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# Weyl formulae for some singular metrics with application to acoustic modes in gas giants

Yves Colin de Verdière\*, Charlotte Dietze†, Maarten V. de Hoop‡  
Emmanuel Trélat§

## Abstract

This paper is motivated by recent works on inverse problems for acoustic wave propagation in the interior of gas giant planets. In such planets, the speed of sound is isotropic and tends to zero at the surface. Geometrically, this corresponds to a Riemannian manifold with boundary whose metric blows up near the boundary. Here, the spectral analysis of the corresponding Laplace-Beltrami operator is presented and the Weyl law is derived. The involved exponents depend on the Hausdorff dimension which, in the supercritical case, is larger than the topological dimension.

**AMS classification:** 11F72, 58C40.

## 1 Introduction

### 1.1 Seismology on gas giant planets

Seismology has played an important role in revealing the (deep) interiors of gas giant planets in our solar system [6, 24]. Indeed, the acoustic spectra and free oscillations have been studied for Saturn and Jupiter over the past few decades [33, 20, 18]. The excitation of acoustic modes in gas giant planets presumably occurs through convection in their interiors. The observation of acoustic eigenfrequencies, that is, the discrete spectrum can be realized, in principle, through visible photometry, thermal infrared photometry, Doppler spectrometry, and ring seismology for nonradial oscillations [25, 26] (in particular, in the case of Saturn). In ring seismology and with the Cassini mission, one measured the “resonances” in the inner C ring of Saturn with visual and infrared mapping spectrometer (VIMS) stellar occultations [22, 14, 17]. The rings are gravitationally coupled to the acoustic modes of the planet (taking self gravitation into account). Detection of Jupiter’s acoustic eigenvalues has been attempted with ground-based imaging-spectrometry (seismographic imaging interferometer for monitoring of planetary atmospheres or SYMPA) by measuring line of sight velocity [32, 18]. Recently, Juno spacecraft gravity measurements have provided evidence for normal modes of Jupiter [15].

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## 1.2 Singular Riemannian metrics

On a gas giant planet, unlike a rocky planet, the speed of sound goes to zero at the boundary. In the geometric mathematical model that we employ hereafter, the rate at which this happens follows a power law which determines a specific conformal blow-up rate of a Riemannian metric, thus defining a singular metric. This rate happens to be slower than on asymptotically hyperbolic manifolds and the boundary is at a finite distance from interior points. The rate is implied by an equation of state in the upper part of the planet, in general, in the sense of a fit. (For some models of the speed of sound of Jupiter and Saturn showing this behavior, see [20, Figure 1] and [23, Figure 1].) Only for a polytrope is the rate exact. Polytropes, for which the pressure is proportional to a power of the density of mass, have been viewed as relevant simplifications; models with variable polytrope index have indeed been applied to planet and material models [34]. Typically, an equation of state is computed numerically using density functional molecular dynamics simulations with mixtures of chemical elements: The dominant elements in terms of mass fraction are hydrogen and helium, but also heavy elements are important. The equation of state is different for the upper part and the deep interior as the helium fraction can be higher in the interior due to helium rain (helium becoming immiscible with hydrogen at high pressure). Equations of state play a vital role in the evolution and realization of structure of gas giant planets [28, 29].

More specifically, if  $g_e$  is the Euclidean Riemannian metric on a smooth domain  $X \subset \mathbb{R}^{n+1}$ , then the speed of sound  $c$  can be encoded by the conformally Euclidean Riemannian metric  $g = c^{-2}g_e$ . In local coordinates where the boundary of  $X$  is (locally) described by  $u = 0$ , the polytropic model suggests that  $c \sim u^{1/2}$ . Indeed, the natural generalization is  $c \sim u^{\alpha/2}$ , that is,  $c^{-2} \sim u^{-\alpha}$ ; through previous analysis [13] it appears that restricting  $\alpha$  according to  $\alpha \in (0, 2)$  guarantees the presence of a discrete spectrum as it has been observed. Thus, the Riemannian geometry lies between standard geometry with boundary and asymptotically hyperbolic geometry. Some of the phenomena in this geometry are unlike those seen at either end. The extreme case  $\alpha = 0$  corresponds physically to solid bodies and mathematically to manifolds with boundary, and the other extreme  $\alpha = 2$  corresponds to asymptotically hyperbolic geometry but is far from all planetary models.

Therefore, following [13, Section 1.1], we model a gas giant planet as a smooth manifold  $X$  with a boundary, endowed with a Riemannian metric  $g$  on  $X \setminus \partial X$  such that, near  $\partial X$ , we have  $g = \bar{g}/u^\alpha$  where  $\bar{g}$  is a well-defined Riemannian metric up to the boundary, and  $\partial X = \{u = 0\}$  locally. The fact that  $\bar{g}$  is neither zero nor infinite at  $\partial X$  implies a specific blow-up rate for  $g$  near  $\partial X$ . This conformal power-law blow-up is the key geometric feature of gas giant metrics. The speed of sound might contain jump discontinuities where phase transitions occur (see [27]), that is, the metric can contain conormal singularities while the manifold consists of multiple “layers”. A key interior boundary in gas giants corresponds with the transition from molecular to metallic hydrogen. Accounting for discontinuities in an asymptotic formalism for gas giant seismology was developed a few decades ago (see [30]).

The mathematical study of the spectrum associated with gas giants’ acoustic modes was initiated in [13]. In this paper, we analyze the relevant Laplace-Beltrami operator and we compute the Weyl law. The study of Weyl asymptotics, which reflects some properties of the singular metric, is a preliminary step towards analyzing some inverse problems, in view of reconstructing some features of the internal structure of gas giant planets.

## 2 Mathematical model and main results

### 2.1 Mathematical model

Let  $X$  be a smooth compact manifold of dimension  $n + 1$  with a boundary  $\partial X$ . Near  $\partial X$ ,  $X$  is diffeomorphic to  $[0, 1) \times M$ , where  $M$  is a smooth compact manifold of dimension  $n \geq 1$  and  $\partial X$  is identified with  $\{0\} \times M$  and also with  $u = 0$  where  $u$  is a transverse coordinate, locally near  $\partial X$ , ranging over  $[0, 1)$ . As discussed in Section 1.2, we consider on  $X$  a singular Riemannian metric  $g$  that is a smooth metric on  $X \setminus \partial X$ , written near  $\partial X$  as

$$g = \bar{g}/u^\alpha$$

where  $0 < \alpha < 2$  and  $\bar{g}$  is a smooth (non-singular) Riemannian metric on  $X$ , up to the boundary. Following [13, Proposition 2], which uses a normal form for the metric near the boundary, due to [19, Lemma 5.2], we have

$$g = u^{-\alpha}(du^2 + g_0(u))$$

where  $g_0(u)$  is a smooth Riemannian metric on  $M$  (pulled back to the level set  $u = \text{Cst}$ ) depending smoothly on  $u \in [0, 1)$ .

We make a change of variable. Setting  $x = x(u) = \int_0^u s^{-\alpha/2} ds = (1 - \frac{\alpha}{2})^{-1} u^{1-\frac{\alpha}{2}}$ , we get

$$g = dx^2 + x^{-\beta} g_1(x) \text{ where } \beta = \frac{2\alpha}{2-\alpha} \quad (1)$$

and  $g_1 = g_1(x)$  is a smooth Riemannian metric on  $M$  (pulled back to the level set  $x = \text{Cst}$ ) depending smoothly on  $x \in [0, 1)$ . We note that, since  $\alpha \in (0, 2)$ ,  $\beta$  can take any positive value. We also note that a polytrope (for any index) corresponds to  $\beta = \beta_{\text{poly}} = 2$ . We have that  $g_1(x) = C(\alpha)g_0(u)$  for some constant  $C(\alpha) > 0$ .

For any  $x \in [0, 1)$ , denoting by  $dv_1^x$  the volume measure on  $M$  associated to the metric  $g_1(x)$ , the  $g$ -volume is  $dv_g = x^{-\beta n/2} |dx| dv_1^x$ . The volume is finite if and only if  $\beta < \beta_c$ , where

$$\beta_c = \frac{2}{n} \quad (2)$$

is a critical value of  $\beta$ . We will see later that this critical value plays a role in the Weyl asymptotics. At this point, we can note that  $\beta_{\text{poly}} > \beta_c$  for  $n = 2$ .

The following three propositions were proved in [13]. The first proposition concerns the Hausdorff dimension.

