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# $L^p$ -estimates, controllability and local-well posedness for backward SPDEs

Víctor Hernández-Santamaría\*      Kévin Le Balc'h†      Liliana Peralta‡

June 27, 2024

## Abstract

In this paper, we study linear backward parabolic SPDEs and present new a priori estimates for their weak solutions. Inspired by the seminal work of Y. Hu, J. Ma and J. Yong from 2002 on strong solutions, we establish  $L^p$ -estimates requiring minimal assumptions on the regularity of the coefficients, the terminal data, and the external force. To this end, we derive a new Itô's formula for the  $L^p$ -norm of the solution, extending the classical result in the  $L^2$ -setting. This formula is then used to improve further the regularity of the first component of the solution up to  $L^\infty$ . Additionally, we present applications such as the controllability of backward SPDEs with  $L^p$ -controls and a local existence result for a semilinear equation without imposing any growth condition on the nonlinear term.

*Keywords:* Backward SPDEs, weak solutions, Itô's formula, controllability, Banach fixed-point.

## 1 Introduction

### 1.1 Notation and main results

Backward stochastic partial differential equations (BSPDEs, for short) arise in many applications of probability theory and stochastic processes. In the literature, BSPDEs can be found in various contexts, including control under incomplete information (see [Ben83, Tan98]), as adjoint equations in optimal control (see e.g. [NN90, HP91, Zho93]), in controllability problems (see e.g. [BRT03, TZ09, HSLBP23]), in mathematical finance for the formulation of the stochastic Feynman-Kac formula and the Black-Scholes equation (see [MY97, MY99]), among other applications.

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space on which is defined a one-dimensional standard Brownian motion  $W = \{W(t) : t \geq 0\}$  and let  $\{\mathcal{F}_t\}_{t \geq 0}$  be the natural filtration generated by  $W$ , augmented by all the  $\mathbb{P}$ -null sets in  $\mathcal{F}$ . For any given (deterministic) time  $T > 0$ , let us consider the following linear BSPDE

$$\begin{cases} dy = -(\Delta y + \alpha y + \beta Y + F) dt + Y dW(t) & \text{in } (0, T) \times \mathcal{O}, \\ y = 0 & \text{on } (0, T) \times \partial\mathcal{O}, \\ y(T) = y_T & \text{in } \mathcal{O}, \end{cases} \quad (1)$$

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where  $\mathcal{O} \subset \mathbb{R}^n$  is an open and bounded set with a  $C^2$  boundary  $\partial\mathcal{O}$ . In (1), the coefficients  $\alpha$  and  $\beta$ , the forcing term  $F$  and the terminal datum  $y_T$  are suitable random fields verifying appropriate measurability and regularity conditions.

A solution to (1) consists of a pair of random fields  $(y, Y)$  that are  $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted and satisfy (1) in a specified sense. At this point, it is important to note a fundamental difference from standard SPDEs: the noise term in a BSPDE like (1) arises from the intrinsic randomness in the coefficients and terminal data—making it endogenous—and comes from martingale representation theorems.

A classical question for BSPDEs consists in understanding the notion of the solution (for instance strong, mild, or weak in the PDE sense) and determining the best regularity available for the given data of the problem. This issue has been studied in the literature in different contexts and frameworks. In [HP91, LvN19], using semigroup theory, the authors establish the existence and uniqueness of mild solutions for abstract backward evolution equations in the  $L^2$  and  $L^p$  settings, respectively. In [HMY02, Dok12, DT12], the problem of existence and regularity of strong solutions (i.e., solutions that can be evaluated point-wisely in the spatial variable) is studied in bounded and/or unbounded domains. In the context of weak solutions (i.e., the PDE satisfies a scalar product equation), the works [DQT12, DTZ13] tackle the existence and regularity of solutions posed in the whole space within the  $L^p$ -setting and in higher order Sobolev spaces  $W^{m,p}$  ( $m \geq 1, p \geq 2$ ), respectively. Finally, existence and regularity of solutions in Hölder spaces are established in [TW16].

In this direction, the first goal of this paper is to establish  $L^p$ - and  $L^\infty$ -estimates for weak solutions to (1) (see Definition 1.1 below) without imposing additional regularity and/or differentiability assumptions on the coefficients, the terminal data or the source term. Our aim is to use natural hypotheses corresponding to the desired function space: for instance, obtaining  $L^p$  estimates requires  $L^p$  terminal datum and source term, and similarly,  $L^\infty$  solutions are obtained with  $L^\infty$  data.

To make this precise, we begin by introducing the following notations. In what follows, we denote  $\{\mathcal{F}_t\}_{t \geq 0}$  by  $\mathbb{F}$  unless we want to emphasize a precise  $\mathcal{F}_t$ . Let  $(X, \|\cdot\|_X)$  be a Banach space, for any  $p \in [2, +\infty)$  and any  $s \in \{0, T\}$ , we denote by  $L^p_{\mathcal{F}_s}(\Omega; X)$  the set of all  $\mathcal{F}_s$ -measurable  $X$ -valued random variables  $\xi : \Omega \rightarrow X$  such that  $\mathbb{E}(|\xi|_X^p) < +\infty$ . For any  $p, q \in [2, \infty)$  we define the spaces

$$L^p_{\mathbb{F}}(\Omega; L^q(0, T; X)) := \left\{ \psi : \Omega \times [0, T] \rightarrow X \mid \begin{aligned} &\psi(\cdot) \text{ is an } \mathbb{F}\text{-adapted process} \\ &\text{on } [0, T] \text{ and } \mathbb{E} \left[ \left( \int_0^T \|\psi(t)\|_X^q dt \right)^{\frac{p}{q}} \right] < +\infty \right\}, \end{aligned} \quad (2)$$

$$L^q_{\mathbb{F}}(0, T; L^p(\Omega; X)) := \left\{ \psi : \Omega \times [0, T] \rightarrow X \mid \begin{aligned} &\psi(\cdot) \text{ is an } \mathbb{F}\text{-adapted process} \\ &\text{on } [0, T] \text{ and } \int_0^T \left[ \mathbb{E} \left( \|\psi(t)\|_X^p \right)^{\frac{q}{p}} \right] dt < +\infty \right\}, \end{aligned} \quad (3)$$

endowed with their natural norms. In a clear and analogous way, we define the spaces (2) and (3) when  $p$  and/or  $q$  equal to  $\infty$ . When  $p = q$  with  $p \in [2, +\infty)$ , from Fubini's theorem we have that  $L^p_{\mathbb{F}}(\Omega; L^p(0, T; X)) = L^p_{\mathbb{F}}(0, T; L^p(\Omega; X))$  and the norms coincide and for simplicity we just write  $L^p_{\mathbb{F}}(0, T; X)$ . When  $p = q = \infty$ , we also have that  $L^\infty_{\mathbb{F}}(\Omega; L^\infty(0, T; X)) = L^\infty_{\mathbb{F}}(0, T; L^\infty(\Omega; X))$  and the norms coincide. Lastly, by  $L^p_{\mathbb{F}}(\Omega; C([0, T]; X))$  we denote the Banach space consisting of all  $X$ -valued  $\mathbb{F}$ -adapted continuous processes  $\psi(\cdot)$  such that  $\mathbb{E} \left( \|\psi\|_{C([0, T]; X)}^p \right) < +\infty$ , also equipped with the canonical norm.

In the remainder of this document, we will make the following instrumental assumption on the coefficients of system (1)

**(H)**  $\alpha, \beta \in L^\infty_{\mathbb{F}}(0, T; L^\infty(\mathcal{O}))$ .

We recall the following notion of solution for (1).

**Definition 1.1.** A pair of random fields  $(y, Y)$  is called a weak solution to (1) if

- 1)  $(y, Y)$  is  $L^2(\mathcal{O}) \times L^2(\mathcal{O})$ -valued and  $\mathcal{F}_t$ -measurable for each  $t \in [0, T]$ ,
- 2)  $(y, Y) \in [L^2_{\mathbb{F}}(\Omega; C([0, T]); L^2(\mathcal{O})) \cap L^2_{\mathbb{F}}(0, T; H^1_0(\mathcal{O}))] \times L^2_{\mathbb{F}}(0, T; L^2(\mathcal{O}))$ , and
- 3) for any  $t \in [0, T]$  and  $\phi \in H^1_0(\mathcal{O})$  it holds

$$\begin{aligned} (y(t), \phi)_{L^2(\mathcal{O})} &= (y_T, \phi)_{L^2(\mathcal{O})} - \int_t^T (\nabla y, \nabla \phi)_{L^2(\mathcal{O})} ds \\ &\quad + \int_t^T (\alpha y + \beta Y + F, \phi)_{L^2(\mathcal{O})} ds - \int_t^T (Y, \phi)_{L^2(\mathcal{O})} dW(s), \quad \text{a.s.} \end{aligned}$$

The following result ensures the existence and uniqueness of such weak solutions.

**Theorem 1.2.** Let  $\alpha, \beta$  be given coefficients satisfying **(H)**. If  $y_T \in L^2_{\mathcal{F}_T}(\Omega; L^2(\mathcal{O}))$  and  $F \in L^2_{\mathbb{F}}(0, T; L^2(\mathcal{O}))$ , then (1) admits a unique weak solution in the sense of Definition 1.1. Moreover, the following energy estimate holds

$$\begin{aligned} &\mathbb{E} \left( \sup_{t \in [0, T]} \|y(t)\|_{L^2(\mathcal{O})}^2 \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |\nabla y|^2 dx dt \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |Y|^2 dx dt \right) \\ &\leq C \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^2 dx + \int_0^T \int_{\mathcal{O}} |F|^2 dx dt \right), \end{aligned} \quad (4)$$

for a positive constant  $C > 0$  independent of  $y_T$  and  $F$ .

This theorem is well-known in the literature and the proof can be derived in two different ways: by a duality analysis similar to [Zho92] or by a more direct and constructive approach relying on Galerkin method (see, e.g., [Gao18, Proposition 2.1] or [SY09, Section 3]).

Our first main result says that the integrability of the weak solution  $(y, Y)$  can be improved by taking more regular data. More precisely, we have the following.

**Theorem 1.3.** Let  $(y, Y)$  be a weak solution to (1). Assume that  $y_T \in L^p_{\mathcal{F}_T}(\Omega; L^p(\mathcal{O}))$  and  $F \in L^p_{\mathbb{F}}(0, T; L^p(\mathcal{O}))$  for some  $p \in [2, +\infty)$ . Then,

$$(y, Y) \in L^p_{\mathbb{F}}(\Omega; C([0, T]; L^p(\mathcal{O}))) \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; L^2(\mathcal{O})))$$

and satisfies

$$\begin{aligned} &\mathbb{E} \left( \sup_{t \in [0, T]} \|y(t)\|_{L^p(\mathcal{O})}^p \right) + \mathbb{E} \left[ \left( \int_0^T \|Y(t)\|_{L^2(\mathcal{O})}^2 dt \right)^{p/2} \right] \\ &\leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p dx + \int_0^T \int_{\mathcal{O}} |F|^p dx dt \right), \end{aligned}$$

for a constant  $C > 0$  only depending on  $\mathcal{O}$ ,  $\alpha$ ,  $\beta$  and  $p$ .

Our second result says that we can go up to  $L^\infty$  for the first component  $y$  of the solution  $(y, Y)$  to (1). In more detail, we have the following.

**Theorem 1.4.** *Let  $(y, Y)$  be a weak solution to (1). Assume that  $y_T \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  and  $F \in L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))$ . Then the process  $y$  taken from the solution  $(y, Y)$  to (1) belongs to  $L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))$  and satisfies*

$$\operatorname{ess\,sup}_{(\omega, t) \in \Omega \times [0, T]} \|y(t)\|_{L^\infty(\mathcal{O})} \leq \exp(CT) \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \right), \quad (5)$$

for some  $C > 0$  only depending on  $\mathcal{O}$ ,  $\alpha$  and  $\beta$ .

