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Different notions of Passivity and their characterization in the port-Hamiltonian formalism

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Abstract

In this work, we give an overview of different notions of stability, dissipativity and passivity. In particular, we first focus our attention to incremental stability, convergence dynamics, and contractive systems. We then consider their extension to the case of open systems, i.e., dynamical systems that interact with the surrounding environment via inputs and output signals, hence we address different concepts of dissipativity/passivity, such as incremental, convergence and differential dissipativity/passivity. We write the notions into an unifying framework in order to make an easier comparison among them by leveraging coordinate-free definitions. We then obtain conditions to determine the related storage functions for a class of passive system, i.e., port-Hamiltonian dynamics. In particular, we show what additional conditions must be satisfied by a passive system with its associated storage function in order to fit the non-standard passivity definitions. We obtain the relative storage functions based on the knowledge of the system Hamiltonian that for these systems plays the role of storage function for the standard passivity notion.

Keywords: Incremental Stability, incremental Passivity, Convergent dynamics, Contraction, differential passivity, port-Hamiltonian systems, incremental port-Hamiltonian systems

1. Introduction

The aim of this work is to provide in Sections 2 and 3 an overview of different notions of stability that have had a relevant impact into the control community, such as Lyapunov and incremental stability, convergent dynamics, contractive system, and a new stability concept for dissipative Hamiltonian systems that has been inspired by maximal monotone structures. The extended notion of Lyapunov stability to open systems, i.e., systems with inputs and outputs, is the so-called *Dissipativity* property, and a particular case of the latter is the so-called *Passivity*. They are recalled in Section 4.

The former stability definitions are directly related to some passivity properties when we deal with a input-output relationship in the state space. We thus aim at providing a general overview of different dissipativity notions in Section 5, and then characterize alternative passivity notions in Section 6. In particular, we aim to find an uniform framework to compare: incremental passivity, convergent passivity, differential passivity, and the incremental port-Hamiltonian systems (here called maximal monotone passivity).

The discussed notions are applied to the class of dissipative and port-Hamiltonian systems. In particular, we analyze the stability conditions in the framework of dissipative Hamiltonian systems in 7. Furthermore, because port-Hamiltonian are a particular class of passive systems, we aim to describe in Section 8, the conditions under which a passive system can satisfy the other passivity notions, while seeking the relative storage function based on the knowledge of the system Hamiltonian. We then summarize the paper content in Sec. 9. We also provide, in Appendix A, the concepts of variational and prolonged

systems that are used when discussing contractive systems and differential dissipativity/passivity.

The introduction is intentionally left short because the literature review is given and detailed throughout the text when introducing the corresponding concepts.

Notation: Given a real-valued function $H : \mathbb{R}^n \rightarrow \mathbb{R}$, we denote its gradient (in covector form) as $\frac{\partial H}{\partial x} = \partial_x H$, and for the sake of simplicity we write its column vector form as $\nabla H(x) = \frac{\partial H^\top}{\partial x}(x)$. Its Hessian will then be written as $\nabla^2 H(x) = \frac{\partial}{\partial x} \cdot \frac{\partial H^\top}{\partial x}(x)$. For a matrix-valued function $J(x) \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, we consider, with some abuse of notation, that its time derivative, to be written as $\dot{J}(x) = \frac{\partial J(x)}{\partial x} \dot{x}$, or, column-wise, we write

$\dot{J}_i(x) = \frac{\partial J_i}{\partial x} \dot{x}$, where $J_i(x)$ refers to the i -th column of $J(x)$.

When we evaluate $\dot{J}(x)\nabla H(x)$, we write it in matrix form as $\dot{J}(x)\nabla H(x) = \frac{\partial H}{\partial x} \frac{\partial^\top J}{\partial x} \dot{x}$, meaning that the resulting vector will

have as i -th element $\dot{J}_i \nabla H = \left(\dot{x}^\top \frac{\partial J_i^\top}{\partial x} \right)^\top \nabla H(x) = \frac{\partial H}{\partial x} \frac{\partial^\top J_i}{\partial x} \dot{x}$.

Along the text, for the sake of readability, we might use the following contract notation $\nabla H_i = \nabla H(x_i)$, $J_i = J(x_i)$, $R_i = R(x_i)$, and $g_i = g(x_i)$, $i = 1, 2$.

We denote the set of nonnegative real as $\mathbb{R}_{\geq 0}$. Given a subset $D \subset \mathbb{R}_{\geq 0}$, a continuous function $\alpha : D \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to class \mathcal{K}_∞

if it belongs to class \mathcal{K} and $\alpha(\infty) = \infty$. A continuous function $\beta : D \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to belong to the class \mathcal{KL} if, for each fixed s , $\beta(r, s)$ belongs to class \mathcal{K} and, for each fixed r , $\beta(r, s)$ is decreasing in s with $\beta(r, \infty) = 0$. Given a domain $\mathcal{X} \subseteq \mathbb{R}^n$, a continuous function $V(x, t) : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is called a positive definite function if $V(0, t) = 0$ and for some K_∞ function α , $V(x(t), t) \geq \alpha(|x(t)|)$ for all $x \in \mathcal{X}$ and $t \in \mathbb{R}$.

2. Stability Theory

Stability theory plays a significant role in engineering and control theory, although the approach can be used to study closed dynamical systems in any domain. Lyapunov stability criterion is a general and useful approach to analyze the stability of nonlinear systems. In particular, the Lyapunov direct method is the most used tool for design and analysis of nonlinear *closed* systems. The fundamental concept of the Lyapunov direct method is that if we consider the distance between the system trajectory and the final (desired) equilibrium point to be an energy-like function (for which the equilibrium point might be its global minimum) for the system under consideration and this energy-like function is continuously dissipating, then the system will eventually reach such an equilibrium point and remain there. The reader can be interested in referring to [1, 2, 3] for additional details.

In this work, we consider a *forward complete*¹ dynamical system described by the following differential equations

$$\dot{x} = f(x, t), \quad x(t_0) = x_0 \quad (1)$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ and $t \geq t_0 \in \mathbb{R}$, where \mathcal{X} is considered to be the forward invariant system manifold while x are some local set of coordinates. System (1) is said to be *autonomous* if we drop the explicit dependence on time in $f(x, t)$, i.e., the system dynamics can be written as

$$\dot{x} = f(x), \quad x(t_0) = x_0. \quad (2)$$

A particular case of (2) is described by the class of autonomous systems whose vectorfield $f(x)$ presents a particular structure, i.e.,

$$\dot{x} = (J(x) - R(x))\nabla H(x), \quad x(t_0) = x_0 \quad (3)$$

where $J : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ is a skew-symmetric matrix that describes the underlying Dirac structure (i.e., the energy transfer between different physical domains), $R : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ is a positive semi-definite matrix describing the dissipative elements of the system, and the C^2 function $H : \mathcal{X} \rightarrow \mathbb{R}$ is the Hamiltonian function associated to the system energy. Systems belonging to this class are called *dissipative Hamiltonian* systems.

The standard definition of asymptotic stability in the sense of Lyapunov or attractiveness of an equilibrium point² can be expressed, as in classical approaches, by the following ‘regional’ characterization.

¹A system is called forward complete if for every initial condition the corresponding solution is defined for all forward time [4].

²Where x^* is said to be an equilibrium point for (1) if $f(x^*, t) = 0$ for all $t \in \mathbb{R}$.

Definition 2.1. *The equilibrium point $x^* \in \mathcal{X}$ is called a stable equilibrium point of (1) if for all t_0 and $\varepsilon > 0$, there exists a $\delta(t_0, \varepsilon)$ such that the following implication holds true*

$$|x_0 - x^*| < \delta(t_0, \varepsilon) \implies |x(t) - x^*| < \varepsilon, \quad \forall t \geq t_0,$$

where $x(t) \in \mathcal{X}$ is the solution of (1) starting from x_0 at t_0 . Furthermore, x^* is said to be an asymptotically stable equilibrium point if in addition it is attractive for the system trajectory $x(t)$, i.e., for all $t_0 \in \mathbb{R}$ there exists a $\delta(t_0)$ such that

$$|x_0 - x^*| < \delta(t_0) \implies \lim_{t \rightarrow \infty} |x(t) - x^*| = 0 \quad (4)$$

A more general definition of asymptotic stability/attractiveness of an equilibrium point can be formalized by involving the definition of a weaker, non-euclidean, distance, i.e., by defining a positive definite function, such as a Lyapunov function candidate, $V_\star : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$.

