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HYPERBOLICITY OF GENERIC HYPERSURFACES OF POLYNOMIAL DEGREE VIA GREEN-GRIFFITHS JET DIFFERENTIALS

BENOÎT CADOREL

ABSTRACT. We give a new version of a recent result of Bérczi-Kirwan, proving the Kobayashi and Green-Griffiths-Lang conjectures for generic hypersurfaces in \mathbb{P}^{n+1} , with a polynomial lower bound on the degree. Our strategy again relies on Siu’s technique of slanted vector fields and the use of holomorphic Morse inequalities to prove the existence of a jet differential equation with a negative twist – however, instead of using a space of invariant jet differentials, we base our computations on the classical Green-Griffiths jet spaces.

1. INTRODUCTION

The Green-Griffiths-Lang conjecture [GG80, Lan87] predicts that any projective manifold of general type X should be *quasi-Brody hyperbolic*, namely there should exist a proper algebraic subset $Z \subsetneq X$ containing the image of any non-constant holomorphic map $f : \mathbb{C} \rightarrow X$. Studying this question in the case of hypersurfaces is already a very difficult problem – in this situation, Kobayashi also conjectured that one should be able to obtain genuine Brody hyperbolicity for higher degrees and *generic* hypersurfaces:

Conjecture 1.1. *Let $n \geq 2$ be an integer.*

- (1) (Green-Griffiths-Lang conjecture for smooth hypersurfaces) *Any smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq n + 3$ is quasi-Brody hyperbolic;*
- (2) (Kobayashi conjecture [Kob05]) ¹ *A generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq 2n$ (or $d \geq 2n + 1$ if $n \leq 4$) is Brody hyperbolic i.e. there exists no non-constant holomorphic map $f : \mathbb{C} \rightarrow X$.*

The previous conjecture has attracted a lot of attention in the last few years, and we now know that the two items hold if we consider *generic* hypersurfaces of high enough degree:

Theorem 1. *There exists two sequences of integers d_n, d'_n such that the following hold:*

- (1) (Diverio-Merker-Rousseau [DMR10]) *a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d_n$ is quasi-Brody hyperbolic;*
- (2) (Brotbek [Bro17]) *a generic hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \geq d'_n$ is Brody hyperbolic.*

Obtaining effective bounds for the sequences d_n, d'_n is no easy task, and we are still very far from the linear bounds of Conjecture 1.1. However, a significant breakthrough has been made recently by Bérczi and Kirwan [BK24], who managed to obtain polynomial bounds $d_n \approx d'_n \approx O(n^4)$ in both items – thus substantially improving the previously known bounds, that were all growing at least as $e^{O(n \log n)}$ (see e.g. [Den20, Dem20, MT19]).

¹The bound in the second item did not appear in Kobayashi’s original article: it would follow naturally from results by Clemens-Ein-Voisin-Pacienza (see [Cle86, Ein88, Ein91, Voi96, Voi98, Pac04]) – at least for a very general hypersurface – if the Green-Griffiths-Lang conjecture were known to hold in full generality.

The strategy of [BK24] relies on the technique of *slanted vector fields* introduced by Siu [Siu04]. Eventually, everything boils down to proving the bigness of a well-chosen line bundle on an projective *jet space* $X_k \rightarrow X$ sitting above X (see Section 3 below). The classical strategy to prove this bigness is to apply Siu's *algebraic Morse inequalities*, that requires in turn to show the positivity of an adequate intersection number.

There are several possible choices for the jet space X_k , but not all seem to give very satisfactory bounds on d_n or d'_n : most of the previous exponential bounds were obtained for example using the Demailly-Semple jets spaces $X_k = X_k^{DS}$. The novelty in [BK24] was to introduce a jet space $X_k = X_k^{BK}$ on which the intersection theory is much more favorable, by means of the *non-reductive Geometric Invariant Theory*.

The previous spaces X_k^{DS} and X_k^{BK} are jet bundles naturally associated to the so-called *invariant jet differentials* – their definition is quite elaborate compared to the *Green-Griffiths jet bundles* X_k^{GG} introduced more than 40 years ago (see [GG80]). Quite surprisingly, these latter jet spaces seem to have been a bit overlooked in their potential applications to the problem at hand.

In these notes, we will show that it is indeed possible to use $X_k = X_k^{GG}$ and that following the strategy described above also yields polynomial degree bounds. More precisely, one can show the following:

Theorem 2. *Let $n \geq 2$ be an integer.*

- (1) *a generic hypersurface $X \geq \mathbb{P}^{n+1}$ of degree $d > \frac{153}{4}n^5$ is quasi-Brody hyperbolic;*
- (2) *a generic hypersurface $X \geq \mathbb{P}^{n+1}$ of degree $d > \frac{153}{4}(2n-1)^5$ is Brody hyperbolic.*

The fact that the bound looks similar in the second item is no mystery: as in [BK24], it follows from the work of Riedl-Yang [RY22] that if the first item of Theorem 1 has been proved using e.g. the jet differentials techniques of [DMR10], then the second item must also hold with $d'_n = d_{2n-1}$.

As we explained above, we make no change to the strategy of slanted vector fields: the only new input is the computation of the intersection number coming from the algebraic Morse inequalities. To perform these computations, we will use the theory of *weighted projective bundles* and their associated Segre classes, in a manner very similar to some earlier work of the author on jet differentials on compactifications of ball quotients (see [Cad20]).

1.1. Organization of the article. These notes are divided in three parts and an annex:

- (1) Section 2: we gather a few facts on weighted vector bundles, jets spaces and the holomorphic Morse inequalities;
- (2) Section 3: we recall the main criterion for hyperbolicity of generic hypersurfaces, that sums up the strategy of slanted vector fields (see Theorem 5). We then present the positivity statement that is needed to apply the holomorphic Morse inequalities (Proposition 3.1).
- (3) Section 4: we prove Proposition 3.1.
- (4) Section 5: In an annex to this article, we give a quite elementary proof of the numerical version of the Whitney formula employed in Section 4 (see the equation (10)). The author hopes this proof is even simpler than the one he presented in his thesis; in the end, it is based on straightforward computations of integrals on simplexes (in a manner very similar to the seminal work of Green-Griffiths [GG80]).

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2. WEIGHTED PROJECTIVE BUNDLES AND GREEN-GRIFFITHS JET DIFFERENTIALS

We recall here some of the results and notation of [Cad20] and [Cad18, Chapitre 3] pertaining to weighted projective bundles and their intersection theory.

2.0.1. Weighted projective bundles. Let X be a complex projective manifold. By a *weighted vector bundle* on X , we mean the data of finitely many couples $(E_i, a_i)_{1 \leq i \leq s}$, where $E_i \rightarrow X$ are vector bundles, and the $a_i \geq 1$ are integers. We will often write this data under the form

$$E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}.$$

Given a weighted vector bundle, we can construct several associated objects on X :

Definition 2.1. Let $\mathbf{E} := E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}$ be a weighted vector bundle over X .

- (1) The dual of \mathbf{E} is $\mathbf{E}^* := (E_1^*)^{(a_1)} \oplus \dots \oplus (E_s^*)^{(a_s)}$;
- (2) The symmetric algebra of \mathbf{E} is the graded \mathcal{O}_X -algebra $S^\bullet \mathbf{E} = \bigoplus_{m \in \mathbb{N}} S^m \mathbf{E}$ whose pieces are the vector bundles

$$S^m(E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}) := \bigoplus_{a_1 l_1 + \dots + a_s l_s = m} S^{l_1} E_1 \otimes \dots \otimes S^{l_s} E_s,$$

endowed with its natural product law $(S^{l_1} E_1 \otimes \dots \otimes S^{l_s} E_s) \otimes (S^{l'_1} E_1 \otimes \dots \otimes S^{l'_s} E_s) \longrightarrow S^{l_1+l'_1} E_1 \otimes \dots \otimes S^{l_s+l'_s} E_s$.

