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Stationary Mean Field Games on networks with sticky transition conditions*

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Abstract

We study stochastic Mean Field Games on networks with sticky transition conditions. In this setting, the diffusion process governing the agent's dynamics can spend finite time both in the interior of the edges and at the vertices. The corresponding generator is subject to limitations concerning second-order derivatives and the invariant measure breaks down into a combination of an absolutely continuous measure within the edges and a sum of Dirac measures positioned at the vertices. Additionally, the value function, solution to the Hamilton-Jacobi-Bellman equation, satisfies generalized Kirchhoff conditions at the vertices.

Keywords: Mean Field Games, networks, Markov processes, sticky points.

AMS subject classification: 35R02, 49N80, 91A16.

1 Introduction

The theory of Mean Field Games (MFG in short), introduced in [16, 17] delves into analyzing differential games as the number of agents approaches infinity. The adaptation of the MFG theory to network has been considered in [1, 2, 8, 9]. It revolves around a PDE system comprising two differential equations: a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation, each defined within the network's edges. These equations are complemented with appropriate boundary conditions, as well as initial-final conditions for evolutive problems, and transition conditions at the vertices. The transition conditions are pivotal both in modeling, describing how agents behave upon reaching a vertex, and theoretically, ensuring the problem possesses properties that allow it to be studied, for example the Maximum Principle for the Hamilton-Jacobi-Bellman equation.

In previous research on MFG on networks, the vertex transition condition for the Hamilton-Jacobi-Bellman equation typically adheres to the classical Kirchhoff condition. This condition

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implies that the agent spend zero time at the vertices and enter one of the adjacent edges with a probability determined by specific coefficients, as dictated by the associated Markov process. In terms of duality, the vertex transition condition for the Fokker-Planck equation aligns with flux conservation principles. These conditions collectively indicate that the distribution of agents remains absolutely continuous, with no mass concentration occurring at the vertices. Nevertheless, in various physical models implemented on networks, such as those dealing with traffic flow or data transmission, congestion at vertices is a common occurrence. This congestion arises from the delay in distributing agents, whether they be vehicles or data packets, along the corresponding incident edges. Consequently, it becomes important to develop a mathematical framework that can capture and describe these congestion phenomena.

In our investigation, we explore MFG on networks governed by Markov processes featuring sticky transition conditions. Unlike the traditional diffusion processes governed by Kirchhoff conditions, in this framework an agent is slowed down at vertices, which results in a positive finite occupation time there. This behavior leads to the accumulation of agents at these vertices and their distribution across the network splitting into two distinct components. One component is absolutely continuous along the arcs, while the other concentrates at the vertices. We demonstrate that the invariant measure of the sticky process can be characterized similarly to the problem discussed in [1] for the absolute continuous component inside the edges, while the singular part is represented as a sum of Dirac masses at the vertices with coefficients proportional to the stickiness of the process. A similar decomposition has also been observed in [7, 10].

Once the characterization of the invariant measure is obtained, we turn to the study the well-posedness of Hamilton-Jacobi equations associated to discounted infinite horizon and long-term averaged optimal control problems of the sticky diffusion process. These equations involve a general Kirchhoff condition incorporating the stickiness coefficient and the cost incurred at the vertices. Differential equations on networks with generalized transition conditions at the vertices have been also studied in [4, 20, 21]. We also make use of the Itô formula as detailed in [22, 5] to prove a verification theorem for the discounted Hamilton-Jacobi equation, providing a justification of the generalized Kirchhoff condition.

Subsequently, we demonstrate the existence of solutions for the Mean Field Game system through a fixed point argument, supplemented by establishing uniqueness leveraging a classical monotonicity assumption on the cost. In examining the Mean Field Games system, a pivotal aspect is the duality between the linearized Hamilton-Jacobi-Bellman equation and the Fokker-Planck equation, which is satisfied also in this case.

The paper is structured as follows. Section 2 provides basic definitions for functions on networks and related functional spaces. Section 3 examines diffusion processes on networks and the Fokker-Planck equation for the corresponding invariant distribution. In Section 4, we focus on the Hamilton-Jacobi-Bellman equation associated with discount and ergodic control problems. Lastly, Section 5 presents the proofs of the main results concerning the existence and uniqueness for the Mean Field Game system.

2 Networks and functions spaces

This section presents the basic notions and properties of a network and some function spaces associated to it [1, 2, 9]. We consider a finite set \mathcal{V} of points in \mathbb{R}^d which will be the set of vertices of the network. We then consider a subset $P \subset \mathcal{V} \times \mathcal{V}$ satisfying

- if $v \in \mathcal{V}$ then $(v, v) \notin P$,
- for every $v_1 \in \mathcal{V}$, there exists a $v_2 \in \mathcal{V}$ such that either $(v_1, v_2) \in P$ or $(v_2, v_1) \in P$.

We say that v is a boundary vertex if there a unique $\bar{v} \in \mathcal{V}$ for which either (v, \bar{v}) or (\bar{v}, v) belongs to P and we denote with $\partial\mathcal{V}$ the set of the boundary vertices. We define the set of edges of the network \mathcal{E} as the set of all segments between the two points of a pair in P

$$\mathcal{E} = \{ \{ \theta v_1 + (1 - \theta)v_2 : \theta \in [0, 1] \} : (v_1, v_2) \in P \}. \quad (1)$$

Since \mathcal{E} is a finite collection of subsets of \mathbb{R}^d , we can index it by a finite set \mathcal{A} so that $\mathcal{E} = \{ \Gamma_\alpha \subset \mathbb{R}^d : \alpha \in \mathcal{A} \}$, each Γ_α being one the the segments defined in (1). Finally the network is given by $\Gamma = \cup_{\alpha \in \mathcal{A}} \Gamma_\alpha$ and we endow it with the topology induced by the geodesic distance on Γ . In all of this work we assume that the network Γ is connected.

For $v \in \mathcal{V}$, we define $\mathcal{A}_v = \{ \alpha \in \mathcal{A} : v \in \Gamma_\alpha \}$, that is the set of all indices $\alpha \in \mathcal{A}$ such that the vertex v belongs to the edge Γ_α . We assume that, for each pair $\alpha, \beta \in \mathcal{A}$ with $\alpha \neq \beta$, one has $\Gamma_\alpha \cap \Gamma_\beta = \{v\}$ whenever $\alpha, \beta \in \mathcal{A}_v$ and $\Gamma_\alpha \cap \Gamma_\beta = \emptyset$ otherwise. Let Γ_α be the edge with vertices v_i and v_j . We denote by $L_\alpha \in (0, \infty)$ the length of the edge and we define a unit vector $e_\alpha = (v_j - v_i)/L_\alpha$. Then Γ_α admits a parametrization $\pi_\alpha : [0, L_\alpha] \rightarrow \Gamma_\alpha$ defined by $\pi_\alpha(s) = v_i + se_\alpha$.

To a function $u : \Gamma \rightarrow \mathbb{R}$, we associate the function $u_\alpha : (0, L_\alpha) \rightarrow \mathbb{R}$, $\alpha \in \mathcal{A}$, defined by

$$u_\alpha(y) = (u \circ \pi_\alpha)(y), \quad \text{for every } y \in (0, L_\alpha).$$

For $x \in \Gamma_\alpha$, we set

$$u|_{\Gamma_\alpha}(x) = \begin{cases} u_\alpha \circ \pi_\alpha^{-1}(x) & \text{for } x \in \Gamma_\alpha \setminus \mathcal{V}, \\ \lim_{y \rightarrow 0^+} u_\alpha(y) & \text{if } x = \pi_\alpha(0), \\ \lim_{y \rightarrow L_\alpha^-} u_\alpha(y) & \text{if } x = \pi_\alpha(L_\alpha), \end{cases} \quad (2)$$

if the previous limits exist.

For every Borel set $A \in \mathcal{B}(\Gamma)$, we define the Lebesgue measure of A by $\mathcal{L}(A) = \sum_{\alpha \in \mathcal{A}} \pi_\alpha \# \mathcal{L}_\alpha(A \cap \Gamma_\alpha)$ where \mathcal{L}_α is the usual one dimensional Lebesgue measure on $[0, L_\alpha]$. Clearly $\mathcal{L}(\Gamma) = \sum_{\alpha \in \mathcal{A}} L_\alpha$. For a \mathcal{L} -integrable function $f : \Gamma \rightarrow \mathbb{R}$ we then have

$$\int_\Gamma f(x) dx := \int_\Gamma f(x) \mathcal{L}(dx) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} f(x) \pi_\alpha \# \mathcal{L}_\alpha(dx) = \sum_{\alpha \in \mathcal{A}} \int_0^{L_\alpha} f_\alpha(y) dy.$$

We denote by $\mathcal{C}(\Gamma)$ the space of continuous real valued functions on Γ , which is a Banach space when equipped with the norm $\|u\|_{\mathcal{C}(\Gamma)} = \sup_{x \in \Gamma} |u(x)|$. It is also convenient to allow functions to be discontinuous at the junctions but continuous in each edge. We first define

$$\mathcal{PC}(\Gamma) = \{ u : \Gamma \rightarrow \mathbb{R} : u_\alpha \in \mathcal{C}(0, L_\alpha) \text{ and the limits in (2) exist for each } \alpha \in \mathcal{A} \}.$$

endowed with the norm of uniform convergence on each edge

$$\|u\|_{\mathcal{PC}(\Gamma)} = \max_{\alpha \in \mathcal{A}} \|u_\alpha\|_{L^\infty((0, L_\alpha))},$$

We then consider the following equivalence relation

$$\forall u, v \in \mathcal{PC}(\Gamma) \quad (u \sim v) \Leftrightarrow (u_\alpha(x) = v_\alpha(x), \forall x \in (0, L_\alpha), \forall \alpha \in \mathcal{A}),$$

and finally we define $PC(\Gamma) = \mathcal{PC}(\Gamma)/\sim$. We observe that $PC(\Gamma)$ with the induced norm is a Banach space. We also denote by $PC^\varsigma(\Gamma)$, for $\varsigma \in (0, 1]$ the subspace of all $u \in PC(\Gamma)$ such that $u_\alpha \in \mathcal{C}^\varsigma([0, L_\alpha])$ for all $\alpha \in \mathcal{A}$, i.e. such that

$$\sup_{\substack{x, y \in [0, L_\alpha] \\ x \neq y}} \frac{|u_\alpha(x) - u_\alpha(y)|}{|x - y|^\varsigma} < \infty \quad \text{for all } \alpha \in \mathcal{A}.$$

For $x \in \Gamma_\alpha$, $\alpha \in \mathcal{A}$, we define the derivative of u at $x \in \Gamma_\alpha \setminus \mathcal{V}$ as the directional derivative

$$\partial_\alpha u(x) = \lim_{h \rightarrow 0} \frac{u(x + h\mathbf{e}_\alpha) - u(x)}{h} = \lim_{h \rightarrow 0} \frac{u_\alpha(\pi_\alpha^{-1}(x + h\mathbf{e}_\alpha)) - u_\alpha(\pi_\alpha^{-1}(x))}{h}.$$

Higher order derivatives are defined in the same way. At the vertices, we define $\partial_\alpha u(v)$ as the outward directional derivative of u at $v \in \mathcal{V}$ for each $\alpha \in \mathcal{A}_v$, i.e.

$$\partial_\alpha u(v) = \begin{cases} \lim_{h \downarrow 0} \frac{u_\alpha(0) - u_\alpha(h)}{h} & \text{if } v = \pi_\alpha(0), \\ \lim_{h \downarrow 0} \frac{u_\alpha(L_\alpha) - u_\alpha(L_\alpha - h)}{h} & \text{if } v = \pi_\alpha(L_\alpha), \end{cases}$$

when the above limits exist. Notice that if we define

$$n_{v, \alpha} = \begin{cases} 1 & \text{if } v = \pi_\alpha(L_\alpha), \\ -1 & \text{if } v = \pi_\alpha(0), \end{cases} \quad (3)$$

then $\partial_\alpha u(v) = n_{v, \alpha} \partial u_\alpha(\pi_\alpha^{-1}(v))$.

