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Sharp local Bernstein estimates for Laplace eigenfunctions on Riemannian manifolds

Kévin Le Balc'h*

September 26, 2024

This actual version contains a bad mistake, located in Section 2.3 that invalidates the proofs of the main results.

Abstract

In this paper we focus on local growth properties of Laplace eigenfunctions on a compact Riemannian manifold. The principal theme is that a Laplace eigenfunction behaves locally as a polynomial function of degree proportional to the square root of the eigenvalue. In this direction, we notably prove sharp local L^∞ -Bernstein estimates, conjectured by Donnelly and Fefferman in 1990. As a byproduct we also obtain analogous inequalities for A -harmonic functions where the square root of the eigenvalue is replaced by the doubling index of the solution. Our proof is based on a refinement of the original proof of L^2 -Bernstein estimates by Donnelly and Fefferman, based on L^2 -Carleman estimates, with a suitable bootstrap argument involving elliptic regularity estimates and Gagliardo–Nirenberg interpolation inequalities.

Keywords: Bernstein estimates, Laplace eigenfunctions.

Classifications: 58J50, 35J05, 35P20, 35R01.

1 Introduction

Let M be a C^∞ -smooth, compact, connected, Riemannian manifold of dimension $d \geq 2$, without boundary, equipped with a Riemannian metric g . In this article we are interested in growth properties of Laplace eigenfunctions $\varphi_\lambda \in C^\infty(M)$ associated to the eigenvalue $\lambda \geq 0$,

$$-\Delta_g \varphi_\lambda = \lambda \varphi_\lambda \text{ in } M, \quad (1.1)$$

where $\Delta_g = \operatorname{div}_g \circ \nabla_g$ is the Laplace-Beltrami operator.

One may distinguish between global growth and local growth.

The famous classical Bernstein's estimate on trigonometric polynomials is typically a global growth estimate on linear combination of Laplace eigenfunctions on the one-dimensional torus. Let $n \geq 0$ and $T = \sum_{k=-n}^n a_k e^{ikx}$ for $x \in (0, 2\pi)$, then

$$\sup_{x \in (0, 2\pi)} |T'(x)| \leq n \sup_{x \in (0, 2\pi)} |T(x)|. \quad (1.2)$$

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For a survey around (1.2), one can read [QZ19]. Global L^∞ -Bernstein estimates also hold for a single Laplace eigenfunction on M , i.e. there exist $C > 0$ depending only on M such that for every Laplace eigenfunction φ_λ ,

$$\sup_M |\nabla \varphi_\lambda| \leq C\sqrt{\lambda} \sup_M |\varphi_\lambda|. \quad (1.3)$$

This estimate (1.3) is actually a consequence of standard elliptic estimates for harmonic functions, see [OCP13, Corollary 3.3] for a proof. Note that one can actually extend global Bernstein estimates for a single Laplace eigenfunction that is (1.3) to a linear combination of Laplace eigenfunctions, i.e. there exist $C > 0$ depending only on M such that for $\Phi_\Lambda = \sum_{\lambda_k \leq \Lambda} a_k \varphi_{\lambda_k}$, then

$$\sup_M |\nabla \Phi_\Lambda| \leq C\sqrt{\Lambda} \sup_M |\Phi_\Lambda|, \quad (1.4)$$

see for instance [FM10, Theorem 2.1] and [IO22, Theorem 1.2]. In this latter case, the proofs are considerably more involved.

Concerning local growth, from the breakthrough work of Donnelly, Fefferman [DF88], we know that φ_λ also shares local growth properties with a polynomial function of degree proportional to $\sqrt{\lambda}$. One of the most celebrated result is the following bound on the doubling index of Laplace eigenfunctions, see [DF88, Theorem 4.2].

• There exist $r_0, C > 0$ depending only on M , such that for every Laplace eigenfunction $\varphi_\lambda \in C^\infty(M)$, i.e. satisfying (1.1), for every $x \in M$, $r \in (0, r_0)$,

$$\sup_{B_g(x, 2r)} |\varphi_\lambda| \leq e^{C\sqrt{\lambda}} \sup_{B_g(x, r)} |\varphi_\lambda|. \quad (1.5)$$

Note that (1.5) is in perfect agreement to the previous heuristics because

$$\sup_{t \in (-2r, 2r)} t^{\sqrt{\lambda}} = 2^{\sqrt{\lambda}} \sup_{t \in (-r, r)} t^{\sqrt{\lambda}}.$$

For $f \in C^\infty(M)$, $x \in M$, $r > 0$, the number

$$N_f(B_g(x, r)) := \log \left(\frac{\sup_{B_g(x, 2r)} |f|}{\sup_{B_g(x, r)} |f|} \right),$$

is usually called the doubling index of f in the ball $B_g(x, r)$. Note that for $x \in M$,

$$\lim_{r \rightarrow 0} N_f(B_g(x, r)) = \text{vanishing order of } f \text{ at } x,$$

where the vanishing order of f at x is the smallest integer k such that the derivatives of f of order smaller than k vanish while there is some non-zero derivative of order k . As a consequence, the doubling index estimate (1.5) tells us that the vanishing order of φ_λ is bounded by $C\sqrt{\lambda}$. This last result is sharp if we do not make extra assumptions on the Riemannian manifold because the vanishing order of spherical harmonics is comparable to $\sqrt{\lambda}$.

In [DF90a], the authors pursue the analogy between Laplace eigenfunctions and polynomial functions. They obtain the following local L^2 -Bernstein estimates, see [DF90a, Theorem 1].

• There exist $r_0, C > 0$ depending only on M , such that for every Laplace eigenfunction $\varphi_\lambda \in C^\infty(M)$, i.e. satisfying (1.1), for every $x \in M$, $r \in (0, r_0)$, $\lambda \geq 1$,

$$\|\varphi_\lambda\|_{L^2(B_g(x, r(1 + \frac{1}{\sqrt{\lambda}})))} \leq C \|\varphi_\lambda\|_{L^2(B_g(x, r))}, \quad (1.6)$$

and

$$\|\nabla\varphi_\lambda\|_{L^2(B_g(x,r))} \leq C \frac{\sqrt{\lambda}}{r} \|\varphi_\lambda\|_{L^2(B_g(x,r))}. \quad (1.7)$$

The inequality (1.7) is called a local L^2 -Bernstein estimate due to the common feature with the standard global L^∞ -Bernstein estimate (1.3).

• In [DF90a], starting from (1.6) and an elementary elliptic regularity result, the authors also obtain the following local L^∞ -Bernstein inequality

$$\sup_{B_g(x,r)} |\nabla\varphi_\lambda| \leq C \frac{\lambda^{\frac{d+2}{4}}}{r} \sup_{B_g(x,r)} |\varphi_\lambda|. \quad (1.8)$$

The authors also formulate the following conjecture.

Conjecture 1.1 ([DF90a]). The Bernstein inequality (1.8) still holds replacing $\frac{d+2}{4}$ by $\frac{1}{2}$.

Conjecture 1.1 is again motivated by the heuristics that φ_λ behaves as $t^{\sqrt{\lambda}}$ because

$$\sup_{t \in (-r,r)} \left(\frac{d}{dt} t^{\sqrt{\lambda}} \right) = \sup_{t \in (-r,r)} \sqrt{\lambda} t^{\sqrt{\lambda}-1} = \frac{\sqrt{\lambda}}{r} \sup_{t \in (-r,r)} t^{\sqrt{\lambda}}.$$

• In [Don95], Dong refines (1.8) for surfaces, i.e. when $d = 2$, using powerful geometric idea from [Don92], by obtaining

$$\sup_{B_g(x,r)} |\nabla\varphi_\lambda| \leq C \max \left(\frac{\sqrt{\lambda}}{r}, \lambda^{\frac{3}{4}} \right) \sup_{B_g(x,r)} |\varphi_\lambda| \quad (d = 2). \quad (1.9)$$

• In the very recent preprint [DM23], Decio and Malinnikova prove the following estimates

$$\sup_{B_g(x,r)} |\nabla\varphi_\lambda| \leq C \max \left(\frac{\sqrt{\lambda}}{r}, \sqrt{\lambda} \log(\lambda) \right) \sup_{B_g(x,r)} |\varphi_\lambda| \quad (d = 2), \quad (1.10)$$

and for $\delta > 0$ arbitrary,

$$\sup_{B_g(x,r)} |\nabla\varphi_\lambda| \leq C(\delta) \max \left(\frac{\sqrt{\lambda}}{r} \log^{2+\delta}(\lambda), \lambda \log^{2+\delta}(\lambda) \right) \sup_{B_g(x,r)} |\varphi_\lambda| \quad (d \geq 2). \quad (1.11)$$

In particular, (1.10) is a strong refinement of (1.9) and gives Conjecture 1.1 for surfaces up to a logarithm loss and (1.11) gives Conjecture 1.1 up to a logarithm loss at the wavelength scale, i.e. $r \in (0, r_0 \sqrt{\lambda}^{-1})$ while (1.11) resembles more to the Markov's inequality for polynomials at larger scales in any dimension. Recall that for an algebraic polynomial of degree n , the Markov inequality holds

$$\sup_{x \in (-1,+1)} |P'_n(x)| \leq n^2 \sup_{x \in (-1,+1)} |P_n(x)|. \quad (1.12)$$

Note that (1.12) is sharp because Chebychev polynomials are extremizers of this inequality.

The first main result of this paper is the establishment of Conjecture 1.1 on L^∞ -Bernstein estimates for Laplace eigenfunctions.

Theorem 1.2. *There exist $r_0, C > 0$ depending only on M , such that for every $\lambda \geq 1$, for every Laplace eigenfunction $\varphi_\lambda \in C^\infty(M)$, i.e. satisfying (1.1), for every $x \in M$, $r \in (0, r_0)$,*

$$\sup_{B_g(x, r(1 + \frac{1}{\sqrt{\lambda}}))} |\varphi_\lambda| \leq C \sup_{B_g(x, r)} |\varphi_\lambda|, \quad (1.13)$$

and

$$\sup_{B_g(x, r)} |\nabla \varphi_\lambda| \leq C \frac{\sqrt{\lambda}}{r} \sup_{B_g(x, r)} |\varphi_\lambda|. \quad (1.14)$$

The sharp doubling index estimate (1.5) is an easy consequence of (1.13), so we immediately deduce the sharpness of Theorem 1.2.

We actually prove a similar result for solutions to harmonic functions in the Euclidean space where the role of the square root of the eigenvalue is played by the doubling index, that serves as a local degree of the solution. More precisely, we look at L^∞ -Bernstein estimates for A -harmonic functions with a bounded doubling index.

The matrix $A = (a^{ij}(x))_{1 \leq i, j \leq d}$ is supposed to be symmetric, uniformly elliptic, with Lipschitz entries

$$\Lambda_1^{-1} |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda_1 |\xi|^2, \quad |a^{ij}(x) - a^{ij}(y)| \leq \Lambda_2 |x - y|, \quad x, y \in B_2, \quad \xi \in \mathbb{R}^d, \quad (1.15)$$

for some $\Lambda_1, \Lambda_2 > 0$.