- (3) the weighted projective space $\mathbb{P}(\mathbf{E})$ is the projectivized scheme

$$\mathbb{P}(\mathbf{E}) = \text{Proj}_X(S^\bullet \mathbf{E}).$$

With the notation of the previous definition, one can also define dually $\mathbb{P}(\mathbf{E})$ as a \mathbb{C}^* -quotient:

$$P(\mathbf{E}^*) = P(E_1^{*(a_1)} \oplus \dots \oplus E_s^{*(a_s)}) := (E_1^* \oplus \dots \oplus E_s^*) - \{0\} / \mathbb{C}^*,$$

where by $\{0\}$ we denote the zero section, and the action of $\lambda \in \mathbb{C}^*$ on the total space of \mathbf{E}^* is given fiberwise by

$$\lambda \cdot (v_1, \dots, v_r) = (\lambda^{a_1} v_1, \dots, \lambda^{a_s} v_s).$$

This implies that $\pi : \mathbb{P}(\mathbf{E}) \rightarrow X$ is a bundle in weighted projective spaces ; it is endowed with tautological sheaves $\mathcal{O}^{\text{sh}}(m)$ (with $m \in \mathbb{N}$) for which one has

$$\pi_* \mathcal{O}^{\text{sh}}(m) = S^m(\mathbf{E}).$$

In general, the $\mathcal{O}^{\text{sh}}(m)$ are not locally trivial, but this is however the case if $\gcd(a_1, \dots, a_s)$ divides m . If one has also $m > 0$, then $\mathcal{O}^{\text{sh}}(m)$ is a relatively ample line bundle with respect to π . In all the following, we will use the notation $\mathcal{O}(1)$ to denote the m th-root of $\mathcal{O}^{\text{sh}}(m)$ as a \mathbb{Q} -line bundle, i.e. the element $\mathcal{O}(1) := \frac{1}{m} \mathcal{O}^{\text{sh}}(m)$ in the rational Picard group $\text{Pic} \mathbb{P}(\mathbf{E}) \otimes \mathbb{Q}$. Accordingly, we let $\mathcal{O}(d) := \mathcal{O}(1)^{\otimes d}$ for any integer $d \geq 1$; this element coincides with $\mathcal{O}^{\text{sh}}(d)$ (up to \mathbb{Q} -linear equivalence) if d is divisible enough.

Remark 2.2. Alternatively, one could also see $\mathbb{P}(\mathbf{E})$ as a smooth Deligne-Mumford stack, endowed with a natural tautological (stacky) line bundle $\mathcal{O}(1)$. In this case, the bundles $\mathcal{O}(m)$ can all be seen as line bundles on the corresponding stack, and one has naturally $\mathcal{O}(1)^{\otimes m} = \mathcal{O}(m)$.

2.0.2. Weighted Segre classes. If $\mathbf{E} = E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}$ is a weighted vector bundle, we gave in [Cad20] a definition of the Segre classes of E as endomorphisms of the rational Chow rings $(A_* X)_{\mathbb{Q}}$, as follows. Let $\alpha \in A_* X$ be any class on X . Then one lets

$$(1) \quad s_j(\mathbf{E}) \cap \alpha = \frac{1}{m^{j+r-1}} \pi_* (c_1 \mathcal{O}(m)^{j+r-1} \cap \pi^* \alpha),$$

where $\pi : \mathbb{P}(\mathbf{E}^*) \rightarrow X$ is the natural projection, $r = \sum_j \text{rk} E_j$ and $m := \text{lcm}(a_1, \dots, a_s)$ ². In this situation, we proved the following Whitney formula in [Cad18, Proposition 3.2.11], that makes sense as an equality between endomorphisms of $(A_* X)_{\mathbb{Q}}$:

$$(2) \quad s_{\bullet}(E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}) = \frac{\gcd(a_1, \dots, a_s)}{a_1 \dots a_s} \prod_j s_{\bullet}(E_j^{(a_j)}),$$

where $s_{\bullet}(E^{(a)}) = \frac{1}{a^{\text{rk} E - 1}} \sum_l \frac{s_l(E)}{a^l}$ for a single vector bundle E and any integer $a > 0$. The reader can refer to the annex (see Section 5) for a proof of a numerical version of this formula, based on straightforward computations of Euler characteristics.

2.0.3. Green-Griffiths jet differentials. Let X be a complex projective manifold. We refer to [Dem12, Section 7] for all definitions related to Green-Griffiths jet differentials. For our purposes, it will be enough to know that for any order $k \in \mathbb{N}$, we may define the *Green-Griffiths algebra of holomorphic jet differentials*

$$E_{k, \bullet}^{GG} \Omega_X = \bigoplus_{m \in \mathbb{N}} E_{k, m}^{GG} \Omega_X,$$

which is an \mathcal{O}_X -algebra whose sections represent holomorphic differential equations of order k on X . The Green-Griffiths jet bundles are the projective schemes associated to these algebras:

$$X_k^{GG} := \mathbf{Proj}_X (E_{k, \bullet}^{GG} \Omega) \xrightarrow{p_k} X$$

These spaces are endowed with natural tautological \mathbb{Q} -line bundles $\mathcal{O}_{GG, k}(1)$ such that

$$E_{k, m}^{GG} \Omega_X = p_k^* \mathcal{O}_{GG, k}(m).$$

One of the crucial properties of the Green-Griffiths algebra is the existence of a natural filtration whose graded object is easy to describe in terms of weighted vector bundles.

Theorem 3 (Green-Griffiths [GG80], see also [Dem12, Section 7]). *Let $k \in \mathbb{N}$. Then there exists a filtration F on $E_{k, \bullet}^{GG} \Omega_X$, compatible with its structure of \mathcal{O}_X -algebra, and whose associated graded algebra satisfies*

$$\text{Gr}_F(E_{k, \bullet}^{GG} \Omega_X) \cong S^{\bullet} \Omega_k,$$

²In [Cad20], we used the notation $r = \sum_j \text{rk} E_j - 1$ instead.

where $\Omega_k := \Omega_X^{(1)} \oplus \dots \oplus \Omega_X^{(k)}$.

By elementary considerations on short exact sequences ([Mer15, Lemma 2.15], see also [Cad19, Proposition 2.2]), one has, for any line bundle $L \rightarrow X$:

$$h^0(E_{k,m}^{GG} \Omega_X \otimes L) \geq h^0(X, S^m \Omega_k \otimes L) - h^1(X, S^m \Omega_k \otimes L).$$

We will use this inequality jointly with the following result:

Proposition 2.3. *Let $P_k := \mathbb{P}(\Omega_k) \xrightarrow{\pi_k} X$, and let L be a line bundle on X . Then one has, for any $1 \leq i \leq \dim X$ and any $m \geq 1$ divisible by $\gcd(1, 2, \dots, k)$:*

$$h^i(X, S^m \Omega_k \otimes L) = h^i(P_k, \mathcal{O}_{P_k}(m) \otimes \pi_k^* L),$$

where $\mathcal{O}_{P_k}(m)$ are the tautological line bundles on P_k .

Proof. By the Leray spectral sequence and the projection formula, it suffices to show that $R^i(\pi_k)_* \mathcal{O}_{P_k}(m) = 0$ for any $i > 0$ and any m divisible enough. This can be checked fiberwise, and immediately follows from the corresponding results for the cohomology of weighted projective spaces (see e.g. [Dol82, 1.4 Theorem]). \square

In particular, if we consider a very ample line bundle $\mathcal{O}_X(1)$ on X and any positive $\epsilon > 0$, one has

$$(3) \quad h^0(X_k^{GG}, \mathcal{O}_{GG,k}(m) \otimes p_k^* \mathcal{O}(-m\epsilon)) \geq (h^0 - h^1)(P_k, \mathcal{O}_{P_k}(m) \otimes \pi_k^* \mathcal{O}(-m\epsilon))$$

2.0.4. *Holomorphic Morse inequalities.* Let us recall the statement of the famous holomorphic Morse inequalities, in the version proved by Siu:

Theorem 4. [Siu [Siu93], Demailly [Dem96, §12], see also [Laz04, Remark 2.2.20]] *Let Y be a complex projective variety of dimension n . Let A, B be two nef line bundles on Y , and let $L := A \otimes B^{-1}$. Then one has*

$$(h^0 - h^1)(Y, L^{\otimes m}) \geq (A^n - nA^{n-1} \cdot B) \frac{m^n}{n!} + O(m^{n-1}).$$

2.0.5. *Nefness of adequate twists.* With the notation of the previous section, our wish in Section 3 will be to apply the holomorphic Morse inequalities to a line bundle of the form $\mathcal{O}_{P_k}(m) \otimes \mathcal{O}_X(-m\epsilon)$. To do this, the following statement will be quite useful.