Theorems 1.3 and 1.4 should be compared with [HMY02, Theorem 3.2], where similar results are presented for one-dimensional BSPDEs. However, our contributions differ in several ways. Firstly, while [HMY02] is restricted to the one-dimensional case, our results are established for BSPDEs in any dimension. Secondly, we address weak solutions rather than strong solutions. This distinction is important since weak solutions are less regular in the spatial variable and cannot be evaluated pointwise, preventing the direct application of Itô's formula to the function  $u^{2p}$  as in [HMY02]. Instead, we establish an Itô's formula for the  $L^p$ -norm in the backward case (see Proposition 2.1 below), using some ideas from [DG15] for SPDEs, which involve truncations and limit procedures. Finally, our work considers coefficients that are merely bounded, as opposed to some differentiability conditions imposed in [HMY02, eq. (2.4)]. Similarly, we relax the conditions on the initial datum and the source term, which are less restrictive compared to those required by [HMY02, eqs. (2.2)–(2.3)].

As in [HMY02], we also obtain a uniform bound on the first component of the solution of the BSPDE (see Theorem 1.4), which initially can be surprising due to the presence of the stochastic integral. Nevertheless, as mentioned before, the noise term in (1) is endogenous to the equation, and as noted in [DT12], this special feature makes the theory for these equations closer to deterministic PDEs than to SPDEs.

## 1.2 Applications

Once we have established the regularity of the BSPDE (1), we are interested in studying the following applications:

1. Controllability.
2. Local existence of semilinear equations.

Controllability is a qualitative property of dynamical systems that refers to the ability to steer the system from any initial state to any desired final state within a finite time period, using an appropriate control input. In our context, let us consider the control system

$$\begin{cases} dy = -(\Delta y + \alpha y + \beta Y + \chi_{\mathcal{O}_0} h) dt + Y dW(t) & \text{in } (0, T) \times \mathcal{O}, \\ y = 0 & \text{on } (0, T) \times \partial\mathcal{O}, \\ y(T) = y_T & \text{in } \mathcal{O}, \end{cases} \quad (6)$$

where  $h$  is an external control force localized on a set  $\mathcal{O}_0 \subset \mathcal{O}$ . In this part we assume that  $\mathcal{O}$  is connected.

Due to Theorems 1.2, 1.3 and 1.4, system (6) is well-posed for any terminal datum  $y_T \in L_{\mathbb{F}_T}^p(\Omega; L^p(\mathcal{O}))$  and any function  $h \in L_{\mathbb{F}}^p(0, T; L^p(\mathcal{O}_0))$  with  $p \in [2, +\infty]$ . This motivates us to find controls  $h$  such that the following controllability condition is fulfilled.

**Definition 1.5.** Let  $p \in [2, \infty]$ . System (6) is said to be null-controllable in  $L^p$  if for any terminal datum  $y_T \in L_{\mathcal{F}_T}^p(\Omega; L^p(\mathcal{O}))$ , there exists a control  $h \in L_{\mathbb{F}}^p(0, T; L^p(\mathcal{O}_0))$  such that the corresponding weak solution  $(y, Y)$  of (6) satisfies

$$y(0, \cdot) = 0 \quad \text{a.s.}$$

In the case  $p = 2$ , this problem has been studied extensively, most notably in [BRT03] and [TZ09] (see [HSLBP23] for the extension to the semilinear case). Using duality arguments, the controllability for (6) is established via Carleman estimates, which have been adapted from their classical deterministic form (see [FI96]) to the stochastic setting. However, to our knowledge, there are no results in the literature addressing the case when  $p > 2$ . To bridge this gap, we present a result covering the full range  $p \in (2, +\infty]$  for a simplified yet interesting case.

**Theorem 1.6.** *Let  $p \in (2, +\infty]$  and assume that  $\beta \equiv 0$ . Then, system (6) is null-controllable in  $L^p$ . Moreover, if  $T \in (0, 1)$ , the following estimate on the control holds*

$$\|h\|_{L^p_{\mathbb{F}}(0,T;L^p(\mathcal{O}))} \leq \exp(C/T) \|y_T\|_{L^p_{\mathcal{F}_T}(\Omega;L^p(\mathcal{O}))}, \quad (7)$$

for some  $C > 0$  depending at most on  $\mathcal{O}$ ,  $\mathcal{O}_0$ ,  $p$  and  $\alpha$ .

By simplifying the original system, we employ the duality approach used in [TZ09, Section 7], but following the spirit of [Liu14], where we avoid the need to prove stochastic Carleman estimates and use instead known results in the deterministic setting. Our result extends those in [BRT03] and [TZ09] to a wider range of values of  $p$ , circumventing technical challenges typically associated with stochastic control systems.

We remark that this  $L^\infty$  controllability result and, in particular, the estimate of the control (7) in small-time, i.e.  $T \in (0, 1)$  is very useful for obtaining local controllability of semilinear equations. We refer to Remark 4.1 for more details. We also remark that an estimate similar to (7) holds in the case  $T \geq 1$  but with a different exponential bound.

Our second application is concerned about the local existence of solutions to the semilinear system

$$\begin{cases} dy = -(\Delta y + \alpha y + \beta Y + f(y)) dt + Y dW(t) & \text{in } (0, T) \times \mathcal{O}, \\ y = 0 & \text{on } (0, T) \times \partial\mathcal{O}, \\ y(T) = y_T & \text{in } \mathcal{O}, \end{cases} \quad (8)$$

where we assume that

(F)  $f \in C^\infty(\mathbb{R})$  such that  $f(0) = 0$ .

A pair of random fields  $(y, Y)$  is called a weak solution to (8) if

- 1)  $(y, Y)$  is  $L^2(\mathcal{O}) \times L^2(\mathcal{O})$ -valued and  $\mathcal{F}_t$ -measurable for each  $t \in [0, T]$ ,
- 2)  $(y, Y) \in [L^2_{\mathbb{F}}(\Omega; C([0, T]); L^2(\mathcal{O})) \cap L^2_{\mathbb{F}}(0, T; H_0^1(\mathcal{O}))] \times L^2_{\mathbb{F}}(0, T; L^2(\mathcal{O}))$ , and
- 3) for any  $t \in [0, T]$  and  $\phi \in H_0^1(\mathcal{O})$  it holds

$$\begin{aligned} (y(t), \phi)_{L^2(\mathcal{O})} &= (y_T, \phi)_{L^2(\mathcal{O})} - \int_t^T (\nabla y, \nabla \phi)_{L^2(\mathcal{O})} ds \\ &\quad + \int_t^T (\alpha y + \beta Y + f(y), \phi)_{L^2(\mathcal{O})} ds - \int_t^T (Y, \phi)_{L^2(\mathcal{O})} dW(s), \quad \text{a.s.} \end{aligned}$$

To state our result, let us define the functional space  $\mathcal{X} := L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(\mathcal{O}))) \cap L^2_{\mathbb{F}}(\Omega; L^2(0, T; H_0^1(\mathcal{O}))) \cap L^\infty_{\mathbb{F}}(0, T; L^\infty(\mathcal{O}))$  and for any  $p \in [2, +\infty)$  we set

$$\mathcal{X}_p := \{(y, Y) \in \mathcal{X} \times L^p_{\mathbb{F}}(\Omega; L^2(0, T; L^2(\mathcal{O})))\}$$

endowed with its natural norm, that is,  $\|(y, Y)\|_{\mathcal{X}_p} = \|y\|_{\mathcal{X}} + \|Y\|_{L^p_{\mathbb{F}}(\Omega; L^2(0, T; L^2(\mathcal{O}))}$ .

**Theorem 1.7.** *Let  $T > 0$ ,  $p \in [2, +\infty)$  and assume that **(F)** holds. There exists  $\delta > 0$  only depending on  $\mathcal{O}$ ,  $T$ ,  $p$  and  $f$  such that for any  $y_0 \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  satisfying  $\|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} \leq \delta$ , there is a unique weak solution  $(y, Y)$  to (8) that belongs to  $\mathcal{Y}_p$ .*

The proof of Theorem 1.7 relies on a standard Banach fixed-point procedure and exploits the boundedness of the solution provided by Theorem 1.4 to prove the contractivity of a suitable nonlinear map.

Note that we do not impose any growth or sign conditions on the nonlinear term. As a result, we obtain only a local well-posedness result for (8), applicable to sufficiently small terminal data. Nonetheless, this result extends existing works, which typically require global Lipschitz conditions (see, e.g., [HP91, HMY02] or [LZ21, Chapter 4]) and avoids the need for restrictive conditions on  $f$ , providing a more general framework for the analysis of semilinear systems.

### 1.3 Outline of the paper

The rest of the paper is organized as follows. In Section 2, we present and prove an Itô formula for the  $L^p$ -norm of the solution of the BSPDE (1). As a consequence, we will deduce the proof of Theorem 1.3. Then, in Section 3 we use the Itô formula to show uniform  $L^p$ -estimates then Theorem 1.4 concerning  $L^\infty$ -estimates for the solution of the BSPDE (1). Section 4 is devoted to prove the control result in Theorem 1.6, while Section 5 focuses on the study of the semilinear equation (8) and its well-posedness.

## 2 Itô's formula for the $L^p$ -norm

The goal of this section is to state the Itô's formula for the  $L^p$ -norm and deduce  $L^p$ -estimates for the weak solution  $(y, Y)$  of (1). Note that during this section, the  $L^p$ -estimates are not uniform with respect to  $p$ . The final conclusion of this section will be the proof of Theorem 1.3.

The goal of this section is to prove the following result, which is essential for the proofs of Theorem 1.3 and Theorem 1.4. This result includes an initial a priori estimate and an Itô's formula for the  $L^p$ -norm of the solution to the backward SPDE (1). More precisely, we have the following.

**Proposition 2.1** (Itô's formula for  $L^p$ -norm). *Let  $(y, Y)$  be a weak solution to (1) and assume that  $y_T \in L_{\mathcal{F}_T}^p(\Omega; L^p(\mathcal{O}))$  and  $F \in L_{\mathbb{F}}^p(0, T; L^p(\mathcal{O}))$  for some  $p \in [2, +\infty)$ . Then, there is a constant  $C > 0$  only depending on  $\mathcal{O}$ ,  $\alpha$ ,  $\beta$ , and  $p$  such that*

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [0, T]} \|y(t)\|_{L^p(\mathcal{O})}^p \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |y|^{p-2} |\nabla y|^2 \, dx dt \right) \\ & + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |y|^{p-2} Y^2 \, dx dt \right) \leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p \, dx + \int_0^T \int_{\mathcal{O}} |F|^p \, dx dt \right). \end{aligned} \quad (9)$$

Moreover, for any  $t \in [0, T]$ , we have

$$\begin{aligned} & \int_{\mathcal{O}} |y(t)|^p \, dx + p(p-1) \int_t^T \int_{\mathcal{O}} |y|^{p-2} |\nabla y|^2 \, dx ds + \frac{p(p-1)}{2} \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 \, dx ds \\ & = \int_{\mathcal{O}} |y_T|^p \, dx + p \int_t^T \int_{\mathcal{O}} |y|^{p-2} y (\alpha y + \beta Y + F) \, dx - p \int_t^T \int_{\mathcal{O}} |y|^{p-2} y Y dW(s), \quad \text{a.s.} \end{aligned} \quad (10)$$

For the proof, we follow the methodology in [DG15] adapted to the backward case and for the reader's convenience we present it in several steps.

## 2.1 Technical lemmas

We begin by presenting some auxiliary functions. For each  $n \in \mathbb{N}$  and some  $p \geq 2$ , consider the twice continuously differentiable function  $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\phi_n(r) := \begin{cases} |r|^p & \text{if } |r| < n, \\ n^{p-2} \frac{p(p-1)}{2} (|r| - n)^2 + pn^{p-1} (|r| - n) + n^p & \text{if } |r| \geq n. \end{cases} \quad (11)$$

It is not difficult to check that  $\phi_n$  satisfy

$$|\phi_n(r)| \leq M|r|^2, \quad |\phi'_n(r)| \leq M|r|, \quad |\phi''_n(r)| \leq M, \quad r \in \mathbb{R}, \quad (12)$$

where  $M > 0$  depends only on  $p$  and  $n$ . Furthermore, we have

$$|\phi_n(r)| \leq N|r|^p, \quad |\phi'_n(r)| \leq N|r|^{p-1}, \quad |\phi''_n(r)| \leq N|r|^{p-2}, \quad r \in \mathbb{R}, \quad (13)$$

for a positive constant  $N$  only depending on  $p$ , and if  $n \rightarrow \infty$ , we have

$$\phi_n(r) \rightarrow |r|^p, \quad \phi'_n(r) \rightarrow p|r|^{p-2}r, \quad \phi''_n(r) \rightarrow p(p-1)|r|^{p-2}, \quad r \in \mathbb{R}. \quad (14)$$

Additionally, we can check that the following inequalities hold

$$|r\phi'_n(r)| \leq p\phi_n(r) \quad (15)$$

$$|\phi'_n(r)|^2 \leq 4p\phi''_n(r)\phi_n(r), \quad (16)$$

$$[\phi''_n(r)]^{p/(p-2)} \leq [p(p-1)]^{p/(p-2)} \phi_n(r) \quad \text{for all } r \in \mathbb{R}, \quad (17)$$

and, by the definition of (11), we can see that

$$\phi''_n(r) \geq 0 \quad \text{for all } r \in \mathbb{R}. \quad (18)$$

We start with the following lemma.