For every integer $k \geq 1$, the function space

$$\mathcal{C}^k(\Gamma) = \left\{ u \in \mathcal{C}(\Gamma) : u_\alpha \in \mathcal{C}^k([0, L_\alpha]), \forall \alpha \in \mathcal{A} \right\},$$

equipped with the norm $\|u\|_{\mathcal{C}^k(\Gamma)} = \sum_{\alpha \in \mathcal{A}} \sum_{0 \leq j \leq k} \|\partial^j u_\alpha\|_{L^\infty(0, L_\alpha)}$ is a Banach space. We will also need to consider Hölder continuous functions on Γ . For each $k \in \mathbb{N}$ and $\varsigma \in (0, 1]$ we define

$$\mathcal{C}^{k, \varsigma}(\Gamma) = \left\{ u \in \mathcal{C}^k(\Gamma) : \|u\|_{\mathcal{C}^{k, \varsigma}(\Gamma)} < \infty \right\}, \quad (4)$$

where

$$\|u\|_{\mathcal{C}^{k, \varsigma}(\Gamma)} = \|u\|_{\mathcal{C}^k(\Gamma)} + \max_{\alpha \in \mathcal{A}} \sup_{\substack{x, y \in [0, L_\alpha] \\ x \neq y}} \frac{|\partial^k u_\alpha(x) - \partial^k u_\alpha(y)|}{|x - y|^\varsigma}.$$

For $p \in [1, \infty]$, we denote by $L^p(\Gamma) := L^p(\Gamma, \mathcal{B}(\Gamma), \mathcal{L})$ and we notice that

$$L^p(\Gamma) = \{u : \Gamma \rightarrow \mathbb{R} : u_\alpha \in L^p(0, L_\alpha) \text{ for every } \alpha \in \mathcal{A}\}.$$

with the equivalent norm $\|u\|_{L^p(\Gamma)} = \left(\sum_{\alpha \in \mathcal{A}} \|u_\alpha\|_{L^p(0, L_\alpha)}^p\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|u\|_{L^\infty(\Gamma)} = \max_{\alpha \in \mathcal{A}} \|u_\alpha\|_{L^\infty(0, L_\alpha)}$. For any integer $k \geq 1$ and every $p \in [1, \infty]$, we define the Sobolev space

$$W^{k,p}(\Gamma) = \{u \in L^p(\Gamma) : u_\alpha \in W^{k,p}((0, L_\alpha)), \forall \alpha \in \mathcal{A}\}.$$

endowed with the norm $\|u\|_{W^{k,p}} = (\|u\|_{L^p(\Gamma)}^p + \sum_{j=1}^k \|\partial^j u\|_{L^p(\Gamma)}^p)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|u\|_{W^{k,\infty}} = \|u\|_{L^\infty(\Gamma)} + \sum_{j=1}^k \|\partial^j u\|_{L^\infty(\Gamma)}$.

We also consider Sobolev space with prescribed value at the vertices. For a given set of strictly positive real numbers $\gamma = \{\gamma_{v,\alpha} \in (0, +\infty) : v \in \mathcal{V}, \alpha \in \mathcal{A}_v\}$, we define

$$V_\gamma^{k,p}(\Gamma) = \left\{u \in W^{k,p}(\Gamma) : \frac{u|_{\Gamma_\alpha}(v)}{\gamma_{v,\alpha}} = \frac{u|_{\Gamma_\beta}(v)}{\gamma_{v,\beta}}, \forall v \in \mathcal{V}, \forall \alpha, \beta \in \mathcal{A}_v\right\}.$$

Moreover if $\gamma_{v,\alpha} = 1$ for every $\alpha \in \mathcal{A}_v$ and each $v \in \mathcal{V}$ we simply write $V^{k,p}(\Gamma)$ and one has the following identification $V^{k,p}(\Gamma) = \mathcal{C}(\Gamma) \cap W^{k,p}(\Gamma)$. These spaces will be alternatively used as the space in which we look for a solution of a given PDE on Γ or as a set of test functions. For any $k \geq 1$ and every $p \in [1, \infty]$, the spaces $W^{k,p}(\Gamma)$ and $V_\gamma^{k,p}(\Gamma)$ are Banach space and $W^{k,2}(\Gamma)$ and $V_\gamma^{k,2}(\Gamma)$ are Hilbert space with obvious inner product.

We finally denote by $\mathcal{P}_1(\Gamma)$ the space of Borel probability measures on Γ endowed with the Kantorovich-Rubinstein metric d_1 (we refer the reader to [23, Chapter 6] for precise definitions).

3 Diffusion process on networks with sticky transition conditions

We introduce diffusion processes on networks with sticky transition conditions, or transition conditions with spatial delay, at the vertices. We consider the linear differential operator \mathcal{G} on Γ defined on each edge Γ_α by

$$\mathcal{G}_\alpha f(x) = \mu_\alpha \partial^2 f(x) + b_\alpha(x) \partial f(x) \quad \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, f \in D(\mathcal{G}), \quad (5)$$

with domain

$$D(\mathcal{G}) = \left\{ f \in \mathcal{C}^2(\Gamma) : \left. \begin{array}{l} \mathcal{G}f \in \mathcal{C}(\Gamma), \eta_v \mathcal{G}f(v) + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha f(v) = 0 \\ \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \partial_\alpha f(v) = 0 \text{ for all } v \in \partial\mathcal{V}, \alpha \in \mathcal{A}_v \end{array} \right\}, \quad (6)$$

where $b \in PC(\Gamma)$, $\eta_v \geq 0$, $\mu_\alpha, \gamma_{v,\alpha} > 0$, $\alpha \in \mathcal{A}$, and $\sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} = 1$ for all $v \in \mathcal{V} \setminus \partial\mathcal{V}$ (note that \mathcal{G} does not depend on the choice of b in the corresponding equivalence class). According to [14, Theorem 3.1], there exists a Feller process X , with continuous paths, defined on Γ with

generator \mathcal{G} . However, in general, there exists no path-wise description of such process, except for the case $\eta_v = 0$ for all $v \in \mathcal{V}$, which corresponds to a Kirchhoff transition condition at the vertices (see [13]), and for sticky Brownian motion (see [6]).

Recently, in [22], a path-wise characterization of the process associated to \mathcal{G} has been obtained for the case of a simple graph Γ^0 composed of one vertex $\mathcal{V} = \{0\}$ and two semi-infinite edges $\mathcal{E} = \{\Gamma_+^0, \Gamma_-^0\}$, where $\Gamma_+^0 = [0, +\infty)$ and $\Gamma_-^0 = (-\infty, 0]$. In this case, we can rewrite the domain of \mathcal{G} as

$$D(\mathcal{G}) = \left\{ f \in \mathcal{C}^2(\Gamma^0) : \mathcal{G}f \in \mathcal{C}(\Gamma^0), \eta \mathcal{G}(0) + p_+ \partial_+ f(0) + p_- \partial_- f(0) = 0 \right\} \quad (7)$$

where $p_{\pm} = \mu_{\pm} \gamma_{0,\pm}$ and ∂_{\pm} denote the outward directional derivatives in $x = 0$. The coefficient p_{\pm} are the probability that the process, being at $x = 0$, enters respectively in Γ_{\pm}^0 . Set $b(x) = b_{\pm}(x)$ and $\sigma = \sqrt{2\mu_{\pm}}$ for $x \in \Gamma_{\pm}^0$ and consider the stochastic differential equation (SDE)

$$\begin{cases} dX(t) &= (b(X(t))dt + \sigma dW(t))1_{\{X(t) \neq 0\}} + (p_+ - p_-)L^X(t, 0) \\ X(0) &= x \end{cases} \quad (8)$$

where

$$\eta L^X(t, 0) = \int_0^t 1_{\{X(s)=0\}} ds \quad (9)$$

is the local time of the process X at 0, which is an increasing process with probability 1 and constant in any interval on which $X(t) \neq 0$.

In the following theorem, we summarize the results in [22] concerning the relationship between the infinitesimal generator defined in (5) and the SDE (8).

Theorem 3.1. *Assume that $\sigma > 0$ and b is uniformly Lipschitz continuous in Γ_{\pm} .*

- (i) *Let $X(t)$ be a diffusion associated with the infinitesimal operator \mathcal{G} defined in (5),(7). Then $X(t)$ is a weak solution to (8) for some Brownian motion $W(t)$.*
- (ii) *The pair given by the SDE (8) and the local time (9) has a solution which is unique in law.*

Observe that solutions to (8) are not unique. For example, the SDE is satisfied by an undelayed process ($\eta = 0$) since this process spends almost no time at 0. Uniqueness, in weak sense, is recovered adding to the process the corresponding local time at $x = 0$.