Proposition 2.4. *Let $\mathcal{O}_X(1)$ be a very ample line bundle on X . Then the \mathbb{Q} -line bundle $L_k = \mathcal{O}_{P_k}(1) \otimes \pi_k^* \mathcal{O}_X(2)$ is nef on P_k .*

Proof. It follows from Definition 2.1 (1) that for any $m \geq 0$, one has

$$S^m(\Omega_k) \otimes \mathcal{O}(2m) \cong S^m\left(\bigoplus_{1 \leq l \leq k} \Omega_X(2l)^{(l)}\right)$$

for any $m \in \mathbb{N}$. This implies that L_k can be seen as the \mathbb{Q} -tautological line bundle of the weighted vector bundle

$$\mathbb{P}\left(\bigoplus_{1 \leq l \leq k} \Omega_X(2l)^{(l)}\right),$$

which is naturally isomorphic to P_k as a scheme above X . However, Lemma 2.5 below implies that each of the pieces $\Omega_X(2l)$ is globally generated. This implies that for m divisible enough, the line bundle $L_k^{\otimes m}$ is globally generated as well, and hence is nef. \square

The following very classical lemma was used in the proof of the previous proposition.

Lemma 2.5. *Let $X \subset \mathbb{P}^n$ be a submanifold, and let $\mathcal{O}_X(1)$ be the associated very ample line bundle. Then $\Omega_X(2)$ is globally generated.*

Proof. Since the restriction map $\Omega_{\mathbb{P}^n}|_X \rightarrow \Omega_X$ is onto, it suffices to prove the result for $X = \mathbb{P}^n$. Let Z_0, \dots, Z_n be homogeneous coordinates, and $z_i := \frac{Z_i}{Z_0}$ be the associated inhomogeneous coordinates on the chart $U_0 := \{Z_0 \neq 0\}$. Then the elements

$$Z_0^2 dz_i = Z_0 dZ_i - Z_i dZ_0 \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(2))$$

generate $\Omega_{\mathbb{P}^n}(2)$ on U_0 . The same reasoning also holds for the other charts U_j . \square

Remark 2.6. Using the semi-continuity of the nef property for the countable Zariski topology, we can see as in [Cad20, Proposition 4.4] that the \mathbb{Q} -line bundle $\mathcal{O}_{GG,k}(1) \otimes p_k^* \mathcal{O}(2)$ is nef on X_k^{GG} . The fact that we may obtain a nef line bundle on X_k^{GG} by taking a twist on the base *independently of k* is in stark contrast with the case of the Demailly-Semple tower, where we need an twist on the base growing exponentially fast as we climb the jet tower (see [Dem12, Proposition 7.19]). As Bérczi-Kirwan remarked in [BK24], it is also possible to use a constant twist for their non-reductive GIT quotient X_k^{BK} , which makes the holomorphic Morse inequalities much easier to satisfy.

3. STATEMENT OF THE MAIN RESULTS

In this section, we state the main estimates which, joint with Siu's strategy of slanted vector fields and Riedl-Yang's work [RY22], allow to derive the main result. All of this has become quite classical, so we will only quote the necessary statements, and refer to the original articles for more details.

We fix an integer $n \geq 2$. The general slanted jet techniques give the following:

Theorem 5 ([DMR10, Mer09, Dar16]). *Fix $d \geq n$. Assume that for any smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d , we have proven that the \mathbb{Q} -line bundle*

$$\mathcal{O}_{n,GG}(1) \otimes p_n^* \mathcal{O}_X(-(5n+3))$$

is big. Then the generic hypersurface of degree d is quasi-Brody hyperbolic.

Let $\epsilon > 0$ and $k \in \mathbb{N}_{\geq 1}$ to be fixed later. To prove the bigness of the \mathbb{Q} -line bundle $\mathcal{O}_{k,GG}(1) \otimes p_k^* \mathcal{O}_X(-\epsilon)$, we see in view of (3) and Theorem 4 that it suffices to apply the Morse inequalities to the following \mathbb{Q} -line bundle on $P_k = \mathbb{P}(\Omega_k)$:

$$M := \mathcal{O}_{P_k}(1) \otimes \pi_k^* \mathcal{O}_X(-\epsilon)$$

To do this, use first Proposition 2.4 to write it as a difference of two nef line bundles

$$M = A - B,$$

with $A = \mathcal{O}_{P_k,GG}(1) \otimes \pi_k^* \mathcal{O}_X(2)$ and $B = p_k^* \mathcal{O}_X(2 + \epsilon)$. The Morse inequalities then ask to show the positivity of

$$P(n, d, \epsilon) := A^{N_k} - N_k A^{N_k} \cdot B.$$

where $N_k = \dim P_k = n + nk - 1$. To satisfy the hypothesis of Theorem 5, we will just have to specialize to the case $k = n$ and $\epsilon = 5n + 3$.

This positivity of $P(n, d, \epsilon)$ can be achieved thanks to the following proposition, that will be proved in Section 4. The idea of using the Fujiwara estimates (see Lemma 4.4) originally stems from [DMR10] and has been used again e.g. in [Dar15, BK24].

Proposition 3.1 (Main estimates). *Assume $n \geq 2$, and fix $k = n$. Let $\epsilon > 0$ be a rational number.*

(1) *One may write*

$$(4) \quad P(n, d, \epsilon) = d \sum_{j=0}^n Q_j(n, \epsilon) d^j.$$

for some polynomials with rational coefficients $Q_j(n, \epsilon)$. The leading term $Q_n(n, \epsilon) > 0$ actually depends only on n .

(2) *There is a number $D_\epsilon > 0$, depending only on ϵ such that*

$$|Q_j(n, \epsilon)| < (D_\epsilon n^4)^{n-j} Q_n(n, \epsilon)$$

for all $j \geq 1$. One may actually take $D_\epsilon = \max(\frac{27}{2}, 9(1 + \frac{\epsilon}{4}))$.

(3) *(Fujiwara bound) As a consequence, if*

$$d > 2D_\epsilon \cdot n^4$$

then $P(n, d, \epsilon) > 0$. For such values of n, d, ϵ , the line bundle

$$\mathcal{O}_{X_n^{GG}}(1) \otimes \pi_n^* \mathcal{O}_X(-\epsilon)$$

is big.

Thus, if we fix first $n \geq 2$, and then take $\epsilon := 5n + 3$ in Proposition 3.1, we get the bound

$$d > 18 \left(\frac{5n+3}{4} + 1 \right) n^4.$$

Let us simply take a monomial lower bound that ensures positivity for all $n \geq 2$:

Theorem 6 (=Theorem 2 (1)). *For $n \geq 2$ and $d > \frac{153}{4} n^5$, the generic hypersurface of degree d in \mathbb{P}^{n+1} is quasi-hyperbolic.*

This proves the first item of Theorem 2. The second one follows from the work of Riedl-Yang [RY22], who showed that if the first item has been proven for $d \geq d_n$ using the jet differential techniques discussed here, then the Kobayashi conjecture must hold with the lower bound $d'_n = d_{2n-1}$.

4. MAIN COMPUTATIONS

In this section, we prove Proposition 3.1. We will first drop the hypothesis $k = n$ to perform the beginning of our computations; we will only resume this hypothesis after Step 4.

