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Simpson's quadrature for a nonlinear variational symplectic scheme

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Abstract

We propose a variational symplectic numerical method for the time integration of dynamical systems issued from the least action principle. We assume a quadratic internal interpolation of the state between two time steps and we approximate the action in one time step by the Simpson's quadrature formula. The resulting scheme is nonlinear and symplectic. First numerical experiments concern a nonlinear pendulum and we have observed experimentally very good convergence properties.

1) Introduction

The principle of least action is a key point for establishing evolution equations and partial differential equations, from classical to quantum mechanics and electromagnetisms [2, 12]. An important application of this principle is proposed with the finite element method and it is used for engineering applications since the 1950's. The principle of least action is also the starting point for the conception of symplectic numerical schemes for dynamical systems (see

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e.g. [5, 6, 11]). In particular, the Newmark scheme [7], which is very popular in structural dynamics, is symplectic. In [4], we have proposed a linear Simpson symplectic scheme that extends the previous works [6, 11] for quadratic interpolation.

In this contribution, we study a nonlinear Simpson symplectic scheme, as an alternative to Newmark's scheme. One motivation is to solve stiff problems in robotics [9]. We first recall in Section 2 the fundamental statements relative to Lagrangian and Hamiltonian mechanics and focus our attention on the nonlinear pendulum. Then the classical discrete dynamical system obtained directly with a second order discretization of the Lagrangian is presented in Section 3. Then we define and study in Section 4 a symplectic Simpson numerical scheme based on a quadrature with internal quadratic interpolation. First numerical results are presented in Section 5 and the exact solution of the nonlinear pendulum is recalled in the Annex.

2) Continuous approach

We consider a dynamical system described by a state $q(t)$ of constant mass m composed by a simple real variable to fix the ideas, and for $0 \leq t \leq T$. The continuous action S_c introduces a Lagrangian L

$$(1) \quad L\left(q, \frac{dq}{dt}\right) = \frac{m}{2} \left(\frac{dq}{dt}\right)^2 - V(q)$$

and we have

$$(2) \quad S_c = \int_0^T L\left(\frac{dq}{dt}, q(t)\right) dt.$$

The trajectories associated with the extremals of the action satisfy the Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial(\frac{dq}{dt})}\right) = \frac{\partial L}{\partial q}$. With the Lagrangian proposed in (1), the differential equation

$$(3) \quad \frac{d}{dt}\left(m \frac{dq}{dt}\right) + \frac{\partial V}{\partial q} = 0$$

of Newtonian mechanics is recovered. With the momentum $p \equiv \frac{\partial L}{\partial(\frac{dq}{dt})} = m \frac{dq}{dt}$ and the Hamilton function $H(p, q) \equiv p \frac{dq}{dt} - L\left(q, \frac{dq}{dt}\right)$, we obtain the first order system of Hamilton's equations $\frac{dp}{dt} + \frac{\partial H}{\partial q} = 0$, $\frac{dq}{dt} - \frac{\partial H}{\partial p} = 0$. In the case of the Lagrangian function introduced in (1), we have $H(p, q) = \frac{1}{2m} p^2 + V(q)$ and

$$(4) \quad \frac{dp}{dt} + \frac{\partial V}{\partial q} = 0, \quad \frac{dq}{dt} - \frac{1}{m} p = 0.$$

In this contribution, we consider the case of the nonlinear pendulum that corresponds to the potential $V(q) = m\omega^2(1 - \cos q)$. Then we have $\frac{dp}{dt} + m\omega^2 \sin q = 0$ and finally the second order dynamics $\frac{d^2q}{dt^2} + \omega^2 \sin q = 0$. An analytical solution of this problem is established in Annex. This exact solution is also shown in Figure 1.