The second main result of this paper is the following one.

Theorem 1.3. *There exist $r_0, C > 0$ depending on A such that for every $u \in H_{loc}^1(B_2) \cap L^\infty(B_2)$ satisfying*

$$-\operatorname{div}(A(x)\nabla u) = 0 \text{ in } B_2, \quad (1.16)$$

and

$$N_u(B(0, 1)) := \log \left(\frac{\sup_{B(0, 2)} |u|}{\sup_{B(0, 1)} |u|} \right) \leq N, \quad N \geq 1, \quad (1.17)$$

then for every $r \in (0, r_0)$,

$$\sup_{B(0, r(1 + \frac{1}{N}))} |u| \leq C \sup_{B(0, r)} |u|, \quad (1.18)$$

and

$$\sup_{B(0, r)} |\nabla u| \leq C \frac{N}{r} \sup_{B(0, r)} |u|. \quad (1.19)$$

Theorem 1.3 has to be compared to [DM23, Theorem 2] where the authors obtain a similar result with stronger regularity assumptions on the matrix A and stronger smallness assumptions on the radius r , that has to be small in function of the doubling index.

On the one hand, Theorem 1.2 and Theorem 1.3 are related by the standard lifting trick that allows to pass from Laplace eigenfunctions to harmonic functions. If φ_λ satisfies (1.1) then the function

$$u(x, t) = \varphi_\lambda(x) e^{\sqrt{\lambda}t} \quad (x, t) \in M \times \mathbb{R}, \quad (1.20)$$

is harmonic on the product manifold $M \times \mathbb{R}$ and by using (1.5) its doubling index is bounded by $C\sqrt{\lambda}$. This standard trick was first observed by [Lin91] in the study of the nodal volume for Laplace eigenfunctions on compact Riemannian manifolds, it has other applications like for instance the obtaining of the bound on the doubling index of Laplace eigenfunctions in (1.5),

see [LM20, Proposition 2.4.1]. On the other hand, it is worth mentioning that Theorem 1.2 is not a direct consequence of Theorem 1.3, as it is the case in [DM23] where the authors deduce L^∞ -Bernstein estimates for Laplace eigenfunctions from L^∞ -Bernstein estimates for A -harmonic functions because they are working at the wavelength scale, i.e. $r \leq C\sqrt{\lambda}^{-1}$. In our case, the same phenomenon appears, we can only deduce Theorem 1.2 from Theorem 1.3 for $r \leq C\sqrt{\lambda}^{-1}$. This is why we will actually prove Theorem 1.2 in an independent way by following the same strategy of the proof of Theorem 1.3 even if the some new technical difficulties appear.

New ingredient. The proof of Theorem 1.3, and therefore the one of Theorem 1.2, takes its source inside the proof of L^2 -Bernstein estimates for Laplace eigenfunctions, i.e. (1.6) and (1.7) from [DF90a], that is based on an adequate L^2 -Carleman estimate. Note that if we start directly from (1.6) and (1.7) then use elliptic regularity estimates and Sobolev embeddings, we cannot obtain better than the weak L^∞ -Bernstein estimate from [DF90a] i.e. (1.8). The new ingredient consists in implementing a suitable bootstrap argument involving scaled versions of elliptic regularity estimates and Gagliardo-Nirenberg interpolations inequalities in the Carleman's strategy.

Organization of the paper. In Section 2, we prove the main results i.e. Theorem 1.2 and Theorem 1.3. In Section 3, we state generalizations of Theorem 1.2, Theorem 1.3 and mention some related open problems.

2 Proof of the growth estimates

The goal of this part is to prove Theorem 1.2 and Theorem 1.3.

The first four parts are dedicated to the proof of Theorem 1.3 while the last part is devoted to the proof of Theorem 1.2. Recall that the proof of the growth estimates for Laplace eigenfunctions on Riemannian manifolds stated in Theorem 1.2 is a small adaptation of the one of the growth estimates for A -harmonic functions on the Euclidean space stated in Theorem 1.3, this is why we will only insist on the new difficulties that appear in the fifth part. The first part consists in stating L^2 -Carleman estimates for the operator $\operatorname{div}(A\nabla\cdot)$, the second part proves vanishing order estimates for A -harmonic functions with bounded doubling index, the third part establishes scaled versions of elliptic regularity estimates and Gagliardo-Nirenberg interpolations inequalities, the fourth part consists in applying the first three parts together with a suitable bootstrap argument to deduce the estimate (1.18) and then (1.19).

In the next four parts, the positive constants $C > 0$, $c > 0$ depend on A and d while in the last part, the positive constants $C > 0$, $c > 0$ are allowed to depend on M , g and d . To insist on the dependence of a positive constant C in function of some parameter s , we will sometimes use the notation $C = C(s)$. Moreover, the constants can vary from one line to another without explicitly mentioning it.

2.1 L^2 -Carleman estimates

The goal of this part is to state L^2 -Carleman estimates.

To simplify the notations in the next, we set

$$\operatorname{div}(A(x)\nabla f) = \Delta_A f.$$

We also introduce the spherical coordinates of a point $x = (x_1, \dots, x_d) \in \mathbb{R}^d \setminus \{0\}$ by

$$(\rho, \theta) = (\rho, \theta_1, \dots, \theta_{d-1}) \in (0, +\infty) \times \mathbb{S}^{d-1},$$

with

$$\begin{aligned} x_1 &= \rho \cos(\theta_1), \quad x_2 = \rho \sin(\theta_1) \cos(\theta_2), \quad x_3 = \rho \sin(\theta_1) \sin(\theta_2) \cos(\theta_3), \quad \dots, \\ x_{d-1} &= \rho \sin(\theta_1) \cdots \sin(\theta_{d-2}) \cos(\theta_{d-1}), \quad x_d = \rho \sin(\theta_1) \cdots \sin(\theta_{d-2}) \sin(\theta_{d-1}), \end{aligned}$$

for

$$\theta_1, \dots, \theta_{d-2} \in [0, \pi) \text{ and } \theta_{d-1} \in [0, 2\pi).$$

The orthonormal spherical basis will be denoted by (e_ρ, e_θ) and for a given vector field ξ in \mathbb{R}^d , its components with respect to this basis will be denoted respectively by (ξ_ρ, ξ_θ) .

First, we have the following standard L^2 -Carleman estimate.

Lemma 2.1. *There exists a positive constant $C = C(A) > 0$ such that for every $\alpha \geq C$, $f \in C_c^\infty(B_2 \setminus \{0\})$, the following estimate holds*

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} |f|^2 dx + \alpha \int_{B_2} \rho^{1-2\alpha} |\nabla f|^2 dx \leq C \int_{B_2} \rho^{2-2\alpha} |\Delta_A f|^2 dx. \quad (2.1)$$

Note that Lemma 2.1 is a direct application of [EV03, Theorem 2], stated in the parabolic case.

The next result tells us how the Carleman estimate (2.1) from Lemma 2.1 translates when the function vanishes in a small ball centered at 0.

Lemma 2.2. *There exists a positive constant $C = C(A) > 0$ and $c = c(A) > 0$ such that for every $r \in (0, c)$, $\alpha \geq C$, for all $f \in C_c^\infty(B_2 \setminus B_r)$, the following estimate holds*

$$\begin{aligned} \alpha^3 \int_{B_2} \rho^{-1-2\alpha} |f|^2 dx + \alpha \int_{B_2} \rho^{1-2\alpha} |\nabla f|^2 dx + \frac{\alpha^4}{r^2} \int_{B(0, r(1+2\alpha^{-1}))} \rho^{-2\alpha} |f|^2 dx \\ \leq C \int_{B_2} \rho^{2-2\alpha} |\Delta_A f|^2 dx. \end{aligned} \quad (2.2)$$

Lemma 2.2 can be obtained by adapting the arguments of the proof of the Carleman estimate in [DF90a, Lemma A], stated for the operator $-\Delta_g u - \lambda u$, in the Riemannian case.

For the sake of completeness and because we will need some small modifications of these Carleman estimates in the sequel of the paper, we decide to give a complete self-contained proof of Lemma 2.1 and Lemma 2.2 in Appendix A.

2.2 Vanishing order estimate

Before proving Theorem 1.3, one needs to prove a result on the vanishing order estimate for A -harmonic functions with bounded doubling index.

First, we have the following modification of the Carleman estimate (2.1) from Lemma 2.1.

Lemma 2.3. *There exists a positive constant $C = C(A) > 0$ such that for every $x_0 \in B_{1/4}$, for every $\alpha \geq C$, $f \in C_c^\infty(B_2 \setminus \{x_0\})$, the following estimate holds*

$$\alpha^3 \int_{B_2} |x - x_0|^{-1-2\alpha} |f|^2 dx + \alpha \int_{B_2} |x - x_0|^{1-2\alpha} |\nabla f|^2 dx \leq C \int_{B_2} |x - x_0|^{2-2\alpha} |\Delta_A f|^2 dx. \quad (2.3)$$

The proof of Lemma 2.3 is postponed in Appendix A, it is an adaptation of Lemma 2.1.

As a consequence of Lemma 2.3, we deduce the following result.

Lemma 2.4. *There exists $C = C(A) > 0$ such that for every $u \in H_{loc}^1(B_2) \cap L^\infty(B_2)$ satisfying (1.16) and (1.17), the following estimate holds*

$$\sup_{B(x,1/4)} |u| \geq \exp(-CN) \sup_{B_2} |u| \quad \forall x \in B_{1/2}. \quad (2.4)$$

Proof. Let us fix $x_0 \in B(0, 1/2)$.

Let $x_{\max} \in B(0, 1)$ be such that

$$|u(x_{\max})| = \sup_{B(0,1)} |u|.$$

We distinguish two cases.

First case: $x_{\max} \in B(x_0, 1/4)$. Then we have by (1.17),

$$\sup_{B(x_0,1/4)} |u| = \sup_{B(0,1)} |u| \geq \exp(-N) \sup_{B_2} |u|,$$

so (2.4) holds.

Second case: $x_{\max} \notin B(x_0, 1/4)$. Let $\chi \in C_c^\infty(B_2; [0, 1])$ be a cut-off function such that

$$\begin{aligned} \chi &= 0 \text{ in } B(x_0, 1/16), \\ \chi &= 1 \text{ in } B(0, 15/8) \setminus B(x_0, 1/8), \\ \chi &= 0 \text{ in } B(0, 2) \setminus B(0, 31/16). \end{aligned}$$

In particular $\chi \equiv 1$ in $B(x_{\max}, 1/8)$.