Step 1. Expression of A^{N_k} and $A^{N_k-1} \cdot B$ in terms of weighted Segre classes. Recall that $P_k = \mathbb{P}(\mathbf{O}_k)$. Let us denote by $\mathcal{O}_k(1)$ its tautological \mathbb{Q} -line bundle. Dually, we may also write $P_k = \mathbb{P}(\mathbf{T}_k)$ with

$$\mathbf{T}_k := T_X^{(1)} \oplus \dots \oplus T_X^{(k)}.$$

Thus, one may expand the Newton binomial and use the definition of weighted Segre classes (1) to obtain the following (pullbacks to P_k are implied):

$$(5) \quad \begin{aligned} A^{N_k} &= \int_{P_k} (u + 2h)^{N_k} && (u = c_1(\mathcal{O}_k(1)), h = c_1(\mathcal{O}_X(1))) \\ &= \sum_{l=0}^n 2^l \binom{N_k}{l} \int_X s_{n-l}(\mathbf{T}_k) h^l, \end{aligned}$$

On the other hand, we also have:

$$\begin{aligned}
(6) \quad A^{N_k-1} \cdot B &= \int_{P_k} (u+2h)^{N_k-1} (2+\epsilon) h \\
&= \sum_{l=1}^n 2^{l-1} (2+\epsilon) \binom{N_k-1}{l-1} \int_X s_{n-l}(\mathbf{T}_k) h^l.
\end{aligned}$$

Step 2. Application of the Whitney formula. To compute the previous numbers, we will apply the Whitney formula (10) to express the weighted Segre classes of \mathbf{T}_k in terms of the hyperplane class h . In our context, since $X \subset \mathbb{P}^{n+1}$ is a smooth degree d hypersurface, we have

$$s_{\bullet}(T_X) = s_{\bullet}(T_{\mathbb{P}^{n+1}}) c_{\bullet}(N_X) = \left[\sum_{j=0}^n (-h)^j \right]^{n+2} (1+hd).$$

Thus the Whitney formula yields

$$(7) \quad s_{\bullet}(\mathbf{T}_k) = \frac{1}{(k!)^n} \prod_{1 \leq l \leq k} \left[\sum_{j=0}^n \left(-\frac{h}{l}\right)^j \right]^{n+2} \left[1 + \frac{hd}{l} \right]$$

Step 3. Expressions of (5) and (6) as polynomials in d . Let us start by writing the coefficient $\Lambda_{\alpha,\beta}$ of $d^{\alpha} h^{\beta}$ in the expression $s_{\beta}(\mathbf{T}_k)$ for any integers α, β . Inspection of (7) shows that $\Lambda_{\alpha,\beta} = 0$ unless $\beta \geq \alpha$. If this holds, let us write $\beta = \alpha + \gamma$ with $\gamma \geq 0$, in which case one then has

$$(k!)^n \Lambda_{\alpha,\beta} = (-1)^{\gamma} B_{\gamma} C_{\alpha} \quad (\gamma = \beta - \alpha \geq 0)$$

where B_{γ} is the coefficient of h^{γ} in $\prod_{1 \leq l \leq k} \left(\sum_{j=0}^n \frac{h}{l}\right)^{n+2}$, and C_{α} is the coefficient of h^{α} in $\prod_{1 \leq l \leq k} \left(1 + \frac{h}{l}\right)$.

Proposition 4.1. *We have the following formulas:*

(1) $B_0 = 1$ and for all $\gamma \geq 1$, one has

$$B_{\gamma} := \sum_{\{u_1 \leq \dots \leq u_{\gamma}\} \subset S_{k,n+2}} \frac{1}{u_1 \dots u_{\gamma}}$$

In this expression, the sum runs over all non-decreasing sequences u_1, \dots, u_{γ} in the ordered set

$$S_{k,n+2} := \{1_1 < \dots < 1_{n+2} < 2_1 < \dots < 2_{n+2} < \dots < k_1 < \dots < k_{n+2}\},$$

and the product is computed by forgetting the indexes (see also [Cad18, p.77]).

(2) $C_0 = 1$ and for all $\alpha \geq 1$, one has

$$C_{\alpha} := \sum_{1 \leq l_1 < \dots < l_{\alpha} \leq k} \frac{1}{l_1 \dots l_{\alpha}}$$

Proof. Only the first item needs explaining. If $\gamma \geq 1$, the definition of B_{γ} gives

$$B_{\gamma} = \sum_{p_{1,1} + \dots + p_{1,n+2} + \dots + p_{k,1} + \dots + p_{k,n+2} = \gamma} \frac{1}{1^{p_{1,1}} \dots 1^{p_{1,n+2}} 2^{p_{2,1}} \dots 2^{p_{2,n+2}} \dots k^{p_{k,1}} \dots k^{p_{k,n+2}}}$$

Now, there is a bijection between the set of $k(n+2)$ -uples of integers $(p_{l,j})_{1 \leq l \leq k, 1 \leq j \leq n+2}$ summing to γ , and the set of sequences $\{u_1 \leq \dots \leq u_{\gamma}\} \subset S_{k,n+2}$: to any such tuple, we associate the sequence obtained by taking each element $l_j \in S_{k,n+2}$ repeated $p_{l,j}$ times. Expressing the previous sum in terms of the sequences $(u_i)_{1 \leq i \leq \gamma}$ instead gives the requested expression. \square

We may now rewrite the two intersection numbers as polynomials in d :

$$\begin{aligned}
 (k!)^n A^{N_k} &= \sum_{l=0}^n 2^l \binom{N_k}{l} \left(\sum_{0 \leq \alpha \leq n-l} \Lambda_{\alpha, n-l} d^\alpha \right) \cdot \left(\int_X h^n \right) \\
 &= \sum_{l=0}^n 2^l \binom{N_k}{l} \left(\sum_{\alpha+\gamma=n-l} (-1)^\gamma B_\gamma C_\alpha d^\alpha \right) \cdot \left(\int_X h^n \right) \\
 &= d \sum_{\alpha=0}^n \left[\sum_{l=0}^{n-\alpha} \binom{N_k}{l} 2^l (-1)^{n-\alpha-l} B_{n-\alpha-l} \right] C_\alpha d^\alpha \quad \left(\text{since } \int_X h^n = d \right)
 \end{aligned}$$

Similarly, one has

$$\begin{aligned}
 (k!)^n A^{N_k-1} \cdot B &= \sum_{l=1}^n \binom{N_k-1}{l-1} 2^{l-1} (2+\epsilon) \left(\sum_{0 \leq \alpha \leq n-l} \Lambda_{\alpha, n-l} d^\alpha \right) h^l \\
 &= \sum_{l=1}^n \binom{N_k-1}{l-1} 2^{l-1} (2+\epsilon) \left[\sum_{\alpha+\gamma=n-l} (-1)^\gamma B_\gamma C_\alpha d^\alpha \right] \cdot \left(\int_X h^n \right) \\
 &= d \sum_{\alpha=0}^n \left[\sum_{l=1}^{n-\alpha} \binom{N_k-1}{l-1} 2^{l-1} (2+\epsilon) (-1)^{n-l-\alpha} B_{n-\alpha-l} \right] C_\alpha d^\alpha
 \end{aligned}$$

This shows that:

$$A^{N_k} - N_k A^{N_k-1} \cdot B = d \sum_{\alpha=0}^n Q_\alpha(n, \epsilon) d^\alpha$$

with

$$Q_\alpha(n, \epsilon) = C_\alpha \left[\sum_{l=0}^{n-\alpha} \binom{N_k}{l} 2^l (-1)^{n-\alpha-l} B_{n-\alpha-l} - \sum_{l=1}^{n-\alpha} N_k \binom{N_k-1}{l-1} 2^{l-1} (2+\epsilon) (-1)^{n-\alpha-l} B_{n-\alpha-l} \right]$$

Let us rewrite this by singling out the term $l=0$ and merging the two sums using the formula $a \binom{b}{a} = b \binom{b-1}{a-1}$:

$$Q_\alpha(n, \epsilon) = (-1)^{n-\alpha} C_\alpha \left[B_{n-\alpha} + \sum_{l=1}^{n-\alpha} (2 - (2+\epsilon)l) \binom{N_k}{l} (-1)^l 2^{l-1} B_{n-\alpha-l} \right]$$

Step 4. Bounds on the coefficients. We now fix $k=n$. To obtain the bound (2), we are simply going to drop all the signs in the expression of Q_α to define

$$R_\alpha = C_\alpha \left[B_{n-\alpha} + \sum_{l=1}^{n-\alpha} (2 + (2+\epsilon)l) \binom{N_k}{l} 2^{l-1} B_{n-\alpha-l} \right]$$

and then use the very coarse inequality $|Q_\alpha| \leq R_\alpha$. We may rewrite R_α as

$$(8) \quad R_\alpha = C_\alpha \left[B_{n-\alpha} + \sum_{l=1}^{n-\alpha} D_l B_{n-\alpha-l} \right]$$

with

$$D_l = (2 + (2+\epsilon)l) \binom{N_k}{l} 2^{l-1}$$

The main estimates on the R_α will come from (8) and the following inequalities:

Lemma 4.2. *We have the following upper bounds:*

$$(1) \quad B_{\alpha+1} \leq 2n^2 B_\alpha$$

- (2) $C_\alpha \leq \frac{3}{2}n^2 C_{\alpha+1}$ if $\alpha + 1 \leq n$
(3) $D_{\alpha+1} \leq 9n^2 D_\alpha$ for $\alpha \geq 1$.