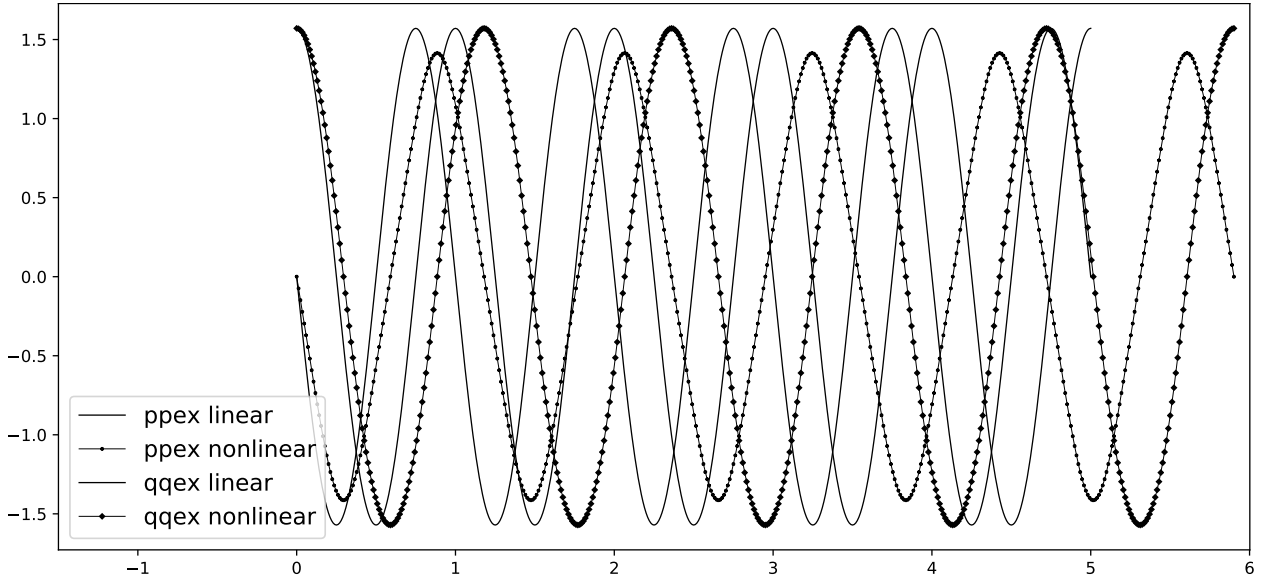


Figure 1. Typical evolution for five periods of a nonlinear pendulum satisfying the dynamics (4) with $V(q) = m\omega^2(1 - \cos q)$, $\omega = 2\pi$, $p(0) = 0$ and $q(0) = \frac{\pi}{2}$. We observe that the nonlinear period is not given by the linear evaluation $T = \frac{2\pi}{\omega}$ anymore but follows the relation (25).

3) Newmark scheme

In this section we recall the classical Newmark scheme [7] which we use as the reference scheme for benchmarking purposes. A discretization of the relation (2) is obtained by splitting the interval $[0, T]$ into N elements and we set $h = \frac{T}{N}$. At the discrete time $t_j = jh$, an approximation q_j of $q(t_j)$ is introduced and a discrete form S_d of the continuous action S_c can be defined: $S_d = \sum_{j=1}^{N-1} L_N(q_j, q_{j+1})$. The discrete Lagrangian $L_N(q_\ell, q_r)$ is derived from the relation (1) with a centered finite difference approximation $\frac{dq}{dt} \simeq \frac{q_r - q_\ell}{h}$ and a midpoint quadrature formula $\int_0^h V(q(t)) dt \simeq hV(\frac{q_\ell + q_r}{2})$:

$$(5) \quad L_N(q_\ell, q_r) = \frac{mh}{2} \left(\frac{q_r - q_\ell}{h} \right)^2 - hV\left(\frac{q_\ell + q_r}{2}\right).$$

