{T_i} \in \{0\} \cup \bigcup_{\ell \leq c} E_0^\ell \cup \bigcup_{c(j) \leq c} E_j \text{ almost surely,}$$

and the induction assumption. We then obtain

$$\begin{aligned} \mathbb{E}_z(H(Z_n)1_{Z_n \neq 0}) &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i > \ell-1} C_\rho \rho^{n-\ell} (\theta_a H(X_{\ell-1}) + C_a)) \\ &\leq C\rho^n H(z) + \sum_{\ell=0}^n \mathbb{E}_x(1_{T_i > \ell-1} C_\rho \rho^{n-\ell} (\theta_a + C_a) H(X_{\ell-1})) \end{aligned}$$

where we used the Markov property at time $\ell - 1$ and Proposition 1.5. We deduce from (23) that

$$\begin{aligned} \mathbb{P}_z(Z_n \neq 0) &\leq C\rho^n H(z) + C_\rho(\theta_a + C_a) C \sum_{\ell=0}^n \rho^n H(z) \\ &\leq C(n+1)\rho^n H(z). \end{aligned}$$

Since this is true for any $\rho > \bar{\theta} \vee \gamma$, this proves (21) for all $z \in E_i$ with $c(i) = c + 1$.

The same arguments show that this also holds true for all $z \in E_0^{c+1}$.

This concludes the proof of Lemma 4.1. □

From now on, we assume without loss of generality that there exists $k \in \{1, \dots, k_0\}$ such that, for all $i \in \{k+1, \dots, k\}$, $\theta_{0,i} < \bar{\theta}$, and, for all $i \in \{1, \dots, k\}$, $\theta_{0,i} = \bar{\theta} = \theta_0$. For any $i \in \{k+1, \dots, k_0\}$, we have, fixing any $\rho \in (\max_{j \in \{k'+1, \dots, k\}} \theta_{0,j}, \bar{\theta})$ and using the same method as in Step 1 from the proof of Lemma 4.1,

$$\mathbb{E}_z(H(Z_n)1_{n < T_i}) \leq C_\rho \rho^n H(z), \quad \forall z \in E_i,$$

where we recall that T_i is the first exit time from E_i by $(Z_n)_{n \in \mathbb{N}}$ and where C_ρ is a positive constant which may depend on ρ . Setting

$$E_0 := E_0' \cup \bigcup_{i=k+1}^{k_0} E_i,$$

one easily checks, using the last inequality and (19), that, for any $r \in (\gamma \vee \rho, \bar{\theta})$, there exists $C_r > 0$ such that

$$\mathbb{E}_z(H(Z_n)1_{Z_n \in E_0}) \leq C_r r^n H(z), \quad \forall z \in E_0.$$

In order to apply Theorem 4.1 in [3], which implies the first part of Theorem 1.3, it remains to prove that, for all $i \in \{1, \dots, k'\}$ and all $z \in E_i$, we have

$$|\theta_0^{-n} \mathbb{E}_z(f(Z_n)1_{Z_n \in E_i}) - h_i(z)\mu_i(f)| \leq C\alpha^n H_i(x) \|f\|_{H_i}, \quad \forall n \geq 0, \forall f \in L^\infty(H_i), \quad (27)$$

for some $C > 0, \alpha \in (0, 1)$, some non-negative non-zero function h_i and some probability measure μ_i on E_i . In order to do so, it is sufficient to check Assumptions (E1-4) in Section 2 of [4]. These conditions are recalled in the Appendix and we will more precisely check Assumptions (E1), (E2'), (E3) and (E4), which is sufficient to obtain (E1-4) according to Proposition A.1. Fixing $i \in \{1, \dots, k'\}$, we define the set

$$K_i := \left\{ z \in E_i, \frac{\mathbb{E}_z(H_i(Z_1)\mathbf{1}_{Z_1 \neq 0})}{H_i(z)} > \gamma \right\},$$

which is finite according to (18).

Since E_i is a communication class and the process is assumed to be aperiodic, there exists $n_1^i \geq 0$ such that

$$c_1^i := \inf_{x, y \in K_i} \mathbb{P}_x(Z_{n_1^i} = y) > 0.$$

Note that $K_i \neq \emptyset$, since it would imply that $\theta_{0,i} = \theta_0 \leq \gamma$. Fix $z_i \in K_i$. Then, for all $z \in K_i$,

$$\mathbb{P}_z(Z_{n_1^i} \in \cdot) \geq \mathbb{P}_z(Z_{n_1^i} = z_i)\delta_{z_i}(\cdot) \geq c_1^i \nu_i(\cdot \cap K),$$

for $\nu_i = \delta_{z_i}$. This entails (E1). In addition, taking $\varphi_1 = H_i$ and $\theta_1 = \gamma < \theta_0 = \theta_{0,i}$, the condition (E'2) is an immediate consequence of the definition of K_i .