By local elliptic regularity, we have that $u \in H_{loc}^2(B_2)$ so by a straightforward density argument one can apply the modified version of the Carleman estimate (2.3) of Lemma 2.3 to $f := \chi u$. By denoting $w(x) = |x - x_0|$, we then obtain

$$\alpha^3 \int_{B_2} w^{-1-2\alpha} |f|^2 dx \leq C \int_{B_2} w^{2-2\alpha} |\Delta_A f|^2 dx.$$

By using the equation (1.16), we deduce that

$$\alpha^3 \int_{B_2} w^{-1-2\alpha} \chi^2 |u|^2 dx \leq C \int_{B_2} w^{2-2\alpha} (|\nabla \chi|^2 |\nabla u|^2 + [|\nabla \chi|^2 + |D^2 \chi|^2] |u|^2) dx.$$

By using the definition of χ and local elliptic regularity estimates, it is straightforward to get that there exist universal positive constants $C_3 > C_2 > C_1 > 0$ and a positive constant $C = C(A) > 0$ such that

$$\exp(-C_2 \alpha) |u(x_{\max})|^2 \leq C \exp(-C_3 \alpha) (\sup_{B_2} |u|)^2 + C \exp(-C_1 \alpha) \left(\sup_{B(x_0, 1/8)} |u| \right)^2.$$

By using the definition of x_{\max} and the doubling index estimate (1.17), we therefore obtain that

$$\exp(-C_2 \alpha) (\sup_{B_1} |u|)^2 \leq C \exp(-C_3 \alpha) \exp(2N) (\sup_{B_1} |u|)^2 + C \exp(-C_1 \alpha) \left(\sup_{B(x_0, 1/8)} |u| \right)^2.$$

Now the punchline, by taking $\alpha \geq C(C_2, C_3, A)N$, the first right hand side term can be hidden in the left hand side term to deduce that

$$\sup_{B_1} |u| \leq \exp(CN) \sup_{B(x_0, 1/8)} |u|.$$

This concludes the proof of (2.4) recalling again (1.17). \square

2.3 Elliptic regularity estimates and interpolation inequalities

There is a mistake in this part, because the scaling in α parameter is wrong, so the proof of the next lemma does not work. This point is crucial for the next.

The goal of this part is to establish scaled versions of local elliptic regularity estimates for the operator $\operatorname{div}(A\nabla\cdot)$ and Gagliardo-Nirenberg interpolations inequalities.

We have the following relations for the divergence and gradient operators in spherical coordinates

$$\operatorname{div}(\xi) = \frac{1}{\rho^{n-1}}\partial_\rho(\rho^{n-1}\xi_\rho) + \frac{1}{\rho}\operatorname{div}_\theta(\xi_\theta), \quad (2.5)$$

and

$$\nabla v = (\partial_\rho v)e_\rho + \frac{1}{\rho}\nabla_\theta v, \quad (2.6)$$

In the sequel, we need the following notation for the annulus

$$\mathcal{C}(r_1, r_2) = B(0, r_2) \setminus B(0, r_1) \quad \forall 0 < r_1 < r_2. \quad (2.7)$$

Lemma 2.5. *Let $C_3 > C_2 > C_1 > C_0 > 0$, $r \in (0, c)$ and $\alpha \geq C$.*

1. *Let $p \in (1, +\infty)$, then there exists a positive constant $C = C(A, p, C_0, C_1, C_2, C_3) > 0$ such that for every $u \in W_{loc}^{2,p}(B_2)$, the following estimate holds*

$$\begin{aligned} r^{-1}\alpha \|\nabla u\|_{L^p(\mathcal{C}(r(1+C_1\alpha^{-1}), r(1+C_2\alpha^{-1})))} \\ \leq C \left(\|\Delta_A u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} \right. \\ \left. + r^{-2}\alpha^2 \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} \right). \quad (2.8) \end{aligned}$$

2. *Let $p \in [2, +\infty)$, $q \in (2, +\infty]$ be such that*

$$\frac{1}{q} = \frac{1}{2} \left(\frac{1}{p} - \frac{1}{d} \right) + \frac{1}{2p}. \quad (2.9)$$

Then there exists a positive constant $C = C(A, p, C_0, C_3) > 0$ such that for every $u \in W_{loc}^{1,p}(B_2)$ then $u \in L_{loc}^q(B_2)$ and

$$\begin{aligned} \|u\|_{L^q(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} \\ \leq C \alpha^{\frac{1-d}{2d}} \|\nabla u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}^{1/2} \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}^{1/2} \\ + C(r^d\alpha^{-1})^{-1/2d} \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}. \quad (2.10) \end{aligned}$$

Proof. We prove the first point. We set

$$\hat{u}(\rho, \theta) = u(r\rho, \theta), \quad \hat{A}(\rho, \theta) = A(r\rho, \theta) \quad (\rho, \theta) \in (1 + C_0\alpha^{-1}, 1 + C_3\alpha^{-1}) \times \mathbb{S}^{d-1}. \quad (2.11)$$

Then, we find from (2.6)

$$\nabla \hat{u}(\rho, \theta) = r\partial_\rho u(r\rho, \theta)e_\rho + \frac{1}{\rho}\nabla_\theta u(r\rho, \theta) = r(\partial_\rho u(r\rho, \theta)e_{r\rho} + \frac{1}{r\rho}\nabla_\theta u(r\rho, \theta)) = r\nabla u(r\rho, \theta),$$

and then from (2.5)

$$\begin{aligned}\operatorname{div}(\hat{A}\nabla\hat{u}) &= \frac{1}{\rho^{n-1}}\partial_\rho(\rho^{n-1}(\hat{A}\nabla\hat{u})_\rho) + \frac{1}{\rho}\operatorname{div}_\theta((\hat{A}\nabla\hat{u})_\theta) \\ &= r^2\left(\frac{1}{(r\rho)^{n-1}}\partial_\rho((r\rho)^{n-1}(A\nabla u)_\rho) + \frac{1}{r\rho}\operatorname{div}_\theta((A\nabla u)_\theta)\right) = r^2\operatorname{div}(A(r\rho, \theta)\nabla u(r\rho, \theta)).\end{aligned}$$

Then we set

$$\tilde{u}(\rho, \theta) = \hat{u}(1 + \alpha^{-1}\rho, \theta), \quad \tilde{A}(\rho, \theta) = \hat{A}(1 + \alpha^{-1}\rho, \theta) \quad (\rho, \theta) \in (C_0, C_3) \times \mathbb{S}^{d-1}. \quad (2.12)$$

In the same way, we find

$$\nabla\tilde{u}(\rho, \theta) = \alpha^{-1}\nabla\hat{u}(1 + \alpha^{-1}\rho, \theta),$$

and then

$$\operatorname{div}(\tilde{A}\nabla\tilde{u})(\rho, \theta) = \alpha^{-2}\operatorname{div}(\hat{A}(1 + \alpha^{-1}\rho, \theta)\nabla\hat{u}(1 + \alpha^{-1}\rho, \theta)).$$

Therefore, we find successively by changes of variable

$$\begin{aligned}\|\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))} &\approx (r^{-d}\alpha)^{1/p}\|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}, \\ \|\nabla\tilde{u}\|_{L^p(\mathcal{C}(C_1, C_2))} &\approx (r^{-d}\alpha)^{1/p}(r\alpha^{-1})\|\nabla u\|_{L^p(\mathcal{C}(r(1+C_1\alpha^{-1}), r(1+C_2\alpha^{-1})))}, \\ \|\Delta_{\tilde{A}}\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))} &\approx (r^{-d}\alpha)^{1/p}(r\alpha^{-1})^2\|\Delta_{A}u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}.\end{aligned}$$

In the three above estimates and in the sequel of the proof, for two positive quantities f and g , we denote by $f \approx g$ the fact that there exist two positive constants $c, C > 0$ depending on A, p, C_0, C_1, C_2, C_3 such that $cf \leq g \leq Cf$.

By applying local elliptic regularity estimates, in particular [GT01, Theorem 9.11] and by deducing from (1.15),

$$\Lambda_1^{-1}|\xi|^2 \leq \langle \tilde{A}(x)\xi, \xi \rangle \leq \Lambda_1|\xi|^2, \quad |\tilde{A}^{ij}(x) - \tilde{A}^{ij}(y)| \leq C\Lambda_2|x - y|.$$

we obtain that there exists $C = C(A, p, C_0, C_1, C_2, C_3) > 0$ such that

$$\|\nabla\tilde{u}\|_{L^p(\mathcal{C}(C_1, C_2))} \leq C\left(\|\Delta_{\tilde{A}}\tilde{u}\|_{L^p(\mathcal{C}(1+C_0, 1+C_3))} + \|\tilde{u}\|_{L^p(\mathcal{C}(1+C_0, 1+C_3))}\right).$$

Then we use the previous relations between the norms of \tilde{u} and u and their derivatives to get

$$\begin{aligned}r\alpha^{-1}\|\nabla u\|_{L^p(\mathcal{C}(r(1+C_1\alpha^{-1}), r(1+C_2\alpha^{-1})))} \\ \leq C\left((r\alpha^{-1})^2\|\Delta_{A}u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} \right. \\ \left. + \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}\right).\end{aligned}$$

We multiply the above estimate by $(r\alpha^{-1})^{-2}$ to get the expected result (2.8).

We prove the second point, we apply Gagliardo-Nirenberg interpolation's inequality, see [Nir66], to \tilde{u} defined by the relations (2.11) and (2.12) to get

$$\|\tilde{u}\|_{L^q(\mathcal{C}(C_0, C_3))} \leq C\|\nabla\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))}^{1/2}\|\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))}^{1/2} + C\|\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))}.$$

Moreover, we have as before

$$\|\tilde{u}\|_{L^q(\mathcal{C}(C_0, C_3))} \approx (r^{-d}\alpha)^{1/q}\|u\|_{L^q(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))},$$

$$\begin{aligned}\|\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))} &\approx (r^{-d}\alpha)^{1/p} \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}, \\ \|\nabla\tilde{u}\|_{L^p(\mathcal{C}(C_0, C_3))} &\approx (r^{-d}\alpha)^{1/p} (r\alpha^{-1}) \|\nabla u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}.\end{aligned}$$

By putting these relations in the previous Gagliardo-Nirenberg interpolation's inequality, we obtain

$$\begin{aligned}\|u\|_{L^q(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} (r^{-d}\alpha)^{1/q} \\ \leq C \|\nabla u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}^{1/2} (r^{-d}\alpha)^{1/2p} (r\alpha^{-1})^{1/2} \\ \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))}^{1/2} (r^{-d}\alpha)^{1/2p} \\ + C \|u\|_{L^p(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C_3\alpha^{-1})))} (r^{-d}\alpha)^{1/p},\end{aligned}$$

that simplifies into (2.10) by using (2.9). \square

2.4 Proof of the growth estimate for A -harmonic functions

The goal of this part is to prove Theorem 1.3.

The proof of (1.18) will be quite long and rather technical, while the proof of (1.19) will be a straightforward corollary of a generalization of (1.18) that we state below, see Lemma 2.6.

Proof of (1.18) of Theorem 1.3. We split the proof into several steps. It is worth mentioning that the first three steps are strongly inspired by the original proof of L^2 -Bernstein estimates for Laplace eigenfunctions from [DF90a].