Definition 2.2. *An equilibrium point x^* is said to be stable if there exists a positive definite Lyapunov function $V_\star(x(t), t)$, where $V_\star(x, t) = 0$ for all real t implies $x(t) = x^*$, such that*

$$V_\star(x(t_1), t_1) - V_\star(x(t_0), t_0) \leq 0 \quad (5)$$

for any $t_1 > t_0 \in \mathbb{R}$. Furthermore, x^* is said to be an asymptotically stable equilibrium point if such a Lyapunov function V_\star satisfies

$$V_\star(x(t_1), t_1) - V_\star(x(t_0), t_0) < 0 \quad (6)$$

for all $t_1 > t_0 \in \mathbb{R}$.

3. Additional notions of stability

In the following, we give other notions of stability such as incremental stability [5], contraction analysis [6] and convergent dynamics [7] that have been received increased interest in recent years due to their potential application in synchronization [8, 9, 10, 11], nonlinear output regulation [12, 13], steady-state analysis of nonlinear systems [14], observer design [15], and many other nonlinear control problems such as the stability analysis of stochastic systems [16].

Incrementally stable system has been introduced in [17], while contraction analysis in [6] (see also [18] for a more historical overview), and the convergent dynamics in [19, 20]. They have also been further investigated in [21, 22, 23, 24, 25, 26].

3.1. Incremental stability

The incremental stability describes the property of any two system trajectories $x_1(t), x_2(t)$ to convergence one on the other.

Definition 3.1 (Incremental stability [5]). *System (1) is incrementally asymptotically stable in a positively invariant set $\mathcal{X} \subset \mathbb{R}^n$ if there exists a class \mathcal{KL} function β such that for any $x_{10}, x_{20} \in \mathcal{X}$ and $t \geq t_0$,*

$$\|x_1(t) - x_2(t)\| \leq \beta(\|x_{10} - x_{20}\|, t - t_0) \quad (7)$$

In this note, we will use the following more general definition of incrementally stable systems.

Definition 3.2 (Incremental stability). *The system (12) is called incrementally stable if there exists an incremental Lyapunov function $\mathfrak{S} : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}_{\geq 0}$, with $\mathfrak{S}(x_1, x_2) = 0$ implies $x_1 = x_2$, such that for any two distinct state trajectories $(x_1, x_2) \in \mathcal{X} \times \mathcal{X}$, it holds that*

$$\mathfrak{S}(x_1(t_1), x_2(t_1)) - \mathfrak{S}(x_1(t_0), x_2(t_0)) < 0 \quad (8)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0$, or equivalently

$$\lim_{t \rightarrow \infty} x_1(t) - x_2(t) = 0.$$

In the above definition, the incremental Lyapunov function \mathfrak{S} is used with the same meaning of a distance d as used in the coordinated-independent definition of incremental stability as employed in [25] or in [27], reported here for completeness.

Definition 3.3 (Definition 1 in [27]). *Consider system (2) on a given manifold \mathcal{X} . Let \mathcal{X} be a forward invariant set and $d : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ a continuous distance on \mathcal{X} . The system (1) is incrementally stable on \mathcal{X} wrt d if there exists a class \mathcal{K} function α such that $\forall x_{10}, x_{20} \in \mathcal{X}$,*

$$d(x_1(t), x_2(t)) \leq \alpha(d(x_{10}, x_{20}))$$

and $\forall t_0 \in \mathbb{R}, \forall t \geq t_0$

$$\lim_{t \rightarrow \infty} d(x_1(t), x_2(t)) = 0.$$

3.2. Convergent system

A particular case of an incrementally stable system is provided by the case when the final trajectory is bounded for all time $t \in \mathbb{R}$, it is unique and attractive for all system initial conditions. This is described by the following definition.

Definition 3.4 (Convergent dynamics [7]). *System (1) is uniformly convergent in a positively invariant set $\mathcal{X} \subset \mathbb{R}^n$ if*

- there exists a unique solution $x^*(t)$ in \mathcal{X} defined and bounded for all $t \in \mathbb{R}$;
- such a solution $x^*(t)$ is uniformly asymptotically stable in \mathcal{X} , i.e., there exists a class \mathcal{KL} function β such that for any system trajectory $x(t)$ initialized at $(t_0, x_0) \in \mathbb{R} \times \mathcal{X}$ and $t \geq t_0$

$$\|x(t) - x^*(t)\| \leq \beta(\|x_0 - x^*(t_0)\|, t - t_0). \quad (9)$$

In this note, we will use the following coordinate independent definition of convergent dynamics.

Definition 3.5 (Convergent dynamics). *The system (12) has a convergent dynamics if there exist*

- a unique solution $x^*(t) \in \mathcal{X}$ defined and bounded for all $t \in \mathbb{R}$;
- a Lyapunov-like function $\mathfrak{S} : \mathcal{X} \times \mathcal{X} \mapsto \mathbb{R}_{\geq 0}$, where $\mathfrak{S}(x(t), x^*(t)) = 0$ implies $x(t) = x^*(t)$ for all $t \in \mathbb{R}$, such that for any state trajectories $x(t) \neq x^*(t)$ with $x(t) \in \mathcal{X}$, it holds that

$$\mathfrak{S}(x(t_1), x^*(t_1)) - \mathfrak{S}(x(t_0), x^*(t_0)) < 0 \quad (10)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 > t_0$, or equivalently

$$\lim_{t \rightarrow \infty} x(t) - x^*(t) = 0.$$

3.3. Contractive systems

The first results of contraction theory can be found in [19] and [20] (see also [7]), and they are based on a constant Euclidean metric, see also [3, Problem 5.2] and [28] for a more recent reference.

It plays a crucial role in studying the stability and/or the attractiveness of invariant manifolds for nonlinear autonomous systems. More specifically, this attractiveness property is characterized by the existence of a positive definite quadratic form (called Lyapunov-Finsler metric in [27]) which is decreasing along the flow of the autonomous prolonged dynamics, see Appendix A for a discussion on the prolonged dynamics.

The contraction approach was born as a tool to study and describe the stability of nonlinear systems as a counterpart of Lyapunov approach, see [27]. Only later on, in [6] the theory has been extended to nonlinear state dependent metric, while further developments can be found in [5, 27, 15].

Definition 3.6 (Contractive system). *An autonomous system (2) on a smooth manifold \mathcal{X} , is said to be contractive if there exists a C^1 function $P : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$, two strictly positive real numbers \underline{p} and \bar{p} such that P has a well-defined time derivative and it satisfies*

$$\underline{p}I \leq P(x) \leq \bar{p}I, \\ \dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x) + \frac{\partial f^\top}{\partial x}(x) P(x) < 0, \quad \forall x \in \mathcal{X}.$$

Definition 3.7 (Contractive time-varying system). *An autonomous system (1) on a smooth manifold \mathcal{X} , is said to be contractive if there exists a C^1 function $P : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$, two strictly positive real numbers \underline{p} and \bar{p} such that P has a well-defined time derivative and it satisfies*

$$\underline{p}I \leq P(x) \leq \bar{p}I, \\ \dot{P}(x) + P(x) \frac{\partial f}{\partial x}(x, t) + \frac{\partial f^\top}{\partial x}(x, t) P(x) < 0, \quad \forall x \in \mathcal{X}.$$

Despite the fact that for the nonlinear autonomous dynamics we can give an interpretation to the contraction property of a system (see the appendix), we cannot conclude anything about the contraction property of a time varying system except for its virtual time-varying dynamics (that in principle does not have any straightforward interpretation).

3.4. Maximal monotone stability

A very specific stability notion, that recalls that of incremental stability was inspired by the definition of incremental port-Hamiltonian system in [29].

Definition 3.8. *A dissipative Hamiltonian system (3) is said to be stable in a maximal monotone sense (or maximally monotone stable) if for any two distinct trajectories $x_1(t)$ and $x_2(t)$ initialized, respectively, on $x_{10} \in \mathcal{X}$ and $x_{20} \in \mathcal{X}$ at $t_0 \in \mathbb{R}$, it holds*

$$(\nabla H(x_1) - \nabla H(x_2))^\top (\dot{x}_1 - \dot{x}_2) \leq 0 \quad (11)$$

for all $t \geq t_0$ and any $x_{10} \neq x_{20}$.