Remark 3.2. For a undelayed diffusions, i.e. $\eta = 0$, the amount of time that the process X spend at $x = 0$ has zero measure, i.e. $\int_0^t 1_{\{X(s)=0\}} ds = 0$ for all $t > 0$ with probability 1. A sticky diffusion process can be always rewritten as

$$X(t) = Y(r^{-1}(t))$$

where $Y(t)$ is a standard diffusion process and r^{-1} is the functional inverse of the strictly increasing function $r(t) = t + \eta L^X(t, 0)$. Since $r^{-1}(t) < t$ for every $t > 0$, the process $Y(t)$ is slowed down when it is at $x = 0$. Because of this property, $\int_0^t 1_{\{X(s)=0\}} ds$ has positive probability of being strictly positive for all $t > 0$ and, yet, the process does not stay at zero for any positive interval of time. Note that, away from the vertex, X behaves as a standard diffusion process.

The extension of the previous results to the case of a general network is considered in [5]. We summarize in the following theorem the properties that are relevant to this work.

Theorem 3.3. *Let X be the Feller process on the network Γ generated by (5), (6). Then*

- (i) *There exists a one-dimensional Brownian motion W such that, for every $T > 0$, $f \in \mathcal{C}([0, T] \times \Gamma)$ with $\partial_t f \in \mathcal{C}([0, T] \times \Gamma)$ and $\partial_x f, \partial_x^2 f \in \mathcal{C}([0, T], PC(\Gamma))$, and $0 < t \leq T$, we have*

$$\begin{aligned} f(t, X(t)) &= f(0, X(0)) + \int_0^t (\partial_s f(s, X(s)) + \mathcal{G}_\alpha f(s, X(s))) 1_{\{X(s) \in \Gamma_\alpha \setminus \mathcal{V}\}} ds \\ &\quad + 2 \int_0^t \sqrt{\mu_\alpha} \partial_x f(s, X(s)) 1_{\{X(s) \in \Gamma_\alpha \setminus \mathcal{V}\}} dW_s \\ &\quad + \sum_{v \in \mathcal{V}} \int_0^t \left(\eta_v \partial_s f(s, v) - \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v, \alpha} \partial_\alpha f(s, v) \right) dL^X(s, v), \end{aligned}$$

where $L^X(\cdot, v)$ is the local time of the process at $v \in \mathcal{V}$, defined by

$$L^X(t, v) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{d(X(s), v) < \varepsilon\}} ds, \quad (10)$$

and we use the convention that $\eta_v = 0$ if $v \in \partial\mathcal{V}$;

- (ii) *for every bounded measurable function $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ we have*

$$\int_0^t g(s) 1_{\{X(s) = v\}} ds = \eta_v \int_0^t g(s) dL^X(s, v).$$

We come back to the study of the Markov process associated to (5)-(6) and we are interested in characterizing the corresponding stationary distribution, i.e. a probability measure $\mathbf{m} \in \mathcal{P}_1(\Gamma)$ such that (see [11, Chapter 4] for instance)

$$\int \mathcal{G}f(x) \mathbf{m}(dx) = 0 \quad \text{for all } f \in D(\mathcal{G}).$$

We will show that the measure \mathbf{m} splits into an absolutely continuous part and a sum of Dirac masses concentrated at the vertices, i.e.

$$\mathbf{m} = m \mathcal{L} + \sum_{v \in \mathcal{V} \setminus \partial\mathcal{V}} \eta_v T_v[m] \delta_v, \quad (11)$$

where the density $m \in V_\gamma^{1,2}(\Gamma)$ is a weak solution (see Definition 3.4 below) to

$$\begin{cases} -\mu_\alpha \partial^2 m(x) - \partial(b(x)m(x)) = 0 & \text{for all } x \in \Gamma_\alpha, \alpha \in \mathcal{A}, \\ \frac{m|_{\Gamma_\alpha}(v)}{\gamma_{v, \alpha}} = \frac{m|_{\Gamma_\beta}(v)}{\gamma_{v, \beta}} =: T_v[m] & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \partial_\alpha m|_{\Gamma_\alpha}(v) + n_{v, \alpha} m|_{\Gamma_\alpha}(v) b|_{\Gamma_\alpha}(v) = 0 & \text{for all } v \in \mathcal{V}, \\ m \geq 0, 1 \geq \int_\Gamma m dx = 1 - \sum_{v \in \mathcal{V} \setminus \partial\mathcal{V}} \eta_v T_v[m] \geq 0. \end{cases} \quad (12)$$

Note that the second line of the previous system define, up to the factors η_v , the coefficients of the Dirac masses at the vertices.

Definition 3.4. A weak solution to (12) is a function $m \in V_\gamma^{1,2}(\Gamma)$ such that

$$\int_{\Gamma} \mu \partial m \partial v + b m \partial v \, dx = 0 \quad \text{for every } v \in V^{1,2}(\Gamma) \quad (13)$$

and $m \geq 0$, $1 \geq \int_{\Gamma} m \, dx = 1 - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \geq 0$.

We refer the reader to [1] for a justification of the weak formulation of (12). Recall the notation $\gamma := \{\gamma_{v,\alpha} : \alpha \in \mathcal{A}_v, v \in \mathcal{V} \setminus \partial \mathcal{V}\}$.

Lemma 3.5. Assume that $b \in L^\infty(\Gamma)$ and let $m \in V_\gamma^{1,2}(\Gamma)$ satisfy (13). Then there exists a positive constant $C = C(\mu, \|b\|_{L^\infty}, \gamma, \Gamma)$ such that

$$\|m\|_{V_\gamma^{1,2}(\Gamma)} \leq C \|m\|_{L^1}.$$

Proof. We consider the function $\psi \in PC(\Gamma)$ uniquely determined by $\psi|_{\Gamma_\alpha}(v) = \gamma_{v,\alpha}$ for every $\alpha \in \mathcal{A}_v$ and $v \in \mathcal{V} \setminus \partial \mathcal{V}$, ψ_α is affine if $\Gamma_\alpha \cap \partial \mathcal{V} = \emptyset$ and constant otherwise. Then, the function $m\psi$ belongs to $V^{1,2}(\Gamma)$. Using $m\psi$ as a test-function in (13), we obtain, for any $\varepsilon > 0$,

$$\begin{aligned} (\min \gamma) \int_{\Gamma} \mu |\partial m|^2 \, dx &\leq \int_{\Gamma} \mu |m \partial m \partial \psi| + |b m (m \partial \psi + \partial m \psi)| \, dx \\ &\leq C \int_{\Gamma} |m \partial m| + |m|^2 \, dx \leq C \left(\varepsilon \|\partial m\|_{L^2}^2 + C_\varepsilon \|m\|_{L^2} \right). \end{aligned} \quad (14)$$

As a consequence of the continuous embedding $W^{1,1}(\Gamma) \hookrightarrow L^\infty(\Gamma)$ we have

$$\|m\|_{L^2} \leq \|m\|_{L^\infty}^{\frac{1}{2}} \|m\|_{L^1}^{\frac{1}{2}} \leq C_1 \|m\|_{W^{1,1}}^{\frac{1}{2}} \|m\|_{L^1}^{\frac{1}{2}} \leq C_2 (\|\partial m\|_{L^2} + \|m\|_{L^1})^{\frac{1}{2}} \|m\|_{L^1}^{\frac{1}{2}}. \quad (15)$$

Combining (14) and (15) and using Young's inequality we deduce

$$(\min \gamma) \underline{\mu} \|\partial m\|_{L^2}^2 \leq 2\varepsilon \|\partial m\|_{L^2}^2 + \tilde{C}_\varepsilon \|m\|_{L^1},$$

where $\underline{\mu} = \min_{\alpha \in \mathcal{A}} \mu_\alpha > 0$. Choosing ε small enough we obtain the desired inequality. \square

Lemma 3.6. Let $b \in PC(\Gamma)$. Then, for every $\vartheta \geq 0$ there exists unique $m^\vartheta \in V_\gamma^{1,2}(\Gamma)$ such that $m^\vartheta \geq 0$, $\int_{\Gamma} m^\vartheta \, dx = \vartheta$ and satisfying (13). Furthermore $m^0 = 0$ and $m^\vartheta > 0$ for $\vartheta > 0$.

Proof. From Lemma 3.5 it is clear that $m^0 = 0$ is the unique element in $V_\gamma^{1,2}(\Gamma)$ satisfying the conditions in the statement. From [1, Theorem 2.7] we know that there exists a unique m^1 satisfying (13) with $\int_{\Gamma} m^1 \, dx = 1$. Moreover we know that $m^1 > 0$, in particular $T_v[m^1] > 0$ for every $v \in \mathcal{V} \setminus \partial \mathcal{V}$. Then, setting $m^\vartheta = \vartheta m^1$, we obtain an element in $V_\gamma^{1,2}(\Gamma)$ satisfying (13), $m^\vartheta > 0$ and $\int_{\Gamma} m^\vartheta \, dx = \vartheta$. \square

Proposition 3.7. Let $b \in PC(\Gamma)$. Then there exists a unique weak solution $m \in V_\gamma^{1,2}(\Gamma)$ to (12).

Proof. For existence, we consider the mapping $\Phi: [0, 1] \rightarrow \mathbb{R}_+$ defined by

$$\Phi(\vartheta) = \vartheta + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m^\vartheta],$$

where m^ϑ is given in Lemma 3.6. We claim that the mapping Φ is continuous. Indeed, let $(\vartheta_n)_{n \geq 0}$ be a sequence in $[0, 1]$ converging to some $\vartheta \in [0, 1]$ as n tends to infinity. Using Lemma 3.6, for each $n \geq 0$, there exists $m^n \in V_\gamma^{1,2}(\Gamma)$ such that $\int m^n dx = \vartheta_n$, $m^n \geq 0$ and satisfies (13). Using Lemma 3.5, we see that the sequence $(m^n)_{n \geq 0}$ is bounded in $V_\gamma^{1,2}(\Gamma)$ and we may therefore extract a subsequence, which we still denote by m^n , converging weakly in $V_\gamma^{1,2}(\Gamma)$ and strongly in $PC(\Gamma)$ to some $m \in V_\gamma^{1,2}(\Gamma)$ as n tends to infinity. It follows that m also satisfies (13), is non-negative and $\int m dx = \vartheta$. From uniqueness in Lemma 3.6 we conclude to $m = m^\vartheta$ and that the whole sequence converges to m^ϑ . This proves the continuity of the mapping $\vartheta \mapsto m^\vartheta$ from $[0, 1]$ to $PC(\Gamma)$. The continuity of Φ then follows from the continuity of $PC(\Gamma) \ni \tilde{m} \mapsto T_v[\tilde{m}] \in \mathbb{R}$ for every $v \in \mathcal{V} \setminus \partial \mathcal{V}$.

