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# Existence of Nash Equilibria in Preference Priority Games Proven in Isabelle

Stéphane Le Roux, Érik Martin-Dorel, and Jan-Georg Šmaus

<sup>1</sup> Université Paris-Saclay, ENS Paris-Saclay, LMF, CNRS, France

<sup>2</sup> IRIT, Université de Toulouse, France

<sup>3</sup> IRIT, Université de Toulouse, France

## 1 Introduction

This work is about formalising game-theoretic results using proof assistants [EJ91, LeR09, Nip09]. In previous work [LMS17], we have formalised [LeR14, Lemma 2.4] both in Coq and Isabelle. The result is as follows: starting from a two-player game with finitely many outcomes, one may derive a game by rewriting each of these outcomes with either the basic outcome “Player 1 wins” or “Player 2 wins”. If all ways of deriving such a *win/lose (w/l) game* lead to a game where one player has a winning strategy, then the original game has a Nash equilibrium (NE).

Here, we present an application of this work to parity games and priority games using Isabelle. Dittmann has proven in Isabelle that parity games are positionally determined. First, we generalise this result to priority games, where parity is replaced by an arbitrary winning set  $W$ . Secondly, we consider *preference priority games*, i.e., sequential games where players have preferences over outcomes. We show that such games have an NE.

## 2 Positional Determinacy of Priority Games

We consider *sequential* games: there is a graph partitioned so that each vertex is owned by one of the two players, and a play is a path through this graph. The path starts in the initial vertex, and in each vertex, the owner decides where to go next according to some strategy.

**Definition 1.** An *arena* is a tuple  $(V_1, V_2, v_0, E)$  where  $V_1 \cap V_2 = \emptyset$ , and  $v_0 \in V := V_1 \cup V_2$ , and  $E \subseteq V^2$  is such that for all  $v \in V$ , the set  $vE := \{u \in V \mid (v, u) \in E\}$  is non-empty.

A *positional strategy* of **Player 1** in an arena  $(V_1, V_2, v_0, E)$  is a function  $s : V_1 \rightarrow V$  such that  $(v, s(v)) \in E$  for all  $v \in V_1$  (“positional” because the history is ignored; in the sequel, we only consider positional strategies).

In a straightforward way, a strategy pair induces a unique infinite path denoted by  $\langle s_1, s_2 \rangle$ .

**Definition 2.** A *priority game form* is an arena together with a priority function  $\pi : V \rightarrow \mathbb{N}$ .

For an infinite path with bounded  $\pi$ , the least priority occurring infinitely often as a label of a visited vertex is called *induced priority*.

**Definition 3.** A *w/l priority game* consists of a priority game form and a subset  $W \subseteq \mathbb{N}$ . A run  $\rho$  is winning for **Player 1** iff the induced priority of  $\rho$  is in  $W$ .

If  $W := 2\mathbb{N}$ , the w/l priority game is called a *parity game*.

**Definition 4.** Given a w/l priority game  $(V_1, V_2, v_0, E, \pi, W)$ , a **Player 1 winning strategy**  $s_1$  is such that for all **Player 2** strategy  $s_2$ , the induced priority of  $\langle s_1, s_2 \rangle$  is in  $W$ . A w/l priority game such that one player has a winning strategy is said to be *weakly positionally determined*.

The term “weakly” indicates that a winning (positional) strategy runs against a strategy of the opponent that is itself also positional. Usually, positional determinacy means that a positional strategy wins even against more general (non-positional) strategies.

Dittmann [Dit16] has shown in Isabelle that parity games are (not just weakly!) positionally determined [EJ91]. Based on a transformation from priority games to parity games, we can extend the statement to the priority games:

**Lemma 5.** W/l priority games with bounded  $\pi$  are positionally determined.

We have also shown the result in Isabelle but only *weak* determinacy. Note that Lemma 5 could be proved by applying [GZ05, Thm. 2, Cor. 7], but the proof that we formalize is more direct when assuming positional determinacy of parity games.