Step 1: Carleman estimate to a truncated version of u . Let us take $\chi \in C_c^\infty(B_2; [0, 1])$

be a cut-off function such that

$$\begin{aligned}\chi &= 0 \text{ in } B(0, r(1 + (1/4)\alpha^{-1})), \\ \chi &= 1 \text{ in } B(0, 15/8) \setminus B(0, r(1 + (1/2)\alpha^{-1})), \\ \chi &= 0 \text{ in } B(0, 2) \setminus B(0, 31/16),\end{aligned}$$

satisfying the estimates

$$|\nabla\chi| \leq C, \quad |D^2\chi| \leq C, \quad x \in B(0, 31/16) \setminus B(0, 15/8), \quad (2.13)$$

and

$$|\nabla\chi| \leq Cr^{-1}\alpha, \quad |D^2\chi| \leq Cr^{-2}\alpha^{-2}, \quad x \in B(0, r(1+(1/2)\alpha^{-1})) \setminus B(0, r(1+(1/4)\alpha^{-1})). \quad (2.14)$$

We define $f = \chi u$, note that $f \in H_{\text{loc}}^2(B_2)$ with $\text{supp}(f) \subset\subset B_2 \setminus B(0, r)$ so we can apply the Carleman estimate (2.2) from Lemma 2.2 to f to get

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} |f|^2 dx + \frac{\alpha^4}{r^2} \int_{B(0, r(1+2\alpha^{-1}))} \rho^{-2\alpha} |f|^2 dx \leq C \int_{B_2} \rho^{2-2\alpha} |\Delta_A f|^2 dx. \quad (2.15)$$

We now use the elliptic equation (1.16) satisfied by u to deduce from (2.15)

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} \chi^2 |u|^2 dx + \frac{\alpha^4}{r^2} \int_{B(0, r(1+\alpha^{-1}))} \rho^{-2\alpha} \chi^2 |u|^2 dx \leq C(I_1 + I_2), \quad (2.16)$$

where

$$I_1 = \int_{B(0, r(1+(1/2)\alpha^{-1})) \setminus B(0, r(1+(1/4)\alpha^{-1}))} \rho^{2-2\alpha} (|\nabla\chi|^2 |\nabla u|^2 + [|\nabla\chi|^2 + |D^2\chi|^2] |u|^2) dx, \quad (2.17)$$

and

$$I_2 = \int_{B(0, 31/16) \setminus B(0, 15/8)} \rho^{2-2\alpha} (|\nabla\chi|^2 |\nabla u|^2 + [|\nabla\chi|^2 + |D^2\chi|^2] |u|^2) dx. \quad (2.18)$$

Step 2: Absorption of the boundary terms. The goal of this step would be to absorb the cut-off terms from (2.16) that are located near the boundary of $B(0, 2)$ that is the term I_2 defined in (2.18). By (2.13) and from local elliptic regularity estimates, it is straightforward to get that exist a universal positive constant $C_3 > 0$ and a positive constant $C = C(A) > 0$ such that

$$I_2 \leq C \exp(-C_3\alpha) (\sup_{B_2} |u|)^2. \quad (2.19)$$

Moreover, we can give a lower bound of the first term in the left hand side of (2.16) of the following form by using the definition of χ and local elliptic regularity estimates

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} \chi^2 |u|^2 dx \geq C^{-1} \exp(-C_2\alpha) (\sup_{B(x_0, 1/4)} |u|)^2, \quad (2.20)$$

where $0 < C_2 < C_3$ and $x_0 \in B_2$ be such that $|x_0| = 3/8$. Now by using the vanishing order estimate (2.4) from Lemma 2.4, note in particular that $\chi \equiv 1$ in $B(x_0, 1/4)$ for $r \in (0, c)$ with $c > 0$ small enough, we deduce from (2.20) that

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} \chi^2 |u|^2 dx \geq C^{-1} \exp(-C_2\alpha) \exp(-CN) (\sup_{B_2} |u|)^2, \quad (2.21)$$

Now the punchline, by taking $\alpha \geq C(C_2, C_3, A)N$, we obtain from (2.16), (2.19) and (2.21) that I_2 can be hidden in the left hand side of (2.16), that is there exists $C = C(A) > 0$ such that

$$\alpha^3 \int_{B_2} \rho^{-1-2\alpha} \chi^2 |u|^2 dx + \frac{\alpha^4}{r^2} \int_{B(0, r(1+2\alpha^{-1}))} \rho^{-2\alpha} \chi^2 |u|^2 dx \leq C I_1. \quad (2.22)$$

Step 3: Growth estimate at L^2 -regularity. The goal of this step would be to compare the second left hand side term in (2.22) and the right hand side term in (2.22) to first obtain growth estimate at L^2 -regularity. First note that $\rho^{-2\alpha}$ has the same amplitude in $B(0, r(1+2\alpha^{-1})) \setminus B(0, r)$ so one can simplify (2.22) by using (2.17) into

$$\begin{aligned} & \frac{\alpha^4}{r^2} \int_{B(0, r(1+2\alpha^{-1}))} \chi^2 |u|^2 dx \\ & \leq C \int_{B(0, r(1+(1/2)\alpha^{-1})) \setminus B(0, r(1+(1/4)\alpha^{-1}))} r^2 (|\nabla\chi|^2 |\nabla u|^2 + [|\nabla\chi|^2 + |D^2\chi|^2] |u|^2) dx. \end{aligned} \quad (2.23)$$

Now we use (2.14) to deduce from (2.23) that

$$\begin{aligned} & \frac{\alpha^4}{r^2} \int_{B(0, r(1+2\alpha^{-1}))} \chi^2 |u|^2 dx \\ & \leq C \int_{B(0, r(1+(1/2)\alpha^{-1})) \setminus B(0, r(1+(1/4)\alpha^{-1}))} r^2 ((r^{-1}\alpha)^2 |\nabla u|^2 + (r^{-2}\alpha^2)^2 |u|^2) dx. \end{aligned} \quad (2.24)$$

By using the scaled local elliptic regularity estimate (2.8) for $p = 2$ from Lemma 2.5 to the right hand side of (2.24) we get for some $C_0, C'_0 > 0$ such that $C_0 < 1/4 < 1/2 < C'_0 < 3/4$,

$$\frac{\alpha^4}{r^2} \int_{B(0,r(1+2\alpha^{-1}))} \chi^2 |u|^2 dx \leq C \frac{\alpha^4}{r^2} \int_{B(0,r(1+C'_0\alpha)) \setminus B(0,r(1+C_0\alpha))} |u|^2 dx. \quad (2.25)$$

In particular, (2.25) translates into

$$\int_{B(0,r(1+2\alpha^{-1})) \setminus B(0,r(1+2^{-1}\alpha^{-1}))} |u|^2 dx \leq C \int_{B(0,r(1+C'_0\alpha)) \setminus B(0,r(1+C_0\alpha))} |u|^2 dx. \quad (2.26)$$

Note that this type of strategy already appears in [DF90a] to obtain growth estimate at L^2 -regularity leading in particular to the Bernstein estimates at L^2 -regularity for Laplace eigenfunctions, i.e. (1.6) and (1.7).

Step 4: Initialization of the bootstrap argument. The goal of this step would be to improve the growth estimate at L^2 -regularity (2.26) by employing scaled versions of local elliptic regularity estimates and Gagliardo-Nirenberg interpolation inequalities from Lemma 2.5 by using the equation satisfied by u , i.e. (1.16). First we have from (2.26) and (1.16) i.e. $\Delta_A u = 0$,

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0,r(1+2\alpha^{-1})) \setminus B(0,r(1+2^{-1}\alpha^{-1}))} |u|^2 dx + r^2 \int_{B(0,r(1+2\alpha^{-1})) \setminus B(0,r(1+2^{-1}\alpha^{-1}))} |\Delta_A u|^2 dx \\ \leq C \alpha^4 r^{-2} \int_{B(0,r(1+C'_0\alpha)) \setminus B(0,r(1+C_0\alpha))} |u|^2 dx. \end{aligned} \quad (2.27)$$

We now use the local elliptic regularity estimate (2.8) for $p = 2$ to obtain from (2.27) that

$$\begin{aligned} \alpha^2 \int_{B(0,r(1+C_3\alpha^{-1})) \setminus B(0,r(1+C_2\alpha^{-1}))} |\nabla u|^2 dx \\ \leq C \alpha^4 r^{-2} \int_{B(0,r(1+C'_0\alpha)) \setminus B(0,r(1+C_0\alpha))} |u|^2 dx, \end{aligned} \quad (2.28)$$

for some positive constants $1/2 < C_2 < 3/4 < 1 < C_3 < 2$. We now use the interpolation inequality (2.10) for $p = 2$ with $q > 2$ defined by (2.9) to deduce from (2.27), (2.28) that

$$\alpha^{2-\frac{1}{2d}} r^{-1/2} \|u\|_{L^q(\mathcal{C}(r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \leq C \alpha^2 r^{-1} \|u\|_{L^2(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.29)$$

In particular, by using Hölder's estimate on the right hand side of (2.29) and the fact that the Lebesgue measure of $\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1}))$ is bounded by $C r^d \alpha^{-1}$,

$$\begin{aligned} \alpha^{2-\frac{1}{2d}} r^{-1/2} \|u\|_{L^q(\mathcal{C}(r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \\ \leq C \alpha^2 r^{-1} (r^d \alpha^{-1})^{(1/2-1/q)} \|u\|_{L^q(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \end{aligned} \quad (2.30)$$

By recalling (2.9) with $p = 2$, one can check that (2.30) simplifies into

$$\|u\|_{L^q(\mathcal{C}(r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \leq C \|u\|_{L^q(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.31)$$

In particular, (2.31) leads to a growth-estimate at L^q -regularity that is

$$\|u\|_{L^q(B(0,r(1+C_3\alpha^{-1})))} \leq C \|u\|_{L^q(B(0,r(1+\max(C'_0, C_2)\alpha^{-1})))}, \quad (2.32)$$

by adding $\|u\|_{L^q(B(0,r(1+\max(C'_0, C_2)\alpha^{-1})))}$ in both sides of (2.31). Note that $C_3 > 1 > 3/4 > \max(C'_0, C_2)$. The estimate (2.32) will not be used in the next, it only furnishes a growth estimate at L^q -regularity leading in particular to the Bernstein estimates at L^q -regularity for A -harmonic functions. We need to iterate such an argument to reach L^∞ .