4. Dissipative and passive systems

Concurrently to the development of the contraction analysis, the concept of dissipativity was introduced by J.C. Willems in [30] and has played a relevant role in systems theory, in particular in the analysis of open and interconnected systems, as it generalizes the concept of Lyapunov function to system interfacing the surrounding environment, as described in [31]. It also gives the foundations to dissipativity-based nonlinear control designs, see, e.g., [32, 33, 34].

Consider a nonlinear system, in the state space (local) coordinates of a smooth manifold \mathcal{X} , expressed by a model of the form

$$\Sigma : \begin{cases} \dot{x} = f(x, u), & x(0) = x_0 \\ y = h(x, u) \end{cases} \quad (12)$$

with the input value $u \in \mathcal{U} \subseteq \mathbb{R}^m$, the output values $y \in \mathcal{Y} \subseteq \mathbb{R}^p$, and the system's state $x \in \mathcal{X} \subseteq \mathbb{R}^n$, where \mathcal{U} , \mathcal{Y} , and \mathcal{X} are the input, output, and state spaces, while $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{T}\mathcal{X}$ and $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$. With some abuse of notation \mathcal{U} refers, for the sake of brevity of exposition, also to the space of admissible inputs.

A particular case of (12) is when the functions f and h are input affine, i.e., with some abuse of notation we can write

$$\begin{cases} \dot{x} = f(x) + g(x)u, & x(0) = x_0 \\ y = h(x) + j(x)u \end{cases} \quad (13)$$

where $g : \mathcal{X} \rightarrow \mathbb{R}^{n \times m}$ and $j : \mathcal{X} \rightarrow \mathbb{R}^{p \times m}$. Furthermore, whenever the system output map h does not depend on the input, the system is called *input affine strictly proper*, and it has dynamics

$$\begin{cases} \dot{x} = f(x) + g(x)u, & x(0) = x_0 \\ y = h(x). \end{cases} \quad (14)$$

Otherwise, when the system dynamics f is a nonlinear function of the input u , the system is simply called *strictly proper* and its dynamics reads as

$$\begin{cases} \dot{x} = f(x, u), & x(0) = x_0 \\ y = h(x). \end{cases} \quad (15)$$

The definition of dissipative system for Σ , introduced in [30], employs two real-valued functions, i.e., the supply rate denoted by $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$, and the storage function $S : \mathcal{X} \rightarrow \mathbb{R}$.

Definition 4.1 (Dissipation inequality). *System Σ is said to satisfy the dissipation inequality with respect to the supply rate s and storage function S if*

$$S(x(t_2)) - S(x(t_1)) \leq \int_{t_1}^{t_2} s(u(t), y(t)) dt \quad (16)$$

for all $t_1, t_2 \in \mathbb{R}$, with $t_2 \geq t_1$ and all admissible state-input-output trajectory $(x(t), u(t), y(t)) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$.

To be more precise, the tuple $(x(t), u(t), y(t))$ should belong to the behavior of Σ , see [31] for more details.

The well-known and accepted interpretation of the dissipation inequality (16), is that (in general) not all the supplied

quantity ($\int s dt$) is stored (in S), and thus the amount that is supplied but not stored is *dissipated*.

The definition of the dissipation inequality allows us to introduce the definition of dissipative and cyclo-dissipative systems as follows.

Definition 4.2 (Cyclo-dissipative systems). *A system Σ is said to be cyclo-dissipative with respect to the supply rate $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$ if there exists a real-valued storage function $S : \mathcal{X} \rightarrow \mathbb{R}$ such that the dissipation inequality (16) holds.*

Definition 4.3 (Dissipative systems). *A system Σ is said to be dissipative with respect to the supply rate $s : \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}$ if it is cyclo-dissipative with a non-negative storage function $S : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the dissipation inequality (16).*

As described in [31], the notion of storage function for dissipative systems is a generalization to open systems of the concept of Lyapunov function for closed systems.

A particular case of dissipative system is given when the supply rate takes the form of an inner product between the input and the output of the system, i.e., $s(u, y) = y^\top u$. We then have the following definition.

Definition 4.4 ((Cyclo-)Passive system). *A square (i.e., $m = p$) dynamical system Σ is said to be (cyclo-)passive if it is (cyclo-)dissipative with respect to the supply rate $s(u, y) = y^\top u$.*

On the latter, a special class of nonlinear physical systems has been built, i.e., port-Hamiltonian systems, as described in the following subsection.

Note that, for the sake of simplicity, in the port-Hamiltonian formalism discussed in the following (see also Section 8) we will employ the differential form of the dissipation inequality (16), i.e., if S is C^1 then (16) can be written as $\dot{S}(x) \leq s(u(t), y(t))$.

4.1. Port-Hamiltonian systems

The port-Hamiltonian approach to modeling and control of complex physical systems is a well-established framework, and it started/pioneered with the work by [35], [36], see also [37, 32, 38] for a general overview of the topic including also control techniques. A particular feature of port-Hamiltonian systems is that of describing all the main physical properties of the system under consideration, such as energy dissipation, passivity, and power conservation laws. Moreover, the formalism is particularly well suited for the interconnection of physical systems, preserving the passivity, stability, and structure in a larger port-Hamiltonian system, [39].

The concept of passive system is explicitly expressed by the dynamical equations of such a class of nonlinear systems. In particular, a nonlinear system Σ is port-Hamiltonian when its dynamical equations present a predefined structure of the form

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \nabla H(x) + g(x)u \\ y &= g(x)^\top \nabla H(x) \end{aligned} \quad (17)$$

where J, R, H, g are as described above. The particular structure of y defines an output that is power-conjugated to the input u ,

i.e., their inner product provides an equivalent power quantity. We say that the passivity concept is explicitly expressed by the dynamics of a port-Hamiltonian system (17) because, via the energy balance, we retrieve the differential form of the dissipation inequality (16), i.e.,

$$\begin{aligned}\dot{H}(x) &= \nabla H^\top(x)(J(x) - R(x))\nabla H(x) + \nabla^\top H(x)g(x)u \\ &= -\nabla H^\top(x)R(x)\nabla H(x) + y^\top u \leq y^\top u.\end{aligned}$$

Thus the system dynamics explicitly satisfies the dissipation inequality of a passive system if H is lower bounded, or of a cyclo-passive system otherwise.

5. Additional notions of dissipativity

Each definition of stability introduced in Section 3 can be extended to a dissipativity definition.

5.1. Incremental dissipativity

Incremental dissipativity is considered by [10] and partly by [12], as a methodology to study synchronization of nonlinear oscillators. A more recent result can be found in [40].

Definition 5.1 (Incremental Dissipativity [40]). *The system (12) is called incrementally dissipative w.r.t. the supply function $\mathfrak{s} : \mathcal{U} \times \mathcal{U} \times \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$, if there exists an incremental storage function $\mathfrak{S} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, with $\mathfrak{S}(x_1, x_2) = 0$ implies $x_1 = x_2$, such that for any two state-input-output trajectories $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, it holds that*

$$\begin{aligned}\mathfrak{S}(x_1(t_1), x_2(t_1)) - \mathfrak{S}(x_1(t_0), x_2(t_0)) \\ \leq \int_{t_0}^{t_1} \mathfrak{s}(u_1(t), u_2(t), y_1(t), y_2(t))dt\end{aligned}\quad (18)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 \geq t_0$.

5.2. Differential dissipativity

A strictly related concept to contraction theory is that of differential dissipativity. As the contraction theory gives tools for studying the neighborhood of autonomous systems trajectories, analogously the notion of differential dissipativity can be employed as a tool to study the variational state-input-output behavior.

The theory of differential dissipativity was thus first developed in [41] in which the authors introduce the concept of differential storage function $\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R}$ and of differential supply rate $\mathcal{f} : \mathcal{TU} \times \mathcal{TU} \rightarrow \mathbb{R}$, so to ‘lift’ the standard dissipativity notion to the system’s manifold tangent bundle, when employing the state trajectory $x(t)$ as an exogenous signal. We recall the definitions of differential dissipativity given in [41] (see also [40] for a more recent development), here extended to the weaker notion of differential cyclo-dissipativity.