We observe that $S_d = \dots + L_N(q_{j-1}, q_j) + L_N(q_j, q_{j+1}) + \dots$. Then the discrete Euler Lagrange equation is obtained when the action is stationary, $\delta S_d = 0$ for an arbitrary variation δq_j of the discrete variable q_j . It takes the form

$$(6) \quad \frac{\partial L_N}{\partial q_r}(q_{j-1}, q_j) + \frac{\partial L_N}{\partial q_\ell}(q_j, q_{j+1}) = 0.$$

Taking into account the relation (5), we obtain

$$(7) \quad \frac{q_{j+1} - 2q_j + q_{j-1}}{h^2} + \frac{1}{2m} [V'_{j+1/2} + V'_{j-1/2}] = 0$$

with $V'_{j+1/2} \equiv \frac{\partial V}{\partial q}\left(\frac{q_j + q_{j+1}}{2}\right)$. Observe first that the scheme (7) is implicit. Secondly, this numerical scheme is clearly consistent with the second order differential equation (3) associated with the Lagrangian proposed in (1). The momentum p_r is defined by

$$(8) \quad p_r = \frac{\partial L_N}{\partial q_r}(q_\ell, q_r).$$

We have $p_{j+1} = m \frac{q_{j+1} - q_j}{h} - \frac{h}{2} V'_{j+1/2}$ and an analogous relation for p_j . Then after two lines of algebra, we obtain the two relations $p_{j+1} - p_j = -h V'_{j+1/2}$ and $p_{j+1} + p_j = \frac{2m}{h} (q_{j+1} - q_j)$ and a discrete system involving the momentum and the state:

$$(9) \quad p_{j+1} - p_j + h \frac{\partial V}{\partial q} \left(\frac{q_j + q_{j+1}}{2} \right) = 0, \quad q_{j+1} - q_j - \frac{h}{2m} (p_{j+1} + p_j) = 0.$$

These relations are consistent with the first order Hamilton equations (4).

The implicit scheme (9) is implemented as follows. We write the equations that have to be solved at each time step under the form $F_N(p_{j+1}, q_{j+1}) = 0$. Then we consider the Newton algorithm $(p, q) \rightarrow (p^*, q^*)$:

$$F_N(p, q) + dF_N(p, q) \cdot (p^* - p, q^* - q) = 0$$

with the initialization $p = p_j, q = q_j$. Observe that the inverse of the Jacobian matrix can be evaluated easily:

$$(dF_N(p, q))^{-1} = \frac{1}{1 + \frac{h^2}{4m} V''(\frac{q_j + q}{2})} \begin{pmatrix} 1 & -\frac{h}{2} V''(\frac{q_j + q}{2}) \\ \frac{h}{2m} & 1 \end{pmatrix}.$$

In our experiments, this algorithm is converging to machine accuracy in five iterations, which is typical of Newton's algorithm.

By definition (see [11]), symplectic schemes are area-preserving. Therefore, it is sufficient to establish that the jacobian $J \equiv \frac{\partial p_{j+1}}{\partial p_j} \frac{\partial q_{j+1}}{\partial q_j} - \frac{\partial p_{j+1}}{\partial q_j} \frac{\partial q_{j+1}}{\partial p_j}$ is equal to 1. From (9), with $V''_{j+1/2} \equiv \frac{\partial V}{\partial q}(q_{j+1/2})$ and $q_{j+1/2} = \frac{1}{2}(q_j + q_{j+1})$, we have

$$\begin{aligned} \delta p_{j+1} + \frac{h}{2} V''_{j+1/2} \delta q_{j+1} &= \delta p_j - \frac{h}{2} V''_{j+1/2} \delta q_j \\ -\frac{h}{2m} \delta p_{j+1} + \delta q_{j+1} &= \frac{h}{2m} \delta p_j + \delta q_j. \end{aligned}$$

This system can be rewritten

$$(10) \quad A \begin{pmatrix} \delta p \\ \delta q \end{pmatrix}_{j+1} = B \begin{pmatrix} \delta p \\ \delta q \end{pmatrix}_j.$$

We have $\det A = 1 + \frac{h^2}{4m} V''_{j+1/2} = \det B$ and in consequence $J = 1$. Observe that although the Newmark scheme (9) is symplectic, it does not preserve the energy. While the quantity $H_j \equiv \frac{1}{2m} p_j^2 + V(q_j)$ remained constant for the linear version of the scheme (see [4]), it is not strictly constant in this nonlinear case.