Step 5: Bootstrap argument. We iterate the previous argument starting from (2.29). Let us define by induction

$$p_0 = 2, \quad \frac{1}{p_{n+1}} = \frac{1}{2} \left(\frac{1}{p_n} - \frac{1}{d} \right) + \frac{1}{2p_n} \quad \forall n \in \{0, \dots, d-1\}. \quad (2.33)$$

One can check that

$$\frac{1}{p_n} = \frac{1}{2} - \frac{n}{2d}, \quad \forall n \in \{0, \dots, d-1\} \text{ and } p_d = +\infty. \quad (2.34)$$

In the same way, we define

$$\beta_0 = 2, \quad \beta_{n+1} = \beta_n - \frac{1}{2d} \quad \forall n \in \{0, \dots, d-1\} \Rightarrow \beta_d = \frac{3}{2}, \quad (2.35)$$

$$s_0 = -1, \quad s_{n+1} = s_n + \frac{1}{2} \quad \forall n \in \{0, \dots, d-1\} \Rightarrow s_d = -1 + \frac{d}{2}. \quad (2.36)$$

From an easy induction applied to the previous step, conjugated with the scaled version of elliptic regularity estimates and Gagliardo-Nirenberg interpolation inequalities from Lemma 2.5 we obtain after d iterations that

$$\alpha^{\beta_d r^{s_d}} \|u\|_{L^{p_d}(\mathcal{C}(r(1+C_d\alpha^{-1}), r(1+C'_d\alpha^{-1})))} \leq C\alpha^2 r^{-1} \|u\|_{L^2(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.37)$$

for some positive constants $2^{-1} < C_d < 3/4 < 1 < C'_d < 2$. So (2.37) gives

$$\alpha^{\frac{3}{2}r^{-1+\frac{d}{2}}} \|u\|_{L^\infty(\mathcal{C}(r(1+C_d\alpha^{-1}), r(1+C'_d\alpha^{-1})))} \leq C\alpha^2 r^{-1} \|u\|_{L^2(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.38)$$

We now apply Hölder's estimate to the right hand side of (2.38) and the fact that the Lebesgue measure of $\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1}))$ is bounded by $Cr^d\alpha^{-1}$ to get

$$\begin{aligned} \alpha^{\frac{3}{2}r^{-1+\frac{d}{2}}} \|u\|_{L^\infty(\mathcal{C}(r(1+C_d\alpha^{-1}), r(1+C'_d\alpha^{-1})))} \\ \leq C\alpha^2 r^{-1} (r^d\alpha^{-1})^{1/2} \|u\|_{L^\infty(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}, \end{aligned}$$

that simplifies into

$$\|u\|_{L^\infty(\mathcal{C}(r(1+C_d\alpha^{-1}), r(1+C'_d\alpha^{-1})))} \leq C \|u\|_{L^\infty(\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.39)$$

Recall that the constants are such that $1/2 < C_d < 3/4 < 1 < C'_d < 2$. In particular, this leads to a growth-estimate at L^∞ -regularity that is

$$\|u\|_{L^\infty(B(0, r(1+C'_d\alpha^{-1})))} \leq C \|u\|_{L^\infty(B(0, r(1+\max(C'_0, C_d)\alpha^{-1})))}, \quad (2.40)$$

by adding $\|u\|_{L^\infty(B(0, r(1+\max(C'_0, C_d)\alpha^{-1})))}$ in both sides of (2.39). If we replace r by $r/((1+\max(C'_0, C_d)\alpha^{-1}))$, we then get from (2.40)

$$\|u\|_{L^\infty(B(0, r(1+C'_d\alpha^{-1})))} \leq C \|u\|_{L^\infty(B(0, r))}, \quad (2.41)$$

for another constant $C'_d > 0$. We then recall that $\alpha \geq CN$ so one has from (2.41)

$$\|u\|_{L^\infty(B(0, r(1+C'_d N^{-1})))} \leq C \|u\|_{L^\infty(B(0, r))}. \quad (2.42)$$

We then iterate (2.42) to finally deduce (1.18). \square

From the proof of (1.18) of Theorem 1.3, one can also deduce the following result.

Lemma 2.6. *There exist $r_0, C > 0$ such that for every $u \in H_{loc}^1(B_2) \cap L^\infty(B_2)$ satisfying (1.16) and (1.17), then for every $r \in (0, r_0)$,*

$$\sup_{B(x, r(1+\frac{1}{N}))} |u| \leq C \sup_{B(x, r)} |u|, \quad \forall x \in (0, 2r). \quad (2.43)$$

Indeed, the proof of (2.43) consists in applying a modified version of the Carleman estimate (2.2) in the punctured domain $B(0, 2) \setminus B(x, r)$. Details are omitted.

We can now pass to the proof of (1.19) of Theorem 1.3.

Proof of (1.19) of Theorem 1.3. Let us then take $y \in B(0, r)$ be such that

$$|\nabla u(y)| = \sup_{B(0, r)} |\nabla u|.$$

By a scaled local elliptic estimate applied to the equation (1.16) and by using (2.43), we have

$$|\nabla u(y)| \leq C \frac{N}{r} \sup_{B(y, rN^{-1})} |u| \leq C \frac{N}{r} \sup_{B(x, r(1+N^{-1}))} |u| \leq C \frac{N}{r} \sup_{B(x, r)} |u|.$$

This ends the proof of (1.19). \square

2.5 Proof of the growth estimate for Laplace eigenfunctions

The goal of this part is to prove Theorem 1.2.

To simplify the notations, in all the following, the geodesic balls $B_g(x, r)$ will be denoted by $B(x, r)$ and the Riemannian volume of integration will be denoted by dx .

The following crucial L^2 -Carleman estimate for the operator $-\Delta_g - \lambda$ will be one of the main ingredient of the proof.

Lemma 2.7. *There exists a positive constant $C = C(M, g) > 0$ and $c = c(M, g) > 0$ such that for every $r_M \in (0, c)$, $r \in (0, c)$, $x_0 \in M$, $\alpha \geq C(\sqrt{\lambda} + 1)$, for all $f \in C_c^\infty(B_{r_M} \setminus B_r)$, the following estimate holds*

$$\begin{aligned} \alpha^3 \int_{B(x_0, r_M)} \rho^{-1-2\alpha} |f|^2 dx + \alpha \int_{B(x_0, r_M)} \rho^{1-2\alpha} |\nabla f|^2 dx + \frac{\alpha^4}{r^2} \int_{B(x_0, r(1+2\alpha^{-1}))} \rho^{-2\alpha} |f|^2 dx \\ \leq C \int_{B(x_0, r_M)} \rho^{2-2\alpha} |\Delta_g f + \lambda f|^2 dx, \quad (2.44) \end{aligned}$$

where $\rho(x) = \text{dist}(x, x_0)$.

The proof of Lemma 2.7 is similar to the one of Lemma 2.2, it is actually proved in [DF90a, Lemma A]. The crucial difference between the proof of Lemma 2.2 is the presence of the zero-order term λf in the elliptic operator $-\Delta_g f - \lambda f$. This new term cannot be treated as a source term and absorbed by the Carleman parameter α because such a strategy would lead to (2.44) for $\alpha \geq C(\lambda^{2/3} + 1)$. This is why it has to be directly included in the symmetric part of the conjugated operator in the Carleman's strategy, note that here we crucially use the assumption that λ is constant then does not depend on the x -variable.

The following classical vanishing order estimate holds for Laplace eigenfunctions.

Lemma 2.8. *There exist $C = C(M, g) > 0$, $c = c(M, g) > 0$ and $r_M, r_0 \in (0, c)$, such that for every $\lambda \geq 1$, for every $\varphi_\lambda \in C^\infty(M)$ satisfying (1.1), the following estimate holds*

$$\sup_{B(x_0, \frac{r_0}{4})} |\varphi_\lambda| \geq \exp(-C\sqrt{\lambda}) \sup_{B(x_0, r_M)} |\varphi_\lambda| \quad \forall x \in B\left(x_0, \frac{r_0}{2}\right). \quad (2.45)$$

The proof of Lemma 2.8 is similar to the one of Lemma 2.4 so we omit it.

With Lemma 2.7 and Lemma 2.8, we can now deduce the proof of the growth estimate (1.13) by adapting the arguments of the proof of the growth estimate (1.18).

Proof of (1.13) from Theorem 1.2. The first three steps are the same. We apply the Carleman estimate (2.44) to a truncated version of φ_λ , the boundary terms are then absorbed by the vanishing order estimate (2.45), we finally obtain a Bernstein estimate at L^2 -regularity

$$\int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\varphi_\lambda|^2 dx \leq C \int_{B(x_0, r(1+C'_0\alpha)) \setminus B(x_0, r(1+C_0\alpha))} |\varphi_\lambda|^2 dx. \quad (2.46)$$

Note that this type of strategy already appears in [DF90a] to obtain growth estimate at L^2 -regularity leading in particular to the Bernstein estimates at L^2 -regularity for Laplace eigenfunctions, i.e. (1.6) and (1.7).

Step 4: Initialization of the bootstrap argument. The goal of this step would be to improve the growth estimate at L^2 -regularity (2.46) by employing scaled versions of local elliptic regularity estimates and Gagliardo-Nirenberg interpolation inequalities from Lemma 2.5 by using the equation satisfied by φ_λ , i.e. (1.1). First we have from (2.46) and (1.1) i.e. $\Delta_g \varphi_\lambda + \lambda \varphi_\lambda = 0$,

$$\begin{aligned} & \alpha^4 r^{-2} \int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\varphi_\lambda|^2 dx + r^2 \int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\Delta_g \varphi_\lambda|^2 dx \\ & \leq C \alpha^4 r^{-2} \int_{B(x_0, r(1+C'_0\alpha)) \setminus B(x_0, r(1+C_0\alpha))} |\varphi_\lambda|^2 dx \\ & \quad + C r^2 \lambda^2 \int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\varphi_\lambda|^2 dx. \end{aligned} \quad (2.47)$$

The last right hand side term of (2.47) can be absorbed by the first left hand side term recalling that $\alpha \geq C\sqrt{\lambda}$ to get

$$\begin{aligned} & \alpha^4 r^{-2} \int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\varphi_\lambda|^2 dx + r^2 \int_{B(x_0, r(1+2\alpha^{-1})) \setminus B(x_0, r(1+2^{-1}\alpha^{-1}))} |\Delta_g \varphi_\lambda|^2 dx \\ & \leq C \alpha^4 r^{-2} \int_{B(x_0, r(1+C'_0\alpha)) \setminus B(x_0, r(1+C_0\alpha))} |\varphi_\lambda|^2 dx. \end{aligned} \quad (2.48)$$

We now use the local elliptic regularity estimate (2.8) for $p = 2$ to obtain from (2.48) that

$$\begin{aligned} & \alpha^2 \int_{B(x_0, r(1+C_3\alpha^{-1})) \setminus B(x_0, r(1+C_2\alpha^{-1}))} |\nabla \varphi_\lambda|^2 dx \\ & \leq C \alpha^4 r^{-2} \int_{B(x_0, r(1+C'_0\alpha)) \setminus B(x_0, r(1+C_0\alpha))} |u|^2 dx, \end{aligned} \quad (2.49)$$

for some positive constants $1/2 < C_2 < 3/4 < 1 < C_3 < 2$. We now use the interpolation inequality (2.10) for $p = 2$ with $q > 2$ defined by (2.9) to deduce from (2.48), (2.49) that

$$\alpha^{2-\frac{1}{2d}} r^{-1/2} \|\varphi_\lambda\|_{L^q(\mathcal{C}(x_0, r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \leq C\alpha^2 r^{-1} \|\varphi_\lambda\|_{L^2(\mathcal{C}(x_0, r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.50)$$

In particular, by using Hölder's estimate on the right hand side of (2.50) and the fact that the Lebesgue measure of $\mathcal{C}(r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1}))$ is bounded by $Cr^d\alpha^{-1}$,

$$\begin{aligned} \alpha^{2-\frac{1}{2d}} r^{-1/2} \|\varphi_\lambda\|_{L^q(\mathcal{C}(x_0, r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \\ \leq C\alpha^2 r^{-1} (r^d\alpha^{-1})^{(1/2-1/q)} \|\varphi_\lambda\|_{L^q(\mathcal{C}(x_0, r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \end{aligned} \quad (2.51)$$

By recalling (2.9) with $p = 2$, one can check that (2.51) simplifies into

$$\|\varphi_\lambda\|_{L^q(\mathcal{C}(x_0, r(1+C_2\alpha^{-1}), r(1+C_3\alpha^{-1})))} \leq C \|\varphi_\lambda\|_{L^q(\mathcal{C}(x_0, r(1+C_0\alpha^{-1}), r(1+C'_0\alpha^{-1})))}. \quad (2.52)$$

In particular, (2.31) leads to a growth-estimate at L^q -regularity that is

$$\|\varphi_\lambda\|_{L^q(B(x_0, r(1+C_3\alpha^{-1})))} \leq C \|\varphi_\lambda\|_{L^q(B(x_0, r(1+\max(C'_0, C_2)\alpha^{-1})))}, \quad (2.53)$$

by adding $\|\varphi_\lambda\|_{L^q(B(x_0, r(1+\max(C'_0, C_2)\alpha^{-1})))}$ in both sides of (2.52). Note that $C_3 > 1 > 3/4 > \max(C'_0, C_2)$. The estimate (2.53) will not be used in the next, it only furnishes a growth estimate at L^q -regularity leading in particular to the Bernstein estimates at L^q -regularity for Laplace eigenfunctions. We need to iterate such an argument to reach L^∞ .