Definition 5.2 (Diff. (cyclo-) dissipative system). *The dynamical system Σ is said to be differentially (cyclo-) dissipative with respect to the differential supply rate $\mathcal{f} : \mathcal{TX} \times \mathcal{TU} \times \mathcal{TU} \rightarrow \mathbb{R}$*

if there exists a differential storage function $(\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R})$ $\mathcal{S} : \mathcal{TX} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\mathcal{S}(x(t_1), \xi(t_1)) - \mathcal{S}(x(t_0), \xi(t_0)) \leq \int_{t_0}^{t_1} \mathcal{f}(x(t), u(t), v(t), y(t), \zeta(t))dt\quad (19)$$

for all $t_1, t_0 \in \mathbb{R}$, $t_1 \geq t_0$, and for any admissible state-input-output trajectory of $\mathcal{T}\Sigma$, i.e., $(x, \xi, u, v, y, \zeta) \in \mathcal{TX} \times \mathcal{TU} \times \mathcal{TU}$.

By exploiting the assumption that $\mathcal{S} \in C^1$, (19) is equivalent to

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \mathcal{f}(x(t), u(t), v(t), y(t), \zeta(t))\quad (20)$$

for all admissible state-input-output trajectories (x, ξ, u, v, y, ζ) of $\mathcal{T}\Sigma$.

6. Additional notions of passivity

Along with the different definitions of dissipativity we find the corresponding definitions of passivity.

6.1. Incremental passivity

The incremental passivity property instead refers to the passivity characterization of the system with state, input, and output signals substituted by their increments (see [42]), i.e., $\tilde{x} = x_1(t) - x_2(t)$, $\tilde{u} = u_1(t) - u_2(t)$, and $\tilde{y} = y_1(t) - y_2(t)$, where $x_i(t)$ is the trajectory at time t initialized in x_{i0} at time t_0 , influenced by the input signal $u_1(\tau)$, for $t \in [t_0, t)$, and y_i its relative output $y_i(t) = h(x_i(t))$, $i = 1, 2$. Some applications can be found in [43] for controller design of convex gradient systems, or output regulation as in [12].

Definition 6.1 (Incremental Passivity [12]). *A square (i.e., $m = p$) dynamical system (12) is called incrementally passive if there exists an incremental storage function $\mathfrak{S} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, with $\mathfrak{S}(x_1, x_2) = 0$ implying $x_1 = x_2$, such that for any two state-input-output trajectories $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, it holds that*

$$\begin{aligned}\mathfrak{S}(x_1(t_1), x_2(t_1)) - \mathfrak{S}(x_1(t_0), x_2(t_0)) \\ \leq \int_{t_0}^{t_1} (u_1(t) - u_2(t))^\top (y_1(t) - y_2(t))dt\end{aligned}\quad (21)$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 \geq t_0$.

6.2. Convergence passivity

We introduce the concept of convergence passivity as a particular case of incremental passivity.

Definition 6.2 (Convergence Passivity). *The system (12) is called passively convergent if there exists a unique state-input-output trajectory $(x^*(t), u^*(t), y^*(t))$, with $x^*(t)$ bounded in backward and forward time and an incremental storage function $\mathfrak{S}_* : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$, with $\mathfrak{S}_*(x(t)) = 0$ implies $x(t) = x^*(t)$,*

such that for any state-input-output trajectories $(x, u, y) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, it holds that

$$\begin{aligned} & \mathfrak{S}_\star(x(t_1)) - \mathfrak{S}_\star(x(t_0)) \\ & \leq \int_{t_0}^{t_1} (u(t) - u^\star(t))^\top (y(t) - y^\star(t)) dt \quad (22) \end{aligned}$$

for all $t_0, t_1 \in \mathbb{R}$ with $t_1 \geq t_0$.

6.3. Differential passivity

The concept of differential dissipativity has then been specialized to differential passivity in [44] and described in a geometric and coordinate-free framework in [45]. See [46] for an application to the control of flexible-joints robots. Thus, following the standard definition of passivity introduced above and lifted to the tangent bundle \mathcal{TX} , we have the following definition of differential passivity. We recall the definitions of differential passivity given in [44] (see also [40] for a recent development), here extended to the weaker notion of differential cyclo-passivity.

Definition 6.3 (Diff. (cyclo-) passivity [44]). *A square (i.e., $m = p$) dynamical system Σ is differentially (cyclo-) passive if it is differentially (cyclo-) dissipative with respect to the differential supply rate $\int(v, \zeta) = \zeta^\top v$, i.e., if there exists a differential storage function \mathcal{S} such that*

$$\mathcal{S}(x(t_2), \xi(t_2)) - \mathcal{S}(x(t_1), \xi(t_1)) \leq \int_{t_1}^{t_2} \zeta^\top(\tau) v(\tau) d\tau \quad (23)$$

for all $t_2 \geq t_1$ and all admissible state-input-output trajectory (x, ξ, u, v, y, ζ) to the prolonged system $\mathcal{T}\Sigma$.

Equivalently, with the additional assumption $\mathcal{S} \in C^1$, the differential passivity inequality can be written as

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \zeta(t)^\top v(t). \quad (24)$$

It is worth noting the similarity with the definition of strictly Krasovskii's passivity in [47, Def. 3.1], that however is a particular case of the differential passivity defined here and in [44].

The definition of differential passivity given in [41] is different from the one given in [44], [45], and the one considered here. In particular, there the supply rate for passivity is given by $\int = \zeta^\top W(x)v$ for some metric $W(x)$ and the differential storage function has the more general meaning of a Finsler metric (see also [48] for an application to exponential stability of nonlinear systems).

In the remainder of the paper, we focus our attention on the particular direction, on the tangent bundle, determined by the system vectorfield, see some motivations in the appendix. In this case, the differential dissipativity notion earns a more concrete and tangible meaning, where $\xi(t) = \dot{x}(t)$, $v(t) = \dot{u}(t)$, and $\zeta(t) = \dot{y}(t)$.

With this particular choice, the differential dissipative inequality (19), reads as

$$\begin{aligned} & \mathcal{S}(x(t_1), \xi(t_1)) - \mathcal{S}(x(t_0), \xi(t_0)) \leq \\ & \int_{t_0}^{t_1} \int(x(t), u(t), \dot{u}(t), y(t), \dot{y}(t)) dt \end{aligned}$$

or equivalently, under the assumption $\mathcal{S} \in C^1$,

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \int(x(t), u(t), \dot{u}(t), y(t), \dot{y}(t)).$$

For the differential passivity notion we have instead

$$\mathcal{S}(x(t_1), \xi(t_1)) - \mathcal{S}(x(t_0), \xi(t_0)) \leq \int_{t_0}^{t_1} \dot{y}(t)^\top \dot{u}(t) dt$$

or equivalently, under the assumption $\mathcal{S} \in C^1$,

$$\dot{\mathcal{S}}(x(t), \xi(t)) \leq \dot{y}(t)^\top \dot{u}(t).$$

In order to make a direct connection to the contractivity property described in Section 3.3, we usually consider as differential storage function a Riemannian distance, i.e., the differential storage function \mathcal{S} as a quadratic form $\mathcal{S} = \xi^\top P(x)\xi$, with $P(x)$ being a positive definite matrix as given in Definition 3.6.

6.4. Maximal monotone passivity

A slightly different definition of incremental passivity has been adopted in [49, 29], where the proposed definition characterizes a class of port-Hamiltonian systems, i.e., incremental port-Hamiltonian, although such a definition is weaker than that of incrementally passive systems as defined in Section 6.1. In order to differentiate the two notions here we call the property of incremental port-Hamiltonian systems the *maximal monotone passivity*.

Definition 6.4 (Maximal monotone passivity [49]). *A square (i.e., $m = p$) dynamical system (12) is called maximally monotone passive if for any two state-input-output trajectories $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, it holds that*

$$\begin{aligned} & (\nabla H(x_1(t)) - \nabla H(x_2(t)))^\top (\dot{x}_1 - \dot{x}_2) \leq \\ & (y_1(t) - y_2(t))^\top (u_1(t) - u_2(t)) \quad (25) \end{aligned}$$

for all $t \geq t_0 \in \mathbb{R}$.

Remark 1. *This definition of passivity refers specifically to the increments of effort $\tilde{e} = \nabla H(x_1) - \nabla H(x_2)$ and flows $\tilde{f} = \dot{x}_1 - \dot{x}_2$ and it is related to the power of the input/output increments (which is a sort of power of the increments).*

We treat the two definition separately because they only coincide when $H(x)$ is a quadratic form in x up to a constant, i.e., $H(x) = x^\top Qx + c$ for some real c , as discussed in Section 8. Note that indeed, the two characterization of passivity come from different concepts. In particular, the latter has been introduced starting from the definition of maximal monotone relation that generalizes the concept of Dirac structure.