**Lemma 2.2.** *Let  $n \in \mathbb{N}$ ,  $p \in [2, \infty)$ ,  $t \in [0, T]$ , and  $(y, Y)$  be a weak solution to (1). Then*

$$\mathbb{E} \left( \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y \, dx \, dW(s) \right) = 0.$$

*Proof.* By Cauchy-Schwarz and Hölder inequalities together with (12) we see that

$$\begin{aligned} \int_t^T \left( \int_{\mathcal{O}} \phi'_n(y) Y \, dx \right)^2 ds &\leq \int_t^T \|\phi'_n(y)\|_{L^2(\mathcal{O})}^2 \|Y\|_{L^2(\mathcal{O})}^2 ds \\ &\leq M \int_t^T \|y\|_{L^2(\mathcal{O})}^2 \|Y\|_{L^2(\mathcal{O})}^2 ds \\ &\leq M \sup_{s \in (t, T]} \|y(s)\|_{L^2(\mathcal{O})}^2 \int_t^T \|Y\|_{L^2(\mathcal{O})}^2 ds, \quad \text{a.s.} \end{aligned}$$

for any  $t \in [0, T]$ . Taking expectation and using again Cauchy-Schwarz's inequality we get

$$\mathbb{E} \left( \left[ \int_t^T \left( \int_{\mathcal{O}} \phi'_n(y) Y \, dx \right)^2 ds \right]^{1/2} \right) \leq M \mathbb{E} \left( \sup_{s \in (t, T]} \|y(s)\|_{L^2(\mathcal{O})}^2 \right)^{1/2} \mathbb{E} \left( \int_t^T \|Y\|_{L^2(\mathcal{O})}^2 ds \right)^{1/2}$$



whence

$$\begin{aligned} & \mathbb{E} \left( \left[ \int_t^T \left( \int_{\mathcal{O}} \phi'_n(y) Y \, dx \right)^2 ds \right]^{1/2} \right) \\ & \leq \frac{M}{2} \left[ \mathbb{E} \left( \sup_{s \in (t, T]} \|y(s)\|_{L^2(\mathcal{O})}^2 \right) + \mathbb{E} \left( \int_t^T \|Y\|_{L^2(\mathcal{O})}^2 ds \right) \right] < +\infty. \end{aligned}$$

by the regularity estimate (4). Defining  $H(s) := \int_{\mathcal{O}} \phi'_n(y(s)) Y(s) \, dx$ , we have proved that  $H(\cdot) \in L^2(\Omega \times (0, T))$ . Therefore,  $\mathbb{E} \left( \int_t^T H(s) dW(s) \right) = \mathbb{E} \left( \int_0^T H(s) dW(s) - \int_0^t H(s) dW(s) \right) = 0$ . This ends the proof.  $\square$

We continue with the following identity.

**Lemma 2.3.** *Let  $n \in \mathbb{N}$  and  $(y, Y)$  be a weak solution to (1). For any  $t \in [0, T]$ , the following identity holds*

$$\begin{aligned} & \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi''_n(y) |\nabla y|^2 \, dx ds + \frac{1}{2} \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y^2 \, dx ds \\ & = \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_t^T \int_{\mathcal{O}} \phi'_n(y) (\alpha y + \beta Y + F) \, dx ds - \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y \, dx dW(s), \quad \text{a.s.} \end{aligned} \tag{19}$$

*Proof.* Since  $\phi_n$  is a twice continuously differentiable function, we can apply Itô's formula and use equation (1) to get

$$\begin{aligned} \int_{\mathcal{O}} \phi_n(y(T)) \, dx &= \int_{\mathcal{O}} \phi_n(y(t)) \, dx - \int_t^T \int_{\mathcal{O}} \phi'_n(y) \Delta y \, dx ds - \int_t^T \int_{\mathcal{O}} \left( \phi'_n(y) F - \frac{1}{2} \phi''_n(y) Y^2 \right) \, dx ds \\ & \quad - \int_t^T \int_{\mathcal{O}} \phi'_n(y) (\alpha y + \beta Y) \, dx ds + \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y \, dx dW(s), \quad \text{a.s.} \end{aligned}$$

for any  $t \in [0, T]$ . Note that for  $i = 1, \dots, d$ ,  $\partial_{x_i}(\phi'_n(y)) = \phi''_n(y) \partial_{x_i} y$ , whence integrating by parts in the space variable and using the homogeneous Dirichlet boundary condition of (1) yields

$$\begin{aligned} \int_{\mathcal{O}} \phi_n(y(T)) \, dx &= \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi''_n(y) |\nabla y|^2 \, dx ds - \int_t^T \int_{\mathcal{O}} \left( \phi'_n(y) F - \frac{1}{2} \phi''_n(y) Y^2 \right) \, dx ds \\ & \quad - \int_t^T \int_{\mathcal{O}} \phi'_n(y) (\alpha y + \beta Y) \, dx ds + \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y \, dx dW(s), \quad \text{a.s.} \end{aligned}$$

After rearranging some terms, we obtain the desired equality.  $\square$

In the remainder of this section,  $C$  denotes a generic positive constant uniform with respect to  $n$  and that may change from line to line. We have the following estimate.

**Lemma 2.4.** *Let  $n \in \mathbb{N}$ ,  $p \in [2, +\infty)$ ,  $(y, Y)$  be a weak solution to (1) and assume that  $y_T \in L^p_{\mathcal{F}_T}(\Omega; L^p(\mathcal{O}))$  and  $F \in L^p_{\mathbb{F}}(0, T; L^p(\mathcal{O}))$ . Then, the following estimate holds*

$$\begin{aligned} & \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \phi''_n(y) |\nabla y|^2 \, dx ds \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \phi''_n(y) Y^2 \, dx ds \right) \\ & \leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p \, dx + \int_0^T \int_{\mathcal{O}} |F|^p \, dx ds \right), \end{aligned} \tag{20}$$

for some constant  $C > 0$  that only depends on  $p$ ,  $\|\alpha\|_{L^\infty(0, T; L^\infty(\mathcal{O}))}$  and  $\|\beta\|_{L^\infty(0, T; L^\infty(\mathcal{O}))}$ .

*Proof.* Let  $t > 0$  and  $0 < r \leq t$ . From identity (19) of Lemma 2.3, we take conditional expectation to obtain

$$\begin{aligned} & \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 \, dx ds + \frac{1}{2} \int_t^T \int_{\mathcal{O}} \phi_n''(y) Y^2 \, dx ds \mid \mathcal{F}_r \right) \\ &= \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_t^T \int_{\mathcal{O}} \phi_n'(y) (\alpha y + \beta Y + F) \, dx ds \mid \mathcal{F}_r \right) \\ & \quad - \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} \phi_n'(y) Y \, dx dW(s) \mid \mathcal{F}_r \right). \end{aligned}$$

From (15)-(17), the positivity property (18) and Young's inequality, we can deduce

$$|\phi_n'(y)F| \leq C|F|\phi_n(y)^{(p-1)/p} \leq C(|F|^p + \phi_n(y)), \quad (21)$$

$$|\phi_n'(y)\alpha y| \leq C|\alpha|\phi_n(y), \quad (22)$$

$$|\phi_n'(y)\beta Y| \leq \delta \phi_n''(y)|Y|^2 + \frac{C|\beta|^2}{\delta} \phi_n(y), \quad (23)$$

almost surely, for a constant  $C > 0$  only depending on  $p$  and any  $\delta \in (0, 1)$ . Therefore, taking  $\delta$  small enough

$$\begin{aligned} & \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 \, dx ds + \int_t^T \int_{\mathcal{O}} \phi_n''(y) Y^2 \, dx ds \mid \mathcal{F}_r \right) \\ & \leq C \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_t^T \int_{\mathcal{O}} |F|^p \, dx ds + \int_t^T \int_{\mathcal{O}} \phi_n(y) \, dx ds \mid \mathcal{F}_r \right) \\ & \quad - \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} \phi_n'(y) Y \, dx dW(s) \mid \mathcal{F}_r \right), \end{aligned}$$

where  $C > 0$  depends on  $p$  and the norms  $\|\alpha\|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(\mathcal{O}))}$  and  $\|\beta\|_{L_{\mathbb{F}}^{\infty}(0,T;L^{\infty}(\mathcal{O}))}$ .

Taking expectation in the above inequality and by Fubini's theorem and Lemma 2.2, we get

$$\begin{aligned} & \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 \, dx ds + \int_t^T \int_{\mathcal{O}} \phi_n''(y) Y^2 \, dx ds \right) \\ & \leq C \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_0^T \int_{\mathcal{O}} |F|^p \, dx ds \right) + C \int_t^T \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y) \, dx ds \right). \quad (24) \end{aligned}$$

By (18), we can drop the second and third terms in the above estimate and, by the backward Grönwall inequality (see Lemma A.1), we obtain

$$\mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(t)) \, dx \right) \leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_0^T \int_{\mathcal{O}} |F|^p \, dx ds \right). \quad (25)$$

Integrating in time (25) and combining with (24) entails

$$\begin{aligned} & \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(t)) \, dx \right) + \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 \, dx ds \right) + \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} \phi_n''(y) Y^2 \, dx ds \right) \\ & \leq C e^{CT} \left[ \mathbb{E} \left( \int_{\mathcal{O}} \phi_n(y(T)) \, dx \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |F|^p \, dx ds \right) \right], \quad (26) \end{aligned}$$

where we have replaced the terminal data of (1). Dropping the first term in (26) and applying the monotone convergence theorem yield the desired result.  $\square$

The following result tells us that we can improve Lemma 2.4 by adding a supremum (in time) estimate on the left-hand side of (20).

**Lemma 2.5.** *Under the assumptions of Lemma 2.4, there is  $C > 0$  only depending on  $p$  such that*

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} \phi_n(y(t)) \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 dx ds \right) + \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} \phi_n''(y) Y^2 dx ds \right) \\ & \leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p dx + \int_0^T \int_{\mathcal{O}} |F|^p dx ds \right). \end{aligned} \quad (27)$$

*Proof.* Using as a starting point identity (19) of Lemma 2.3, we use (21)–(23) to deduce that almost surely

$$\begin{aligned} & \int_{\mathcal{O}} \phi_n(y(t)) dx + \int_t^T \int_{\mathcal{O}} \phi_n''(y) |\nabla y|^2 dx ds + \int_t^T \int_{\mathcal{O}} \phi_n''(y) Y^2 dx ds \\ & \leq C \left( \int_{\mathcal{O}} |y_T|^p dx + \int_t^T \int_{\mathcal{O}} |F|^p dx ds + \int_t^T \int_{\mathcal{O}} \phi_n(y) dx ds \right) - \int_t^T \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s), \end{aligned}$$

for any  $t \in [0, T]$  and some  $C > 0$  only depending on  $p$ ,  $\|\alpha\|_{L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(\mathcal{O}))}$  and  $\|\beta\|_{L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(\mathcal{O}))}$ . Hence

$$\begin{aligned} & \int_{\mathcal{O}} \phi_n(y(t)) dx \leq C \left( \int_{\mathcal{O}} |y_T|^p dx + \int_0^T \int_{\mathcal{O}} |F|^p dx ds \right) + \sup_{t \in [0, T]} \left| \int_t^T \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right| \\ & + C \int_t^T \int_{\mathcal{O}} \phi_n(y) dx ds, \quad \text{a.s.} \end{aligned} \quad (28)$$

From the backward Grönwall inequality (see Lemma A.1) and taking supremum in  $[0, T]$ , we have that a.s.