**Proposition 1.** *The Hausdorff dimension of  $(X, g)$  is*

$$d_H = \max \left( n + 1, n \left( 1 + \frac{\beta}{2} \right) \right).$$

We define  $\delta_H = n(1 + \frac{\beta}{2})$ , and note that  $d_H > n + 1$  ( $n + 1$  is the topological dimension of  $X$ ) if and only if  $\beta > \beta_c$ . We give in Appendix A.5 a sketch of the proof of Proposition 1, in which we also show that  $d_H$  coincides with the Minkowski dimension of  $(X, g)$ .

**Proposition 2.** *The Laplace-Beltrami operator  $\Delta_g$ , with core  $C_0^\infty(X \setminus \partial X)$ , is essentially self-adjoint if and only if  $\beta \geq \beta_c$ .*

For  $\beta < \beta_c$ , there exist several extensions of  $\Delta_g$ , with core  $C_0^\infty(X \setminus \partial X)$ . In the further analysis, we consider its Friedrichs extension (that is, ‘‘Dirichlet extension’’).

**Proposition 3.** *For every  $\beta > 0$ , the spectrum of  $\Delta_g$  is discrete.*

We denote the eigenvalues of  $\Delta_g$  by  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots$  with associated eigenfunctions  $\phi_j$ ,  $j \in \mathbf{N}^*$ , normalized in  $L^2(X, dv_g)$ . We note that, if the volume of  $X$  is infinite, i.e., if  $\beta \geq \beta_c$ , we have  $\lambda_1 > 0$ , in contrast to the usual Riemannian case. The Weyl counting function is defined by

$$N(\lambda) = \#\{j \in \mathbf{N}^* \mid \lambda_j \leq \lambda\}$$

where the eigenvalues are counted with their multiplicity. Our objective is to derive a Weyl law describing the asymptotics of  $N(\lambda)$  as  $\lambda \rightarrow +\infty$ .

**Remark 1.** The following fact will be used in Section 3.5. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the metric  $g$  is  $\varepsilon$ -quasi-isometric (see Appendix A.1) to a singular Riemannian metric  $\tilde{g}$  on  $X$ , smooth on  $X \setminus \partial X$  and given by  $\tilde{g} = dx^2 + x^{-\beta}g_1(0)$  on  $(0, \delta) \times M$ . In order to derive a Weyl law for  $(X, g)$  it suffices to derive the corresponding Weyl law for  $\tilde{g}$  for any  $\varepsilon > 0$  (see, again, Appendix A.1 for details). This remark is important, because it implies that we mainly have to work within the so-called *separable* case.

**Separable case.** We say that we are in the *separable* case if the metric  $g_1(x)$  on  $M$  (defined by (1)) does not depend on  $x$ , i.e.,  $g_1(x) = g_1(0)$  for any  $x \in (0, 1)$ ; we still denote this metric by  $g_1$ . In the sequel, we consider  $[0, 1) \times M$  instead of  $[0, \delta) \times M$  for simplicity of notation, while the proofs are similar in both cases.

We denote by  $\Delta_M$  the Laplace-Beltrami operator on  $(M, g_1)$ . We denote the eigenvalues of  $\Delta_M$  by  $0 \leq \omega_1 \leq \omega_2 \leq \dots \leq \omega_j \leq \dots$  with an associated orthonormal basis of eigenfunctions  $(\psi_j)_{j \in \mathbf{N}^*}$ . The Weyl counting function for  $\Delta_M$  is defined by

$$N_M(\omega) = \#\{k \in \mathbf{N}^* \mid \omega_k \leq \omega\}.$$

Since  $g_1$  is a smooth Riemannian metric on  $M$ , the classical Weyl law for  $(M, g_1)$  yields that  $N_M(\omega) = \gamma_n \text{Vol}_{g_1}(M) \omega^{n/2} + O(\omega^{n/2})$  as  $\omega \rightarrow +\infty$  where

$$\gamma_n = \frac{1}{(4\pi)^{n/2} \Gamma(\frac{n}{2} + 1)} \quad (3)$$

(see [3, Chapter 3E] for the heat trace and then apply the Karamata tauberian theorem, i.e., Theorem 3 in Appendix A.2).

Denoting by  $dv_1$  the volume measure on  $M$  associated to the metric  $g_1 = g_1(0)$ , the  $g$ -volume is  $dv_g = x^{-\beta n/2} |dx| dv_1$ . Making the change of function  $f \mapsto x^{-\beta n/4} f$ , we get the new volume form  $|dx| dv_1$ ; the Laplace-Beltrami operator on  $X_1 = (0, 1) \times M$  is now given by

$$\Delta_g = -\partial_x^2 + \frac{C_\beta}{x^2} + x^\beta \Delta_M$$

where  $x \in (0, 1)$  and

$$C_\beta = \frac{\beta n}{4} \left( \frac{\beta n}{4} + 1 \right).$$

The proof is straightforward by performing an integration by parts with respect to  $x$  in the Dirichlet form defining the Laplace-Beltrami operator, using the Dirichlet boundary condition at  $x = 0$ . We note that  $C_\beta \geq 3/4$  if and only if the volume of  $X$  is infinite. Using the Weyl criterion (see Appendix A.3), this inequality also implies that  $P_\omega$  defined by (5) below is essentially self-adjoint for any  $\omega > 0$ , but not for  $\omega = 0$ .

We will need to work first on the non-compact conic manifold  $X_\infty = (0, +\infty) \times M$  endowed with the metric  $g = dx^2 + x^{-\beta}g_1$ . Let  $\Delta_\infty$  stand for the Laplace-Beltrami operator on  $(X_\infty, g)$ . Invoking a separation of variables, we have

$$\Delta_\infty = \bigoplus_{k=1}^{+\infty} (\text{id} \otimes \pi_k) (P_{\omega_k} \otimes \text{id}) (\text{id} \otimes \pi_k) \quad (4)$$

where

$$P_\omega = -\partial_x^2 + \frac{C_\beta}{x^2} + \omega x^\beta \quad (5)$$

is a Schrödinger operator on  $L^2((0, +\infty), dx)$  for any  $\omega \geq 0$ , and where  $\pi_k$  is the orthogonal projection of  $L^2(M, dv_1)$  onto the subspace generated by  $\psi_k$  and  $\text{id}$  denotes the identity operator on  $L^2(X_\infty, |dx| dv_1)$  (resp., on  $L^2(M, dv_1)$ ). Hence,  $\Delta_\infty$  is unitarily equivalent to  $\bigoplus_{k=1}^{+\infty} P_{\omega_k}$ .

## 2.2 Main results

Recalling that  $g_1$  is defined by (1), we set  $G = g_1(0)$  and denote by  $v_G$  the corresponding volume form on  $M$ . We also recall that  $\beta_c$  is defined in (2) and that  $\gamma_n$  is defined in (3).

### Theorem 1. (Weyl asymptotics)

- If  $\beta > \beta_c$  then

$$N(\lambda) \sim A(\beta, n) v_G(M) \lambda^{d_H/2}$$

as  $\lambda \rightarrow +\infty$ , with

$$A(\beta, n) = \frac{n\gamma_n(\beta + 2)}{4\Gamma(1 + d_H/2)} \int_0^{+\infty} \mathcal{Z}_1(\tau) \tau^{\frac{d_H}{2}-1} d\tau$$

where  $\mathcal{Z}_1(\tau) = \text{Tr}(\exp(-\tau P_1))$  and  $P_1$  is the Schrödinger operator on  $L^2((0, +\infty), dx)$  defined by (5).

- If  $\beta = \beta_c = 2/n$  then

$$N(\lambda) \sim C_n v_G(M) \lambda^{(n+1)/2} \ln \lambda$$

as  $\lambda \rightarrow +\infty$ , with

$$C_n = \frac{1}{(n+1)(4\pi)^{(n+1)/2} \Gamma((n+1)/2)}.$$

In particular,  $C_1 = 1/8\pi$ .

- If  $\beta < \beta_c$  then

$$N(\lambda) \sim \gamma_{n+1} v_g(X) \lambda^{(n+1)/2}$$

as  $\lambda \rightarrow +\infty$ .

### Remark 2.

- When  $M = \mathbb{R}/2\pi\mathbb{Z}$  and  $X$  is diffeomorphic to the hemisphere, endowed with the so-called Grushin metric, the authors of [4] derived the Weyl law using an explicit computation of the spectrum. We recover their result as a particular case with  $n = 1$  and  $\beta = \beta_c = 2$ .
- To prove Theorem 1, we make use of heat kernels. Alternatively, it is possible to use Dirichlet-Neumann bracketing. Both methods allow to treat conormal jump singularities of the metric  $\bar{g}$  inside  $X$  that model layering in the gas planet.

**Remark 3.** A natural question is whether  $\beta$  can be determined from the Weyl asymptotics. Indeed, when  $n$  is known,  $\beta$  can be determined in the case where  $\beta \geq \beta_c$ . When  $\beta < \beta_c$  the question remains open. To shed light on this, it would be useful and interesting to get the next term in the small-time heat trace expansion (see Section 3.2.2) when  $\beta \leq \beta_c$ .

