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► **To cite this version:**

Yves Colin de Verdière, Charlotte Dietze, Maarten V. de Hoop, Emmanuel Trélat. Weyl formulae for some singular metrics with application to acoustic modes in gas giants. 2024. <hal-04627841v2>

HAL Id: hal-04627841

<https://hal.science/hal-04627841v2>

Preprint submitted on 5 Feb 2025

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Weyl formulae for some singular metrics with application to acoustic modes in gas giants

Yves Colin de Verdière* Charlotte Dietze† Maarten V. de Hoop‡
Emmanuel Trélat§

Abstract

This paper is motivated by recent works on inverse problems for acoustic wave propagation in the interior of gas giant planets. In such planets, the speed of sound is isotropic and tends to zero at the surface. Geometrically, this corresponds to a Riemannian manifold with boundary whose metric blows up near the boundary. Here, the spectral analysis of the corresponding Laplace-Beltrami operator is presented and the Weyl law is derived. The involved exponents depend on the Hausdorff dimension which, in the supercritical case, is larger than the topological dimension.

Keywords: Weyl law, singular metrics, heat asymptotics, gas giants.

AMS classification: 11F72, 58C40.

1 Introduction

1.1 Seismology on gas giant planets

Seismology has played an important role in revealing the (deep) interiors of gas giant planets in our solar system (see [8, 28]). Indeed, the acoustic spectra and free oscillations have been studied for Saturn and Jupiter over the past few decades (see [21, 23, 38]). The excitation of acoustic modes in gas giant planets presumably occurs through convection in their interiors. The observation of acoustic eigenfrequencies, that is, the discrete spectrum can be realized, in principle, through visible photometry, thermal infrared photometry, Doppler spectrometry, and ring seismology for nonradial oscillations (see [29, 30]), in particular, in the case of Saturn. In ring seismology and with the Cassini mission, one measured the “resonances” in the inner C ring of Saturn with visual and infrared mapping spectrometer (VIMS) stellar occultations (see [16, 20, 26]). The rings are gravitationally coupled to the acoustic modes of the planet (taking self gravitation into account). Detection of Jupiter’s acoustic eigenvalues has been attempted with ground-based imaging-spectrometry (seismographic imaging interferometer for monitoring of planetary atmospheres or SYMPA) by measuring line of sight velocity (see [21, 37]). Recently,

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Juno spacecraft gravity measurements have provided evidence for normal modes of Jupiter (see [17]).

1.2 Singular Riemannian metrics

On a gas giant planet, unlike a rocky planet, the speed of sound goes to zero at the boundary. In the geometric mathematical model that we employ hereafter, the rate at which this happens follows a power law which determines a specific conformal blow-up rate of a Riemannian metric, thus defining a singular metric. This rate happens to be slower than on asymptotically hyperbolic manifolds and the boundary is at a finite distance from interior points. The rate is implied by an equation of state in the upper part of the planet, in general, in the sense of a fit. (For some models of the speed of sound of Jupiter and Saturn showing this behavior, see [23, Figure 1] and [27, Figure 1].) Only for a polytrope is the rate exact. Polytropes, for which the pressure is proportional to a power of the density of mass, have been viewed as relevant simplifications; models with variable polytrope index have indeed been applied to planet and material models (see [40]). Typically, an equation of state is computed numerically using density functional molecular dynamics simulations with mixtures of chemical elements: The dominant elements in terms of mass fraction are hydrogen and helium, but also heavy elements are important. The equation of state is different for the upper part and the deep interior as the helium fraction can be higher in the interior due to helium rain (helium becoming immiscible with hydrogen at high pressure). Equations of state play a vital role in the evolution and realization of structure of gas giant planets (see [33, 34]).

More specifically, if g_e is the Euclidean Riemannian metric on a smooth domain $X \subset \mathbb{R}^{n+1}$, then the speed of sound c can be encoded by the conformally Euclidean Riemannian metric $g = c^{-2}g_e$. In local coordinates where the boundary of X is (locally) described by $u = 0$, the polytropic model suggests that $c \sim u^{1/2}$. Indeed, the natural generalization is $c \sim u^{\alpha/2}$, that is, $c^{-2} \sim u^{-\alpha}$; through previous analysis done in [15] it appears that restricting α according to $\alpha \in (0, 2)$ guarantees the presence of a discrete spectrum as it has been observed. Thus, the Riemannian geometry lies between standard geometry with boundary and asymptotically hyperbolic geometry. Some of the phenomena in this geometry are unlike those seen at either end. The extreme case $\alpha = 0$ corresponds physically to solid bodies and mathematically to manifolds with boundary, and the other extreme $\alpha = 2$ corresponds to asymptotically hyperbolic geometry but is far from all planetary models.

Therefore, following [15, Section 1.1], we model a gas giant planet as a smooth manifold X with a boundary, endowed with a Riemannian metric g on $X \setminus \partial X$ such that, near ∂X , we have $g = \bar{g}/u^\alpha$ where \bar{g} is a well-defined Riemannian metric up to the boundary, and $\partial X = \{u = 0\}$ locally. The fact that \bar{g} is neither zero nor infinite at ∂X implies a specific blow-up rate for g near ∂X . This conformal power-law blow-up is the key geometric feature of gas giant metrics. The speed of sound might contain jump discontinuities where phase transitions occur (see [32]), that is, the metric can contain conormal singularities while the manifold consists of multiple “layers”. A key interior boundary in gas giants corresponds with the transition from molecular to metallic hydrogen. Accounting for discontinuities in an asymptotic formalism for gas giant seismology was developed a few decades ago (see [35]).

The mathematical study of the spectrum associated with gas giants’ acoustic modes was initiated in [15]. In this paper, we analyze the relevant Laplace-Beltrami operator and we compute the Weyl law. The study of Weyl asymptotics, which reflects some properties of the singular metric, is a preliminary step towards analyzing some inverse problems, in view of reconstructing some features of the internal structure of gas giant planets.

For a discussion on how the singular metrics of the type considered here arise from equations of state on gas giants, we refer to [15, Section 1.2]. The basic, polytropic equation of state leads precisely to a blow up at a rate $\alpha = 1$, above the critical value introduced in the main text. Observations show the presence of a discrete acoustic spectrum; this can only be ensured if $\alpha < 2$.

2 Mathematical model and main results

2.1 Mathematical model

Let X be a smooth compact manifold of dimension $n + 1$ with a boundary ∂X . Near ∂X , X is diffeomorphic to $[0, 1) \times M$, where M is a smooth compact manifold of dimension $n \geq 1$ and ∂X is identified with $\{0\} \times M$ and also with $u = 0$ where u is a transverse coordinate, locally near ∂X , ranging over $[0, 1)$. As discussed in Section 1.2, we consider on X a singular Riemannian metric g that is a smooth metric on $X \setminus \partial X$, written near ∂X as

$$g = \bar{g}/u^\alpha \tag{1}$$

where $0 < \alpha < 2$ and \bar{g} is a smooth (non-singular) Riemannian metric on X , up to the boundary. Following [15, Proposition 2], which uses a normal form for the metric near the boundary, due to [22, Lemma 5.2], we have

$$g = u^{-\alpha}(du^2 + g_0(u))$$

where $g_0(u)$ is a Riemannian metric on M (pulled back to the level set $u = \text{Cst}$) depending smoothly on $u \in [0, 1)$.

We make a change of variable. Setting $x = x(u) = \int_0^u s^{-\alpha/2} ds = (1 - \frac{\alpha}{2})^{-1} u^{1 - \frac{\alpha}{2}}$, we get

$$g = dx^2 + x^{-\beta} g_1(x) \text{ where } \beta = \frac{2\alpha}{2 - \alpha} \tag{2}$$

and $g_1(x)$ is a Riemannian metric on M (pulled back to the level set $x = \text{Cst}$), depending smoothly on $x \in (0, 1)$ and that is continuous at $x = 0$. We note that, since $\alpha \in (0, 2)$, β can take any positive value. We also note that a polytrope (for any index) corresponds to $\beta = \beta_{\text{poly}} = 2$. We have that $g_1(x) = C(\alpha)g_0(u)$ for some constant $C(\alpha) > 0$.

For any $x \in [0, 1)$, denoting by dv_1^x the volume measure on M associated to the metric $g_1(x)$, the g -volume is $dv_g = x^{-\beta n/2} |dx| dv_1^x$. The volume is finite if and only if $\beta < \beta_c$, where

$$\beta_c = \frac{2}{n} \tag{3}$$

is a critical value of β . We will see later that this critical value plays a role in the Weyl asymptotics. At this point, we can note that $\beta_{\text{poly}} > \beta_c$ for $n = 2$.