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathcal{O}} \phi_n(y(t)) dx \\ & \leq C e^{CT} \left( \int_{\mathcal{O}} |y_T|^p dx + \int_0^T \int_{\mathcal{O}} |F|^p dx ds + 2 \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right| \right), \end{aligned} \quad (29)$$

where we have used that

$$\begin{aligned} & \sup_{t \in [0, T]} \left| \int_t^T \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right| \\ & \leq \left| \int_0^T \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right| + \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right| \\ & \leq 2 \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} \phi_n'(y) Y dx dW(s) \right|. \end{aligned} \quad (30)$$

We take expectation in (29) and apply Burkholder-Davis-Gundy inequality to get

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} \phi_n(y(t)) dx \right) \\ & \leq C e^{CT} \left[ \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p dx + \int_0^T \int_{\mathcal{O}} |F|^p dx ds \right) + \mathbb{E} \left( \left| \int_0^T \left( \int_{\mathcal{O}} \phi_n'(y) Y dx \right)^2 ds \right|^{1/2} \right) \right]. \end{aligned} \quad (31)$$

To estimate the last term in the above inequality we note that from (16)

$$|\phi'_n(y)Y| \leq 2|Y|\phi''_n(y)^{1/2}\phi_n(y)^{1/2} \quad \text{a.s.}$$

and applying Hölder and Young inequalities we have that there exists a numerical constant  $C > 0$  such that

$$\begin{aligned} & \mathbb{E} \left( \left| \int_0^T \left( \int_{\mathcal{O}} \phi'_n(y)Y \, dx \right)^2 ds \right|^{1/2} \right) \\ & \leq \delta \mathbb{E} \left( \sup_{s \in [0, T]} \int_{\mathcal{O}} \phi_n(y) \, dx \right) + \frac{C}{\delta} \mathbb{E} \left( \int_0^T \int_{\mathcal{O}} |Y|^2 |\phi''_n(y)| \, dx ds \right), \end{aligned} \quad (32)$$

for any  $\delta \in (0, 1)$ . Replacing (32) in (31) and taking  $\delta$  small enough we get

$$\begin{aligned} & \mathbb{E} \left( \sup_{t \in [0, T]} \int_{\mathcal{O}} \phi_n(y(t)) \, dx \right) \\ & \leq C e^{CT} \mathbb{E} \left( \int_{\mathcal{O}} |y_T|^p \, dx + \int_0^T \int_{\mathcal{O}} |F|^p \, dx ds + \int_0^T \int_{\mathcal{O}} |Y|^2 |\phi''_n(y)| \, dx ds \right). \end{aligned} \quad (33)$$

Finally, putting together estimate (20) of Lemma 2.4 and inequality (33) we obtain the desired result. This ends the proof.  $\square$

## 2.2 Proof of the Ito's formula for the $L^p$ -norm

Now we are in position to prove the Ito's formula, that is, Proposition 2.1.

*Proof of Proposition 2.1.* Estimate (9), follows from (14), estimate (27) in Lemma 2.5 and a straightforward application of Fatou's Lemma. This proves the first assertion of Theorem 1.3.

To prove the Itô's formula (10), we proceed as follows. From Lemma 2.3, we have that the following identity holds

$$\begin{aligned} & \int_{\mathcal{O}} \phi_n(y(t)) \, dx + \int_t^T \int_{\mathcal{O}} \phi''_n(y) |\nabla y|^2 \, dx ds + \frac{1}{2} \int_t^T \int_{\mathcal{O}} \phi''_n(y) Y^2 \, dx ds \\ & = \int_{\mathcal{O}} \phi_n(y(T)) \, dx + \int_t^T \int_{\mathcal{O}} \phi'_n(y) (\alpha y + \beta Y + F) \, dx ds - \int_t^T \int_{\mathcal{O}} \phi'_n(y) Y \, dx dW(s) \quad \text{a.s.}, \end{aligned} \quad (34)$$

for all  $t \in [0, T]$ . We will see that each term in (34) converges to the corresponding one in (10). At this point, we will make use of estimate (9).

From estimate (9), we readily deduce that

$$\sup_{t \in [0, T]} \|y(t)\|_{L^p(\mathcal{O})}^p < +\infty, \quad \text{a.s.} \quad (35)$$

Then, by (13), (14), (35), and dominated convergence theorem we obtain that

$$\int_{\mathcal{O}} \phi_n(y(t)) \, dx \rightarrow \int_{\mathcal{O}} |y(t)|^p \, dx \quad \text{a.s. for all } t \in [0, T].$$

This proves the convergence of the first terms in both sides of identity (34).

Now we note from (13) that a.s.

$$|\phi'_n(y)F| \leq N|y|^{p-1}|F|$$

and, moreover, by successive applications of Hölder inequality we have for all  $t \in [0, T]$

$$\int_t^T \int_{\mathcal{O}} |y|^{p-1}|F| \, dx ds \leq \|y\|_{L^p((t,T];L^p(\mathcal{O}))}^{p-1} \|F\|_{L^p((t,T];L^p(\mathcal{O}))} < +\infty \quad \text{a.s.},$$

where the boundedness follows from (35) and the regularity of  $F$ . Therefore, dominated convergence theorem together with (14) imply that

$$\int_t^T \int_{\mathcal{O}} \phi'_n(y)F \, dx ds \rightarrow p \int_t^T \int_{\mathcal{O}} |y|^{p-2}yF \, dx ds \quad \text{a.s. for all } t \in [0, T].$$

Similar arguments using the energy estimate (9) yield that

$$\begin{aligned} \int_t^T \int_{\mathcal{O}} \phi''_n(y)|\nabla y|^2 \, dx ds &\rightarrow p(p-1) \int_t^T \int_{\mathcal{O}} |y|^{p-2}|\nabla y|^2 \, dx ds \quad \text{a.s.}, \\ \frac{1}{2} \int_t^T \int_{\mathcal{O}} \phi''_n(y)Y^2 \, dx ds &\rightarrow \frac{p(p-1)}{2} \int_t^T \int_{\mathcal{O}} |y|^{p-2}|Y|^2 \, dx ds \quad \text{a.s.}, \\ \int_t^T \int_{\mathcal{O}} \phi'_n(y)\alpha y \, dx ds &\rightarrow p \int_t^T \int_{\mathcal{O}} \alpha|y|^p \, dx ds \quad \text{a.s.}, \\ \int_t^T \int_{\mathcal{O}} \phi'_n(y)\beta Y \, dx ds &\rightarrow p \int_t^T \int_{\mathcal{O}} |y|^{p-2}\beta y Y \, dx ds \quad \text{a.s.}, \end{aligned}$$

for all  $t \in [0, T]$ .

Finally, for the term  $\int_t^T \int_{\mathcal{O}} \phi'_n(y)Y \, dx dW(s)$ , first we will prove that

$$\int_t^T \left| \langle \phi'_n(y), Y \rangle_{L^2(\mathcal{O})} - \langle p|y|^{p-2}y, Y \rangle_{L^2(\mathcal{O})} \right|^2 ds \rightarrow 0 \quad \text{a.s. for all } t \in [0, T]. \quad (36)$$

For this, we recall from (13) that  $|\phi'_n(r)| \leq N|r|^{p-1}$  and by Cauchy-Schwarz inequality

$$\left| \langle \phi'_n(y), Y \rangle_{L^2(\mathcal{O})} - \langle p|y|^{p-2}y, Y \rangle_{L^2(\mathcal{O})} \right|^2 \leq C \left( \int_{\mathcal{O}} |y|^{p-2}|Y|^2 \, dx \right) \left( \int_{\mathcal{O}} |y|^p \, dx \right) \quad \text{a.s.}$$

On the other hand, by estimate (9), we have

$$\begin{aligned} &\int_t^T \left( \int_{\mathcal{O}} |y|^{p-2}|Y|^2 \, dx \right) \left( \int_{\mathcal{O}} |y|^p \, dx \right) ds \\ &\leq C \left( \sup_{s \in [t, T]} \int_{\mathcal{O}} |y(s)|^p \, dx \right) \left( \int_t^T \int_{\mathcal{O}} |y|^{p-2}|Y|^2 \, dx ds \right), \\ &< +\infty, \quad \text{a.s. for all } t \in [0, T]. \end{aligned}$$

From (14), the continuity of the  $L^2$ -inner product, and dominated convergence theorem allow us to conclude (36) and therefore

$$\int_t^T \int_{\mathcal{O}} \phi'_n(y)Y dW(s) \rightarrow p \int_t^T \int_{\mathcal{O}} |y|^{p-2}yY dW(s) \quad \text{a.s. for all } t \in [0, T].$$

The proof is complete.  $\square$

### 2.3 Extra regularity for the process $Y$ and proof of Theorem 1.3

The goal of this part is to establish extra-regularity for the process  $Y$ .

**Proposition 2.6.** *Let  $(y, Y)$  be a weak solution to (1) and assume that  $y_T \in L^p_{\mathbb{F}_T}(\Omega; L^p(\mathcal{O}))$  and  $F \in L^p_{\mathbb{F}}(0, T; L^p(\mathcal{O}))$  for some  $p \in (2, +\infty)$ . Then  $Y \in L^p_{\mathbb{F}}(\Omega; L^2(0, T; L^2(\mathcal{O})))$  and there is a constant  $C > 0$  only depending on  $\mathcal{O}$ ,  $\alpha$ ,  $\beta$ , and  $p$  such that*

$$\mathbb{E} \left[ \left( \int_0^T \|Y(t)\|_{L^2(\mathcal{O})}^2 dt \right)^{p/2} \right] \leq C e^{CT} \mathbb{E} \left( \|y_T\|_{L^p(\mathcal{O})}^p + \int_0^T \|F(t)\|_{L^p(\mathcal{O})}^p dt \right). \quad (37)$$

*Proof.* Let  $(y, Y)$  be a weak solution to (1). Along the proof,  $C$  stands for a generic positive constant depending at most on  $\mathcal{O}$ ,  $\|\alpha\|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(\mathcal{O}))}$ ,  $\|\beta\|_{L^\infty_{\mathbb{F}}(0, T; L^\infty(\mathcal{O}))}$ , and  $p$  but which is uniform with respect to  $T$ . This constant may change from line to line.

Using formula (10) in Proposition 2.1 with  $p = 2$  (which amounts to the usual Ito's formula) and dropping some of the positive terms we have

$$\int_t^T \int_{\mathcal{O}} |Y|^2 dx ds \leq \int_{\mathcal{O}} |y_T|^2 dx + 2 \int_t^T \int_{\mathcal{O}} y(\alpha y + \beta Y + F) dx ds - 2 \int_t^T \int_{\mathcal{O}} y Y dx dW(s), \quad \text{a.s.}$$

for any  $t \in [0, T]$ . Since  $y \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2(\mathcal{O})))$ , then  $\int_{\mathcal{O}} |y_T|^2 \leq \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^2$  a.s. From this, and taking the power  $p/2$  with  $p > 2$  in both sides of the above inequality, we deduce

$$\begin{aligned} & \left( \int_t^T \int_{\mathcal{O}} |Y|^2 dx ds \right)^{p/2} \\ & \leq C \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p + \left( \int_0^T \int_{\mathcal{O}} |\alpha| |y|^2 dx ds \right)^{p/2} + \left( \int_0^T \int_{\mathcal{O}} |\beta y Y| dx ds \right)^{p/2} \right) \\ & \quad + C \left( \int_0^T \int_{\mathcal{O}} |y F| dx ds + \left| \int_t^T \int_{\mathcal{O}} y Y dx dW(s) \right|^{p/2} \right), \end{aligned} \quad (38)$$

almost surely for any  $t \in [0, T]$ .