Notice that $\Phi(0) = 0$ and that $\Phi(\vartheta) > \vartheta$ for $\vartheta > 0$ since each m^ϑ is strictly positive. From the intermediate value theorem, we conclude that there exists $\bar{\vartheta} \in [0, 1]$ such that

$$\Phi(\bar{\vartheta}) = \bar{\vartheta} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m^{\bar{\vartheta}}] = \int_\Gamma m^{\bar{\vartheta}} dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m^{\bar{\vartheta}}] = 1.$$

This proves existence.

We now prove uniqueness. Notice first that we cannot have $\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m] = 1$ since, otherwise, Lemma 3.6 implies $m = 0$, which is a contradiction. Let m_1 and m_2 be two weak solutions to (12). Up to relabelling, we may assume that

$$1 > \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_1] \geq \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_2]. \quad (16)$$

Set $w_i = \left(1 - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_i]\right)^{-1} m_i$ for $i = 1, 2$. Then w_i satisfies (12), $w_i \geq 0$ and $\int w_i dx = 1$. From uniqueness in [1, Theorem 2.7] we deduce that $w_1 = w_2 =: \bar{m}$. In particular

$$m_i = \left(1 - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_i]\right) \bar{m}. \quad (17)$$

From (16) we obtain $m_1 \leq m_2$ and therefore

$$\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_1] = \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v [m_2].$$

Uniqueness then follows from (17). □

Theorem 3.8. *The measure \mathfrak{m} defined by (11) is a stationary distribution for the Markov process generated by (5) and (6).*

Proof. Let $f \in D(\mathcal{G})$. Integrating by part and using the fact that $m \in V_\gamma^{1,2}(\Gamma)$, we compute

$$\begin{aligned} - \int_\Gamma \mu \partial^2 f(x) m(x) dx &= - \sum_{\alpha \in \mathcal{A}} \int_0^{L_\alpha} \mu_\alpha \partial^2 f_\alpha(x) m_\alpha(x) dx \\ &= - \sum_{\alpha \in \mathcal{A}} \mu_\alpha [\partial f_\alpha(x) m_\alpha(x)]_{x=0}^{x=L_\alpha} + \sum_{\alpha \in \mathcal{A}} \int_0^{L_\alpha} \mu_\alpha \partial f_\alpha(x) \partial m_\alpha(x) dx \\ &= - \left[\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha f(v) T_v[m] \right] + \int_\Gamma \mu \partial f(x) \partial m(x) dx. \end{aligned}$$

Since $f \in D(\mathcal{G})$ and m satisfies (13), it follows that

$$\begin{aligned} - \int_\Gamma \mathcal{G}f(x) \mathbf{m}(dx) &= - \int_\Gamma \mathcal{G}f(x) m(x) dx - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \mathcal{G}f(v) \\ &= \int_\Gamma (\mu \partial f \partial m + b \partial f m) dx - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} T_v[m] \left(\eta_v \mathcal{G}f(v) + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha f(v) \right) = 0. \end{aligned}$$

□

Remark 3.9. Observe that in the undelayed case, i.e. $\eta_v = 0$ for every $v \in \mathcal{V} \setminus \partial \mathcal{V}$, then we recover the stationary distribution obtained in [1].

Proposition 3.10 (Stability of m). *Given $b^n \in PC(\Gamma)$, $n \in \mathbb{N}$, let $m^n \in V_\gamma^{1,2}(\Gamma)$ be the corresponding solution to (12), given by Proposition 3.7. Assume that there exists $b \in PC(\Gamma)$ such that b_α^n converges uniformly to b_α for every $\alpha \in \mathcal{A}$. Then there exists $m \in V_\gamma^{1,2}(\Gamma)$ such that*

$$\begin{cases} m_\alpha^n \rightarrow m_\alpha & \text{uniformly, for every } \alpha \in \mathcal{A}, \\ m^n \rightharpoonup m & \text{weakly in } V_\gamma^{1,2}(\Gamma). \end{cases} \quad (18)$$

Proof. We first consider arbitrary subsequences, still denoted by b^n and m^n . We know that $m^n \geq 0$ and $0 \leq \int_\Gamma m^n dx \leq 1$ for every $n \in \mathbb{N}$. In particular $(m^n)_{n \in \mathbb{N}}$ is bounded in $L^1(\Gamma)$. It then follows from Lemma 3.5 that it is also bounded in $V_\gamma^{1,2}(\Gamma)$. We deduce that there exists a strictly increasing map $\iota: \mathbb{N} \rightarrow \mathbb{N}$ and $m \in V_\gamma^{1,2}(\Gamma)$ such that $m^{\iota(n)} \rightharpoonup m$ weakly in $V_\gamma^{1,2}(\Gamma)$ and $m_\alpha^{\iota(n)} \rightarrow m_\alpha$ uniformly. We can pass to the limit in the weak formulation (13) to deduce that m also satisfies (13). In addition we also have

$$1 = \lim_{n \rightarrow \infty} \left[\int_\Gamma m^{\iota(n)} dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m^{\iota(n)}] \right] = \int_\Gamma m dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m].$$

This proves that m is a solution (12). Since this solution is unique we have proven that every subsequence of $(m^n)_{n \in \mathbb{N}}$ has a further subsequence converging to a unique limit. We conclude that the full sequence converges to m in the topologies mentioned in (18).

□

Proposition 3.11 (Stability of \mathbf{m}). *Let $(m^n)_{n \in \mathbb{N}}$ be a sequence in $PC(\Gamma)$ satisfying*

$$m^n \geq 0 \quad \text{and} \quad \int_{\Gamma} m^n dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m^n] = 1 \quad \text{for all } n \in \mathbb{N},$$

and converging uniformly to some $m \in PC(\Gamma)$. Define $(\mathbf{m}^n)_{n \in \mathbb{N}}$ and \mathbf{m} according to (11). Then \mathbf{m}_n converges to \mathbf{m} in $\mathcal{P}_1(\Gamma)$.

Proof. Notice first that the uniform convergence of m^n to m implies that $m \geq 0$ and that

$$1 = \lim_{n \rightarrow \infty} \left[\int_{\Gamma} m^n dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m^n] \right] = \int_{\Gamma} m dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m].$$

It follows that we indeed have $\mathbf{m} \in \mathcal{P}_1(\Gamma)$. Let $\psi \in \text{Lip}(\Gamma)$, the space of functions from Γ to \mathbb{R} which are Lipschitz continuous with respect to the geodesic metric on Γ . We compute

$$\begin{aligned} \int_{\Gamma} \psi(x)(\mathbf{m} - m^n)(dx) &= \int_{\Gamma} \psi(x)(m(x) - m^n(x))dx + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v \psi(v) (T_v[m] - T_v[m^n]) \\ &\leq \|\psi\|_{\text{Lip}(\Gamma)} \text{diam}(\Gamma) \left(\|m - m^n\|_{L^1(\Gamma)} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v |T_v[m] - T_v[m^n]| \right). \end{aligned}$$

From standard properties of the Kantorovich-Rubinstein metric (see [23, Remark 6.5] for instance) we conclude that $\lim_{n \rightarrow \infty} \mathbf{d}_1(\mathbf{m}^n, \mathbf{m}) = 0$. \square

4 The Hamilton-Jacobi-Bellman equation

This section is devoted to the study of the Hamilton-Jacobi-Bellman equation entering in the Mean Field Game system. Moreover, we provide a justification of the transition condition at the vertices by proving a verification theorem for a discounted infinite horizon optimal control problem, whose dynamics is given by a controlled sticky diffusion process on the network.

We consider the discounted Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\mu_{\alpha} \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x) = F(x) & \text{for all } x \in \Gamma_{\alpha} \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_{\alpha}}(v) = u|_{\Gamma_{\beta}}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_{\alpha} \gamma_{v, \alpha} \partial_{\alpha} u(v) = \eta_v (\theta_v - \lambda u(v)) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_{\alpha} u(v) = 0 & \text{for all } v \in \partial \mathcal{V}, \end{cases} \quad (19)$$

and the corresponding ergodic problem

$$\begin{cases} -\mu_{\alpha} \partial^2 u(x) + H(x, \partial u(x)) + \rho = F(x) & \text{for all } x \in \Gamma_{\alpha} \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_{\alpha}}(v) = u|_{\Gamma_{\beta}}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_{\alpha} \gamma_{v, \alpha} \partial_{\alpha} u(v) = \eta_v (\theta_v - \rho) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_{\alpha} u(v) = 0 & \text{for all } v \in \partial \mathcal{V}. \end{cases} \quad (20)$$

The second line in (19) and (20) gives the continuity of u at the vertices, while the following two lines are a generalized Kirchhoff condition at the internal vertices and a Neumann condition at the boundary vertices.

We will make the following assumptions.

(H1) We assume that $\lambda > 0$ and that, for each $v \in \mathcal{V} \setminus \partial\mathcal{V}$ and $\alpha \in \mathcal{A}$, $\mu_\alpha > 0$, $\gamma_{v,\alpha} > 0$, $\eta_v \geq 0$ and $\theta_v \in \mathbb{R}$. Moreover

$$\sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} = 1 \quad \text{for all } v \in \mathcal{V}.$$

(H2) For each $\alpha \in \mathcal{A}$,

- The mapping $H_\alpha : \Gamma_\alpha \times \mathbb{R} \rightarrow \mathbb{R}$, defined by

$$H_\alpha(x, p) = \begin{cases} H(x, p) & \text{if } x \in \Gamma_\alpha \setminus \mathcal{V}, \\ \lim_{\substack{y \rightarrow x \\ y \in \Gamma_\alpha \setminus \mathcal{V}}} H(y, p) & \text{if } x \in \mathcal{V}, \end{cases}$$

is well-defined and continuous; moreover, for each $x \in \Gamma_\alpha \setminus \mathcal{V}$, the mapping $p \mapsto H_\alpha(x, p)$ is continuously differentiable and $\partial_p H_\alpha$ can be extended to a continuous mapping on $\Gamma_\alpha \times \mathbb{R}$.

- There exists a constant $C_H > 0$ and $q \in (1, 2]$ such that

$$|H_\alpha(x, p)| \leq C_H (1 + |p|^q) \quad \text{for all } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \quad (21)$$

$$|\partial_p H_\alpha(x, p)| \leq C_H (1 + |p|^{q-1}) \quad \text{for all } (x, p) \in \Gamma_\alpha \times \mathbb{R}, \quad (22)$$

$$H_\alpha(x, p) \geq C_H^{-1} |p|^q - C_H \quad \text{for all } (x, p) \in \Gamma_\alpha \times \mathbb{R}. \quad (23)$$

(H3) F belongs to $PC^\varsigma(\Gamma)$ for some $\varsigma \in (0, 1)$.