The following three propositions were proved in [15]. The first proposition concerns the Hausdorff dimension.

Proposition 1. *The Hausdorff dimension of (X, g) is*

$$d_H = \max \left(n + 1, n \left(1 + \frac{\beta}{2} \right) \right).$$

We define $\delta_H = n(1 + \frac{\beta}{2})$, and note that $d_H > n + 1$ ($n + 1$ is the topological dimension of X) if and only if $\beta > \beta_c$. We give in Appendix A.5 a sketch of the proof of Proposition 1, in which we also show that d_H coincides with the Minkowski dimension of (X, g) .

Proposition 2. *The Laplace-Beltrami operator Δ_g , with core $C_0^\infty(X \setminus \partial X)$, is essentially self-adjoint if and only if $\beta \geq \beta_c$.*

For $\beta < \beta_c$, there exist several extensions of Δ_g , with core $C_0^\infty(X \setminus \partial X)$. In the further analysis, we consider its Friedrichs extension (that is, “Dirichlet extension”). As we consider “pressure” as the physical quantity satisfying the partial differential equation, the natural boundary condition is the Dirichlet one — this corresponds with a pressure-free boundary condition, like for oceans.

Proposition 3. *For every $\beta > 0$, the spectrum of Δ_g is discrete.*

We denote the eigenvalues of Δ_g by $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots$ with associated eigenfunctions ϕ_j , $j \in \mathbb{N}^*$, normalized in $L^2(X, dv_g)$. We note that, if the volume of X is infinite, i.e., if $\beta \geq \beta_c$, we have $\lambda_1 > 0$, in contrast to the usual Riemannian case. The Weyl counting function is defined by

$$N(\lambda) = \#\{j \in \mathbb{N}^* \mid \lambda_j \leq \lambda\}$$

where the eigenvalues are counted with their multiplicity. Our objective is to derive a Weyl law describing the asymptotics of $N(\lambda)$ as $\lambda \rightarrow +\infty$.

Remark 1. The following fact will be used in Section 3.5. For any $\varepsilon > 0$, there exists $\delta > 0$ such that the metric g is ε -quasi-isometric (see Appendix A.1) to a singular Riemannian metric \tilde{g} on X , smooth on $X \setminus \partial X$ and given by $\tilde{g} = dx^2 + x^{-\beta}g_1(0)$ on $(0, \delta) \times M$. In order to derive a Weyl law for (X, g) it suffices to derive the corresponding Weyl law for \tilde{g} for any $\varepsilon > 0$ (see, again, Appendix A.1 for details). This remark is important, because it implies that we mainly have to work within the so-called *separable* case.

Separable case. We say that we are in the *separable* case if the metric $g_1(x)$ on M (defined by (2)) does not depend on x , i.e., $g_1(x) = g_1(0)$ for any $x \in (0, 1)$; we still denote this metric by g_1 . In the sequel, we consider $[0, 1) \times M$ instead of $[0, \delta) \times M$ for simplicity of notation, while the proofs are similar in both cases.

We denote by Δ_M the Laplace-Beltrami operator on (M, g_1) . We denote the eigenvalues of Δ_M by $0 \leq \omega_1 \leq \omega_2 \leq \dots \leq \omega_j \leq \dots$ with an associated orthonormal basis of eigenfunctions $(\psi_j)_{j \in \mathbb{N}^*}$. The Weyl counting function for Δ_M is defined by

$$N_M(\omega) = \#\{k \in \mathbb{N}^* \mid \omega_k \leq \omega\}.$$

Since g_1 is a smooth Riemannian metric on M , the classical Weyl law for (M, g_1) yields that $N_M(\omega) = \gamma_n \text{Vol}_{g_1}(M)\omega^{n/2} + O(\omega^{n/2})$ as $\omega \rightarrow +\infty$ where

$$\gamma_n = \frac{1}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \quad (4)$$

(see [3, Chapter 3E] for the heat trace and then apply the Karamata tauberian theorem, i.e., Theorem 3 in Appendix A.2).

Denoting by dv_1 the volume measure on M associated to the metric $g_1 = g_1(0)$, the g -volume is $dv_g = x^{-\beta n/2} |dx| dv_1$. Making the change of function $f \mapsto x^{-\beta n/4} f$, we get the new volume form $|dx| dv_1$; the Laplace-Beltrami operator on $X_1 = (0, 1) \times M$ is now given by

$$\Delta_g = -\partial_x^2 + \frac{C_\beta}{x^2} + x^\beta \Delta_M$$

where $x \in (0, 1)$ and

$$C_\beta = \frac{\beta n}{4} \left(\frac{\beta n}{4} + 1 \right).$$

The proof is straightforward by performing an integration by parts with respect to x in the Dirichlet form defining the Laplace-Beltrami operator, using the Dirichlet boundary condition at $x = 0$. We note that $C_\beta \geq 3/4$ if and only if the volume of X is infinite. Using the Weyl criterion (see Appendix A.3), this inequality also implies that P_ω defined by (6) below is essentially self-adjoint for any $\omega > 0$, but not for $\omega = 0$.

We will need to work first on the non-compact conic manifold $X_\infty = (0, +\infty) \times M$ endowed with the metric $g = dx^2 + x^{-\beta} g_1$. Let Δ_∞ stand for the Laplace-Beltrami operator on (X_∞, g) . Invoking a separation of variables, we have

$$\Delta_\infty = \bigoplus_{k=1}^{+\infty} (\text{id} \otimes \pi_k) (P_{\omega_k} \otimes \text{id}) (\text{id} \otimes \pi_k) \quad (5)$$

where

$$P_\omega = -\partial_x^2 + \frac{C_\beta}{x^2} + \omega x^\beta \quad (6)$$

is a Schrödinger operator on $L^2((0, +\infty), dx)$ for any $\omega \geq 0$, and where π_k is the orthogonal projection of $L^2(M, dv_1)$ onto the subspace generated by ψ_k and id denotes the identity operator on $L^2(X_\infty, |dx| dv_1)$ (resp., on $L^2(M, dv_1)$). Hence, Δ_∞ is unitarily equivalent to $\bigoplus_{k=1}^{+\infty} P_{\omega_k}$.

2.2 Main results

Recalling that g_1 is defined by (2), we set $G = g_1(0)$ and denote by v_G the corresponding volume form on M . We also recall that β_c is defined in (3) and that γ_n is defined in (4).

2.2.1 Weyl asymptotics

Theorem 1. • If $\beta > \beta_c$ then

$$N(\lambda) \sim A(\beta, n) v_G(M) \lambda^{d_H/2}$$

as $\lambda \rightarrow +\infty$, with

$$A(\beta, n) = \frac{n\gamma_n(\beta + 2)}{4\Gamma(1 + d_H/2)} \int_0^{+\infty} \mathcal{Z}_1(\tau) \tau^{\frac{d_H}{2} - 1} d\tau$$

where $\mathcal{Z}_1(\tau) = \text{Tr}(\exp(-\tau P_1))$ and P_1 is the Schrödinger operator on $L^2((0, +\infty), dx)$ defined by (6).

• If $\beta = \beta_c = 2/n$ then

$$N(\lambda) \sim C_n v_G(M) \lambda^{(n+1)/2} \ln \lambda$$

as $\lambda \rightarrow +\infty$, with

$$C_n = \frac{1}{(n+1)(4\pi)^{(n+1)/2} \Gamma((n+1)/2)}.$$

In particular, $C_1 = 1/8\pi$.

- If $\beta < \beta_c$ then

$$N(\lambda) \sim \gamma_{n+1} v_g(X) \lambda^{(n+1)/2}$$

as $\lambda \rightarrow +\infty$.

Remark 2. When $M = \mathbb{R}/2\pi\mathbb{Z}$ and X is diffeomorphic to the hemisphere, endowed with the so-called Grushin metric, the authors of [5] derived the Weyl law using an explicit computation of the spectrum. We recover their result as a particular case with $n = 1$ and $\beta = \beta_c = 2$.