4) Simpson's quadrature with quadratic interpolation

Internal interpolation between 0 and h is written in terms of quadratic finite elements (see *e.g.* [8]). For $0 \leq \theta \leq 1$, we first set

$$(11) \quad \varphi_0(\theta) = (1 - \theta)(1 - 2\theta), \quad \varphi_{1/2}(\theta) = 4\theta(1 - \theta), \quad \varphi_1(\theta) = \theta(2\theta - 1).$$

With $t = h\theta$, we consider the polynomial function

$$(12) \quad q(t) = q_\ell \varphi_0(\theta) + q_m \varphi_{1/2}(\theta) + q_r \varphi_1(\theta).$$

Then $q(0) = q_\ell$, $q(\frac{h}{2}) = q_m$ and $q(h) = q_r$ and the basis functions (11) are well adapted to these degrees of freedom. We have also $\frac{dq}{dt} = g_\ell(1 - \theta) + g_r \theta$ with the derivatives g_ℓ and g_r given by the relations

$$(13) \quad g_\ell = \frac{dq}{dt}(0) = \frac{1}{h} (-3q_\ell + 4q_m - q_r), \quad g_r = \frac{dq}{dt}(h) = \frac{1}{h} (q_\ell - 4q_m + 3q_r).$$

We remark also that

$$(14) \quad g_m = \frac{dq}{dt} \left(\frac{h}{2} \right) = \frac{1}{2} (g_\ell + g_r) = \frac{q_r - q_\ell}{h}.$$

Once the interpolation is defined in an interval of length h , we use it by splitting the range $[0, T]$ into N pieces, and $h = \frac{T}{N}$. With $t_j = jh$, we set $q_j \simeq q(t_j)$ for $0 \leq j \leq N$ and $q_{j+1/2} \simeq q(t_j + \frac{h}{2})$ with $0 \leq j \leq N-1$. In the interval $[t_j, t_{j+1}]$, the function $q(t)$ is a polynomial of degree 2, represented by the relation (12) with $t = t_j + \theta h$, $q_\ell = q_j$, $q_m = q_{j+1/2}$ and $q_r = q_{j+1}$.

For the numerical integration of a regular function ψ on the interval $[0, 1]$, the Simpson method is very popular: $\int_0^1 \psi(\theta) d\theta \simeq \frac{1}{6} [\psi(0) + 4\psi(\frac{1}{2}) + \psi(1)]$. This quadrature formula is exact up to polynomials of degree three. Then a discrete Lagrangian $L_S(q_\ell, q_m, q_r) \simeq \int_0^h [\frac{m}{2} (\frac{dq}{dt})^2 - V(q)] dt$ can be defined with the Simpson quadrature formula associated with an internal polynomial approximation $q(t)$ of degree 2 presented in (12):

$$(15) \quad L_S(q_\ell, q_m, q_r) = \frac{mh}{12} (g_\ell^2 + 4g_m^2 + g_r^2) - \frac{h}{6} (V_\ell + 4V_m + V_r)$$

with $V_\ell = V(q_\ell)$, $V_m = V(q_m)$ and $V_r = V(q_r)$. The discrete action Σ_d for a motion $t \mapsto q(t)$ between the initial time and a given time $T > 0$ is discretized with N regular intervals and take the form $\Sigma_d = \sum_{j=1}^{N-1} L_S(q_j, q_{j+1/2}, q_{j+1})$.