Proof of the lemma. (1). One has

$$\begin{aligned}
B_{\alpha+1} &= \sum_{\{u_1 \leq \dots \leq u_\alpha \leq u_{\alpha+1}\} \subset S_{n,n+2}} \frac{1}{u_1 \dots u_\alpha u_{\alpha+1}} \\
&\leq \sum_{\{u_1 \leq \dots \leq u_\alpha\} \subset S_{n,n+2}} \sum_{u_{\alpha+1} \in S_{n,n+2}} \frac{1}{u_1 \dots u_\alpha} \frac{1}{u_{\alpha+1}} \\
&\leq n(n+2) \sum_{\{u_1 \leq \dots \leq u_\alpha\} \subset S_{n,n+2}} \frac{1}{u_1 \dots u_\alpha} \\
&\leq n(n+2) B_\alpha \leq 2n^2 B_\alpha.
\end{aligned}$$

To obtain the last inequality, recall that $n \geq 2$.

(2). One has, if $\alpha + 1 \leq n$:

$$\begin{aligned}
C_{\alpha+1} &= \sum_{1 \leq l_1 < \dots < l_\alpha < l_{\alpha+1} \leq n} \frac{1}{l_1 \dots l_\alpha l_{\alpha+1}} \\
&= \frac{1}{\alpha+1} \sum_{S \subset \llbracket 1, n \rrbracket, |S|=\alpha} \sum_{l \notin S} \frac{1}{\text{prod}(S)} \frac{1}{l} \\
&\geq \frac{1}{n+1} \sum_{S \subset \llbracket 1, n \rrbracket, |S|=\alpha} \sum_{l \notin S} \frac{1}{\text{prod}(S)} \frac{1}{n} \\
&= \frac{n-\alpha}{n(n+1)} C_\alpha \geq \frac{1}{n(n+1)} C_\alpha \geq \frac{2}{3n^2} C_\alpha.
\end{aligned}$$

(3). One has

$$\begin{aligned}
D_{\alpha+1} &= (2 + (2 + \epsilon)(\alpha + 1)) \binom{N_n}{\alpha + 1} 2^\alpha \\
&\leq 6N_n (2 + (2 + \epsilon)\alpha) \binom{N_n}{\alpha} 2^{\alpha-1} = 6N_n D_\alpha
\end{aligned}$$

where we used $\binom{N_n}{\alpha+1} = \frac{N_n - \alpha}{\alpha + 1} \binom{N_n}{\alpha} \leq N_n \binom{N_n}{\alpha}$ and

$$\frac{2 + (2 + \epsilon)(\alpha + 1)}{2 + (2 + \epsilon)\alpha} \leq \frac{2 + (2 + \epsilon)(\alpha + 1)}{(2 + \epsilon)\alpha} \leq 1 + \frac{\alpha + 1}{\alpha} \leq 3.$$

Note that $N_n = n + n^2 - 1 \leq \frac{3}{2}n^2$, so finally we find

$$D_{\alpha+1} \leq 9n^2 D_\alpha.$$

□

Finally, we obtain:

Lemma 4.3. *There exists a constant D independent of n, d, ϵ , such that*

- (1) $R_{n-1} \leq 9n^4 (1 + \frac{\epsilon}{4}) R_n$

(2) for all $1 \leq \alpha \leq n-1$, we have

$$R_{\alpha-1} \leq \frac{27}{2} n^4 R_\alpha.$$

Proof. (1) One has

$$R_n = C_n B_0 = C_n > 0$$

and

$$\begin{aligned} R_{n-1} &= C_{n-1}(B_1 + D_1 B_0) \\ &= C_{n-1}(B_1 + (4 + \epsilon)N_n) \\ &\leq \frac{3}{2} n^2 C_n \cdot (2n^2 + (4 + \epsilon)\frac{3}{2}n^2) = 9n^4(1 + \frac{\epsilon}{4})R_n. \end{aligned}$$

(2) One has

$$\begin{aligned} R_{\alpha-1} &= C_{\alpha-1} \left[B_{n-\alpha+1} + \sum_{l=1}^{n-\alpha+1} D_l B_{n-\alpha+1-l} \right] \\ &= C_{\alpha-1} \left[B_{n-\alpha+1} + \sum_{l=1}^{n-\alpha} D_l B_{n-\alpha+1-l} + D_{n-\alpha+1} \right] \\ &\leq \frac{3}{2} n^2 C_\alpha \left[2n^2 B_{n-\alpha} + 2n^2 \sum_{l=1}^{n-\alpha} D_l B_{n-\alpha-l} + 9n^2 D_{n-\alpha} \right] \\ &= \frac{27}{2} n^4 C_\alpha \left[\frac{2}{9} B_{n-\alpha} + \frac{2}{9} \sum_{l=1}^{n-\alpha-1} D_l B_{n-\alpha-l} + D_{n-\alpha} \right] \\ &\leq \frac{27}{2} n^4 R_\alpha. \end{aligned}$$

Again, one gets the result. \square

This now proves item (2) of Proposition 3.1: the previous lemma shows inductively that

$$R_\alpha \leq D_\epsilon^{n-j} n^{4(n-j)} R_n,$$

with $D_\epsilon = \max(9(1 + \frac{\epsilon}{4}), \frac{27}{4})$. To obtain the result, just use the fact that $Q_n = C_n = R_n > 0$ and the inequalities $|Q_\alpha| \leq R_\alpha$.

Step 5. Fujiwara bound. Recall the following elementary result:

Lemma 4.4. *Let $Q(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_0$ be a polynomial with real coefficients such that $a_n > 0$. Assume there exists a constant $M > 0$ such that*

$$(9) \quad |a_{n-j}| \leq M^j a_n$$

for all $j > 0$. Then for $t > 2M$, one has $Q(t) > 0$.

In our situation, the polynomial Q is given by

$$Q(t) = \sum_{j=1}^n Q_j(n, \epsilon) t^j,$$

(see (4)), where n, ϵ are seen as constants. Then by item (2) of Proposition 3.1, one has the bound (9) with

$$M = D_\epsilon n^4.$$

This gives the result.

5. ANNEX. INTERSECTION THEORY ON WEIGHTED PROJECTIVE BUNDLES

The purpose of this annex is mainly to give an elementary proof of the Whitney formula (10), as an equality between numerical classes (which is enough to perform the computations presented in this article). The route we follow will be closely related to the original computations of Green-Griffiths [GG80] (i.e. estimating Euler characteristics by integrals of polynomials on adequate simplexes) – we also used this kind of computations in the work [Cad19].

Let X be a complex projective manifold, endowed with a weighted vector bundle $\mathbf{E} := E_1^{(a_1)} \oplus \dots \oplus E_r^{(a_r)}$ (the reader can refer to Section 2 for the terminology). Then the Whitney formula that was proved in the author's thesis can be stated as follows:

Proposition 5.1 ([Cad18, Proposition 3.2.11]). *We have the following equality of endomorphisms of $(A_*X)_{\mathbb{Q}}$:*

$$(10) \quad s_{\bullet}(E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}) = \frac{\gcd(a_1, \dots, a_s)}{a_1 \dots a_s} \prod_j s_{\bullet}(E_j^{(a_j)})$$

where $s_{\bullet}(E^{(a)}) = \frac{1}{a^{\text{rk}E-1}} \sum_l \frac{s_l(E)}{a^l}$. for a single vector bundle E and any integer $a > 0$.