Remark 3. There exists an abundant literature on the asymptotic spectral study of singular metrics, of Laplace-Beltrami operators that degenerate at the boundary of the domain, of singular Schrödinger operators with potentials blowing up at the boundary, etc (see, e.g., [4, 24] and references therein). An article that is close to ours, and that is complementary, is the recent one [7], in which the authors investigate the asymptotic behavior of the Weyl counting function on d -dimensional singular Riemannian manifolds, showing that, under some curvature and convexity assumptions, $N(\lambda) \sim \frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)} \lambda^{d/2} v(\lambda)$ as $\lambda \rightarrow +\infty$, where the volume $v(\lambda)$ of the set at distance greater than $1/\sqrt{\lambda}$ from the boundary is assumed to be a slowly-varying function of λ . Following [7, Section 7], their result¹ covers our subcritical and critical cases, i.e., $\beta \leq \beta_c$, and indeed yields the same constants as in Theorem 1. In the supercritical case $\beta > \beta_c$, the result of [7] only yields an asymptotic double inequality $C_1 \lambda^{d_H/2} \leq N(\lambda) \leq C_2 \lambda^{d_H/2}$ (while Theorem 1 gives an equivalent). It is noticeable anyway that their result covers also cases where β may vary along the boundary.

After a first version of our article was submitted end of 2024, first of all, Bernard Helffer pointed out the reference [31], and second, an anonymous referee mentioned the existence of the article [39], which we did not know. We are very much indebted to B. Helffer and to the referee for these crucial indications. It is a fact that both papers [31, 39] actually contain our Theorem 1 (to apply them, keep the metric in the form (1) and express the Dirichlet form). But, on the one part, the constants in [31] are not as explicit as they are in our result; on the other part, the paper [39] is difficult to follow and we think that the proof that we elaborate in the present article is easier (although our result is less general) and, in any case, it is a different method.

Remark 4. To prove Theorem 1, we make an intensive use of heat kernels. Alternatively, it is possible to use Dirichlet-Neumann bracketing (as in [7]). Both methods allow to treat conormal jump singularities of the metric \bar{g} inside X that model layering in the gas planet.

Remark 5. A natural question is whether β can be determined from the Weyl asymptotics. Indeed, when n is known, β can be determined in the case where $\beta \geq \beta_c$. When $\beta < \beta_c$ the question remains open. To shed light on this, it would be useful and interesting to get the next term in the small-time heat trace expansion (see Section 3.2.2) when $\beta \leq \beta_c$. We refer to Section 4 for other open problems.

2.2.2 Weyl measures

We next compute the *Weyl measures*, which are the probability measures w_g on X , defined, if the limit exists, by

$$\int_X f dw_g = \lim_{\lambda \rightarrow +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \int_X f |\phi_j|^2 dv_g$$

¹It should be noted that, with the notations of [7, Section 7], it is actually not required that m be an integer. Therefore, indeed, their result can be applied to the case of a real number $\beta \leq \beta_c$. We thank the authors for interesting discussions regarding this issue.

for any function $f : X \rightarrow \mathbb{R}$ that is continuous up to the boundary of X . Such measures have been introduced in [11, 13] in the framework of sub-Riemannian geometry in order to provide an account of how the high-frequency eigenfunctions concentrate.

Theorem 2. • If $\beta \geq \beta_c$ then the Weyl measure is $\delta_{x=0} \otimes dv_G/v_G(M)$.

- If $\beta < \beta_c$ then the Weyl measure is the uniform probability distribution given by the normalized volume of (X, g) , that is $dv_g/v_g(X)$.

Using [13, Corollary 7.1], we obtain the following consequence.

Corollary 1. If $\beta \geq \beta_c$ then there exists a density-one subsequence $(\phi_{j_k})_{k \in \mathbb{N}^*}$ of the sequence of eigenfunctions that concentrates on ∂X , meaning that for any compact subset $K \subset X \setminus \partial X$, we have

$$\lim_{k \rightarrow +\infty} \int_K |\phi_{j_k}|^2 dv_g = 0.$$

3 Proofs of Theorems 1 and 2

Our strategy of proof is the following. We first treat the separable case (Sections 3.1 to 3.4). As a preliminary, we perform in Section 3.1 a spectral study of the 1D Schrödinger operator P_ω defined by (6), deriving exponential estimates for truncated heat traces. Then, in Section 3.2, we estimate the small-time asymptotics of the truncated heat trace of Δ_g , near the boundary (actually, on a cone); the three cases $\beta > \beta_c$ (supercritical), $\beta = \beta_c$ (critical), $\beta \leq \beta_c$ (subcritical), must be treated in different ways. In Section 3.3, using a heat parametrix, we glue together the heat kernel near the boundary and the Riemannian heat kernel far from the boundary. Finally in Section 3.4 we prove Theorem 1 in the separable case.

In Section 3.5, we show how to pass from the separable to the general case by using the fact that the metric g is quasi-isometric to a separable metric. In Section 3.6, we prove Theorem 2. Our approach uses again heat traces.

3.1 Spectral study of the 1D Schrödinger operators

We consider the family of Schrödinger operators,

$$P_1 = -\partial_x^2 + q_{C,\beta}(x)$$

where $q_{C,\beta}(x) = Cx^{-2} + x^\beta$, $C > 0$ and $\beta > 0$, acting on $L^2((0, +\infty), dx)$. The operators P_1 are essentially self-adjoint if and only if $C \geq 3/4$; when $C < 3/4$ we consider the Friedrichs extension of P_1 with core $C_0^\infty((0, +\infty))$ (see Appendix A.3). The spectrum of P_1 is discrete; we denote it by $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$. We derive precise semi-classical asymptotics for the associated truncated heat trace.

Let $\chi : [0, +\infty) \rightarrow [0, 1]$ be a smooth decreasing function with $\chi \equiv 1$ on $[0, a]$ with $a > 0$ and $\chi' \leq 0$ everywhere. We note that $\chi \equiv 1$ is included. We define the corresponding truncated heat trace by

$$\mathcal{Z}_\chi(\tau) = \text{Tr} (e^{-\tau P_1} \chi) \quad \forall \tau > 0.$$

Let $\gamma = \max(1/\beta, 1/2)$.

Proposition 4. Given any $0 < \tau \leq 1$, we have

$$\mathcal{Z}_\chi(\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx + \mathcal{O}(\tau^{-\gamma})$$

and for $\tau \geq 1$,

$$\mathcal{Z}_\chi(\tau) = O(e^{-\mu_1 \tau})$$

uniformly with respect to χ in both cases.

The counting function $N_1(\mu) = \#\{j \in \mathbb{N}^* \mid \mu_j \leq \mu\}$ satisfies $N_1(\mu) \sim A\mu^{\frac{1}{2} + \frac{1}{\beta}}$ as $\mu \rightarrow +\infty$ with $A = \sqrt{\frac{2}{\pi}} \frac{1}{\beta} B(3/2, 1 + 1/\beta)$ where B is the Beta function.

Proof of Proposition 4. We first establish an elementary lemma. We denote by Q^N (resp., Q^D) the self-adjoint operator $-\partial_x^2$ on an interval of length 1 with Neumann (resp., Dirichlet) boundary condition.

Lemma 1. For $0 < \tau \leq 1$ and $\star \in \{D, N\}$, we have $\text{Tr}(\exp(-\tau Q^\star)) = (4\pi\tau)^{-1/2} + O(1)$.

Proof of Lemma 1. The estimate does not depend on the chosen interval. The spectrum of Q^N is $\{n^2\pi^2 \mid n \in \mathbb{N}\}$ and the spectrum of Q^D is $\{n^2\pi^2 \mid n \in \mathbb{N}^*\}$. Hence both traces differ by 1, and it suffices to prove the estimate for Q^N . Writing

$$\text{Tr}(\exp(-\tau Q^N)) = \frac{1}{2} \left(1 + \sum_{n \in \mathbb{Z}} e^{-\tau n^2 \pi^2} \right)$$

and applying the Poisson summation formula gives the result. \square

We now prove the proposition. We first consider the case where $\tau \leq 1$. We are going to apply Dirichlet-Neumann bracketing with the decomposition $(0, +\infty) = \cup_{j=0}^{+\infty} J_k$ where the intervals J_k are defined below.