7. Stability notions of dissipative Hamiltonian systems

The different stability notions introduced in Section 3 can be applied to the case of dissipative Hamiltonian systems.

7.1. Incremental dissipative Hamiltonian systems

It is well-known that the concepts of incremental stability and contracting systems overlap [27], it is not absurd to find similar (if not the same) conditions to contractive systems when studying the incremental stability property.

Assumption 1. *There exists a positive real q such that $qI \leq \nabla^2 H(x)$ for all $x \in \mathcal{X}$ and $\nabla^2 H(x)$ is locally invertible for all $x \in \mathcal{X}$.*

Then the following holds.

Theorem 7.1. *Under Assumption 1, the dissipative Hamiltonian system (3) is incrementally stable with incremental Lyapunov function $\mathfrak{H}(x_1, x_2) = (x_1 - x_2)^\top \nabla^2 H(x_2)(x_1 - x_2)$, for any two system trajectories $x_1, x_2 \in \mathcal{X}$, if and only if*

$$\frac{d}{dt} \nabla^2 H(x) - 2\nabla^2 H(x)R'(x)\nabla^2 H(x) < 0 \quad (26)$$

for all $x \in \mathcal{X}$, where

$$R'(x) = R(x) - \text{sym} \left[\frac{\partial H}{\partial x} \left(\frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right) \nabla^2 H^{-1}(x) \right]. \quad (27)$$

Proof. The necessity part of the theorem comes from the definition of incremental stability, i.e., Definition 3.2. Here we focus on the sufficiency part. To prove the incremental stability property of an autonomous system (3), we need to consider two distinct trajectories x_i , $i = 1, 2$, both satisfying (3), and study the properties of convergence of one on the other. In order to do so, we introduce the incremental variable $\tilde{x} = x_1 - x_2$, whose dynamics is given by

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x}_1 - \dot{x}_2 \\ &= [J(x_1) - R(x_1)]\nabla H(x_1) - [J(x_2) - R(x_2)]\nabla H(x_2) \\ &= [J(\tilde{x} + x_2) - R(\tilde{x} + x_2)]\nabla H(\tilde{x} + x_2) - \\ &\quad [J(x_2) - R(x_2)]\nabla H(x_2). \end{aligned}$$

Consider now the case \tilde{x} is sufficiently small for all t , then its dynamics can be analyzed by exploiting the Taylor expansion. In particular, we can write

$$\begin{aligned} J(\tilde{x} + x_2)\nabla H(\tilde{x} + x_2) &= J(x_2)\nabla H(x_2) + \\ &\quad \frac{\partial J(x)\nabla H(x)}{\partial x}(x_2) \cdot [\tilde{x} + x_2 - x_2] + \text{h.o.t.} \end{aligned}$$

and analogously for the term in R . Thus, by neglecting the higher order terms, around the origin for \tilde{x} , its dynamics reads as

$$\begin{aligned} \dot{\tilde{x}} &\approx \left[\frac{\partial J(x)\nabla H(x)}{\partial x}(x_2) - \frac{\partial R(x)\nabla H(x)}{\partial x}(x_2) \right] \tilde{x} \\ &= \left[\nabla H^\top(x_2) \frac{\partial J(x)}{\partial x}(x_2) - \nabla H^\top(x_2) \frac{\partial R(x)}{\partial x}(x_2) \right] \tilde{x} + \\ &\quad [J(x_2) - R(x_2)]\nabla^2 H(x_2)\tilde{x} \\ &= [J'(x_2) - R'(x_2)]\nabla^2 H(x_2)\tilde{x}. \end{aligned}$$

To prove the local asymptotic stability of \tilde{x} we consider as incremental Lyapunov function $\mathfrak{H}(x_1, x_2) = \frac{1}{2}\tilde{x}^\top \nabla^2 H(x)\tilde{x}$, and by assumption its time derivative is strictly negative for all x (and thus also for all x_2), i.e., for $\tilde{x} \neq 0$

$$\dot{\mathfrak{H}}(x_1, x_2) = \frac{1}{2}\tilde{x}^\top \left(\frac{d}{dt} \nabla^2 H(x_2) - 2\nabla^2 H(x_2)R'(x_2)\nabla^2 H(x_2) \right) \tilde{x} < 0.$$

Moreover, any locally incrementally stable system is also globally incrementally stable [5] and we thus proved the theorem. \square

What if we use another incremental Lyapunov function candidate such as (33)?

Remark 2. *In the introduced framework, the case of convergent dissipative Hamiltonian systems will then be just a particular case of the incrementally stable system when considering x_1 to be a generic evolution of the autonomous system and x_2 to be the unique bounded solution $x^*(t)$ that exists for all $t \in \mathbb{R}$.*

7.2. Contractive and dissipative Hamiltonian systems

In the context of port-Hamiltonian systems, some results on contraction theory are available but only for the case of constant metric, see [50, 51, 52]. To the best of the authors' knowledge, none of the works on contractive port-Hamiltonian systems consider a nonlinear state-dependent metric, nor exploit the knowledge of the available Hamiltonian function. Only recently in [53], contraction and differential passivity were developed in the context of port-Hamiltonian systems by exploiting the Hamiltonian Hessian as state dependent contractive metric. In the remainder of this subsection we give a brief recap of the result in [53], reported here for the sake of completeness. In particular, we first show an original way to apply and exploit the contraction theory approach to dissipative Hamiltonian systems by defining the concept to dissipative differential Hamiltonian dynamics, i.e., the variational dynamics of (3) has again the structure of a dissipative Hamiltonian system.

Definition 7.1 ((Diss.) Diff. Hamiltonian system). *A dissipative Hamiltonian system (3) on the smooth manifold \mathcal{X} , is said differentially Hamiltonian if there exists a 'lifted' Hamiltonian function $\mathcal{H} : \mathcal{TX} \rightarrow \mathbb{R}$, a skew symmetric matrix $J' : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$ and a symmetric matrix $R' : \mathcal{X} \rightarrow \mathbb{R}^{n \times n}$, such that the variational dynamics of (3) reads as*

$$\dot{\xi} = (J'(x) - R'(x))\nabla_{\xi} \mathcal{H}(x, \xi), \quad (x, \xi) \in \mathcal{TX}. \quad (28)$$

Furthermore, we say that it is dissipative differentially Hamiltonian, if additionally \mathcal{H} is lower bounded and $\dot{\mathcal{H}} \leq 0$ along the flow of prolonged system \mathcal{TS} .

We then provide the conditions under which a dissipative Hamiltonian system is also dissipative differential Hamiltonian.

Theorem 7.2. *Under Assumption 1, a dissipative Hamiltonian system (3) is dissipative differentially Hamiltonian with respect to the Hamiltonian function $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$, if and only if for all $x \in \mathcal{X}$,*

$$\frac{d}{dt} \nabla^2 H(x) - 2\nabla^2 H(x)R'(x)\nabla^2 H(x) \leq 0 \quad (29)$$

where

$$R'(x) = R(x) - \text{sym} \left[\frac{\partial H}{\partial x} \left(\frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right) \nabla^2 H^{-1}(x) \right]. \quad (30)$$

Proof. We first show that under the assumption of everywhere locally invertibility of Hamiltonian Hessian $\nabla^2 H(x)$, any dissipative Hamiltonian system (3) is differentially Hamiltonian with Hamiltonian function

$$\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi.$$

We thus compute the variational system of (3), i.e., by defining $\xi = \dot{x}$, we write its dynamics as

$$\begin{aligned} \dot{\xi} &= (J(x) - \hat{R}(x)) \nabla H(x) + (J(x) - R(x)) \check{\nabla} H(x) \\ &= \frac{\partial H}{\partial x} \left[\frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right] \xi + (J(x) - R(x)) \nabla^2 H(x) \xi \\ &= (J'(x) - R'(x)) \nabla^2 H(x) \xi, \end{aligned}$$

where R' is as defined in (30) and

$$J'(x) = J(x) - \text{skew} \left[\frac{\partial H}{\partial x} \left(\frac{\partial^\top J}{\partial x} - \frac{\partial^\top R}{\partial x} \right) \nabla^2 H^{-1}(x) \right].$$

We now show that the time derivative of $\mathcal{H}(x, \xi) = \frac{1}{2} \xi^\top \nabla^2 H(x) \xi$ is nonpositive under condition (29), since it is given by

$$\dot{\mathcal{H}} = \frac{1}{2} \xi^\top \left(\frac{d}{dt} \nabla^2 H(x) \right) \xi - \xi^\top \nabla^2 H(x) R'(x) \nabla^2 H(x) \xi$$

thus condition (29) provides the necessary and sufficient to guarantee $\dot{\mathcal{H}} \leq 0$ along the prolonged system's trajectory $(x(t), \xi(t)) \in \mathcal{TX}$. \square

From the above theorem, one can notice that every dissipative Hamiltonian system (3) is differentially Hamiltonian with a Hamiltonian function $\mathcal{H}(x, \xi)$, but in order to be also dissipative, the structural matrices must satisfy the additional requirement (29).