The proof presented in [Cad18] essentially copies Fulton's presentation in [Ful98]. As announced above, we will show that this formula holds at least after quotienting by the numerical equivalence relation – the idea will be to see the top Segre class as the leading coefficient in the asymptotic expansion of the Euler characteristic of the symmetric products of \mathbf{E}^* . As the reader will see, this type of computation is very close in spirit to the ones of [GG80]. We will need several combinatorial lemmas, all gathered in Section 5.1 below.

*Proof of the identity as endomorphisms of $(A_*X)_{\mathbb{Q}, \text{num}}$.* We will prove that for any integer $k \in \llbracket 1, n \rrbracket$ and any cycle $\alpha \in A_k X$, one has for $1 \leq l \leq k$:

$$s_l(E_1^{(a_1)} \oplus \dots \oplus E_s^{(a_s)}) \cap \alpha \equiv_{\text{num}} \left(\frac{\gcd(a_1, \dots, a_s)}{a_1 \dots a_s} \prod_j s_{\bullet}(E_j^{(a_j)}) \right)_l \cap \alpha$$

Step 1. Reduction steps. The previous formula means that intersecting both sides with any $\beta \in A_{k-l}(X)$ must give the same intersection number; we see right away that replacing α by $\alpha \cap \beta$, allows to reduce to the case $l = k$, where one has to show equality between the two sides of the equation, which are seen as elements of \mathbb{Z} . Also, breaking α into its irreducible components, we see that we may finally assume $\alpha = [X]$ and $j = n$. By the classical splitting principle (see [Ful98, p.51]), one may also assume that each E_i is split as a sum of line bundles.

Step 2. Proof of the formula in the split case. Let us then assume that each $E_i = L_{i,1} \oplus \dots \oplus L_{i,r_i}$ is split as a direct sum of line bundles for all $1 \leq i \leq s$, and let $\alpha_{i,j} = c_1(L_{i,j})$ denote the corresponding Chern roots. We will rather prove the dual formula for $s_n(\mathbf{E}^*)$ to make the signs easier to track.

Let $m_0 := \gcd(a_1, \dots, a_r)$. Then, applying the asymptotic Riemann-Roch theorem, one has that

$$\begin{aligned} \int_X s_n(\mathbf{E}^*) &= \frac{1}{m_0^{n+r-1}} \int_{\mathbb{P}(\mathbf{E})} c_1 \mathcal{O}(m_0)^{n+r-1} \quad (\text{by definition of } s_n(\mathbf{E}^*)) \\ &= \lim_{\substack{m \rightarrow +\infty \\ m_0 | m}} \frac{\chi(\mathbb{P}(\mathbf{E}), \mathcal{O}(m))}{m^{n+r-1} / (n+r-1)!} \end{aligned}$$

By the Leray spectral sequence, one also has $\chi(\mathbb{P}(\mathbf{E}), \mathcal{O}(m)) = \chi(X, \pi_* \mathcal{O}(m)) = \chi(X, S^m \mathbf{E})$. But then, since each E_i is split, Proposition 5.5 below implies that

$$\int_X s_n(\mathbf{E}^*) = \frac{m_0}{a_1^{r_1} \dots a_s^{r_s}} \int_X \left(\prod_{i=1}^s \prod_{j=1}^{r_i} \sum_{p=0}^n \left(\frac{\alpha_{i,j}}{a_i} \right)^p \right)_n.$$

Now, we see that for all $i = 1, \dots, s$, the expression $\frac{1}{a_i^{r_i-1}} \prod_{j=1}^{r_i} \sum_{p=0}^n \left(\frac{\alpha_{i,j}}{a_i} \right)^p$ identifies with the formula for $s_\bullet((E_i^*)^{(a_i)})$ that is given in the statement of the proposition. Thus, one obtains

$$\int_X s_n(\mathbf{E}^*) = \frac{m_0}{a_1 \dots a_r} \int_X \left(\prod_{i=1}^s s_\bullet((E_i^*)^{(a_i)}) \right)_n,$$

which gives the result. \square

5.1. Combinatorial lemmas. The purpose of the following discussion is to prove Proposition 5.5, that was used in the proof of the Whitney formula. We need several combinatorial lemmas, closely related to several computations that appeared in [Cad19].

Notation.

- (1) For all n , we let vol_n denote the n -dimensional euclidian volume measure.
- (2) Let $m \in \mathbb{N}$. A m -dimensional simplex Δ is a metric space isomorphic to the convex envelop of $m+1$ points in \mathbb{R}^m , such that each p of them generate an affine $(p-1)$ -space. We will sometimes write $\Delta \overset{\circ}{\subset} \mathbb{R}^m$ to emphasize the fact that Δ has non-empty interior in \mathbb{R}^m (or equivalently, that $\dim \Delta = m$) and to oppose this situation to the case of a $(m-1)$ -dimensional simplex included in \mathbb{R}^m .
- (3) Recall that for any m -dimensional simplex $\Delta \overset{\circ}{\subset} \mathbb{R}^m$, the *uniform probability measure* of Δ is the measure $d\mathbf{P}_\Delta = \frac{1}{\text{vol}_m(\Delta)} d\text{vol}_m$.

Since this measure on Δ is the unique probability measure which is the restriction of a translation invariant measure on \mathbb{R}^m , we see that if $\Delta_1, \Delta_2 \subseteq \mathbb{R}^m$ are m -dimensional simplexes, and if $\Psi \in \text{GL}(\mathbb{R}^m)$ is such that $\Delta_2 = \Psi(\Delta_1)$, then Ψ sends the uniform measure of Δ_1 on the uniform measure of Δ_2 .

5.2. Lattices and volumes of fundamental domains. Let $a_1, \dots, a_r \in \mathbb{N}$. Let

$$H = \{(t_1, \dots, t_r) \in \mathbb{Z}^r \mid \sum_i a_i t_i = 0\}.$$

Then $H \subseteq \mathbb{Z}^r$ is a *primitive* sublattice, meaning that \mathbb{Z}^r/H is torsion-free. Hence, by the adapted basis theorem, there exists a basis (f_1, \dots, f_r) of \mathbb{Z}^r such that (f_1, \dots, f_{r-1}) is in turn a basis for H . Let $C_H = \sum_{1 \leq i \leq r-1} [0, 1] \cdot f_i$ denote the associated fundamental domain of H . Note that all fundamental domains are image of one another by an element of $\text{SL}(H)$ so they all have the same $(n-1)$ -volume.

Lemma 5.2. *Any fundamental domain of H has volume $\text{vol}_{r-1}(C_H) = \frac{\sqrt{\sum_{1 \leq i \leq r} a_i^2}}{\text{gcd}(a_1, \dots, a_r)}$.*

Proof. The lattice H and the proposed formula for the volume do not change if we replace a_i by $\frac{a_i}{\text{gcd}(a_1, \dots, a_r)}$, hence we can suppose that $\text{gcd}(a_1, \dots, a_r) = 1$. In this case, there exist $u_1, \dots, u_r \in \mathbb{Z}$ such that $\sum_i a_i u_i = 1$. Replacing f_r by the vector (u_1, \dots, u_r) in (f_1, \dots, f_r) still gives a basis of \mathbb{Z}^r , so we can assume that $f_r = (u_1, \dots, u_r)$.