Euler-Lagrange equations, coming from the stationary action $\delta \Sigma_d = 0$, are first established for an arbitrary variation $\delta q_{j+1/2}$ of the internal degree of freedom in the interval $[t_j, t_{j+1}]$:

$$(16) \quad \frac{\partial L_S}{\partial q_m} = 0.$$

Due to the relations (13)(14), we first observe that $\frac{\partial g_\ell}{\partial q_m} = \frac{4}{h}$, $\frac{\partial g_m}{\partial q_m} = 0$ and $\frac{\partial g_r}{\partial q_m} = -\frac{4}{h}$. Then, due to the expression (15) of the discrete Lagrangian, we have, with $V'_m \equiv \frac{\partial V}{\partial q}(q_m)$,

$$\begin{aligned} \frac{\partial L_S}{\partial q_m} &= \frac{mh}{6} (g_\ell \frac{4}{h} + g_r \frac{4}{h}) - \frac{2}{3} h V'_m = \frac{2}{3} \frac{m}{h} (-4q_\ell + 8q_m - 4q_r) - \frac{2}{3} h V'_m \\ &= \frac{16}{3} \frac{m}{h} [q_m - \frac{1}{2}(q_\ell + q_r) - \frac{h^2}{8m} V'_m] \end{aligned}$$

$$(17) \quad q_m - \frac{h^2}{8m} \frac{\partial V}{\partial q}(q_m) = \frac{1}{2} (q_\ell + q_r).$$

It defines implicitly the value q_m at the middle of the interval as a function of the extremities q_ℓ and q_r . Under the form $\frac{4m}{h^2} (q_\ell - 2q_m + q_r) + \frac{\partial V}{\partial q}(q_m) = 0$, the relation (17) is clearly consistent with the differential equation (3).

The discrete action takes the form $\Sigma_d = \dots + L_S(q_{j-1}, q_{j-1/2}, q_j) + L_S(q_j, q_{j+1/2}, q_{j+1}) + \dots$. The variation of this discrete action is equal to zero. We have in consequence the following discrete Euler-Lagrange equations: $\frac{\partial L_S}{\partial q_r}(q_{j-1}, q_{j-1/2}, q_j) + \frac{\partial L_S}{\partial q_\ell}(q_j, q_{j+1/2}, q_{j+1}) = 0$. We obtain after some lines of elementary calculus

$$(18) \quad \frac{1}{h^2} (q_{j-1} - 2q_j + q_{j+1}) + \frac{1}{3m} (V'_{j-1/2} + V'_j + V'_{j+1/2}) = 0.$$

Due to the condition (16), the right momentum $p_r = \frac{\partial L_S}{\partial q_r}$ can be evaluated as follows:

$$p_r = \frac{mh}{6} [g_\ell (-\frac{1}{h}) + 4g_m (\frac{1}{h}) + g_r (\frac{3}{h})] - \frac{h}{6} V'_r$$

$$\begin{aligned}
 &= \frac{m}{6} \left[-\frac{1}{h} (-3q_\ell + 4q_m - q_r) + \frac{1}{h} (q_r - q_\ell) + \frac{3}{h} (q_\ell - 4q_m + 3q_r) \right] - \frac{h}{6} V'_r \\
 &= \frac{m}{6h} [2q_\ell - 16q_m + 14q_r] - \frac{h}{6} V'_r = \frac{1}{6} \left[\frac{2m}{h} (q_\ell - 8q_m + 7q_r) - h V'_r \right].
 \end{aligned}$$

We replace the intermediate value q_m with the relation (17) and we have

$$p_r = \frac{1}{6} \left[\frac{2m}{h} (q_\ell - 8(\frac{1}{2}(q_\ell + q_r) + \frac{h^2}{8m} V'_m) + 7q_r) - h V'_r \right] = \frac{m}{h} (q_r - q_\ell) - \frac{h}{6} (2V'_m + V'_r).$$