We next compute the *Weyl measures*, which are the probability measures  $w_g$  on  $X$ , defined, if the limit exists, by

$$\int_X f dw_g = \lim_{\lambda \rightarrow +\infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \int_X f |\phi_j|^2 dv_g$$

for any function  $f : X \rightarrow \mathbb{R}$  that is continuous up to the boundary of  $X$ . Such measures have been introduced in [9, 11] in the framework of sub-Riemannian geometry in order to provide an account of how the high-frequency eigenfunctions concentrate.

**Theorem 2. (Weyl measures)**

- If  $\beta \geq \beta_c$  then the Weyl measure is  $\delta_{x=0} \otimes dv_G/v_G(M)$ .
- If  $\beta < \beta_c$  then the Weyl measure is the uniform probability distribution given by the normalized volume of  $(X, g)$ , that is  $dv_g/v_g(X)$ .

Using [11, Corollary 7.1], we obtain the following consequence.

**Corollary 1.** If  $\beta \geq \beta_c$  then there exists a density-one subsequence  $(\phi_{j_k})_{k \in \mathbb{N}^*}$  of the sequence of eigenfunctions that concentrates on  $\partial X$ , meaning that for any compact subset  $K \subset X \setminus \partial X$ , we have

$$\lim_{k \rightarrow +\infty} \int_K |\phi_{j_k}|^2 dv_g = 0.$$

### 3 Proofs of Theorems 1 and 2

Our strategy of proof is the following. We first treat the separable case (Sections 3.1 to 3.4). As a preliminary, we perform in Section 3.1 a spectral study of the 1D Schrödinger operator  $P_\omega$  defined by (5), deriving exponential estimates for truncated heat traces. Then, in Section 3.2, we estimate the small-time asymptotics of the truncated heat trace of  $\Delta_g$ , near the boundary (actually, on a cone); the three cases  $\beta > \beta_c$ ,  $\beta = \beta_c$ ,  $\beta \leq \beta_c$ , must be treated in different ways. In Section 3.3, using a heat parametrix, we glue together the heat kernel near the boundary and the Riemannian heat kernel far from the boundary. Finally in Section 3.4 we prove Theorem 1 in the separable case.

In Section 3.5, we show how to pass from the separable to the general case by using the fact that the metric  $g$  is quasi-isometric to a separable metric. In Section 3.6, we prove Theorem 2. Our approach uses again heat traces.

#### 3.1 Spectral study of the 1D Schrödinger operators

We consider the family of Schrödinger operators,

$$P_1 = -\partial_x^2 + q_{C,\beta}(x)$$

where  $q_{C,\beta}(x) = Cx^{-2} + x^\beta$ ,  $C > 0$  and  $\beta > 0$ , acting on  $L^2((0, +\infty), dx)$ . The operators  $P_1$  are essentially self-adjoint if and only if  $C \geq 3/4$ ; when  $C < 3/4$  we consider the Friedrichs extension of  $P_1$  with core  $C_0^\infty((0, +\infty))$  (see Appendix A.3). The spectrum of  $P_1$  is discrete; we denote it by

$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ . We derive precise semi-classical asymptotics for the associated truncated heat trace.

Let  $\chi : [0, +\infty) \rightarrow [0, 1]$  be a smooth decreasing function with  $\chi \equiv 1$  on  $[0, a]$  with  $a > 0$  and  $\chi' \leq 0$  everywhere. We note that  $\chi \equiv 1$  is included. We define the corresponding truncated heat trace by

$$\mathcal{Z}_\chi(\tau) = \text{Tr} \left( e^{-\tau P_1} \chi \right) \quad \forall \tau > 0.$$

Let  $\gamma = \max(1/\beta, 1/2)$ .

**Proposition 4.** *Given any  $0 < \tau \leq 1$ , we have*

$$\mathcal{Z}_\chi(\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx + O(\tau^{-\gamma})$$

and for  $\tau \geq 1$ ,

$$\mathcal{Z}_\chi(\tau) = O(e^{-\mu_1 \tau})$$

uniformly with respect to  $\chi$  in both cases.

The counting function  $N_1(\mu) = \#\{j \in \mathbb{N}^* \mid \mu_j \leq \mu\}$  satisfies  $N_1(\mu) \sim A\mu^{\frac{1}{2} + \frac{1}{\beta}}$  as  $\mu \rightarrow +\infty$  with  $A = \sqrt{\frac{2}{\pi}} \frac{1}{\beta} B(3/2, 1 + 1/\beta)$  where  $B$  is the Beta function.

*Proof of Proposition 4.* We first establish an elementary lemma. We denote by  $Q^N$  (resp.,  $Q^D$ ) the self-adjoint operator  $-\partial_x^2$  on an interval of length 1 with Neumann (resp., Dirichlet) boundary condition.

**Lemma 1.** *For  $0 < \tau \leq 1$  and  $\star \in \{D, N\}$ , we have  $\text{Tr}(\exp(-\tau Q^\star)) = (4\pi\tau)^{-1/2} + O(1)$ .*

*Proof of Lemma 1.* The estimate does not depend on the chosen interval. The spectrum of  $Q^N$  is  $\{n^2\pi^2 \mid n \in \mathbb{N}\}$  and the spectrum of  $Q^D$  is  $\{n^2\pi^2 \mid n \in \mathbb{N}^*\}$ . Hence both traces differ by 1, and it suffices to prove the estimate for  $Q^N$ . Writing

$$\text{Tr}(\exp(-\tau Q^N)) = \frac{1}{2} \left( 1 + \sum_{n \in \mathbb{Z}} e^{-\tau n^2 \pi^2} \right)$$

and applying the Poisson summation formula gives the result.  $\square$

We now prove the proposition. We first consider the case where  $\tau \leq 1$ . We are going to apply Dirichlet-Neumann bracketing with the decomposition  $(0, +\infty) = \cup_{j=0}^{+\infty} J_k$  where the intervals  $J_k$  are defined below.

Let  $x_0$  be defined by  $q_{C,\beta}(x_0) = \min q_{C,\beta}(x)$ . Then  $q_{C,\beta}(x) = Cx^{-2} + x^\beta$  is increasing on  $[x_0, +\infty)$ . Let  $J_k = [x_0 + k, x_0 + k + 1]$  with  $k \geq 1$  and  $J_0 = ]0, x_0 + 1]$ . We have the following estimates for the Dirichlet and Neumann heat traces  $\mathcal{Z}_{k,\chi}^\star$  on  $J_k$ : for  $k \geq 1$ ,

$$\mathcal{Z}_{k,1}^N(\tau) \leq \left( \frac{1}{\sqrt{4\pi\tau}} + O(1) \right) e^{-\tau q(x_0+k)} \leq \left( \frac{1}{\sqrt{4\pi\tau}} + O(1) \right) \int_{x_0+k-1}^{x_0+k} e^{-\tau q_{C,\beta}(x)} dx$$

and

$$\begin{aligned} \mathcal{Z}_{k,\chi}^D(\tau) &\geq \left( \frac{1}{\sqrt{4\pi\tau}} + O(1) \right) e^{-\tau q(x_0+k+1)} \chi(x_0 + k + 1) \\ &\geq \left( \frac{1}{\sqrt{4\pi\tau}} + O(1) \right) \int_{x_0+k+1}^{x_0+k+2} e^{-\tau q_{C,\beta}(x)} \chi(x) dx, \end{aligned} \tag{6}$$



while

$$\mathcal{Z}_0^N(\tau) = O(\tau^{-1/2}).$$

The minimax principle implies that each eigenvalue  $\mu_j$  is larger than the  $j^{\text{th}}$ -eigenvalue of the union for all  $k$  of the Neumann problem on the intervals  $J_k$ . In this way, we obtain the following upper bound for the trace with  $\chi = 1$ :

$$\mathcal{Z}_1(\tau) \leq O\left(\frac{1}{\sqrt{\tau}}\right) + \left(\frac{1}{\sqrt{4\pi\tau}} + O(1)\right) \int_{x_0}^{+\infty} e^{-\tau q_{C,\beta}(x)} dx. \quad (7)$$

Noting that

$$\left| \int_0^{+\infty} \left( e^{-\tau q_{C,\beta}(x)} - e^{-\tau x^\beta} \right) dx \right| = O(1),$$

we infer that

$$\mathcal{Z}_1(\tau) \leq \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} dx + O(\tau^{-\gamma}).$$

Similarly, using the fact that the Dirichlet heat kernel is smaller than the global kernel (see [12, Theorem 2.1.6]), we get from (6) the following lower bound:

$$\mathcal{Z}_\chi(\tau) \geq \frac{1}{\sqrt{4\pi\tau}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx + O(\tau^{-\gamma}). \quad (8)$$

Note that the same lower bound is valid when replacing  $\chi$  by  $1 - \chi$ .