In the remainder of this subsection, we link the introduced definition of dissipative differential Hamiltonian system to the notion of contractive dynamics. In order to do so, we exploit the Hessian of the Hamiltonian function H as a contraction metric (called P in Definition 3.6), and we thus introduce the following assumption.

Assumption 2. *There exist positive real values \underline{q} and \bar{q} , such that*

$$\underline{q}I \leq \nabla^2 H \leq \bar{q}I, \quad \forall x \in \mathcal{X}.$$

Then the following corollary is immediate.

Corollary 7.1. *Under Assumption 2, system (3) is contractive on \mathcal{X} with metric $\nabla^2 H(x)$ under Assumption 2, if and only if it is dissipative differentially Hamiltonian and (29) holds with strict inequality for all $x \in \mathcal{X}$.*

The sufficient part of the proof is just an application of Theorem 7.2, while the necessary part comes from the definition of Contractive systems, we thus omit the proof of this corollary.

Remark 3. *This result is not surprising because the notions of incremental stability and contractivity are equal for autonomous systems, what changes is their correlated notions of passivity, as will be detailed in next section.*

7.3. Maximal monotone stability

For the maximal monotone stability we have the following Lemma that comes directly from the definition.

Lemma 1. *The dissipative Hamiltonian system (3) is maximally monotone stable if and only if*

$$\left[\begin{array}{cc} R_1 & \frac{(J_1 - R_1)^\top + (J_2 - R_2)}{2} \\ \frac{(J_1 - R_1) + (J_2 - R_2)^\top}{2} & R_2 \end{array} \right] \geq 0 \quad (31)$$

for all $x_1, x_2 \in \mathcal{X}$ solution of (3).

Proof. By considering the structure of a dissipative Hamiltonian system $\dot{x}_i = (J_i - R_i) \nabla H_i$, for $i = 1, 2$, we can check the stability conditions from Definition 3.8, we have

$$\begin{aligned} (\nabla H_1 - \nabla H_2)^\top (\dot{x}_1 - \dot{x}_2) &= -\nabla H_1^\top R_1 \nabla H_1 - \nabla H_2^\top R_2 \nabla H_2 \\ &\quad - \nabla H_1^\top (J_2 - R_2) \nabla H_2 - \nabla H_2^\top (J_1 - R_1) \nabla H_1 \leq 0 \end{aligned}$$

providing

$$-\left[\nabla H_1^\top \quad \nabla H_2^\top \right] \begin{bmatrix} R_1 & J_2 - R_2 \\ J_1 - R_1 & R_2 \end{bmatrix} \begin{bmatrix} \nabla H_1 \\ \nabla H_2 \end{bmatrix} \leq 0$$

which is satisfied under the condition (31). \square

It is easy to see that a sufficient condition for (31) to be satisfied for any pair of trajectory x_1, x_2 , is that $J(x_1) = J(x_2) = J$ and $R(x_1) = R(x_2) = R$.

8. Passivity notions in pHs formalism

When talking about passivity we need to consider the inputs and outputs interfaces with the environment, such as the input-output ports in the port-Hamiltonian system dynamics (17). In particular, as already shown before, the ports signals, i.e., the input u and the output y , are power conjugated and they lead to an intrinsic passivity property of this class of systems. However, the system dynamics, although passive in the standard sense, may not satisfy the conditions of the other passivity notions introduced in Section 6. In the following, we study such particular cases in the port-Hamiltonian formalism.

8.1. Incremental passivity for port-Hamiltonian systems

The incremental passivity property comes directly by checking the passivity condition on the increments of the state $\tilde{x} = x_1 - x_2$, of the input $\tilde{u} = u_1 - u_2$, and of the output $\tilde{y} = y_1 - y_2$. Although port-Hamiltonian systems are passive by construction (it is actually a necessary requirement for a system to be port-Hamiltonian), they may not be incrementally passive. In particular, for port-Hamiltonian systems we have $\dot{H}(x) = \partial H(x)\dot{x} \leq y^\top u$ and to obtain the incremental property we provide the condition in terms of two general trajectories $x_1, x_2 \in \mathcal{X}$, i.e., for any two trajectories $x_1, x_2 \in \mathcal{X}$ we seek conditions under which there exists a function $\mathfrak{H} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ such that

$$\dot{\mathfrak{H}} = \partial_{x_1} \mathfrak{H}(x_1, x_2)\dot{x}_1 + \partial_{x_2} \mathfrak{H}(x_1, x_2)\dot{x}_2 \leq \tilde{y}^\top \tilde{u}. \quad (32)$$

We then have the following theorem

Theorem 8.1. *A port-Hamiltonian system (17) is incrementally passive with respect to the incremental storage function*

$$\mathfrak{H}(x_1, x_2) = H(x_1) + H(x_2) + h(x_1, x_2), \quad (33)$$

for some $h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ if and only if? for any two distinct state-input-output trajectories $(x_i(t), u_i(t), y_i(t)) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, $i = 1, 2$, there exists a coupling storage function $h(x_1, x_2)$ such that

$$\begin{aligned} \frac{\partial h(x_1, x_2)}{\partial x_1} &= -\frac{\partial H(x_2)}{\partial x_2} = -\nabla H^\top(x_2) \\ \frac{\partial h(x_1, x_2)}{\partial x_2} &= -\frac{\partial H(x_1)}{\partial x_1} = -\nabla H^\top(x_1), \end{aligned} \quad (34)$$

the input matrix is constant $g(x_1) = g(x_2) = g$ and (31) holds.

Proof. According to the definition of incremental passivity, there must exist a storage function $\mathfrak{H}(x_1, x_2)$ such that for any trajectories x_i , $i = 1, 2$, with associated inputs u_i and outputs y_i , $i = 1, 2$, we have (32). In particular, we have³

$$\begin{aligned} \dot{\mathfrak{H}}(x_1, x_2) &= \frac{\partial \mathfrak{H}(x_1, x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial \mathfrak{H}(x_1, x_2)}{\partial x_2}\dot{x}_2 \\ &\leq \tilde{y}^\top \tilde{u} = y_1^\top u_1 + y_2^\top u_2 - y_1^\top u_2 - y_2^\top u_1 \\ &= \nabla H_1^\top g_1 u_1 + \nabla H_2^\top g_2 u_2 - \\ &\quad \nabla H_1^\top g_1 u_2 - \nabla H_2^\top g_2 u_1. \end{aligned} \quad (35)$$

We prove the sufficiency part of the theorem by imposing the incremental storage function form in (33) and substituting it in the inequality above yields

$$\begin{aligned} \frac{\partial \mathfrak{H}(x_1, x_2)}{\partial x_1}\dot{x}_1 + \frac{\partial \mathfrak{H}(x_1, x_2)}{\partial x_2}\dot{x}_2 &= \\ &- \nabla H_1^\top R_1 \nabla H_1 + \nabla H_1^\top g_1 u_1 - \nabla H_2^\top R_2 \nabla H_2 + \nabla H_2^\top g_2 u_2 + \\ &\frac{\partial h(x_1, x_2)}{\partial x_1}(J_1 - R_1)\nabla H_1 + \frac{\partial h(x_1, x_2)}{\partial x_1}g_1 u_1 + \\ &\frac{\partial h(x_1, x_2)}{\partial x_2}(J_2 - R_2)\nabla H_2 + \frac{\partial h(x_1, x_2)}{\partial x_2}g_2 u_2 \\ &\leq \nabla H_1^\top g_1 u_1 + \nabla H_2^\top g_2 u_2 - \nabla H_1^\top g_1 u_2 - \nabla H_2^\top g_2 u_1. \end{aligned}$$

³For the sake of readability, we use the subscript $i = 1, 2$ to mean that we consider that matrix-valued function in x_1 and x_2 , respectively, i.e., $\nabla H_i = \nabla H(x_i)$, $J_i = J(x_i)$, $R_i = R(x_i)$, and $g_i = g(x_i)$, $i = 1, 2$.