For all $x, y \in K_i$ and $n \geq 0$, we have, using the Markov property at time n_1^i and the fact that $\mathbb{P}_x(Z_n \neq 0)$ is non-increasing in n (as $\{0\}$ is absorbing),

$$\mathbb{P}_x(Z_n \neq 0) \geq \mathbb{P}_x(Z_{n \vee n_1^i} \neq 0) \geq \mathbb{P}_x(Z_{n_1^i} = y) \mathbb{P}_y(Z_{n \vee n_1^i - n_1^i} \neq 0) \geq c_1^i \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic. This concludes the proof of the first part of Theorem 1.3.

To conclude the proof of Theorem 1.3, it remains to prove that the set $E = \{x \in \mathbb{N}^p \setminus \{0\}, \sum_{i \in I} \eta_i(x) = 0\}$ is finite and that any quasi-stationary distribution with absorption parameter strictly smaller than θ_0 is supported by E .

We first observe that Theorem 4.1 in [3] also implies that $j \equiv 0$ on the set E . Assume on the contrary that E is not finite and fix $z_0 \in \mathbb{N}^p \setminus \{0\}$ such that $\sum_{i=1}^\ell \eta_i(z_0) > 0$ and $j(z_0) = 0$ (existence of such a point z_0 is guaranteed by Proposition 2.3 in [3]). We observe that

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_{z_0}(Z_n \neq 0) > 0.$$

Then, by primitivity of \mathfrak{M} , there exists n_0 such that for all $x \in \mathbb{N}^p$ we have $\mathfrak{M}^{n_0}(x) > 0$. This implies that for $k > 0$ large enough we have $\mathbb{P}_{kx}(Z_{n_0} \geq z_0) > 0$. Indeed, if it is not the case, we can conclude using Theorem 6 in [9] that

$$\mathfrak{M}^{n_0}(x) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_{n_0} | Z_0 = kx)}{k} \geq \lim_{k \rightarrow +\infty} \frac{z_0}{k} = 0,$$

which is a contradiction. Then, we have for any z large enough $\mathbb{P}_z(Z_{n_0} \geq z_0) > 0$. Hence, using the fact that E is not finite and hence contains arbitrarily large points, there exists $z \in E$ and $z'_0 \geq z_0$ such that $\mathbb{P}_z(Z_{n_0} = z'_0) > 0$. By super-additivity of ξ , we have $\mathbb{P}_{z'_0}(Z_n \neq 0) \geq \mathbb{P}_z(Z_n \neq 0)$ and hence

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_z(Z_{n_0+n} \neq 0) > 0.$$

This is not compatible with the definition of E and (5). We have thus proved by contradiction that E is finite.

Let ν_{QS} be a quasi-stationary distribution for Z in $\mathcal{M}(Q_a)$ with absorption parameter $\theta_{QS} < \theta_0$. Then (5) implies that $\sum_{i=1}^{\ell} \nu_{QS}(\eta_i) = 0$ and hence the support of ν_{QS} is included in E . Since E is finite and since E is a closed set for the Markov chain, we deduce, by usual finite dimensional linear algebra arguments, that ν_{QS} belongs to the convex hull of any maximal family of positive left eigenvector of the transition matrix of Z restricted to E . This concludes the proof of the penultimate assertion of Theorem 1.3.

To conclude, it remains to prove that there are no quasi-stationary distributions for Z in $\mathcal{M}(Q_a)$ with absorption parameter strictly larger than θ_0 . This is a direct consequence of (5). Indeed, since j is bounded, integrating this inequality with respect to a quasi-stationary distribution ν_{QS} with absorption parameter θ_{QS} , and taking $f \equiv 1$, shows that

$$\limsup_{n \rightarrow +\infty} \theta_0^{-n} n^{-\|j\|_{\infty}} \theta_{QS}^n \mathbb{P}_z(Z_n \neq 0) < +\infty,$$

and hence $\theta_{QS} \leq \theta_0$.