Now, we use a variant of the fact that

$$(A + B = A' + B', A \geq A', B \geq B') \Rightarrow (A = A', B = B').$$

We take  $A = \mathcal{Z}_\chi$ ,  $B = \mathcal{Z}_{1-\chi}$ ,  $A' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} \chi(x) dx$  and  $B' = (4\pi\tau)^{-\frac{1}{2}} \int_0^{+\infty} e^{-\tau x^\beta} (1 - \chi(x)) dx$ . By (7), we have

$$A + B = A' + B' + O(\tau^{-\gamma})$$

and, from (8),

$$A \geq A' + O(\tau^{-\gamma}), \quad B \geq B' + O(\tau^{-\gamma}).$$

It follows that  $A = A' + O(\tau^{-\gamma})$  and  $B = B' + O(\tau^{-\gamma})$ . In particular,

$$\mathcal{Z}_1(\tau) \sim \frac{1}{(4\pi\tau)^{\frac{1}{2}}} \int_0^{+\infty} e^{-\tau x^\beta} dx.$$

Using the Karamata tauberian Theorem (recalled in Appendix A.2), we get

$$N_1(\mu) \sim A\mu^{\frac{1}{2} + \frac{1}{\beta}}$$

as  $\mu \rightarrow +\infty$ , with the constant  $A$  defined in Proposition 4.

We now prove the exponential upper bound for  $\tau \geq 1$ . We note that  $\mathcal{Z}_\chi(\tau) \leq \mathcal{Z}_1(\tau)$ . Moreover all eigenvalues  $\mu_j$  of  $P_1$  are larger than the minimum  $q(x_0)$  of  $q$ . Then, for  $\tau \geq 1$ , we have

$$\mathcal{Z}_1(\tau) = e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-\tau(\mu_j - \mu_1)} \leq e^{-\tau\mu_1} \sum_{j=1}^{+\infty} e^{-(\mu_j - \mu_1)}.$$

The Weyl law applied to  $P_1$  implies that the sum at the right-hand side converges and thus  $\mathcal{Z}_\chi(\tau) \leq ce^{-\tau\mu_1}$  for some  $c > 0$ .  $\square$

**Corollary 2.** *There exists  $C_1 > 0$  only depending on  $C$  and  $\beta$  but not on  $\chi$ , such that, setting  $J = \int_0^{+\infty} \chi(x) dx$ , we have, for every  $\tau > 0$ ,*

$$\mathcal{Z}_\chi(\tau) \leq \frac{C_1}{\sqrt{\tau}} \min\left(J, \tau^{-1/\beta}\right).$$

*Proof.* The bound involving  $J$  follows from the estimate  $e(\tau, x, x) \leq (4\pi\tau)^{-\frac{1}{2}}$  which is valid for any positive potential  $q$  (see [12, Theorem 2.1.6]). The other bound follows from Proposition 4 and from the estimate  $\int_0^{+\infty} e^{-\tau x^\beta} dx = O(\tau^{-1/\beta})$ .  $\square$

Given any  $\omega > 0$ , we set

$$P_\omega = -\partial_x^2 + Cx^{-2} + \omega x^\beta. \quad (9)$$

**Proposition 5.** *For any  $\omega > 0$ , the operator  $P_\omega$  is unitarily equivalent to  $\omega^{2/(2+\beta)} P_1$ . In particular, the spectrum of  $P_\omega$  is  $\omega^{2/(2+\beta)}$  times the spectrum of  $P_1$ .*

*Proof.* Considering the unitary map  $U : L^2(\mathbb{R}^+, dx) \rightarrow L^2(\mathbb{R}^+, dx)$  defined by

$$Uf(x) = \omega^{\frac{1}{2(2+\beta)}} f(\omega^{\frac{1}{2+\beta}} x),$$

we find that  $U^* P_\omega U = \omega^{2/(2+\beta)} P_1$ .  $\square$

### 3.2 Truncated heat asymptotics for the cone $X_\infty$

In this subsection, we compute the small-time asymptotics of the truncated heat trace,

$$Z_{\infty, \chi}(t) = \text{Tr}(e^{-t\Delta_g} \chi)$$

where  $\chi$  is as in Section 3.1 and moreover is compactly supported in  $[0, +\infty)$ , and  $g$  is the metric  $g = dx^2 + x^{-\beta} g_1$  on the cone  $X_\infty = (0, +\infty) \times M$ . The manifold  $M$  is equipped with the metric  $g_1$  that is independent of  $x$ . Here, we do not assume that  $\partial M$  is empty: this will be useful in the proof of Theorem 2. We will only use the Weyl asymptotics on  $M$ .

Using the direct sum decomposition given in (4), we have

$$Z_{\infty, \chi}(t) = \sum_{k=1}^{+\infty} \text{Tr}(e^{-tP_{\omega_k}} \chi) \quad (10)$$

where we recall that the  $\omega_k$  are the eigenvalues of  $\Delta_M$  and  $P_\omega$  is defined by (9).

We make the following two preliminary observations:

- For  $k$  fixed and  $t \rightarrow 0^+$ , we have  $\text{Tr}(e^{-tP_{\omega_k}} \chi) = O(t^{-1/2})$ . This term will be negligible in the sequel because the global trace is not less than  $C/t$  for some  $C > 0$  (since  $\dim(X) \geq 2$ ).
- For  $t > 0$  fixed, the smooth function  $f : \omega \mapsto \text{Tr}(e^{-tP_\omega} \chi)$  has a fast decay at infinity: by Proposition 5,

$$f(\omega) \leq \text{Tr}(e^{-tP_\omega}) = \text{Tr}(e^{-t\omega^{2/(2+\beta)} P_1}).$$

The claim then follows from the second assertion given in Proposition 4.

We split the sum (10) into two parts,

$$Z_{\infty, \chi}(t) = \sum_{\omega_k < 1} + \sum_{\omega_k \geq 1} = Z_{\infty, \chi}^0(t) + Z_{\infty, \chi}^1(t).$$

The first part,  $Z_{\infty,\chi}^0(t)$ , is  $O(t^{-1/2})$  by the first preliminary observation and we thus only have to estimate the second part,  $Z_{\infty,\chi}^1(t)$ . Using Proposition 5 and its proof, we have

$$Z_{\infty,\chi}^1(t) = \sum_{\omega_k \geq 1} \text{Tr} \left( e^{-t\omega_k^{2/(2+\beta)} P_1} \chi(\cdot/\omega_k^{1/(2+\beta)}) \right) = \sum_{\omega_k \geq 1} \mathcal{Z}_{\chi(\cdot/\omega_k^{1/(2+\beta)})}(t\omega_k^{2/(2+\beta)})$$

and we remark that, for  $\omega \geq 1$ , the function  $\chi(\cdot/\omega^{1/(2+\beta)})$  is identically equal to 1 on  $[0, a]$  (where  $a$  was introduced Section 3.1), so that we can use the estimate of Section 3.1.

Converting this sum into an integral (see Appendix A.2), using the Weyl law on  $M$ , we obtain

$$\#\{\omega_k \leq \omega\} \sim \gamma_n \text{Vol}(M) \omega^{n/2} \quad \text{as } \omega \rightarrow +\infty.$$

Using Proposition 7 and the definition of  $f$ , we get

$$Z_{\infty,\chi}^1(t) \sim \frac{n\gamma_n \text{Vol}(M)}{2} \int_1^{+\infty} \mathcal{Z}_{\chi(\cdot/\omega^{1/(2+\beta)})}(t\omega^{2/(2+\beta)}) \omega^{\frac{n}{2}-1} d\omega.$$

Making the change of variable  $\tau = t\omega^{2/(2+\beta)}$ , we arrive at the following lemma, recalling the  $\delta_H$  was introduced below Proposition 1.

**Lemma 2.** *The following holds,*

$$Z_{\infty,\chi}^1(t) \sim \frac{n\gamma_n(\beta+2)\text{Vol}(M)}{4t^{\delta_H/2}} \int_t^{+\infty} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau) \tau^{\frac{\delta_H}{2}-1} d\tau \quad \text{as } t \rightarrow 0^+.$$

The integral,

$$I(t) = \int_t^{+\infty} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau) \tau^{\frac{\delta_H}{2}-1} d\tau \tag{11}$$

is convergent at  $\tau = \infty$  for all  $\beta > 0$  but, in general, not at  $\tau = 0$  because if  $\beta \leq \beta_c$  then  $\frac{\delta_H}{2} - 1 \leq \frac{1}{2} + \frac{1}{\beta} - 1$ . We can compare this with the estimate in Corollary 2.