Let us estimate the expectation of the last four terms in the right-hand side of (38). For the first one, from Hölder inequality and since  $\alpha$  verifies (H), we readily deduce

$$\mathbb{E} \left[ \left( \int_0^T \int_{\mathcal{O}} |\alpha| |y|^2 dx ds \right)^{p/2} \right] \leq CT^{p/2} \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right). \quad (39)$$

For the second one, recalling (H) and using Cauchy-Schwarz inequality successively, we have

$$\left( \int_0^T \int_{\mathcal{O}} |\beta y Y| dx ds \right)^{p/2} \leq C \left( \int_0^T \|y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/4} \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/4} \quad \text{a.s.}$$

Therefore, taking expectation and using Cauchy-Schwarz and Young inequalities we obtain

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_0^T \int_{\mathcal{O}} |\beta y Y| dx ds \right)^{p/2} \right] \\ & \leq \delta \mathbb{E} \left[ \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] + \frac{C}{\delta} \mathbb{E} \left[ \left( \int_0^T \|y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] \end{aligned}$$

for any  $\delta > 0$ . Using Hölder inequality in the last term, we can further estimate

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_0^T \int_{\mathcal{O}} |\beta y Y| \, dx ds \right)^{p/2} \right] \\ & \leq \delta \mathbb{E} \left[ \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] + \frac{CT^{p/2}}{\delta} \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right). \end{aligned} \quad (40)$$

To estimate the third term in the right-hand side of (38), we proceed as follows. Using Cauchy-Schwarz and Hölder inequalities, we get

$$\left( \int_0^T \int_{\mathcal{O}} |yF| \, dx dt \right)^{p/2} \leq \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^{p/2} \left( \int_0^T \|F(s)\|_{L^2(\mathcal{O})} ds \right)^{p/2}, \quad \text{a.s.}$$

Taking expectation and using Cauchy-Schwarz and Young inequalities we obtain

$$\mathbb{E} \left[ \left( \int_0^T \int_{\mathcal{O}} |yF| \, dx dt \right)^{p/2} \right] \leq \frac{1}{2} \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right) + \frac{1}{2} \mathbb{E} \left[ \left( \int_0^T \|F(s)\|_{L^2(\mathcal{O})} ds \right)^p \right]. \quad (41)$$

To estimate the last term in (38), arguing as we did in (30), we have that

$$\sup_{t \in [0, T]} \left| \int_t^T \int_{\mathcal{O}} yY \, dx dW(s) \right|^{p/2} \leq 2 \sup_{t \in [0, T]} \left| \int_0^t \int_{\mathcal{O}} yY \, dx dW(s) \right|^{p/2},$$

whence, by Burkholder-Davis-Gundy inequality, we deduce that

$$\mathbb{E} \left[ \left| \int_t^T \int_{\mathcal{O}} yY \, dx dW(s) \right|^{p/2} \right] \leq C \mathbb{E} \left[ \left| \int_0^T \left( \int_{\mathcal{O}} yY \, dx \right)^2 ds \right|^{p/4} \right]. \quad (42)$$

Using Cauchy-Schwarz and Hölder inequalities we can estimate the right-hand side of (42) as

$$\mathbb{E} \left[ \left| \int_t^T \int_{\mathcal{O}} yY \, dx dW(s) \right|^{p/2} \right] \leq C \mathbb{E} \left[ \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^{p/2} \left| \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right|^{p/4} \right],$$

and from Young inequality, we deduce that for any  $\delta > 0$

$$\mathbb{E} \left[ \left| \int_t^T \int_{\mathcal{O}} yY \, dx dW(s) \right|^{p/2} \right] \leq \delta \mathbb{E} \left[ \left| \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right|^{p/2} \right] + \frac{C}{\delta} \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right). \quad (43)$$

Taking expectation in (38) and putting together with (39), (40), (41), and (43) we have that for any  $t \in [0, T]$  the following holds

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_t^T \int_{\mathcal{O}} |Y|^2 \, dx ds \right)^{p/2} \right] \\ & \leq C \left( 1 + \frac{2T^{p/2}}{\delta} \right) \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right) + 2\delta \mathbb{E} \left[ \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] \\ & \quad + \frac{1}{2} \mathbb{E} \left[ \left( \int_0^T \|F(s)\|_{L^2(\mathcal{O})} ds \right)^p \right]. \end{aligned}$$

In particular, taking  $t = 0$  in the above estimate and setting  $\delta > 0$  small enough yields

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] \\ & \leq C(1 + T^{p/2}) \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right) + C \mathbb{E} \left[ \left( \int_0^T \|F(s)\|_{L^2(\mathcal{O})} ds \right)^p \right]. \end{aligned} \quad (44)$$

To conclude, using Hölder inequality and since  $\|z\|_{L^2(\mathcal{O})}^p \leq |\mathcal{O}|^{\frac{p-2}{2}} \|z\|_{L^p(\mathcal{O})}^p$  for any  $z \in L^p(\mathcal{O})$ , we obtain

$$\begin{aligned} & \mathbb{E} \left[ \left( \int_0^T \|Y(s)\|_{L^2(\mathcal{O})}^2 ds \right)^{p/2} \right] \\ & \leq C(1 + T^{p/2}) \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^2(\mathcal{O})}^p \right) + CT^{p-1} \mathbb{E} \left( \int_0^T \|F(s)\|_{L^2(\mathcal{O})}^p ds \right) \\ & \leq C' e^{C'T} \mathbb{E} \left( \sup_{s \in [0, T]} \|y(s)\|_{L^p(\mathcal{O})}^p + \int_0^T \|F(s)\|_{L^p(\mathcal{O})}^p ds \right), \end{aligned} \quad (45)$$

for some constant  $C' > 0$  only depending on  $\mathcal{O}$ ,  $\alpha$ ,  $\beta$ , and  $p$ . To conclude, we use estimate (9) in (45) to obtain the desired inequality (37). This ends the proof.  $\square$

*Proof of Theorem 1.3.* The result follows directly from Proposition 2.1 and Proposition 2.6.  $\square$

### 3 Uniform $L^p$ - and $L^\infty$ -estimates

The goal of this section is to show Theorem 1.4. In the first step towards Theorem 1.4, we will use Itô's formula for the  $L^p$ -norm (see Proposition 2.1) to obtain a precise uniform estimate for the solution to (1). The result reads as follows.

**Proposition 3.1.** *Let  $(y, Y)$  be a weak solution to (1). Assume that  $y_T \in L^p_{\mathcal{F}_T}(\Omega; L^p(\mathcal{O}))$  and  $F \in L^p_{\mathbb{F}}(0, T; L^p(\mathcal{O}))$ . Then the process  $y$  taken from the solution  $(y, Y)$  to (1) satisfies*

$$\|y\|_{L^\infty_{\mathbb{F}}(0, T; L^p(\Omega; L^p(\mathcal{O})))} \leq \exp(CT) \left( \|y_T\|_{L^p_{\mathcal{F}_T}(\Omega; L^p(\mathcal{O}))} + \|F\|_{L^p_{\mathbb{F}}(0, T; L^p(\mathcal{O}))} \right),$$

for a constant  $C > 0$  independent of  $y_T$ ,  $F$  and  $p$ .

*Proof.* Let  $(y, Y)$  be a weak solution to (1). We start from formula (10) in Proposition 2.1, namely

$$\begin{aligned} & \int_{\mathcal{O}} |y(t)|^p dx + p(p-1) \int_t^T \int_{\mathcal{O}} |y|^{p-2} |\nabla y|^2 dx ds + \frac{p(p-1)}{2} \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 dx ds \\ & = \int_{\mathcal{O}} |y_T|^p dx + p \int_t^T \int_{\mathcal{O}} |y|^{p-2} y (\alpha y + \beta Y + F) dx - p \int_t^T \int_{\mathcal{O}} |y|^{p-2} y Y dW(s), \quad \text{a.s.}, \end{aligned}$$

that holds for any  $t \in [0, T]$ .

In particular, by taking into account that the third term in the left-hand side is nonnegative, we deduce that

$$\begin{aligned} & \int_{\mathcal{O}} |y(t)|^p dx + \frac{p(p-1)}{2} \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 dx ds \\ & \leq \int_{\mathcal{O}} |y_T|^p dx + p \int_t^T \int_{\mathcal{O}} |y|^{p-1} |\alpha y + \beta Y + F| dx - p \int_t^T \int_{\mathcal{O}} |y|^{p-2} y Y dW(s), \quad \text{a.s.} \end{aligned} \quad (46)$$



Let  $r \leq t$  and take conditional expectation with respect to  $\mathcal{F}_r$  in (46) to get

$$\begin{aligned} & \mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_r \right) + \frac{p(p-1)}{2} \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 dx ds \mid \mathcal{F}_r \right) \\ & \leq \mathbb{E} \left( \|y_T\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_r \right) + p \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-1} [|\alpha y| + |\beta Y| + |F|] dx ds \mid \mathcal{F}_r \right) \\ & \quad - \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} p |y|^{p-2} y Y dW(s) \mid \mathcal{F}_r \right). \end{aligned} \quad (47)$$

Let us estimate the second integral in (47). By a direct computation using the monotonicity of the conditional expectation, we have

$$p \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-1} |\alpha y| dx ds \mid \mathcal{F}_r \right) \leq p \|\alpha\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \mid \mathcal{F}_r \right). \quad (48)$$

For the second one, by Cauchy-Schwarz and Young inequality we have

$$\begin{aligned} & p \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-1} |\beta Y| dx ds \mid \mathcal{F}_r \right) \\ & \leq \frac{p}{2} \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 dx ds \mid \mathcal{F}_r \right) + \frac{p \|\beta\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}^2}{2} \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \mid \mathcal{F}_r \right). \end{aligned} \quad (49)$$

For the last one, using Young inequality  $ab \leq (1/p)a^p + (1/q)b^q$  with  $q = p/(p-1)$ , so  $pab \leq a^p + (p-1)b^{p/(p-1)}$  with  $a = |F|$  and  $b = |y|^{p-1}$  we obtain that

$$\begin{aligned} & p \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-1} |F| dx ds \mid \mathcal{F}_r \right) \\ & \leq \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |F|^p dx ds \mid \mathcal{F}_r \right) + (p-1) \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \mid \mathcal{F}_r \right). \end{aligned} \quad (50)$$

Therefore, combining (47) with estimates (48)–(50) we get

$$\begin{aligned} & \mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_r \right) + \frac{p(p-2)}{2} \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^{p-2} |Y|^2 dx ds \mid \mathcal{F}_r \right) \\ & \leq \mathbb{E} \left( \|y_T\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_r \right) + \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |F|^p dx ds \mid \mathcal{F}_r \right) \\ & \quad + [pK + (p-1)] \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \mid \mathcal{F}_r \right) \\ & \quad - \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} p |y|^{p-2} y Y dW(s) \mid \mathcal{F}_r \right), \end{aligned}$$

where  $K := \|\alpha\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} + \|\beta\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}^2$ . Note that since  $p \geq 2$ , the second term in the left-hand side of the above inequality is always nonnegative.

Now, we take  $r = t$  in the previous estimate and use the fact that  $\mathbb{E}(\|y(t)\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_t) = \|y(t)\|_{L^p(\mathcal{O})}^p$  since  $y$  is  $\mathcal{F}_t$ -adapted to obtain

$$\begin{aligned} & \|y(t)\|_{L^p(\mathcal{O})}^p \\ & \leq \mathbb{E} \left( \|y_T\|_{L^p(\mathcal{O})}^p \mid \mathcal{F}_t \right) + \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |F|^p dx ds \mid \mathcal{F}_t \right) + [pK + (p-1)] \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \mid \mathcal{F}_t \right) \\ & \quad - \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} p |y|^{p-2} y Y dW(s) \mid \mathcal{F}_t \right) \end{aligned} \quad (51)$$

whence, by taking expectation and using Fubini's theorem yield, we get

$$\mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \right) \leq \|y_T\|_{L_{\mathcal{F}_T}^p(\Omega; L^p(\mathcal{O}))}^p + \|F\|_{L_{\mathbb{F}}^p(0, T; L^p(\mathcal{O}))}^p + [pK + (p-1)] \int_t^T \mathbb{E} \left( \|y(s)\|_{L^p(\mathcal{O})}^p \right) ds,$$

for all  $t \in [0, T]$ , because

$$\mathbb{E} \left( \int_t^T \int_{\mathcal{O}} p|y|^{p-2} y Y dW(s) \right) = 0.$$

This can be shown by means of estimate (9) and following the steps in Lemma 2.2. For brevity, we skip the details.