Under the conditions $g(x) = g$ and (34), the inequality reduces to

$$\begin{aligned} &- \nabla H_1^\top R_1 \nabla H_1 - \nabla H_2^\top R_2 \nabla H_2 - \\ &\quad \nabla H_2^\top (J_1 - R_1)\nabla H_1 - \nabla H_1^\top (J_2 - R_2)\nabla H_2 \leq 0 \end{aligned}$$

which is equivalent to condition (31).

Necessity part follows by contradiction to have an incremental storage function \mathfrak{H} that does not depend on the input. In particular, due to the output map of a port-Hamiltonian system, in inequality (35), after the appropriate simplifications of the terms $y_1^\top u_1$ and $y_2^\top u_2$, in order to lose the dependence on the input signals u_1 and u_2 , it must hold that $g(x_1) = g(x_2)$ for any trajectories pair (x_1, x_2) , i.e., $g(x) = g$, and that $h(x_1, x_2)$ satisfies (34). We thus proved the theorem. \square

In general, it is not easy to obtain a solution for condition (33) although it is trivially satisfied for $H(x) = \frac{1}{2}x^\top Qx + c$, for some real c .

Somehow, the result of this theorem provides some answers to the questions recently posed in [42] about the incremental dissipativity of physical systems, for the particular case of passivity. With the port-Hamiltonian formalism, we can treat a large class of physical systems with input and output conjugated signals. To obtain incremental passivity with respect to the incremental storage function (33) for such class of passive systems, we need to introduce additional and limiting conditions, such as those on the input matrix, on the Hamiltonian function, as well as on the system's structural matrices.

Remark 4. *The conditions to get convergence passivity, i.e., Definition 6.2 are the same as for incremental stability just discussed, with the only difference that the state-input-output trajectory $(x_2, u_2, y_2) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$ should be considered as the state-input-output trajectory $(x^*(t), u^*(t), y^*(t)) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$ defined for all $t \in \mathbb{R}$.*

8.2. Differential Passivity for port-Hamiltonian systems

In order to discuss differential passivity we have to lift the port-Hamiltonian system onto the tangent bundle \mathcal{TX} , thus obtaining the prolonged system on the tangent bundle \mathcal{TX} , as also described in the appendix.

Hence, to deal with the differential passivity properties of (17), we focus our attention on its variational dynamics, i.e., the differential Hamiltonian system with the additional input vectorfield $(g(x)u)$ variation, augmented with the variation of the output terms as given in \dot{y} . We thus consider the prolonged port-Hamiltonian system

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\nabla H(x) + g(x)u, \\ y &= g(x)^\top \nabla H(x), \quad x(0) = x_0 \\ \dot{\xi} &= (J'(x) - R'(x))\nabla^2 H(x)\xi + \dot{g}(x)u + g(x)\dot{u}, \\ \xi(0) &= (J(x_0) - R(x_0))\nabla H(x_0) + g(x_0)u(0), \\ \dot{y} &= \dot{g}(x)^\top \nabla H(x) + g(x)^\top \nabla^2 H(x)\xi \end{aligned} \quad (36)$$

where $\xi(t) \in \mathcal{TX}$ here refers to the controlled system vectorfield, $\xi(t) = \dot{x}(t)$.

We then have the following conditions to guarantee differential passivity for (17) with storage function $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$.

Theorem 8.2. *Consider the port-Hamiltonian system (17) on the smooth manifold \mathcal{X} . It is differentially passive with storage function $\mathcal{H} : \mathcal{TX} \rightarrow \mathbb{R}$, $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$, if and only if the input matrix $g(x)$ is constant and the following inequality⁴ holds for all $x \in \mathcal{X}$*

$$\frac{d\nabla^2 H(x)}{dt} - 2\nabla^2 HR'(x)\nabla^2 H \leq 0. \quad (37)$$

Proof. To prove passivity with storage function $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$, one can first notice that the variational dynamics with state-input-output trajectory (ξ, \dot{u}, \dot{y}) has a port-Hamiltonian structure with Hamiltonian function $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$ and input-output conjugate pair if and only if $\dot{g}(x) = 0$. However, to guarantee differential passivity, a sufficient condition is that the equivalent differential Hamiltonian system is dissipative, i.e., inequality (37) holds, so that

$$\dot{\mathcal{H}} = \frac{1}{2}\xi^\top \left[\frac{d\nabla^2 H(x)}{dt} - 2\nabla^2 H(x)R'(x)\nabla^2 H(x) \right] \xi + \xi^\top \nabla^2 H(x)g(x)\dot{u} \leq \dot{y}^\top \dot{u}.$$

This proves the sufficient part.

To obtain the necessary part we find conditions to satisfy

$$\dot{\mathcal{H}} \leq \dot{y}^\top \dot{u}$$

for all admissible $(x, \xi, u, \dot{u}, y, \dot{y}) \in \mathcal{TX} \times \mathcal{TU} \times \mathcal{TY}$. The time derivative of $\mathcal{H}(x, \xi)$ along the prolonged dynamics reads as

$$\begin{aligned} \dot{\mathcal{H}} &= \frac{1}{2}\xi^\top \left[\frac{d\nabla^2 H(x)}{dt} - 2\nabla^2 HR'(x)\nabla^2 H \right] \xi + \\ &\quad + \xi^\top \nabla^2 H(x)g(x)\dot{u} + \xi^\top \nabla^2 H(x)\dot{g}(x)u \\ &= \frac{1}{2}\xi^\top \left[\frac{d\nabla^2 H(x)}{dt} - 2\nabla^2 HR'(x)\nabla^2 H \right] \xi + \\ &\quad + \dot{y}^\top \dot{u} - \nabla H(x)^\top \dot{g}(x)\dot{u} + \xi^\top \nabla^2 H(x)\dot{g}(x)u \end{aligned}$$

that is, in order to fit in the definition,

$$\begin{aligned} \frac{1}{2}\xi^\top \left[\frac{d\nabla^2 H(x)}{dt} - 2\nabla^2 HR'(x)\nabla^2 H \right] \xi + \\ - \nabla H(x)^\top \dot{g}(x)\dot{u} + \xi^\top \nabla^2 H(x)\dot{g}(x)u \leq 0 \end{aligned}$$

must hold for all admissible (x, ξ, u, \dot{u}) , this implies that $\dot{g}(x) = 0$ and that inequality (37) hold. Thus proving the theorem. \square

With this approach, we let the differential Hamiltonian $\mathcal{H}(x, \xi) = \frac{1}{2}\xi^\top \nabla^2 H(x)\xi$ play the role of a differential storage function, denoted by \mathcal{S} in the Sections 5 and 6, under appropriate conditions. We refer to this property to be a ‘differential’ passivity-preservation, since the role of \mathcal{H} , in the differential framework, is the analogous of the standard role played

by the system Hamiltonian H for passivity in standard port-Hamiltonian systems. Moreover, when we compare our result to the conditions in [45, eq. (16)], we notice that our conditions are in a sense weaker because we do not require the storage function \mathcal{S} to have a zero Lie derivative along the complete lift of the input vectorfield $g(x)$. On the other hand, some limitations on the input class might be considered so to satisfy the inequality (37). However, when the Hessian of H is constant, such a condition is easier to satisfy as shown in the following illustrative example.

8.3. Maximal monotone passivity

A slightly different definition of incremental passivity, in the framework of port-Hamiltonian Systems, has been introduced in [29] and the condition to be satisfied in this case is described by the relationship (25).