5 Proof of Theorem 1.4

Fix $a \in (1, \zeta)$ such that $(\lambda^*)^a < \theta_0$, and consider the Lyapunov type property with constants θ_a and C_a from Proposition 1.5. We make use of Section 2 in [4]. In a first step, we check that Assumption E therein (recalled in the Appendix below) holds true for the process under consideration. In a second step, we prove (6), (7) and (8) and (9) using the results of [4].

Step 1. Assumption (E) holds true. Fix $\theta_1 \in (\theta_a, \theta_0)$. As \mathcal{P} is strictly positive on $\mathbb{R}_+^p \setminus \{0\}$ and positively homogeneous, we can fix $r_1 \geq p$ large enough so that $Q_a(z) \geq \frac{C_a}{\theta_1 - \theta_a}$ for all $|z| \geq r_1$, so that

$$\mathbb{E}_z(Q_a(Z_1)) \leq \theta_a Q_a(z) + C_a \leq \theta_1 Q_a(z), \quad \forall z \in \mathbb{N}^p \text{ such that } |z| \geq r_1. \quad (28)$$

Let us set $K = \{z \in \mathbb{N}^p \setminus \{0\}, |z| \leq r_1\}$ and $\varphi_1 = Q_a / \inf Q_a$ (note that $\inf Q_a > 0$ by Assumption (P)).

We deduce from the irreducibility and aperiodicity properties that there exists $n_1 \geq 0$ such that

$$c_1 := \inf_{x, y \in K} \mathbb{P}_x(Z_{n_1} = y) > 0. \quad (29)$$

Setting $\nu = \delta_{(1, \dots, 1)}$, this entails property (E1).

We have $\inf \varphi_1 = 1$, so that the first line of (E2) is satisfied. Moreover, according to (28), the third line of (E2) also holds true.

Choosing $\theta_2 \in (\theta_1, \theta_0)$, we deduce from the definition of θ_0 and the irreducibility of Z that

$$\theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty,$$

where e_i is the i^{th} element of the canonical basis of \mathbb{N}^p . Since, by the super-additivity of ξ , the process Z is stochastically non-decreasing in the initial condition and since, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $i \in \{1, \dots, p\}$ such that $z \geq e_i$, we deduce that

$$\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \geq \theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty.$$

Let $n \geq 1$ be large enough so that $\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \geq 1$, and set, for all $z \in \mathbb{N}^p \setminus \{0\}$

$$\varphi_2(z) = \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0).$$

We have, for all $z \in \mathbb{N}^p \setminus \{0\}$ and using the Markov property at time 1,

$$\begin{aligned}
\left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq 0}) &= \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_{k+1} \neq 0) \\
&= \theta_2 \sum_{k=1}^n \theta_2^{-k} \mathbb{P}_z(Z_k \neq 0) \\
&= \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \varphi_2(z) + \theta_2 (\theta_2^{-n} \mathbb{P}_z(Z_n \neq 0) - 1) \\
&\geq \theta_2 \left(\sum_{k=0}^{n-1} \theta_2^{-k} \right) \varphi_2(z).
\end{aligned}$$

As a consequence the second and fourth lines of (E2) are satisfied. Note also that we have $\inf_{\mathbb{N}^p \setminus \{0\}} \varphi_2 \geq \frac{1}{\sum_{k=0}^{n-1} \theta_2^{-k}}$.

For all $x, y \in K$ and $n \geq 0$, we have, using the Markov property at time n_1 and the fact that $\mathbb{P}_x(Z_n \neq 0)$ is non-increasing in n (as $\{0\}$ is absorbing),

$$\mathbb{P}_x(Z_n \neq 0) \geq \mathbb{P}_x(Z_{n \vee n_1} \neq 0) \geq \mathbb{P}_x(Z_{n_1} = y) \mathbb{P}_y(Z_{n \vee n_1 - n_1} \neq 0) \geq c_1 \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process Z is aperiodic and irreducible.

We have thus proved that Assumption (E) holds true for Z absorbed when it leaves $\mathbb{N}^p \setminus \{0\}$. This concludes the first step.