Proposition 4.1. *Assume **(H1)**-**(H3)**. Then, there exists a solution $u \in \mathcal{C}^{2,\varsigma}(\Gamma)$ to (19).*

Proof. The following argument was introduced for star-shaped networks in [21] and extended to general networks with Kirchhoff conditions in [3]. Let us write $\mathbb{V} = \mathcal{V} \setminus \partial\mathcal{V}$. Let $\Phi : \mathbb{R}^{\mathbb{V}} \rightarrow \mathcal{C}^2(\Gamma)$ with $\Phi(z) = u^z$, where

$$\begin{cases} -\mu_\alpha \partial^2 u_\alpha^z(x) + H_\alpha(x, \partial u_\alpha^z(x)) + \lambda u_\alpha^z(x) = F_\alpha(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \\ u_\alpha^z(0) = z_v \text{ if } v = \pi_\alpha(0) \in \mathbb{V} \text{ and } \partial u_\alpha^z(0) = 0 \text{ if } v \in \partial\mathcal{V}, \\ u_\alpha^z(L_\alpha) = z_v \text{ if } v = \pi_\alpha(L_\alpha) \in \mathbb{V} \text{ and } \partial u_\alpha^z(L_\alpha) = 0 \text{ if } v \in \partial\mathcal{V}. \end{cases} \quad (24)$$

System (24) is family of PDEs defined for each edge Γ_α coupled by means of the boundary value $u_\alpha^z(v) = z_v$ for all $\alpha \in \mathcal{A}_v$, $v \in \mathbb{V}$. From the standard theory of quasi-linear elliptic equations we know that this mapping is well-defined and that

$$\max_{\alpha \in \mathcal{A}} \|u_\alpha^z\|_{\mathcal{C}^{2,\varsigma}([0, L_\alpha])} \leq C \left(|z|_\infty + \max_{\alpha \in \mathcal{A}} \|F_\alpha\|_{\mathcal{C}^{0,\varsigma}([0, L_\alpha])} \right).$$

A straightforward adaptation of [21, Proposition B.1] shows that Φ is continuous. We then set

$$K := \max \left\{ \|F\|_{L^\infty} + \|H(\cdot, 0)\|_{L^\infty}, \max_{v \in \mathcal{V}} |\theta_v| \right\}.$$

For every $z \in \mathbb{R}^\mathbb{V}$, one can see that the constant function

$$\phi_\alpha^z := \max \left\{ K/\lambda, \max_{v \in \mathcal{V}} |z| \right\}$$

is a super-solution to (24) for every $\alpha \in \mathcal{A}$. From a standard comparison principle in each Γ_α we deduce that $u_\alpha^z \leq \phi_\alpha^z$ for every $\alpha \in \mathcal{A}$. Fix $v_0 \in \mathcal{V} \setminus \partial\mathcal{V}$, $M = \max\{K/\lambda, K\}$ and z^0 be such that

$$\begin{cases} z_v^0 = M & \text{if } v = v_0, \\ z_v^0 \leq M & \text{otherwise.} \end{cases}$$

In this case we have $\phi_\alpha^{z^0} \equiv M$ for every $\alpha \in \mathcal{A}$. Since $u_\alpha^{z^0} \leq \phi_\alpha^{z^0}$ and $u_\alpha^{z^0}(\pi_\alpha^{-1}(v_0)) = M = \phi_\alpha^{z^0}(\pi_\alpha^{-1}(v_0))$, it follows that $\pi_\alpha^{-1}(v_0)$ is a maximum point of $u_\alpha^{z^0}$ for every $\alpha \in \mathcal{A}_{v_0}$, and therefore $\partial u_\alpha^{z^0}(v_0) \geq 0$. Hence, recalling the definition of K , we have

$$\eta_v \lambda z_v^0 + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial u_\alpha^{z^0}(\pi_\alpha^{-1}(v)) \geq \eta_v \theta_v.$$

An analogous conclusion can be repeated for every $v \in \mathbb{V}$. Similarly, if z^0 is such that

$$\begin{cases} z_v^0 = -M & \text{if } v = v_0, \\ z_v^0 \geq -M & \text{otherwise.} \end{cases}$$

we obtain, by replacing ϕ_α^z with $-\phi_\alpha^z$ which is a sub-solution to (24) for every $\alpha \in \mathcal{A}$, that

$$\eta_v \lambda z_{v_0}^0 + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial u_\alpha^{z^0}(\pi_\alpha^{-1}(v_0)) \leq \eta_v \theta_v.$$

For each $v \in \mathbb{V}$ we consider the continuous mapping $\Psi_v: \mathcal{C}^1(\Gamma) \rightarrow \mathbb{R}$ defined by

$$\Psi_v(u) = \eta_v \lambda u(v) + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v).$$

We have proven that, for each $v \in \mathbb{V}$, the mapping $\Psi_v \circ \Phi: [-M, M]^\mathbb{V} \rightarrow \mathbb{R}$ is continuous and satisfies

$$\begin{cases} \Psi_v \circ \Phi(z) \geq \eta_v \theta_v & \text{if } z_v = M, \\ \Psi_v \circ \Phi(z) \leq \eta_v \theta_v & \text{if } z_v = -M. \end{cases}$$

We can apply the Poincaré-Miranda theorem [19, 18] to the mapping $[-M, M]^\mathbb{V} \ni z \mapsto (\Psi_v \circ \Phi(z))_{v \in \mathbb{V}} \in \mathbb{R}^\mathbb{V}$ to conclude that there exists $z^* \in [-M, M]^\mathbb{V}$ such that $\Psi_v \circ \Phi(z^*) = \eta_v \theta_v$ for every $v \in \mathbb{V}$. This concludes the proof. \square

To show the uniqueness of the solution to (19), we prove a comparison principle.

Definition 4.2 (Sub- and Super-solutions).

1. We say that $u \in \mathcal{C}^2(\Gamma)$ is a sub-solution to (19) if

$$\begin{cases} -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x) \leq F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) \leq \eta_v (\theta_v - \lambda u(v)) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha u(v) \leq 0 & \text{for all } v \in \partial \mathcal{V}. \end{cases}$$

2. We say that $v \in \mathcal{C}^2(\Gamma)$ is a super-solution to (19) if

$$\begin{cases} -\mu_\alpha \partial^2 v(x) + H(x, \partial v(x)) + \lambda v(x) \geq F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha v(v) \geq \eta_v (\theta_v - \lambda v(v)) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha v(v) \geq 0 & \text{for all } v \in \partial \mathcal{V}. \end{cases}$$

Proposition 4.3 (Comparison principle). *Assume (H1)-(H2) and let $u, v \in \mathcal{C}^2(\Gamma)$ be sub- and super-solution to (19), respectively. Then $u \leq v$ in Γ . In particular there exists at most one classical solution to (19).*

Proof. Set $w = u - v$. Clearly $w \in \mathcal{C}^2(\Gamma)$ satisfies

$$\begin{cases} -\mu_\alpha \partial^2 w_\alpha + b_\alpha \partial w_\alpha + \lambda w_\alpha \leq 0 & \text{in } (0, L_\alpha) \text{ for all } \alpha \in \mathcal{A}, \\ \eta_v \lambda w(v) + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha w(v) \leq 0 & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha w(v) \leq 0 & \text{for all } v \in \partial \mathcal{V}, \end{cases} \quad (25)$$

where

$$b_\alpha(x) = \int_0^1 \partial_p H_\alpha(x, t u_\alpha(x) + (1-t)v_\alpha(x)) dt \quad \text{for all } x \in (0, L_\alpha).$$

Let x_0 be a maximum point of w , such a point exists since Γ is a compact metric space and w is continuous on Γ . We may assume that $M := w(x_0) > 0$, since otherwise there is nothing to prove. Moreover, because of the strong maximum principle [15, Theorem 3.5] and Hopf's lemma [15, Lemma 3.4] we know that either w is constant in Γ_α , where $\alpha \in \mathcal{A}$ is such that $x_0 \in \Gamma_\alpha$ or $x_0 \in \mathcal{V}$ and $\partial_\alpha w(x_0) > 0$. In the first case, (25) directly yields $M \leq 0$, a contradiction. We must therefore have $x_0 = v \in \mathcal{V}$. Using the same argument for each $\alpha \in \mathcal{A}_v$, we are left with the case $\partial_\alpha w(v) > 0$ for every $\alpha \in \mathcal{A}_v$. In the case where $v \in \partial \mathcal{V}$, we also directly obtain a contradiction from (25). We are henceforth left with the case $v \in \mathcal{V} \setminus \partial \mathcal{V}$. We have

$$\sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha w(v) > 0$$

so that (25) implies $\eta_v \lambda M = \eta_v \lambda w(v) < 0$, also a contradiction. We conclude that $M \leq 0$ and, in particular, that $u \leq v$ in Γ . \square

We now turn to the study of (20). As usual, we show existence of a solution passing to the limit for $\lambda \rightarrow 0^+$ in (19).

Lemma 4.4. *Assume (H1)-(H3) and let $u \in \mathcal{C}^2(\Gamma)$ be the solution to (19). There exists $C_1 > 0$, depending only on $\|F\|_{L^\infty}$, $(\theta_v)_{v \in \mathcal{V} \setminus \partial \mathcal{V}}$ and $\|H(\cdot, 0)\|_{L^\infty}$, such that*

$$\|\lambda u\|_{L^\infty} \leq C_1 \quad (26)$$

Proof. Setting $C = \max_{x \in \Gamma} |H_\alpha(x, 0) - F(x)|$ and $C_1 = \max\{C, \max\{\theta_v\}_{v \in \mathcal{V} \setminus \partial \mathcal{V}}\}$, we see that the constant functions $-C_1/\lambda$ and C_1/λ are respectively a sub and a supersolution to (19). Hence the results follows immediately from Proposition 4.3. \square