By simply following the fifth step of the proof of the growth estimate (1.18), consisting in implementing a bootstrap argument, we easily obtain the desired estimate (1.13). \square

We then note that (1.14) is a consequence of (1.13).

Proof of (1.14) of Theorem 1.2. Let us take $y \in B_g(x, r)$ be such that

$$|\nabla\varphi_\lambda(y)| = \sup_{B_g(x, r)} |\nabla\varphi_\lambda|.$$

By a scaled local elliptic estimate applied to the equation satisfied by φ_λ , i.e. (1.1), and (1.13), we have

$$|\nabla\varphi_\lambda(y)| \leq C \frac{\sqrt{\lambda}}{r} \sup_{B_g(y, r\sqrt{\lambda}^{-1})} |\varphi_\lambda| \leq C \frac{\sqrt{\lambda}}{r} \sup_{B_g(x, r(1+\sqrt{\lambda}^{-1}))} |\varphi_\lambda| \leq C \frac{\sqrt{\lambda}}{r} \sup_{B_g(x, r)} |\varphi_\lambda|,$$

that is exactly (1.14). \square

3 Extensions

The goal of this part is to state several generalizations of our main results Theorem 1.2 and Theorem 1.3. For the sake of simplicity, we only give sketches of the proof, the details will be omitted. We also propose some open problems that can be investigated in the future.

3.1 Manifolds with boundaries

The treatment of C^∞ -manifolds M , possibly with boundaries i.e. $\partial M \neq \emptyset$, are treated by the following result.

Theorem 3.1. *There exist $r_0, C > 0$ depending only on M , such that for every Laplace eigenfunction $\varphi_\lambda \in C^\infty(M)$, i.e. satisfying*

$$-\Delta_g \varphi_\lambda = \lambda \varphi_\lambda \text{ in } M, \quad (\varphi_\lambda = 0 \text{ on } \partial M) \text{ or } (\partial_\nu \varphi_\lambda = 0 \text{ on } \partial M),$$

then, for every $x \in M$, $r \in (0, r_0)$, $\lambda \geq 1$, (1.13) and (1.14) hold.

The proof of Theorem 3.1 can be obtained from Theorem 1.2 and the double manifold trick, see for instance [DF90b] or more precisely [BM23, Section 3], that consists in reducing the question to the case of a manifold without boundary by gluing two copies of M along the boundary in such a way that the new double manifold \tilde{M} inherits a Lipschitz metric, which allows one to apply the previous results (without boundary) to this double manifold.

3.2 On elliptic differential inequalities

First, we have the following generalization of Theorem 1.2.

Theorem 3.2. *There exist $r_0, C > 0$ depending only on M , such that for function $\varphi \in C^\infty(M)$, satisfying the elliptic differential inequality*

$$|-\Delta_g \varphi| \leq \lambda |\varphi| + \mu |\nabla \varphi| \text{ in } M, \quad (3.1)$$

where $\lambda, \mu \geq 1$, then for every $x \in M$, $r \in (0, r_0)$,

$$\sup_{B_g(x, r(1 + \min(\frac{1}{\lambda^{2/3}}, \frac{1}{\mu^2})))} |\varphi| \leq C \sup_{B_g(x, r)} |\varphi|, \quad (3.2)$$

and

$$\sup_{B_g(x, r)} |\nabla \varphi| \leq C \frac{\max(\lambda^{2/3}, \mu^2)}{r} \sup_{B_g(x, r)} |\varphi|. \quad (3.3)$$

The proof of Theorem 3.2 cannot use the standard lifting trick from (1.20) so one needs to proceed differently.

First, by a standard L^2 -Carleman estimate and the arguments of [DF88], the proof consists in establishing the following vanishing order estimate for φ .

• There exist $r_0, C > 0$ depending only on M , such that for every function $\varphi \in C^\infty(M)$ satisfying (3.1), for every $x \in M$, $r \in (0, r_0)$,

$$\sup_{B_g(x, 2r)} |\varphi| \leq e^{C \max(\lambda^{2/3}, \mu^2)} \sup_{B_g(x, r)} |\varphi|. \quad (3.4)$$

Secondly, by the application of a L^2 -Carleman estimate on a punctured geodesic ball, taking $\alpha \geq C \max(\lambda^{2/3}, \mu^2)$ to absorb the right hand side terms and by using the arguments of the proof of (1.18) together with (3.4), one is able to prove (3.2) then (3.3). An extra technical difficulty appears in Step 4 and consequently in Step 5, because the adding of terms involving L^p -norms of $\Delta_g u$ in the left hand side introduces terms in function of u and ∇u that needed to be absorbed. This is indeed the case by a bootstrap argument using scaled versions of local elliptic regularity estimates and Gagliardo-Nirenberg interpolations inequalities applied to u and ∇u .

Note that (3.2) and (3.3) are crucially related to the vanishing order estimate (3.4) that is itself related to the possible rate of decay to the solutions of second order differential inequalities in the Euclidean space \mathbb{R}^d inspired by the so-called Landis conjecture [KL88]. According to the Meshkov type counterexamples to the Landis conjecture for complex-valued functions [Mes91] and [Dav14], (3.4) is probably sharp so are (3.2) and (3.3) if we consider $\varphi \in C^\infty(M; \mathbb{C})$. But, one can probably sharpen (3.2) and (3.3) when $d = 2$, assuming that φ is real-valued by using the recent paper on vanishing order estimates of real-valued solutions to second order elliptic equations in the plane, [LMNN20] or [LBS23]. When $d \geq 3$, and assuming that φ is real-valued, we do not know if one can sharpen (3.2) and (3.3) but in that direction, one can read the very recent preprint [FK24] that constructs a real-valued counterexample to the quantitative Landis conjecture in \mathbb{R}^d for $d \geq 4$.

3.3 Solutions to elliptic equations with bounded doubling index

Let us take a matrix $A = (a^{ij}(x))_{1 \leq i, j \leq d}$ symmetric, uniformly elliptic, with Lipschitz entries, that is satisfying (1.15). The lower order terms are given by $W = W(x) \in L^\infty(B_2; \mathbb{C}^d)$ and $V \in L^\infty(B_2; \mathbb{C})$ satisfying

$$|W(x)| + |V(x)| \leq \Lambda_3, \quad x \in B_2,$$

for some $\Lambda_3 > 0$.

We have the following generalization of Theorem 1.3.

Theorem 3.3. *There exist $r_0, C > 0$ depending on A, W, V such that for every $u \in H_{loc}^1(B_2) \cap L^\infty(B_2)$ satisfying*

$$-\operatorname{div}(A(x)\nabla u) + W \cdot \nabla u + Vu = 0 \text{ in } B_2,$$

and

$$N_u(B(0, 1)) \leq N, \quad N \geq 1,$$

then for every $r \in (0, r_0)$,

$$\sup_{B(0, r(1 + \frac{1}{N}))} |u| \leq C \sup_{B(0, r)} |u|,$$

and

$$\sup_{B(0, r)} |\nabla u| \leq C \frac{N}{r} \sup_{B(0, r)} |u|.$$

The proof of Theorem 3.3 follows the lines of the one of Theorem 1.3, the lower order terms $W \cdot \nabla u + Vu$ are absorbed by the use of the Carleman parameter $\alpha \geq C\|W\|_\infty^2 + C\|V\|_\infty^{2/3}$. Again Steps 4 and 5 are modified to include a bootstrap argument applied to ∇u .

3.4 Linear combination of eigenfunctions

An interesting open problem is the generalization of local L^∞ -Bernstein estimates for linear combination of eigenfunctions. While global L^∞ -Bernstein estimates (1.4) have been established, it seems that its local counterpart has not been investigated yet. Let us recall that from [JL99, Theorem 14.3], the following result holds.

• There exist $r_0, C > 0$ depending only on M , such that for every linear combination of Laplace eigenfunctions

$$\Phi_\Lambda = \sum_{\lambda_k \leq \Lambda} a_k \varphi_{\lambda_k}, \quad a_k \in \mathbb{C}, \quad \Lambda \geq 1, \quad \text{with } -\Delta_g \varphi_{\lambda_k} = \lambda_k \varphi_{\lambda_k} \text{ in } M, \quad (3.5)$$

then for every $x \in M$, $r \in (0, r_0)$,

$$\sup_{B_g(x, 2r)} |\Phi_\Lambda| \leq e^{C\sqrt{\Lambda}} \sup_{B_g(x, r)} |\Phi_\Lambda|. \quad (3.6)$$

It is then natural to conjecture the following result.

Conjecture 3.4. There exist $r_0, C > 0$ depending only on M such that for every linear combination of Laplace eigenfunctions Φ_Λ , that is satisfying (3.5), the following Bernstein estimates hold

$$\sup_{B_g(x, r(1 + \frac{1}{\sqrt{\Lambda}}))} |\Phi| \leq C \sup_{B_g(x, r)} |\Phi|, \quad (3.7)$$

and

$$\sup_{B_g(x, r)} |\nabla \Phi| \leq C \frac{\sqrt{\Lambda}}{r} \sup_{B_g(x, r)} |\Phi|. \quad (3.8)$$

The main difficulty for obtaining Conjecture 3.4 comes from the fact that Φ does not satisfy an elliptic equation. The standard trick to remove this difficulty consists in setting

$$\tilde{\Phi}(x, t) = \sum_{\lambda_k \leq \Lambda} a_k \frac{\sinh(\sqrt{\lambda_k t})}{\sqrt{\lambda_k}} \varphi_{\lambda_k} \quad (x, t) \in M \times \mathbb{R}, \quad (3.9)$$

because $\Delta \tilde{\Phi}$ is harmonic in $M \times \mathbb{R}$, with respect to the metric $\tilde{g} = g \otimes dt$. This transformation (3.9) is crucially used for proving (3.6). A first attempt for proving (3.7) would be to adapt the proof of (1.18) then (3.8) with a boundary L^2 -Carleman-type inequality in the spirit of [JL99, Lemma 14.5]. Indeed, this boundary type estimate is useful for deducing an estimate of Φ from an estimate of $\tilde{\Phi}$.