Step 2. Conclusion of the proof. Theorem 2.1 in [4] implies that there exists a unique quasi-stationary distribution ν_{QS} in $\mathcal{M}(Q_a)$ and that (8) holds true (using the fact that in our case, φ_2 is lower bounded away from 0). Theorem 2.3 in this reference also implies that there exists an associated non-negative eigenfunction $\eta_{QS} \in L^\infty(Q_a)$ with eigenvalue $\theta_{QS} > 0$, where θ_{QS} is the absorption parameter associated to ν_{QS} , such that $\inf_K \eta_{QS} > 0$ and

$$\theta_{QS}^{-n} \mathbb{P}_\cdot(Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{L^\infty(Q_a)} \eta_{QS}(\cdot). \quad (30)$$

Since the process is irreducible, for all $z \in \mathbb{N}^p \setminus \{0\}$, there exists $n \geq 0$ such that $\mathbb{P}_z(Z_n = (1, \dots, 1)) > 0$ and hence

$$\theta_{QS}^n \eta_{QS}(z) = \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) \geq \eta_{QS}((1, \dots, 1)) \mathbb{P}_z(Z_n = (1, \dots, 1)) > 0,$$

so that η_{QS} is positive. We thus proved (6) (for the absorption parameter θ_{QS} which we prove below to be equal to θ_0). Finally, since ξ is super-additive, $\theta_{QS}^{-n} \mathbb{P}_z(Z_n \neq 0)$ and hence η_{QS} increases with z , so that η_{QS} is lower bounded away from 0. Corollary 2.7 in [4] shows that (7) holds true, and, taking $f \equiv 1$, also implies that $\theta_{QS} = \theta_0$. Finally, (9) is an immediate consequence of Corollaire 2.11 in [4]. This concludes the proof of Theorem 1.4.

A Sufficient criterion for the exponential convergence to a quasi-stationary distribution

In Section 2 of [4], the authors state that Assumption (E) is sufficient to prove exponential convergence toward a quasi-stationary distribution. Let us recall this assumption, with the notations and settings of the present paper.

In what follows, we consider a process $(Z_n)_{n \in \mathbb{N}}$ evolving in a measurable state space $E \cup \{\partial\}$, where $\partial \notin E$ is an absorbing point.

Assumption (E). There exist a positive integer n_1 , positive real constants $\theta_1, \theta_2, c_1, c_2, c_3$, two functions $\varphi_1, \varphi_2 : E \rightarrow \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that

(E1) (*Local Dobrushin coefficient*). $\forall z \in K$,

$$\mathbb{P}_z(Z_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(E2) (*Global Lyapunov criterion*). We have $\theta_1 < \theta_2$ and

$$\begin{aligned} \inf_{z \in E} \varphi_1(z) &\geq 1, \quad \sup_{z \in K} \varphi_1(z) < \infty \\ \inf_{z \in K} \varphi_2(z) &> 0, \quad \sup_{z \in E} \varphi_2(z) \leq 1, \\ \mathbb{E}_z(\varphi_1(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\leq \theta_1 \varphi_1(z) + c_2 \mathbf{1}_K(z), \quad \forall z \in E \\ \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\geq \theta_2 \varphi_2(z), \quad \forall z \in E. \end{aligned}$$

(E3) (*Local Harnack inequality*). We have

$$\sup_{n \in \mathbb{Z}_+} \frac{\sup_{y \in K} \mathbb{P}_y(Z_n \neq \partial)}{\inf_{y \in K} \mathbb{P}_y(Z_n \neq \partial)} \leq c_3.$$

(E4) (*Aperiodicity*). For all $z \in K$, there exists $n_4(z)$ such that, for all $n \geq n_4(z)$,

$$\mathbb{P}_z(Z_n \in K) > 0.$$

Our aim is to show that the condition involving the function φ_2 can be replaced by a condition involving θ_0 , where

$$\theta_0 = \sup_{z \in E} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n \neq \partial) > 0\}, \quad (31)$$

More precisely, we consider the following assumption.

(E2') We have $\theta_1 < \theta_0$ and

$$\begin{aligned} \inf_{z \in E} \varphi_1(z) &\geq 1, \quad \sup_{z \in K} \varphi_1(z) < \infty \\ \mathbb{E}_z(\varphi_1(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &\leq \theta_1 \varphi_1(z) + c_2 \mathbf{1}_K(z), \quad \forall z \in E. \end{aligned} \quad (32)$$

Proposition A.1. *Assume that there exist a positive integer n_1 , positive real constants θ_1, c_1, c_2, c_3 , a function $\varphi_1 : E \rightarrow \mathbb{R}_+$ and a probability measure ν on a measurable subset $K \subset E$ such that (E1), (E2'), (E3) and (E4) hold true. Then, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists φ_2 such that Assumption (E2) holds true.*

Proof. In a first step, we prove that, for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$. In a second step, we conclude by building a function φ_2 which satisfies Assumption (E2).