Lemma 4.5. *Assume (H1)-(H3) and let $u \in \mathcal{C}^2(\Gamma)$ the solution to (19). There exists $C_2 > 0$, depending only on $\|F\|_{L^\infty}$, $(|\theta_v|)_{v \in \mathcal{V} \setminus \partial \mathcal{V}}$, $\|H(\cdot, 0)\|_{L^\infty}$, C_H , γ , Γ and q , such that*

$$\|\partial u\|_{L^q} \leq C_2.$$

Proof. Let $\psi: \Gamma \rightarrow \mathbb{R}_+$ be the unique function affine in Γ_α for each $\alpha \in \mathcal{A}$, $\psi|_{\Gamma_\alpha}(v) = \gamma_{v,\alpha}$ for all $v \in \mathcal{V} \setminus \partial \mathcal{V}$ and $\alpha \in \mathcal{A}_v$ and ψ_α constant if the edge Γ_α touches the boundary. Multiplying by ψ in the equation satisfied by u and integrating by parts we obtain

$$\begin{aligned} \int_{\Gamma} F \psi \, dx &= \int_{\Gamma} (-\mu \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x)) \psi \, dx \\ &= \int_{\Gamma} \mu \partial u \partial \psi + H(x, \partial u) \psi + \lambda u \psi \, dx - \sum_{\alpha \in \mathcal{A}} [\mu_\alpha \partial u_\alpha(x) \psi_\alpha(x)]_{x=0}^{x=L_\alpha} \\ &= \int_{\Gamma} \mu \partial u \partial \psi + H(x, \partial u) \psi + \lambda u \psi \, dx - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} n_{v,\alpha} \mu_\alpha \gamma_{v,\alpha} \partial u|_{\Gamma_\alpha}(v) \\ &= \int_{\Gamma} \mu \partial u \partial \psi + H(x, \partial u) \psi + \lambda u \psi \, dx - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) \\ &= \int_{\Gamma} \mu \partial u \partial \psi + H(x, \partial u) \psi + \lambda u \psi \, dx + \left(\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \lambda u(v) - \theta_v \right). \end{aligned}$$

Notice that

$$0 < \min \gamma \leq \psi \leq \max \gamma < +\infty.$$

Using (H2) and (26), we deduce that there exists a positive constant $C > 0$, independent of λ , such that

$$\begin{aligned} \int_{\Gamma} |\partial u|^q \, dx &\leq C \left(\int_{\Gamma} |\partial u| \, dx + C_1 + \max_v |\theta_v| + \|F\|_{L^\infty} \right) \\ &\leq C \left(\|\partial u\|_{L^q}^{1/q} \mathcal{L}(\Gamma)^{(q-1)/q} + C_1 + \max_v |\theta_v| + \|F\|_{L^\infty} \right). \end{aligned}$$

The conclusion then follows from Young's inequality. \square

Theorem 4.6. *Under assumptions (H1)-(H3), there exists a solution $(u, \rho) \in \mathcal{C}^{2,\varsigma}(\Gamma) \times \mathbb{R}$ to (20). Furthermore, this solution is unique up to translation of u by a constant. Finally there exists $\hat{C} > 0$, depending only on $\|F\|_{L^\infty}$, $(|\theta_v|)_{v \in \mathcal{V} \setminus \partial \mathcal{V}}$ and $\|H(\cdot, 0)\|_{L^\infty}$, C_H , γ , Γ and q , such that*

$$\|\partial u\|_{L^q(\Gamma)} + \|u\|_{W^{2,1}(\Gamma)} \leq \hat{C}.$$

Proof. The argument follows the lines of [1, Theorem 3.4].

Proof of Existence : For each $k \in \mathbb{N}$ we set $\lambda_k = 2^{-k}$ and consider $v_k \in \mathcal{C}^{2,\varsigma}(\Gamma)$ the unique solution to (19) with λ replaced by λ_k . For each $k \in \mathbb{N}$ there exists $x_k \in \Gamma$ such that $\min_\Gamma v_k = v(x_k)$ and we set $u_k = v_k - v_k(x_k)$. Each u_k satisfies

$$\begin{cases} -\mu_\alpha \partial^2 u_k(x) + H(x, \partial u_k(x)) + \lambda_k u_k(x) = F(x) - \lambda_k v_k(x_k) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ u_k|_{\Gamma_\alpha}(v) = u_k|_{\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V}, \\ \lambda_k u_k(v) + \eta^{-1} \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u_k(v) = \theta_v - \lambda_k v_k(x_k) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha u_k(v) = 0 & \text{for all } v \in \partial \mathcal{V}. \end{cases} \quad (27)$$

In particular, using (H2) and Lemma 4.4, we have

$$\begin{aligned} |\partial^2 u_k| &\leq \underline{\mu}^{-1} (|H(x, \partial u_k)| + |F| + \lambda_k (|u_k| + |v_k(x_k)|)) \\ &\leq \underline{\mu}^{-1} (C_H (1 + |\partial v_k|^q) + \|F\|_{L^\infty} + 2C_1), \end{aligned}$$

where $\underline{\mu} = \min_{\alpha \in \mathcal{A}} \mu_\alpha$. From Lemma 4.5 we deduce that ∂u_k is bounded in $W^{1,1}(\Gamma)$, uniformly in k .

Let $x \in \Gamma$. Since Γ is connected and because each v_k is continuous, there exist $N \in \mathbb{N}$, $(v_1, \dots, v_N) \in \mathcal{V}^N$ and $(\alpha_1, \dots, \alpha_{N+1}) \in \mathcal{A}^{N+1}$, such that

$$\begin{cases} x_k \in \Gamma_{\alpha_1}, \\ x \in \Gamma_{\alpha_{N+1}}, \\ v_n \in \Gamma_{\alpha_n} \cap \Gamma_{\alpha_{n+1}} \text{ for all } 1 \leq n \leq N, \end{cases}$$

and, also using Lemma 4.5,

$$\begin{aligned} |v_k(x) - v_k(x_k)| &\leq \int_{\pi_{\alpha_1}^{-1}(x_k)}^{\pi_{\alpha_1}^{-1}(v_1)} |\partial v_k(y)| dy + \left[\sum_{n=2}^N \int_{\pi_{\alpha_n}^{-1}(v_{n-1})}^{\pi_{\alpha_n}^{-1}(v_n)} |\partial v_k(y)| dy \right] \\ &\quad + \int_{\pi_{\alpha_{N+1}}^{-1}(v_N)}^{\pi_{\alpha_{N+1}}^{-1}(x)} |\partial v_k(y)| dy \leq \|\partial v_k\|_{L^1} \leq C_2 \mathcal{L}(\Gamma)^{(q-1)/q}. \end{aligned}$$

It follows that

$$\|u_k\|_{L^\infty} \leq C_2 \mathcal{L}(\Gamma)^{(q-1)/q} \quad \text{for all } k \in \mathbb{N}.$$

This proves that the sequence $(u_k)_{k \in \mathbb{N}}$ is bounded in $W^{2,1}(\Gamma)$. From the continuous embedding $W^{2,1}(\Gamma) \hookrightarrow \mathcal{C}^1(\Gamma)$ we deduce from a bootstrap argument that the sequence is also bounded in $\mathcal{C}^2(\Gamma)$.

It follows that, up to extraction of a subsequence, we have

- the sequence $(\lambda_k v_k(x_k))_{k \in \mathbb{N}}$ converges to some constant $\rho \in \mathbb{R}$,
- the sequence $(u_k)_{k \in \mathbb{N}}$ converges in $\mathcal{C}^1(\Gamma)$ to some u , with $u \in V^{2,\infty}(\Gamma)$,
- the sequence $(\partial^2 u_k)_{k \in \mathbb{N}}$ converges weakly in $L^p(\Gamma)$ to $\partial^2 u$, for any $1 < p < \infty$.

Passing to the limit in (27) we obtain that

$$\begin{cases} \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta(\theta_v - \rho) & \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial\mathcal{V}. \end{cases}$$

In addition, if we fix $\alpha \in \mathcal{A}$ and consider $\varphi \in \mathcal{C}^\infty(\Gamma)$, compactly supported in the interior of Γ_α , we obtain

$$\int_\Gamma (-\mu \partial^2 u + H(x, \partial u)) \varphi dx = \int_\Gamma (F - \rho) \varphi dx.$$

It follows that

$$-\mu \partial^2 u + H(x, \partial u) = F - \rho \quad \text{almost everywhere in } \Gamma \setminus \mathcal{V}.$$

We then deduce that $u \in \mathcal{C}^{2,\varsigma}(\Gamma)$ and is a classical solution to (20).

Proof of Uniqueness : Let (u_1, ρ_1) and (u_2, ρ_2) be two solutions to (20). Set $w = u_1 - u_2$ and $\bar{\rho} = \rho_1 - \rho_2$. Up to relabelling, we may assume that $\bar{\rho} \geq 0$. Then $(w, \bar{\rho})$ satisfies

$$\begin{cases} -\mu_\alpha \partial^2 w_\alpha + b_\alpha \partial w_\alpha = -\bar{\rho} \leq 0 & \text{in } (0, L_\alpha) \text{ for all } \alpha \in \mathcal{A}, \\ \bar{\rho} + \eta_v^{-1} \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha w(v) = 0 & \text{for all } v \in \mathcal{V} \setminus \partial\mathcal{V}, \\ \partial_\alpha w(v) = 0 & \text{for all } v \in \partial\mathcal{V}, \end{cases} \quad (28)$$

where

$$b_\alpha(x) = \int_0^1 \partial_p H_\alpha(x, t u_\alpha(x) + (1-t)v_\alpha(x)) dt \quad \text{for all } x \in (0, L_\alpha).$$

We first claim that $\bar{\rho} = 0$. Indeed, since w is continuous and Γ compact, we may choose $x_0 \in \operatorname{argmax}_\Gamma w$. Assume first that x_0 belongs to the interior of Γ_α for some $\alpha \in \mathcal{A}$. Using the optimality conditions at x_0 in (28) we obtain $\bar{\rho} \leq 0$. Assume now that $x_0 = v \in \mathcal{V}$. From Hopf's lemma we know that we must have $v \in \mathcal{V} \setminus \partial\mathcal{V}$. In this case we have $\partial_\alpha w(v) \geq 0$ for all $\alpha \in \mathcal{A}_v$, so that the junction condition in (28) also yields $\bar{\rho} \leq 0$. This proves the claim.