In other words,

$$(19) \quad p_{j+1} = \frac{m}{h} (q_{j+1} - q_j) - \frac{h}{6} (2V'_{j+1/2} + V'_{j+1}).$$

Similarly, we have $p_j = \frac{m}{h} (q_j - q_{j-1}) - \frac{h}{6} (2V'_{j-1/2} + V'_j)$ and taking into account the discrete Euler-Lagrange equations (18), we find

$$(20) \quad p_j = \frac{m}{h} (q_{j+1} - q_j) + \frac{h}{6} (V'_j + 2V'_{j+1/2}).$$

From the relations (19) and (20), we deduce the discrete Hamiltonian dynamics

$$(21) \quad \begin{cases} p_{j+1} - p_j + \frac{h}{6} (V'_j + 4V'_{j+1/2} + V'_{j+1}) = 0 \\ q_{j+1} - q_j - \frac{h^2}{12m} (V'_{j+1} - V'_j) - \frac{h}{2m} (p_{j+1} + p_j) = 0 \end{cases}$$

We write the system (17)(21) under the form $F_S(q_{j+1/2}, p_{j+1}, q_{j+1}) = 0$. We have

$$dF_S(q_m, p, q) = \begin{pmatrix} 1 - \frac{h\theta}{8} & 0 & -\frac{1}{2} \\ \frac{2}{3} m \theta & 1 & \frac{1}{6} m \varphi \\ 0 & -\frac{h}{2m} & 1 - \frac{h\varphi}{12} \end{pmatrix}$$

with $\theta \equiv \frac{h}{m} V''(q_m)$ and $\varphi \equiv \frac{h}{m} V''(q)$. After a formal calculation with the help of the free software ‘‘SageMath’’ [10], we explicit the inverse of this Jacobian matrix:

$$\begin{cases} (dF_S(q_m, p, q))^{-1} \\ = \begin{pmatrix} 1 & & & & \\ -\frac{2}{3} m \theta + \frac{1}{18} h m \theta \varphi & 1 - \frac{1}{8} h \theta - \frac{1}{12} h \varphi + \frac{1}{96} h^2 \theta \varphi & -\frac{1}{3} m \theta - \frac{1}{6} m \varphi + \frac{1}{48} h m \theta \varphi & & \\ & -\frac{1}{3} h \theta & & \frac{1}{2} \frac{h}{m} - \frac{1}{16} \frac{h^2 \theta}{m} & \\ & & & & 1 - \frac{1}{8} h \theta \end{pmatrix}. \end{cases}$$

At each time step, the numerical resolution of the nonlinear system of three equations $F_S(q_{j+1/2}, p_{j+1}, q_{j+1}) = 0$ is conducted with a Newton algorithm. As with Newmark’s algorithm, we have observed machine precision convergence at the fifth iteration with the proposed scheme.

The Simpson scheme (17)(21) is symplectic. With the notations $V_j'' \equiv \frac{\partial^2 V}{\partial q^2}(q_j)$ and $\tilde{V}_{j+1/2}'' \equiv \frac{2V_{j+1/2}''}{1 - \frac{h^2}{8m} V_{j+1/2}''}$, we have by differentiation of the relations (21):

$$\begin{aligned}
 \delta p_{j+1} + \frac{h}{6} (V_{j+1}'' + \tilde{V}_{j+1/2}'') \delta q_{j+1} &= \delta p_j - \frac{h}{6} (V_j'' + \tilde{V}_{j+1/2}'') \delta q_j \\
 -\frac{h}{2m} \delta p_{j+1} + (1 - \frac{h^2}{12m} V_{j+1}'') \delta q_{j+1} &= \frac{h}{2m} \delta p_j + (1 - \frac{h^2}{12m} V_j'') \delta q_j.
 \end{aligned}$$

As for the Newmark scheme, this system can be written with a relation (10). We have in this case $\det A = 1 + \frac{h^2}{12m} \tilde{V}_{j+1/2}'' = \det B$ and the relation $\frac{\partial p_{j+1}}{\partial p_j} \frac{\partial q_{j+1}}{\partial q_j} - \frac{\partial p_{j+1}}{\partial q_j} \frac{\partial q_{j+1}}{\partial p_j} = 1$ is established.