Step 1. Assume on the contrary that there exists $\theta_2 \in (\theta_1, \theta_0)$ such that, for all $n \geq 1$, $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) < 1$. Then, using (E1), we deduce that, for all $n \geq n_1$,

$$1 > \inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq \theta_2^{-n} c_1 \mathbb{P}_\nu(Z_{n-n_1} \in K)$$

and hence that, for all $n \geq 0$,

$$\mathbb{P}_v(Z_n \in K) < \frac{\theta_2^{n_1}}{c_1} \theta_2^n.$$

Applying (32) iteratively, we deduce that, for all $z \in E$, for all $n \geq 1$,

$$\mathbb{E}_z(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial}) \leq \theta_1^n \varphi_1(z) + c_2 \sum_{k=1}^n \theta_1^{n-k} \mathbb{P}_z(Z_{k-1} \in K).$$

Integrating with respect to v and using the two previous inequalities, we get

$$\mathbb{E}_v(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial}) \leq \theta_1^n v(\varphi_1) + c_2 \sum_{k=1}^n \theta_1^{n-k} \mathbb{P}_v(Z_{k-1} \in K) \leq \theta_1^n v(\varphi_1) + c_2 \sum_{k=1}^n \theta_1^{n-k} \frac{\theta_2^{n_1-1}}{c_1} \theta_2^k \leq C' \theta_2^n, \quad (33)$$

for some constant C' where we used the fact that $v(\varphi_1) \leq \sup_K \varphi_1 < +\infty$ and $\theta_1 < \theta_2$.

For all $z \in E$ and all $n \geq 0$, we have, denoting $\tau_K = \min\{k \geq 0, Z_k \in K\}$ and using the strong Markov property at time τ_K ,

$$\begin{aligned} \mathbb{P}_z(Z_n \neq \partial) &\leq \mathbb{E}_z(\varphi_1(Z_n) \mathbf{1}_{Z_n \neq \partial} \mathbf{1}_{n < \tau_K}) + \sum_{k=0}^n \mathbb{P}_z(\tau_K = k) \sup_{y \in K} \mathbb{P}_y(Z_{n-k} \neq \partial) \\ &\leq \theta_1^n \varphi_1(z) + \sum_{k=0}^n \mathbb{P}_z(\tau_K = k) c_3 \mathbb{P}_v(Z_{n-k} \neq \partial), \end{aligned}$$

where we used (32) for the first term in the right hand side, and (E3) for the second term. Since we also have, by (32) and for all $k \geq 1$ and trivially for $k = 1$,

$$\mathbb{P}_z(\tau_K = k) \leq \mathbb{P}_z(Z_{k-1} \neq \partial, k-1 < \tau_K) \leq \mathbb{E}_z(\varphi_1(Z_{k-1}) \mathbf{1}_{Z_{k-1} \neq \partial} \mathbf{1}_{k-1 < \tau_K}) \leq \theta_1^{k-1} \varphi_1(z),$$

and since $\mathbb{P}_v(Z_{n-k} \neq \partial) \leq \mathbb{E}_v(\varphi_1(Z_{n-k}) \mathbf{1}_{Z_{n-k} \neq \partial})$, we deduce from the previous inequality and from (33) that

$$\mathbb{P}_z(Z_n \neq \partial) \leq \left(\theta_1^n + \sum_{k=0}^n \theta_1^{k-1} c_3 C' \theta_2^{n-k} \right) \varphi_1(z).$$

In particular, for all $z \in E$, for all $\theta > \theta_2 > \theta_1$,

$$\theta^{-n} \mathbb{P}_z(Z_n \neq \partial) \xrightarrow{n \rightarrow +\infty} 0,$$

then, by definition of θ_0 , we get $\theta_0 \leq \theta_2$, which contradicts assumption $\theta_2 \in (\theta_1, \theta_0)$.

This concludes the proof of our first step: for any $\theta_2 \in (\theta_1, \theta_0)$, there exists $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$.