### 3.2.1 Case $\beta > \beta_c$

We estimate the small-time behavior of  $I(t)$  defined by (11). By the monotone convergence theorem, we have

$$\lim_{\varepsilon \rightarrow 0^+} \int_0^{+\infty} e(\tau, x, x) \chi(\varepsilon x) dx = \int_0^{+\infty} e(\tau, x, x) dx$$

where  $e$  is the heat kernel of  $P_1$ . Hence, for any  $\tau > 0$ ,

$$\lim_{t \rightarrow 0^+} \mathcal{Z}_{\chi(\cdot/\sqrt{t/\tau})}(\tau) = \mathcal{Z}_1(\tau).$$

Using, again, the monotone convergence theorem, we conclude that

$$\lim_{t \rightarrow 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau) \tau^{\frac{\delta_H}{2}-1} d\tau.$$

From Corollary 2, we get

$$\mathcal{Z}_1(\tau) \tau^{\delta_H/2-1} \leq C \tau^{\frac{n}{2}(1+\frac{\beta}{2})-1-\frac{1}{2}-\frac{1}{\beta}} e^{-\mu_1 \tau}$$

and  $\beta > \beta_c = 2/n$  implies  $\frac{n}{2}(1+\frac{\beta}{2})-1-\frac{1}{2}-\frac{1}{\beta} > -1$ . Thus, the corresponding limit is finite:

$$\lim_{t \rightarrow 0^+} I(t) = \int_0^{+\infty} \mathcal{Z}_1(\tau) \tau^{\frac{\delta_H}{2}-1} d\tau < +\infty.$$

### 3.2.2 Case $\beta \leq \beta_c$

By the second estimate in Proposition 4, the contribution to the integral from 1 to  $+\infty$  in the expression for  $I(t)$  (which was introduced in Lemma 2) is  $O(1)$  uniformly with respect to  $t$  and, hence, the corresponding part of  $Z_{\infty, \chi}^1(t)$  is  $O(t^{-\delta_H/2})$ , which will be negligible. We only need to estimate the asymptotics of

$$J(t) = \int_t^1 \mathcal{Z}_{\chi(\cdot\sqrt{t/\tau})}(\tau) \tau^{\frac{\delta_H}{2}-1} d\tau.$$

**Sub-case  $\beta < \beta_c$ .** We prove that there exists a  $\delta > 0$  such that

$$\mathrm{Tr} \left( e^{-t\Delta_g} \chi \right) = O \left( J^\delta t^{-(n+1)/2} \right) \quad (12)$$

as  $t \rightarrow 0^+$ . We split the integral,

$$J(t) = \int_t^1 \mathrm{Tr} \left( e^{-\tau P_1} \chi \left( \sqrt{t/\tau} \right) \right) \tau^{\frac{\delta_H}{2}-1} d\tau = J_1(t) + J_2(t) = \int_t^{\tau_0} + \int_{\tau_0}^1$$

where  $\tau_0$  satisfies  $\tau_0^{-1/\beta} = J\sqrt{\tau_0/t}$ , i.e.,  $\tau_0 = (t/J^2)^{\beta/(2+\beta)}$  with  $J = \int_0^{+\infty} \chi(x) dx$  as in Corollary 2. We get upper bounds for  $J_1$  and  $J_2$  using the upper bounds given in Corollary 2 as follows. Using the first argument in the minimum, we have

$$J_1(t) \leq C \frac{J}{\sqrt{t}} \tau_0^{\delta_H/2} = C J^{1-(n\beta/2)} t^{(n\beta/4)-(1/2)}$$

Similarly, using the second argument in the minimum, we find that

$$J_2(t) \leq C \int_{\tau_0}^1 \tau^{-1/\beta} \tau^{(\delta_H/2)-(3/2)} d\tau \leq C J^{1-(n\beta/2)} t^{(n\beta/4)-(1/2)}.$$

Finally,

$$t^{-\delta_H/2} I(t) \leq C J^{1-(n\beta/2)} t^{-(n+1)/2}$$

so that we can take  $\delta = 1 - (n\beta/2)$ . We will use this further in Section 3.4 by choosing  $J$  small.

**Sub-case  $\beta = \beta_c$ .** When  $\beta = 2/n$ , we have to estimate the asymptotics of

$$J(t) = \int_t^1 \mathrm{Tr} \left( e^{-\tau P_1} \chi(\cdot\sqrt{t/\tau}) \right) \tau^{(n-1)/2} d\tau.$$

Using the estimate of Proposition 4, we get

$$J(t) \sim \frac{1}{\sqrt{4\pi}} \int_t^1 \tau^{\frac{n}{2}-1} d\tau \int_0^\infty e^{-\tau x^{2/n}} \chi(x\sqrt{t/\tau}) dx$$

modulo terms of smaller order in  $\tau$ . Using the change of variable  $y = \tau x^{2/n}$ , we get

$$J(t) \sim \frac{n}{4\sqrt{\pi}} \int_t^1 \frac{d\tau}{\tau} \int_0^\infty e^{-y} y^{\frac{n}{2}-1} \chi \left( \sqrt{t\tau}^{-(n+1)/2} y^{n/2} \right) dy = \frac{n}{4\sqrt{\pi}} \int_t^1 \frac{d\tau}{\tau} F \left( \tau^{(n+1)/2} / t^{1/2} \right)$$

where the function  $F$ , defined by

$$F(X) = \int_0^{+\infty} e^{-y} y^{\frac{n}{2}-1} \chi \left( y^{n/2} / X \right) dy,$$

is smooth and satisfies  $F(X) = O_{X \rightarrow 0}(X)$  and  $\lim_{X \rightarrow +\infty} F(X) = \Gamma(n/2)$ . Using the new variable  $u = \tau^{(n+1)/2}/t^{1/2}$ , we get

$$J(t) \sim \frac{n}{2(n+1)\sqrt{\pi}} \int_{t^{n/2}}^{t^{-1/2}} F(u) \frac{du}{u},$$

and finally

$$J(t) \sim \frac{n\Gamma(n/2)}{4(n+1)\sqrt{\pi}} |\ln t|$$

as  $t \rightarrow 0^+$ .

### 3.3 The heat parametrix in the separable metric case

We adapt the method of [5] and we use Appendix A.4. We denote by  $z, z'$  some generic points of  $X$  and by  $z = (x, m)$ ,  $z' = (x', m')$  generic points of  $[0, 1) \times M \subset X$ . Let  $\chi$  be as in the previous sections, vanishing near  $x = 1$  and extended by 0 inside  $X$ . Let  $\eta \in C_0^\infty([0, 1))$  so that  $\eta = 1$  near the support of  $\chi$ , and  $\eta_0 \in C_0^\infty(X)$ , vanishing near  $\partial X$  and equal to 1 near the support of  $1 - \chi$ . We choose  $a > 0$  so that  $\eta_0$  vanishes for  $x \leq 2a$ . We claim that

$$p(t; z, z') = \eta(x)e_\infty(t; z, z')\chi(x') + \eta_0(x)e_0(t; z; z')(1 - \chi(x'))$$

where  $e_\infty$  is the heat kernel on the cone  $X_\infty$  and  $e_0$  the Riemannian heat kernel generated by the Laplacian  $\Delta_g$  on  $X \setminus \{x \leq a\}$  with Dirichlet boundary conditions, is a good approximation of the heat kernel on  $X$  as  $t \rightarrow 0$ .

**Proposition 6.** *Let  $P(t)$  be the operator of Schwartz kernel  $p(t, \cdot, \cdot)$ . We have*

$$\text{Tr}(P(t) - e^{-t\Delta_g}) = O(t^\infty)$$

as  $t \rightarrow 0^+$ .