Let x_0 be defined by $q_{C,\beta}(x_0) = \min q_{C,\beta}(x)$. Then $q_{C,\beta}(x) = Cx^{-2} + x^\beta$ is increasing on $[x_0, +\infty)$. Let $J_k = [x_0 + k, x_0 + k + 1]$ with $k \geq 1$ and $J_0 =]0, x_0 + 1]$. We have the following estimates for the Dirichlet and Neumann heat traces $\mathcal{Z}_{k,\chi}^\star$ on J_k : for $k \geq 1$,

$$\mathcal{Z}_{k,1}^N(\tau) \leq \left(\frac{1}{\sqrt{4\pi\tau}} + O(1) \right) e^{-\tau q(x_0+k)} \leq \left(\frac{1}{\sqrt{4\pi\tau}} + O(1) \right) \int_{x_0+k-1}^{x_0+k} e^{-\tau q_{C,\beta}(x)} dx$$

and

$$\begin{aligned} \mathcal{Z}_{k,\chi}^D(\tau) &\geq \left(\frac{1}{\sqrt{4\pi\tau}} + O(1) \right) e^{-\tau q(x_0+k+1)} \chi(x_0 + k + 1) \\ &\geq \left(\frac{1}{\sqrt{4\pi\tau}} + O(1) \right) \int_{x_0+k+1}^{x_0+k+2} e^{-\tau q_{C,\beta}(x)} \chi(x) dx, \end{aligned} \quad (7)$$

while

$$\mathcal{Z}_0^N(\tau) = O(\tau^{-1/2}).$$

The minimax principle implies that each eigenvalue μ_j is larger than the j^{th} -eigenvalue of the union for all k of the Neumann problem on the intervals J_k . In this way, we obtain the following upper bound for the trace with $\chi = 1$:

$$\mathcal{Z}_1(\tau) \leq O\left(\frac{1}{\sqrt{\tau}}\right) + \left(\frac{1}{\sqrt{4\pi\tau}} + O(1)\right) \int_{x_0}^{+\infty} e^{-\tau q_{C,\beta}(x)} dx. \quad (8)$$

Noting that

$$\left| \int_0^{+\infty} \left(e^{-\tau q_{C,\beta}(x)} - e^{-\tau x^\beta} \right) dx \right| = O(1),$$

we infer that

$$\mathcal{Z}_1(\tau) \leq \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} dx + O(\tau^{-\gamma}).$$

Similarly, using the fact that the Dirichlet heat kernel is smaller than the global kernel (see [14, Theorem 2.1.6]), we get from (7) the following lower bound:

$$\mathcal{Z}_\chi(\tau) \geq \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx + O(\tau^{-\gamma}). \quad (9)$$

Note that the same lower bound is valid when replacing χ by $1 - \chi$.

Now, we use a variant of the fact that

$$(A + B = A' + B', A \geq A', B \geq B') \Rightarrow (A = A', B = B').$$

We take $A = \mathcal{Z}_\chi$, $B = \mathcal{Z}_{1-\chi}$, $A' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx$ and $B' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} (1 - \chi(x)) dx$. By (8), we have

$$A + B = A' + B' + O(\tau^{-\gamma})$$

and, from (9),

$$A \geq A' + O(\tau^{-\gamma}), B \geq B' + O(\tau^{-\gamma}).$$

It follows that $A = A' + O(\tau^{-\gamma})$ and $B = B' + O(\tau^{-\gamma})$. In particular,

$$\mathcal{Z}_1(\tau) \sim \frac{1}{(4\pi\tau)^{\frac{1}{2}}} \int_0^{+\infty} e^{-\tau x^\beta} dx.$$

Using the Karamata tauberian Theorem (recalled in Appendix A.2), we get

$$N_1(\mu) \sim A\mu^{\frac{1}{2} + \frac{1}{\beta}}$$

as $\mu \rightarrow +\infty$, with the constant A defined in Proposition 4.

We now prove the exponential upper bound for $\tau \geq 1$. We note that $\mathcal{Z}_\chi(\tau) \leq \mathcal{Z}_1(\tau)$. Moreover all eigenvalues μ_j of P_1 are larger than the minimum $q(x_0)$ of q . Then, for $\tau \geq 1$, we have

$$\mathcal{Z}_1(\tau) = e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-\tau(\mu_j - \mu_1)} \leq e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-(\mu_j - \mu_1)}.$$

The Weyl law applied to P_1 implies that the sum at the right-hand side converges and thus $\mathcal{Z}_\chi(\tau) \leq ce^{-\tau\mu_1}$ for some $c > 0$. \square

Corollary 2. *There exists $C_1 > 0$ only depending on C and β but not on χ , such that, setting $J = \int_0^{+\infty} \chi(x) dx$, we have, for every $\tau > 0$,*

$$\mathcal{Z}_\chi(\tau) \leq \frac{C_1}{\sqrt{\tau}} \min\left(J, \tau^{-1/\beta}\right).$$

Proof. The bound involving J follows from the estimate $e(\tau, x, x) \leq (4\pi\tau)^{-\frac{1}{2}}$ which is valid for any positive potential q (see [14, Theorem 2.1.6]). The other bound follows from Proposition 4 and from the estimate $\int_0^{+\infty} e^{-\tau x^\beta} dx = O(\tau^{-1/\beta})$. \square

Given any $\omega > 0$, we set

$$P_\omega = -\partial_x^2 + Cx^{-2} + \omega x^\beta. \quad (10)$$

Proposition 5. *For any $\omega > 0$, the operator P_ω is unitarily equivalent to $\omega^{2/(2+\beta)}P_1$. In particular, the spectrum of P_ω is $\omega^{2/(2+\beta)}$ times the spectrum of P_1 .*

Proof. Considering the unitary map $U : L^2(\mathbb{R}^+, dx) \rightarrow L^2(\mathbb{R}^+, dx)$ defined by

$$Uf(x) = \omega^{\frac{1}{2(2+\beta)}} f(\omega^{\frac{1}{2+\beta}} x),$$

we find that $U^*P_\omega U = \omega^{2/(2+\beta)}P_1$. □

3.2 Truncated heat asymptotics for the cone X_∞

In this subsection, we compute the small-time asymptotics of the truncated heat trace,

$$Z_{\infty, \chi}(t) = \text{Tr} \left(e^{-t\Delta_g} \chi \right)$$

where χ is as in Section 3.1 and moreover is compactly supported in $[0, +\infty)$, and g is the metric $g = dx^2 + x^{-\beta}g_1$ on the cone $X_\infty = (0, +\infty) \times M$. The manifold M is equipped with the metric g_1 that is independent of x . Here, we do not assume that ∂M is empty: this will be useful in the proof of Theorem 2. We will only use the Weyl asymptotics on M .

Using the direct sum decomposition given in (5), we have

$$Z_{\infty, \chi}(t) = \sum_{k=1}^{+\infty} \text{Tr} \left(e^{-tP_{\omega_k}} \chi \right) \quad (11)$$

where we recall that the ω_k are the eigenvalues of Δ_M and P_ω is defined by (10).

We make the following two preliminary observations:

- For k fixed and $t \rightarrow 0^+$, we have $\text{Tr}(e^{-tP_{\omega_k}} \chi) = O(t^{-1/2})$. This term will be negligible in the sequel because the global trace is not less than C/t for some $C > 0$ (since $\dim(X) \geq 2$).
- For $t > 0$ fixed, the smooth function $f : \omega \mapsto \text{Tr}(e^{-tP_\omega} \chi)$ has a fast decay at infinity: by Proposition 5,

$$f(\omega) \leq \text{Tr}(e^{-tP_\omega}) = \text{Tr}(e^{-t\omega^{2/(2+\beta)}P_1}).$$

The claim then follows from the second assertion given in Proposition 4.

We split the sum (11) into two parts,

$$Z_{\infty, \chi}(t) = \sum_{\omega_k < 1} + \sum_{\omega_k \geq 1} = Z_{\infty, \chi}^0(t) + Z_{\infty, \chi}^1(t).$$

The first part, $Z_{\infty, \chi}^0(t)$, is $O(t^{-1/2})$ by the first preliminary observation and we thus only have to estimate the second part, $Z_{\infty, \chi}^1(t)$. Using Proposition 5 and its proof, we have

$$Z_{\infty, \chi}^1(t) = \sum_{\omega_k \geq 1} \text{Tr} \left(e^{-t\omega_k^{2/(2+\beta)}P_1} \chi(\cdot/\omega_k^{1/(2+\beta)}) \right) = \sum_{\omega_k \geq 1} \mathcal{Z}_{\chi(\cdot/\omega_k^{1/(2+\beta)})}(t\omega_k^{2/(2+\beta)})$$

and we remark that, for $\omega \geq 1$, the function $\chi(\cdot/\omega^{1/(2+\beta)})$ is identically equal to 1 on $[0, a]$ (where a was introduced Section 3.1), so that we can use the estimate of Section 3.1.