A Proof of the Carleman estimates

In this part, we introduce standard notations inspired by Riemannian geometry. We do this because it simplifies the formulas appearing in the proof of the next lemmas. We set

$$\partial_i f = \frac{\partial f}{\partial x_i}, \quad \nabla_x f = (\partial_1 f, \dots, \partial_d f), \quad \nabla f = A \nabla_x f, \quad \operatorname{div}(\xi) = \sum_{i=1}^d \partial_i \xi_i \quad \text{and} \quad \Delta f = \operatorname{div}(\nabla f),$$

Let $A^{-1} = (a_{ij}(x))_{1 \leq i, j \leq d}$ the inverse matrix of coefficients of A .

For two vector fields ξ and η , we have

$$\xi \cdot \eta = \sum_{i, j=1}^d a_{ij}(x) \xi_i \eta_j, \quad |\xi|^2 = \xi \cdot \xi, \quad (A.1)$$

With these notations at hand, when f, h are smooth compactly supported functions, we have

$$\operatorname{div}(A(x) \nabla f) = \Delta f, \quad \Delta(f^2) = 2f \Delta f + 2|\nabla f|^2, \quad \int f \Delta h dx = \int h \Delta f dx = - \int \nabla f \cdot \nabla h dx.$$

The proof of Lemma 2.1 will follow the lines of the one of [EV03, Theorem 2], by keeping in the left-hand-side the whole anti-symmetric term of the conjugated operator, see Lemma A.1 for a precise formulation. Indeed, this term will then be exploited to give a more direct proof of Lemma 2.2 than the one of [DF90a, Lemma A].

A.1 The standard Carleman estimate

The main result of this section is the following Carleman estimate, that leads to Lemma 2.1.

Lemma A.1. *There exists a positive constant $C = C(A) > 0$, a radial increasing function $w = w(r)$ for $0 < \rho < 2$ satisfying*

$$C^{-1} \leq \frac{w(\rho)}{\rho} \leq C, \quad C^{-1} \leq |\partial_\rho w(\rho)| \leq C \quad \forall \rho \in (0, 2), \quad (\text{A.2})$$

such that for every $\alpha \geq C$, $f \in C_c^\infty(B_2 \setminus \{0\})$, the following estimate holds

$$\alpha^3 \int_{B_2} w^{-1-2\alpha} |f|^2 dx + \alpha \int_{B_2} w^{1-2\alpha} |\nabla_x f|^2 dx + \alpha^2 \int_{B_2} \frac{|\nabla w|^2}{w^2} |\mathcal{A}(g)|^2 dx \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx, \quad (\text{A.3})$$

where $g = w^{-\alpha} f$ and

$$\mathcal{A}(g) = \frac{w \nabla w \cdot \nabla g}{|\nabla w|^2} + \frac{1}{2} F_w^A g, \quad F_w^A = \frac{w \Delta w - |\nabla w|^2}{|\nabla w|^2}, \quad (\text{A.4})$$

together with the bound

$$|F_w^A| \leq C. \quad (\text{A.5})$$

Proof. We follow [EV03, Theorem 2], sometimes line by line.

Let $g = w^{-\alpha} f$ and we compute

$$w^{-\alpha} \Delta f = \Delta g + \frac{\alpha^2 |\nabla w|^2}{w^2} g + 2\alpha \frac{|\nabla w|^2}{w^2} \mathcal{A}(g), \quad (\text{A.6})$$

where the definition of $\mathcal{A}(g)$ is recalled in (A.4). We also set M_w^A the $d \times d$ symmetric matrix

$$M_w^A = \frac{1}{2} (M_{ij} + M_{ji})_{1 \leq i, j \leq d}, \quad (\text{A.7})$$

where, using the summation notation of repeated indices,

$$M_{ij} = \frac{1}{2} \operatorname{div} \left(\frac{w \nabla w}{|\nabla w|^2} \right) \delta_{ij} - \partial_{x_j} \left(\frac{w a^{ik} \partial_{x_k} w}{|\nabla w|^2} \right) + \frac{1}{2} a_{ik} \frac{w a^{kl} \partial_l w}{|\nabla w|^2} \partial_k a^{hi} - \frac{1}{2} F_w^A \delta_{ij}. \quad (\text{A.8})$$

We split the proof in several steps.

Step 1: A first identity. The goal of this step is to prove that

$$\int \frac{w^2}{|\nabla w|^2} (w^{-\alpha} \Delta f)^2 \geq 4\alpha \int M_w^A \nabla g \cdot \nabla g + \alpha \int F_w^A \Delta(g^2) + 4\alpha^2 \int \frac{|\nabla w|^2}{w^2} \mathcal{A}(g)^2. \quad (\text{A.9})$$

Let

$$J_1 = \int \left(2\alpha \frac{|\nabla w|}{w} \mathcal{A}(g) \right)^2, \quad J_2 = 2 \int \left[2\alpha \mathcal{A}(g) \left(\Delta g + \alpha^2 \frac{|\nabla w|^2}{w^2} g \right) \right]. \quad (\text{A.10})$$

Then, we have

$$\int \frac{w^2}{|\nabla w|^2} (w^{-\alpha} \Delta f)^2 \geq J_1 + J_2. \quad (\text{A.11})$$

First note that

$$\int \frac{|\nabla w|^2}{w^2} \mathcal{A}(g) \cdot g = 0, \quad (\text{A.12})$$

so that from (A.4) and the identity $\Delta(g^2) = 2g\Delta g + 2|\nabla g|^2$,

$$J_2 = 4\alpha \int \mathcal{A}(g)\Delta g = 2\alpha \int \left(\frac{2w\nabla w \cdot \nabla g}{|\nabla w|^2} \Delta g - F_w^A |\nabla g|^2 \right) + \alpha \int F_w \Delta(g^2). \quad (\text{A.13})$$

We now have the following identity

$$\begin{aligned} 2(\beta \cdot \nabla g)\Delta g &= 2\operatorname{div}((\beta \cdot \nabla g)\nabla g) - \operatorname{div}(\beta|\nabla g|^2) + \operatorname{div}(\beta)|\nabla g|^2 \\ &\quad - 2\partial_i \beta^k a^{ij} \partial_j g \partial_k g + \beta^k \partial_k a^{ij} \partial_i g \partial_j g, \quad \forall \beta \in C^\infty(B_2 \setminus \{0\}; \mathbb{R}^d), \end{aligned}$$

and we choose

$$\beta = \frac{w\nabla w}{|\nabla w|^2},$$

to get from (A.13) and the divergence theorem

$$4\alpha \int \mathcal{A}(g)\Delta g = 4\alpha \int M_w^A \nabla g \cdot \nabla g + \alpha \int F_w^A \Delta(g^2), \quad (\text{A.14})$$

where M_w^A is defined in (A.7) and (A.8). By gathering (A.10), (A.11), (A.12) and (A.14) we obtain (A.9) so the conclusion of Step 1.

Step 2: Choice of w . For $\mu \geq 1$, a parameter to be chosen later, let

$$\sigma(x) = \left(\sum_{i,j=1}^d a_{ij}(0)x_i x_j \right)^{1/2}, \quad \varphi(s) = s \exp\left(\int_0^s \frac{e^{-\mu t} - 1}{t} dt \right), \quad \phi(s) = \frac{\varphi(s)}{s\varphi'(s)} = e^{\mu s}. \quad (\text{A.15})$$

We define

$$w(x) = \varphi(\sigma(x)).$$

With this definition, we can now compute

$$M_\sigma^A \nabla \sigma = 0, \quad (\text{A.16})$$

$$F_w^A = F_\sigma^A \phi(\sigma) - \sigma \phi'(\sigma), \quad M_w^A = \phi(\sigma) M_\sigma^A + \sigma \phi'(\sigma) \left(I - \frac{\nabla \sigma \otimes \nabla \sigma}{|\nabla \sigma|^2} \right), \quad (\text{A.17})$$

and

$$F_\sigma^{A(0)} = n - 2, \quad M_\sigma^{A(0)} = 0. \quad (\text{A.18})$$

Notice that the following properties hold

$$\varphi'(r) > 0, \quad cr \leq \varphi(r) \leq Cr,$$

so we have

$$c\sigma \leq w(x) \leq C\sigma, \quad c \leq |\nabla w(x)| \leq C, \quad |\nabla |\nabla w|^2| \leq C, \quad |\Delta \phi| \leq Cw^{-1}, \quad |F_w| \leq C.$$

Now we estimate the first two terms appearing in the right hand side of (A.9).

Let us treat the first term. From the second part of (A.17), we have

$$M_w^A \nabla g \cdot \nabla g = \sigma \phi' \left(|\nabla g|^2 - \frac{(\nabla \sigma \cdot \nabla g)^2}{|\nabla \sigma|^2} \right),$$

and denoting by $\tilde{\nabla} g$ the tangential components of the gradient of g along the level sets of $\sigma(x)$ with respect to the metric $\sum_{i,j=1}^d a_{ij}(x) dx_i dx_j$, we have

$$\tilde{\nabla} g = \nabla g - \frac{\nabla \sigma \cdot \nabla g}{|\nabla \sigma|^2} \nabla \sigma = \nabla g - \frac{\nabla w \cdot \nabla g}{|\nabla w|^2} \nabla w. \quad (\text{A.19})$$

From (A.16), (A.18) and (A.19), we have that

$$M_\sigma^A \nabla g \cdot \nabla g = (M_\sigma^A - M_\sigma^{A(0)}) \tilde{\nabla} g \cdot \tilde{\nabla} g.$$

On the other hand, a computation and the Lipschitz condition on the matrix A give that there exists $C > 0$ depending on d and Λ_1, Λ_2 such that

$$|M_\sigma^A - M_\sigma^{A(0)}| \leq C\sigma,$$

so that

$$\int M_\sigma^A \nabla g \cdot \nabla g \geq \int \sigma (\phi' - C\phi) |\tilde{\nabla} g|^2 \quad (\text{A.20})$$

Now we estimate from below the second term appearing in (A.9). We observe that

$$F_w^A = (d-2)\phi + (B\phi - \sigma\phi'),$$

where $B = F_\sigma^A - F_\sigma^{A(0)}$ that satisfies by using the Lipschitz condition on the matrix A ,

$$|B(x)| \leq C\sigma.$$

So we have from the identities

$$\Delta g^2 = 2g\Delta g + 2|\nabla g|^2, \quad |\nabla g|^2 = |\tilde{\nabla} g|^2 + \frac{(\nabla w \cdot \nabla g)^2}{|\nabla w|^2},$$

that

$$\begin{aligned} \int F_w^A \Delta(g^2) &= (d-2) \int (\Delta\phi)g^2 + 2 \int (B\phi - \sigma\phi')g\Delta g \\ &\quad + 2 \int (B\phi - \sigma\phi')|\tilde{\nabla} g|^2 + 2 \int (B\phi - \sigma\phi') \frac{(\nabla \sigma \cdot \nabla g)^2}{|\nabla \sigma|^2}. \end{aligned} \quad (\text{A.21})$$