5) First numerical experiments and conclusions

We have implemented the Simpson symplectic scheme (17)(21) and have compared it with the Newmark scheme (9). Typical results for $N = 10$ meshes and one period are displayed

in Figure 2. They are compared with the exact solution presented in Figure 1. Quantitative errors with the maximum norm are also presented in Table 1 below. An asymptotic order of convergence can be estimated for the momentum, the state and various energies.

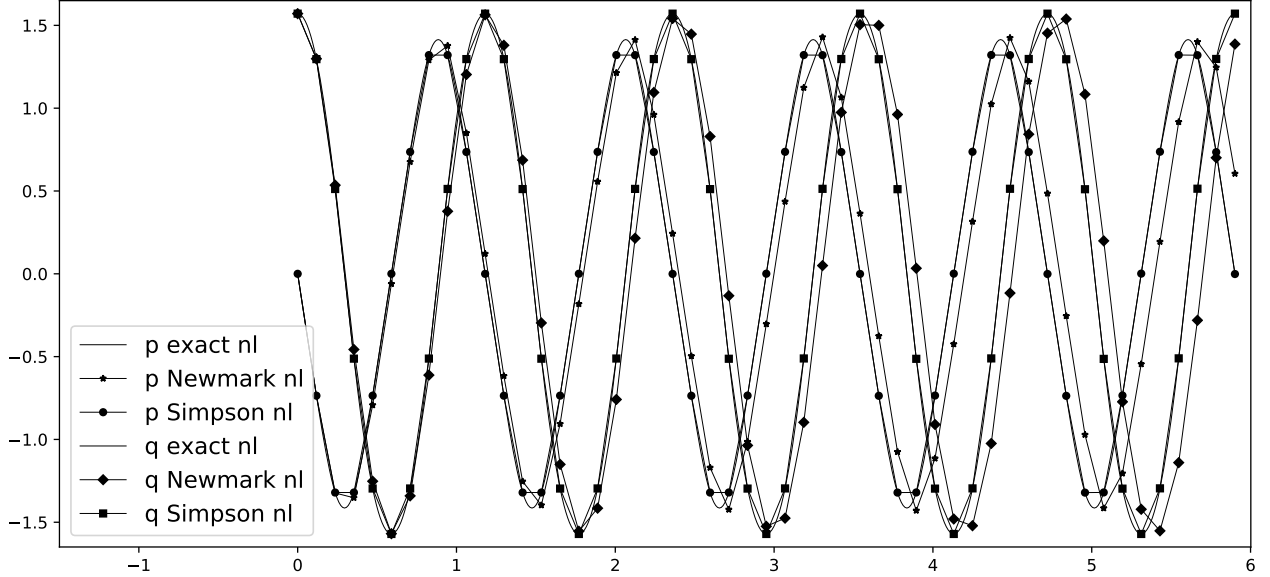


Figure 2. Comparing Newmark and Simpson schemes with 10 points per period and a total time of 5 periods. The exact solution is presented in Figure 1. The errors for the Newmark scheme are clearly visible whereas the Simpson scheme is still very precise despite the reduced number of time steps.

	number of meshes	50	100	200	order
Newmark	momentum	$2.93 \cdot 10^{-2}$	$7.32 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$	2.0
Symplectic Simpson	momentum	$6.08 \cdot 10^{-6}$	$3.78 \cdot 10^{-7}$	$2.36 \cdot 10^{-8}$	4.0
Newmark	state	$5.26 \cdot 10^{-3}$	$1.31 \cdot 10^{-3}$	$3.29 \cdot 10^{-4}$	2.0
Symplectic Simpson	state	$1.05 \cdot 10^{-6}$	$6.51 \cdot 10^{-8}$	$4.06 \cdot 10^{-9}$	4.0
Newmark	relative energy	$9.06 \cdot 10^{-4}$	$2.29 \cdot 10^{-4}$	$5.73 \cdot 10^{-5}$	2.0
Symplectic Simpson	relative energy	$1.30 \cdot 10^{-6}$	$8.42 \cdot 10^{-8}$	$5.25 \cdot 10^{-9}$	4.0