Converting this sum into an integral (see Appendix A.2), using the Weyl law on M , we obtain

$$\#\{\omega_k \leq \omega\} \sim \gamma_n \text{Vol}(M)\omega^{n/2} \quad \text{as } \omega \rightarrow +\infty.$$

Using Proposition 7 and the definition of f , we get

$$Z_{\infty, \chi}^1(t) \sim \frac{n\gamma_n \text{Vol}(M)}{2} \int_1^{+\infty} \mathcal{Z}_{\chi(\cdot/\omega^{1/(2+\beta)})}(t\omega^{2/(2+\beta)})\omega^{\frac{n}{2}-1} d\omega.$$

Making the change of variable $\tau = t\omega^{2/(2+\beta)}$, we arrive at the following lemma, recalling the δ_H was introduced below Proposition 1.

Lemma 2. *The following holds,*

$$Z_{\infty, \chi}^1(t) \sim \frac{n\gamma_n(\beta+2)\text{Vol}(M)}{4t^{\delta_H/2}} \int_t^{+\infty} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau \text{ as } t \rightarrow 0^+.$$

The integral

$$I(t) = \int_t^{+\infty} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau \quad (12)$$

is convergent at $\tau = \infty$ for all $\beta > 0$ but, in general, not at $\tau = 0$ because if $\beta \leq \beta_c$ then $\frac{\delta_H}{2} - 1 \leq \frac{1}{2} + \frac{1}{\beta} - 1$. We can compare this with the estimate in Corollary 2.

3.2.1 Case $\beta > \beta_c$

We estimate the small-time behavior of $I(t)$ defined by (12). By the monotone convergence theorem, we have

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{+\infty} e(\tau, x, x)\chi(\varepsilon x) dx = \int_0^{+\infty} e(\tau, x, x) dx$$

where e is the heat kernel of P_1 . Hence, for any $\tau > 0$,

$$\lim_{t \rightarrow 0^+} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau) = \mathcal{Z}_1(\tau).$$

Using, again, the monotone convergence theorem, we conclude that

$$\lim_{t \rightarrow 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau.$$

From Corollary 2, we get

$$\mathcal{Z}_1(\tau)\tau^{\delta_H/2-1} \leq C\tau^{\frac{n}{2}(1+\frac{\beta}{2})-1-\frac{1}{2}-\frac{1}{\beta}}e^{-\mu_1\tau}$$

and $\beta > \beta_c = 2/n$ implies $\frac{n}{2}(1+\frac{\beta}{2})-1-\frac{1}{2}-\frac{1}{\beta} > -1$. Thus, the corresponding limit is finite:

$$\lim_{t \rightarrow 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau < +\infty.$$

3.2.2 Case $\beta \leq \beta_c$

By the second estimate in Proposition 4, the contribution to the integral from 1 to $+\infty$ in the integral $I(t)$ defined by (12) is $O(1)$ uniformly with respect to t and hence the corresponding part of $Z_{\infty, \chi}^1(t)$ is $O(t^{-\delta_H/2})$ as $t \rightarrow 0^+$, which will be negligible. We only need to estimate the small-time asymptotics of

$$I_1(t) = \int_t^1 \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau)\tau^{\frac{\delta_H}{2}-1} d\tau.$$

Sub-case $\beta < \beta_c$. We prove that there exists a $\delta > 0$ such that

$$\mathrm{Tr} \left(e^{-t\Delta_g} \chi \right) = \mathcal{O} \left(J^\delta t^{-(n+1)/2} \right) \quad (13)$$

as $t \rightarrow 0^+$. We split the integral

$$I_1(t) = \int_t^1 \mathrm{Tr} \left(e^{-\tau P_1} \chi \left(\sqrt{t/\tau} \right) \right) \tau^{\frac{\delta_H}{2}-1} d\tau = I_{1,1}(t) + I_{1,2}(t) = \int_t^{\tau_0} + \int_{\tau_0}^1$$

where τ_0 satisfies $\tau_0^{-1/\beta} = J\sqrt{\tau_0}/t$, i.e., $\tau_0 = (t/J^2)^{\beta/(2+\beta)}$ with $J = \int_0^{+\infty} \chi(x) dx$ as in Corollary 2. We get upper bounds for $I_{1,1}(t)$ and $I_{1,2}(t)$ using the upper bounds given in Corollary 2 as follows. Using the first argument in the minimum, we have

$$I_{1,1}(t) \leq C \frac{J}{\sqrt{t}} \tau_0^{\delta_H/2} = C J^{1-n\beta/2} t^{n\beta/4-1/2}$$

Similarly, using the second argument in the minimum, we find that

$$I_{1,2}(t) \leq C \int_{\tau_0}^1 \tau^{-1/\beta} \tau^{\delta_H/2-3/2} d\tau \leq C J^{1-n\beta/2} t^{n\beta/4-1/2}.$$

Finally,

$$t^{-\delta_H/2} I(t) \leq C J^{1-n\beta/2} t^{-(n+1)/2}$$

so that we can take $\delta = 1 - n\beta/2$. We will use this further in Section 3.4 by choosing J small.

Sub-case $\beta = \beta_c$. When $\beta = 2/n$, we have to estimate the asymptotics of

$$I_1(t) = \int_t^1 \mathrm{Tr} \left(e^{-\tau P_1} \chi(\cdot \sqrt{t/\tau}) \right) \tau^{(n-1)/2} d\tau.$$

Using the estimate of Proposition 4, we get

$$I_1(t) \sim \frac{1}{\sqrt{4\pi}} \int_t^1 \tau^{\frac{n}{2}-1} d\tau \int_0^\infty e^{-\tau x^{2/n}} \chi(x \sqrt{t/\tau}) dx$$

modulo terms of smaller order in τ . Using the change of variable $y = \tau x^{2/n}$, we get

$$I_1(t) \sim \frac{n}{4\sqrt{\pi}} \int_t^1 \frac{d\tau}{\tau} \int_0^\infty e^{-y} y^{\frac{n}{2}-1} \chi \left(\sqrt{t\tau}^{-(n+1)/2} y^{n/2} \right) dy = \frac{n}{4\sqrt{\pi}} \int_t^1 \frac{d\tau}{\tau} F \left(\tau^{(n+1)/2} / t^{1/2} \right)$$

where the function F , defined by

$$F(X) = \int_0^{+\infty} e^{-y} y^{\frac{n}{2}-1} \chi \left(y^{n/2} / X \right) dy,$$

is smooth and satisfies $F(X) = \mathcal{O}_{X \rightarrow 0}(X)$ and $\lim_{X \rightarrow +\infty} F(X) = \Gamma(n/2)$. Using the new variable $u = \tau^{(n+1)/2} / t^{1/2}$, we get

$$I_1(t) \sim \frac{n}{2(n+1)\sqrt{\pi}} \int_{t^{n/2}}^{t^{-1/2}} F(u) \frac{du}{u},$$

and finally

$$I_1(t) \sim \frac{n\Gamma(n/2)}{4(n+1)\sqrt{\pi}} |\ln t|$$

as $t \rightarrow 0^+$.

3.3 The heat parametrix in the separable metric case

We adapt the method of [6] and we use Appendix A.4. We denote by z, z' some generic points of X and by $z = (x, m), z' = (x', m')$ generic points of $[0, 1) \times M \subset X$. Let χ be as in the previous sections, vanishing near $x = 1$ and extended by 0 inside X . Let $\eta \in C_0^\infty([0, 1])$ so that $\eta = 1$ near the support of χ , and $\eta_0 \in C_0^\infty(X)$, vanishing near ∂X and equal to 1 near the support of $1 - \chi$. We choose $a > 0$ so that η_0 vanishes for $x \leq 2a$. In the next result, we show that

$$p(t; z, z') = \eta(x)e_\infty(t; z, z')\chi(x') + \eta_0(x)e_0(t; z; z')(1 - \chi(x'))$$

where e_∞ is the heat kernel on the cone X_∞ and e_0 the Riemannian heat kernel generated by the Laplacian Δ_g on $X \setminus \{x \leq a\}$ with Dirichlet boundary conditions, is a good approximation of the heat kernel on X as $t \rightarrow 0$.

Proposition 6. *Let $P(t)$ be the operator of Schwartz kernel $p(t, \cdot, \cdot)$. We have*

$$\text{Tr}(P(t) - e^{-t\Delta_g}) = O(t^\infty)$$

as $t \rightarrow 0^+$.