*Proof.* We set  $r(t, z, z') = (\partial_t + (\Delta_g)_z)p(t, z, z')$ . The kernel  $r$  vanishes if  $x$  is small enough. By the local nature of the small-time asymptotics of Riemannian heat kernels (see Appendix A.4),  $e_\infty(t, \cdot, \cdot)$  and  $e_0(t, \cdot, \cdot)$  are  $O(t^\infty)$  close in  $C^\infty$  topology on  $[0, 1) \times M$ . Moreover, if  $x \in \text{supp}(\eta')$  or  $x \in \text{supp}(\eta_0')$ ,  $p(t, \cdot, \cdot)$  and  $e_0(t, \cdot, \cdot)$  are  $O(t^\infty)$  in the  $C^\infty$  topology, because  $z \neq z'$ . It follows that  $r(t, \cdot, \cdot) = O(t^\infty)$  in  $C^\infty$  topology. Therefore, denoting by  $R(t)$  the operator of Schwartz kernel  $r(t, \cdot, \cdot)$ , the trace norm of  $R(t)$  is a  $O(t^\infty)$ . By the Duhamel formula, using that  $P(t) \rightarrow \text{id}$  as  $t \rightarrow 0^+$ , we have

$$P(t) - e^{-t\Delta_g} = \int_0^t e^{-(t-s)\Delta_g} r(s) ds.$$

The result follows because the operator norm of  $e^{-t\Delta_g}$  is not greater than 1.  $\square$

### 3.4 Completion of the proof of Theorem 1 in the separable metric case

Thanks to the previous section, we only have to estimate the trace of  $P(t)$ . We use the local nature of the heat asymptotics to show that the contribution of the term  $\eta_0 e_0(t)(1 - \chi)$  is equivalent to  $(4\pi t)^{-(n+1)/2} \int_X (1 - \chi) dv_g$ . We are left to estimate the term that corresponds to the truncated cone as in Section 3.2. This gives the conclusion when  $\beta \geq \beta_c$ .

When  $\beta < \beta_c$ , the first term can be made smaller than  $\varepsilon t^{-(n+1)/2}$  for any  $\varepsilon > 0$  by choosing  $J = \int_0^{+\infty} \chi(x) dx$  small enough as mentioned in Section 3.2.2.

### 3.5 From the separable to the general case

We prove that, for any given  $\varepsilon > 0$ , the metric  $g$  on  $X$  is  $\varepsilon$ -quasi-isometric to a separable metric  $g_s$ . We choose  $\delta > 0$  so that

$$|g_0(u) - g_0(0)| \leq \varepsilon(du^2 + g_0(u))$$

for any  $u \in [0, \delta]$ . Then, we choose  $\eta \in C_0^\infty([0, \delta])$ , identically equal to 1 near  $u = 0$ . We consider the separable metric  $g_s$  which coincides with  $g$  outside  $u \leq \delta$  and is given near  $\partial X$  by

$$g_s = \eta u^{-\alpha} (du^2 + g_0(0)) + (1 - \eta)g.$$

Then

$$\left| \frac{g_s}{g} - 1 \right| = \left| \eta \frac{g_0(u) - g_0(0)}{du^2 + g_0(u)} \right| \leq \varepsilon.$$

Using Appendix A.1, this concludes the proof of Theorem 1 in the general (non-separable) case.

### 3.6 Proof of Theorem 2

#### 3.6.1 Case $\beta < \beta_c$

We consider the heat traces  $Z_f(t) = \text{Tr}(e^{-t\Delta_g} f)$  where  $f : X \rightarrow \mathbb{R}$  is continuous. Let  $\varepsilon > 0$ . We choose a smooth function  $\chi : X \rightarrow [0, 1]$  that is identically equal to 1 near  $\partial X$  and such that  $\int_X \chi dv_g \leq \varepsilon$ . Writing  $g = \chi f + (1 - \chi)f$ , we get

$$\text{Tr}(e^{-t\Delta_g} f) = \text{Tr}(e^{-t\Delta_g}(1 - \chi)f) + \text{Tr}(e^{-t\Delta_g}\chi f) = J(t) + K(t).$$

By the local nature of the heat trace asymptotics (see Appendix A.4), we have

$$J(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X (1 - \chi)f dv_g$$

as  $t \rightarrow 0^+$ . Besides,

$$K(t) \leq \|f\|_\infty \text{Tr}(e^{-t\Delta_g}\chi).$$

By (12), and since we can choose  $\varepsilon > 0$  arbitrarily small, it follows that

$$Z_f(t) \sim \frac{1}{(4\pi t)^{(n+1)/2}} \int_X f dv_g$$

as  $t \rightarrow 0^+$ , which gives the expected result.

#### 3.6.2 Case $\beta \geq \beta_c$

We give the proof in the case  $\beta > \beta_c$ . The case  $\beta = \beta_c$  is treated similarly. Let us first prove that the support of the Weyl measure is contained in  $\partial X$ . If  $\text{supp}(f) \cap \partial X = \emptyset$ , we get again by the local nature of the heat asymptotics that

$$\text{Tr}(e^{-t\Delta_g} f) = O(t^{-(n+1)/2})$$

while, by the Weyl law given in Theorem 1, we have

$$t^{-(n+1)/2} = o(\text{Tr}(e^{-t\Delta_g}))$$

as  $t \rightarrow 0^+$ . Hence, it suffices to consider functions  $f$  of the form  $f = \mathbf{1}_D$  where  $D = [0, a] \times D_1$  with  $D_1$  a piecewise smooth domain in  $M$ .

We recall that, according to [12, Chapter 5, Theorem 2.1.6], we have

$$0 \leq e_D(t, m, m) \leq e_X(t, m, m) \quad \forall m \in D \quad (13)$$

where  $e_D$  is the Dirichlet heat kernel on  $D$ .

We set  $D' = [0, a] \times (M \setminus D)$  and  $D'' = X \setminus [0, a] \times M$ . For any domain  $K$ , we denote by  $Z_K$  the Dirichlet heat trace and by  $Z'_K(t) = \int_K Z_X(t, m, m) dv_g(m)$ . We have

$$Z_D(t) \sim C \text{Vol}(D_1) t^{-d_H/2}, \quad Z_{D'}(t) \sim C \text{Vol}(M \setminus D_1) t^{-d_H/2} \quad (14)$$

and  $Z_{D''}(t) = o(t^{-d_H/2})$  as  $t \rightarrow 0^+$ . Note that for (14), we used the proofs in Section 3.2 for the case where the  $n$ -dimensional manifold  $M$  (here:  $D_1$  and  $M \setminus D_1$ ) can have a boundary. The sum  $(Z'_D + Z'_{D'} + Z'_{D''})(t) = Z'_X(t)$  is equivalent to  $C \text{Vol}(M) t^{-d_H/2}$  by Theorem 1. Hence,  $(Z'_D + Z'_{D'})(t) \sim C \text{Vol}(M) t^{-d_H/2}$ . On the other hand, thanks to (13) we have

$$Z_D \leq Z'_D, \quad Z_{D'} \leq Z'_{D'}, \quad Z_{D''} \leq Z'_{D''}.$$

It follows that

$$Z'_D(t) \sim C \text{Vol}(D_1) t^{-d_H/2}$$

as  $t \rightarrow 0^+$ , which yields the desired result.

## 4 Discussion and open problems

In this article, motivated by the propagation of acoustic waves in gas giant planets, we derived the Weyl law for the Laplace-Beltrami operator on a smooth compact Riemannian  $(n+1)$ -dimensional manifold  $X$  with boundary whose metric blows up near the boundary. Many new questions emerge. We present some of them.

**Quantum Ergodicity and Quantum Limits.** We have seen in Corollary 1 that, if  $\beta \geq \beta_c$ , then a density-one subsequence of eigenfunctions concentrates on  $\partial X$ . This is a preliminary result towards Quantum Ergodicity (QE).

Recall that, on a locally compact space  $U$  endowed with a probability Radon measure  $\mu$ , given a self-adjoint nonnegative operator  $T$  on  $L^2(U, \mu)$ , of discrete spectrum  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \leq \dots \rightarrow +\infty$  associated with an orthonormal eigenbasis  $\Phi = (\phi_j)_{j \in \mathbb{N}^*}$  of  $L^2(U, \mu)$ , a Quantum Limit (QL) of  $\Phi$  is a probability Radon measure  $\nu$  on  $U$  that is a weak limit of a subsequence of the probability measures  $|\phi_j|^2 \mu$ , i.e., there exists a subsequence  $(j_k)_{k \in \mathbb{N}^*}$  such that

$$\int_U f |\phi_{j_k}|^2 d\mu \xrightarrow{k \rightarrow +\infty} \int_U f d\nu \quad \forall f \in C_c^0(U). \quad (15)$$

We say that QE holds for  $(T, \Phi)$  if there exists a QL  $\nu$  on  $U$  and a subsequence  $(j_k)_{k \in \mathbb{N}^*}$  of density one such that (15) holds.

One may wonder whether, when  $\beta \geq \beta_c$ , a QE property on  $M$  would imply a QE property on  $X$ . Proving this fact certainly requires fine spectral properties of Schrödinger operators (see [1]). Besides, inspired by [9, Theorem B], we wonder what can be said on QLs supported on  $\partial X = \{0\} \times M$ : are they invariant under the geodesic flow of  $(M, G)$  (where  $G = g_1(0)$ )? Defining QLs on  $T^*M$  will already be a challenge.