To finish, we apply backward Grönwall estimate (see Lemma A.1) to deduce

$$\mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \right) \leq \exp([pK + (p-1)](T-t)) \left( \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))}^p + \|F\|_{L^p(\Omega; L^p(0, T; L^p(\mathcal{O}))}^p \right),$$

that is

$$\|y(t)\|_{L^p(\Omega; L^p(\mathcal{O}))}^p \leq \exp([pK + (p-1)](T-t)) \left( \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))}^p + \|F\|_{L^p(\Omega; L^p(0, T; L^p(\mathcal{O}))}^p \right).$$

We take the power  $1/p$  and we get

$$\begin{aligned} \|y(t)\|_{L^p(\Omega; L^p(\mathcal{O}))} &\leq \exp((K + (p-1)/p)(T-t)) \left( \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))}^p + \|F\|_{L^p(\Omega; L^p(0, T; L^p(\mathcal{O}))}^p \right)^{1/p} \\ &\leq \exp([K + 1](T-t)) \left( \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))} + \|F\|_{L^p(\Omega; L^p(0, T; L^p(\mathcal{O}))} \right), \end{aligned}$$

where we have used that  $(p-1)/p < 1$ . Taking  $L^\infty$ -norm in time yields the desired result.  $\square$

We present the proof of our main result.

*Proof of Theorem 1.4.* From Proposition 3.1, we have that the following estimate holds

$$\|y(t)\|_{L^p(\Omega; L^p(\mathcal{O}))} \leq \exp(CT) \left( \|y_T\|_{L_{\mathcal{F}_T}^p(\Omega; L^p(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^p(0, T; L^p(\mathcal{O}))} \right) \quad (52)$$

for all  $t \in [0, T]$ , where  $y$  can be found from  $(y, Y)$  solution to (1) and  $C > 0$  is a constant independent of  $p$ .

By Hölder inequality, the right-hand side of the above expression is bounded as

$$\begin{aligned} R.H.S. &\leq \exp(CT) \left( |\mathcal{O}|^{1/p} \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + |\mathcal{O}|^{1/p} |T|^{1/p} \|F\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \right) \\ &\leq \exp(C'(1+T)) \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \right) \end{aligned} \quad (53)$$

for a constant  $C' > 0$  depending on  $\mathcal{O}$  but uniform with respect to  $T$  and  $p$ . Estimate (53) and Lemma A.2 allows us to pass to the limit as  $p \rightarrow +\infty$  in (52) and obtain

$$\|y(t)\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))} \leq \exp(CT) \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \right)$$

for every  $t \in [0, T]$ . Therefore,

$$\|y\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\Omega; L^\infty(\mathcal{O}))} \leq \exp(CT) \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \right).$$

The result (5) then follows by interchanging the  $L^\infty$ -norms.  $\square$

## 4 Applications to controllability

The goal of this part is to present some applications to the control of backward stochastic parabolic equations at the linear level first then at the nonlinear level. In particular, we will prove Theorem 1.6.

Hereinafter, we set  $Q_T := (0, T) \times \mathcal{O}$  and  $\Sigma_T := (0, T) \times \partial\mathcal{O}$  to abridge the notation. We consider the following backward stochastic heat equation

$$\begin{cases} dy = -(\Delta y + \alpha y + \chi_{\mathcal{O}_0} h) dt + Y dW(t) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(T) = y_T & \text{in } \mathcal{O}. \end{cases} \quad (54)$$

In (54), we recall that  $(y, Y)$  the state variable while  $h$  is the control variable located in  $\mathcal{O}_0$  where  $\mathcal{O}_0$  is a nonempty open subset of  $\mathcal{O}$ .

We now present the proof of the null-controllability result of Theorem 1.6.

*Proof of Theorem 1.6.* We take  $p \in (2, +\infty)$  so  $p' \in (1, 2)$ .

The adjoint of (54) is given by the random heat equation

$$\begin{cases} dq = (\Delta q + \alpha q) dt & \text{in } Q_T, \\ q = 0 & \text{on } \Sigma_T, \\ q(0) = q_0 & \text{in } \mathcal{O}. \end{cases} \quad (55)$$

Note that the randomness in (55) appears through  $\alpha \in L^\infty_{\mathbb{F}}(\Omega; L^\infty(0, T; L^\infty(\mathcal{O})))$  and  $q_0 \in L^{p'}(\Omega; L^{p'}(\mathcal{O}))$ . For  $\omega \in \Omega$  fixed, we have from [FCZ00, Proposition 3.2]

$$\|q(\omega, T)\|_{L^2(\mathcal{O})} \leq C \|q(\omega)\|_{L^1((0, T) \times \mathcal{O}_0)}, \quad (56)$$

from which we directly deduce, by using  $L^2(\mathcal{O}) \hookrightarrow L^{p'}(\mathcal{O})$  and  $L^{p'}((0, T) \times \mathcal{O}_0) \hookrightarrow L^1((0, T) \times \mathcal{O}_0)$ ,

$$\|q(\omega, T)\|_{L^{p'}(\mathcal{O})} \leq C \|q(\omega)\|_{L^{p'}((0, T) \times \mathcal{O}_0)}.$$

By taking the power  $p'$  and by taking the expectation we finally get

$$\mathbb{E} \left[ \|q(T)\|_{L^{p'}(\mathcal{O})}^{p'} \right] \leq C \mathbb{E} \left[ \|q\|_{L^{p'}((0, T) \times \mathcal{O}_0)}^{p'} \right].$$

This rewrites as

$$\|q(T)\|_{L^{p'}(\Omega; L^{p'}(\mathcal{O}))} \leq C \|q\|_{L^{p'}(\Omega; L^{p'}((0, T) \times \mathcal{O}_0))}. \quad (57)$$

Note that the constant  $C > 0$  in (57) is of the form  $\exp(C/T)$  for  $T \in (0, 1)$  for some  $C$  depending on  $\mathcal{O}$ ,  $\mathcal{O}_0$ ,  $\alpha$  and  $p$  because the observability constant  $C$  in (56) appearing in the deterministic observability inequality is of the form  $\exp(C/T)$ .

Now, we will follow an argument given by [TZ09], adapted to the  $L^{p'}$ -setting. For any  $y_T \in L^p(\Omega; L^p(\mathcal{O}))$ , the goal is to find a control  $h \in L^p(\Omega; L^p((0, T) \times \mathcal{O}_0))$  such that the solution  $y$  of (54) satisfies  $y(0) = 0$ .

We introduce the following linear subspace of  $L^p_{\mathcal{F}}(0, T; L^p(\mathcal{O}_0))$ ,

$$\mathcal{X} = \{q|_{\Omega \times (0, T) \times \mathcal{O}_0} \in L^p_{\mathcal{F}}(0, T; L^p(\mathcal{O})) ; q \text{ satisfies (55) for some } q_0 \in L^p(\Omega; L^p(\mathcal{O}))\}.$$

Then we define the following linear functional on  $\mathcal{X}$  by

$$\mathcal{L}(q) = -\mathbb{E} \int_{\mathcal{O}} y_T q(T) dx.$$

The observability estimates (57) leads to the fact that  $\mathcal{L}$  is a bounded linear functional on  $\mathcal{X}$  because

$$\left| \mathbb{E} \int_{\mathcal{O}} y_T q(T) dx \right| \leq \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))} \|q(T)\|_{L^{p'}(\Omega; L^{p'}(\mathcal{O}))} \leq C \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))} \|q\|_{L^{p'}(\Omega; L^{p'}((0,T) \times \mathcal{O}_0))}. \quad (58)$$

By the Hahn-Banach theorem,  $\mathcal{L}$  can be extended to a bounded linear functional on  $L_{\mathcal{F}}^{p'}(0, T; L^{p'}(\mathcal{O}_0))$  with

$$\|\mathcal{L}\|_{\mathcal{L}(L_{\mathcal{F}}^{p'}(0, T; L^{p'}(\mathcal{O}_0)), \mathbb{R})} \leq C \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))}, \quad (59)$$

where  $C$  is the same as in (57). Now, we use the following fact coming from [LYZ12, Corollary 2.3]

$$(L_{\mathcal{F}}^{p'}(\Omega; L^{p'}(0, T; L^{p'}(\mathcal{O}_0))))' = L_{\mathcal{F}}^p(\Omega; L^p(0, T; L^p(\mathcal{O}_0))).$$

This means that there exists  $h \in L_{\mathcal{F}}^p(\Omega; L^p(0, T; L^p(\mathcal{O}_0)))$  such that

$$\mathcal{L}(q) = \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt \quad \forall q \in L^{p'}(\Omega; L^{p'}(0, T; L^{p'}(\mathcal{O}_0))), \quad (60)$$

in particular for  $q$  solution to (55)

$$\mathcal{L}(q) = -\mathbb{E} \int_{\mathcal{O}} y_T q(T) dx = \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt \quad \forall q_0 \in L^{p'}(\Omega; L^{p'}(\mathcal{O})). \quad (61)$$

Now we use Itô's formula to get

$$d(yq) = ydq + qdy + dydq.$$

So we get

$$\mathbb{E} \int_{\mathcal{O}} y_T q(T) dx - \mathbb{E} \int_{\mathcal{O}} y(0) q_0 dx = \mathbb{E} \int_{\mathcal{O}_T} ydq + qdy + dydq = \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt.$$

Therefore, we have

$$\mathbb{E} \int_{\mathcal{O}} y(0) q_0 dx = 0 \quad \forall q_0 \in L^{p'}(\Omega; L^{p'}(\mathcal{O})).$$

This implies that  $y(0) = 0$  in  $\mathcal{O}$  a.s. Moreover, we have from (59) and (60) that

$$\|h\|_{L_{\mathcal{F}}^p(\Omega; L^p(0, T; L^p(\mathcal{O}_0)))} \leq C \|y_T\|_{L^p(\Omega; L^p(\mathcal{O}))},$$

where

$$C = C(T) \leq \exp(C/T),$$

hence the expected bound on  $h$  in (7). This concludes the proof.

Now we deal with the more subtle case  $p = +\infty$ . The difficulty is that one cannot use [LYZ12, Corollary 2.3] because the so-called Radon-Nikodym property is not satisfied by  $L^\infty$ , in particular  $(L_{\mathcal{F}}^1(\Omega; L^1(0, T; L^1(\mathcal{O}_0))))'$  is not equal to  $L_{\mathcal{F}}^\infty(\Omega; L^\infty(0, T; L^\infty(\mathcal{O}_0)))$ . We start from the  $L^1$ -observability estimate for solution  $q$  to (55)

$$\|q(T)\|_{L^1(\Omega; L^1(\mathcal{O}))} \leq C \|q\|_{L^1(\Omega; L^1((0, T) \times \mathcal{O}_0))}, \quad (62)$$

that is an easy consequence of (56). We then introduce the following linear subspace of  $L^1_{\mathcal{F}}(0, T; L^1(\mathcal{O}_0))$ ,

$$\mathcal{X} = \{q_{|\Omega \times (0, T) \times \mathcal{O}_0} \in L^1_{\mathcal{F}}(0, T; L^1(\mathcal{O})) ; q \text{ satisfies (55) for some } q_0 \in L^1(\Omega; L^1(\mathcal{O}))\}.$$

Then we define the following linear functional on  $\mathcal{X}$  by

$$\mathcal{L}(q) = -\mathbb{E} \int_{\mathcal{O}} y_T q(T) dx.$$

The observability estimate (62) leads to the fact that  $\mathcal{L}$  is a bounded linear functional on  $\mathcal{X}$ . By the Hahn-Banach theorem,  $\mathcal{L}$  can be extended to a bounded linear functional on  $L^1_{\mathcal{F}}(0, T; L^1(\mathcal{O}_0))$  with

$$\|\mathcal{L}\|_{\mathcal{L}(L^1_{\mathcal{F}}(0, T; L^1(\mathcal{O}_0)), \mathbb{R})} \leq C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))}, \quad (63)$$

where  $C$  is the same as in (62). In particular,  $\mathcal{L}$  is a bounded linear functional on  $L^2_{\mathcal{F}}(0, T; L^2(\mathcal{O}_0))$  because  $L^2_{\mathcal{F}}(0, T; L^2(\mathcal{O}_0)) \hookrightarrow L^1_{\mathcal{F}}(0, T; L^1(\mathcal{O}_0))$ , therefore we can use the following fact coming from [LYZ12, Corollary 2.3],

$$(L^2_{\mathcal{F}}(\Omega; L^2(0, T; L^2(\mathcal{O}_0))))' = L^2_{\mathcal{F}}(\Omega; L^2(0, T; L^2(\mathcal{O}_0))).$$

This means that there exists  $h \in L^2_{\mathcal{F}}(\Omega; L^2(0, T; L^2(\mathcal{O}_0)))$  such that

$$\mathcal{L}(q) = \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt \quad \forall q \in L^2(\Omega; L^2(0, T; L^2(\mathcal{O}_0))), \quad (64)$$

in particular for  $q$  solution to (55)

$$\mathcal{L}(q) = -\mathbb{E} \int_{\mathcal{O}} y_T q(T) dx = \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt \quad \forall q_0 \in L^2(\Omega; L^2(\mathcal{O})).$$

As before we easily deduce by Itô's formula that  $y(0) = 0$  in  $\mathcal{O}$  a.s.