Proof. We set $r(t, z, z') = (\partial_t + (\Delta_g)_z)p(t, z, z')$. The kernel r vanishes if x is small enough. By the local nature of the small-time asymptotics of Riemannian heat kernels (see Appendix A.4), $e_\infty(t, \cdot, \cdot)$ and $e_0(t, \cdot, \cdot)$ are $O(t^\infty)$ close in C^∞ topology on $[0, 1) \times M$. Moreover, if $x \in \text{supp}(\eta')$ or $x \in \text{supp}(\eta_0)$, $p(t, \cdot, \cdot)$ and $e_0(t, \cdot, \cdot)$ are $O(t^\infty)$ in the C^∞ topology, because $z \neq z'$. It follows that $r(t, \cdot, \cdot) = O(t^\infty)$ in C^∞ topology. Therefore, denoting by $R(t)$ the operator of Schwartz kernel $r(t, \cdot, \cdot)$, the trace norm of $R(t)$ is a $O(t^\infty)$. By the Duhamel formula, using that $P(t) \rightarrow \text{id}$ as $t \rightarrow 0^+$, we have

$$P(t) - e^{-t\Delta_g} = \int_0^t e^{-(t-s)\Delta_g} r(s) ds.$$

The result follows because the operator norm of $e^{-t\Delta_g}$ is not greater than 1. \square

3.4 Completion of the proof of Theorem 1 in the separable metric case

Thanks to the previous section, we only have to estimate the trace of $P(t)$. We use the local nature of the heat asymptotics to show that the contribution of the term $\eta_0 e_0(t)(1 - \chi)$ is equivalent to $(4\pi t)^{-(n+1)/2} \int_X (1 - \chi) dv_g$. We are left to estimate the term that corresponds to the truncated cone as in Section 3.2. This gives the conclusion when $\beta \geq \beta_c$.

When $\beta < \beta_c$, the first term can be made smaller than $\varepsilon t^{-(n+1)/2}$ for any $\varepsilon > 0$ by choosing $J = \int_0^{+\infty} \chi(x) dx$ small enough as mentioned in Section 3.2.2.

3.5 From the separable to the general case

We prove that, for any given $\varepsilon > 0$, the metric g on X is ε -quasi-isometric to a separable metric g_s . We choose $\delta > 0$ so that

$$|g_0(u) - g_0(0)| \leq \varepsilon(du^2 + g_0(u))$$

for any $u \in [0, \delta]$. Then, we choose $\eta \in C_0^\infty([0, \delta])$, identically equal to 1 near $u = 0$. We consider the separable metric g_s which coincides with g outside $u \leq \delta$ and is given near ∂X by

$$g_s = \eta u^{-\alpha} (du^2 + g_0(0)) + (1 - \eta)g.$$

Then

$$\left| \frac{g_s}{g} - 1 \right| = \left| \eta \frac{g_0(u) - g_0(0)}{du^2 + g_0(u)} \right| \leq \varepsilon.$$

Using Appendix A.1, this concludes the proof of Theorem 1 in the general (non-separable) case.

3.6 Proof of Theorem 2

3.6.1 Case $\beta < \beta_c$

We consider the heat traces $Z_f(t) = \text{Tr}(e^{-t\Delta_g} f)$ where $f : X \rightarrow \mathbb{R}$ is continuous. Let $\varepsilon > 0$. We choose a smooth function $\chi : X \rightarrow [0, 1]$ that is identically equal to 1 near ∂X and such that $\int_X \chi dv_g \leq \varepsilon$. Writing $g = \chi f + (1 - \chi)f$, we get

$$\text{Tr}(e^{-t\Delta_g} f) = \text{Tr}(e^{-t\Delta_g} (1 - \chi)f) + \text{Tr}(e^{-t\Delta_g} \chi f) = J(t) + K(t).$$

By the local nature of the heat trace asymptotics (see Appendix A.4), we have

$$J(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X (1 - \chi) f dv_g$$

as $t \rightarrow 0^+$. Besides,

$$K(t) \leq \|f\|_\infty \text{Tr}(e^{-t\Delta_g} \chi).$$

By (13), and since we can choose $\varepsilon > 0$ arbitrarily small, it follows that

$$Z_f(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X f dv_g$$

as $t \rightarrow 0^+$, which gives the expected result.

3.6.2 Case $\beta \geq \beta_c$

We give the proof in the case $\beta > \beta_c$. The case $\beta = \beta_c$ is treated similarly. Let us first prove that the support of the Weyl measure is contained in ∂X . If $\text{supp}(f) \cap \partial X = \emptyset$, we get again by the local nature of the heat asymptotics that

$$\text{Tr}(e^{-t\Delta_g} f) = O(t^{-(n+1)/2})$$

while, by the Weyl law given in Theorem 1, we have

$$t^{-(n+1)/2} = o(\text{Tr}(e^{-t\Delta_g}))$$

as $t \rightarrow 0^+$. Hence, it suffices to consider functions f of the form $f = \mathbb{1}_D$ where $D = [0, a] \times D_1$ with D_1 a piecewise smooth domain in M .

We recall that, according to [14, Chapter 5, Theorem 2.1.6], we have

$$0 \leq e_D(t, m, m) \leq e_X(t, m, m) \quad \forall m \in D \tag{14}$$

where e_D is the Dirichlet heat kernel on D .

We set $D' = [0, a] \times (M \setminus D)$ and $D'' = X \setminus [0, a] \times M$. For any domain K , we denote by Z_K the Dirichlet heat trace and by $Z'_K(t) = \int_K Z_X(t, m, m) dv_g(m)$. We have

$$Z_D(t) \sim C \text{Vol}(D_1) t^{-d_H/2}, \quad Z_{D'}(t) \sim C \text{Vol}(M \setminus D_1) t^{-d_H/2} \tag{15}$$

and $Z_{D''}(t) = o(t^{-d_H/2})$ as $t \rightarrow 0^+$. Note that for (15), we used the proofs in Section 3.2 for the case where the n -dimensional manifold M (here: D_1 and $M \setminus D_1$) can have a boundary. The sum $(Z'_D + Z'_{D'} + Z'_{D''})(t) = Z_X(t)$ is equivalent to $C \text{Vol}(M)t^{-d_H/2}$ by Theorem 1. Hence, $(Z'_D + Z'_{D'})(t) \sim C \text{Vol}(M)t^{-d_H/2}$. On the other hand, thanks to (14) we have

$$Z_D \leq Z'_D, \quad Z_{D'} \leq Z'_{D'}, \quad Z_{D''} \leq Z'_{D''}.$$

It follows that

$$Z'_D(t) \sim C \text{Vol}(D_1)t^{-d_H/2}$$

as $t \rightarrow 0^+$, which yields the desired result.

4 Discussion and open problems

In this article, motivated by the propagation of acoustic waves in gas giant planets, we derived the Weyl law for the Laplace-Beltrami operator on a smooth compact Riemannian $(n + 1)$ -dimensional manifold X with boundary whose metric blows up near the boundary. Many new questions emerge. We present some of them.

Quantum Ergodicity and Quantum Limits. We have seen in Corollary 1 that, if $\beta \geq \beta_c$, then a density-one subsequence of eigenfunctions concentrates on ∂X . This is a preliminary result towards Quantum Ergodicity (QE).

Recall that, on a locally compact space U endowed with a probability Radon measure μ , given a self-adjoint nonnegative operator T on $L^2(U, \mu)$, of discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \rightarrow +\infty$ associated with an orthonormal eigenbasis $\Phi = (\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(U, \mu)$, a Quantum Limit (QL) of Φ is a probability Radon measure ν on U that is a weak limit of a subsequence of the probability measures $|\phi_j|^2 \mu$, i.e., there exists a subsequence $(j_k)_{k \in \mathbb{N}^*}$ such that

$$\int_U f |\phi_{j_k}|^2 d\mu \xrightarrow{k \rightarrow +\infty} \int_U f d\nu \quad \forall f \in C_c^0(U). \quad (16)$$

We say that QE holds for (T, Φ) if there exists a QL ν on U and a subsequence $(j_k)_{k \in \mathbb{N}^*}$ of density one such that (16) holds.

One may wonder whether, when $\beta \geq \beta_c$, a QE property on M would imply a QE property on X . Proving this fact certainly requires fine spectral properties of Schrödinger operators (see [1]). Besides, inspired by [11, Theorem B], we wonder what can be said on QLs supported on $\partial X = \{0\} \times M$: are they invariant under the geodesic flow of (M, G) (where $G = g_1(0)$)? Defining QLs on T^*M will already be a challenge.