Step 2. Fix $\theta_2 \in (\theta_1, \theta_0)$ and let $n \geq 1$ such that $\inf_{z \in K} \theta_2^{-n} \mathbb{P}_z(Z_n \in K) \geq 1$, and define the function

$$\varphi_2 : z \in E \mapsto C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_k \in K)$$

with $C_{\theta_2} := \left(\frac{1-\theta_2^n}{1-\theta_2} \right)^{-1}$. Then, $\inf_{z \in K} \varphi_2(z) > 0$ and $\sup_{z \in E} \varphi_2(z) \leq 1$. In addition,

$$\begin{aligned} \mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) &= C_{\theta_2} \sum_{k=0}^{n-1} \theta_2^{-k} \mathbb{P}_z(Z_{k+1} \in K) = C_{\theta_2} \theta_2 \sum_{k=1}^n \theta_2^{-k} \mathbb{P}_z(Z_k \in K) \\ &= \theta_2 \varphi_2(z) + C_{\theta_2} \theta_2 (\theta_2^{-n} \mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z)). \end{aligned}$$

Since $\theta_2^{-n} \mathbb{P}_z(Z_n \in K) - \mathbf{1}_K(z) \geq 0$ for all $z \in K$ (by definition of n) and for all $z \in E \setminus K$ (since for such z , we have $\mathbf{1}_K(z) = 0$), we deduce that

$$\mathbb{E}_z(\varphi_2(Z_1) \mathbf{1}_{Z_1 \neq \partial}) \geq \theta_2 \varphi_2(z), \quad \forall z \in E.$$

This concludes the proof of Proposition A.1. \square

References

- [1] G. Alsmeyer. Bisexual Galton-Watson processes: A survey. [https://www.uni-muenster.de/Stochastik/alsmeyer/bisex\(survey\).pdf](https://www.uni-muenster.de/Stochastik/alsmeyer/bisex(survey).pdf), 2002.
- [2] K. B. Athreya and P. E. Ney. Branching Processes. Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 196.
- [3] N. Champagnat and D. Villemonais. Quasi-stationary distributions in reducible state spaces. arXiv preprint arXiv:2201.10151, 2022.
- [4] N. Champagnat and D. Villemonais. General criteria for the study of quasi-stationarity. Electronic Journal of Probability, 28:1 – 84, 2023.
- [5] P. Collet, S. Martínez, and J. San Martín. Quasi-Stationary Distributions. Probability and its Applications (New York). Springer, Heidelberg, 2013. Markov chains, diffusions and dynamical systems.
- [6] D. J. Daley. Extinction conditions for certain bisexual Galton-Watson branching processes. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 9(4):315–322, 1968.
- [7] D. J. Daley, D. M. Hull, and J. M. Taylor. Bisexual Galton-Watson branching processes with superadditive mating functions. Journal of Applied Probability, pages 585–600, 1986.
- [8] P. A. Ferrari, H. Kesten, S. Martinez, and P. Picco. Existence of Quasi-Stationary Distributions. A Renewal Dynamical Approach. The Annals of Probability, 23(2):501 – 521, 1995.
- [9] C. Fritsch, D. Villemonais, and N. Zaldueño. The multi-type bisexual Galton-Watson branching process. Annales de l’Institut Henri Poincaré, in press.
- [10] M. González, D. M. Hull, R. Martínez, and M. Mota. Bisexual branching processes in a genetic context: The extinction problem for y-linked genes. Mathematical Biosciences, 202(2):227–247, 2006.
- [11] M. González and M. Molina. On the limit behaviour of a superadditive bisexual Galton-Watson branching process. Journal of Applied Probability, 33(4):960–967, 1996.
- [12] D. M. Hull. A survey of the literature associated with the bisexual Galton-Watson branching process. Extracta Mathematicae, 18(3):321–343, 2003.
- [13] U. Krause. Relative stability for ascending and positively homogeneous operators on Banach spaces. Journal of Mathematical Analysis and Applications, 188(1):182–202, 1994.
- [14] P. Maillard. The λ -invariant measures of subcritical Bienaymé-Galton-Watson processes. Bernoulli, 24(1):297–315, 2018.
- [15] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. Probability Surveys, 9:340–410, 2012.
- [16] M. Molina. Two-sex branching process literature. In Workshop on Branching Processes and Their Applications, pages 279–293. Springer, 2010.
- [17] E. Seneta and D. Vere-Jones. On quasi-stationary distributions in discrete-time Markov chains with a denumerable infinity of states. Journal of Applied Probability, 3:403–434, 1966.

- [18] E. A. van Doorn and P. K. Pollett. Quasi-stationary distributions for discrete-state models. European Journal of Operational Research, 230(1):1–14, 2013.
- [19] A. M. Yaglom. Certain limit theorems of the theory of branching random processes. Doklady Akademii Nauk (N.S.), 56:795–798, 1947.