Now, from the strong maximum principle, we know that w cannot attain a maximum point outside \mathcal{V} unless it is constant. On the other hand, as a consequence of Hopf's lemma, if $v \in \mathcal{V}$ is a strict maximum of w we must have $\partial_\alpha w(v) > 0$ for all $\alpha \in \mathcal{A}$, which contradicts the junction condition in (28). This concludes the proof. \square

4.1 Verification Theorem

Following [12, Section III.8], we introduce an optimal control problem for a sticky diffusion process on the network and we prove a verification theorem for the corresponding value function in term of the solution to (19). We consider $\mu_\alpha, \gamma_\alpha, \eta_v$ as in (H1), the set of controls

$$\mathfrak{A} = \{a : \Gamma \rightarrow \mathbb{R} : a \in \mathcal{PC}(\Gamma), \|a\|_{\mathcal{PC}(\Gamma)} \leq R\},$$

for some positive constant R , and a function $b : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for every $\alpha \in \mathcal{A}$, the function $b|_{\Gamma_\alpha \times \mathbb{R}}$ is continuous for the topology of $\Gamma_\alpha \times \mathbb{R}$ induced by $\Gamma \times \mathbb{R}$. In this case, and for every $a \in \mathfrak{A}$, we have that $b(\cdot, a(\cdot))$ also belongs to $\mathcal{PC}(\Gamma)$ and we may therefore consider the generator of a sticky Markov process

$$\mathcal{G}_\alpha^a f(x) = \mu_\alpha \partial^2 f(x) + b_\alpha(x, a_\alpha(x)) \partial f(x) \quad \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \quad (29)$$

with domain

$$D(\mathcal{G}^a) = \left\{ f \in \mathcal{C}^2(\Gamma) : \begin{array}{l} \mathcal{G}^a f \in \mathcal{C}(\Gamma), \eta_v \mathcal{G}^a f(v) + \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha f(v) = 0 \\ \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \partial_\alpha f(v) = 0 \text{ for all } v \in \partial \mathcal{V} \end{array} \right\}. \quad (30)$$

For every $x \in \Gamma$, let $X^{a,x} = X^a$ be the strong Markov process corresponding to \mathcal{G}^a with initial distribution δ_x . Given the cost function $\ell : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}$, continuous in $\Gamma_\alpha \times \mathbb{R}$ for $\alpha \in \mathcal{A}$, $F \in \mathcal{PC}(\Gamma)$, and $\theta_v \in \mathbb{R}$ for $v \in \mathcal{V} \setminus \partial \mathcal{V}$, define the cost functional $J : \Gamma \times \mathfrak{A} \rightarrow \mathbb{R}$

$$J(x, a) = \mathbb{E}_x \left[\int_0^{+\infty} e^{-\lambda s} \left[(\ell(X^a(s), a(X^a(s))) + F(X^a(s))) 1_{\{X(s) \notin \mathcal{V}\}} \right. \right. \\ \left. \left. + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \theta_v 1_{\{X^a(s)=v\}} \right] ds \right],$$

where $\lambda > 0$. For each $x \in \Gamma$, consider the value function

$$V(x) = \inf_{a \in \mathfrak{A}} J(x, a). \quad (31)$$

Our goal is to identify V as the unique solution to the following discounted Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \lambda u(x) = F(x) & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_\alpha}(v) = u|_{\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v (\theta_v - \lambda u(v)) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial \mathcal{V}, \end{cases} \quad (32)$$

where

$$H_\alpha(x, p) := \sup_{|a| \leq R} \{-b_\alpha(x, a)p - \ell_\alpha(x, a)\}, \quad \alpha \in \mathcal{A}.$$

Theorem 4.7. Let $u \in \mathcal{C}^2(\Gamma)$ be a solution to (32). Then, for every $a \in \mathfrak{A}$, we have $u(x) \leq J(x, a)$. In particular we have $u(x) \leq V(x)$.

Proof. Let $u \in \mathcal{C}^2(\Gamma)$ be a solution to (32) and let $a \in \mathfrak{A}$. Consider X^a the Feller process generated by (29), (30). We define a function $f: \mathbb{R}_+ \times \Gamma \rightarrow \mathbb{R}$ by setting $f(t, x) = u(x)e^{-\lambda t}$. Let also $(t_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R}_+ converging to $+\infty$. From Theorem 3.3 we deduce that, for every $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{E}_x[f(t_n, X^a(t_n))] &= f(0, x) + \mathbb{E}_x \left[\int_0^{t_n} (\partial_s + \mathcal{G}^a) f(s, X^a(s)) \mathbf{1}_{\{X(s) \notin \mathcal{V}\}} ds \right] \\ &+ \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \mathbb{E}_x \left[\int_0^{t_n} \left(\eta_v \partial_s f(s, v) - \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v, \alpha} \partial_\alpha f(s, v) \right) dL^{X^a}(s, v) \right]. \end{aligned} \quad (33)$$

Notice that, for $s > 0$ and $X^a(s) \notin \mathcal{V}$,

$$\begin{aligned} (\partial_s + \mathcal{G}^a) f(s, X^a(s)) &= [-\lambda u(X^a(s)) + \mu \partial^2 u(X^a(s)) + b(X^a(s), a(X^a(s))) \partial u(X^a(s))] e^{-\lambda s} \\ &\geq [-\lambda u(X^a(s)) + \mu \partial^2 u(X^a(s)) - H(X^a(s), \partial u(X^a(s))) - \ell(X^a(s), a(X^a(s)))] e^{-\lambda s} \\ &= -[F(X^a(s)) + \ell(X^a(s), a(X^a(s)))] e^{-\lambda s}. \end{aligned}$$

From the junction condition in (32) we also have

$$\eta_v \partial_s f(s, v) - \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v, \alpha} \partial_\alpha f(s, v) = \left(-\eta_v \lambda u(v) - \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v, \alpha} \partial_\alpha u(v) \right) e^{-\lambda s} = -\eta_v \theta_v e^{-\lambda s}.$$

So that (33) and Theorem 3.3 imply

$$\begin{aligned} \mathbb{E}_x \left[u(X^a(t_n)) e^{-\lambda t_n} \right] &\geq u(x) - \mathbb{E}_x \left[\int_0^{t_n} (F(X^a(s)) + \ell(X^a(s), a(X^a(s)))) \mathbf{1}_{\{X^a(s) \notin \mathcal{V}\}} e^{-\lambda s} ds \right] \\ &- \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \mathbb{E}_x \left[\int_0^{t_n} \eta_v \theta_v e^{-\lambda s} dL^{X^a}(s, v) \right] \\ &= u(x) - \mathbb{E}_x \left[\int_0^{t_n} (F(X^a(s)) + \ell(X^a(s), a(X^a(s)))) \mathbf{1}_{\{X(s) \notin \mathcal{V}\}} e^{-\lambda s} ds \right] \\ &- \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \mathbb{E}_x \left[\int_0^{t_n} \theta_v \mathbf{1}_{\{X^a(s) = v\}} e^{-\lambda s} ds \right]. \end{aligned}$$

Since u , F and $\ell(\cdot, a(\cdot))$ are bounded, we may apply Lebesgue's convergence theorem to pass to the limit $n \rightarrow \infty$ and obtain $u(x) \leq J(x, a)$. \square

To get the reverse inequality, we consider the following assumption

(H4) For each $(x, p) \in \Gamma_\alpha \times \mathbb{R}$, there exists a unique $a_\alpha^* = a_\alpha^*(x, p)$, with $|\alpha_\alpha^*| \leq R$ for every $\alpha \in \mathcal{A}$, such that

$$H_\alpha(x, p) = -b_\alpha(x, a_\alpha^*)p - \ell_\alpha(x, a_\alpha^*)$$

and the map $a_\alpha^*: \Gamma_\alpha \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Proposition 4.8. Assume **(H4)** and let $u \in \mathcal{C}^2(\Gamma)$ be a solution to (32). Then $u(x) \geq V(x)$ for all $x \in \Gamma$.

Proof. Using **(H4)**, we have

$$H(x, \partial u(x)) = \partial_p H(x, \partial u(x)) \partial u(x) - \ell(x, a^*(x)).$$

We may consider the generator \mathcal{G}^* defined as in (29) with b replaced by $\partial_p H(\cdot, \partial u(\cdot))$ and the corresponding Markov process X^* . Then the same computation as in Theorem 4.7 yields

$$\begin{aligned} u(x) &= \mathbb{E}_x \left[\int_0^{+\infty} (F(X^*(s)) + \ell(X^*(s), a^*(X^*(s)))) \mathbf{1}_{\{X^*(s) \notin \mathcal{V}\}} e^{-\lambda s} ds \right] \\ &\quad + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \mathbb{E}_x \left[\int_0^{+\infty} \theta_v \mathbf{1}_{\{X^*(s) = v\}} e^{-\lambda s} ds \right] \geq V(x) \end{aligned}$$

□

5 The MFG system

Aim of this section is to study the Mean Field Games model with sticky transition condition at the vertices. We consider the following MFG system

$$\left\{ \begin{array}{ll} (i) \quad -\mu_\alpha \partial^2 u(x) + H(x, \partial u(x)) + \rho = F[\mathbf{m}] & \text{for all } x \in \Gamma_\alpha \setminus \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_\alpha}(v) = u|_{\Gamma_\beta}(v) & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \gamma_{v,\alpha} \partial_\alpha u(v) = \eta_v (\theta_v + F[\mathbf{m}](v) - \rho) & \text{for all } v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \partial_\alpha u(v) = 0 & \text{for all } v \in \partial \mathcal{V}, \\ \int_\Gamma u dx = 0 \\ (ii) \quad \mathbf{m} = m \mathcal{L} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \delta_v, \\ -\mu_\alpha \partial^2 m(x) - \partial(\partial_p H(x, \partial u(x)) m(x)) = 0 & \text{for all } x \in \Gamma_\alpha, \alpha \in \mathcal{A}, \\ \frac{m|_{\Gamma_\alpha}(v)}{\gamma_{v,\alpha}} = \frac{m|_{\Gamma_\beta}(v)}{\gamma_{v,\beta}} =: T_v[m] & \text{for all } \alpha, \beta \in \mathcal{A}_v, v \in \mathcal{V} \setminus \partial \mathcal{V}, \\ \sum_{\alpha \in \mathcal{A}_v} \mu_\alpha \partial_\alpha m|_{\Gamma_\alpha}(v) + n_{v,\alpha} m|_{\Gamma_\alpha}(v) \partial_p H_\alpha(v, \partial u|_{\Gamma_\alpha}(v)) = 0 & \forall v \in \mathcal{V}, \\ m \geq 0, 1 \geq \int_\Gamma m dx = 1 - \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \geq 0. \end{array} \right. \quad (34)$$

In (34), the item (i) corresponds to the Hamilton-Jacobi-Bellman equation, (ii) to the Fokker-Planck equation. Note that the coupling and the ergodic constant also appear in the right hand

side of the Kirchhoff condition, in addition to the equation. We observe that, given the solution m of the Fokker-Planck equation inside the edges with the corresponding transition conditions at the vertices, the measure \mathfrak{m} is completely determined by the coefficients $T_v[m]$. Hence we consider as unknowns in (34) u , ρ and m .

