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On Non-uniqueness of Phase Retrieval for Functions with Disconnected Support

by Roman G. Novikov and Tianli Xu

Abstract: We show that the phase retrieval problem is not uniquely solvable even for functions with strongly disconnected compact support in multidimensions.

1. Introduction

The phase retrieval problem consists in finding a function $v : \mathbb{R}^d \rightarrow \mathbb{C}$ from the magnitude $|\hat{v}|$ of its Fourier transform

$$\hat{v}(p) = Fv(p) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad x \in \mathbb{R}^d. \quad (1)$$

This problem naturally arises in quantum mechanics, optics, and related areas such as electron tomography and X-ray imaging; see, for example, [1] - [8] and references therein.

In general, many different functions have the same Fourier modulus. These different solutions can be obtained by multiplying $|\hat{v}|$ by measurable complex-valued functions with modulus one and taking the inverse Fourier transform; see, for example, [2].

When v is compactly supported, the degree of ambiguity is reduced. In particular, for $d = 1$, all solutions with compact support could be obtained from any one of them by flipping (conjugating) non-real zeros of its Fourier transform extended by analyticity to the complex plane; see [8], [3].

When v is a sum of functions with sufficiently disconnected compact supports, the degree of ambiguity is further reduced. In particular, for $d = 1$, this ambiguity is completely described in [2]. Roughly speaking, in this case, the phase retrieval problem almost always has essentially a unique solution.

In addition, it is also mentioned in the literature that for functions with sufficiently disconnected compact support, the degree of non-uniqueness of phase retrieval is further reduced in dimension $d \geq 2$; see [2].

Moreover, the recent important work [6] suggests an efficient numerical phase retrieval algorithm for functions with sufficiently disconnected compact support. This algorithm works very well numerically and possible non-trivial non-uniqueness is not even discussed in [6].

Recall that the non-uniqueness in phase retrieval for compactly supported v with possible additional assumptions is non-trivial (and of interest) if it does not reduce to the functions $v_{\alpha,y}$ and $\tilde{v}_{\alpha,y}$ associated to v , where

$$v_{\alpha,y} = e^{i\alpha} v(x-y) \text{ and } \tilde{v}_{\alpha,y} = e^{i\alpha} \overline{v(-x+y)}, \quad \alpha \in \mathbb{R}, y \in \mathbb{R}^d, \quad (2)$$

where bar denotes the complex conjugation; see, for example, [1].

Nevertheless, the purpose of the present note is to pay attention to interesting non-trivial non-uniqueness in phase retrieval for functions with strongly disconnected compact support in multidimensions.

2. The Main Result

We consider complex-valued functions v on \mathbb{R}^d of the form

$$v = \sum_{k=1}^n v_k, \quad \text{supp } v_k \subset D_k, \quad v_k \neq 0, \quad (3)$$

where

$$D_k \text{ are open convex bounded domains in } \mathbb{R}^d, \quad \text{dist}(D_i, D_j) \geq r > 0 \text{ for } i \neq j. \quad (4)$$

Here, $\text{dist}(\mathcal{A}, \mathcal{B})$ denotes the distance between sets \mathcal{A} and \mathcal{B} in \mathbb{R}^d .

In order to present interesting non-uniqueness in phase retrieval for functions v in (3) for $n \geq 2$, $d \geq 2$

with arbitrarily large r , we consider first non-uniqueness in phase retrieval for functions v in (3) for $n = 1$. To our knowledge, the latter non-uniqueness for $n = 1$ is not yet completely described in the literature for $d \geq 2$. However, examples of this non-uniqueness can be constructed as follows.