Our goal is then to prove that  $h \in L^\infty_{\mathcal{F}}(\Omega; L^\infty(0, T; L^\infty(\mathcal{O}_0)))$  and

$$\|h\|_{L^\infty_{\mathcal{F}}(\Omega; L^\infty(0, T; L^\infty(\mathcal{O}_0)))} \leq 2C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))}. \quad (65)$$

From (63) and (64), we deduce that

$$\left| \mathbb{E} \int_0^T \int_{\mathcal{O}_0} h q dx dt \right| \leq C \|q\|_{L^1(\Omega; L^1(0, T; L^1(\mathcal{O}))} \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))} \quad \forall q \in L^2(\Omega; L^2(0, T; L^2(\mathcal{O}_0))). \quad (66)$$

Then we take

$$q = \text{sign}(h) 1_{|h| > 2C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))}} 1_{\mathcal{O}_0},$$

so the left hand side of (66) is bounded from below by

$$2C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))} \mu_{\Omega, [0, T], \mathcal{O}_0}(|h| > 2C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))}),$$

while the right hand side of (66) translates into

$$C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))} \mu_{\Omega, [0, T], \mathcal{O}_0}(|h| > 2C \|y_T\|_{L^\infty(\Omega; L^\infty(\mathcal{O}))}),$$

where  $\mu_{\Omega,[0,T],\mathcal{O}_0}$  denotes the product measure on  $\Omega \times [0, T] \times \mathcal{O}$ , so we get

$$\begin{aligned} & 2C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))}\mu_{\Omega,[0,T],\mathcal{O}_0}(|h| > 2C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))}) \\ & \leq C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))}\mu_{\Omega,[0,T],\mathcal{O}_0}(|h| > 2C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))}). \end{aligned}$$

That is a contradiction unless we have

$$\mu_{\Omega,[0,T],\mathcal{O}_0}(|h| > 2C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))}) = 0.$$

This exactly means that we have

$$|h| \leq 2C\|y_T\|_{L^\infty(\Omega;L^\infty(\mathcal{O}))} \quad \text{a.e. } (t, x) \in Q_T, \text{ a.s.}$$

This concludes the proof of (65), then the one of Theorem 1.6.  $\square$

**Remark 4.1.** *In view of the results in the nonlinear setting stated in Section 1.2, we can also consider the following controlled backward stochastic semilinear heat equation*

$$\begin{cases} dy = (-\Delta y + f(y) + \chi_{\mathcal{O}_0}h) dt + YdW(t) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(T) = y_T & \text{in } \mathcal{O}. \end{cases} \quad (67)$$

Using the control result in the linear setting stated in Theorem 1.6 for  $p = +\infty$  and in view of Theorem 1.7, we can state the following result.

**Proposition 4.2.** *There is  $\delta > 0$  depending only on  $T$ ,  $\mathcal{O}$ , and  $\mathcal{O}_0$  such that for every  $y_T \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  satisfying  $\|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega;L^\infty(\mathcal{O}))} \leq \delta$ , there exists a control  $h \in L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}_0))$  such that the weak solution  $(y, Y)$  of (67) satisfies  $y(0) = 0$ .*

*To prove this result, we first prove null-controllability results for the heat equation with a  $L^\infty$ -weighted source term by using crucially the cost estimate (7). With this at hand, a standard Banach fixed-point theorem leads to the proof of the previous proposition. We refer to [LTT13] for details in the deterministic setting and [HSLBP22] in the stochastic setting for forward equations. Nonetheless, due to the paper's length and the additional notation and complexity that such a detailed exposition would introduce, we omit further details.*

## 5 Local well-posedness

In this section, we present the proof of Theorem 1.7. We begin by stating the following result.

**Proposition 5.1.** *Let  $p \in [2, +\infty)$ ,  $(y, Y)$  be a weak solution to (1) and assume that  $y_T \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  and  $F \in L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))$ . Then,  $(y, Y) \in \mathcal{Y}_p$  and satisfies*

$$\|(y, Y)\|_{\mathcal{Y}_p} \leq C \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega;L^\infty(\mathcal{O}))} + \|F\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \right),$$

for a constant  $C > 0$  only depending on  $\mathcal{O}$ ,  $T$ , and  $p$ .

*Proof.* The proof is a direct consequence of Hölder inequality, Theorem 1.2, point (ii) of Theorem 1.3, Proposition 2.6.  $\square$

*Proof of Theorem 1.7.* We will see that if the terminal data  $y_T$  is small enough, then there is a unique solution  $(y, Y)$  to (8) in  $\mathcal{Y}_p$ . We split the proof in two parts.

-*Existence.* By Taylor's formula at second order, we have that

$$\forall s \in \mathbb{R}, \quad f(s) = f'(0)s + s^2G(s), \quad (68)$$

where  $G(s) = \int_0^1 (1-\sigma)f''(\sigma s)d\sigma$ . For any  $r \in (0, 1)$ , let

$$\mathcal{Y}_p^r := \{(y, Y) \in \mathcal{Y}_p : \|(y, Y)\|_{\mathcal{Y}_p} \leq r\}, \quad (69)$$

and consider the linear equation

$$\begin{cases} dy = -(\Delta y + \alpha y + \beta Y + f'(0)y + \bar{y}^2 G(\bar{y})) dt + Y dW(t) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(T) = y_T & \text{in } \mathcal{O}, \end{cases} \quad (70)$$

where  $\bar{y}$  is the first entry of any given pair  $(\bar{y}, \bar{Y}) \in \mathcal{Y}_p^r$  and  $y_0 \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$ . Note that this is a linearized version of (8).

Let us denote by  $(y, Y)$  the solution to (70). We will check that the mapping

$$\begin{aligned} \mathcal{N} : \mathcal{Y}_p^r &\longrightarrow \mathcal{Y}_p^r \\ (\bar{y}, \bar{Y}) &\longmapsto (y, Y) \end{aligned}$$

is well-defined if we consider terminal data  $y_T$  small enough. Since  $f \in C^\infty(\mathbb{R})$ , from point i) of Lemma A.3, we have that  $|G(s)| \leq M$  for all  $s \in [-2, 2]$  where the constant  $M > 0$  only depends on  $f$ . Hence, defining  $\bar{F} := \bar{y}^2 G(\bar{y})$  we have

$$\begin{aligned} \|\bar{F}\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} &= \|\bar{y}^2 G(\bar{y})\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} \\ &\leq M \|\bar{y}^2\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))} = M \|\bar{y}\|_{L_{\mathbb{F}}^\infty(0, T; L^\infty(\mathcal{O}))}^2 \\ &\leq M \|(\bar{y}, \bar{Y})\|_{\mathcal{Y}_p}^2 \leq Mr^2 < +\infty, \end{aligned} \quad (71)$$

for any  $(\bar{y}, \bar{Y}) \in \mathcal{Y}_p^r$ .

Thus, using Proposition 5.1 with this particular  $\bar{F}$  and estimate (71) yield that there is a solution  $(y, Y) \in \mathcal{Y}_p$  to (70) such that

$$\|(y, Y)\|_{\mathcal{Y}_p} \leq C \left( \|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} + Mr^2 \right) \quad (72)$$

for some  $C > 0$  only depending on  $\mathcal{O}$ ,  $T$  and  $p$ . Let us take

$$r \leq \frac{1}{8CM},$$

where  $C$  is the constant appearing in (72) and fix  $\delta > 0$  such that  $\delta \leq r/2C$ . Thus for any  $y_T \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  verifying  $\|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} \leq \delta$  we have

$$\|(y, Y)\|_{\mathcal{Y}_p} \leq \frac{r}{2} + \frac{r}{8} = \frac{5r}{8} \leq r. \quad (73)$$

This shows that the map  $\mathcal{N} : \mathcal{Y}_p^r \rightarrow \mathcal{Y}_p^r$  is well-defined.

To conclude, we will verify that  $\mathcal{N}$  is a contraction on  $\mathscr{Y}_p^r$  and use Banach fixed point theorem. By point ii) of Lemma A.3 with  $\mathcal{I} = [-2, 2]$ , we have that for any  $\bar{y}_1, \bar{y}_2$  taken from the pairs  $(\bar{y}_i, \bar{Y}_i) \in \mathscr{Y}_p^r$ ,  $i = 1, 2$ ,

$$\begin{aligned} & \|\bar{y}_1 G(\bar{y}_1) - \bar{y}_2 G(\bar{y}_2)\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \\ & \leq M \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \left( \|\bar{y}_1\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} + \|\bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \right) \\ & \quad + M_1 \|\bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}^2 \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}, \end{aligned} \quad (74)$$

for some positive constants  $M, M_1$  only depending on  $f$ .

Denote by  $(y_i, Y_i)$  the solution of (70) for the corresponding  $\bar{y}_i$ ,  $i = 1, 2$ . From the linearity of (70), Proposition 5.1 with  $\bar{F} = \bar{y}_1 G(\bar{y}_1) - \bar{y}_2 G(\bar{y}_2)$  and  $y_T \equiv 0$ , and estimate (74), we can compute

$$\begin{aligned} & \|\mathcal{N}(\bar{y}_1, \bar{Y}_1) - \mathcal{N}(\bar{y}_2, \bar{Y}_2)\|_{\mathscr{Y}_p} = \|(y_1, Y_1) - (y_2, Y_2)\|_{\mathscr{Y}_p} \\ & \leq CM \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \left( \|\bar{y}_1\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} + \|\bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \right) \\ & \quad + CM_1 \|\bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}^2 \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \\ & \leq 2CMr \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} + CM_1 r^2 \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))}. \end{aligned} \quad (75)$$

From (75) and decreasing (if necessary) the value of  $r$  and taking  $r \leq \min\{\frac{1}{8CM}, \frac{1}{\sqrt{4CM_1}}\}$ , we get

$$\|\mathcal{N}(\bar{y}_1, \bar{Y}_1) - \mathcal{N}(\bar{y}_2, \bar{Y}_2)\|_{\mathscr{Y}_p} \leq \frac{1}{2} \|\bar{y}_1 - \bar{y}_2\|_{L_{\mathbb{F}}^\infty(0,T;L^\infty(\mathcal{O}))} \leq \frac{1}{2} \|(\bar{y}_1, \bar{Y}_1) - (\bar{y}_2, \bar{Y}_2)\|_{\mathscr{Y}_p}.$$

Thus  $\mathcal{N}$  is a contraction on the set  $\mathscr{Y}_p^r$ . A direct application of Banach fixed point theorem yields that there is a unique fixed point for  $\mathcal{N}$  which is a weak solution to (8) thanks to (68) and that belongs to  $\mathscr{Y}_p^r$  since (73) holds.