The Isabelle formalisation of this lemma with all the preliminaries comprises approximately 1350 lines of proof script, about as much as our entire Isabelle development of [LMS17]. The difficulty is that infinite paths are defined coinductively, and thus statements relating different priority and parity games must be proven by coinduction.

### 3 Nash Equilibria for Preference Priority Games

We now consider *simultaneous* or *one-shot* games, as opposed to sequential games.

**Definition 6.** A *game form* is a tuple  $\langle S_1, S_2, O, v \rangle$  where  $S_1$  and  $S_2$  are the strategies of Players 1 and 2, resp.;  $O$  is a nonempty set (of possible outcomes);  $v : S_1 \times S_2 \rightarrow O$  is the outcome function. A game form endowed with two binary relations  $\prec_1, \prec_2$  over  $O$  for each player (modeling her preference) is called a *game*.

A *w/l game* is a game where  $O = \{True, False\}$  and the preferences are  $False \prec_1 True$  and  $True \prec_2 False$ . If one player has a winning strategy the game is said to be *determined*.

**Definition 7.** Let  $\langle S_1, S_2, O, v, \prec_1, \prec_2 \rangle$  be a game. A strategy profile  $(s_1, s_2)$  in  $S_1 \times S_2$  is a *Nash equilibrium* (NE) if it makes both players stable:

$$(\forall s'_1 \in S_1, v(s_1, s_2) \not\prec_1 v(s'_1, s_2)) \wedge (\forall s'_2 \in S_2, v(s_1, s_2) \not\prec_2 v(s_1, s'_2))$$

Given a game form and a set  $W \subseteq O$ , one can *derive* a w/l game in a straightforward way: all outcomes in  $W$  are mapped to *True* (Player 1 wins). If a game form is such that for *every*  $W$ , the derived w/l game is determined, we call the game form itself *determined*. Fig. 1 shows an example with  $S_1 = \{1_t, 1_b\}$ ,  $S_2 = \{2_l, 2_r\}$ , and  $O = \{\heartsuit, \clubsuit, \diamond\}$ .

In [LMS17], we have formalised in Isabelle and Coq a theorem [LeR14] stating that a game  $g$  whose game form is determined has an NE. Fig. 1 shows the main theorem in Isabelle code.





We now link sequential games to simultaneous games by putting a black box around the process of constructing an infinite sequence using strategies and then extracting a number from it. The black box is a (simultaneous) game form: it takes two strategies and returns a number.

At the same time, we can define *preference-priority games*, which are sequential games where rather than having a winning set  $W$ , we have preferences of the players on the outcomes in  $\mathbb{N}$ . By linking preference-priority games to the simultaneous setting, we can apply the main theorem above to show that preference-priority games also have an NE:

```

theorem equilibrium_transfer_finite :
  assumes finite0 : "finite (range (form g))"
    and trans1 : "trans (pref1 g)"
    and irref1 : "irrefl (pref1 g)"
    and trans2 : "trans (pref2 g)"
    and irref2 : "irrefl (pref2 g)"
    and det : "determinedForm (form g) R1 R2"
  shows "∃ s1 ∈ R1. ∃ s2 ∈ R2. isNash g s1 s2"

```

	$2_l$	$2_r$
$1_t$		
$1_b$		

- $\diamond \mapsto \text{True} : 1_b \text{ wins}$
- $\diamond, \heartsuit \mapsto \text{False} : 2_l \text{ wins}$
- ...

Figure 1: The main theorem of [LMS17], and a game whose game form is determined

```

lemma equilibriumTransferPreferencePriority :
  assumes "snd (fst PPG) ∈ V_fst (fst PPG)"
  and "ParityGame (fst (fst PPG))"
  and "deadendFree (fst (fst PPG))"
  and acyclic1 : "acyclic ((fst ∘ snd) PPG)"
  and acyclic2 : "acyclic ((snd ∘ snd) PPG)"
  shows "∃ s1. ∃ s2. isNashPPG PPG s1 s2"

```

We have proven this result on paper and using Isabelle. The Isabelle formalisation of this lemma with all the preliminaries comprises approximately 450 lines of proof script.

As a future work, we would like to lift the restriction of *weak* positional determinacy.

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