From (A.6), we then have that the second term of (A.21) writes as follows

$$\begin{aligned} 2 \int (B\phi - \sigma\phi')g\Delta g &= 2 \int (B\phi - \sigma\phi')g\Delta f w^{-\alpha} \\ &\quad + 2\alpha^2 \int (\sigma\phi' - B\phi) \frac{|\nabla w|^2}{w^2} g^2 + 2 \int (\sigma\phi' - B\phi) 2\alpha \frac{|\nabla w|^2}{w^2} g \mathcal{A}(g). \end{aligned} \quad (\text{A.22})$$

Thus, from (A.20), (A.21) and (A.22) we have for $\alpha \geq 1$,

$$\begin{aligned} 4\alpha \int M_w^A \nabla g \cdot \nabla g + \alpha \int F_w \Delta(g^2) \\ \geq 2\alpha \int (\sigma\phi' - 2C\sigma\phi + B\phi) |\tilde{\nabla} g|^2 + 2\alpha^3 \int (\sigma\phi' - B\phi) \frac{|\nabla w|^2}{w^2} g^2 - R_1, \end{aligned} \quad (\text{A.23})$$

where for some constant $C > 0$ depending on d, Λ_1, Λ_2 and μ ,

$$R_1 \leq C \left(\alpha \int w^{-1} g^2 + \alpha \int w^{1-\alpha} |g| |\Delta g| + \alpha^2 \int w^{-1} |\mathcal{A}(g)| |g| + \alpha \int w \frac{|\nabla \sigma \cdot \nabla g|^2}{|\nabla \sigma|^2} \right). \quad (\text{A.24})$$

By choosing $\mu = C(d, \Lambda_1, \Lambda_2) \geq 1$ sufficiently large, we then get that for some $c = c(d, \Lambda_1, \Lambda_2, \mu) > 0$ by using (A.23),

$$4\alpha \int M_w^A \nabla g \cdot \nabla g + \alpha \int F_w \Delta(g^2) \geq c \left(\alpha \int \sigma |\tilde{\nabla} g|^2 + 2\alpha^3 \int \sigma \frac{|\nabla w|^2}{w^2} g^2 \right) - R_1 \quad (\text{A.25})$$

Once we combine (A.9), (A.25) and (A.24), we obtain the following

$$\begin{aligned} 4\alpha^2 \int \frac{|\nabla w|^2}{w^2} A(g)^2 + 2\alpha \int \sigma \phi' |\tilde{\nabla} g|^2 + 2\alpha^3 \int \sigma \phi' \frac{|\nabla w|^2}{w^2} g^2 &\leq \int \frac{w^2}{|\nabla w|^2} (w^{-\alpha} \Delta f)^2 \\ &+ C \left(\alpha \int w^{-1} g^2 + \alpha \int w^{1-\alpha} |g| |\Delta g| + \alpha^2 \int w^{-1} |\mathcal{A}(g)| |g| + \alpha \int w \frac{|\nabla \sigma \cdot \nabla g|^2}{|\nabla \sigma|^2} \right). \end{aligned} \quad (\text{A.26})$$

This ends Step 2.

Step 3: Absorption. To conclude the proof, recall the form of $\mathcal{A}(g)$ and then

$$|F_w^A| \leq C. \quad (\text{A.27})$$

Thus

$$\frac{w}{|\nabla w|^2} \frac{|\nabla w \cdot \nabla g|^2}{|\nabla w|^2} \leq C w^{-1} |\mathcal{A}(g)|^2 + C \frac{|g|^2}{w}, \quad (\text{A.28})$$

so that

$$\begin{aligned} \alpha \int w \frac{|\nabla \sigma \cdot \nabla g|^2}{|\nabla \sigma|^2} &= \alpha \int w \frac{|\nabla w \cdot \nabla g|^2}{|\nabla \sigma|^2} \leq C\alpha \int w^{-1} |\mathcal{A}(g)|^2 + C\alpha \int w^{-1} |g|^2 \\ &\leq C\alpha \int \frac{|\nabla w|^2}{w} |\mathcal{A}(g)|^2 + C\alpha \int w^{-1} |g|^2. \end{aligned} \quad (\text{A.29})$$

Also,

$$C\alpha^2 \int w^{-1} |\mathcal{A}(g)| |g| \leq \frac{c}{2} \alpha^2 \int \frac{|\nabla w|^2}{w} |\mathcal{A}(g)|^2 + C\alpha^2 \int w^{-1} g^2, \quad (\text{A.30})$$

and

$$\alpha \int w^{(1-\alpha)} |g| |\Delta f| \leq \int w^2 (w^{-\alpha} |\Delta f|)^2 + C\alpha^2 \int w^{-1} g^2. \quad (\text{A.31})$$

From (A.26), (A.29), (A.30) and (A.31), conjugated with

$$\alpha^2 \int \frac{|\nabla w|^2}{w} |\mathcal{A}(g)|^2 \geq c\alpha^2 \int \frac{|\nabla w|^2}{w} |\mathcal{A}(g)|^2,$$

we obtain the expected inequality (A.3) and the bound (A.5) comes from (A.27). This concludes the proof. \square

A.2 The L^2 -Carleman estimate in a punctured domain

The goal of this part consists in proving Lemma 2.2. In the proof we assume that $A(0) = I_d$. Note that this is not a restriction because by a local change of variables we can drop the assumption that $A(0) = I_d$, replacing balls by ellipses, technical details are omitted here for an accurate argument.

Proof. We start from (2.1) to obtain first

$$\alpha^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} \frac{|\nabla w|^2}{w^2} |\mathcal{A}(g)|^2 dx \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx.$$

By using (A.2) and the assumption on f that vanishes in $B(0,r)$, this translates into

$$\alpha^2 r^{-2} \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\mathcal{A}(g)|^2 dx \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx.$$

Then we develop $|\mathcal{A}(g)|^2$ by (A.4) to get that

$$\alpha^2 r^{-2} \int_{B(0,r(1+2^{-1}\alpha^{-1}))} \left(\frac{w \nabla w \cdot \nabla g}{|\nabla w|^2} \right)^2 \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^{-2} \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |F_w^A g|^2.$$

By using that $A(0) = I_d$, the definition of σ in (A.15), the definition of the scalar product between two vector fields in (A.1) we have that

$$\nabla w \cdot \nabla g = A(\nabla_x |\rho|) \cdot A \nabla_x g = \nabla_x |\rho| \cdot A^2 \nabla_x g = \partial_\rho g + \nabla_x |\rho| \cdot (A^2(x) - I_d) \nabla_x g$$

By using the fact that A is Lipschitz, i.e. the assumption (1.15), we deduce from the two previous estimates that

$$\begin{aligned} \alpha^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\partial_\rho g|^2 \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^{-2} \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |F_w^A g|^2 \\ + C \alpha^2 r^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\nabla g|^2. \end{aligned} \quad (\text{A.32})$$

By integrating along a radial line and by using that g vanishes in $B(0,r)$, we obtain that

$$\alpha^4 r^{-2} \int_{B(0,r(1+2\alpha^{-1}))} |g|^2 \leq C \alpha^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\partial_\rho g|^2,$$

so from the two previous estimates

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0,r(1+2\alpha^{-1}))} |g|^2 \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^{-2} \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |F_w^A g|^2 + C \alpha^2 r^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\nabla g|^2. \end{aligned}$$

Then, for $\alpha \geq C$, using (A.5) one can absorb the second right hand side term to obtain

$$\alpha^4 r^{-2} \int_{B(0,r(1+2\alpha^{-1}))} |g|^2 \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^2 \int_{B(0,r(1+2^{-1}\alpha^{-1}))} |\nabla g|^2.$$

We now come back to the variable f to deduce

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0, r(1+2\alpha^{-1}))} |f|^2 w^{-2\alpha} \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^2 \int_{B(0, r(1+2^{-1}\alpha^{-1}))} w^{-2\alpha} (|\nabla f|^2 + \alpha^2 |f|^2). \end{aligned}$$

The second term can be absorbed recalling that $r \in (0, r_0)$ with $r_0 > 0$ sufficiently small. Then, we add to the left hand side

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0, r(1+2\alpha^{-1}))} |f|^2 w^{-2\alpha} + \int_{B(0, r(1+2\alpha^{-1}))} w^{2-2\alpha} |\Delta f|^2 dx \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^2 \int_{B(0, r(1+2^{-1}\alpha^{-1}))} w^{-2\alpha} |\nabla f|^2. \end{aligned}$$

So, by a L^2 local elliptic regularity estimate, using that $w^{-2\alpha}$ has the same amplitude in the annulus $B(0, r(1+2\alpha^{-1})) \setminus B(0, r)$, recalling that f vanishes in $B(0, r)$, we then obtain

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0, r(1+2\alpha^{-1}))} |f|^2 w^{-2\alpha} + \alpha^2 \int_{B(0, r(1+\alpha^{-1}))} w^{-2\alpha} |\nabla_x f|^2 dx \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx + C \alpha^2 r^2 \int_{B(0, r(1+2^{-1}\alpha^{-1}))} w^{-2\alpha} |\nabla f|^2. \end{aligned}$$

Therefore, one can absorb the last right hand side term recalling that $r \in (0, r_0)$ with $r_0 > 0$ sufficiently small to get

$$\begin{aligned} \alpha^4 r^{-2} \int_{B(0, r(1+2\alpha^{-1}))} |f|^2 w^{-2\alpha} + \alpha^2 \int_{B(0, r(1+\alpha^{-1}))} w^{-2\alpha} |\nabla_x f|^2 dx \\ \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx. \quad (\text{A.33}) \end{aligned}$$

By gathering (2.1) and (A.33) we get the expected result (2.2). \square

A.3 The modified version of the standard Carleman estimate

We have the following Carleman estimate whose proof is an easy adaptation of the one of Lemma A.1.

Lemma A.2. *There exists a positive constant $C = C(A) > 0$ such that for every $x_0 \in B(0, 1/4)$, there exists an increasing function w such that*

$$C^{-1} \leq \frac{w(|x - x_0|)}{|x - x_0|} \leq C,$$

such that for every $\alpha \geq C$, $f \in C_c^\infty(B_2 \setminus \{x_0\})$, the following estimate holds

$$\alpha^3 \int_{B_2} w^{-1-2\alpha} |f|^2 dx + \alpha \int_{B_2} w^{1-2\alpha} |\nabla_x f|^2 dx \leq C \int_{B_2} w^{2-2\alpha} |\Delta f|^2 dx.$$

Proof. The proof exactly follows the lines of Lemma A.1. Only the choice of w , coming from (A.15), is different. We instead set

$$\sigma(x) = \left(\sum_{i,j=1}^d a_{ij}(x_0) (x_i - x_{0,i})(x_j - x_{0,j}) \right)^{1/2}.$$

Then the next estimates are established in function of w that behaves as $|x - x_0|$. \square

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