-*Uniqueness.* Let  $\delta > 0$  be fixed and small enough as in the previous part. Let  $y_T \in L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))$  be such that  $\|y_T\|_{L_{\mathcal{F}_T}^\infty(\Omega; L^\infty(\mathcal{O}))} \leq \delta$  and assume that there are two weak solutions  $(y_1, Y_1)$  and  $(y_2, Y_2)$  to (8) for this same data that belong to some ball  $\mathscr{Y}_p^r$  (recall (69)). Defining  $(y, Y) := (y_1 - y_2, Y_1 - Y_2)$ , it is not difficult to see that these new variables verify

$$\begin{cases} dy = -(\Delta y + \alpha y + \beta Y + (f(y_1) - f(y_2))) dt + Y dW(t) & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(T) = 0 & \text{in } \mathcal{O}. \end{cases}$$

Arguing as we did for obtaining (51) and taking expectation, we can deduce

$$\mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \right) \leq \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |f(y_1) - f(y_2)|^p dx ds \right) + C \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \right),$$

where  $C > 0$  is a constant only depending on  $p, \alpha$  and  $\beta$ . Since  $(y_i, Y_i) \in \mathscr{Y}_p^r$ ,  $i = 1, 2$ , and using the fact that  $f$  is locally Lipschitz on  $[-r, r]$ , there is a constant  $C' > 0$  depending on  $r$  such that  $|f(s_1) - f(s_2)| \leq C'|s_1 - s_2|$ ,  $s_1, s_2 \in [-r, r]$ , therefore

$$\mathbb{E} \left( \|y(t)\|_{L^p(\mathcal{O})}^p \right) \leq C'' \mathbb{E} \left( \int_t^T \int_{\mathcal{O}} |y|^p dx ds \right),$$

for a constant  $C'' > 0$ . A standard Grönwall argument using Lemma A.1, yields that  $y_1 = y_2$  a.e.  $(t, x) \in Q_T$ , a.s. This ends the proof.  $\square$



## A Auxiliary results

**Lemma A.1** (Backward Grönwall inequality). *Let  $g(t)$ ,  $\alpha(t)$ ,  $\beta(t)$  and  $\gamma(t)$  be integrable functions with  $\beta(t), \gamma(t) \geq 0$ . For any  $t \in [0, T]$ , if*

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \gamma(s)g(s)ds,$$

then

$$g(t) \leq \alpha(t) + \beta(t) \int_t^T \alpha(u)\gamma(u)e^{\int_t^u \beta(s)\gamma(s)ds}du.$$

In particular, if  $\alpha(t) \equiv \alpha$ ,  $\beta(t) \equiv \beta$  and  $\gamma(t) \equiv 1$ , then

$$g(t) \leq \alpha e^{\beta(T-t)}.$$

**Lemma A.2.** *Let  $(X, \mathcal{M}, \mu)$  be a finite measure space and  $f : X \rightarrow \mathbb{R}$  be a measurable function. We assume that there exists  $K > 0$  such that*

$$\forall p \geq 1, \|f\|_p \leq K. \quad (76)$$

Then  $f \in L^\infty(X)$  and

$$\|f\|_\infty \leq K.$$

*Proof.* First, we prove that  $f \in L^\infty(X)$  by proceeding as follows. We argue by contradiction. We assume that  $\|f\|_\infty = +\infty$ . Let  $M > 0$ , and let us define  $A_M = \{|f| \geq M\}$  then we have that  $\mu(A_M) > 0$ , then take  $p$  large enough such that  $\mu(A_M)^{1/p} \geq 1/2$ , then  $\|f\|_p \geq (\mu(A_M)M^p)^{1/p} \geq M/2$  and since  $M$  is arbitrary, this is contradiction. Thus, this proves that  $f \in L^\infty(X)$ .

Then, we prove the bound. This is a well-known fact that because  $f \in L^\infty(X)$ ,

$$\lim_{p \rightarrow +\infty} \|f\|_p = \|f\|_\infty.$$

Therefore, by using the uniform bound on  $L^p$  spaces of  $f$  i.e. (76), we deduce the result.  $\square$

**Lemma A.3.** *Let  $f \in C^\infty(\mathbb{R})$  and define  $G(s) = \int_0^1 (1 - \sigma)f''(\sigma s)d\sigma$ . For any bounded closed interval  $\mathcal{I} \subset \mathbb{R}$  containing the origin, there are constants  $M, M_1 > 0$  only depending on  $f$  and  $\mathcal{I}$  such that*

$$i) |G(s)| \leq M \text{ for all } s \in \mathcal{I}.$$

$$ii) \text{ For any } s_1, s_2 \in \mathcal{I}, |s_1^2 G(s_1) - s_2^2 G(s_2)| \leq M|s_1 - s_2|(|s_1| + |s_2|) + M_1|s_2|^2|s_1 - s_2|.$$

*Proof.* Point **i)** is direct with  $M := \max_{\tau \in \mathcal{I}} |f''(\tau)|$  since  $f \in C^\infty(\mathbb{R})$  and  $0 \in \mathcal{I}$ . For **ii)**, let  $s_1, s_2 \in \mathcal{I}$ , then we can write

$$s_1^2 G(s_1) - s_2^2 G(s_2) = (s_1^2 - s_2^2)G(s_1) + s_2^2(G(s_1) - G(s_2)) =: J_1 + J_2. \quad (77)$$

Using point **i)**, we can estimate  $J_1$  as follows

$$|J_1| = |(s_1^2 - s_2^2)| |G(s_1)| = |(s_1 - s_2)(s_1 + s_2)| |G(s_1)| \leq M|s_1 - s_2|(|s_1| + |s_2|). \quad (78)$$

On the other hand, w.l.o.g assume that  $s_1 > s_2$ . For any  $\sigma \in [0, 1]$ , note that since  $f \in C^\infty(\mathbb{R})$ , we have from mean value theorem that

$$|f''(\sigma s_1) - f''(\sigma s_2)| = \left| \int_{\sigma s_2}^{\sigma s_1} f'(u)du \right| \leq \sigma \max_{\tau \in \mathcal{I}} |f'(\tau)| |s_1 - s_2| \leq M_1 |s_1 - s_2|, \quad (79)$$

where we have used that  $[\sigma s_2, \sigma s_1] \subset \mathcal{I}$  since  $0 \in \mathcal{I}$  and where  $M_1 := \max_{\tau \in \mathcal{I}} |f'(\tau)|$ . Thus, from (79), we can estimate  $J_2$  as

$$\begin{aligned} |J_2| &= |s_2|^2 |G(s-1) - G(s_2)| = |s_2|^2 \left| \int_0^1 (1-\sigma) [f''(\sigma s_1) - f''(\sigma s_2)] d\sigma \right| \\ &\leq |s_2|^2 \int_0^1 |1-\sigma| |f''(\sigma s_1) - f''(\sigma s_2)| d\sigma \leq M_1 |s_2|^2 |s_1 - s_2|. \end{aligned} \quad (80)$$

Putting together (77), (78) and (80) yields the desired result.  $\square$

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## References

- [Ben83] Alain Bensoussan. Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. *Stochastics*, 9(3):169–222, 1983.
- [BRT03] Viorel Barbu, Aurel Răşcanu, and Gianmario Tessitore. Carleman estimates and controllability of linear stochastic heat equations. *Appl. Math. Optim.*, 47(2):97–120, 2003.
- [DG15] Konstantinos Dareiotis and Máté Gerencsér. On the boundedness of solutions of spdes. *Stochastic Partial Differential Equations: Analysis and Computations*, 3(1):84–102, 2015.
- [Dok12] Nikolai Dokuchaev. Backward parabolic Ito equations and the second fundamental inequality. *Random Oper. Stoch. Equ.*, 20(1):69–102, 2012.
- [DQT12] Kai Du, Jinniao Qiu, and Shanjian Tang.  $L^p$  theory for super-parabolic backward stochastic partial differential equations in the whole space. *Appl. Math. Optim.*, 65(2):175–219, 2012.
- [DT12] Kai Du and Shanjian Tang. Strong solution of backward stochastic partial differential equations in  $C^2$  domains. *Probab. Theory Related Fields*, 154(1-2):255–285, 2012.
- [DTZ13] Kai Du, Shanjian Tang, and Qi Zhang.  $W^{m,p}$ -solution ( $p \geq 2$ ) of linear degenerate backward stochastic partial differential equations in the whole space. *J. Differential Equations*, 254(7):2877–2904, 2013.
- [FCZ00] Enrique Fernández-Cara and Enrique Zuazua. Null and approximate controllability for weakly blowing up semilinear heat equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 17(5):583–616, 2000.
- [FI96] Andrei V. Fursikov and Oleg Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
- [Gao18] Peng Gao. Global Carleman estimates for the linear stochastic Kuramoto-Sivashinsky equations and their applications. *J. Math. Anal. Appl.*, 464(1):725–748, 2018.
- [HMY02] Ying Hu, Jin Ma, and Jiongmin Yong. On semi-linear degenerate backward stochastic partial differential equations. *Probab. Theory Related Fields*, 123(3):381–411, 2002.
- [HP91] Ying Hu and Shi Ge Peng. Adapted solution of a backward semilinear stochastic evolution equation. *Stochastic Anal. Appl.*, 9(4):445–459, 1991.
- [HSLBP22] Víctor Hernández-Santamaría, Kévin Le Balc’h, and Liliana Peralta. Statistical null-controllability of stochastic nonlinear parabolic equations. *Stoch. Partial Differ. Equ. Anal. Comput.*, 10(1):190–222, 2022.

- [HSLBP23] Víctor Hernández-Santamaría, Kévin Le Balc’h, and Liliana Peralta. Global null-controllability for stochastic semilinear parabolic equations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 40(6):1415–1455, 2023.
- [Liu14] Xu Liu. Global Carleman estimate for stochastic parabolic equations, and its application. *ESAIM Control Optim. Calc. Var.*, 20(3):823–839, 2014.
- [LTT13] Yuning Liu, Takéo Takahashi, and Marius Tucsnak. Single input controllability of a simplified fluid-structure interaction model. *ESAIM Control Optim. Calc. Var.*, 19(1):20–42, 2013.
- [LvN19] Qi Lü and Jan van Neerven. Backward stochastic evolution equations in UMD Banach spaces. In *Positivity and noncommutative analysis*, Trends Math., pages 381–404. Birkhäuser/Springer, Cham, [2019] ©2019.
- [LYZ12] Qi Lü, Jiongmin Yong, and Xu Zhang. Representation of Itô integrals by Lebesgue/Bochner integrals. *J. Eur. Math. Soc. (JEMS)*, 14(6):1795–1823, 2012.
- [LZ21] Qi Lü and Xu Zhang. *Mathematical control theory for stochastic partial differential equations*, volume 101 of *Probability Theory and Stochastic Modelling*. Springer, Cham, [2021] ©2021.
- [MY97] Jin Ma and Jiongmin Yong. Adapted solution of a degenerate backward SPDE, with applications. *Stochastic Process. Appl.*, 70(1):59–84, 1997.
- [MY99] Jin Ma and Jiongmin Yong. On linear, degenerate backward stochastic partial differential equations. *Probab. Theory Related Fields*, 113(2):135–170, 1999.
- [NN90] Noriaki Nagase and Makiko Nisio. Optimal controls for stochastic partial differential equations. *SIAM J. Control Optim.*, 28(1):186–213, 1990.
- [SY09] Padmanabhan Sundar and Hong Yin. Existence and uniqueness of solutions to the backward 2D stochastic Navier-Stokes equations. *Stochastic Process. Appl.*, 119(4):1216–1234, 2009.
- [Tan98] Shanjian Tang. The maximum principle for partially observed optimal control of stochastic differential equations. *SIAM J. Control Optim.*, 36(5):1596–1617, 1998.
- [TW16] Shanjian Tang and Wenning Wei. On the Cauchy problem for backward stochastic partial differential equations in Hölder spaces. *Ann. Probab.*, 44(1):360–398, 2016.
- [TZ09] Shanjian Tang and Xu Zhang. Null controllability for forward and backward stochastic parabolic equations. *SIAM J. Control Optim.*, 48(4):2191–2216, 2009.
- [Zho92] Xun Yu Zhou. A duality analysis on stochastic partial differential equations. *J. Funct. Anal.*, 103(2):275–293, 1992.
- [Zho93] Xun Yu Zhou. On the necessary conditions of optimal controls for stochastic partial differential equations. *SIAM J. Control Optim.*, 31(6):1462–1478, 1993.

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