Since (f_1, \dots, f_r) is a basis of \mathbb{Z}^r , we have $\text{vol}_r(\sum_{1 \leq i \leq r} [0, 1] \cdot f_i) = 1$. Moreover,

$$\begin{aligned} \text{vol}_r(\sum_{1 \leq i \leq r} [0, 1] \cdot f_i) &= \text{vol}_{r-1}(\sum_{1 \leq i \leq r-1} [0, 1] \cdot f_i) \cdot \|\pi_{H^\perp}(f_r)\|_{\text{eucl}} \\ &= \text{vol}_{r-1}(C_H) \cdot \|\pi_{H^\perp}(f_r)\|_{\text{eucl}} \end{aligned}$$

where $\pi_{H^\perp}(f_r)$ is the orthogonal projection of f_r on H^\perp , and $\|\cdot\|_{\text{eucl}}$ is the euclidian norm. We obtain

$$\text{vol}_{r-1}(C_H) = \frac{1}{\|\pi_{H^\perp}(f_r)\|}.$$

Let us now compute $\|\pi_{H^\perp}(f_r)\|$. Since $H^\perp = \mathbb{R} \cdot (a_1, \dots, a_r)$ by definition of H , one can write

$$f_r = (u_1, \dots, u_r) = \lambda \cdot (a_1, \dots, a_r) + (b_1, \dots, b_r),$$

with $\lambda \in \mathbb{R}$ and $(b_1, \dots, b_r) \in H_{\mathbb{R}}$. Thus, one has

$$0 = \sum_j a_j b_j = \sum_j a_j u_j - \lambda \sum_j a_j^2.$$

Since $\sum_j a_j u_j = 1$, this gives $\lambda = \frac{1}{\sum_j a_j^2}$ and thus $\|\pi_{H^\perp}(f_r)\|_{\text{eucl}}^2 = \lambda^2 \sum_j a_j^2 = \frac{1}{\sum_j a_j^2}$. We obtain the result. \square

Lemma 5.3. *Let $\underline{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, and let $\Delta_{\underline{a}} = \{(t_i) \in \mathbb{R}_+^r \mid \sum_i a_i t_i = 1\}$. Then the volume of $\Delta_{\underline{a}}$ is $\text{vol}_{r-1}(\Delta_{\underline{a}}) = \frac{1}{(r-1)!} \frac{\text{gcd}(a_1, \dots, a_r)}{a_1 \dots a_r} \text{vol}_{r-1}(C_H)$;*

Proof. By Lemma 5.2, it suffices to show that $\text{vol}_{r-1}(\Delta_{\underline{a}}) = \frac{1}{(r-1)!} \frac{\sqrt{\sum_{1 \leq i \leq r} a_i^2}}{a_1 \dots a_r}$. To perform this computation, we can for example use the parametrization of $\Delta_{\underline{a}}$ given by $\psi : t \in \Delta \mapsto (\frac{1}{a_1} t_1, \dots, \frac{1}{a_{r-1}} t_{r-1}, \frac{1}{a_r} (1 - \sum_{1 \leq i \leq r-1} t_i))$, where $\Delta = \{(t_i) \in [0, 1]^{r-1} \mid \sum_i t_i \leq 1\}$ is the standard $(r-1)$ -dimensional simplex in \mathbb{R}^{r-1} . We have then $\psi^*(d\text{vol}_{r-1}) = \sqrt{\det G} d\text{vol}_{r-1}$, where $G = (\langle \psi_*(e_i), \psi_*(e_j) \rangle)_{i,j}$ is the Gram matrix of the vectors $\psi_*(e_i)$ ($(e_i)_i$ being the canonical basis of \mathbb{R}^{r-1}). A simple computation shows that $\det G = \frac{1}{\prod_i a_i^2} \sum_i a_i^2$. Thus, we have $\text{vol}_{r-1}(\Delta_{\underline{a}}) = \frac{\sqrt{\sum_i a_i^2}}{\prod_i a_i} \text{vol}_{r-1}(\Delta)$. To conclude, it suffices to compute $\text{vol}_{r-1}(\Delta) = \frac{1}{(r-1)!}$, which is easy. \square

5.3. Some integrals. We are now going to compute the integral of some monomial functions on simplexes with respect to the uniform probability measure. The goal is to prove the following.

Lemma 5.4. *Let $\underline{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$, and let $\Delta_{\underline{a}} = \{(t_i) \in \mathbb{R}_+^r \mid \sum_i a_i t_i = 1\}$. Let $p_1, \dots, p_r \in \mathbb{N}$. Then*

$$\int_{\Delta_{\underline{a}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{a}}}(t) = \frac{(r-1)! p_1! \dots p_r!}{(p_1 + p_2 + \dots + p_r + r - 1)! a_1^{p_1} \dots a_r^{p_r}}.$$

First, let us remark that we can easily get back to the case where \underline{a} is equal to $\underline{1} := (1, \dots, 1)$. Indeed, letting $\Psi(t_1, \dots, t_r) = (a_1 t_1, \dots, a_r t_r)$, one has $\Psi_* d\mathbf{P}_{\Delta_{\underline{1}}} = d\mathbf{P}_{\Delta_{\underline{a}}}$, so that

$$\begin{aligned} \int_{\Delta_{\underline{a}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{a}}}(t) &= \int_{\Delta_{\underline{1}}} \left(\frac{t_1}{a_1}\right)^{a_1} \dots \left(\frac{t_r}{a_r}\right)^{a_r} d\mathbf{P}_{\Delta_{\underline{1}}}(t) \\ &= \frac{1}{a_1^{p_1} \dots a_r^{p_r}} \int_{\Delta_{\underline{1}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{1}}}(t) \end{aligned}$$

For any $r \in \mathbb{N}$ and $p_1, \dots, p_r \in \mathbb{N}$, let $C_{p_1, \dots, p_r} := \int_{\Delta_{\underline{1}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{1}}}(t)$. By the remark above, the proof of Lemma 5.4 will then be complete with the following result.

Lemma 5.5. *We have $C_{p_1, \dots, p_r} = \frac{(r-1)! p_1! \dots p_r!}{(p_1 + p_2 + \dots + p_r + r - 1)!}$.*

Proof. By induction on r . Letting $\Delta_r := \Delta_{(1, \dots, 1)}$ to simplify the notation (1 is repeated r times), one has

$$\begin{aligned} C_{p_1, \dots, p_r} &= \int_{t_1 + \dots + t_r = 1} t_1^{p_1} \dots t_r^{p_r} \frac{dt_1 \wedge \dots \wedge dt_{r-1}}{\text{vol}_{dt_1 \wedge \dots \wedge dt_{r-1}}(\Delta_r)} \\ &= \int_{t_1 + s = 1} dt_1 t_1^{p_1} \frac{\text{vol}_{dt_2 \wedge \dots \wedge dt_{r-1}}(s\Delta_{r-1})}{\text{vol}_{dt_1 \wedge \dots \wedge dt_{r-1}}(\Delta_r)} \int_{t_2 + \dots + t_r = s} t_2^{p_2} \dots t_r^{p_r} \frac{dt_2 \wedge \dots \wedge dt_{r-1}}{\text{vol}_{dt_2 \wedge \dots \wedge dt_{r-1}}(s\Delta_{r-1})} \\ &= \int_{t_1 + s = 1} dt_1 t_1^{p_1} \frac{s^{r-2}/(r-2)!}{1/(r-1)!} \cdot s^{p_2 + \dots + p_r} \int_{t_2 + \dots + t_r = 1} t_2^{p_2} \dots t_r^{p_r} \frac{dt_2 \wedge \dots \wedge dt_{r-1}}{\text{vol}_{dt_2 \wedge \dots \wedge dt_{r-1}}(\Delta_{r-1})} \\ &= (r-1) \int_{t_1 + s = 1} dt_1 t_1^{p_1} s^{r-2+p_2+\dots+p_r} C_{p_2, \dots, p_r} = (r-1) \frac{p_1!(p_2 + \dots + p_r + r - 2)!}{(p_1 + p_2 + \dots + p_r + r - 1)!} C_{p_2, \dots, p_r} \end{aligned}$$

where at the last line, we used Lemma 5.6 below. This permits to prove the formula by induction. \square

Lemma 5.6. *Let $a, b \in \mathbb{N}$. One has*

$$\int_0^1 t^a (1-t)^b dt = \frac{a!b!}{(a+b+1)!}.$$

Proof. This comes from the Beta function identity $\int_0^1 t^a (1-t)^b dt = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$. \square

5.4. Riemann integrals and asymptotic estimates. Using the previous results, we can now give the following asymptotic estimates that will prove useful to compute the asymptotics of Euler characteristics.