We define

$$f_j(x) = \prod_{k=1}^d f_{j,k}(x_k), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad (5)$$

$$g_j(x) = \prod_{k=1}^d g_{j,k}(x_k), \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d, \quad (6)$$

where $f_{j,k}, g_{j,k} \in L^2(\mathbb{R})$, $\text{supp } f_{j,k}, \text{supp } g_{j,k} \subseteq [-\epsilon, \epsilon]$, $|\hat{f}_{j,k}|^2 = |\hat{g}_{j,k}|^2$ on \mathbb{R} , $j = 1, \dots, J$, $k = 1, \dots, d$, and $f_j \neq g_j$ in $L^2(\mathbb{R}^d)$. Here, $f_{j,k}$ and $g_{j,k}$ are constructed, using results of [8] on non-uniqueness in phase retrieval in dimension $d = 1$.

Thus, we have

$$|\hat{f}_j|^2 = |\hat{g}_j|^2 \text{ on } \mathbb{R}^d, \quad f_j \neq g_j \text{ in } L^2(\mathbb{R}^d). \quad (7)$$

For the equality in (7), we also used the formula

$$F\left(\prod_{k=1}^d \phi_k\right)(p) = \prod_{k=1}^d \hat{\phi}_k(p_k), \quad p = (p_1, \dots, p_d) \in \mathbb{R}^d, \quad (8)$$

where F is defined by (1), $\phi_k = \phi_k(x_k)$ are functions on \mathbb{R} and $\hat{\phi}_k$ are their one-dimensional Fourier transforms.

We define f, g as

$$f = F^{-1}\hat{f}, \quad \hat{f}(p) = \prod_{j=1}^J \hat{f}_j(A_j p), \quad p \in \mathbb{R}^d, \quad (9)$$

$$g = F^{-1}\hat{g}, \quad \hat{g}(p) = \prod_{j=1}^J \hat{g}_j(A_j p), \quad p \in \mathbb{R}^d, \quad (10)$$

where F^{-1} denotes the inverse Fourier transform, $A_j \in SO(d)$, the group of orthogonal matrices in dimension d with determinant 1.

Let

$$B_r = \{x \in \mathbb{R}^d : |x| \leq r\}, \quad r > 0; \quad (11)$$

$$N_\epsilon(U) = \{x \in \mathbb{R}^d : \text{dist}(x, U) < \epsilon\}, \quad \epsilon > 0, U \subset \mathbb{R}^d; \quad (12)$$

$$u_1 * u_2(x) = \int_{\mathbb{R}^d} u_1(x-y)u_2(y)dy, \quad x \in \mathbb{R}^d, \quad (13)$$

where u_1, u_2 are test functions on \mathbb{R}^d .

Lemma 1: Let f, g be defined by formulas (5), (6), (9), (10). Then: $\text{supp } f, \text{supp } g \subset B_r$, where $r = J\sqrt{d}\epsilon$; $|\hat{f}|^2 = |\hat{g}|^2$ on \mathbb{R}^d ; and $f \neq g$ in $L^2(\mathbb{R}^d)$, in general, and, for example, for $J = 1$.

Proceeding from non-uniqueness in phase retrieval for functions v in (3) for $n = 1$, $d \geq 2$, as in Lemma 1, our non-uniqueness result for phase retrieval for functions v in (3) for $n \geq 2$, $d \geq 2$ with arbitrarily large r is as follows.

Theorem 1: Let f, g be two complex-valued functions on \mathbb{R}^d such that $\text{supp } f, \text{supp } g \subset B_\delta$, $|\hat{f}|^2 = |\hat{g}|^2$ on \mathbb{R}^d , but $f \neq g$ in $L^2(\mathbb{R}^d)$. Let v be as in (3), (4), where $r > 2\delta$. Let $v_f = f * v$, $v_g = g * v$. Then:

- (i) v_f, v_g are of the form (3), (4) with $N_\delta(D_k)$ in place of D_k and $r_\delta = r - 2\delta$ in place of r ;
- (ii) $|\hat{v}_f|^2 = |\hat{v}_g|^2$ on \mathbb{R}^d , but $v_f \neq v_g$ in $L^2(\mathbb{R}^d)$.