Inverse problems on spectra. A natural question is: does the spectrum of X determine the spectrum of M ? Attacking this problem certainly requires developing appropriate trace formulas, as in [10].

Closed geodesics. Recalling that $G = g_1(0)$ where g_1 is defined by (2), it is natural to view geodesics on (M, G) as limits, in an appropriate sense, of geodesics on (X, g) . A natural question is then: do there exist some closed geodesics of X accumulating on (converging to) closed geodesics of $\partial X = M$? We refer to [9] for a similar question investigated in the framework of contact sub-Riemannian 3D manifolds. Here again, having appropriate trace formulas might be useful.

Observability properties. The study of the Weyl asymptotics is a first step towards solving some inverse problems. As explained in Section 1, the knowledge of spectrum properties can already be used to check the validity of some models, but the main objective in the physical context would be the ability to reconstruct some features of the internal structure of the planets, based on the observation of acoustic waves. The feasibility of such an inverse problem is mathematically modeled by an observability inequality, which can be settled as follows for half-waves. Given any $T > 0$ and any subset ω of X , we say that the observability property holds true for (ω, T) if there exists a positive constant $C_T(\omega)$ such that

$$\int_0^T \left\| \mathbb{1}_\omega e^{it\sqrt{\Delta_g}} \phi \right\|_{L^2(X, dv_g)}^2 dt \geq C_T(\omega) \|\phi\|_{L^2(X, dv_g)}^2 \quad \forall \phi \in L^2(X, dv_g). \quad (17)$$

- When $\beta < \beta_c$, we expect that (17) holds as soon as ω is open and (ω, T) satisfies the Geometric Control Condition (GCC, see [2]), like in the classical case of a non-singular Riemannian metric.
- When $\beta \geq \beta_c$, an obvious necessary condition for (17) to hold is that ω contain an open neighborhood of a subset of ∂X . Indeed, take ϕ in (17) to be a highfrequency eigenfunction and apply Corollary 1. We think that this condition is sufficient if moreover (ω, T) satisfies GCC.

We note that, when X is a closed ball in \mathbb{R}^{n+1} (an idealized situation for an exactly round planet), GCC is never satisfied unless ω contains an open neighborhood of the *whole* boundary of X , which is certainly not relevant for applications from the physical point of view. In this case where X is a round ball, it is more interesting to take a small observation subset ω , containing a small open subset of ∂X . But, as soon as ω is a proper subset of a half-ball, GCC (and thus (17)) obviously fails due to trapped rays, propagating along a diameter never meeting ω . In this deteriorated context, we wonder, however, whether (17) is anyway satisfied if we restrict the inequality to radial waves or to surface waves, which are the most physically meaningful waves to be observed.

Metrics that are singular on larger codimension submanifolds. In this paper, we have considered a class of singular metrics blowing up at the boundary of X , where the boundary can be seen as a codimension-one submanifold of X .

In more general, let X be a smooth compact manifold and let Z be a submanifold of X of codimension $m \in \mathbb{N}^*$, and consider the class of singular metrics g on X that are smooth on $X \setminus Z$ and that, near Z , are written as

$$g = h + g_Z(x)r^{-\beta}$$

in a neighborhood of Z assumed to be diffeomorphic to $Z \times B_m$ where B_m is the unit ball of \mathbb{R}_x^m (this holds if the normal bundle of Z is trivial) equipped with the Euclidean metric h and the polar coordinates (r, σ) , and $g_Z(x)$ is a metric on Z , parametrized by $x \in B_m$ and depending smoothly on x . The techniques developed in our paper can certainly be extended to compute the Weyl asymptotics in such cases.

Two-term expansion. In Theorem 1 we have computed an equivalent of the Weyl counting function $N(\lambda)$ as $\lambda \rightarrow +\infty$. As alluded in Remark 5, an interesting open issue is to compute the second term of the expansion. When $\beta < \beta_c$, the main term is the same as in the nonsingular case, and we wonder whether the second term would reflect the singularity and would be larger than the usual second term obtained in the classical smooth case (as, for instance, in the recent article [19]). When $\beta \geq \beta_c$, we wonder whether the second term would reflect only what happens in the interior, or would still be affected by the singularity.

A Appendix

A.1 Quasi-isometries

Let X be a smooth manifold of dimension $n+1$, with boundary. Two metrics g_1 and g_2 , smooth on $X \setminus \partial X$, are said to be ε -quasi-isometric if

$$\left| \frac{g_1}{g_2} - 1 \right| \leq \varepsilon$$

uniformly on $X \setminus \partial X$. For $i \in \{1, 2\}$, let Δ_{g_i} be the Friedrichs extension of the Laplace-Beltrami operator on (X, g_i) with core $C_0^\infty(X \setminus \partial X)$.

If Δ_{g_1} has a discrete spectrum $(\lambda_j^1)_{j \in \mathbb{N}^*}$ then Δ_{g_2} has also a discrete spectrum $(\lambda_j^2)_{j \in \mathbb{N}^*}$ and, for $\varepsilon \leq \frac{1}{2}$ (this condition is to get bounds on the inverse of g_i), there exists $C(n) > 0$ such that, for every $j \in \mathbb{N}^*$,

$$\left| \frac{\lambda_j^1}{\lambda_j^2} - 1 \right| \leq C(n)\varepsilon.$$

Indeed, this estimate follows from the minimax characterization of the eigenvalues and from the comparison of the Rayleigh quotients, i.e., of the volumes and co-metrics.

A.2 Karamata tauberian theorem and converse

We recall the Karamata tauberian theorem (see [18, Chapter XIII, Theorem 2]).

Theorem 3. *Let μ be a positive Radon measure on \mathbb{R}^+ . If there exists $\alpha > 0$ such that*

$$\int_0^{+\infty} e^{-t\lambda} d\mu(\lambda) \sim At^{-\alpha} \quad (\text{resp., } A|\ln t|t^{-\alpha})$$

as $t \rightarrow 0^+$, then

$$\mu([0, \lambda]) \sim \frac{A}{\Gamma(\alpha+1)} \lambda^\alpha \quad \left(\text{resp., } \frac{A}{\Gamma(\alpha+1)} \lambda^\alpha \ln \lambda \right)$$

as $\lambda \rightarrow +\infty$.

We need a converse of Theorem 3. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a nonincreasing function of class C^1 , such that f and f' have a fast decay at infinity. Let $(\lambda_j)_{j \in \mathbb{N}^*}$ be a nondecreasing sequence of positive real numbers. We define the counting function $N(\lambda) = \#\{j \in \mathbb{N}^* \mid \lambda_j \leq \lambda\}$, for any $\lambda \in \mathbb{R}$. The objective is to estimate the sum

$$S = \sum_{j=1}^{+\infty} f(\lambda_j).$$

Proposition 7. *Assume that there exist $C > 0$ and $\alpha > 0$ such that $N(\lambda) \sim C\lambda^\alpha$ as $\lambda \rightarrow +\infty$. For any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$, depending on the counting function N but not on f , such that*

$$\left| S - C\alpha \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda \right| \leq K(\varepsilon)f(\lambda_1) + \varepsilon \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda.$$

Proof. Given any $\varepsilon > 0$, let $\Lambda_0 > 0$ such that, for every $\lambda \geq \Lambda_0$,

$$(1 - \varepsilon)C\lambda^\alpha \leq N(\lambda) \leq (1 + \varepsilon)C\lambda^\alpha. \quad (18)$$

Noting that $dN(\lambda) = \sum_{j=1}^{+\infty} \delta_{\lambda_j}$, using the Stieltjes integral, we have

$$S = \sum_{\lambda_j < \Lambda_0} f(\lambda_j) + \int_{\Lambda_0}^{+\infty} f(\lambda) dN(\lambda).$$

Now, since $\sum_{\lambda_j < \Lambda_0} f(\lambda_j) \leq N(\Lambda_0)f(\lambda_1)$, we get by integration by parts, using the fast decay of f at infinity, that

$$S \leq N(\Lambda_0)(f(\lambda_1) - f(\Lambda_0)) - \int_{\Lambda_0}^{+\infty} f'(\lambda)N(\lambda)d\lambda.$$

We derive an upper bound for S . A lower bound is obtained similarly. Using (18), integrating by parts and using that $f(\Lambda_0) \leq f(\lambda_1)$, we obtain

$$\begin{aligned} - \int_{\Lambda_0}^{+\infty} f'(\lambda)N(\lambda) d\lambda &\leq -(1 + \varepsilon)C \int_{\Lambda_0}^{+\infty} f'(\lambda)\lambda^\alpha d\lambda \\ &\leq (1 + \varepsilon)C \left(f(\lambda_1)\Lambda_0^\alpha + \alpha \int_{\Lambda_0}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda \right). \end{aligned}$$

Therefore,

$$S \leq f(\lambda_1)(N(\Lambda_0) + C(1 + \varepsilon)\Lambda_0^\alpha) + (1 + \varepsilon)C\alpha \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda$$

and the result follows with $K(\varepsilon) = N(\Lambda_0) + C(1 + \varepsilon)\Lambda_0^\alpha$. \square

A.3 Weyl circle-point limit criterion

We consider the Schrödinger operator $P = -\partial_x^2 + q(x)$ on $C_0^\infty((0, +\infty))$, where q is a smooth function on $(0, +\infty)$. According to the Weyl circle-point limit criterion (see [36, Theorem X.7]), P is essentially self-adjoint if and only if there exists at least one solution of $Pu = 0$ that is not square integrable at 0 and at least one solution of $Pu = 0$ that is not square integrable at $+\infty$.

