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# On Non-uniqueness of Phase Retrieval for Functions with Disconnected Support

by Roman G. Novikov and Tianli Xu

**Abstract:** We show that the phase retrieval problem is not uniquely solvable even for functions with strongly disconnected compact support in multidimensions.

## 1. Introduction

The phase retrieval problem consists in finding a function  $v : \mathbb{R}^d \rightarrow \mathbb{C}$  from the magnitude  $|\hat{v}|$  of its Fourier transform

$$\hat{v}(p) = Fv(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad x \in \mathbb{R}^d. \quad (1)$$

This problem naturally arises in quantum mechanics, optics, and related areas such as electron tomography and X-ray imaging; see, for example, [1] - [8] and references therein.

In general, many different functions have the same Fourier modulus. These different solutions can be obtained by multiplying  $|\hat{v}|$  by measurable complex-valued functions with modulus one and taking the inverse Fourier transform; see, for example, [2].

When  $v$  is compactly supported, the degree of ambiguity is reduced. In particular, for  $d = 1$ , all solutions with compact support could be obtained from any one of them by flipping (conjugating) non-real zeros of its Fourier transform extended by analyticity to the complex plane; see [8], [3].

When  $v$  is a sum of functions with sufficiently disconnected compact supports, the degree of ambiguity is further reduced. In particular, for  $d = 1$ , this ambiguity is completely described in [2]. Roughly speaking, in this case, the phase retrieval problem almost always has essentially a unique solution.

In addition, it is also mentioned in the literature that for functions with sufficiently disconnected compact support, the degree of non-uniqueness of phase retrieval is further reduced in dimension  $d \geq 2$ ; see [2].

Moreover, the recent important work [6] suggests an efficient numerical phase retrieval algorithm for functions with sufficiently disconnected compact support. This algorithm works very well numerically and possible non-trivial non-uniqueness is not even discussed in [6].

Recall that the non-uniqueness in phase retrieval for compactly supported  $v$  with possible additional assumptions is non-trivial (and of interest) if it does not reduce to the functions  $v_{\alpha,y}$  and  $\tilde{v}_{\alpha,y}$  associated to  $v$ , where

$$v_{\alpha,y} = e^{i\alpha} v(x - y) \text{ and } \tilde{v}_{\alpha,y} = e^{i\alpha} \overline{v(-x + y)}, \quad \alpha \in \mathbb{R}, y \in \mathbb{R}^d, \quad (2)$$

where bar denotes the complex conjugation; see, for example, [1].

Nevertheless, the purpose of the present note is to pay attention to interesting non-trivial non-uniqueness in phase retrieval for functions with strongly disconnected compact support in multidimensions.

## 2. The Main Result

We consider complex-valued functions  $v$  on  $\mathbb{R}^d$  of the form

$$v = \sum_{k=1}^n v_k, \quad \text{supp } v_k \subset D_k, \quad v_k \neq 0, \quad (3)$$

where

$$D_k \text{ are open convex bounded domains in } \mathbb{R}^d, \quad \text{dist}(D_i, D_j) \geq r > 0 \text{ for } i \neq j. \quad (4)$$

Here,  $\text{dist}(\mathcal{A}, \mathcal{B})$  denotes the distance between sets  $\mathcal{A}$  and  $\mathcal{B}$  in  $\mathbb{R}^d$ .

In order to present interesting non-uniqueness in phase retrieval for functions  $v$  in (3) for  $n \geq 2$ ,  $d \geq 2$

with arbitrarily large  $r$ , we consider first non-uniqueness in phase retrieval for functions  $v$  in (3) for  $n = 1$ . To our knowledge, the latter non-uniqueness for  $n = 1$  is not yet completely described in the literature for  $d \geq 2$ . However, examples of this non-uniqueness can be constructed as follows.