Lemma 1 and Theorem 1 are of interest, at least, for $d \geq 2$. The non-uniqueness of Theorem 1 for $d = 1$ is a particular case of the non-uniqueness considered in the theorem in [2].

Theorem 1 is of interest even when all v_k in (3) are Dirac delta functions, i.e.,

$$v_k(x) = C_k \delta(x - y_k), \quad y_k \in D_k, \quad C_k \in \mathbb{C}, \quad k = 1, \dots, n. \quad (14)$$

Note that the non-uniqueness of Lemma 1 and Theorem 1 is not reduced (at least, in general) to associated functions in the sense of formula (2), in a similar way with non-uniqueness of [8].

Thus, in order to have complete uniqueness in phase retrieval for functions v of the form (3), (4), even with strongly disconnected support and even modulo associated functions, additional a priori information is necessary. In connection with natural theoretical and numerical results in this direction, see, for example, [4], [7] and references therein.

3. Proof of Lemma 1 and Theorem 1

Recall that

$$F^{-1}(\phi_1\phi_2) = (2\pi)^{-d}F^{-1}\phi_1 * F^{-1}\phi_2, \quad (15)$$

$$(2\pi)^d F(u_1 * u_2) = Fu_1Fu_2, \quad (16)$$

where F is defined by (1), ϕ_1, ϕ_2, u_1, u_2 are test functions on \mathbb{R}^d , and v_k are test functions on \mathbb{R} .

Recall also that if $\text{supp } u_1 \subseteq \mathcal{U}_1, \text{supp } u_2 \subseteq \mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are closed bounded domains in \mathbb{R}^d , then

$$\text{supp } u_1 * u_2 \subseteq \mathcal{U}_1 + \mathcal{U}_2, \quad (17)$$

where

$$\mathcal{U}_1 + \mathcal{U}_2 = \{x + y : x \in \mathcal{U}_1, y \in \mathcal{U}_2\}. \quad (18)$$

In addition,

$$B_{r_1}(a_1) + B_{r_2}(a_2) = B_{r_1+r_2}(a_1 + a_2), \quad (19)$$

where

$$B_r(a) = \{x \in \mathbb{R}^d : |x - a| \leq r\}, \quad a \in \mathbb{R}^d, \quad r > 0. \quad (20)$$

Lemma 1 follows from formulas (7), (15), (17) and (19) with $a_1 = a_2 = 0$.

To prove Theorem 1, we consider

$$v_f = f * v = \sum_{k=1}^n f * v_k, \quad v_g = g * v = \sum_{k=1}^n g * v_k. \quad (21)$$

Due to formula (16), we have that

$$\hat{v}_f = (2\pi)^{-d}\hat{f}\hat{v}, \quad \hat{v}_g = (2\pi)^{-d}\hat{g}\hat{v}. \quad (22)$$

Using also that $|\hat{f}|^2 = |\hat{g}|^2$ on \mathbb{R}^d , we get

$$|\hat{v}_f|^2 = |\hat{v}_g|^2 \text{ on } \mathbb{R}^d. \quad (23)$$

In addition, using formulas (22), the property that $\hat{f} \neq \hat{g}$ in $L^2(\mathbb{R}^d)$ and the property that \hat{v} is non-zero almost everywhere on \mathbb{R}^d , we get that

$$\hat{v}_f \neq \hat{v}_g \text{ in } L^2(\mathbb{R}^d). \quad (24)$$

The properties that $\text{supp } f, \text{supp } g \subset B_\delta, \text{supp } v_k \subset D_k$ and formula (17) imply that

$$\text{supp } f * v_k \subset N_\delta(D_k), \quad \text{supp } g * v_k \subset N_\delta(D_k). \quad (25)$$

In addition, since $\text{dist}(D_i, D_j) \geq r > 2\delta, i \neq j$, we have that

$$\text{dist}(N_\delta(D_i), N_\delta(D_j)) \geq r_\delta, \quad i \neq j. \quad (26)$$

Theorem 1 follows from formulas (21) - (26).

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