Theorem 8.3. *System (17) is maximally monotone passive if and only if $g(x) = g$ and for any two distinct state-input-output trajectories $(x_i, u_i, y_i) \in \mathcal{X} \times \mathcal{U} \times \mathcal{Y}$, $i = 1, 2$, of the system dynamics, (31) holds.*

Proof. As for the incremental passivity, we introduce two general tuples (x_i, u_i, y_i) , $i = 1, 2$, both satisfy the system dynamics (17). By considering $H(x_i)$, $i = 1, 2$, as storage functions in (25), we can directly check the conditions to obtain passivity, i.e., (25) reads as

$$\begin{aligned} (\nabla H_1 - \nabla H_2)^\top [(J_1 - R_1)\nabla H_1 + g_1 u_1 - (J_2 - R_2)\nabla H_2 - g_2 u_2] = \\ = \nabla H_1^\top (J_1 - R_1)\nabla H_1 + \nabla H_1^\top g_1 u_1 - \nabla H_1^\top (J_2 - R_2)\nabla H_2 \\ - \nabla H_1^\top g_2 u_2 - \nabla H_2^\top (J_1 - R_1)\nabla H_1 - \nabla H_2^\top g_1 u_1 + \\ \nabla H_2^\top (J_2 - R_2)\nabla H_2 + \nabla H_2^\top g_2 u_2 = \\ = -(\nabla H_1^\top \quad \nabla H_2^\top) \begin{bmatrix} R_1 & J_2 - R_2 \\ J_1 - R_1 & R_2 \end{bmatrix} \begin{pmatrix} \nabla H_1 \\ \nabla H_2 \end{pmatrix} + y_1^\top u_1 + \\ y_2^\top u_2 - \nabla H_1^\top g_2 u_2 - \nabla H_2^\top g_1 u_1. \end{aligned}$$

In order for the right hand side of the equation to be less than $(y_1 - y_2)^\top (u_1 - u_2)$ for all (x_i, u_i, y_i) , $i = 1, 2$, it must be

$$\text{sym} \begin{bmatrix} R_1 & J_2 - R_2 \\ J_1 - R_1 & R_2 \end{bmatrix} \geq 0$$

for any trajectories x_1 and x_2 , i.e., the condition (31), and that $g_1 = g_2 = g$. Indeed, for $g_1 = g_2 = g$ we have

$$y_1^\top u_1 + y_2^\top u_2 - \nabla H_1^\top g u_2 - \nabla H_2^\top g u_1 = (y_1 - y_2)^\top (u_1 - u_2).$$

And thus we proved the theorem. \square

The example considered in [29] actually shows that the incremental passivity is different from maximal monotone passivity, as also discussed in its final version compared to the archive version [54]. Indeed, the system

$$\begin{aligned} \dot{x} &= u \\ y &= \nabla H(x) \end{aligned}$$

is maximally monotone passive for any differentiable H but it is not incremental passive.

⁴Or equivalently one can check $\frac{d}{dt}\nabla^2 H(x)^{-1} + 2R'(x) \geq 0$.

Remark 5. *It is unavoidable to see the link between the incremental and maximal monotone passivity. Although the two definitions describe in general two different concepts, they are equivalent whenever $H(x) = x^T Qx + c$ for some real c . Moreover, with such a Hamiltonian function, the additional conditions to be satisfied are then equal, i.e., $g(x_1) = g(x_2) = g$ and the positive semidefiniteness of the coupling matrix as in (31).*

9. Conclusions

In this paper, we introduced different notions of stability under an unified framework so to better compare them. We then introduced the concept of dissipative and passive systems in the classical sense and the different definitions, such as incremental, convergent, and differential dissipativity/passivity, so to extend the definitions of stability to open systems. We also introduce the port-Hamiltonian formalism, i.e., nonlinear open systems whose Hamiltonian function plays the role of storage function satisfying the standard passivity property.

For this class of systems, we determine the additional conditions to be satisfied in order to fit the other definitions of passivity (or stability in the case of dissipative Hamiltonian systems). We were able to a priori determine some properties of the storage functions, or to obtained it, from the system's Hamiltonian.

The conclusion of the work is that, by implying the port-Hamiltonian formalism (which describes a class of passive system with a well-known storage function in the classical sense), the notions of passivity are in general all different since different conditions must be employed in order to fit in each definition. While, instead, for the case of linear systems all definitions are equivalent.

Appendix A. Variational and prolonged system

As introduced in [55] and exploited in [27], we show how a system Σ , defined on the smooth manifold \mathcal{X} , can be extended/prolonged (or 'lifted' as mentioned in [27]) to a system on the $2n$ -dimensional tangent bundle \mathcal{TX} of the manifold \mathcal{X} , with $2m$ inputs and $2p$ outputs.

Given an admissible state-input-output trajectory $t \rightarrow (x, u, y)$ for Σ , the *variational system* along such a trajectory is given by the following time-varying system

$$D\Sigma : \begin{cases} \dot{\xi} = \frac{\partial f}{\partial x}(x, u)\xi + \frac{\partial f}{\partial u}(x, u)v \\ \zeta = \frac{\partial h}{\partial x}(x, u)\xi + \frac{\partial h}{\partial u}(x, u)v \end{cases}$$

with state $\xi \in \mathbb{R}^n$, where $v = (v_1, \dots, v_m)$ and $\zeta = (\zeta_1, \dots, \zeta_p)$ denote the inputs and the outputs of the variational system. As highlighted in [45], the terminology 'variational' system comes from considering the infinitesimal variations of a family of state-input-output trajectories of Σ , i.e., $(u(t, \varepsilon), y(t, \varepsilon), x(t, \varepsilon))$ parameterized by $\varepsilon \in (-\delta, \delta)$, with $x(t, 0) = x(t)$, $u(t, 0) = u(t)$, and $y(t, 0) = y(t)$. Thus the infinitesimal variations

$$\xi(t) = \frac{\partial x}{\partial \varepsilon}(t, 0), \quad v(t) = \frac{\partial u}{\partial \varepsilon}(t, 0), \quad \zeta(t) = \frac{\partial y}{\partial \varepsilon}(t, 0),$$

satisfy the dynamical equations of $D\Sigma$. In other words, the state-input-output variational trajectory (ξ, v, ζ) describes any 'virtual displacement' of the state-input-output trajectory of Σ . The prolonged system of Σ then corresponds to the augmented $2n$ -dimensional system on the tangent bundle \mathcal{TX} composed of Σ together with $D\Sigma$, i.e., the system dynamics

$$\mathcal{TS} : \begin{cases} \dot{x} = f(x, u), & x(0) = x_0 \\ y = h(x, u) \\ \dot{\xi} = \frac{\partial f}{\partial x}(x, u)\xi + \frac{\partial f}{\partial u}(x, u)v \\ \zeta = \frac{\partial h}{\partial x}(x, u)\xi + \frac{\partial h}{\partial u}(x, u)v. \end{cases}$$

Note that in the standard notation, as exploited e.g. in [45], $\xi = \delta x$, $v = \delta u$, and $\zeta = \delta y$. Furthermore, in [27], in the case of closed system, the variational state $\xi = \delta x$ refers to the tangent vector to the parameterized curve connecting any two system trajectories. The infinitesimal variations $\delta x(t)$ on the state $x(t)$ can also be interpreted as being a generic 'virtual displacement' along any possible direction on the tangent space $\mathcal{T}_x\mathcal{X}$. However, if we specialize such a direction to be the one of the 'state velocity', i.e., we obtain the dynamics of $\xi = \dot{x}$, we describe the time evolution of the vectorfield $f(x, u)$. By doing so, we can properly determine the initial conditions of the variational system, i.e., $\xi(0) = f(x_0, u(0))$. This choice of the variational system direction, recalls and extends the definition of Forward Contraction introduced in [27, sec. III.C].

It has been shown in [15, Prop. 1] that if the vectorfield $f(x)$ is globally Lipschitz with bounded second derivative, the two concepts of contractive and incrementally stable systems are *equivalent*. Additionally, they are both equivalent, see [15, Prop. 1], to the global exponential attractiveness of the manifold $\mathcal{E} = \{(x, \xi) : \xi = 0\}$ for the prolonged systems of $\dot{x} = f(x)$, i.e.,

$$\dot{x} = f(x), \quad \dot{\xi} = \frac{\partial f}{\partial x}(x)\xi.$$

By the characterization we gave to the virtual state ξ in the previous section, i.e., considering $\xi = \dot{x}$, we can additionally conclude that the manifold \mathcal{E} , as introduced in [15, Property P2], is the set of all equilibrium points of the autonomous system $\dot{x} = f(x)$. As a consequence, because the contraction and the incremental stability property are assumed to hold globally in [15], we can conclude that there is a single attractive equilibrium point, i.e., $\mathcal{E} = \{(\bar{x}, 0)\}$ where \bar{x} is such that $f(\bar{x}) = 0$. Whenever the system is not incrementally stable, then the invariant manifold \mathcal{E} refers to (possibly) a continuum of attractive equilibrium points.

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