We define

$$f_j(x) = \prod_{k=1}^d f_{j,k}(x_k), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad (5)$$

$$g_j(x) = \prod_{k=1}^d g_{j,k}(x_k), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad (6)$$

where  $f_{j,k}, g_{j,k} \in L^2(\mathbb{R})$ ,  $\text{supp } f_{j,k}, \text{supp } g_{j,k} \subseteq [-\epsilon, \epsilon]$ ,  $|\hat{f}_{j,k}|^2 = |\hat{g}_{j,k}|^2$  on  $\mathbb{R}$ ,  $j = 1, \dots, J$ ,  $k = 1, \dots, d$ , and  $f_j \neq g_j$  in  $L^2(\mathbb{R}^d)$ . Here,  $f_{j,k}$  and  $g_{j,k}$  are constructed, using results of [8] on non-uniqueness in phase retrieval in dimension  $d = 1$ .

Thus, we have

$$|\hat{f}_j|^2 = |\hat{g}_j|^2 \text{ on } \mathbb{R}^d, \quad f_j \neq g_j \text{ in } L^2(\mathbb{R}^d). \quad (7)$$

For the equality in (7), we also used the formula

$$F\left(\prod_{k=1}^d \phi_k\right)(p) = \prod_{k=1}^d \hat{\phi}_k(p_k), \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d, \quad (8)$$

where  $F$  is defined by (1),  $\phi_k = \phi_k(x_k)$  are functions on  $\mathbb{R}$  and  $\hat{\phi}_k$  are their one-dimensional Fourier transforms.

We define  $f, g$  as

$$f = F^{-1}\hat{f}, \quad \hat{f}(p) = \prod_{j=1}^J \hat{f}_j(A_j p), \quad p \in \mathbb{R}^d, \quad (9)$$

$$g = F^{-1}\hat{g}, \quad \hat{g}(p) = \prod_{j=1}^J \hat{g}_j(A_j p), \quad p \in \mathbb{R}^d, \quad (10)$$

where  $F^{-1}$  denotes the inverse Fourier transform,  $A_j \in SO(d)$ , the group of orthogonal matrices in dimension  $d$  with determinant 1.

Let

$$B_r = \{x \in \mathbb{R}^d : |x| \leq r\}, \quad r > 0; \quad (11)$$

$$N_\epsilon(U) = \{x \in \mathbb{R}^d : \text{dist}(x, U) < \epsilon\}, \quad \epsilon > 0, U \subset \mathbb{R}^d; \quad (12)$$

$$u_1 * u_2(x) = \int_{\mathbb{R}^d} u_1(x-y)u_2(y)dy, \quad x \in \mathbb{R}^d, \quad (13)$$

where  $u_1, u_2$  are test functions on  $\mathbb{R}^d$ .

**Lemma 1:** Let  $f, g$  be defined by formulas (5), (6), (9), (10). Then:  $\text{supp } f, \text{supp } g \subset B_r$ , where  $r = J\sqrt{d}\epsilon$ ;  $|\hat{f}|^2 = |\hat{g}|^2$  on  $\mathbb{R}^d$ ; and  $f \neq g$  in  $L^2(\mathbb{R}^d)$ , in general, and, for example, for  $J = 1$ .

Proceeding from non-uniqueness in phase retrieval for functions  $v$  in (3) for  $n = 1$ ,  $d \geq 2$ , as in Lemma 1, our non-uniqueness result for phase retrieval for functions  $v$  in (3) for  $n \geq 2$ ,  $d \geq 2$  with arbitrarily large  $r$  is as follows.

**Theorem 1:** Let  $f, g$  be two complex-valued functions on  $\mathbb{R}^d$  such that  $\text{supp } f, \text{supp } g \subset B_\delta$ ,  $|\hat{f}|^2 = |\hat{g}|^2$  on  $\mathbb{R}^d$ , but  $f \neq g$  in  $L^2(\mathbb{R}^d)$ . Let  $v$  be as in (3), (4), where  $r > 2\delta$ . Let  $v_f = f * v$ ,  $v_g = g * v$ . Then:

- (i)  $v_f, v_g$  are of the form (3), (4) with  $N_\delta(D_k)$  in place of  $D_k$  and  $r_\delta = r - 2\delta$  in place of  $r$ ;
- (ii)  $|\hat{v}_f|^2 = |\hat{v}_g|^2$  on  $\mathbb{R}^d$ , but  $v_f \neq v_g$  in  $L^2(\mathbb{R}^d)$ .