Table 1. Errors in the maximum norm (L_∞) for a simulation of a single period over all time steps. For a given discretization level, the Simpson scheme is more precise than the Newmark scheme by approximately three orders of magnitude. The orders of convergence are preliminary estimates.

In this work we have recalled Newmark's classical method which is very popular in some fields of the engineering sciences and is symplectic. We have proposed an alternative symplectic variational integrator based on Simpson's rule, to deal with nonlinear differential equations. The method was tested on the non-trivial nonlinear pendulum for which the analytical solution is known. We will be working on more complex problems in the future, starting with the symmetric spinning top. The authors thank the reviewers for their valuable comments.

Annex. Exact solution of the nonlinear pendulum

In this Annex, we follow essentially the synthesis [3]. We consider an angle $\theta_0 \in (0, \pi)$ and the non linear pendulum problem

$$(22) \quad \frac{d^2q}{dt^2} + \omega^2 \sin q = 0, \quad q(0) = \theta_0, \quad \frac{dq}{dt}(0) = 0.$$

It is easy to verify the conservation of energy: $\frac{d}{dt} \left[\frac{1}{2} \left(\frac{dq}{dt} \right)^2 + \omega^2 (1 - \cos q) \right] = 0$. We introduce the parameter $k \equiv \sin \left(\frac{\theta_0}{2} \right)$ and we have $0 < k < 1$. Due to the initial conditions in (22), we observe that $\frac{dq}{dt} < 0$ for small values of $t > 0$. Then we have $\frac{dq}{dt} = -2\omega k \sqrt{1 - \frac{1}{k^2} \sin^2 \left(\frac{q}{2} \right)}$. We introduce also the change of unknown defined by $\sin \varphi = \frac{1}{k} \sin \left(\frac{q}{2} \right)$. We observe that $\varphi(0) = \frac{\pi}{2}$. We have also $\frac{d\varphi}{dt} = -\omega \sqrt{1 - k^2 \sin^2 \varphi} = -\omega \cos \left(\frac{q}{2} \right)$ and we deduce the differential relation

$$(23) \quad \omega dt = - \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

from the previous calculus.

We introduce the incomplete elliptic integral of the first kind $F(a, m) \equiv \int_0^a \frac{d\varphi}{\sqrt{1-m \sin^2 \varphi}}$ and the complete elliptic integral of the first kind $K(m) \equiv \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-m \sin^2 \varphi}} = F\left(\frac{\pi}{2}, m\right)$ presented *e.g.* in the book of Abramowitz and Stegun [1]. Then by integration of (23),

$$(24) \quad \omega t = - \int_{\pi/2}^{\varphi} \frac{d\xi}{\sqrt{1 - k^2 \sin^2 \xi}} = K(k^2) - F(\varphi, k^2).$$

After a fourth of a period T , the parameter φ changes from $\frac{\pi}{2}$ to zero and we obtain

$$(25) \quad \omega \frac{T}{4} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = K(k^2).$$

The Jacobi amplitude $A(., m)$ is defined (see *e.g.* [1]) as the reciprocal function of the incomplete elliptic integral of the first kind: $F(a, m) = u$ is equivalent to $a = A(u, m)$. Due to (24), we have $\varphi = A(K(k^2) - \omega t, k^2)$; finally, $\sin \left(\frac{q}{2} \right) = k \sin \varphi$ and $\frac{dq}{dt} = -2\omega k \cos \varphi$.

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