**Inverse problems on spectra.** A natural question is: does the spectrum of  $X$  determine the spectrum of  $M$ ? Attacking this problem certainly requires developing appropriate trace formulas, as in [8].

**Closed geodesics.** Recalling that  $G = g_1(0)$  where  $g_1$  is defined by (1), it is natural to view geodesics on  $(M, G)$  as limits, in an appropriate sense, of geodesics on  $(X, g)$ . A natural question is then: do there exist some closed geodesics of  $X$  accumulating on (converging to) closed geodesics of  $\partial X = M$ ? We refer to [7] for a similar question investigated in the framework of contact sub-Riemannian 3D manifolds. Here again, having appropriate trace formulas might be useful.

**Observability properties.** The study of the Weyl asymptotics is a first step towards solving some inverse problems. As explained in Section 1, the knowledge of spectrum properties can already be used to check the validity of some models, but the main objective in the physical context would be the ability to reconstruct some features of the internal structure of the planets, based on the observation of acoustic waves. The feasibility of such an inverse problem is mathematically modeled by an observability inequality, which can be settled as follows for half-waves. Given any  $T > 0$  and any subset  $\omega$  of  $X$ , we say that the observability property holds true for  $(\omega, T)$  if there exists a positive constant  $C_T(\omega)$  such that

$$\int_0^T \left\| \mathbb{1}_\omega e^{it\sqrt{\Delta_g}} \phi \right\|_{L^2(X, dv_g)}^2 dt \geq C_T(\omega) \|\phi\|_{L^2(X, dv_g)}^2 \quad \forall \phi \in L^2(X, dv_g). \quad (16)$$

– When  $\beta < \beta_c$ , we expect that (16) holds as soon as  $\omega$  is open and  $(\omega, T)$  satisfies the Geometric Control Condition (GCC, see [2]), like in the classical case of a non-singular Riemannian metric.

– When  $\beta \geq \beta_c$ , an obvious necessary condition for (16) to hold is that  $\omega$  contain an open neighborhood of a subset of  $\partial X$ . Indeed, take  $\phi$  in (16) to be a highfrequency eigenfunction and apply Corollary 1. We think that this condition is sufficient if moreover  $(\omega, T)$  satisfies GCC.

We note that, when  $X$  is a closed ball in  $\mathbb{R}^{n+1}$  (an idealized situation for an exactly round planet), GCC is never satisfied unless  $\omega$  contains an open neighborhood of the *whole* boundary of  $X$ , which is certainly not relevant for applications from the physical point of view. In this case where  $X$  is a round ball, it is more interesting to take a small observation subset  $\omega$ , containing a small open subset of  $\partial X$ . But, as soon as  $\omega$  is a proper subset of a half-ball, GCC (and thus (16)) obviously fails due to trapped rays, propagating along a diameter never meeting  $\omega$ . In this deteriorated context, we wonder, however, whether (16) is anyway satisfied if we restrict the inequality to radial waves or to surface waves, which are the most physically meaningful waves to be observed.

**Metrics that are singular on larger codimension submanifolds.** In this paper, we have considered a class of singular metrics blowing up at the boundary of  $X$ , where the boundary can be seen as a codimension-one submanifold of  $X$ .

In more general, let  $X$  be a smooth compact manifold and let  $Z$  be a submanifold of  $X$  of codimension  $m \in \mathbb{N}^*$ , and consider the class of singular metrics  $g$  on  $X$  that are smooth on  $X \setminus Z$  and that, near  $Z$ , are written as

$$g = h + g_Z(x)r^{-\beta}$$

in a neighborhood of  $Z$  assumed to be diffeomorphic to  $Z \times B_m$  where  $B_m$  is the unit ball of  $\mathbb{R}_x^m$  (this holds if the normal bundle of  $Z$  is trivial) equipped with the Euclidean metric  $h$  and the polar coordinates  $(r, \sigma)$ , and  $g_Z(x)$  is a metric on  $Z$ , parametrized by  $x \in B_m$  and depending smoothly on  $x$ . The techniques developed in our paper can certainly be extended to compute the Weyl asymptotics in such cases.



## A Appendix

### A.1 Quasi-isometries

Let  $X$  be a smooth manifold of dimension  $n + 1$ , with boundary. Two metrics  $g_1$  and  $g_2$ , smooth on  $X \setminus \partial X$ , are said to be  $\varepsilon$ -quasi-isometric if

$$\left| \frac{g_1}{g_2} - 1 \right| \leq \varepsilon$$

uniformly on  $X \setminus \partial X$ . For  $i \in \{1, 2\}$ , let  $\Delta_{g_i}$  be the Friedrichs extension of the Laplace-Beltrami operator on  $(X, g_i)$  with core  $C_0^\infty(X \setminus \partial X)$ .

If  $\Delta_{g_1}$  has a discrete spectrum  $(\lambda_j^1)_{j \in \mathbb{N}^*}$  then  $\Delta_{g_2}$  has also a discrete spectrum  $(\lambda_j^2)_{j \in \mathbb{N}^*}$  and, for  $\varepsilon \leq \frac{1}{2}$  (this condition is to get bounds on the inverse of  $g_i$ ), there exists  $C(n) > 0$  such that, for every  $j \in \mathbb{N}^*$ ,

$$\left| \frac{\lambda_j^1}{\lambda_j^2} - 1 \right| \leq C(n)\varepsilon.$$

Indeed, this estimate follows from the minimax characterization of the eigenvalues and from the comparison of the Rayleigh quotients, i.e., of the volumes and co-metrics.

### A.2 Karamata tauberian theorem and converse

We recall the Karamata tauberian theorem (see [16, Chapter XIII, Theorem 2]).

**Theorem 3.** *Let  $\mu$  be a positive Radon measure on  $\mathbb{R}^+$ . If there exists  $\alpha > 0$  such that*

$$\int_0^{+\infty} e^{-t\lambda} d\mu(\lambda) \sim At^{-\alpha} \quad (\text{resp.}, A|\ln t|t^{-\alpha})$$

as  $t \rightarrow 0^+$ , then

$$\mu([0, \lambda]) \sim \frac{A}{\Gamma(\alpha + 1)} \lambda^\alpha \quad \left( \text{resp.}, \frac{A}{\Gamma(\alpha + 1)} \lambda^\alpha \ln \lambda \right)$$

as  $\lambda \rightarrow +\infty$ .

We need a converse of Theorem 3. Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a nonincreasing function of class  $C^1$ , such that  $f$  and  $f'$  have a fast decay at infinity. Let  $(\lambda_j)_{j \in \mathbb{N}^*}$  be a nondecreasing sequence of positive real numbers. We define the counting function  $N(\lambda) = \#\{j \in \mathbb{N}^* \mid \lambda_j \leq \lambda\}$ , for any  $\lambda \in \mathbb{R}$ . The objective is to estimate the sum

$$S = \sum_{j=1}^{+\infty} f(\lambda_j).$$

**Proposition 7.** *Assume that there exist  $C > 0$  and  $\alpha > 0$  such that  $N(\lambda) \sim C\lambda^\alpha$  as  $\lambda \rightarrow +\infty$ . For any  $\varepsilon > 0$ , there exists  $K(\varepsilon) > 0$ , depending on the counting function  $N$  but not on  $f$ , such that*

$$\left| S - C\alpha \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda \right| \leq K(\varepsilon)f(\lambda_1) + \varepsilon \int_{\lambda_1}^{+\infty} f(\lambda)\lambda^{\alpha-1} d\lambda.$$

*Proof.* Given any  $\varepsilon > 0$ , let  $\Lambda_0 > 0$  such that, for every  $\lambda \geq \Lambda_0$ ,

$$(1 - \varepsilon)C\lambda^\alpha \leq N(\lambda) \leq (1 + \varepsilon)C\lambda^\alpha. \quad (17)$$

Noting that  $dN(\lambda) = \sum_{j=1}^{+\infty} \delta_{\lambda_j}$ , using the Stieltjes integral, we have

$$S = \sum_{\lambda_j < \Lambda_0} f(\lambda_j) + \int_{\Lambda_0}^{+\infty} f(\lambda) dN(\lambda).$$

Now, since  $\sum_{\lambda_j < \Lambda_0} f(\lambda_j) \leq N(\Lambda_0)f(\lambda_1)$ , we get by integration by parts, using the fast decay of  $f$  at infinity, that

$$S \leq N(\Lambda_0) (f(\lambda_1) - f(\Lambda_0)) - \int_{\Lambda_0}^{+\infty} f'(\lambda) N(\lambda) d\lambda.$$