Theorem 5.1 (Existence of solutions). *Assume (H1), (H2) and that $F: \mathcal{P}_1(\Gamma) \rightarrow PC^s(\Gamma)$ is continuous and takes values in a bounded subset of $PC^s(\Gamma)$. Then there exists a solution $(u, \rho, m) \in \mathcal{C}^{2,s}(\Gamma) \times \mathbb{R} \times V_\gamma^{1,2}(\Gamma)$ to the MFG system (34).*

Proof. The proof follows the by now standard argument for existence in second order MFG and relies on Schauder's fixed point theorem. We define

$$\mathcal{M} := \left\{ m \in PC(\Gamma) : \begin{array}{l} m \geq 0, \frac{m|_{\Gamma_\alpha}(v)}{\gamma_{v,\alpha}} = \frac{m|_{\Gamma_\beta}(v)}{\gamma_{v,\beta}} =: T_v[m] \quad \text{for all } \alpha, \beta \in \mathcal{A}_v, \\ v \in \mathcal{V} \setminus \partial\mathcal{V} \quad \text{and} \quad \int_\Gamma m \, dx + \sum_{v \in \mathcal{V} \setminus \partial\mathcal{V}} \eta_v T_v[m] = 1 \end{array} \right\}.$$

Notice that the previous set is a closed convex subset of $PC(\Gamma)$. Let $m \in \mathcal{M}$ and define $\mathfrak{m} \in \mathcal{P}_1(\Gamma)$ according to (11). Since $F[\mathfrak{m}] \in PC^s(\Gamma)$ we know from Theorem 4.6 that there exists a unique solution $(u, \rho) \in \mathcal{C}^{2,s}(\Gamma) \times \mathbb{R}$ to (20) with F replaced by $F[\mathfrak{m}]$ and θ_v replaced by $\theta_v + F[\mathfrak{m}](v)$. From Proposition 3.7 we then know that there exists a unique weak solution $\bar{m} \in V_\gamma^{1,2}(\Gamma) \subset PC(\Gamma)$ to (12). Since $\bar{m} \in \mathcal{M}$, we can define a mapping $\Phi: \mathcal{M} \rightarrow \mathcal{M}$ by setting $\Phi(m) = \bar{m}$. We claim that Φ satisfies the assumptions of Schauder's fixed point theorem [15, Corollary 11.2].

We first prove that Φ is continuous. Let $(m^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{M} and converging to $m \in \mathcal{M}$. Define \mathfrak{m}^n and \mathfrak{m} according to (11). From Proposition 3.11 we know that \mathfrak{m}^n converges to \mathfrak{m} in $\mathcal{P}_1(\Gamma)$. Since F is continuous, we deduce that $F[\mathfrak{m}^n]$ converges to $F[\mathfrak{m}]$ in $PC^s(\Gamma)$. For each $n \in \mathbb{N}$, from Theorem 4.6, there exists a unique solution $(u^n, \rho^n) \in \mathcal{C}^{2,s}(\Gamma) \times \mathbb{R}$ to (20) with $\int u \, dx = 0$, F replaced by $F[\mathfrak{m}^n]$ and θ_v by $\theta_v + F[\mathfrak{m}^n](v)$.

We claim that the sequence $(\rho^n)_{n \in \mathbb{N}}$ is bounded. Indeed, integrating the equation satisfied by (u^n, ρ^n) , we have

$$|\rho^n| \mathcal{L}(\Gamma) = \int_\Gamma \text{sgn}(\rho^n) \rho^n \, dx \leq \int_\Gamma \mu |\partial^2 u^n| + C_H(1 + |\partial u^n|^q) + |F[\mathfrak{m}^n]| \, dx.$$

From Theorem 4.6 we know that u^n is bounded in $W^{2,1}(\Gamma)$ and that ∂u^n is bounded in $L^q(\Gamma)$, so that the right-hand side in the last inequality is bounded. This proves the claim.

By bootstrapping the regularity of u^n we see that $(u^n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{C}^{2,s}(\Gamma)$. We may extract a subsequence converging to a solution $(u, \rho) \in \mathcal{C}^2(\Gamma) \times \mathbb{R}$ to (20) with $\int u \, dx = 0$, F replaced by $F[\mathfrak{m}]$ and θ_v by $\theta_v + F[\mathfrak{m}](v)$. Since this solution is unique we conclude that the whole sequence must converge to (u, ρ) . Then $\bar{m}^n := \Phi(m^n)$ is the unique solution to (12) with b replaced by $\partial_p H(\cdot, \partial u^n)$. Since $\partial u^n|_{\Gamma_\alpha}$ converges uniformly to $\partial u|_{\Gamma_\alpha}$ and $\partial_p H_\alpha$ is continuous for every $\alpha \in \mathcal{A}$, we may apply Proposition 3.10 to conclude that \bar{m}^n converges to $\bar{m} = \Phi(m)$ in \mathcal{M} . This proves the continuity of Φ .

The compactness of Φ follows from the fact that F takes values in a bounded subset of $PC^s(\Gamma)$ and the compact embedding $V_\gamma^{1,2}(\Gamma) \hookrightarrow PC(\Gamma)$. We can therefore apply Schauder's fixed point theorem to conclude the proof. \square

We now briefly discuss uniqueness of the solution to (34) under monotonicity assumptions on the coupling cost.

Theorem 5.2. *For all for all $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{P}_1(\Gamma)$ with $\mathbf{m}_i = m_i \mathcal{L} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m_i] \delta_v$, assume one of the following assumptions:*

(i) *the map $p \mapsto H(x, p)$ is convex and F is strictly monotone, in the sense that*

$$\int_{\Gamma} (F[\mathbf{m}_1] - F[\mathbf{m}_2])(\mathbf{m}_1 - \mathbf{m}_2)(dx) > 0 \quad ;$$

(ii) *the map $p \mapsto H(x, p)$ is strictly convex and F is monotone, in the sense that*

$$\int_{\Gamma} (F[\mathbf{m}_1] - F[\mathbf{m}_2])(\mathbf{m}_1 - \mathbf{m}_2)(dx) \geq 0 \quad \text{for all } \mathbf{m}_1, \mathbf{m}_2 \in \mathcal{P}_1.$$

Then there exists at most one solution $(u, \rho, m) \in \mathcal{C}^{2,s}(\Gamma) \times \mathbb{R} \times V_{\gamma}^{1,2}(\Gamma)$ to the MFG system (34).

Proof. Let (u_1, m_1, ρ_1) and (u_2, m_2, ρ_2) be two solutions to (34) and define $\mathbf{m}_i, i = 1, 2$, as in (11). Set $\bar{u} = u_1 - u_2$, $\bar{\mathbf{m}} = \mathbf{m}_1 - \mathbf{m}_2$, $\bar{m} = m_1 - m_2$ and $\bar{\rho} = \rho_1 - \rho_2$.

Integrating the equation satisfied by \bar{u} against \bar{m} and using the transition condition for u_1 and u_2 , we obtain

$$\begin{aligned} \int_{\Gamma} F[\mathbf{m}_1] - F[\mathbf{m}_2] \bar{m} dx &= \int_{\Gamma} (-\mu \partial^2 \bar{u} + H(x, \partial u_1) - H(x, \partial u_2) + \bar{\rho}) \bar{m} dx \\ &= \int_{\Gamma} \mu \partial \bar{u} \partial \bar{m} + (H(x, \partial u_1) - H(x, \partial u_2)) \bar{m} dx \\ &\quad - \left[\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} T_v[\bar{m}] \sum_{\alpha \in \mathcal{A}_v} n_{v,\alpha} \mu_{\alpha} \gamma_{v,\alpha} \partial_{\alpha} \bar{u}(v) \right] + \int_{\Gamma} \bar{\rho} \bar{m} dx \\ &= \int_{\Gamma} \mu \partial \bar{u} \partial \bar{m} + (H(x, \partial u_1) - H(x, \partial u_2)) \bar{m} dx \\ &\quad + \left[\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[\bar{m}] (\bar{\rho} - (F[\mathbf{m}_1](v) - F[\mathbf{m}_2](v))) \right] + \int_{\Gamma} \bar{\rho} \bar{m} dx \\ &= \int_{\Gamma} \mu \partial \bar{u} \partial \bar{m} + (H(x, \partial u_1) - H(x, \partial u_2)) \bar{m} dx \\ &\quad - \left[\sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[\bar{m}] ((F[\mathbf{m}_1](v) - F[\mathbf{m}_2](v))) \right] + \int_{\Gamma} \bar{\rho} \bar{m}(dx). \end{aligned}$$

Using the fact that $\int_{\Gamma} \bar{\rho} \bar{m}(dx) = 0$, we conclude that

$$\int_{\Gamma} (F[\mathbf{m}_1] - F[\mathbf{m}_2]) \bar{m}(dx) = \int_{\Gamma} \mu \partial \bar{u} \partial \bar{m} + (H(x, \partial u_1) - H(x, \partial u_2)) \bar{m} dx. \quad (35)$$

On the other hand, using \bar{u} as a test-function in the equation satisfied by \bar{m} we obtain

$$\int_{\Gamma} \mu \partial \bar{m} \partial \bar{u} + (m_1 \partial_p H(x, \partial u_1) - m_2 \partial_p H(x, \partial u_2)) \partial \bar{u} = 0. \quad (36)$$

The rest of the proof follows the usual argument introduced in [17]. Indeed, subtracting (36) from (35), we get

$$\begin{aligned} 0 &= \int_{\Gamma} (F[\mathbf{m}_1] - F[\mathbf{m}_2]) \bar{m}(dx) \\ &+ \int_{\Gamma} m_1 [H(x, \partial u_2) - H(x, \partial u_1) - \partial_p H(x, \partial u_1) \partial \bar{u}] dx \\ &+ \int_{\Gamma} m_2 [H(x, \partial u_1) - H(x, \partial u_2) - \partial_p H(x, \partial u_2) \partial \bar{u}] dx. \end{aligned}$$

Assuming that F is strictly monotone, the first integral is non negative. By convexity of H and positivity of m_i , the other two integrals are non negative. It follows that $m_1 = m_2$ and also $u_1 = u_2$. Recalling (11), we also get $m_1 = m_2$. If F is monotone, we conclude in a similar way. \square

Example 5.3. We give an example of coupling cost satisfying the assumptions of Theorem 5.2. Given $\mathbf{m} = m \mathcal{L} + \sum_{v \in \mathcal{V} \setminus \partial \mathcal{V}} \eta_v T_v[m] \delta_v$ we define

$$F[\mathbf{m}](x) = \begin{cases} F_I[m](x) & x \in \Gamma \setminus \mathcal{V}, \\ F_V(T_v[m]) & x \in \mathcal{V} \setminus \partial \mathcal{V}, \end{cases}$$

where

$$\int_{\Gamma} (F_I[m_1] - F_I[m_2])(m_1 - m_2)(dx) > 0$$

for $m_i \in L^1(\Gamma)$, $m_i \geq 0$, $i = 1, 2$, and $F_V : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an increasing function.

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