Lemma 1 and Theorem 1 are of interest, at least, for  $d \geq 2$ . The non-uniqueness of Theorem 1 for  $d = 1$  is a particular case of the non-uniqueness considered in the theorem in [2].

Theorem 1 is of interest even when all  $v_k$  in (3) are Dirac delta functions, i.e.,

$$v_k(x) = C_k \delta(x - y_k), \quad y_k \in D_k, \quad C_k \in \mathbb{C}, \quad k = 1, \dots, n. \quad (14)$$

Note that the non-uniqueness of Lemma 1 and Theorem 1 is not reduced (at least, in general) to associated functions in the sense of formula (2), in a similar way with non-uniqueness of [8].

Thus, in order to have complete uniqueness in phase retrieval for functions  $v$  of the form (3), (4), even with strongly disconnected support and even modulo associated functions, additional a priori information is necessary. In connection with natural theoretical and numerical results in this direction, see, for example, [4], [7] and references therein.

### 3. Proof of Lemma 1 and Theorem 1

Recall that

$$F^{-1}(\phi_1\phi_2) = (2\pi)^{-d}F^{-1}\phi_1 * F^{-1}\phi_2, \quad (15)$$

$$(2\pi)^d F(u_1 * u_2) = Fu_1Fu_2, \quad (16)$$

where  $F$  is defined by (1),  $\phi_1, \phi_2, u_1, u_2$  are test functions on  $\mathbb{R}^d$ , and  $v_k$  are test functions on  $\mathbb{R}$ .

Recall also that if  $\text{supp } u_1 \subseteq \mathcal{U}_1, \text{supp } u_2 \subseteq \mathcal{U}_2$ , where  $\mathcal{U}_1, \mathcal{U}_2$  are closed bounded domains in  $\mathbb{R}^d$ , then

$$\text{supp } u_1 * u_2 \subseteq \mathcal{U}_1 + \mathcal{U}_2, \quad (17)$$

where

$$\mathcal{U}_1 + \mathcal{U}_2 = \{x + y : x \in \mathcal{U}_1, y \in \mathcal{U}_2\}. \quad (18)$$

In addition,

$$B_{r_1}(a_1) + B_{r_2}(a_2) = B_{r_1+r_2}(a_1 + a_2), \quad (19)$$

where

$$B_r(a) = \{x \in \mathbb{R}^d : |x - a| \leq r\}, \quad a \in \mathbb{R}^d, \quad r > 0. \quad (20)$$

Lemma 1 follows from formulas (7), (15), (17) and (19) with  $a_1 = a_2 = 0$ .

To prove Theorem 1, we consider

$$v_f = f * v = \sum_{k=1}^n f * v_k, \quad v_g = g * v = \sum_{k=1}^n g * v_k. \quad (21)$$

Due to formula (16), we have that

$$\hat{v}_f = (2\pi)^{-d}\hat{f}\hat{v}, \quad \hat{v}_g = (2\pi)^{-d}\hat{g}\hat{v}. \quad (22)$$

Using also that  $|\hat{f}|^2 = |\hat{g}|^2$  on  $\mathbb{R}^d$ , we get

$$|\hat{v}_f|^2 = |\hat{v}_g|^2 \text{ on } \mathbb{R}^d. \quad (23)$$

In addition, using formulas (22), the property that  $\hat{f} \neq \hat{g}$  in  $L^2(\mathbb{R}^d)$  and the property that  $\hat{v}$  is non-zero almost everywhere on  $\mathbb{R}^d$ , we get that

$$\hat{v}_f \neq \hat{v}_g \text{ in } L^2(\mathbb{R}^d). \quad (24)$$

The properties that  $\text{supp } f, \text{supp } g \subset B_\delta, \text{supp } v_k \subset D_k$  and formula (17) imply that

$$\text{supp } f * v_k \subset N_\delta(D_k), \quad \text{supp } g * v_k \subset N_\delta(D_k). \quad (25)$$

In addition, since  $\text{dist}(D_i, D_j) \geq r > 2\delta, i \neq j$ , we have that

$$\text{dist}(N_\delta(D_i), N_\delta(D_j)) \geq r_\delta, \quad i \neq j. \quad (26)$$

Theorem 1 follows from formulas (21) - (26).

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