We derive an upper bound for  $S$ . A lower bound is obtained similarly. Using (17), integrating by parts and using that  $f(\Lambda_0) \leq f(\lambda_1)$ , we obtain

$$\begin{aligned} - \int_{\Lambda_0}^{+\infty} f'(\lambda) N(\lambda) d\lambda &\leq -(1 + \varepsilon) C \int_{\Lambda_0}^{+\infty} f'(\lambda) \lambda^\alpha d\lambda \\ &\leq (1 + \varepsilon) C \left( f(\lambda_1) \Lambda_0^\alpha + \alpha \int_{\Lambda_0}^{+\infty} f(\lambda) \lambda^{\alpha-1} d\lambda \right). \end{aligned}$$

Therefore,

$$S \leq f(\lambda_1) (N(\Lambda_0) + C(1 + \varepsilon) \Lambda_0^\alpha) + (1 + \varepsilon) C \alpha \int_{\lambda_1}^{+\infty} f(\lambda) \lambda^{\alpha-1} d\lambda$$

and the result follows with  $K(\varepsilon) = N(\Lambda_0) + C(1 + \varepsilon) \Lambda_0^\alpha$ .  $\square$

### A.3 Weyl circle-point limit criterion

We consider the Schrödinger operator  $P = -\partial_x^2 + q(x)$  on  $C_0^\infty((0, +\infty))$ , where  $q$  is a smooth function on  $(0, +\infty)$ . According to the Weyl circle-point limit criterion (see [31, Theorem X.7]),  $P$  is essentially self-adjoint if and only if there exists at least one solution of  $Pu = 0$  that is not square integrable at 0 and at least one solution of  $Pu = 0$  that is not square integrable at  $+\infty$ .

When  $q(x) = Cx^{-2} + \omega x^\beta$  for some  $C \geq 0$ ,  $\beta > 0$  and  $\omega > 0$ , there is only one solution of  $Pu = 0$  that is square integrable at  $+\infty$ . Near 0, the solutions of  $Pu = 0$  are equivalent to linear combinations of  $x^{\gamma_+}$  and  $x^{\gamma_-}$  where  $\gamma_+$  and  $\gamma_-$  are the two solutions of  $-\gamma(\gamma - 1) + C = 0$ . It follows that  $P$  is essentially self-adjoint if and only if  $\gamma_- \leq -\frac{1}{2}$ , that is, if and only if  $C \geq 3/4$ .

### A.4 Local nature of the small-time asymptotics of heat kernels

Let  $(U, g)$  be a smooth Riemannian manifold and let  $\Delta$  be the Laplace-Beltrami operator. For our needs (see Section 3.3),  $U = X \setminus \partial X$  with the metric  $g$ .

Let  $e_1$  and  $e_2$  be two solutions of  $(\partial_t + \Delta_x)e_i(t, x, y) = 0$  for  $t > 0$ , satisfying  $e_i(t, x, y) = e_i(t, y, x)$  for all  $t > 0$  and  $(x, y) \in U \times U$  and

$$\lim_{t \rightarrow 0^+} \int_U e_i(t, x, y) f(y) dv_g(y) = f(x) \quad \forall x \in U \quad \forall f \in C_0^\infty(U),$$

for  $i \in \{1, 2\}$ .

**Lemma 3.** *We have  $e_1(t, \cdot, \cdot) - e_2(t, \cdot, \cdot) = O(t^\infty)$  as  $t \rightarrow 0^+$  in  $C^\infty$  topology on  $U \times U$ . Moreover, denoting by  $D$  the diagonal of  $U \times U$ , for  $i \in \{1, 2\}$ , we have  $e_i(t, \cdot, \cdot) = O(t^\infty)$  as  $t \rightarrow 0^+$  in  $C^\infty$  topology on  $U \times U \setminus D$ .*

This result reflects Kac’s principle of “not feeling the boundary”, showing that the small-time asymptotic behavior of heat kernels is purely local. A detailed proof can be found in [10, Section 3.2.1]. The idea comes from the paper [21]. The proof uses the fact that the Hörmander operator  $P = 2\partial_t + \Delta_x + \Delta_y$  is hypoelliptic. Extending the kernels  $e_i$  by 0 for  $t < 0$ , we have  $Pe_i = 0$  on  $\mathbb{R} \times U \times U \setminus D$  and  $P(e_1 - e_2) = 0$  on  $\mathbb{R} \times U \times U$ , in the distributional sense. The result then follows by hypoellipticity.

### A.5 $\Delta_g$ as a nonsmooth Hörmander operator

Based on the mathematical model provided in Section 2.1, near any point of the boundary of  $X$  we have  $X \simeq [0, 1) \times \mathbb{R}^n$  with a local system of coordinates  $(x, y)$ , with  $x \in [0, 1)$  and  $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ , and we can write (locally)

$$\Delta_g = - \sum_{i=0}^n X_i^* X_i + V \quad (18)$$

where  $V(x, y) = \frac{C(x, y)}{x^2}$  is a potential and the  $X_i$ ’s are vector fields given by

$$X_0 = a_0(x, y) \partial_x, \quad X_i = x^{\beta/2} a_i(x, y) \partial_{y_i}, \quad i \in \{1, \dots, n\}.$$

The functions  $C$  and  $a_i$ ,  $i \in \{0, \dots, n\}$  are smooth on  $\mathbb{R} \times \mathbb{R}^n$  and  $C(0, \cdot) = C_\beta$  and  $a_i(0, \cdot) = 1$  (they can be expressed in terms of the coefficients of the smooth Riemannian metric  $g_1(x)$  on  $M$  defined by (1)). The separable case corresponds to  $a_0 = 1$  and  $a_i$  not depending on  $x$ .

Expressed as (18), the operator  $\Delta_g$  is then a Hörmander operator, however nonsmooth unless  $\beta \in 2\mathbb{N}^*$ . Because of this lack of smoothness, many classical results cannot be applied here.

When  $\beta \in 2\mathbb{N}^*$ , the above vector fields are smooth and define an almost-Riemannian geometry, in which the Weyl asymptotics of the almost-Riemannian Laplacian  $\Delta_{aR} = - \sum_{i=0}^n X_i^* X_i$  (i.e., (18) with  $V = 0$ ), of Grushin type, has been established in [11].

With these preliminary remarks in mind, we then mention a few interesting facts hereafter.

**Homogeneity.** In the above local coordinates, given any  $\varepsilon > 0$ , we define the *dilation*

$$\delta_\varepsilon(x, y) = (\varepsilon x, \varepsilon^{1+\beta/2} y) \quad \forall (x, y) \in [0, 1) \times \mathbb{R}^n.$$

In the separable case where  $a_0 = 1$  and  $a_i$  does not depend on  $x$ , for any  $i \in \{1, \dots, n\}$ , we define  $\widehat{X}_i = \lim_{\varepsilon \rightarrow 0} \varepsilon \delta_\varepsilon^* X_i = x^{\beta/2} a_i(0) \partial_{y_i}$ , and we have

$$\varepsilon \delta_\varepsilon^* X_0 = X_0 \quad \text{and} \quad \varepsilon \delta_\varepsilon^* \widehat{X}_i = \widehat{X}_i \quad \forall i \in \{1, \dots, n\}.$$

In sR geometry,  $\widehat{X}_i$  is the *nilpotentization* of the vector field  $X_i$  at the point identified with  $(0, 0)$ . Extrapolating results of sub-Riemannian geometry that one can find in [10] to the case of  $\beta > 0$ , denoting by  $d_g$  the  $g$ -distance on  $X$ , one can show that  $d_g((0, 0), (x, y))$  divided by  $|x| + \sum_{i=1}^n |y_i|^{1/(1+\beta/2)}$  is bounded above and below by some positive constants in a neighborhood of  $(0, 0)$ . Noting that  $1/(1 + \beta/2) = 1 - \alpha/2$ , we thus recover [13, Proposition 13] and thus the result of Proposition 1 and the fact that Hausdorff and Minkowski dimensions coincide. In the non-separable case, we obtain the result by using quasi-isometries.

**Weyl law when  $\beta \in 2\mathbb{N}^*$ .** When  $\beta \in 2\mathbb{N}^*$ , we always have  $\beta \geq \beta_c$ , and  $\beta = \beta_c$  if and only if  $n = 1$ . Since the potential  $1/x^2$  is homogeneous, combining results of [10, 11], we recover the Weyl law established in Theorem 1.

**Weyl law when  $\beta \notin 2\mathbb{N}^*$ .** To establish the Weyl law in general sub-Riemannian cases, the approach developed in [11] consists of estimating singular integrals involving the heat kernel, by performing the so-called  $(J + K)$ -decomposition. Applying this approach to the nonsmooth operator in (18) cannot be done directly because we miss a general hypoellipticity theory, valid for nonsmooth vector fields as above, and a generalization of Lemma 3 (see Appendix A.4) to that context.

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