Lemma 5.7. *Fix integers $a_1, \dots, a_r \in \mathbb{N}$, and $p_1, \dots, p_r \in \mathbb{N}$. We have, for $m \rightarrow +\infty$ divisible by $\text{gcd}(a_1, \dots, a_r)$:*

$$\sum_{a_1 l_1 + \dots + a_r l_r = m} \frac{l_1^{p_1}}{p_1!} \dots \frac{l_r^{p_r}}{p_r!} = \frac{\text{gcd}(a_1, \dots, a_r)}{a_1^{p_1+1} \dots a_r^{p_r+1}} \frac{m^{p_1 + \dots + p_r + r - 1}}{(p_1 + p_2 + \dots + p_r + r - 1)!} + o(m^{p_1 + \dots + p_r + r - 1}).$$

Proof. Let H_m be the set of $(l_1, \dots, l_r) \in \mathbb{Z}^r$ such that $\sum_j l_j a_j = m$. It is non-empty if and only if $\text{gcd}(a_1, \dots, a_r) | m$, and if such is the case, it is then a translate of the lattice $H = \{(l_1, \dots, l_r) \in \mathbb{Z}^r \mid \sum_j a_j l_j = 0\}$. Let C_H denote a fundamental domain for H .

As (l_1, \dots, l_r) varies in $H_m \cap \mathbb{N}^r$, the element $(\frac{l_1}{m}, \dots, \frac{l_r}{m})$ varies in $\Delta_{\underline{a}}$, running in a lattice with cells isometric to $\frac{1}{m} C_H$. Thus, one can use a Riemann sum to obtain

$$\begin{aligned} \text{vol}_{r-1}\left(\frac{1}{m} C_H\right) \cdot \sum_{a_1 l_1 + \dots + a_r l_r = m} \left(\frac{l_1}{m}\right)^{p_1} \dots \left(\frac{l_r}{m}\right)^{p_r} &\xrightarrow{m \rightarrow +\infty} \int_{\Delta_{\underline{a}}} t_1^{p_1} \dots t_r^{p_r} d\text{vol}_{r-1}(t). \\ &= \text{vol}_{r-1}(\Delta_{\underline{a}}) \int_{\Delta_{\underline{a}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{a}}}(t). \end{aligned}$$

Thus we deduce

$$\frac{1}{m^{p_1 + \dots + p_r + r - 1}} \sum_{a_1 l_1 + \dots + a_r l_r = m} l_1^{p_1} \dots l_r^{p_r} \xrightarrow{m \rightarrow +\infty} \frac{\text{vol}_{r-1}(\Delta_{\underline{a}})}{\text{vol}_{r-1}(C_H)} \int_{\Delta_{\underline{a}}} t_1^{p_1} \dots t_r^{p_r} d\mathbf{P}_{\Delta_{\underline{a}}}(t).$$

The right hand side can be computed using Lemmas 5.3 and 5.4. This gives the result. \square

We will need another version of that lemma for our application to the asymptotic Riemann-Roch theorem.

Lemma 5.8. *Let $n, r \in \mathbb{N}$ be two integers. Let $\alpha_1, \dots, \alpha_r$ be indeterminates over \mathbb{C} . Fix integers $a_1, \dots, a_r \in \mathbb{N}$, and $p_1, \dots, p_r \in \mathbb{N}$. We have, for $m \rightarrow +\infty$ divisible by $\gcd(a_1, \dots, a_r)$:*

$$\sum_{a_1 l_1 + \dots + a_r l_r = m} \frac{(\alpha_1 l_1 + \dots + \alpha_r l_r)^n}{n!} = \frac{\gcd(a_1, \dots, a_r)}{a_1 \dots a_r} \left[\sum_{p_1 + \dots + p_r = n} \left(\frac{\alpha_1}{a_1} \right)^{p_1} \dots \left(\frac{\alpha_r}{a_r} \right)^{p_r} \right] \frac{m^{n+r-1}}{(n+r-1)!} + o(m^{n+r-1}).$$

where $o(m^{n+r-1})$ means a homogeneous polynomial of degree n in $\alpha_1, \dots, \alpha_n$, all of whose coefficients are negligible compared to m^{n+r-1} .

Proof. We expand the sum, using the Newton identity, and we apply Lemma 5.7:

$$\begin{aligned} \sum_{a_1 l_1 + \dots + a_r l_r = m} \frac{(\alpha_1 l_1 + \dots + \alpha_r l_r)^n}{n!} &= \sum_{a_1 l_1 + \dots + a_r l_r = m} \sum_{p_1 + \dots + p_r = n} \binom{n}{p_1, \dots, p_r} l_1^{p_1} \dots l_r^{p_r} \alpha_1^{p_1} \dots \alpha_r^{p_r} \\ &= \frac{\gcd(a_1, \dots, a_r)}{a_1 \dots a_r} \left[\sum_{p_1 + \dots + p_r = n} \left(\frac{\alpha_1}{a_1} \right)^{p_1} \dots \left(\frac{\alpha_r}{a_r} \right)^{p_r} \right] \frac{m^{n+r-1}}{(n+r-1)!} + o(m^{n+r-1}). \end{aligned}$$

□

Remark 5.9. In [Cad20, Proposition 3.3.8], the author invoked Toën's orbifold Riemann-Roch theorem to give an asymptotic estimate of a particular case of Lemma 5.8, which might seem a bit disproportionate. Let us use these notes to give here a more down to earth argument. We fix $k, n \in \mathbb{N}$. Identifying $\alpha_1 = \dots = \alpha_r = 1$ in the expression of Lemma 5.8, and taking $r = k$, we get:

$$\begin{aligned} \sum_{l_1 + 2l_2 + \dots + kl_k = m} \frac{(l_1 + \dots + l_k)^n}{n!} &= \left(\frac{1}{k!} \sum_{p_1 + \dots + p_k = n} \frac{1}{1^{p_1}} \dots \frac{1}{k^{p_k}} \right) \frac{m^{n+k-1}}{(n+k-1)!} + o(m^{n+k-1}) \\ &= \frac{1}{k!} \left[\sum_{1 \leq i_1 \leq \dots \leq i_n \leq k} \frac{1}{i_1 \dots i_n} \right] \frac{m^{n+k-1}}{(n+k-1)!} + o(m^{n+k-1}). \end{aligned}$$

The formula holds without restriction of divisibility on m since $\gcd(1, 2, \dots, k) = 1$. This gives back the estimate of [Cad20].

5.5. Asymptotics of Euler characteristics. In this section, we use Lemma 5.8 to determine the asymptotic behaviour of the Euler characteristics of symmetric powers of some weighted projective sums.

Proposition 5.10. *Let X be a complex projective manifold of dimension n . Let L_1, \dots, L_r be line bundles on X , and fix integers $a_1, \dots, a_r \in \mathbb{N}$. Then, one has the asymptotic expansion, as m goes to $+\infty$ while being divisible by $\gcd(a_1, \dots, a_r)$:*

$$\chi(X, S^m(L_1^{(a_1)} \oplus \dots \oplus L_r^{(a_r)})) = \frac{\gcd(a_1, \dots, a_r)}{a_1 \dots a_r} \int_X \left(\prod_{j=1}^r \sum_{p=0}^n \frac{c_1(L_j)^p}{a_j^p} \right)_n \frac{m^{n+r-1}}{(n+r-1)!} + o(m^{n+r-1}).$$

where $(\cdot)_n$ means we take the part of pure degree n of the class between the brackets.

Let $\alpha_i = c_1(L_i)$ for all $1 \leq i \leq r$. One has then, using the Hirzebruch-Riemann-Roch theorem:

$$\begin{aligned} \chi(X, S^m(L_1^{(a_1)} \oplus \dots \oplus L_r^{(a_r)})) &= \chi(X, \bigoplus_{a_1 l_1 + \dots + a_r l_r = m} L_1^{\otimes l_1} \otimes \dots \otimes L_r^{\otimes l_r}) \\ &= \sum_{a_1 l_1 + \dots + a_r l_r = m} \chi(X, L_1^{\otimes l_1} \otimes \dots \otimes L_r^{\otimes l_r}) \\ &= \sum_{a_1 l_1 + \dots + a_r l_r = m} \left[\int_X \frac{(\alpha_1 l_1 + \dots + \alpha_r l_r)^n}{n!} + \sum_{j=0}^n \int_X \beta_j \cdot (\alpha_1 l_1 + \dots + \alpha_r l_r)^{n-j} \right]. \end{aligned}$$

where for all $j = 1, \dots, n$, the symbol $\beta_j \in H^{2j}(X)$ denotes a cohomology class depending only on X , but not on m . One can now apply Lemma 5.8 to obtain the result.

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