When $q(x) = Cx^{-2} + \omega x^\beta$ for some $C \geq 0$, $\beta > 0$ and $\omega > 0$, there is only one solution of $Pu = 0$ that is square integrable at $+\infty$. Near 0, the solutions of $Pu = 0$ are equivalent to linear combinations of x^{γ_+} and x^{γ_-} where γ_+ and γ_- are the two solutions of $-\gamma(\gamma - 1) + C = 0$. It follows that P is essentially self-adjoint if and only if $\gamma_- \leq -\frac{1}{2}$, that is, if and only if $C \geq 3/4$.

A.4 Local nature of the small-time asymptotics of heat kernels

Let (U, g) be a smooth Riemannian manifold and let Δ be the Laplace-Beltrami operator. For our needs (see Section 3.3), $U = X \setminus \partial X$ with the metric g .

Let e_1 and e_2 be two solutions of $(\partial_t + \Delta_x)e_i(t, x, y) = 0$ for $t > 0$, satisfying $e_i(t, x, y) = e_i(t, y, x)$ for all $t > 0$ and $(x, y) \in U \times U$ and

$$\lim_{t \rightarrow 0^+} \int_U e_i(t, x, y)f(y) dv_g(y) = f(x) \quad \forall x \in U \quad \forall f \in C_0^\infty(U),$$

for $i \in \{1, 2\}$.

Lemma 3. *We have $e_1(t, \cdot, \cdot) - e_2(t, \cdot, \cdot) = O(t^\infty)$ as $t \rightarrow 0^+$ in C^∞ topology on $U \times U$. Moreover, denoting by D the diagonal of $U \times U$, for $i \in \{1, 2\}$, we have $e_i(t, \cdot, \cdot) = O(t^\infty)$ as $t \rightarrow 0^+$ in C^∞ topology on $U \times U \setminus D$.*

This result reflects Kac’s principle of “not feeling the boundary”, showing that the small-time asymptotic behavior of heat kernels is purely local. A detailed proof can be found in [12, Section 3.2.1]. The idea comes from the paper [25]. The proof uses the fact that the Hörmander operator $P = 2\partial_t + \Delta_x + \Delta_y$ is hypoelliptic. Extending the kernels e_i by 0 for $t < 0$, we have $Pe_i = 0$ on $\mathbb{R} \times U \times U \setminus D$ and $P(e_1 - e_2) = 0$ on $\mathbb{R} \times U \times U$, in the distributional sense. The result then follows by hypoellipticity.

A.5 Δ_g as a nonsmooth Hörmander operator

Based on the mathematical model provided in Section 2.1, near any point of the boundary of X we have $X \simeq [0, 1) \times \mathbb{R}^n$ with a local system of coordinates (x, y) , with $x \in [0, 1)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, and we can write (locally)

$$\Delta_g = - \sum_{i=0}^n X_i^* X_i + V \quad (19)$$

where $V(x, y) = \frac{C(x, y)}{x^2}$ is a potential and the X_i ’s are vector fields given by

$$X_0 = a_0(x, y) \partial_x, \quad X_i = x^{\beta/2} a_i(x, y) \partial_{y_i}, \quad i \in \{1, \dots, n\}.$$

The functions C and a_i , $i \in \{0, \dots, n\}$ are smooth on $\mathbb{R} \times \mathbb{R}^n$ and $C(0, \cdot) = C_\beta$ and $a_i(0, \cdot) = 1$ (they can be expressed in terms of the coefficients of the smooth Riemannian metric $g_1(x)$ on M defined by (2)). The separable case corresponds to $a_0 = 1$ and a_i not depending on x .

Expressed as (19), the operator Δ_g is then a Hörmander operator, however nonsmooth unless $\beta \in 2\mathbb{N}^*$. Because of this lack of smoothness, many classical results cannot be applied here.

When $\beta \in 2\mathbb{N}^*$, the above vector fields are smooth and define an almost-Riemannian geometry, in which the Weyl asymptotics of the almost-Riemannian Laplacian $\Delta_{aR} = - \sum_{i=0}^n X_i^* X_i$ (i.e., (19) with $V = 0$), of Grushin type, has been established in [13].

With these preliminary remarks in mind, we then mention a few interesting facts hereafter.

Homogeneity. In the above local coordinates, given any $\varepsilon > 0$, we define the *dilation*

$$\delta_\varepsilon(x, y) = (\varepsilon x, \varepsilon^{1+\beta/2} y) \quad \forall (x, y) \in [0, 1) \times \mathbb{R}^n.$$

In the separable case where $a_0 = 1$ and a_i does not depend on x , for any $i \in \{1, \dots, n\}$, we define $\widehat{X}_i = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_\varepsilon^* X_i = x^{\beta/2} a_i(0) \partial_{y_i}$, and we have

$$\varepsilon \delta_\varepsilon^* X_0 = X_0 \quad \text{and} \quad \varepsilon \delta_\varepsilon^* \widehat{X}_i = \widehat{X}_i \quad \forall i \in \{1, \dots, n\}.$$

In sR geometry, \widehat{X}_i is the *nilpotentization* of the vector field X_i at the point identified with $(0, 0)$. Extrapolating results of sub-Riemannian geometry that one can find in [12] to the case of $\beta > 0$, denoting by d_g the g -distance on X , one can show that $d_g((0, 0), (x, y))$ divided by $|x| + \sum_{i=1}^n |y_i|^{1/(1+\beta/2)}$ is bounded above and below by some positive constants in a neighborhood of $(0, 0)$. Noting that $1/(1 + \beta/2) = 1 - \alpha/2$, we thus recover [15, Proposition 13] and thus the result of Proposition 1 and the fact that Hausdorff and Minkowski dimensions coincide. In the non-separable case, we obtain the result by using quasi-isometries.

Weyl law when $\beta \in 2\mathbb{N}^*$. When $\beta \in 2\mathbb{N}^*$, we always have $\beta \geq \beta_c$, and $\beta = \beta_c$ if and only if $n = 1$. Since the potential $1/x^2$ is homogeneous, combining results of [12, 13], we recover the Weyl law established in Theorem 1.

Weyl law when $\beta \notin 2\mathbb{N}^*$. To establish the Weyl law in general sub-Riemannian cases, the approach developed in [13] consists of estimating singular integrals involving the heat kernel, by performing the so-called $(J + K)$ -decomposition. Applying this approach to the nonsmooth operator in (19) cannot be done directly because we miss a general hypoellipticity theory, valid for nonsmooth vector fields as above, and a generalization of Lemma 3 (see Appendix A.4) to that context.

Acknowledgements. The authors thank Bernard Helffer for pointing out the reference [31], and Anton Ermakov for many insightful discussions on gas giants and ring seismology. C. Dietze expresses her deepest gratitude to Phan Thành Nam and Laure Saint-Raymond for their continued support, and thanks Daniel Grieser for pointing out a mistake in the introduction. She acknowledges the support by the European Research Council via ERC CoG RAMBAS, Project No. 101044249 and the Engie Foundation. M.V. de Hoop carried out the work while he was an invited professor at the Centre Sciences des Données at École Normale Supérieure, Paris. He acknowledges the support of the Simons Foundation under the MATH + X Program and the sponsors of the GeoMathematical Imaging Group at Rice University. E. Trélat acknowledges the support of the grant ANR-23-CE40-0010-02 (Einstein-PPF).

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